Posudek vedoucího doktorské práce/Supervisor's report on the doctoral thesis

Jaroslav Hančl: Additive Combinatorics and Number Theory

Doctoral thesis of Jaroslav Hančl revolves around the notion of ideals of combinatorial structures and their growth functions. The structures may be graphs, set partitions, posets, hypergraphs, integer partitions, and so on. In fact, the thesis is concerned with the last two kinds of structures. Each of these families of structures comes with a natural partial ordering on the structures, often called a containment. For example, for graphs this is usually the relation of a subgraph (or sometimes the relation of an induced subgraph). An ideal X of structures is then any set of structures that is downward closed with respect to the containment: if A is contained in B and B lies in X then A lies in X as well. With each ideal X we then associate the growth function that maps n to the number $|X_n|$ of structures with size n in X. Chapter 1 of the thesis concerns integer partitions, Chapter 2 is on k-uniform and, in more details 3-uniform, hypergraphs. The last Chapter 3 views growth functions of ideals more generally, for ideals in weighted posets. We now discuss each of the three chapters.

The topic of Chapter 1 "Ideals of Integer Partitions" is aptly described by its title. It introduces integer partitions and their containment: $\kappa \preceq \lambda$ iff one can obtain κ from λ by deleting some parts. History of integer partitions is briefly reviewed. The main result here is Theorem 1.12 establishing existence of a partition ideal with highly oscillating growth function $f(n)$ ($f(n) = 0$ for infinitely many n but $f(n)$ is also close to the maximum possible growth $p(n)$, the partition function, for infinitely many n). The proof is based on Theorem 1.8, interesting in itself, which gives asymptotics for the number of partitions of n not using parts from a fixed finite set S. An article, written solely by J. Hančl, based on results in this chapter was accepted for publication in the journal *Discrete Mathematics*.

The most substantial part of the thesis is Chapter 2 "Ideals of Ordered Graphs and Uniform Hypergraphs". "Ordered" means that vertex sets are linearly ordered and containment respects these linear orderings and l-colorings of the edges of complete k-uniform hypergraphs. Let n denote the number of vertices. In Theorem 2.11 it is proven that the growth function of any ideal X of edge l-colored complete k-uniform hypergraphs is either constant for large n, or $|X_n| \geq n - k + 2$ for every $n \geq k$. The main result of the chapter and of the whole thesis is Theorem 2.12: if X is an ideal of edge 2-colored complete 3-uniform hypergraphs, then either $|X_n| \leq n^c$ for some constant $c > 0$ and every natural number n (at most polynomial growth), or $|X_n| \geq G_n$ for every $n \geq 23$ (exponential growth) where G_n is determined by the recurrence $G_n = G_{n-1} + G_{n-3}$, $G_1 = G_2 = 1$ and $G_3 = 2$; the lower bound is tight in the sense that it is not hard to find an ideal with growth function equal to G_n. The proof of this theorem takes about 45 pages and is technically quite complicated. I can write this with good knowledge of the proof because this chapter is a result of joint efforts of J. Hančl and myself. I must add that his share on it is far larger than 50%. Getting this complicated proof right was actually the main reason why producing this thesis took some time. Results of Chapter 2 were submitted as a joint article to the *European Journal of Combinatorics*.

I will not say much on Chapter 3 "Wilf Equivalence", although it is from the perspective of a reader probably most attractive and is quite promising. It develops a general theory of Wilf Equivalence, that is of the kind of identities when the growth functions of two different ideals of structures coincide. The chapter contains several interesting results, especially Theorems 3.10, 3.13, 3.17 and 3.18 determining weight-preserving automorphisms of the poset of, respectively, integer partitions, compositions, words and sparse words. Results in the last chapter are in my opinion a solid base for a nice article.

I am quite confident that the results presented in the thesis far exceed requirements for awarding the title Ph.D.

Prague, August 20, 2020 Doc. RNDr. Martin Klazar, Dr.