Report of Doctoral thesis

Optimal control of Lévy-driven
stochastic equations in Hilbert spaces

by Karel Kadlec

Review

In the thesis under review, Karel Kadlec investigates optimal control problems for a class of stochastic evolution equations driven by a Lévy noise. More specifially, he considers equations of the form

$$dX^U(t) = \big(AX^U(t) + BU(t)\big)\,\mathrm{d}t + \Phi\,\mathrm{d}L(t) \qquad \text{for } t \geqslant 0. \tag{1.1}$$

Here, A is the generator of an analytic semigroup on a Hilbert space H and Φ is a linear and continuous operator on H. The random perturbation L is modelled by a cylindrical Lévy process with finite weak second moments or, more restrictive, by a genuine Lévy process with finite strong second moments. The control U is a predictable stochastic process taking values in a Banach space Y, and B is an operator defined on a subset of Y. The optimal control problem is to find an Y-valued predictable stochastic process U via the associated Riccati equation such that U minimises asymptotically the average costs per time unit of a certain given cost function $J(\cdot,t)$ depending on U and on the corresponding solution X^U. Here, the minimisation problem is either formulated in expectation or pathwise. For the Ricatti equation, the usual stabilisabilty and detectability conditions are imposed.

One of the first approaches to stochastic control problems of finite-dimensional systems was introduced by Bismut in [3]. At the same time, Ichikawa initiated the investigation of control problems for evolution equations of the above form with control dependent noise and Gaussian perturbation in [5] and [6]. Since these initial works, numerous publications are devoted to stochastic control problems of evolution equations of the form (1.1) and more general stochastic partial differential equations, however almost all of them are restricted to Gaussian perturbations. These optimal control problems have been attracted much attentions as they arise naturally in various fields such as engineering, economics, population dynamics, meteorology, physics and seismology among many others.

In the Gaussian case, the noise is typically modelled by a genuine Brownian motion in the Hilbert space H or by a cylindrical Brownian motion, i.e. a space-time white noise. In both cases, the noise follows a Gaussian distribution and does not allow for sudden changes of the random perturbation. However, statistical evidence, experimental observations and realistic constraints often suggest that the random noise needs to be more irregular. One might just think of the sudden slip on a fault which causes an earthquake, or sudden spikes in the stock market due to unexpected events. In finite-dimensional dynamical systems, this request for more general models of random perturbations is met by considering Lévy processes as driving noises. In infinite dimensions, the cylindrical Brownian motion is naturally extended to the class of cylindrical Lévy processes, recently introduced by Applebaum and Riedle in [2].

Karel is the first author investigating evolution equations of the form (1.1) driven by a cylindrical or genuine Lévy process. This extension from the classical case, in which the noise is modelled by a cylindrical or genuine Brownian motion, is not only essential for applications, in which a non-continuous and non-Gaussian perturbation is naturally required, but also provides important insights to the stochastic analysis of non-Gausian systems.

More specifically, Karel's contributions are the solutions of the above described control problems either in the case of a cylindrical Lévy process or of a genuine Levy process as driving noises. Naturally, some of the detailed assumptions on the involed coefficients differ in these cases. In the case of a cylindrical Lévy process, the control problem is described by finding a $D \in \mathbb{R}$ and a stochastic process U_0 in a class of admissible controls $\mathscr{U}$ such that $\lim_{t\to\infty} \frac{1}{t} E[J(U_0,t)] = D$, and

$$\liminf_{t\to\infty} \tfrac{1}{t} E[J(U,t)] \geqslant D \text{ for all } U \in \mathscr{U}. \tag{1.2}$$

In the case of a genuine Lévy process, the control problem is described by finding a $D \in \mathbb{R}$ and a stochastic process U_0 in a class of admissible controls $\mathscr{U}$ such that $\lim_{t\to\infty} \frac{1}{t} J(U_0,t) = D$ P-a.s., and

$$\liminf_{t\to\infty} \tfrac{1}{t} J(U,t) \geqslant D \ P\text{-a.s. for all } U \in \mathscr{U}. \tag{1.3}$$

In both cases of driving noises, the main ingredient for deriving the solution of the control problem is an analogue of Itô's formula for the weak (or mild) solution of equation (1.1). This is a highly challenging problem, since the solution exists only in a weak sense and thus, classical forms of Itô's formula in infinite-dimensional spaces are not applicable. Karel solves this elegantly by approximating the solution of (1.1) by the solution of an analogue equation with a more regular noise, similar to a Yoshida approximation of the noise. Since the standard form of Itô's formula can be applied to the solution of the approximating equation, he obtains the desired result by taking the limit. However, due to the fact that the cylindrical Lévy process exists only as a generalised stochastic process, the limit can be taken only in expectation in this case, whereas for a genuine Lévy process, the limit can be taken pathwise. This explains the different notions (1.2) and (1.3) of control problems for the two cases.

The thesis is well written and the mathematical arguments comprehensibly explained. In some parts, the thesis could be improved by a more detailed literature review. For example, as indicated above, stochastic optimal control problems have a long history and various approaches are well known and often investigated. A short summary of other results and approaches would be beneficiary for the thesis, and some related work on non-Gaussian noise could be compared, e.g. [8]. Similarly, the requirement of Itô's formula for weak or mild solutions of stochastic partial differential equations is a common problem in the analysis of infinite-dimensional dynamical systems. For example, the work [1] by Albeverio et al suggests a similar approach by a Yoshida approximation, whereas the publications [4] and [7] discuss Itô's formula in a general form in the variational approach.

Summary

The thesis by Karel Kadlec contains important contributions to the field of optimal control problems of stochastic equations. His results are novel and are based on highly challenging mathematical results and techniques from different areas of modern mathematical research. His results widen the field of possible applications of stochastic optimal control problems to phenomena which require a non-Gaussian and discontinuous random perturbation. The thesis under review clearly indicates the author's capability for independent and creative scientific work.

References

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