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Additive Combinatorics and Number Theory

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Abstract: We present several results for growth functions of ideals of different combinatorial structures. An ideal is a set downward closed under a containment relation, like the relation of subpartition for partitions, or the relation of induced subgraph for graphs etc. Its growth function (GF) counts elements of given size.

For partition ideals we establish an asymptotics for GF of ideals that do not use parts from a finite set S and use this to construct ideal with highly oscillating GF. Then we present application characterising GF of particular partition ideals.

We generalize ideals of ordered graphs to ordered uniform hypergraphs and show two dichotomies for their GF. The first result is a constant to linear jump for k-uniform hypergraphs. The second result establishes the polynomial to exponential jump for 3-uniform hypergraphs. That is, there are no ordered hypergraph ideals with GF strictly inside the constant-linear and polynomial-exponential range. We obtain in both dichotomies tight upper bounds.

Finally, in a quite general setting we present several methods how to generate for various combinatorial structures pairs of sets defining two ideals with identical GF. We call these pairs Wilf equivalent pairs and use the automorphism method and the replacement method to obtain such pairs.

Keywords: Ideal, growth function, asymptotics, Wilf equivalence.

Contents

Notation

We present the notation of this thesis in three parts. First part contains general notation that contains some analytical and combinatorial notation and several constants. Second part is specific notation used in the thesis, divided by the chapter. Last part is a small introduction to the recurrent sequences we use in the thesis.

1. **Analysis**

Notation	Definition	Description
$\mathbb{N}$	$\{1, 2, \dots\}$	positive integers
$\mathbb{N}_0$	$\{0, 1, 2, \dots\}$	non-negative integers
$\mathbb{Z}$	$\{0, 1, -1, 2, -2, \dots\}$	integers
$\mathbb{R}$		real numbers
$\mathbb{R}^+$	$\{r \in \mathbb{R} : r > 0\}$	positive real numbers
(s, t)	$\{r \in \mathbb{R} : s < r < t\}, \ s, t \in \mathbb{R}$	open interval
$\langle s, t \rangle$	$\{r \in \mathbb{R} : s \leq r \leq t\}, \ s, t \in \mathbb{R}$	closed interval
$[a, b]$	$\{a, a+1, \dots, b\}, \ a, b \in \mathbb{Z}$	interval of integers
$\lfloor x \rfloor$	$\max\{n \in \mathbb{Z} : n \leq x\}$	floor function
$\lceil x \rceil$	$\min\{n \in \mathbb{Z} : n \geq x\}$	ceiling function
$\log x$	$\log_e x$	natural logarithm
$f(n) = o(g(n))$	$\lim_{n \to \infty} f(n)/g(n) = 0$	little o notation
$f(n) = O(g(n))$	$\limsup_{n \to \infty} |f(n)/g(n)| < \infty$	big O notation
$f(n) \sim g(n)$	$\lim_{n \to \infty} f(n)/g(n) = 1$	asymptotic equality

2. **Combinatorics**

Notation	Definition	Description
$[n]$	$\{1, 2, \dots, n\}$ and $[0] = \emptyset$	set of first $n \in \mathbb{N}_0$ positive integers
$[a, b]$	$\{a, a+1, \dots, b\}, \ a, b \in \mathbb{Z}$	interval of integers
$n!$	$1 \cdot 2 \cdot \ldots \cdot n$ and $0! = 1$	factorial of $n \in \mathbb{N}_0$
$\min(A)$	$a \in A$ s.t. $\forall b \in A, \ b \not< a$	minimal element of a set $A \subset \mathbb{R}$
$\max(A)$	$a \in A$ s.t. $\forall b \in A, \ b \not> a$	maximal element of a set $A \subset \mathbb{R}$
$A < B$	$a \in A, b \in B \Rightarrow a < b$	ordering of sets $A, B \subset \mathbb{R}$
$A \setminus B$	$\{x \in A : \ x \notin B\}$	difference of sets $A, B \subset \mathbb{R}$
$n + A$	$\{n + a : \ a \in A\}$	addition of $n \in \mathbb{N}$ to a set $A \subset \mathbb{R}$
$|X|$	$\sum_{x \in X} 1$	cardinality of a set X
$\binom{n}{k}$	$\frac{n!}{k!(n-k)!}$	binomial coefficient
$\binom{n}{k_1, k_2, \dots, k_i}$	$\frac{n!}{k_1! k_2! \cdots k_i!}$	multinomial coefficient
$\binom{A}{k}$	$\{X : \ X \subset A, \ |X| = k\}$	all k-element subsets of a set A
$[x^n] p(x)$		coefficient of x^n in a power series
$R_m(\alpha, l)$		Ramsey number for m-tuples, l colors
I_n		identity matrix
U_n		upper triangular matrix
$\cong$		isomorphism of sets

3. Constants

Notation	Value	Formula	Title
e	$2.71828\ldots$	$\sum_{n=0}^{\infty} \frac{1}{n!}$	Euler's number
π	$3.14152\ldots$	$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$	Ludolphian number
φ	$1.61803\ldots$	$\frac{1+\sqrt{5}}{2}$	golden ratio
C	$2.56509\ldots$	$\pi\sqrt{\frac{2}{3}}$	constant in the asymptotics of $p(n)$

4. Our notation for Chapter 1

Notation	Meaning	Numbering
$\mathcal{P}$	set of all partitions	Def. 1.2
$\mathcal{P}_n$	set of partitions of size n	Def. 1.2
$p(n)$	partition function	Def. 1.3
$p_S(n)$	number of partitions with parts in S	Def. 1.5
$p_{-S}(n)$	... with parts not in S	Def. 1.7
$<_s$	subpartition relation	Def. 1.9
$p(n, X)$	growth function of the ideal X	Def. 1.11
$p_{-Z}(n)$	growth function of the ideal F_Z	Def. 1.15

5. Our notation for Chapter 2

Notation	Meaning	Numbering
F_n	Fibonacci sequence	Def. below
G_n	Narayana sequence	Def. below
(n, χ)	coloring of n-vertex graph and k-graph	Def. 2.1 and 2.7
$\preceq$	graph and k-graph subcoloring relation	Def. 2.2 and 2.9
$\mathcal{C}_k$	set of colorings of n-vertex k-graph	Def. 2.8
$T_{f,g,h}$	rich coloring	Def. 2.16
W_1	wealthy colorings of the first type	Def. 2.24
W_2	wealthy colorings of the second type	Def. 2.27
W_3	wealthy colorings of the third type	Def. 2.32
W_4	wealthy colorings of the fourth type	Def. 2.41
$\mathrm{al}(N), \mathrm{al}(M)$	binary switches for lines	Def. 2.45 and 2.57
$R(N), C(N)$	binary switches for matrices	Def. 2.46
$R(M), C(M)$	binary switches for $*$-binary matrices	Def. 2.58
$\preceq$	containment of ($*$-binary) matrices	Def. 2.50 and 2.60
$M_{X,Y,Z}$	crossing matrix	Def. 2.55
$\overline{r}, \overline{c}, \overline{s}$	rows, columns and shafts in a matrix	Def. 2.56
$A(B)$	index set	Def. 2.63
$\mathrm{nu}(H)$	nuclear decomposition of H	Def. 2.75

6. **Our notation for Chapter 3**

Notation	Meaning	Numbering
$(P, \preceq, \lvert \cdot \rvert)$	sized poset	Def. 3.2
$F(B)$	ideal given by a forbidden set B	Def. 3.3
$\sim_W$	Wilf equivalence	Def. 3.4
$\downarrow\lambda$	downset of λ	Def. 3.9
$\uparrow\lambda$	upset of λ	Def. 3.9
$[k]^*$	set of words over alphabet $[k]$	Def. 3.14
$\preceq_t$	tight word containment relation	Def. 3.15
$\preceq_s$	sparse word containment relation	Def. 3.16
$\bigcup A$	union of a set A of partitions	Def. 3.20

7. **Fibonacci numbers (FN)**

The reader is probably familiar with the notion of Fibonacci numbers, but since the literature uses several conventions for the initial terms of FN, let us define them properly.

Let $n \in \mathbb{N}$. Fibonacci number F_n is n-th term in the recurrent sequence $(F_n)_{n \geq 1}$ given by the initial terms $F_1 = F_2 = 1$ and the recurrence

$$F_n = F_{n-1} + F_{n-2},$$

where $n \geq 3$. There are many formulas for FN, we only mention two explicit formulas

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} = \frac{\varphi^n + (1-\varphi)^n}{\sqrt{5}},$$

where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio. First formula interprets F_n as the number of ways how to write $n-1$ as an ordered sum of ones and twos. The second formula establishes that F_n asymptotically behaves like $\varphi^n/\sqrt{5}$. However, the convergence is so fast that $F_n = \lfloor \varphi^n/\sqrt{5} \rceil$ for any $n \in \mathbb{N}_0$. For formal reasons we also define $F_n = 0$ for any $n \leq 0$.

8. **Fibonacci numbers of higher order**

A useful generalization of FN allows any number of terms in the recursive formula. For any $n \in \mathbb{Z}$ and $k \geq 2$ we denote by $F_{n,k}$ the n-th Fibonacci number of order k such that $F_{n,k} = 0$ for $n \leq 0$, $F_{1,k} = 1$ and

$$F_{n,k} = F_{n-1,k} + F_{n-2,k} + \cdots + F_{n-k,k}$$

for any $n \geq 2$. We refer to the sequence $(F_{n,k})_{n \geq 1}$ as the Fibonacci sequence of order k. Observe that for $k = 2$ we have (standard) Fibonacci numbers. The speed of growth of $F_{n,k}$ is exponential with the base equal to the greatest real root of the polynomial $x^k - x^{k-1} - \cdots - x - 1$. For example, $F_{n,3}$ grows as 1.839^n and $F_{n,4}$ grows as 1.927^n.

9. **Narayana cow sequence**

In our result for uniform hypergraph ideals we use the Narayana cow sequence $(G_n)_{n\geq 1}$. Here, for any $n \in \mathbb{N}$, numbers G_n are defined such that $G_1 = G_2 = 1$, $G_3 = 2$ and

$$G_n = G_{n-1} + G_{n-3}$$

for any $n \geq 4$. Like Fibonacci numbers, the G_n are so close the exponential with base equal to the real root $\alpha \approx 1.465$ of $x^3 - x^2 - 1$ that $G_n = \lfloor c\alpha^n \rfloor$, where $c = 0.417\ldots$ is a constant. First terms of $(G_n)_{n\geq 1}$ are

$$(1,\ 1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 13,\ 19,\ 28,\ 41,\ \ldots).$$

For more information on $(G_n)_{n\geq 1}$ we refer to the sequence A000930 in OEIS database [39].

Preface

At the beginning of my PhD studies I was asked for the title of the thesis. As it was hard to say what particular problems I would try and be able to attack, my supervisor proposed rather general title "Additive Combinatorics and Number Theory". Although this title covers my research, one should specify the real content of the thesis.

Generally, you can find there results on growth functions of combinatorial structures. Particularly, the focus is concentrated mainly on two structures: integer partitions and ordered k-uniform hypergraphs. In both structures different aspects of growth functions are studied and the last chapter attempts to view these results globally, on a common ground.

List of author's publications used in this thesis

[A1] Jaroslav Hančl Jr. *Complementary Schur Asymptotics of Partitions.* Submitted to Discrete Mathematics, second revision.

[A2] Jaroslav Hančl Jr. and Martin Klazar. *On Growth Rates of Ordered Hypergraphs.* Submitted to European Journal of Combinatorics.

Original results of the thesis

All theorems presented in this thesis are of two types. On one hand, there are theorems that are properly cited and all authors are listed in the brackets. All other theorems are our own results. I write "our" since I worked with my supervisor on the paper that evolved into the second chapter and we discussed the other chapters thoroughly. This distinction also applies to propositions, corollaries, hypothesis and remarks.

The situation is little tricky with lemmas. Besides latter types, there are also lemmas that are well-known and easy to prove, but is hard to find the real author. For example see Lemma 2.29. Generally, my suggestion to the reader is that any Lemma that does not seem familiar is our original work.

Two submitted papers contain the results given in the thesis. First chapter is based on the results of [A1]. Second chapter presents the results of the paper [A2] that I wrote with my supervisor. Third chapter is new.

More precisely, in the first chapter we find the asymptotics for the number of partitions of n whose parts do not belong to a finite set of positive integers S. Then we show several applications. The most important one, the oscillation theorem, states that there is a partition ideal whose growth function is zero for infinitely many n as well as almost $p(n)$ for infinitely many n. That puts integer partitions in contrast with other combinatorial structures (e.g. graphs and permutations), where such huge initial oscillations are not possible.

Second chapter provides two dichotomies for ideals of ordered k-uniform hypergraphs. Two main results characterize the constant to linear, and the polynomial to exponential jump for the growth function of any given ideal. Particularly, for any $k \geq 2$, there is no ideal of ordered k-uniform hypergraphs with growth function that lies strictly between constant and linear growth. Similarly, there

is no ideal of ordered 3-uniform hypergraphs with the growth function that is superpolynomial but strictly smaller than a recurrent sequence (G_n), where G_n grows roughly as 1.465^n. In more details if X is an ideal of ordered 3-uniform hypergraphs such that its growth function is for some $n \geq 23$ smaller than G_n, then the growth function of X has only at most polynomial growth.

In the third chapter we attack a problem of finding ideals of various structures with identical growth functions. In contrast with the two previous chapters, the presented results may be applied for any ideal in any partially ordered set. We propose two methods that produces such ideals. One is based on the properties of automorphisms of posets and we enlist all automorphisms for permutations, partitions, compositions and words. The second method is based on a general partition identity due to D. I. A. Cohen and, independently, J. B. Remmel that we evolve into a replacement method.

1. Ideals of Integer Partitions

> There is no problem in all mathematics that cannot be solved by direct counting.
>
> *E. Mach, AMM 107, 2000*

1.1 History of integer partitions

The first mention of integer partitions dates back to 1674 when G. W. Leibniz wrote a letter [47] to J. Bernoulli. He found the number of partitions of numbers 3, 4, 5, and 6 and asked for more values. Since those four values $p(3) = 3$, $p(4) = 5$, $p(5) = 7$ and $p(6) = 11$ are primes, he asked if the number of partitions is always a prime number.

It is not hard to disprove this conjecture for $n = 7$, for which $p(7) = 15$. However, that wrong hypothesis opened the question if the number of partitions of n is a prime number for infinitely many positive integers n. Interestingly, while this question is still open today [7], the theory of partitions has became one of the main topics in number theory.

A century afterwards, in a letter [30] to L. Euler, mathematician Ph. Naudé asked for the number of partitions of 50 into seven distinct parts. That inspired Euler to introducing the most powerful tool to count integer partitions, generating functions. With its help, not only he could find the recurrence for the number of partitions of n into m parts (and answer Naudé's question), but he also proved many partition identities. The most famous one is called the Euler identity (we prove it in Theorem 3.5) and, for any positive integer n, equates the number of partitions into distinct parts and the number of partitions into odd parts. After vast correspondence with N. Benoulli and Ch. Goldbach [16], Euler also proved the pentagonal number theorem [32]. It states that, for any $n \neq j(3j \pm 1)/2$, the number of partitions of n with even number of distinct parts equals the number of partitions of n with odd number of distinct parts, and both numbers of partitions differ by ± 1 otherwise. For a modern proof see [6].

In the following years many people were trying to find an explicit formula for the number of partitions into at most k parts. A new insight into the structure of partitions was gained by J. J. Sylvester [68] who introduced Ferrers diagrams, also known as Young diagrams. That improved the combinatorial knowledge of partitions, wonderfully demonstrated by the Franklin's discovery of one-to-one mapping [34] between partitions with an even number of distinct parts and partitions with an odd number of distinct parts. That established a completely new proof of Euler's pentagonal identity.

The theory of integer partitions blossomed in the 20th century with the study of asymptotics. In 1918 Hardy and Ramanujan [37] proved an asymptotics for the number of partitions of n. A deep study of asymptotics of various sets of partitions followed. Soon afterwards Rogers-Ramanujan identities [58] and [59] motivated mathematicians to prove many partition identities. The result of this epoch is summarised in the book [4] written by G. E. Andrews in 1976, which

I personally call "The bible of integer partitions". Moreover, G. E. Andrews [7] also recorded the history of research of integer partitions. That article served as a background for this introduction.

1.2   Partitions and growth function

In this section we describe basic definitions of integer partition theory necessary for latter sections. We start with the definition of integer partitions, explain growth functions of particular sets of partitions and add relevant results.

Definition 1.1 (Integer partition). *Let n, k be positive integers. We say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$ is an integer partition of n if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. We call the numbers $\lambda_1, \ldots, \lambda_k$ parts of the integer partition λ, $n = |\lambda|$ the size of the integer partition λ and k the length of the integer partition λ.*

We usually use only a word "partition" instead of "integer partition". When unimportant or clear from the context, we omit the phrase "of n" and write only "partition". Now we focus our attention on the universe of all partitions and its partition function.

Definition 1.2 (Sets of partitions). *For a positive integer n we define the set of all partitions of n as $\mathcal{P}_n$. Moreover we set $\mathcal{P} = \cup_{n \in \mathbb{N}} \mathcal{P}_n$ to be the set of all partitions.*

Definition 1.3 (Partition function $p(n)$). *Let $n \in \mathbb{N}$. By $p(n)$ we denote the number of all integer partitions of size n. We call the function $n \mapsto p(n)$ the partition function.*

Note that $p(n) = |\mathcal{P}_n|$. For example, as mentioned in introduction, $p(5) = 7$ since 5 has seven partitions

$$
\begin{aligned}
5 &= 5 & &= 2+2+1 \\
&= 4+1 & &= 2+1+1+1 \\
&= 3+2 & &= 1+1+1+1+1 \\
&= 3+1+1.
\end{aligned}
$$

The next theorem due to Hardy and Ramanujan establishes the asymptotic behavior of $p(n)$ as $n \to \infty$. In the original theorem [37, formula (1.55)] Hardy and Ramanujan gave a stronger asymptotics than we present in the following theorem. They stated it as a value of a derivative with a precise error term. We compute the derivative explicitly.

Theorem 1.4 (Hardy–Ramanujan [37], 1918). *Let $n \in \mathbb{N}$,*

$$
\lambda_n := \sqrt{n - \frac{1}{24}}, \quad C := \pi\sqrt{\frac{2}{3}}, \quad \text{and let} \quad D \in (C/2, C)
$$

be an arbitrary constant. Then the partition function $p(n)$ satisfies

$$
p(n) = \frac{e^{C\lambda_n}}{4\pi\sqrt{2}\lambda_n^2}\left(C - \frac{1}{\lambda_n}\right) + O\left(e^{D\sqrt{n}}\right).
$$

Thus the growth of $p(n)$ has square root of n in the exponent. We call this growth "exponential", although we know that it is subexponential. However, the latter theorem implies that

$$p(n) \sim \frac{e^{C\sqrt{n}}}{4n\sqrt{3}}, \quad \text{where } C = \pi\sqrt{2/3} \approx 2.565 . \tag{1.1}$$

The error term in Theorem 1.4 is only a square root of the main term. This resembles strong asymptotic relations for coefficients of power series with unique dominant singularity (see P. Flajolet and R. Sedgewick [33, Chapter V] or the Fibonacci numbers).

After Hardy and Ramanujan published their result, the focus concentrated on the asymptotics of particular families of partitions. Amongst others, families with slowly growing growth functions were of particular interest. By growth function of a family $\mathcal{F}$ of partitions we usually mean the function that assign to any $n \in \mathbb{N}$ the number of partitions of size n that belong to $\mathcal{F}$.

An easy example is the family $\mathcal{F}_1$ of partitions that uses only 1 as a part. Its growth function is $p_1(n) = 1$ for all positive integers n. If we set $\mathcal{F}_2$ to be the family that uses only 1 and 2 as parts, then its growth function is $p_2(n) = \lfloor n/2 \rfloor + 1$. One can adopt the following general approach.

Definition 1.5 (Growth function $p_S(n)$). *Let S be a (finite) set of positive integers. By the number $p_S(n)$ we denote the number of partitions of n with all parts from the set S.*

As an example we denote by $S = \{1, 2, 5, 10, 20, 50\}$ the set of all coins of the Czech currency, called crown. Then the number of ways how to pay for the newspaper that costs 7 Czech crowns is $p_S(7) = 6$ since

$$
\begin{aligned}
7 &= 5 + 2 & &= 2 + 2 + 1 + 1 + 1 \\
  &= 5 + 1 + 1 & &= 2 + 1 + 1 + 1 + 1 + 1 \\
  &= 2 + 2 + 2 + 1 & &= 1 + 1 + 1 + 1 + 1 + 1 + 1.
\end{aligned}
$$

The asymptotic for growth function $p_S(n)$ was studied by Schur [62] who in 1926 proved the following theorem.

Theorem 1.6 (Schur [62], 1926). *Let S be a finite set of positive integers that consists of k relatively prime numbers. Then we have the asymptotics*

$$p_S(n) = \frac{n^{k-1}}{(k-1)! \prod_{s \in S} s} + O(n^{k-2}).$$

In this thesis we study an enumerative problem complementary to the problem of Schur, that is the asymptotics of the number of partitions in which we forbid parts from the set S of positive integers as parts. That is in contrast to the theorem of Schur, who only allowed numbers from S. Thus we need to define the growth function of such partitions.

Definition 1.7 (Growth function $p_{-S}(n)$). *For a (finite) set $S \subset \mathbb{N}$ of integers, let $p_{-S}(n)$ be the number of partitions of n not using any number from S as a part.*

As for $p(n)$ and $p_S(n)$, one can ask also for the asymptotics of $p_{-S}(n)$. We give it in the following theorem.

Theorem 1.8. *Let S be a finite set of integers of size $|S| = t$. Then the number $p_{-S}(n)$ of partitions of n with parts that do not belong to the set S satisfies*

$$p_{-S}(n) \sim p(n) \left(\frac{C}{2\sqrt{n}} \right)^t \prod_{s \in S} s.$$

We present our proof of Theorem 1.8 in the next chapter, alongside with its application on the construction of highly oscillating growth functions of partition ideals. Independently, this asymptotics was also proved in [29, Proposition on page 159], however, we were not aware of this result at the time of writing. So we present our version of the proof via the principle of inclusion and exclusion and the power expansion.

Many other asymptotics for families of partitions were found. Amongst others, the asymptotics for partitions with given multiplicity of parts [23] or partitions with parts congruent to particular values [50], [51], but it is not the object of this thesis. However, the interested reader can read chapter XIV in book [60] written by J. Sándor, D. Mitrinović and B. Crstici, where the recent results on asymptotics of many families of partitions are summarized.

In this chapter we present our main result on asymptotics of growth function of particular integer partition ideals. First we prove Theorem 1.8 that establishes the asymptotics for $p_{-S}(n)$, the number of partitions of n that do not use parts from a finite set S of positive integers. Then we construct a partition ideal with highly oscillating growth function. And finally we show the characterization results for the fast growing growth functions of partition ideals.

Our main result is based on the strong form of the Hardy–Ramanujan asymptotics for $p(n)$ given in Theorem 1.4 that implies asymptotic behavior $p(n) = O(e^{C\sqrt{n}}/n)$. From that we deduce the asymptotics of $p_{-S}(n)$. However, given asymptotics is not sufficient to prove Complementary Schur Theorem. For that we need the bound on error term from Theorem 1.4, power series and algebraic identities as described in Section 1.5.

1.3   Partition ideals

On the set of all partitions one defines a natural relation of being a subpartition and hence the notion of partition ideals.

Definition 1.9 (Subpartition relation)**.** *Let λ, γ be two partitions. We say that λ is a subpartition of γ if no part from λ has more occurrences in λ then in γ. We denote it by $\lambda \preceq \gamma$.*

Definition 1.10 (Partition ideal)**.** *A set X of partitions is called a partition ideal if any subpartition λ of a partition $\gamma \in X$ satisfies $\lambda \in X$.*

To our best knowledge, the notion of partition ideals was first mentioned by G. E. Andrews [4] in his book on partitions. G. E. Andrews also introduced the order of a partition ideal that serves as a local characteristic of the partition ideal meaning how tight are the multiplicities of parts that are close to each other. He

characterized partition ideals of order 1 and enumerated many sets of partitions that are partition ideals. Our main interest in partition ideals is the speed of growth functions.

Definition 1.11 (Growth function of ideal). *Let X be a partition ideal. By $p(n, X)$ we denote the number of partitions of n lying in X and call it the growth function of X.*

Our great goal is to characterise how growth functions of ideals behave. Given a partition ideal X we try to find a growth of $p(n, X)$. Is it eventually constant? Linear? Polynomial? Superpolynomial? "Exponential"? Or anything in between?

In fact, all these growths are possible. Indeed, there is a partition ideal X_1 using only part 1 and a partition ideal X_2 using only parts 1 and 2, their growth functions being $p_1(n) = 1$ and $p_2(n) = \lfloor n/2 \rfloor + 1$, respectively. Schur's asymptotics $p_S(n)$ gives polynomial growth and partition function $p(n)$ gives the "exponential" growth. We can even obtain wide range of results between polynomial and "exponential". As it is shown in [40], setting the set of allowed parts to be the set $S2 = \{1, 2, 4, 8, \dots\}$ of all powers of two, or set $SS = \{1, 4, 9, 16, \dots\}$ of squares of positive integers, or the set $SP = \{2, 3, 5, 7, \dots\}$ of all prime numbers, or the set $SE = \{2, 4, 6, 8, \dots\}$ of even numbers, we obtain

$$\log p_{S2}(n) \sim C_1 \log^2 n, \qquad \log p_{SS}(n) \sim C_2 \sqrt[3]{n},$$

$$\log p_{SP}(n) \sim C_3 \sqrt{\frac{n}{\log n}}, \qquad \log p_{SE}(n) \sim C_4 \sqrt{n}$$

where C_1, C_2, C_3 and C_4 are constants, and the last asymptotics holds only for even n ($p_{SE}(n)$ is zero for odd n).

This suggests that there are no big holes in the all possible growths of growth functions of partition ideals. Thus we focus only on the "exponential" growth; more precisely the top part of "exponential" growths. However, we first need tools to do that. Then we give the characterisation in Section 1.8.

In the following sections we prove the main Theorem 1.12 and show its applications. Next section introduces the main theorem. In Section 1.5 we prove our auxiliary result for $p_{-S}(n)$. Section 1.6 is devoted to the proof of Theorem 1.8. In Section 1.7 we prove our main Theorem 1.12. Last Section 1.8 consists of couple of applications of Theorem 1.8 to growth functions of partition ideals, especially growth functions where almost all partitions are allowed, and topics for further research.

1.4 Ideals with huge oscillations

Easily, one has $p(n, \emptyset) = 0$. As mentioned earlier, $p(n) = p(n, \mathcal{P})$ is the partition function with "exponential" asymptotics (1.1). That is the special case of growth function. Therefore

$$0 \leq p(n, X) \leq p(n) \tag{1.2}$$

holds for any partition ideal X. Let $p_{SE}(n)$ be the number of partitions that consists of only even parts. Then $\log p_{SE}(n) \sim C_4 \sqrt{n}$ holds for any even $n \in \mathbb{N}$,

while $p_{SE}(2n + 1) = 0$. Thus, one may attain lower bound of (1.2) whereas stay close to the upper bound at the same time. Hence, we are interested in finding the partition ideal whose growth function growths as fast as possible on one side, while remain infinitely many times zero on the other side.

Our next Theorem 1.12 shows that the growth function $p(n, X)$ of partition ideal may oscillate a lot. Particularly, we describe a partition ideal whose growth function has values ranging from zero to the $p(n(f(n)))$, where

$$ f(n) = \left(1 - \frac{\log^{1+\varepsilon} n}{\sqrt{n}} \right)^2 . $$

Thus $f(n)$ goes to 1 as fast as $n^{-1/2} \log^{1+\varepsilon} n$ goes to 0, which is a polynomial speed of growth. More precisely:

Theorem 1.12. *Let $\varepsilon > 0$ and $f(n)$ be as above. Then there is a partition ideal X such that both following claims are true.*

1. *There are infinitely many positive integers n such that $p(n, X) = 0$.*

2. *There are infinitely many positive integers n such that $p(n, X) > p(nf(n))$.*

The main tool for the proof of this theorem is Theorem 1.8, particularly its special case for $S = [t]$. Therefore we deeply study the growth function $p_{-[t]}(n)$ counting the number of partitions not using any part from the set $[t]$. Our Theorem 1.8 states that for a finite sets S of positive integers we have

$$ p_{-S}(n) \sim p(n) \left(\frac{C}{2\sqrt{n}} \right)^t \prod_{s \in S} s = p(n) \prod_{s \in S} \frac{Cs}{2\sqrt{n}} . $$

The latter result of Theorem 1.8 is the complementary problem of the so-called Schur asymptotics [62] that deals with the number $p_S(n)$. Schur proved that for finite S with cardinality $|S| = t$ (and such that the $\gcd(S) = 1$) one has the asymptotics

$$ p_S(n) \sim \frac{n^{t-1}}{(t-1)! \prod_{s \in S} s} = \frac{1}{n(t-1)!} \prod_{s \in S} \frac{n}{s}. $$

Note that adding a new element s to S increases $p_S(n)$ by the factor $\frac{n}{s|S|}$ while each element s in the previous asymptotics of $p_{-S}(n)$ decreased $p(n)$ by the factor $\frac{Cs}{2\sqrt{n}}$. This opens the question for the dependence $t = t(n)$ discussed at the end of this chapter.

1.5 Auxiliary results

Let $S \subset \mathbb{N}$ be a finite set and $s, t \in \mathbb{N}$ be positive integers. We first estimate $p(n-s)$ for a fixed $s \in \mathbb{N}$, and then express $p_{-S}(n)$ as a PIE sum of values $p(n-s)$ for various numbers s. At the end of this section we prove an algebraic identity needed later in the proof of the main result and complete the proof.

We set

$$ q(n) := \frac{e^{C\lambda_n}}{\lambda_n^2} \left(C - \frac{1}{\lambda_n} \right) . $$

to be the main term in Theorem 1.4, up to a constant factor. Thus $p(n) = O(q(n))$.

Lemma 1.13. *Let $s, t, n \in \mathbb{N}$ be natural numbers with $n > s$. We have*

$$p(n-s) = \frac{e^{C\sqrt{n}}}{4\pi n \sqrt{2}} \sum_{z=0}^{t} g(z,s) n^{-z/2} + O\left(e^{C\sqrt{n}} n^{-\frac{t+3}{2}}\right) \tag{1.3}$$

where for $z \in [t]$ we denote by $g(z,s)$ a real polynomial in s with degree z and leading term

$$g(z,s) = \frac{(-1)^z C^{z+1}}{2^z z!} s^z + h(z,s) \tag{1.4}$$

where $h(z,s)$ is a real polynomial in s with degree at most $z-1$.

Proof. Let $t \in \mathbb{N}$. We expand the main term in Theorem 1.4 in powers of n. Particularly, for $|x| < c < 1$, we use

$$e^x = \sum_{i \geq 0} \frac{x^i}{i!} = \sum_{i=0}^{l} \frac{x^i}{i!} + O(x^{l+1})$$

in the form of

$$
\begin{aligned}
e^{C\lambda_n} &= \exp\left\{ C\sqrt{n}\left(1 - \frac{1}{24n}\right)^{\frac{1}{2}} \right\} \\
&= \exp\left\{ C\sqrt{n} + c_1 n^{-1/2} + c_2 n^{-2/2} + \cdots + c_l n^{-l/2} + O(n^{-(l+1)/2}) \right\} \\
&= e^{C\sqrt{n}}\left(1 + c_1 n^{-1/2} + c_2 n^{-2/2} + \cdots + c_l n^{-l/2} + O(n^{-(l+1)/2}) \right), \\
\frac{1}{\lambda_n^2} &= \frac{1}{n(1 - 1/24n)} = c_1 n^{-1} + c_2 n^{-2} + \cdots + c_l n^{-l} + O(n^{-l-1}), \\
C - \frac{1}{\lambda_n} &= C - \frac{1}{\sqrt{n}(1 - 1/24n)^{1/2}} \\
&= C + c_1 n^{-1/2} + c_2 n^{-2/2} + \cdots + c_l n^{-l/2} + O(n^{-(l+1)/2}).
\end{aligned}
$$

Here, c_i are some real constants, not necessarily always the same. Therefore, for every $n > n_0(t)$ and some coefficients a_k we get

$$q(n) = \frac{e^{C\lambda_n}}{\lambda_n^2}\left(C - \frac{1}{\lambda_n}\right) = e^{C\sqrt{n}}\left(\sum_{k=0}^{t} a_k n^{-\frac{k}{2}-1} + O(n^{-\frac{t+3}{2}})\right). \tag{1.5}$$

Note that $a_0 = C$. For all integers $n > s > 0$ we have, expanding again

$$
\begin{aligned}
(n-s)^{-\frac{k}{2}-1} &= n^{-\frac{k}{2}-1}(1 - s/n)^{-\frac{k}{2}-1} \\
&= n^{-\frac{k}{2}-1}\left[1 - \frac{s}{n}\binom{-\frac{k}{2}-1}{1} + \cdots + \left(\frac{-s}{n}\right)^l \binom{-\frac{k}{2}-1}{l} + O(n^{-l-1})\right]
\end{aligned}
$$

by I. Newton's binomial theorem, that

$$
\begin{aligned}
&\sum_{k=0}^{2t} a_k (n-s)^{-\frac{k}{2}-1} \\
&= \sum_{k=0}^{2t} a_k \left(\sum_{l=0}^{t} \binom{-1-\frac{k}{2}}{l}(-s)^l n^{-1-\frac{k}{2}-l} + O(n^{-\frac{k}{2}-t-2}) \right) \\
&= \sum_{w=0}^{2t} f(w,s) n^{-w/2-1} + O(n^{-t-3/2}) \tag{1.6}
\end{aligned}
$$

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where

$$f(w, s) = \sum_{k, l \geq 0, \, k+2l=w} (-1)^l a_k \binom{-1 - \frac{k}{2}}{l} s^l \tag{1.7}$$

is a real polynomial in s with $\deg_s f(w, s) = \lfloor w/2 \rfloor$.

Next we combine expansions of the square root and of the exponential function, as above. For all integers $n > s > 0$, where s is fixed, we get

$$
\begin{aligned}
e^{C\sqrt{n-s}} &= \exp\left\{ C\sqrt{n}\left(1 - \frac{s}{n}\right)^{\frac{1}{2}} \right\} \\
&= \exp\left\{ C\sqrt{n} + \frac{c_1 s}{\sqrt{n}} + \frac{c_2 s^2}{n\sqrt{n}} + \frac{c_3 s^3}{n^2\sqrt{n}} + \cdots + \frac{c_{t-1} s^{t-1}}{n^{t-1}\sqrt{n}} + O(n^{-\frac{t+1}{2}}) \right\} \\
&= e^{C\sqrt{n}}\left(\sum_{i=0}^{t} d(i, s)n^{-i/2} + O(n^{-\frac{t+1}{2}}) \right) \tag{1.8}
\end{aligned}
$$

where c_i are some real constants, $c_1 = -C/2$, $d(i, s)$ is a polynomial in s with $\deg_s d(i, s) = i$, $d(0, s) = 1$ and

$$d(i, s) = \frac{\left(-\frac{C}{2}\right)^i}{i!} s^i + c_{i-1} s^{i-1} + \dots. \tag{1.9}$$

Terms $c_{i-1} s^{i-1} + \dots$ are the remaining terms with degree less than i. Combining results (1.5), (1.6) and (1.8) we have, for $n - s > n_0$,

$$
\begin{aligned}
q(n - s) &= e^{C\sqrt{n}}\left(\sum_{i=0}^{t} d(i, s)n^{-i/2} + O(n^{-\frac{t+1}{2}}) \right)\left(\sum_{w=0}^{2t} f(w, s)n^{-\frac{w+2}{2}} + O(n^{-t-2}) \right) \\
&= \frac{e^{C\sqrt{n}}}{n} \sum_{z=0}^{t} g(z, s)n^{-z/2} + O\left(e^{C\sqrt{n}} n^{-\frac{t+3}{2}} \right)
\end{aligned}
$$

where

$$g(z, s) = \sum_{i, w \geq 0, \, i+w=z} d(i, s)f(w, s) = \frac{(-1)^z C^{z+1}}{2^z z!} s^z + h(z, s) \tag{1.10}$$

is a real polynomial in s with $\deg_s g(z, s) = z$ and $h(z, s)$ are the remaining terms with $\deg_s h(z, s) \leq z-1$. Indeed, we have $\max_{i, w \geq 0, \, i+w=z}(i + \lfloor w/2 \rfloor) = z$, attained uniquely for $i = z$ and $w = 0$. Thus (1.7) and (1.9) imply (1.10). The exponent of C increased by one since $a_0 = C$. Therefore for all integers $n > s > 0$,

$$
\begin{aligned}
p(n - s) &= \frac{q(n - s)}{4\pi\sqrt{2}} + O\left(e^{D\sqrt{n}} \right) \\
&= \frac{e^{C\sqrt{n}}}{4\pi n\sqrt{2}} \sum_{z=0}^{t} g(z, s)n^{-z/2} + O\left(e^{C\sqrt{n}} n^{-\frac{t+3}{2}} \right).
\end{aligned}
$$

$\square$

We state and prove an identity needed in the proof of Theorem 1.8.

Lemma 1.14. *Let $t \geq z \geq 0$ be integers and $s_1, s_2, \ldots, s_t$ be variables. Then*

$$\sum_{J \subset [t]} (-1)^{|J|} \left(\sum_{i \in J} s_i \right)^z = \begin{cases} 0 & \text{for } z < t \\ (-1)^t t! \prod_{i=1}^t s_i & \text{for } z = t. \end{cases}$$

In fact, the first case holds more generally for any polynomial in $\sum_{i \in J} s_i$ with degree at most $t - 1$.

Proof. Let $k \leq t$ be positive integers, j_i with $1 \leq j_1 < j_2 < \cdots < j_k \leq t$ be some indices and $\alpha_1, \ldots, \alpha_k$ be positive integers with $\sum \alpha_i = z$, so $k \leq z$. We denote the polynomial on the left side as $f = f(s_1, \ldots, s_t)$ and examine the coefficient $T = [s_{j_1}^{\alpha_1} \cdots s_{j_k}^{\alpha_k}] f$. Clearly, only the sets J containing $\{j_1, \ldots, j_k\}$ contribute to T. Each such set J contributes $(-1)^{|J|} \binom{z}{\alpha_1, \ldots, \alpha_k}$. Summing over all J containing $\{j_1, \ldots, j_k\}$ we obtain

$$[s_{j_1}^{\alpha_1} \cdots s_{j_k}^{\alpha_k}] f = (-1)^k \binom{z}{\alpha_1, \ldots, \alpha_k} \sum_{l=0}^{t-k} (-1)^l \binom{t-k}{l}$$

where $l = |J \setminus \{j_1, \ldots, j_k\}|$. If $z < t$ then $k < t$ and the sum is $(1-1)^{t-k} = 0$ by the binomial theorem. If $z = t$ then only $k = z = t$ yields nonzero contribution for $\alpha_1 = \cdots = \alpha_k = 1$. Thus

$$[s_1 \cdots s_t] f = (-1)^t \binom{z}{1, \ldots, 1} = (-1)^t z! = (-1)^t t! .$$

That proves Lemma 1.14. $\qquad \square$

1.6 The complementary Schur theorem

Recall the statement of Theorem 1.8 that we call complementary Schur theorem.

Theorem 1.8. *Let S be a finite set of integers of size $|S| = t$. Then the number $p_{-S}(n)$ of partitions of n with parts that do not belong to the set S satisfies*

$$p_{-S}(n) \sim p(n) \left(\frac{C}{2\sqrt{n}} \right)^t \prod_{s \in S} s.$$

Proof of Theorem 1.8. Let $S = \{s_1, s_2, \ldots, s_t\}$ and $J \subset [t]$. Note that the partitions of n using each part s_j, $j \in J$, at least once are in bijection with the partitions of $n - \sum_{j \in J} s_j$. Thus, by the principle of inclusion and exclusion,

$$p_{-S}(n) = \sum_{J \subset [t]} (-1)^{|J|} p \left(n - \sum_{j \in J} s_j \right).$$

Lemma 1.13 yields

$$p_{-S}(n) = \sum_{J \subset [t]} (-1)^{|J|} \left(\frac{e^{C\sqrt{n}}}{4\pi n \sqrt{2}} \sum_{z=0}^t g \left(z, \sum_{j \in J} s_j \right) n^{-z/2} + O \left(e^{C\sqrt{n}} n^{-\frac{t+3}{2}} \right) \right)$$

$$= \frac{e^{C\sqrt{n}}}{4\pi n \sqrt{2}} \sum_{z=0}^t \sum_{J \subset [t]} (-1)^{|J|} g \left(z, \sum_{j \in J} s_j \right) n^{-z/2} + O \left(e^{C\sqrt{n}} n^{-\frac{t+3}{2}} \right).$$

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We apply Lemma 1.14 to the polynomial $g(z, \sum_{j \in J} s_j)$ when $z \in [t-1]$—first we understand s_j as variables and only at the end we substitute for them the numbers s_j—and obtain, by the first case of Lemma 1.14,

$$\sum_{J \subset [t]} (-1)^{|J|} g\left(z, \sum_{j \in J} s_j\right) = 0 \ .$$

Hence the only nonzero term of the first sum is for $z = t$. Thus

$$p_{-S}(n) = \frac{e^{C\sqrt{n}}}{4\pi n \sqrt{2}} \sum_{J \subset [t]} (-1)^{|J|} g\left(t, \sum_{j \in J} s_j\right) n^{-t/2} + O\left(e^{C\sqrt{n}} n^{-\frac{t+3}{2}}\right) \ .$$

Finally, we expand $g(t, \sum s_j)$ by equation (1.4) and use Lemma 1.14. By the first case of Lemma 1.14 the contributions of $h(z, \sum s_j)$ sum up to zero. By the second case the contribution of the leading term in $g(z, \sum s_j)$ is

$$\sum_{J \subset [t]} (-1)^{|J|} \frac{(-1)^t C^{t+1}}{2^t t!} \left(\sum_{j \in J} s_j\right)^t = \frac{(-1)^t C^{t+1}}{2^t t!} (-1)^t t! \prod_{i=1}^{t} s_i = \frac{C^{t+1}}{2^t} \prod_{i=1}^{t} s_i \ .$$

Thus

$$p_{-S}(n) = \frac{e^{C\sqrt{n}}}{4\pi n \sqrt{2}} \left(\frac{C^{t+1}}{2^t} \prod_{j=1}^{t} s_j\right) n^{-t/2} + O\left(e^{C\sqrt{n}} n^{-\frac{t+3}{2}}\right) \ ,$$

which in view that $C = \pi\sqrt{2/3}$ and $p(n) \sim e^{C\sqrt{n}}/4n\sqrt{3}$ gives the desired asymptotics

$$p_{-S}(n) \sim \frac{C e^{C\sqrt{n}}}{4\pi n \sqrt{2}} \left(\frac{C}{2}\right)^t n^{-t/2} \prod_{j=1}^{t} s_j = p(n) \left(\frac{C}{2\sqrt{n}}\right)^t \prod_{j=1}^{t} s_j \ .$$

$\square$

1.7 Proof of Theorem 1.12

Recall that in our main theorem we use $\varepsilon > 0$ and

$$f(n) = \left(1 - \frac{\log^{1+\varepsilon} n}{\sqrt{n}}\right)^2 \ .$$

The statement of the theorem is as follows.

Theorem 1.12. *Let $\varepsilon > 0$ and $f(n)$ be as above. Then there is a partition ideal X such that both following claims are true.*

1. *There are infinitely many positive integers n such that $p(n, X) = 0$.*

2. *There are infinitely many positive integers n such that $p(n, X) > p(nf(n))$.*

Proof. We define the sequences $(s_i)_{i=1}^{\infty}$ and $(t_i)_{i=1}^{\infty}$ of positive integers such that $s_1 = 2$,

$$s_{i+1} = t_i^3 + 2$$

and, given s_i, we set

$$t_i = \max\left\{ s_i + 1, \left\lceil \exp\left(\left(\frac{3s_i + 10}{2C} \right)^{1/\varepsilon} \right) \right\rceil, 2n_0 \right\} , \qquad (1.11)$$

where $n_0 = n_0(s_i)$ is such that for any $n \ge n_0$ we have $f(n) \in (1/2, 1)$,

$$2 \cdot \frac{e^{C\sqrt{n}}}{4n\sqrt{3}} > p(n) > \frac{1}{2} \cdot \frac{e^{C\sqrt{n}}}{4n\sqrt{3}} \qquad (1.12)$$

and

$$p_{-[s_i]}(n) > \frac{1}{2}p(n) \prod_{s=1}^{s_i} \frac{Cs}{2\sqrt{n}} . \qquad (1.13)$$

Existence of n_0 follows from the Hardy–Ramanujan asymptotics (1.1) and Theorem 1.8. Trivially, $s_i < t_i < s_{i+1}$.

Let $I_i = \langle s_i, t_i \rangle$. Let X be the partition ideal consisting of the partitions that use parts from any of the intervals I_i with multiplicities at most t_i, and do not use other parts. Our aim is to prove that the first condition is satisfied for $n = s_{i+1} - 1$ and the second condition is satisfied for $n = t_{i+1}$.

Any partition of $s_{i+1} - 1$ lying in X may use only parts $\le t_i$ but, as the multiplicities are restricted, sum of all the parts $\le t_i$ equals

$$\sum_{l=1}^{i} t_l \sum_{j=s_l}^{t_l} j \le t_i \sum_{j=1}^{t_i} j = \frac{t_i^2(t_i + 1)}{2} < s_{i+1} - 1 .$$

Hence easily $p(s_{i+1} - 1, X) = 0$ for any positive integer i.

Let $K = (8\sqrt{3})^{-1}$. To prove the second property we first show that for a fixed positive integer i and any $n \ge t_i$ we have

$$p(n)^{1-\sqrt{f(n)}} > n^{3s_i/2} . \qquad (1.14)$$

Indeed, the definition of $f(n)$ implies $\sqrt{n}(1 - \sqrt{f(n)}) = \log^{1+\varepsilon} n$ and $0 < f(n) < 1$, which, combined with the definition of t_i and n_0, yields for any $n \ge t_i$ bound

$$p(n)^{1-\sqrt{f(n)}} n^{-3s_i/2} \overset{(1.12)}{>} K^{1-\sqrt{f(n)}} e^{C\sqrt{n}(1-\sqrt{f(n)})} n^{-1+\sqrt{f(n)}-3s_i/2}$$

$$> K e^{C\log^{1+\varepsilon} n} n^{-1-3s_i/2}$$

$$= K n^{C\log^{\varepsilon} n - 1 - 3s_i/2} \overset{(1.11)}{\ge} K n^4 > 1,$$

by $1 > K > \frac{1}{16}$ and $n \ge 2$.

Now Theorem 1.8 and (1.14) imply

$$p(t_i, X) \ge p_{-[s_i]}(t_i) \overset{(1.13)}{>} \frac{1}{2}p(t_i) \left(\frac{C}{2\sqrt{t_i}} \right)^{s_i} s_i! \overset{(1.14)}{>} \frac{1}{2}p(t_i)^{\sqrt{f(t_i)}} \left(\frac{Ct_i}{2} \right)^{s_i} s_i!.$$

We apply, again, asymptotics (1.1) for $p(t_i)$ and $p(t_i f(t_i))$ where both $t_i, t_i f(t_i) > n_0$, and have

$$p(t_i)^{\sqrt{f(t_i)}} \overset{(1.12)}{>} \left(\frac{e^{C\sqrt{t_i}}}{8\sqrt{3}t_i} \right)^{\sqrt{f(t_i)}} \overset{(1.12)}{>} \left(\frac{1}{8\sqrt{3}t_i} \right)^{\sqrt{f(t_i)}} 2t_i f(t_i)\sqrt{3}p(t_i f(t_i))$$

$$= \frac{1}{4}(8t_i\sqrt{3})^{1-\sqrt{f(t_i)}} p(t_i f(t_i)) > \frac{1}{4}p(t_i f(t_i)) .$$

Putting these results together we get that

$$p(t_i, X) > \frac{1}{16} p(t_i f(t_i)) \left(\frac{Ct_i}{2}\right)^{s_i} s_i! > p(t_i f(t_i))$$

since $s_i \geq 2, Ct_i \geq 6$. That completes the proof. □

1.8 Growth functions of partition ideals

In this section we present couple of interesting applications of Theorem 1.8. Usually, it is the case of the family where all but a small number of partitions are allowed. We also make use of the Cohen–Remmel theorem [26, 56] that provides equalities between growth functions of partitions ideals satisfying particular union-size condition. Recall that for two partitions λ and γ the notion $\lambda \preceq \gamma$ means that λ is a subpartition of γ.

Definition 1.15 (Forbidden set of an ideal). *Let X be a partition ideal. Let Z be a set of all minimal partitions of $\mathcal{P} \setminus X$ with respect to $\preceq$. Then we say that Z is the set of forbidden partitions of X or the basis of X. We refer to this connection by the notation $X = F_Z$. We denote the growth function of X as*

$$p_{-Z}(n) = p(n, X) .$$

Clearly, any ideal has a unique basis. On the other hand, for any set of incomparable partitions Z there is a partition ideal X such that $X = F_Z$.

Note that the notation $p_{-S}(n)$ differs from notation $p_{-Z}(n)$ since S is a set of positive integers and Z is a set of partitions. However, in the case when Z consists only of mutually different partitions with only one part, then $p_{-S}(n) = p_{-Z}(n)$ for all $n \in \mathbb{N}$.

Definition 1.16 (Union of partitions). *Let $\lambda^1, \ldots, \lambda^i \in \mathcal{P}$ be partitions. Their union is the partition λ such that the multiplicity of any part equals to the maximal multiplicity attained over all λ^i.*

We state the Cohen–Remmel theorem [26, 56] that gives sufficient condition for equality of growth functions of two partition ideals in terms of their bases. Recall that for a partition λ, the size $|\lambda|$ is the sum of all parts of λ.

Theorem 1.17 (Cohen [26], 1981; Remmel [56], 1982). *Let $\Lambda = \{\lambda^1, \lambda^2, \ldots\}$ and $\Gamma = \{\gamma^1, \gamma^2, \ldots\}$ be two finite or infinite sequences of partitions (of the same length), such that for every finite set $I \subset \mathbb{N}$,*

$$\left| \bigcup_{i \in I} \lambda^i \right| = \left| \bigcup_{i \in I} \gamma^i \right| .$$

Then $p_{-\Lambda}(n) = p_{-\Gamma}(n)$ for every positive integer n.

We study this theorem more thoroughly in Section 3.3. The proof immediately follows from Theorem 3.23. Now we use it only for the applications of Complementary Schur Theorem. We define the notion of independent partitions.

Definition 1.18 (Independent partitions)**.** *We say that two partitions $\lambda, \gamma \in \mathcal{P}$ are independent if their parts are pairwise distinct. When two partitions are not independent we call them dependent.*

Easily, two independent partitions are not in a subpartition relation. But we note that dependent partitions may or may not be comparable by $\preceq$. Now we have tools to prove two applications of our Theorem 1.8. We are inspired by similar results of Hančl [40].

Theorem 1.19. *Let F_Z be a partition ideal with basis Z. Let Z consist of infinitely many pairwise independent partitions, and let k be any positive integer. Then*

$$p(n, F_Z) < e^{C\sqrt{n}} n^{-k}$$

for any sufficiently large n and $C = \pi\sqrt{2/3}$.

Proof. Let $k \in \mathbb{N}$ and $t = 2k - 1$. Let $\lambda^1, \lambda^2, \ldots, \lambda^t$ be mutually independent partitions from Z such that $|\lambda^1| < |\lambda^2| < \cdots < |\lambda^t|$. By Theorem 1.17, applied to $\Lambda = \{\lambda^1, \lambda^2, \ldots, \lambda^t\}$ and $\Gamma = \{(|\lambda^1|), (|\lambda^2|), \ldots, (|\lambda^t|)\}$,

$$p_{-Z}(n) \leq p_{-\Lambda}(n) = p_{-\Gamma}(n).$$

Using Theorem 1.8 for $S = \{|\lambda^1|, |\lambda^2|, \ldots, |\lambda^t|\}$ we have for sufficiently large n

$$p_{-\Gamma}(n) = p_{-S}(n) < 2Ke^{C\sqrt{n}} n^{-1-t/2} < e^{C\sqrt{n}} n^{-k}$$

where $K = C^t 2^{-t-2} 3^{-1/2} \prod_{i=1}^{t} |\lambda^i|$ is a constant obtained from Theorem 1.8 and asymptotics (1.1). $\qquad\square$

We note that the independence is a necessary condition since for $W = \{(1)\}$ and $Z = \cup_{i=1}^{\infty} \{(1, i)\}$ we have $p_{-Z}(n) = p_{-W}(n)$ for all $n \in \mathbb{N}$ but $n = 1$.

Theorem 1.20. *Let Z be a finite set of pairwise independent partitions with mutually different lengths. Then the asymptotics for the growth function $p_{-Z}(n)$ is of the form*

$$p_{-Z}(n) \sim Ke^{C\sqrt{n}} n^{-1-\frac{|Z|}{2}},$$

where $K = K(Z)$ is a constant.

Proof. As in the previous proof we first use the Cohen–Remmel Theorem 1.17 to transfer the assumptions such that Theorem 1.8 can be used. Let $k = |Z|$ and $\lambda^1, \lambda^2, \ldots, \lambda^k$ be partitions from Z. Since they are mutually independent and their lengths are mutually different, Theorem 1.17 applied to $\Lambda = \{\lambda^1, \lambda^2, \ldots, \lambda^k\}$ and $\Gamma = \{(|\lambda^1|), (|\lambda^2|), \ldots, (|\lambda^k|)\}$ gives

$$p_{-Z}(n) = p_{-\Lambda}(n) = p_{-\Gamma}(n).$$

The asymptotics of $p_{-\Gamma}(n)$ can be estimated from Theorem 1.8 since Γ consists of only k different positive integers, hence

$$p_{-\Gamma}(n) = p_{-S}(n) \sim Ke^{C\sqrt{n}} n^{-1-k/2}$$

where $S = \{|\lambda^1|, |\lambda^2|, \ldots, |\lambda^k|\}$. That completes the proof. $\qquad\square$

Let us demonstrate outcomes of Theorem 1.20 with an example. We set $Z = \{(2,8),(3,3),(4,5),(7)\}$. Partitions of Z are mutually independent, their lengths are Z are $10,6,9,7$, hence

$$p_{-Z}(n) \sim Ke^{C\sqrt{n}}n^{-3}.$$

The condition on independence is still necessary. That is witnessed by bases $W = \{(1)\}$ and $Z = \{(1,1),(1,2)\}$, for which we easily have $p_{-Z}(n) > p_{-W}(n)$ but Theorem 1.19 would claim asymptotical behaviour $p_{-Z}(n) < p_{-W}(n)$. On the other hand, we believe that the condition on different lengths can be left out. In this case we suggest to generalize Theorem 1.8 for multisets S and use it for the proof of following conjecture.

Hypothesis 1.21. *Let Z be a finite set of pairwise independent partitions. Then there is a constant K such that*

$$p_{-Z}(n) \sim Ke^{C\sqrt{n}}n^{-1-\frac{|Z|}{2}}.$$

As already mentioned, it is also quite challenging to prove results similar to Theorem 1.8 for infinite set S. Then, the asymptotics may depend not only on the size of S but also on the density of S or the structure of S. We start with the simplest situation where $S = [t]$. Dixmier and Nicolas [27] used complex analysis and proved that

$$p_{-[t]}(n) = p(n)\left(\frac{C}{2\sqrt{n}}\right)^t t! \, e^{\frac{-(t+1)^2}{4\sqrt{n}}\left(2C+\frac{1}{2C}\right)}$$

for any $t \leq n^{1/3-\varepsilon}$ and Nicolas and Sarközy [53] improved this type of result even for $t \leq \sqrt{n}$. We note that in the latter result [53] there is a small mistake in the asymptotics of $p_{-[t]}(n)$ on pages 232–233 where they incorrectly base the exponential in $C/\sqrt{n}$ instead of $C/2\sqrt{n}$. However, the paper [29] they refer to does not have same mistake, so we suppose that it is only a typographical mistake.

For more general set S. Our result in Theorem 1.8 was also proved by Erdős, Nicolas and A. Sárközy [29, Proposition on page 159]. They gave the asymptotics of $p_{-S}(n)$ for set S that satisfy $\sum_{s\in S} s = \kappa n$ for a fixed real constant κ. However, we have not found any results depending on the density of S. Thus we propose the following question:

Question 1.22. *Find the largest sublinear function $g(n)$ (e.g. square root, logarithm ...) such that for any set S of integers that satisfy the density condition*

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty} |\{s \in S: \ s < n\}| = O(g(n))$$

the growth function $p_{-S}(n)$ of partitions of n with parts that do not belong to the set S has asymptotics

$$p_{-S}(n) \sim p(n)\left(\frac{C}{2\sqrt{n}}\right)^{S(n)} \prod_{s\in S} s.$$

We proved that in Theorem 1.8 that for $g(n) = const$ the asymptotics is as claimed. But even for $g(n) = \log n$ the answer is unclear. And it is even more challenging to maximize $g(n)$.

An interesting question is where the asymptotics for $p_S(n)$ and $p_{-S}(n)$ meet. Particularly, what is the density of S in terms of n. Hence we pose following question.

Question 1.23. *Find a set of positive integers S such that*

$$p_{-S}(n) \sim p_S(n)$$

for sufficiently large n. Even more challenging is to find S for which the equality instead of asymptotics holds.

2. Ideals of Ordered Graphs and Uniform Hypergraphs

> ... the branch of study we now call "graph theory".
>
> *S. Hollingdale, London, 1989*

In this chapter we describe the behavior of growth function of ordered graphs and ordered k-uniform hypergraphs. We are interested in a phenomenon of their growth function that is called the constant to polynomial jump and polynomial to exponential jump.

Throughout the literature, authors usually call ideals of ordered graphs as hereditary properties of ordered graphs and growth functions as speeds. However, we stay with our notation of ideals and growth because of the connection to ideals of integer partitions described earlier.

An original inspiration for the research of ideals of ordered graphs was found in different combinatorial structure: permutations. Atkinson [8] begun enumerating permutations in permutation ideals and Kaiser and Klazar [41] precisely described all possible growths of ideals of permutations up to the size 2^{n-1}. Later, the combined result of Klazar [45] and Marcus and Tardos [49] established the so called Stanley-Wilf conjecture that was formulated in late 1980. It claims that for any permutation π the number of all permutation of size n that avoids π is at most exponential. Thus, any nontrivial ideal of permutations grows at most exponentially. That is a nice example of exponential to factorial jump.

Similar characterization holds for the ideals of labelled graphs; that is graphs where vertices are labelled but not linearly ordered. We give the full characteristic in the next section. One may also find study of ideals of oriented graphs [3], posets [21, 11] and words [55, 10].

The structure of this chapter is as follows. In Section 2.1 we give an introduction to ordered graphs and sum up known results. Section 2.2 introduces the generalization of ideals of ordered graphs called ideals of ordered k-uniform hypergraphs and states two main results, Theorem 2.11 and Theorem 2.12.

In Section 2.3 we prove the constant to linear jump of Theorem 2.11. Section 2.4 contains the proof of Theorem 2.12 that is divided into three parts. In first part we introduce wealthy colorings in Section 2.4.1. Second part introduces three-dimensional matrices in Section 2.4.2 and tame colorings in Section 2.4.3. And in last part we prepare lemmas in Section 2.4.4 and finalize the proof in Section 2.4.5. Last Section 2.5 analyzes the proof and concludes with several hypothesis.

2.1 Ideals of ordered graphs

Let $n \in \mathbb{N}$ and G be a complete ordered graph on n vertices; that is vertices of G are ordered linearly and labelled by numbers from $[n]$. We color all edges of G with l colors. However, instead of using ordered graphs, we use the notation of a coloring $K = (n, \chi)$ that can be interpreted as an edge 2-coloring of G.
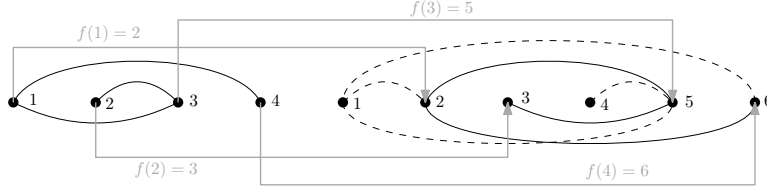
Figure 2.1: An example of containment of two black/white colored graphs. The containment f is marked by gray arrows. Thus $f([4]) = \{2, 3, 5, 6\}$. Dashed edges are those with no pre-image under f.

Definition 2.1 (Coloring). *We say that the pair $K = (n, \chi)$ is a coloring of $[n]$ if n is a positive integer and $\chi : \binom{n}{2} \to [l]$ colors pairs of $[n]$. Let n be the size of K and χ be the coloring function of K. We denote by $\mathcal{C}_2$ the set of all colorings.*

Trivially, the set $\mathcal{C}_2$ of colorings coincides with the set of ordered edge l-colored complete graphs on n vertices. Hence we may work with colorings instead of graphs. On $\mathcal{C}_2$ we define a subcoloring relation that coincides with induced subgraph relation that maintains the order of vertices.

Definition 2.2 (Subcoloring). *We say that a coloring $L = (m, \phi)$ is a subcoloring of a coloring $K = (n, \chi)$, denoted by $L \preceq K$, if there is an increasing mapping $f : [m] \to [n]$ such that $\phi(u, v) = \chi(f(u), f(v))$ for any distinct $u, v \in [m]$.*

Note that one can obtain L from K by deleting some vertices (and all incident edges) and renaming the rest in the same order. One may check the Figure 2.1.

Definition 2.3 (Ideal of colorings, growth function). *Let X be the set of colorings. We say that X is an ideal if it is closed under subcoloring relation $\preceq$. That is if $K \in X$ and $L \preceq K$ then $L \in X$.*

Let X_n be the set of all colorings that belong to X and have size $n \in \mathbb{N}$. By $|X_n|$ we denote the number of elements of X_n and we call the function $n \mapsto |X_n|$ the growth function of X.

Other terms in use for sets of graphs (resp. colorings) closed to a containment relation are hereditary or monotone properties. One may also consider the graph G_0 with no vertices and hence set $|X_0| = 1$ for any graph ideal X. However, from now on, we formulate our results for $n \geq 1$ and do not count G_0 as an element of any ideal of graphs.

Example. Let $F = (5, \theta)$ be the coloring that assign to any pair of $[5]$ the color black. Then there are five mutually different subcolorings of F, namely colorings $C_i = (i, \theta)$, where $i \in [5]$ and θ colors all pairs black. Therefore the set

$$X = \{(i, \theta) : i \in [5]\}$$

is an ideal and $|X_i| = 1$ for any $i \in [5]$. $\triangle$

Balogh, Bollobás and Morris [12, 11] proved an astonishingly precise characterization of growth for ordered graphs. It describes possible growth in the "lower" region below 2^{n-1}. They were inspired by Kaiser and Klazar [41] who proved similar result for permutations.

Theorem 2.4 (Balogh, Bollobás and Morris [11, 12], 2007)**.** *Let X be an ideal of ordered graphs. Then one of the following holds.*

(a) *$|X_n|$ is bounded and eventually constant for any $n \geq n_0$.*

(b) *$|X_n|$ is a polynomial such that $|X_n| \geq n$ for every $n \in \mathbb{N}$ and there exist $k, a_1, \ldots, a_k \in \mathbb{N}$ such that $|X_n| = \sum_{i=0}^{k} a_i \binom{n}{i}$ for any $n \geq n_0$.*

(c) *$|X_n|$ is exponential such that there is $2 \leq k \in \mathbb{N}$ and a polynomial $p(n)$ for which $F_{n-1,k} \leq |X_n| \leq p(n)F_{n-1,k}$ for every $n \in \mathbb{N}$.*

(d) *$|X_n| \geq 2^{n-1}$ for every $n \in \mathbb{N}$.*

In this theorem, $F_{n,k}$ is the generalized Fibonacci number given by the recurrence

$$F_{n,k} = F_{n-1,k} + \cdots + F_{n-k,k}$$

and initial terms $F_{n,k} = 0$ for $n \leq 0$ and $F_{1,k} = 1$. See the notation.

Independently, Klazar [44] proved the linear to polynomial and polynomial to exponential jump from Theorem 2.4 in terms of ideals of colorings. We present these two results here, since our main theorems are its direct generalizations.

Theorem 2.5 (Klazar [44], 2008)**.** *Let $l = 2$ and X be an ideal of colorings (i.e. hereditary property of ordered graphs). Following two statements are true.*

1. *Either $|X_n|$ is eventually constant or $|X_n| \geq n$ for all $n \geq 1$.*

2. *There is a constant $c > 0$ such that either $|X_n| < n^c$ for every $n \geq 2$ or $|X_n| \geq F_{n-1}$ for every $n \geq 1$.*

Recall that F_n is a Fibonacci number, as described in notation. Note that both claims of Theorem 2.5 are also corollaries of Theorem 2.4. Our main results (Theorem 2.11 and Theorem 2.12) are a close analog to the previous theorem. We actually use some parts of the proof of Theorem 2.5 in our proof. However, before we state our results, we need to clear and generalize some notation.

Latter results follows up the study of ideals of graphs that are not ordered. Indeed, omitting the word ordered from previous definitions of ideals of ordered graphs we remain with labelled graphs for which the labels do not form an linearly ordered set.

Scheinerman and Zito [61] were the first to investigate the full spectrum of growth of ideals of labelled graphs. Their results were extended and made more precise by, among others, Balogh, Bollobás and Weinreich [14, 15], Alekseev [2], Bollobás and Thomason [18, 19], and Prömel and Steger [54]. Putting all these results together, one obtain following superresult.

Theorem 2.6 (All authors above [2, 14, 15, 18, 19, 54, 61], 1993-2013)**.** *Let X be an ideal of labelled graphs. Then one of the following growth is true for $|X_n|$.*

(A) *Polynomial to exponential growth $|X_n| = \sum_{i=1}^{k} p_i(n)i^n$, where $p_i(n)$ is a collection of polynomials.*

(B) *Nearly factorial growth $|X_n| = n^{(1-1/k+o(1))n}$ for some $k \geq 2$.*

(C) *Then there is a large gap* $n^{(1+o(1))n} = B_n \leq |X_n| \leq 2^{o(n^2)}$, *where* B_n *is the* n-*th Bell number.*

(D) *Almost maximal growth* $|X_n| = 2^{(1-1/k+o(1))\binom{n}{2}}$ *for some* $k \geq 2$.

(E) *Maximal growth* $|X_n| = 2^{\binom{n}{2}}$.

Even older is the theory of ideals of unlabelled graphs, starting with the results written by Macpherson [48] and more recently by Balogh et al. [13]. Moreover, growth functions of ideals were studied also for oriented graphs [3], posets [21, 11], words [55, 10] and permutations [41, 49].

2.2   Ideals of ordered k-uniform hypergraphs

In this section we state our own results that generalizes a constant to polynomial jump and polynomial to exponential jump in Theorem 2.4, resp. Theorem 2.5 to ordered k-uniform hypergraphs. We start with basic definitions. As in the latter section, instead of ordered edge-colored complete k-uniform hypergraphs we use the notion of coloring $K = (n, \chi)$, where χ colors k-tuples of $[n]$. We also use the term (ordered) hypergraphs or, more precisely, (ordered) k-uniform hypergraphs, but we always mean ordered edge-colored complete k-uniform hypergraphs.

Definition 2.7 (Coloring of k-uniform hypergraphs). *Let* $k, l \geq 2$ *be positive integers. A coloring* $K = (n, \chi)$ *is a pair, where* $n \in \mathbb{N}$ *and* $\chi : \binom{[n]}{k} \to [l]$ *gives to each* k-*element subsets of* $[n]$ *one of the* l *colors.*

Definition 2.8 (Set $\mathcal{C}_k$ of colorings). *Let* $k \geq 2$. *We denote by* $\mathcal{C}_k$ *the set of all colorings* $K = (n, \chi)$ *over all* $n \in \mathbb{N}$ *and all possible coloring functions* $\chi : \binom{[n]}{k} \to [l]$.

The only difference between graph colorings and k-uniform hypergraph colorings is that the coloring function colors k-tuples while graph coloring colors pairs. We usually use term colorings regardless of the value k which should be clear from the context. We need to extend the notation of subcoloring and ideal.

Definition 2.9 (Subcoloring for uniform hypergraphs). *Let* $\preceq$ *be the relation on* $\mathcal{C}_k$ *equivalent to the induced subgraph. That is, we say that* $(m, \phi) \preceq (n, \chi)$ *if there is an increasing mapping* $f : [m] \to [n]$ *such that* $\phi(E) = \chi(f(E))$ *for every* $E \subset [m]$ *with* $|E| = k$.

Definition 2.10 (Ideal of uniform hypergraphs and its growth function). *Let* $X \subset \mathcal{C}_k$ *be a set of colorings,* $n \in \mathbb{N}$ *and* X_n *be the subset of* X *containing all colorings of size* n. *We call* $X \subset \mathcal{C}_k$ *an ideal if it is downward closed to* $\preceq$. *That is,* $K \preceq L \in X$ *implies* $K \in X$. *Moreover, we say that* $n \mapsto |X_n|$ *is the growth function of* X.

Example. Let $k \geq 2$ be an integer. We consider the set $S(k) \subset \mathcal{C}_k$ of colorings (n, χ), where $\chi : \binom{[n]}{k} \to \{0, 1\}$, such that $(n, \chi) \in S(k)$ if and only if the subsets $E \subset [n]$ with $|E| = k$ and $\chi(E) = 1$ are pairwise disjoint intervals in $[n]$. Clearly, $S(k) \subset \mathcal{C}_k$ is an ideal. $\triangle$

We study growth functions of ideals in $\mathcal{C}_k$. In this section we present a generalized version of Theorem 2.5. Our first original result concerns ideals $X \subset \mathcal{C}_k$ for general $k \geq 2$. We prove that the growth of X is either eventually constant or at least linear.

Theorem 2.11. *If $X \subset \mathcal{C}_k$ is an ideal then either $|X_n| = c$ for every $n > n_0$ or $|X_n| \geq n - k + 2$ for every $n \geq k$.*

Both bounds in the latter theorem are tight. On the one hand, the ideal X, that contains all colorings with at most one black edge E that is an interval, has the growth $|X_n| = n - k + 2$ for any $n \geq k$. On the other hand, we take all colorings of X where the black edge E may start only on first $t - 1$ vertices. Then all such colorings form an ideal Y with growth $|Y_n| = t$ for all $n \geq t + k - 2$.

In our second original result we take $k = 3$ and recall Narayana's cow sequence $(G_n)_{n \geq 1} = (1, 1, 2, 3, 4, 6, 9, 13, \dots)$ that is given by the recurrence $G_n = G_{n-1} + G_{n-3}$ for $n \geq 4$ and initial terms $G_1 = G_2 = 1$, $G_3 = 2$. We prove that growth function of ideals of edge 2-colored 3-uniform hypergraphs grow at most polynomially or at least as fast as G_n.

Theorem 2.12. *Let $l = 2$ and $X \subset \mathcal{C}_3$ be an ideal of 2-colored 3-uniform hypergraphs. Then either there is a constant $c > 0$ such that $|X_n| \leq n^c$ for every $n \in \mathbb{N}$, or $|X_n| \geq G_n$ for every $n \geq 23$.*

The upper bound G_n is tight since the latter example shows that $|S(3)_n| = G_n$ because $|S(k)_n|$ equals to the number of ordered tuples $(a_1, a_2, \dots, a_m) \in \mathbb{N}^m$ with $a_i \in \{1, k\}$ and $a_1 + a_2 + \cdots + a_m = n$.

Now we give a proof of Theorem 2.11 in Section 2.3. The proof of Theorem 2.12 is presented in Section 2.4. However, it is qiute difficult so we divide it into five subsections.

2.3   Constant to polynomial jump for k-graphs

She structure of the proof of Theorem 2.11 is as follows. We give a definition of *r*-rich colorings and *c*-simple colorings. Rich colorings contain many subcolorings of a particular size, while simple colorings do not. That is the content of Lemma 2.17 and Lemma 2.19. The rest of the section is devoted to auxiliary lemmas that culminates in Proposition 2.23 that splits ideals into those containing *r*-rich colorings for any $r \geq k$ and those that contain only *c*-simple colorings.

We start with basic notations. Note that $n - E + 1 = \{n - a + 1 \ : \ a \in E\}$.

Definition 2.13 (Reversal coloring)**.** *Let $K = (n, \chi) \in \mathcal{C}_k$ be a coloring. The reversal of K is the coloring (n, ϕ) where $\phi(E) = \chi(n - E + 1)$ for every $E \subset [n]$ with $|E| = k$.*

Definition 2.14 (Monochromatic sets)**.** *Let $(n, \chi) \in \mathcal{C}_k$ be a coloring. We say that a set $A \subset [n]$ is χ-homogeneous or χ-monochromatic if $\chi(E)$ is constant for any $E \in \binom{A}{k}$.*

We omit 'χ-' if it is clear from the context. We write $\chi\left(\binom{A}{k}\right) = \chi(A) = c$ if χ colors every k-set of A with c.

Definition 2.15 (Restriction and normalization). *Let $C = (n, \chi)$ be a coloring and X be a subset of $[n]$. We define a new coloring $D = ([|X|], \chi')$ by restricting χ to k-subsets of X and relabeling the elements of X in increasing order as $1, 2, \ldots, |X|$. Thus $D \preceq C$. We say that D arises by restriction and normalization of C to X.*

Now we approach to r-rich and c-simple colorings.

Definition 2.16 (r-rich coloring of type $T_{f,g,h}$). *Let $f, h \in \mathbb{N}_0$ and $r, g \in \mathbb{N}$ satisfy $k = f + g + h$ and $r \geq k$. We set*

$$E_i = [f] \cup [f + i, f + g + i - 1] \cup [n - h + 1, n], \; i = 1, 2, \ldots, r - k + 2.$$

A coloring (n, χ) is r-rich of type $T_{f,g,h}$ if $n = 2r - k + 1$ and for two colors $a \neq b$ one has $\chi(E_i) = a$ for $i = 1, 2, \ldots, r - k + 1$ and $\chi(E_{r-k+2}) = b$. A coloring (n, χ) is r-rich if it is r-rich of type $T_{f,g,h}$ for some $f, h \in \mathbb{N}_0$ and $g \in \mathbb{N}$.

Note that every E_i is a k-set because $r \geq k$ and hence $n - h + 1 > r - h + 1 \geq f + g + i - 1$. All sets E_i differs only in the middle part $[f + i, f + g + i - 1]$, which "moves" with increasing i.

Colors of other edges in (n, χ) are not restricted. The reason for defining r-rich colorings is that they produce linearly many mutually different subcolorings of given size.

Lemma 2.17. *If an ideal $X \subset \mathcal{C}_k$ contains an r-rich coloring for every $r \geq k$ then $|X_n| \geq n - k + 2$ for every $n \geq k$.*

Proof. Let $X \subset \mathcal{C}_k$ be an ideal, $r \geq k$, $n = 2r - k + 1$, and $K = (n, \chi) \in X$ be an r-rich coloring of type $T_{f,g,h}$. We consider the $r - k + 2$ colorings $C_j = (r, \chi_j) \in X_r$, $j \in [r - k + 1]_0$, obtained from K by deleting $r - k + 1$ numbers from $[n]$, j of them immediately after $[f]$ and $r - k + 1 - j$ of them immediately before $[n - h + 1, n]$, and normalising the remaining elements of K. For $i \in [r - k + 1]$ we set

$$L_i = [f] \cup [f + i, f + i + g - 1] \cup [r - h + 1, r].$$

Therefore

$$(\chi_0(L_i), i \in [r - k + 1]) = (a, a, \ldots, a, a, a),$$
$$(\chi_1(L_i), i \in [r - k + 1]) = (a, a, \ldots, a, a, b),$$
$$(\chi_2(L_i), i \in [r - k]) = (a, a, \ldots, a, b),$$
$$\vdots$$
$$(\chi_{r-k+1}(L_i), i \in [1]) = (b)$$

by the definition of K and C_j. The colorings C_j, $j \in [r - k + 1]_0$, are pairwise distinct and hence $|X_r| \geq r - k + 2$. $\square$

On the other hand, there are colorings that produce at most constantly many different subcolorings of any given size. They may be defined as follows.

Definition 2.18 (c-simple colorings and c-simple ideals). *Let $c \geq k$ be a positive integer. A coloring (n, χ) is c-simple if the following holds.*

C1. The set $[c+1, n-c]$ is χ-homogeneous.

C2. For every $k-1$ distinct vertices $v_1, v_2, \ldots, v_{k-1} \in [n]$ where $v_1 \in [c] \cup \langle n - c + 1, n \rangle$, there is a color c such that $\chi(\{v_1, \ldots, v_{k-1}, w\}) = c$ for all $w \in [2c+1, n-2c]$, $w \neq v_1, \ldots, v_{k-1}$.

An ideal X is c-simple if all its colorings are c-simple.

Any coloring with $n \leq \min(2c+k, 4c+1) = 2c+k$ is trivially c-simple. Indeed, the set in C1 has at most k elements or the set for w in C2 is empty.

Lemma 2.19. *If an ideal X is c-simple then $|X_n|$ is eventually constant.*

Proof. Let X be a c-simple ideal. For a coloring $C = (n, \chi) \in X$ with $n \geq 5c+1$ we consider the coloring $C' = (n-1, \chi')$ obtained by restricting and normalizing C to $[n] \setminus \{2c+1\}$. So $C' \preceq C$ and $C' \in X$. We show that for $n > 5c$ the correspondence $C \mapsto C'$ is injective. Hence for $n > 5c$ the numbers $|X_n|$ weakly decrease and the claim follows.

Let $n > 5c$. Let $C = (n, \chi)$ and $D = (n, \psi)$ be distinct colorings from X. Thus $\chi(E) \neq \psi(E)$ for some edge $E \in \binom{[n]}{k}$. If $2c+1 \notin E$ then E survives in both C' and D' hence $C' \neq D'$. Let $2c+1 \in E$. We have either $E \subset [c+1, n-c]$ or there is an $u \in E \cap ([c] \cup [n-c+1, n])$. In the former case, since $n > 5c$ implies $|[c+1, n-c]| > 3c > k$, we find an $x \in [c+1, n-c] \setminus E$ and set $F = (E \setminus \{2c+1\}) \cup \{x\}$. Similarly, in the latter case, since $|[2c+1, n-2c]| > c \geq k$, there is an $x \in [2c+1, n-2c] \setminus E$ and we set $F = (E \setminus \{2c+1\}) \cup \{x\}$. By c-simplicity of X, in either case $\chi(F) = \chi(E) \neq \psi(E) = \psi(F)$. Since F survives in both C' and D', again $C' \neq D'$. $\qquad\square$

Now we need to prove that the ideal X is either c-simple or contains infinitely many r-rich colorings. That is the ague statement of Proposition 2.23. Before that we need a couple of auxiliary lemmas.

Lemma 2.20. *Let $r \geq k$ be an integer, (n, χ) be a coloring, and $H \subset [n]$ be a χ-homogeneous set of the maximum cardinality. Suppose that $A \subset H$ arises by deleting $k(r-k+1)$ elements both from the beginning and the end of H. Suppose that $A \neq \emptyset$ and that A is not an interval in $[n]$. Then (n, χ) contains an r-rich coloring.*

Proof. By the assumption on A we may assume that $|H| \geq 2k(r-k+1)+2$. We have $2k+1$ pairwise disjoint sets

$$H = B_1 \cup \cdots \cup B_k \cup A \cup C_1 \cup \cdots \cup C_k$$

where $B_1 < \cdots < B_k < A < C_1 < \cdots < C_k$ and $|B_i| = |C_i| = r - k + 1$, $i \in [k]$. Since A is not an interval, an element $e \in [n] \setminus H$ exists with $B_k < e < C_1$. Since $|H|$ is maximum with respect to the monochromaticity, there is a k-set E with $e \in E$, $E \setminus \{e\} \subset H$, and $\chi(E) \neq \chi(H)$. We select such E that has the minimum number of elements greater than e. It follows from $|E \cap H| = k - 1$ that there are two indices $i_0, j_0 \in [k]$ with $E \cap B_{i_0} = E \cap C_{j_0} = \emptyset$. We define

$$E^- = E \cap (\cup_{i=1}^{i_0-1} B_i), \quad E^+ = E \cap (\cup_{j=j_0+1}^{k} C_j), \quad \text{and} \quad E_0 = E \setminus (E^- \cup E^+)$$

be the beginning, ending, and middle part of E, respectively. Clearly $E_0 \neq \emptyset$ as $e \in E_0$. Consider the k-set E' obtained from E by substituting $\max B_{i_0}$ for $\max E_0$. It follows that $\chi(E') = \chi(H)$ — either by the minimality property of E if $\max E_0 > e$, or by $E' \subset H$ if $\max E_0 = e$. Repeatedly shifting the middle part E_0 of E to the left in B_{i_0} (in the second step we substitute $B_{i_0} - 1$ for $\max(E' \setminus (E^- \cup E^+))$, and so on) we obtain $r - k + 1$ sets with size k and the same color $\chi(H)$. We define $D = B_{i_0} \cup E \cup C'$ where C' is any subset of C_{j_0} with size $|C'| = r - k$ that completes D to the right cardinality $|D| = 2r - k + 1$. Then $(|D|, \chi')$, obtained by restriction and normalization of (n, χ) to D, is an r-rich coloring of type $T_{|E^-|, |E_0|, |E^+|}$ and is contained in (n, χ). $\qquad \square$

Lemma 2.21. *Let r with $r \geq k$ be an integer, (n, χ) be a coloring, and s be the maximum size of a χ-homogeneous subset of $[n]$. Let $A \subset [n]$ be χ-homogeneous with size $|A| = s - 2k(r - k + 1)$ and $B \subset [n]$ be χ-homogeneous with $A < B$ or $B < A$. If $|A| \geq k(r - k + 1)$ and $|B| \geq (2k + 2)r$ then (n, χ) contains an r-rich coloring.*

Proof. Let A and B satisfy $|A| \geq k(r - k + 1)$ and $|B| \geq (2k + 2)r$. Note that then both $|A|, |B| \geq k(r - k + 1) \geq r$. We assume that $A < B$, the case $A > B$ is treated by passing to the reverse coloring.

The first case is when $\chi(A) = a \neq b = \chi(B)$. We take the last k vertices of A and the first k vertices of B, $a_k < \cdots < a_1 < b_1 < \cdots < b_k$, and consider the colors $\chi(F_i) = \chi(\{a_{k-i}, \ldots, a_1, b_1, \ldots, b_i\})$ for $0 \leq i \leq k$. Clearly, $\chi(F_0) = a$ and $\chi(F_k) = b$. Let $t \in [k]$ be the first index with $\chi(F_t) \neq a$ and let D consist of the last $r - t + 1$ vertices of A and the first $r - k + t$ vertices of B ($|A|, |B| \geq r$). Then $|D| = 2r - k + 1$ and the first $r - k + 1$ intervals in D of size k have color a but the next one has $b \neq a$. Thus $(|D|, \chi')$, obtained by restriction and normalization of (n, χ) to D, is an r-rich coloring of type $T_{0,k,0}$ and is contained in (n, χ).

The second case is when $\chi(A) = \chi(B) = a$. Consider the coloring $L = (|A \cup B|, \psi)$ obtained by restricting and normalizing (n, χ) to $A \cup B$. Let $A' = [|A|]$ and $B' = [|A| + 1, |A \cup B|]$ be the counterparts of A and B in the domain of L. Since $L \preceq (n, \chi)$, it suffices to find an r-rich coloring in L. We split the first $k(r - k + 1)$ vertices of A' and the last $k(r - k + 1)$ vertices of B' in tuples A_i and B_i, respectively, satisfying

$$A_1 < A_2 < \cdots < A_k < B_1 < B_2 < \cdots < B_k, \ |A_i| = |B_i| = r - k + 1.$$

Since

$$|A'| + |B'| - (r - k + 1) \geq s - 2k(r - k + 1) + (2k + 2)r - (r - k + 1)$$
$$= s - (2k + 1)r + (2k + 2)r > s \,,$$

there is an edge $F = \{f_1 < \cdots < f_k\} \subset A' \cup B'$, such that $f_k < B_k$ and $\psi(F) \neq a$. Note that $f_k \in B'$. Among all such edges F we take one with minimal last element f_k. Since $|F \cap A'| \leq k - 1$, we may take an index $i_0 \in [k]$ such that $F \cap A_{i_0} = \emptyset$. We set $D = A_{i_0} \cup F \cup B_k^-$ and $t = |F \cap (\cup_{l=1}^{i_0-1} A_l)|$ where B_k^- is an arbitrary subset of B_k such that $|B_k^-| = r - k$. Shifting the part of F after A_{i_0} to the left in A_{i_0}, like in the proof of Lemma 2.20, and using minimality of the last element f_k, we get an r-rich coloring of the type $T_{t,k-t,0}$. $\qquad \square$

By $R_m(\alpha, l)$ we denote the Ramsey number for m-tuples and l colors. That is, $R_m(\alpha, l)$ is the smallest $n \in \mathbb{N}$ such that every l-coloring of $\binom{[n]}{m}$ has a homogenous set $A \subset [n]$ of size α.

Lemma 2.22. *Let $r \geq k$, $R = \max\{R_i(r-1, l), i = 1, 2, \ldots, k-1\}$ and (n, χ) be a coloring. Let $A \subset [n]$ be a set with $|A| \geq 2(k-1)R$, $v_1, \ldots, v_{k-1} \in [n]$ be distinct vertices such that $v_1 < A$ or $v_1 > A$ and let $A' \subset A$ arise by deleting both the first and last $(k-1)R$ elements of A. Suppose that not all edges $E_w = \{v_1, \ldots, v_{k-1}, w\}$, where $w \in A' \backslash \{v_1, \ldots, v_{k-1}\}$, have the same color. Then (n, χ) contains an r-rich coloring.*

Proof. Let $v_1 < A$, the case $v_1 > A$ is symmetric. We relabel the vertices so that $v_1 < \cdots < v_{k-1}$. By the assumption we have two (distinct) vertices $w_1, w_2 \in A'$ such that $\chi(E_{w_1}) = a_1 \neq a_2 = \chi(E_{w_2})$. Without loss of generality we assume that $w_1 < w_2$. We divide A into $2k-1$ disjoint sets

$$B_1 < B_2 < \cdots < B_{k-1} < A' < C_1 < C_2 < \cdots < C_{k-1}, \quad |B_i| = |C_i| = R.$$

Clearly, $v_1 < A$ implies that there are indices $i_0, j_0 \in [k-1]$ such that the set $\{v_1, v_2, \ldots, v_{k-1}\}$ is disjoint to both B_{i_0} and C_{j_0}. We set $v_k = n+1$ and define the indices $p, q \in [k-1]$ by $v_p < B_{i_0} < v_{p+1}$ and $v_q < C_{j_0} < v_{q+1}$. Clearly, $p \leq q$. We set $F = \{v_1, \ldots, v_p\} \cup \{v_{q+1}, \ldots, v_{k-1}\}$, $s = q - p + 1$, and $b = s + r - k$. Clearly, $s \in [k-1]$, $b \in [r-k+1, r-1]$, and $|F| + s = k$. From $|B_{i_0}| = R \geq R_s(b, l)$ we conclude that there is a set $X \in \binom{B_{i_0}}{b}$ such that $\chi(F \cup G) = c$ for some color $c \in [l]$ and all $G \in \binom{X}{s}$. Let $X = \{x_1 < x_2 < \cdots < x_b\}$. We set

$$a = \begin{cases} a_1 = \chi(E_{w_1}) \quad \text{and} \quad E = E_{w_1} \backslash F & c \neq a_1, \\ a_2 = \chi(E_{w_2}) \quad \text{and} \quad E = E_{w_2} \backslash F & c = a_1. \end{cases}$$

Thus $a \neq c$. We denote

$$E = \{e_1 < e_2 < \cdots < e_s\} = \{v_{p+1}, \ldots, v_q, w_i\} \quad (i = 1 \text{ or } i = 2)$$

and for $t \in [s]_0$ consider the colors

$$c_t = \chi(F \cup \{x_{b-s+t+1} < \cdots < x_b < e_1 < \cdots < e_t\}) .$$

Since $c_s = a \neq c = c_0$, we may take the minimum index $I \in \{1, \ldots, s\}$ such that $c_I \neq c$. We set
$$D = F \cup \{x_I, \ldots, x_b\} \cup E \cup C_{j_0}^-,$$

where $C_{j_0}^-$ is the set of the first $b - 2s + I$ elements of C_{j_0}. Then by restricting and normalizing of (n, χ) to D we get an r-rich coloring of type $T_{p,s,k-q-1}$ that is contained in (n, χ).

Indeed, the middle part of size s moving to the left in the set $X \cup E$ starts as $\{x_{b-s+I+1} < \cdots < x_b < e_1 < \cdots < e_I\}$ with color $c_I \neq c$, next $I - 1$ edges have color c because of the definition of I, and last $b - s + 1 - I + 1$ edges have color c since their middle parts are all in X. $\qquad \square$

Now we can move on to the key Proposition and the proof of the main theorem follows.

Proposition 2.23. *For every $r \geq k$ there is a constant $c = c(r) \in \mathbb{N}$ such that every ideal X of colorings either contains an r-rich coloring or is c-simple.*

Proof. We assume that $r \geq k$ and that X is an ideal of colorings not containing an r-rich coloring. As in Lemma 2.22 we set $R = \max\{R_i(r-1,l), i = 1, 2, \ldots, k-1\}$, and finally denote

$$d = \max\{(2k+2)r, \, 3k(r-k+1)\} \quad \text{and} \quad c = \max\{(k-1)R, R_k(d,l)\} \, .$$

We prove that X is c-simple. Let $(n, \chi) \in X$ be arbitrary. We may suppose that $n > 2c+k$ since smaller colorings are trivially c-simple. We take a χ-homogenous set $H \subset [n]$ with the maximum cardinality. Thus $|H| \geq d$ since $n > R_k(d,l)$. Let $A \subset H$ be the χ-homogenous set obtained from H by deleting both the first and the last $k(r-k+1)$ elements. By Lemma 2.20, A is a nonempty interval in $[n]$. By Lemma 2.21 we have $\min A < c+1$ and $\max A > n-c$, because $|A| \geq k(r-k+1)$ and $c \geq R_k((2k+2)r,l)$. Thus $[c+1, n-c]$ is a χ-homogenous set and condition C1 in the definition of c-simplicity is satisfied.

Now we assume that $n > 4c+1$, for else the remaining condition C2 in the definition of c-simplicity is satisfied trivially. Let $v_1 \in [c] \cup [n-c+1, n]$ and $v_2, \ldots, v_{k-1} \in [n]$ be arbitrary $k-1$ distinct vertices. Because $|[c+1, n-c]| \geq 2c \geq 2(k-1)R$, we can use Lemma 2.22 with the set $[c+1, n-c]$ and obtain that all the edges $\{v_1, \ldots, v_{k-1}, w\}$, where $w \in [2c+1, n-2c]$ and $w \neq v_i$, have the same color. We see that (n, χ) is c-simple. $\qquad\square$

Proof of Theorem 2.11. Let X be an ideal of colorings. If X contains an r-rich coloring for every $r \geq k$ then $|X_n| \geq n-k+2$ for every $n \geq k$ by Lemma 2.17. Otherwise there is an r_0 such that there is no r-rich coloring in X for any $r \geq r_0$. By Proposition 2.23, the ideal X is c-simple for $c = c(r_0)$. Applying Lemma 2.19 we get that $|X_n|$ is constant for all $n > n_0$. $\qquad\square$

2.4 Polynomial to quasi-Fibonacci jump for 3-graphs

In this section we provide a proof of Theorem 2.12, restated next, which divides the possible speed of growth function of ordered 3-uniform hypergraph ideals into those with at most polynomial growth and those with growth at least G_n. We prove:

Theorem 2.12. *Let $l = 2$ and $X \subset \mathcal{C}_3$ be an ideal of 2-colored 3-uniform hypergraphs. Then either there is a constant $c > 0$ such that $|X_n| \leq n^c$ for every $n \in \mathbb{N}$, or $|X_n| \geq G_n$ for every $n \geq 23$.*

The proof will proceed along similar lines as in [44] but is considerably more complicated. In its first part we define here various "wealthy" colorings in Section 2.4.1 and prove that the growth function of an ideal containing a large number of wealthy coloring grows at least as G_n. We call these colorings r-wealthy colorings of type W_i, $r \in \mathbb{N}$ and $i \in [4]$. Some of them also have subtypes $W_{i,j}$. The meaning of the parameter i is that the underlying set of the coloring is $[ir]$ or $[ir+1]$. In the second part of the proof of Theorem 2.12 in Section 2.4.2 we associate to colorings $(n, \chi) \in \mathcal{C}_3$ three-dimensional "crossing"

matrices $M : [r] \times [s] \times [t] \to \{0, 1, *\}$. Then, in Section 2.4.3 we define p-tame colorings and prove that the growth function of ideals with only p-tame colorings grows at most polynomially. In the last part of the proof we present auxiliary lemmas in Section 2.4.4 and complete the proof in Section 2.4.5 by combining the results on wealthy colorings and crossing matrices.

From now on always $k = 3$ and $l = 2$. In fact, we will use the two colors $\{0, 1\}$, not $[2]$.

2.4.1 Forbidden wealthy colorings

Wealthy colorings W_1

Definition 2.24 (Wealthy coloring of type W_1). *Let $r \geq 3$. A coloring $K = (r, \chi)$ is r-wealthy of type W_1' if, for $a, b \in \{0, 1\}$ with $a \neq b$, K or its reversal satisfies*

$$\chi(\{1, 2, i\}) = \begin{cases} a & \text{for even } i \in [3, r], \\ b & \text{for odd } i \in [3, r]. \end{cases} \tag{2.1}$$

Similarly, K is r-wealthy of type W_1'' if, for $a, b \in \{0, 1\}$ with $a \neq b$, K satisfies

$$\chi(\{1, i, r\}) = \begin{cases} a & \text{for even } i \in [2, r-1], \\ b & \text{for odd } i \in [2, r-1]. \end{cases} \tag{2.2}$$

Other edges may have any color. We call the r-wealthy colorings of type W_1' and W_1'' summarily r-wealthy of type W_1.

For $r = 1, 2$ these are just empty colorings (with no edge). Note that r-wealthy colorings of type W_1 are closed to taking reversals.

Lemma 2.25. *If an ideal X contains for every $r \geq 3$ an r-wealthy coloring of type W_1 then $|X_n| \geq 2^{n-2}$ for every $n \in \mathbb{N}$.*

Proof. Let X be as given. It follows that either for infinitely many $r \in \mathbb{N}$ the ideal X contains an r-wealthy coloring L_r' of type W_1', or for infinitely many $r \in \mathbb{N}$ it contains an r-wealthy coloring L_r'' of type W_1''. In fact, we may replace 'infinitely many' with 'every'.

For $n = 1, 2$ the bound is trivial. Let $n \geq 3$. In the former case, for every $A \subset [3, n]$ there is a coloring $K_A = (n, \chi_A) \preceq L_{2n-2}'$ such that $\chi_A(\{1, 2, i\}) = 0$ if and only if $i \in A$. In the latter case, for every $B \subset [2, n-1]$ there is a coloring $K_B = (n, \chi_B) \preceq L_{2n-2}''$ such that $\chi_B(\{1, i, n\}) = 0$ if and only if $i \in B$. The sets A, resp. B, may be chosen in 2^{n-2} ways and for different As, resp. Bs, the colorings K_A, resp. K_B, are different. Thus the bound follows. $\qquad \square$

The bound is actually a tight one, for consider the set of colorings $X \subset \mathcal{C}_3$ defined by $(n, \chi) \in X$ if and only if for every $E \in \binom{[n]}{3}$ with $E \not\supseteq \{1, 2\}$ one has $\chi(E) = 0$. Then it is easy to see that X is in fact an ideal, that for every $r \in \mathbb{N}$ it contains an r-wealthy coloring of type W_1', and that for every $n \geq 2$ one has $|X_n| = 2^{n-2}$.

Wealthy colorings W_2

Definition 2.26 (Wealthy coloring of type W_2' and W_2''). *Let $r \in \mathbb{N}$. We say that a coloring $K = (n, \chi)$ is r-wealthy of type $W_{2,1}'$ if $n = 2r + 1$ and*

$$\chi(\{i,\, r + j,\, 2r + 1\}) = \begin{cases} 1 & \text{for}\ \ i = j\,, \\ 0 & \text{for}\ \ i \neq j \end{cases} \tag{2.3}$$

where $i, j \in [r]$. Similarly we say that K is r-wealthy of type $W_{2,2}'$ if $n = 2r + 1$ and

$$\chi(\{i,\, r + j,\, 2r + 1\}) = \begin{cases} 1 & \text{for}\ \ i \leq j\,, \\ 0 & \text{for}\ \ i > j \end{cases} \tag{2.4}$$

where $i, j \in [r]$. In both colorings, colors of unspecified edges may be arbitrary.

We visualize the colors of latter edges of K as an $r \times r$ matrix $(a_{i,j})$ such that $\chi(i, r + j, 2r + 1)$ is the entry in row i and column j.

Definition 2.27 (Wealthy coloring of type W_2). *A coloring H is r-wealthy of type $W_{2,1}$ if H can be obtained either from an r-wealthy coloring K of type $W_{2,1}'$ by swapping colors 0 and 1 and/or reversing the order of vertices in the interval $[1, r]$ and/or reversing the order of vertices in the interval $[r + 1, 2r]$ and/or permuting the order of the three intervals $[1, r]$, $[r + 1, 2r]$ and $\{2r + 1\}$. Colorings of type $W_{2,2}$ are defined in an analogous way. A coloring of type W_2 is of type $W_{2,1}$ or type $W_{2,2}$.*

Swapping of the colors simply means that in the above definitions 1 and 0 are exchanged. Reversal of the order in an interval, for example in $[1, r]$ for type $W_{2,1}'$, means that in equation (2.3) we replace the left side with $\chi(\{r - i + 1, r + j, 2r + 1\})$. Permuting the order of the three intervals means, for example for type $W_{2,2}''$ and the reversing permutation (sending 123 to 321), that in equation (2.4) we replace the left side with $\chi(\{1, 1 + j, r + 1 + i\})$. We say more on symmetries of colorings and matrices at the beginning of Section 2.4.2.

Definition 2.28 (Base sets of W_2 colorings). *If $K = (2r + 1, \chi)$ is an r-wealthy coloring of type $W_{2,1}$ (resp. $W_{2,2}$) and if K was obtained from an r-wealthy coloring $K' = (2r + 1, \chi')$ of type $W_{2,1}'$ (resp. $W_{2,2}'$) by the above symmetries so that the intervals $[r]$, $[r + 1, 2r]$ and $\{2r + 1\}$ of K' were permuted in the intervals A, B and $\{c\}$, we say that A, B and $\{c\}$ are the base sets of K.*

Base sets play a role in the end of the proof of Theorem 2.12. Further, we use the next characterizations of the sequences F_n and G_n. One may recall the formal definition of these sequences in the Notation. A binary string is one from $\{0, 1\}^*$.

Lemma 2.29 (Characterisation of Fibonacci numbers). *For the Fibonacci sequence $(F_n)_{n \geq 1}$ and the Narayana cow sequence $(G_n)_{n \geq 1}$ the following holds for $n \geq 2$.*

1. *F_n equals to the number of binary strings $w = w_1 w_2 \ldots w_{n-2}$ of the length $n - 2$ not containing substring 00. The same holds for substring 11.*
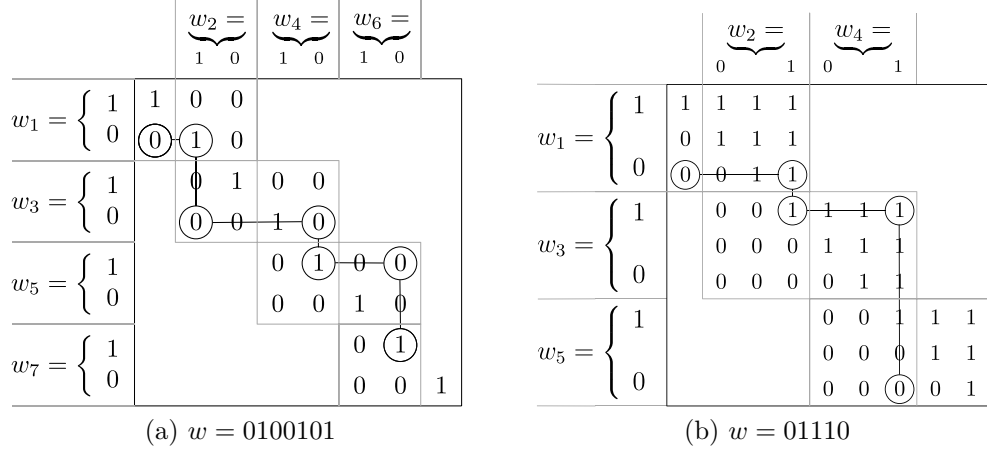
(a) $w = 0100101$ (b) $w = 01110$

Figure 2.2: The submatrices of I_{2n} and U_{3n} for sequences w avoiding some patterns.

2. F_n equals to the number of binary strings $w = w_1 w_2 \ldots w_{n-2}$ of the length $n-2$ not containing substrings $w_{2i} w_{2i+1} = 10$ and $w_{2i-1} w_{2i} = 01$, resp. vice versa.

3. We have $2^n > F_n \geq G_n$ for every $n \in \mathbb{N}$ and the last inequality is strict for $n > 4$.

Proof. All results 1–3 follow easily by induction on n. $\qquad\square$

Recall that the identity matrix I_n has size $n \times n$, has 1s on the main diagonal, and 0s elsewhere. Similarly, the upper triangular matrix U_n has size $n \times n$, has 1s on the main diagonal and above it, and 0s elsewhere.

Lemma 2.30. *For $n \in \mathbb{N}$, let I_n and U_n be the identity matrix and the upper triangular matrix, respectively.*

1. *For any binary string $w = w_1 w_2 \ldots w_{2n-1}$ of the length $2n-1$ avoiding any substring $w_i w_{i+1} = 11$ there exists a matrix M of the size $n \times n$ that is contained in I_{2n} and such that $M(i,i) = w_{2i-1}$ for $i \in [n]$ and $M(i, i+1) = w_{2i}$ for $i \in [n-1]$.*

2. *For any binary string $w = w_1 w_2 \ldots w_{2n-1}$ of the length $2n-1$ avoiding any substring $w_{2i-1} w_{2i} = 10$ and $w_{2i} w_{2i+1} = 01$ there is a matrix M of the size $n \times n$ that is contained in U_{3n} and such that $M(i,i) = w_{2i-1}$ for $i \in [n]$ and $M(i, i+1) = w_{2i}$ for $i \in [n-1]$.*

Proof. 1. Suppose that $w = w_1 w_2 \ldots w_{2n-1}$ avoids consecutive substrings 11. To get M, for each $i \in [n]$ we choose a row and a column of I_{2n}. For different i the chosen rows are different, and so are the chosen columns. The matrix M will consists of the chosen rows and columns.

We proceed as follows. For $w_{2i-1} = a \in \{0,1\}$ we choose the row $2i - a$. For $w_{2i} = b \in \{0,1\}$ we choose the column $2i+1-b$, except for $i = n$ when we choose the column 1. For an example with $n = 4$ see Fig. 2.2 (a). It follows that the resulting matrix M has the stated property.

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2. Suppose that $w = w_1 w_2 \ldots w_{2n-1}$ avoids consecutive substrings $w_{2i-1}w_{2i} = 10$ and $w_{2i}w_{2i+1} = 01$. We choose rows and columns like in part 1. If $w_{2i-1} = a \in \{0,1\}$ we choose the row $3i - 2a$. If $w_{2i} = b \in \{0,1\}$ we choose the column $3i - 1 + 2b$, except for $i = n$ when we choose the column 1. For an example with $n = 3$ see Fig. 2.2 (b) . Again, the resulting matrix M has the stated property. $\qquad \square$

Probably, the sizes of the matrices I_{2n} and U_{3n} in Lemma 2.30 are not minimal. We think that we could prove the lemma with matrices $I_{3n/2}$ and U_{2n}.

Proposition 2.31. *If an ideal X contains for every $r \in \mathbb{N}$ an r-wealthy coloring of type W_2 then $|X_n| \geq F_n$ for every $n \in \mathbb{N}$.*

Proof. Let X be an ideal of colorings. We first assume that for every $r \in \mathbb{N}$, $K_r = (2r + 1, \chi)$ is an r-wealthy coloring of type W_2' such that $K_r \in X$. Let $n = 2m + 1 \in \mathbb{N}$ be odd. We take the $2m$-wealthy coloring $K_{2m} = (4m + 1, \chi) \in X$ of type W_2' and apply part 1 of Lemma 2.30. By it, for each binary string $w = w_1 w_2 \ldots w_{2m-1}$ avoiding consecutive substrings 11 there is a coloring $K_w = (n, \chi_w) = (2m + 1, \chi_w)$ contained in K_{2m}, thus in X, such that for every $i \in [m]$ one has $\chi_w(\{i, m + i, 2m + 1\}) = w_{2i-1}$ and for every $i \in [m - 1]$ one has $\chi_w(\{i, m + 1 + i, 2m + 1\}) = w_{2i}$. For different strings w these colorings are different, and using part 1 of Lemma 2.29 we get the lower bound $|X_{2m+1}| \geq F_{2m+1}$.

To bound $|X_n| = |X_{2m}|$ for even $n \in \mathbb{N}$ we use a variant of the latter justification. We take any binary string $w = w_1 w_2 \ldots w_{2m-2}0$ of length $2m - 1$ that avoids consecutive substrings 11. By Lemma 2.30 there is an $m \times m$ matrix M_ω contained in I_{2m} such that $M_\omega(i, i) = w_{2i-1}$ for $i \in [n]$ and $M_\omega(i, i + 1) = w_{2i}$ for $i \in [n-1]$. Any of these matrices produces a different coloring $K_\omega = (2m+1, \chi_\omega)$ such that $\chi_\omega(\{i, m + i, 2m + 1\}) = w_{2i-1}$ for $i \in [m]$ and $\chi_\omega(\{i, m + 1 + i\}) = w_{2i}$ for $i \in [m - 1]$. We denote, for any latter string w, a colorings K_ω' by restricting and normalizing K_ω to $[2m + 1] \setminus \{m\}$. For different strings ω the colorings $K_\omega' = (n, \chi')$ are different, because $w_{2m-1} = 0$, and hence by part 1 of Lemma 2.29 we have $|X_{2m}| \geq F_{2m}$.

When X contains for every $r \in \mathbb{N}$ an r-wealthy coloring of type W_2'', we argue similarly. For odd $n = 2m + 1$ we take the $3m$-wealthy coloring $K_{3m} = (6m+1, \chi) \in X$ of type W_2'', apply part 2 of Lemma 2.30 and part 2 of Lemma 2.29 and obtain $|X_{2m+1}| \geq F_{2m+1}$. For even $n = 2m$ we argue similarly and restrict to the particular strings w with $w_{2m-1} = 0$ and again apply part 2 of Lemma 2.30 and part 2 of Lemma 2.29. Now we restrict our colorings to the set $[2m+1] \setminus \{m\}$ and obtain $|X_{2m}| \geq F_{2m}$ since $w_{2m-1} = 0$ guarantees that the restricted colorings are mutually different.

In the general case when X contains an r-wealthy coloring of type W_2, we consider one of the above described transformations T transforming the "canonical" coloring W_2' or W_2'' to the given r-wealthy coloring of type W_2. The images of the colorings K_w under T give then the stated lower bound for the general r-wealthy coloring of type W_2. $\qquad \square$

Wealthy colorings W_3

Definition 2.32 (Wealthy coloring of type W_3)**.** *Let $r \in \mathbb{N}$. We introduce wealthy colorings of type W_3. A coloring $K = (n, \chi)$ is*

- *r-wealthy of type $W'_{3,1}$ if $n = 3r$ and*

$$\chi(\{i, r+i, 2r+j\}) = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \tag{2.5}$$

  where $i, j \in [r]$. Colors of the remaining edges are not specified. We say that a coloring H is r-wealthy of type $W_{3,1}$ if H can be obtained from K by swapping colors 0 and 1 and/or reversing the order of vertices in some of the intervals $[1, r]$, $[r + 1, 2r]$ and $[2r + 1, 3r]$ and/or permuting these intervals.

- *r-wealthy of type $W'_{3,2}$ if $n = 3r$ and*

$$\psi(\{i, r+i, 2r+j\}) = \begin{cases} 1 & i \leq j, \\ 0 & i > j, \end{cases} \tag{2.6}$$

  where $i, j \in [r]$. Again, colors of the remaining edges are arbitrary. We say that coloring H is r-wealthy of type $W_{3,2}$ if H can be obtained from K by swapping colors 0 and 1 and/or reversing the order of vertices in some of the intervals $[1, r]$, $[r + 1, 2r]$ and $[2r + 1, 3r]$ and/or permuting these intervals.

- *r-wealthy of type $W_{3,3}$ if $n = 3r + 1$ and for any $i \in [r]$ there are distinct numbers $a_i, b_i, c_i \in \{3i - 2, 3i - 1, 3i\}$ such that $\chi(\{a_i, b_i, 3r + 1\}) \neq \chi(\{a_i, c_i, 3r + 1\})$.*

We call these colorings summarily W_3 type colorings.

Definition 2.33 (Base sets of W_3 colorings)**.** *We define the base sets of an r-wealthy coloring $(3r, \chi)$ of type $W_{3,1}$ or $W_{3,2}$ to be the intervals $[1, r]$, $[r + 1, 2r]$ and $[2r + 1, 3r]$.*

The symmetries of W_3 type colorings are the same as those explained in the definition of type W_2 colorings. However, all given intervals have the same length r and thus the definition of the base sets of W_3 colorings is less technical.

Our aim is to prove that W_3 type colorings contain many subcolorings of given size, as shown in Proposition 2.39. We prepare the background with following definition, lemma and proposition.

Definition 2.34 (n-chain)**.** *Let $k \leq n$ and $A = (a_1, a_2, \ldots, a_k)$ be a k-tuple with $a_i = (c_i, d_i) \in [n]^2$ satisfying*

$$1 \leq c_1 < c_2 < \cdots < c_k \leq n \quad \text{and} \quad 1 \leq d_1 < d_2 < \cdots < d_k \leq n.$$

We call such A an n-chain. We denote by A^* the $n \times n$ binary matrix that has 1s exactly in the positions a_i, $i \in [k]$.
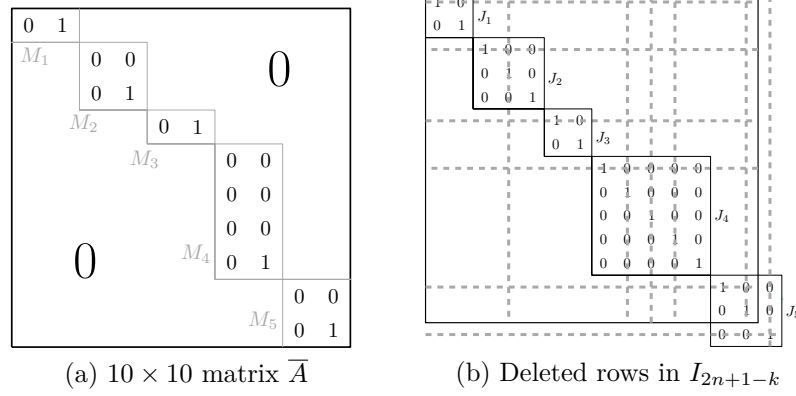
(a) 10×10 matrix $\overline{A}$ (b) Deleted rows in I_{2n+1-k}

Figure 2.3: Finding n-chain $A = \{(1,2),(3,4),(4,6),(8,8)\}$ with $n = 9$ and $k = 4$ as a submatrix $\overline{A}$ in the identity matrix I_{2n+1-k}.

Lemma 2.35. *Let $k \le n$, $A = (a_1, a_2, \ldots, a_k)$ be an n-chain, and A^* be the corresponding $n \times n$ binary matrix. Then A^* is a submatrix of the identity matrix I_{2n}.*

Proof. In fact, we prove that A^* is a submatrix of the identity I_{2n-k}. Let A be an n-chain. We obtain some n rows and n columns of I_{2n} forming the matrix A^* by deleting some rows and columns from I_{2n} as follows. We additionally set $a_0 = (0,0)$, $a_{k+1} = (n+1, n+1)$, and consider the differences

$$(c_i', d_i') := (c_i - c_{i-1}, d_i - d_{i-1}) = a_i - a_{i-1}, \; i \in [k+1].$$

We denote by $\overline{A}$ the $(n+1) \times (n+1)$ matrix obtained from A^* by adding at the bottom of A^* and to the right of it a zero row and a zero column, and changing the zero in their intersection to 1.

To any $a_i = (c_i, d_i) \in A \cup \{(n+1, n+1)\}$ for $i \in [k+1]$ we assign the submatrix J_i of I_{2n+1-k} formed by the rows and columns (with indices) in the interval $[c_{i-1} + d_{i-1} - (i-2), c_i + d_i - i]$, $i \in [k+1]$, and the submatrix M_i of $\overline{A}$ formed by the rows in $[c_{i-1} + 1, c_i]$ and the columns in $[d_{i-1} + 1, d_i]$. Note that every J_i is the identity matrix with size $c_i' + d_i' - 1$, and that the (in general not square) $c_i' \times d_i'$ matrix M_i has 1 in the south-east corner and zeros elsewhere.

We show that each J_i contains M_i. This, together with the fact that the matrices J_i, resp. M_i, cover all ones in I_{2n+1-k}, resp. in $\overline{A}$, and follow one after another, proves the stated claim. One easily checks that deleting the first $d_i' - 1$ rows of J_i and the following $c_i' - 1$ columns of J_i yields M_i. Indeed, since the first $d_i' - 1$ ones of J_i are contained in deleted rows and the following $c_i' - 1$ ones are contained in the deleted columns, only one 1 (the last one) survives for M_i. The size of the resulting matrix is that of M_i as it has $c_i' + d_i' - 1 - (d_i' - 1) = c_i'$ rows and $c_i' + d_i' - 1 - (c_i' - 1) = d_i'$ columns.

Thus $\overline{A}$ is a submatrix of I_{2n+1-k} and A^* is a submatrix of I_{2n-k}. The reader can follow the construction on an example in Figure 2.3, where $A = \{(1,2),(3,4),(4,6),(8,8)\}$ and $n = 9$. $\qquad\square$

Definition 2.36 (Southeast path)**.** *For $m \in \mathbb{N}$ we call $C = (c_1, c_2, \ldots, c_{2m+1}) \subset [m+1]^2$ a southeast path in $[m+1]^2$ if*

$$c_1 = (1,1), \; c_{i+1} - c_i \in \{(0,1),\ (1,0)\}, \text{ and } c_{2m+1} = (m+1, m+1).$$

Clearly, the number of southeast paths in $[m+1]^2$ equals $\binom{2m}{m}$ because they 1-1 correspond to the m-element subsets (of the steps $(0,1)$, say) of the set $[2m]$ (of all steps).

Proposition 2.37. *For $m \in \mathbb{N}$ the set of m-chains is in bijection with the set of southeast paths in $[m+1]^2$, and therefore we have exactly $\binom{2m}{m}$ m-chains.*

Proof. We consider an $m \times m$ matrix drawn in the plane as an $m \times m$ square array of m^2 unit squares which we label by the coordinates $(i,j) \in [m]^2$ in the matrix way, top to bottom and left to right. We label their corners by the elements of $[m+1]^2$ also in the matrix way. Then any southeast path C in $[m+1]^2$ consists of $2m+1$ of these corners, starts in the corner $(1,1)$ and ends in the corner $(m+1, m+1)$. To each C we associate the binary $m \times m$ matrix D^* with 1s exactly in the squares around which C makes a left turn, that is, of the squares whose both bottom and both left side array points lie in C. Clearly, D^* is a matrix of a unique m-chain. In the other way, to any m-chain D we associate a southeast path C in $[m+1]^2$ that starts in the corner $(1,1)$, then goes horizontally until it reaches the left boundary of the column containing the first 1 of D^*, then C makes a right turn and goes vertically until it reaches the bottom boundary of the row containing the first 1, then C makes a left turn around the first 1, then continues in the similar way to make a left turn around the second 1 in D^*, and so on until C finishes in the corner $(m+1, m+1)$. If D^* is the zero matrix or if C made turns around all 1s in D^*, C goes horizontally to the right boundary of column m and then vertically to the corner $(m+1, m+1)$. The two described associations are inverses of one another and give the required bijection. $\square$

Corollary 2.38. *For any $m \geq 1$ there are at least $\binom{2m-2}{m-1}$ m-chains with zero (last) m-th column, or with zero m-th row.*

Proof. This number is at least the number of m-chains that have in both the last column and the last row only zeros. These bijectively correspond to $(m-1)$-chains, and we can use the formula from the previous proposition. $\square$

Proposition 2.39. *If an ideal of colorings X contains for every $r \geq 3$ an r-wealthy coloring of type $W_{3,1}$, or for every $r \geq 3$ an r-wealthy coloring of type $W_{3,2}$, then for every $n \in \mathbb{N}$,*

$$|X_n| > \frac{2^{2(n-2)/3}}{\sqrt{2n}} > \frac{0.28 \cdot 1.587^n}{\sqrt{n}} .$$

Hence $|X_n| \geq G_n$ for every $n \geq 23$.

Proof. First we consider r-wealthy colorings of type $W_{3,1}$. Without loss of generality we consider an ideal X that for every $r \geq 3$ contains an r-wealthy coloring $K_r = (3r, \chi_r)$ of type $W'_{3,1}$. We handle the general case by a transformation argument as in the end of the proof of Proposition 2.31. Let $H = (n, \chi)$ be a coloring such that $n = 2m_1 + m_2$. We define the $m_1 \times m_2$ *partition matrix* $P[H]$ of H by $P[H](i,j) = \chi(\{i, m_1 + i, 2m_1 + j\})$, $i \in [m_1]$ and $j \in [m_2]$. If $H = K_r$ then, clearly, $P[H]$ is the identity matrix I_r.

Now let $n = 3m$ and $m_1 = m_2 = 2m$. We set $r = 2n$ and consider an arbitrary m-chain D. Since D^* is a subset of I_{2m} and $I_{2m} = P[K_r]$ is the partition

matrix of r-wealthy coloring K_r, it follows by Lemma 2.35 that the coloring $L_D = (3m, \chi_D) \in X$ such that $\chi_D(\{i, m+i, 2m+j\}) = D^*(i,j)$ is contained in K_r, thus in X. It is clear that $D \neq D' \implies L_D \neq L_{D'}$. Hence by Proposition 2.37 and by the lower bound $\binom{2m}{m} \geq \frac{1}{2\sqrt{m}} 2^{2m}$ for every $m \in \mathbb{N}$ (see N. D. Kazarinoff [42]) we have that

$$|X_n| = |X_{3m}| \geq \binom{2m}{m} \geq \frac{2^{2m}}{\sqrt{4m}} > \frac{2^{2n/3}}{\sqrt{4n/3}} .$$

To lowerbound $|X_n| = |X_{3m-1}|$, we take the colorings from the set $T = \{L_D \mid \forall i \in [m] : D^*(i,m) = 0\}$, that is, for the D^* having in the last m-th column only zeros. By deleting the vertex $3m$ in each $L_D \in T$ we obtain colorings $T' \subset X_{3m-1}$ that are still mutually different because their partition matrices arise from those D just by deleting the last zero column. By Corollary 2.38,

$$|X_n| = |X_{3m-1}| \geq |A'| = |A| \geq \binom{2m-2}{m-1} \geq \frac{2^{2m-2}}{\sqrt{4m-4}} > \frac{2^{2(n-2)/3}}{\sqrt{4n/3}} .$$

To lowerbound $|X_n| = |X_{3m-2}|$ we argue similarly, we consider the colorings L_D such that D has in the last m-th row only zeros and delete in those L_D the two vertices m and $2m$. Again by Corollary 2.38,

$$|X_n| = |X_{3m-2}| \geq \binom{2m-2}{m-1} \geq \frac{2^{2m-2}}{\sqrt{4m-4}} > \frac{2^{2(n-1)/3}}{\sqrt{4n/3}} .$$

This gives the stated bound.

For the r-wealthy colorings of type $W_{3,2}$ the argument is similar, and we only sketch the necessary modifications to the argument from the previous paragraph. The new n-chains are now the $n \times n$ binary matrices that are obtained from the (old) n-chains by changing every zero above and to the right of any 1 also to 1. Equivalently, the new n-chains are the $n \times n$ binary matrices with no 0 above or to the right of an 1; each such matrix is uniquely determined by the (old) n-chain of 1s with no other 1 to the left or below. We then have an analogy to Lemma 2.35 and can find every new n-chain as a submatrix in the matrix U_{2n} (in the proof we simply mirror the steps in I_{2n} by steps in U_{2n}). We have the same formula for the number of new n-chains as in Proposition 2.37, the southeast paths C are now exactly borders between the area of 0s and the area of 1s. Considering the colorings $L_D = (n, \chi) \in X$ where $n = 3m$, D is a new m-chain, and the partition matrix of the coloring equals D^*, and recalling that each D is determined by a certain (old) m-chain in D, we see that we get the same lower bounds on $|X_{3m}|, |X_{3m-1}|$, and $|X_{3m-2}|$ as before in the previous paragraph.

To bound an n_0 such that $|X_n| \geq G_n$ if $n \geq n_0$, we estimate the numbers G_n from above. Since $G_0 = G_1 = G_2 = 1$ and $G_n = G_{n-1} + G_{n-3}$ for $n \geq 3$, induction shows that for every integer $n \geq 0$ one has $G_n \leq c\alpha^n$ where $\alpha = 1.46557 \cdots < 1.466$ is the only positive root of the polynomial $x^3 - x^2 - 1$ and $c = 0.417 \cdots$ is a constant generated by the initial terms of G_n. Thus we need an n_0 such that if $n \geq n_0$ then

$$\frac{0.343 \cdot 1.587^n}{\sqrt{n}} > 0.418 \cdot 1.466^n ,$$

where $0.343 < 2^{-4/3}/\sqrt{4/3}$. Since $(0.343/0.418)^2 > 0.673$ and $(1.587/1.466)^2 > 1.171$, we need an n_0 such that if $n \geq n_0$ then $0.69 \cdot 1.171^n > n$. It is easy to compute that $n_0 = 23$ suffices, which gives the stated n_0. $\qquad\square$

Lemma 2.40. *If an ideal X contains for every $r \in \mathbb{N}$ an r-wealthy coloring of type $W_{3,3}$ then $|X_n| \geq F_n$ for every $n \in \mathbb{N}$.*

Proof. This bound follows by applying [44, Lemma 3.10]. To any coloring $K = (n, \chi)$, $n \geq 2$, of triples we associate a coloring $K' = (n - 1, \psi)$ of pairs, $\psi : \binom{[n-1]}{2} \to \{0, 1\}$, by

$$\psi(\{x, y\}) = \chi(\{x, y, n\}).$$

It follows that $X' = \{K' \mid K \in X\}$ is an ideal of colorings of pairs. Since we assume that we may take for any r the coloring $K \in X$ to be an r-wealthy coloring of type $W_{3,3}$, it follows that X' contains for every r an r-wealthy coloring of type 2 as defined in [44] before [44, Lemma 3.9] (these are coloring of pairs in $[3r]$ such that no triple $\{3i - 2, 3i - 1, 3i\}$, $i \in [r]$, is monochromatic). By the second claim of [44, Lemma 3.10], $|X'_n| \geq F'_n$ for every $n \geq 1$ where F'_n are the Fibonacci numbers in [44] which relate to the F_n here by $F'_n = F_{n+1}$. Since for every two colorings of triples $K_1 = (n, \chi_1)$ and $K_2 = (n, \chi_2)$, $n \geq 1$, we have $K'_1 \neq K'_2 \Rightarrow K_1 \neq K_2$, we deduce that

$$|X_n| \geq |X'_{n-1}| \geq F'_{n-1} = F_n, \; n \geq 2.$$

For $n = 1$ this bound holds trivially as well. $\qquad\square$

Wealthy colorings W_4

We introduce the last type of wealthy colorings.

Definition 2.41 (Wealthy coloring of type W_4). *A coloring $K = (n, \chi)$ is*

- *r-wealthy of type $W_{4,1}$ if $n = 4r$ and none of the consecutive intervals $\{4i - 3, 4i - 2, 4i - 1, 4i\}$, $i \in [r]$, is monochromatic,*

- *r-wealthy of type $W'_{4,2}$ if $n = 4r$ and for every $i \in [r]$ there are distinct numbers $a_i, b_i, c_i \in \{r + 3i - 2, r + 3i - 1, r + 3i\}$ such that $\chi(\{i, a_i, b_i\}) \neq \chi(\{i, a_i, c_i\})$. A coloring $K = (4r, \chi)$ is r-wealthy of type $W_{4,2}$ if it is obtained from an r-wealthy coloring of type $W'_{4,2}$ by possibly reversing the order of elements in $[r]$ and/or swapping the intervals $[r]$ and $[r + 1, 4r]$ (so that they become $[3r]$ and $[3r + 1, 4r]$, respectively).*

We call these colorings summarily W_4 type colorings.

Observe that both types of W_4 colorings are closed to reversal, swapping colors, reversing the order of the vertices in the interval $[r]$ and permuting (swapping) the intervals $[r]$ and $[r + 1, 4r]$. As for type W_2, $W_{3,1}$ and $W_{3,2}$ colorings we need later the base sets of an r-wealthy coloring of type $W_{4,2}$.

Definition 2.42 (Base sets of W_4 colorings). *Let $K = (4r, \chi)$ be an r-wealthy coloring of type $W_{4,2}$ that is obtained by the above symmetries from an r-wealthy coloring $K' = (4r, \chi)$ of type $W'_{4,2}$. We call the two intervals A and B obtained from the intervals $[r]$ and $[r + 1, 4r]$ of K' by their possible swapping the base sets of coloring K.*

Now we produce two propositions showing that there are many subcolorings to any r-wealthy coloring of type W_4.

Proposition 2.43. *If an ideal of colorings X contains for every $r \in \mathbb{N}$ an r-wealthy coloring of type $W_{4,1}$ then $|X_n| \geq G_n$ for every $n \in \mathbb{N}$, where the numbers G_n are defined at the start of the article.*

Proof. Let $r \in \mathbb{N}$ and $K_r = (4r, \chi_r) \in X$ be an r-wealthy coloring of type $W_{4,1}$. For $i \in [r]$ consider sets

$$S_i = \{4i - 3,\ 4i - 2,\ 4i - 1,\ 4i\} \quad \text{and} \quad V = \{S_i :\ i \in [r]\}\,.$$

To simplify the situation by the Ramsey theorem we consider the coloring

$$c :\ \binom{V}{3} \to \{0, 1\}^{220},$$

where $220 = \binom{12}{3}$, defined by

$$c(S_{i_1},\ S_{i_2},\ S_{i_3}) = (\chi_r(E) \mid E \subset S_{i_1} \cup S_{i_2} \cup S_{i_3},\ |E| = 3) \in \{0,\ 1\}^{220}\,,$$

where $1 \leq i_1 < i_2 < i_3 \leq r$. The triples E are ordered lexicographically according to their vertices: $E = \{a_1 < a_2 < a_3\}$ comes before $E' = \{a_1' < a_2' < a_3'\}$ if and only if $a_1 < a_1'$ or ($a_1 = a_1'$ and $a_2 < a_2'$) or ($a_1 = a_1'$ and $a_2 = a_2'$ and $a_3 < a_3'$). Since r may be as large as we need and X is an ideal, by the Ramsey theorem for 3-uniform hypergraphs we may suppose that the coloring c is constant. This simplification of K_r is called the *shift condition*.

Since in each S_i we have two triples with distinct colors, there exist triples $A \subset S_1$ and $A' \subset A \cup (A + 4)$ such that $\chi_r(A') \neq \chi_r(A)$. Indeed, since S_i is not monochromatic, there are two triples $E_1, E_2 \subset S_i$ such that $\chi_r(E_1) \neq \chi_r(E_2)$ and $|E_1 \cap E_2| = 2$, and easy discussion shows that either $A = E_1$ or $A = E_2$ works. For all $i \in [r]$ we set $T_i = A + 4(i-1) \subset S_i$, and $L_r = (3r, \psi_r)$ to be the restriction and normalization of χ_r to (the triples in) $T_1 \cup T_2 \cup \cdots \cup T_r$. Clearly $L_r \preceq K_r$ for every $r \in \mathbb{N}$. Each L_r is determined by its restriction to $[9]$ because of the shift condition; we set $M = (9, \psi)$ to be this restriction. Without loss of generality, $\psi(A) = \psi(T_1) = 1$.

We reveal the connection of colors of M. We may suppose that all triples $E \subset [9]$ with $|E \cap T_i| = 1$, for $i = 1, 2, 3$, have the same color $\psi(E)$. If not, then we find triples E_1, E_2 such that $|E_1 \cap E_2| = 2$ and $|E_j \cap T_i| = 1$, for $i \in [3]$ and $j \in [2]$. It follows that there is an r-wealthy subcoloring of type W_1 and Lemma 2.25 applies. Let $s = \psi(\{1, 4, 7\})$. We claim that $\psi(E) = s$ for all triples $E \subset [9]$ with $|E \cap T_1| = 1$ and $|E \cap T_2| = 2$ (or equivalently, by the shift condition, $|E \cap T_3| = 2$). Indeed, if there is a triple $E = \{a, b, c\}$ with $a \in T_1$, $b, c \in T_2$, $b < c$, and $\psi(E) = t \neq s$ ($\{s, t\} = \{0, 1\}$) then the shift condition implies

$$\psi_r(\{a,\ b + 3m,\ c + 3m\}) = t,\ m \in \{0,\ 1,\ \ldots,\ r - 2\}\,,$$

and

$$\psi_r(\{a,\ b + 3m,\ c + 3m + 3\}) = \psi_r(\{a,\ c + 3m,\ b + 3m + 3\}) = s\,,$$

where $m \in \{0, 1, \ldots, r-3\}$a. Therefore we can construct for any binary string $w = w_1 w_2 \ldots w_{n-2}$ avoiding the substring tt a coloring $(n, \lambda_w) \preceq L_n$ such that, for $i = 2, 3, \ldots, n-1$, $\lambda_w(\{1, i, i+1\}) = w_{i-1}$. By parts 1 and 3 of Lemma 2.29, $|X_n| \geq F_n \geq G_n$. Switching to the reversals we deduce in a similar way that $\psi(E) = s$ for E satisfying $|E \cap T_1| = 2$ and $|E \cap T_2| = 1$ (or $|E \cap T_3| = 1$).

Thus $\psi(T_1) = \psi(T_2) = \psi(T_3) = 1$ and $\psi(E) = s$ for all other triples $E \subset T_1 \cup T_2 \cup T_3$. We have $s = 0 \neq 1$ by the condition on the colors of A and A'. So in L_r, $\psi_r(T_i) = 1$ for $i \in [r]$ and all other triples have color 0. Thus $S(3) \subset X$, where $S(3)$ is the ideal of colorings mentioned after the statement of Theorem 2.12, and $|X_n| \geq G_n$. $\qquad\square$

As for type W_1 colorings, also this lower bound is tight. Indeed, the ideal $S(3) \subset \mathcal{C}_3$ has growth $|S(3)_n| = G_n$ and for every $r \in \mathbb{N}$ contains an r-wealthy coloring of type $W_{4,1}$. In more details, $S(3)$ consists of the colorings (n, χ) for which there exist 3-intervals $I_1 < I_2 < \cdots < I_r$ in $[n]$ such that $\chi(I_j) = 0$ for every j but $\chi(E) = 1$ for all other edges E. In particular, for every $r \in \mathbb{N}$ one has that $(4r, \chi_r) \in S(3)$ where $\chi_r(\{4j-3, 4j-2, 4j-1\}) = 0$ for $j \in [r]$ and $\chi_r(E) = 1$ for all other edges E. But $(4r, \chi_r)$ is an r-wealthy coloring of type $W_{4,1}$.

Proposition 2.44. *If an ideal of colorings X contains for every $r \in \mathbb{N}$ an r-wealthy coloring of type $W_{4,2}$ then*

$$|X_n| \geq \left(\frac{\lfloor \frac{2(n-4)}{5} \rfloor}{\lfloor \frac{n-4}{5} \rfloor}\right)^2$$

for every $n \geq 9$. Thus $|X_n| \geq F_n$ for all $n \geq 75$ and $|X_n| \geq G_n$ for all $n \geq 20$.

Note that the bound of latter propositions grows as 1.741^n.

Proof. We suppose that X is an ideal of colorings and that $K_r = (4r, \chi_r) \in X$, $r \in \mathbb{N}$, for some r-wealthy colorings K_r of type $W_{4,2}$. We may suppose that for every $i \in [r]$ and for $\{r+3i-2, r+3i-1, r+3i\} = \{a_i, b_i, c_i\}$ we have $\chi(\{i, a_i, b_i\}) \neq \chi(\{i, a_i, c_i\})$. Let

$$S_i = \{i,\, r+3i-2,\, r+3i-1,\, r+3i\},$$

$i \in [r]$. We consider various cases and in all but the last one we show that $|X_n| \geq F_n \geq G_n$ for every $n \geq 1$. In the last case we still have the bound $|X_n| \geq F_n$ but only for $n \geq n_0$. To get a smaller n_0, we therefore compare in this case $|X_n|$ with G_n instead.

Step 1: shift condition. We consider the same coloring

$$c : \binom{V}{3} \to \{0, 1\}^{220} \text{ with } V = \{S_i \mid i \in [r]\}$$

as in the previous proof (note, however, that the quadruples S_i are now different):

$$c(S_{i_1},\, S_{i_2},\, S_{i_3}) = (\chi_r(E) \mid E \subset S_{i_1} \cup S_{i_2} \cup S_{i_3},\, |E| = 3) \in \{0, 1\}^{\binom{12}{3}},$$

where $1 \le i_1 < i_2 < i_3 \le r$ (with the same lexicographic order of the triples E). As before we may suppose using the Ramsey theorem for 3-uniform hypergraphs that for each K_r the coloring c is constant, and again we call this the *shift condition.* Let

$$Z = [r] \text{ and } Y = [r+1, 4r].$$

The shift condition implies that Z is monochromatic.

Step 2: triples E with $|E \cap Z| = 1$ and $|E \cap S_i| \le 1$ for any $i \in [r]$. First we fix a vertex $v \in Z$ and handle the case where, for infinitely many r, there are two triples $E \subset [4r]$ with $v \in E$, $|E \cap Y| = 2$ and $|E \cap S_i| \le 1$ for all $i \in [r]$ have different colors. So let $E = \{v, e_1, e_2\}$ and $F = \{v, f_1, f_2\}$ with $e_i, f_i \in Y$, $e_1 < e_2$, $f_1 < f_2$, and E, F take from each S_i at most one element, be such that $\chi_r(E) \ne \chi_r(F)$. One can take even such triples E and F that either

(a) $x := e_1 = f_1$ and $e_2 < f_2$ (so $E = \{v < x < e_2\}$ and $F = \{v < x < f_2\}$), or

(b) $x := e_2 = f_2$ and $e_1 < f_1$ (so $E = \{v < e_1 < x\}$ and $F = \{v < f_1 < x\}$).

Indeed, for $e_1 = f_1$ we have case (a) (we swap E and F if needed). Suppose that $e_1 < f_1$ (we swap E and F if needed) and set $c = \chi_r(\{v, e_1, f_2\})$. If $c = \chi_r(E)$, we set $x = f_2$, get that $c = \chi_r(\{v, e_1, x\}) \ne \chi_r(\{v, f_1, x\})$, and have case (b). If $c \ne \chi_r(E)$, we set $x = e_1$, get that $\chi_r(\{v, x, e_2\}) \ne c = \chi_r(\{v, x, f_2\})$, and have case (a) (we swap E and F if needed).

In both cases (a) and (b) we show that $|X_n| \ge F_n$ for every n. We consider case (b) in detail, and after that we discuss case (a) more briefly. By the shift condition, we may suppose that the four vertices v, e_1, f_1, x lie in four different sets S_i, thus we set indices $j_0, j_1, j_2, j_3 \in [r]$ such that $v \in S_{j_0}$ (in fact, $v = j_0$), $e_1 \in S_{j_1}$, $f_1 \in S_{j_2}$, and $x \in S_{j_3}$. Since $e_1 < f_1 < x$, also $j_1 < j_2 < j_3$. If $j_0 \notin [j_1, j_2]$ (for infinitely many r) then the shift condition implies that, for any $r \in \mathbb{N}$, K_r contains an r-wealthy coloring of type W_1''. For example, if $j_1 < j_2 < j_0 < j_3$ then we may take $j_0 = r - 1, j_3 = r$, keep v and x fixed, and replace S_{j_1} and S_{j_2} with all $S_{j_1'}$ and $S_{j_2'}$, respectively, for all $j_1' < j_2' < j_0$, the colors of the triples $\{v < e_1' < x\}$ and $\{v < f_1' < x\}$ then create the pattern of an r-wealthy coloring of type W_1''. Hence $|X_n| \ge 2^{n-2} \ge F_n$ for $n \ge 2$ by Lemma 2.25.

We turn to the case when $j_0 \in [j_1, j_2]$, that is, $j_0 \in (j_1, j_2)$ (for infinitely many r). We show that either for any $r \in \mathbb{N}$ the ideal X contains an r-wealthy coloring of type W_1'', or for any $r \in \mathbb{N}$ the ideal X contains an r-wealthy coloring of type W_2 (obtained by symmetries from the W_2'' coloring). Clearly we may assume that $\chi_r(E) = 0$, $\chi_r(F) = 1$, and that x is the least element of the triple $S_j \cap Y$ in which it lies: $x = r + 3j_3 - 2$. If e_1 and f_1 do not share the same order (as the 1st, 2nd, or 3rd element) in the triple $S_j \cap Y$ they lie in, it follows by the shift condition that W_1'' coloring appears and we have the lower bound from the previous paragraph. Thus we may assume that also e_1 and f_1 are the least elements of the triples: $e_1 = r + 3j_1 - 2$ and $f_1 = r + 3j_2 - 2$. We define the partition matrix $M_r(i, j)$ of K_r, $i, j \in [r-1]$, by

$$M_r(i, j) = \chi(\{i, r + 3j - 2, 4r - 2\}) \; (= \chi(E), \chi(F)).$$

By our assumption on the colors of E and F, the matrix M_r has 0s below the main diagonal and 1s above it. Also, by the shift condition, the main diagonal of M_r is monochromatic. One may see the example of an upper diagonal matrix

$U_r = M_r$ in Figure 2.4(a). We see that the restriction and normalization of K_r on the vertex set $S = [r-1] \cup \{r + 3j - 2 \mid j \in [r]\}$ is an $(r-1)$-wealthy coloring of type W_2: if M_r has 1s on the main diagonal, we have directly W_2''', and if it has 0s on the main diagonal, transition to and swapping both colors yields W_2''. Hence X contains an r-wealthy coloring of type W_2 for every $r \in \mathbb{N}$ and by Proposition 2.31, $|X_n| \geq F_n$ for every $n \in \mathbb{N}$.

In the case (a) we proceed similarly to the case (b). Now $E = \{v < x < e_2\}$, $F = \{v < x < f_2\}$, $v \in S_{j_0}$, $x \in S_{j_1}$, $e_2 \in S_{j_2}$, and $f_2 \in S_{j_3}$ for four distinct indices $j_i \in [r]$ with $j_1 < j_2 < j_3$. The case when $j_0 \notin [j_2, j_3]$ leads to W_1' colorings, and the case $j_0 \in [j_2, j_3]$ leads to W_1' colorings or to W_2 colorings (obtained by symmetries from the W_2'' coloring).

The last part of step 2 is to consider triples E such that $|E \cap Z| = 1$ and $|E \cap S_i| \leq 1$ for any $i \in [r]$, but they do not share the vertex of Z. However, by the shift condition it follows that there are two of the described triples E, E' such that $\chi_r(E) \neq \chi_r(E')$, $E \cap Z \neq E' \cap Z$, and $E \cap Y = E' \cap Y$. Hence X contains for every r an r-wealthy coloring of type W_1' and we are done by Lemma 2.25.

So all the described triples have the same color which we call s, and by $t \neq s$ we denote the other color.

Step 3: triples E that for two distinct $i, j \in [r]$ satisfy $|E \cap S_i| = 2$ and $|E \cap S_j| = 1$.

First we show that all triples E that for two distinct $i, j \in [r]$ satisfy $|E \cap Y \cap S_i| = 2$ and $|E \cap Z \cap S_j| = 1$ have the same color $\chi_r(E) = s$ and defer the case $i \in E$, $|E \cap Y \cap S_i| = 1$ (and $|E \cap Y \cap S_j| = 1$) to the end of this step. Suppose not: there is a triple $E = \{v, e_1, e_2\}$ such that for two different indices $i, j \in [r]$ one has $v \in Z \cap S_j$, $e_1, e_2 \in Y \cap S_i$, and $\chi_r(E) = t$. Without loss of generality, $j < i$. Using the shift condition we may assume that $j = v = 1$ and $i = 2$, and further we may take $e_1 = r + 4$ and $e_2 = r + 5$. We show that for infinitely many, and hence for every, $r \geq 3$ and every binary string $w = w_1 w_2 \ldots w_{r-2}$ avoiding consecutive substring tt there exists a coloring $K_w = (r, \chi_w) \in X$ such that $\chi_w(\{1, i, i+1\}) = w_{i-1}$ for $i = 2, 3, \ldots, r - 1$. Indeed, we set

$$C = \{3i + r + 4 \mid w_i = s\}, \ D = \{3i + r + 2 \mid w_i = t\} \text{ and } S = \{1, r + 4\} \cup C \cup D,$$

and consider restriction and normalization of $K_r = (4r, \chi_r)$ to the set S. Since the triples $E = \{v_1, v_2, v_3\} \subset S$ with fixed first vertex $v_1 = 1$ and second vertex v_2 successing by third vertex v_3 satisfy $\chi_r(E) = 1$ if and only if $v_2 \in C$ and $v_3 \in D$, it is easy to see that the result is K_w. By part 1 of Lemma 2.29 we have $|X_n| \geq F_n$ for every n.

An example of the coloring K_w for $r = 13$, $w = 01010001010$, and $t = 1$ is given in Figure 2.4 (b). The elements $j \in [r]$ are in the first row, the triples $Y \cap S_j = \{r + 3j - 2, r + 3j - 1, r + 3j\}$ are placed vertically below them, the elements of S are circled and the mentioned triples of K_w consist of 1 and two elements from $C \cup D$ that are joined in the picture by segments labeled 0 or 1, according to the χ_w-color of the triple E.

Now we show that $\chi_r(E) = s$ for all triples E with $|E \cap Z| = 1, |E \cap Y| = 2$ that for some $i \in [r]$ satisfy $i \in E$ and $|E \cap S_i| = 2$. For contrary, let $\chi_r(E) = t$ for an triple $E = \{i, e_1, e_2\}$ of this form, where we additionally may suppose that $e_1 = r + 3i - 2$, $e_2 = r + 3j - 2$, and $j > i$ (the case $j < i$ is treated similarly). It is clear that for $r \geq 1$ the restriction and normalization of K_r to
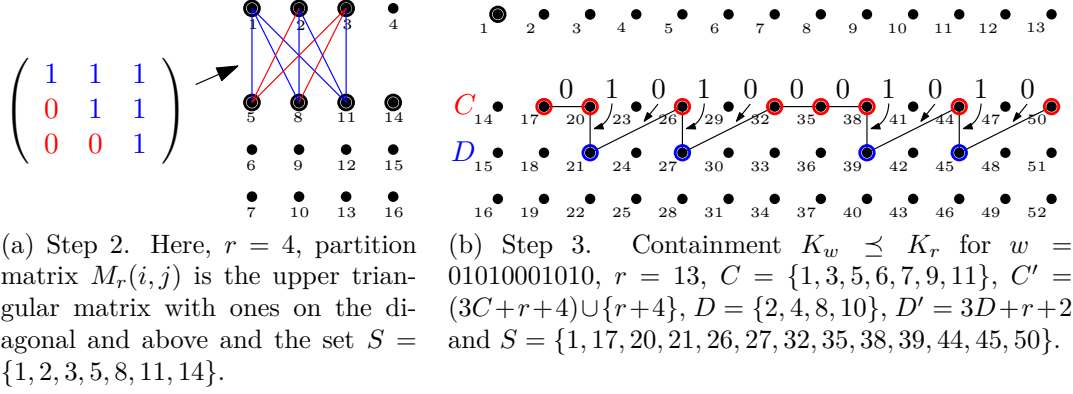
(a) Step 2. Here, $r = 4$, partition matrix $M_r(i,j)$ is the upper triangular matrix with ones on the diagonal and above and the set $S = \{1,2,3,5,8,11,14\}$.

(b) Step 3. Containment $K_w \preceq K_r$ for $w = 01010001010$, $r = 13$, $C = \{1,3,5,6,7,9,11\}$, $C' = (3C+r+4) \cup \{r+4\}$, $D = \{2,4,8,10\}$, $D' = 3D+r+2$ and $S = \{1,17,20,21,26,27,32,35,38,39,44,45,50\}$.

Figure 2.4: Two examples of the restrictions of the original wealthy coloring K_r of type $W_{4,2}$ to the set S (marked by bigger dots).

the set $[r-1] \cup (3[r-1]-2) \cup \{4r-2\}$ is an $(r-1)$-wealthy coloring of type W_2'. By Proposition 2.31, $|X_n| \geq F_n$ for every n.

Step 4: Conclusion. Thus we may assume, due to the shift condition and due to the previous steps, that in the coloring K_r all triples E with $|E \cap Z| = 1$, $|E \cap Y| = 2$, and $|E \cap S_i| \leq 2$ for every $i \in [r]$ have the same color s, say $\chi_r(E) = s = 1$. Since K_r is an r-wealthy coloring of type $W_{4,2}$, for every $i \in [r]$ there is an triple $F \subset S_i$ such that $i \in F$ and $\chi_r(F) = 0$. Without loss of generality we suppose that $F = \{i, r+3i-2, r+3i-1\}$.

We finally show that $|X_n| \geq F_n$ for $n \geq n_0$. Let $n = 5m + \varepsilon$ where $\varepsilon \in \{0,1,2,3,4\}$ and

$$A = \{a_1 < a_2 < \cdots < a_m\} \subset [2m+\varepsilon] \text{ and } B = \{b_1 < b_2 < \cdots < b_m\} \subset [2m]$$

be two m-element sets. A coloring $K = (n, \chi)$ is (A,B)-*disobedient* if $\chi(E) = 1$ for all triples E such that $|E \cap [2m+\varepsilon]| = 1$ and $|E \cap [2m+\varepsilon+1, 5m+\varepsilon]| = 2$, except for the triples

$$F_i = \{a_i, 2m+\varepsilon+b_i+i-1, 2m+\varepsilon+b_i+i\}, \ i \in [m], \text{ with } \chi(F_i) = 0.$$

To mark K as (A,B)-disobedient we write $K = K_{A,B}$ (more precisely it means that $K \in K_{A,B}$). Clearly, for different pairs $(A,B) \neq (A',B')$ of m-element subsets of $[2m]$ we always have $K_{A,B} \neq K_{A',B'}$ (more precisely, $K_{A,B} \cap K_{A',B'} = \emptyset$). Let $r = 3m + \varepsilon$. We prove that for every pair (A,B) of two m-element sets as displayed above the coloring $K_r = K_{3m+\varepsilon} = (12m+4\varepsilon, \chi_r)$ contains an (A,B)-disobedient coloring $K_{A,B}$.

For it we define $a_0 = b_0 = 0$, $a_{m+1} = 2m+\varepsilon+1$ and $b_{m+1} = 2m+1$, and for $i = 0,1,\ldots,m$ the quantities $\alpha_i = a_{i+1} - a_i - 1$ and $\beta_i = b_{i+1} - b_i - 1$ which count the elements of the ground sets $[2m+\varepsilon]$, resp. $[2m]$, in the gaps determined by the elements in A and in B. Further, let $t_0 = 0$ and

$$t_i = i + \sum_{j=0}^{i-1} (\alpha_j + \beta_j) = a_i + b_i - i$$

for $i \in [m]$. For $i = 0,1,\ldots,m$ we define the gap sets $C_i = \{t_i+1, \ldots, t_i+\alpha_i\}$ and $D_i = \{t_i+\alpha_i+1, \ldots, t_i+\alpha_i+\beta_i\}$. Clearly, $|C_i| = \alpha_i$, $|D_i| = \beta_i$, and

$$C_0 < D_0 < \{t_1\} < C_1 < D_1 < \{t_2\} < C_2 < D_2 < \cdots < \{t_m\} < C_m < D_m \quad (2.7)$$

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is a partition of $[r] = [3m+\varepsilon]$ into $3m+2$ (possibly empty) sets. Finally, we label some subsets of $S_i = \{i, r+3i-2, r+3i-1, r+3i\}$: $S_i^1 = \{i\}$, $S_i^2 = \{r+3i-2\}$, $S_i^{23} = \{r+3i-2, r+3i-1\}$, $S_i^{123} = S_i^1 \cup S_i^{23}$, and set

$$S = \bigcup_{i=1}^{m} S_{t_i}^{123} \cup \bigcup_{i=0}^{m} \left(\bigcup_{j \in C_i} S_j^1 \cup \bigcup_{j \in D_i} S_j^2 \right) \subset [12m + 4\varepsilon] = [4r].$$

The partition (2.7) of $[3m+\varepsilon]$ implies that $|S| = 5m+\varepsilon = n$. We show that restriction and normalization of $K_r = (4r, \chi_r)$ to the set S defines an (A, B)-disobedient coloring. Let $f: S \to [n]$ be the increasing bijection. First, $f(t_i) = a_i$, that is, $f(S_{t_i}^1) = \{a_i\}$, because (note that $|S_j^x|$ equals to the length of the word x) t_i has in S exactly

$$\sum_{l=1}^{i-1} |S_{t_l}^1| + \sum_{l=0}^{i-1} \sum_{j \in C_l} |S_j^1| = i - 1 + \sum_{l=0}^{i-1} |C_l| = i - 1 + (a_i - a_0 - i) = a_i - 1$$

predecessors. Also,

$$f(Z \cap S) = f([r] \cap S) = f\left(\bigcup_{i=1}^{m} S_{t_i}^1 \cup \bigcup_{i=0}^{m} \bigcup_{j \in C_i} S_j^1 \right) = [2m + \varepsilon]. \qquad (2.8)$$

because the set inside the large brackets has $m + \sum_{i=0}^{m} |C_i| = 2m + \varepsilon$ elements. Similarly, $f(S_{t_i}^{23}) = \{2m+\varepsilon+b_i+i-1, 2m+\varepsilon+b_i+i\}$ because $S_{t_i}^{23}$ has in $Y = [r+1, 4r] \cap S$ exactly

$$\sum_{l=1}^{i-1} |S_{t_l}^{23}| + \sum_{l=0}^{i-1} \sum_{j \in D_l} |S_j^2| = 2(i-1) + \sum_{l=0}^{i-1} |D_l| = 2(i-1) + b_i - b_0 - i = b_i + i - 2$$

predecessors. Hence $f(S_{t_i}(123)) = F_i$, the i-th triple of an (A, B)-disobedient coloring. In view of all of this (especially recall the form of the coloring χ_r mentioned at the beginning of Step 4), it follows that the restriction and normalization of $K_r = (4r, \chi_r)$ to S (which is the coloring $(n, \chi_r \circ f^{-1})$ where for f^{-1} we abuse notation a little in the obvious way) is an (A, B)-disobedient coloring $K_{A,B}$—see Figure 2.5 for a concrete example (like in Figure 2.4, the quadruples S_i are visualized by columns of four dots, and their first "elements" S_i^1 forming $Y = [r]$ lie in the topmost row).

Different pairs A, B of m-element subsets $A \subset [2m + \varepsilon]$ and $B \subset [2m]$ give distinct (A, B)-disobedient coloring $K_{A,B}$, therefore for every $m \geq 1$ we have

$$|X_n| \geq \binom{2m+\varepsilon}{m} \binom{2m}{m} \geq \binom{2m}{m}^2 \quad \text{where} \quad n = 5m + \varepsilon \leq 5m + 4$$

and the value $m = (n-4)/5$ completes the proof of the first part of the statement.

It remains to find an m_0, resp. m_1, such that if $m \geq m_0$, resp. $m \geq m_1$ then

$$|X_n| \geq \binom{2m}{m}^2 \geq G_{5m+4} \geq G_n, \quad \text{resp.} \quad |X_n| \geq \binom{2m}{m}^2 \geq F_{5m+4} \geq F_n.$$

To find m_0 we again use the bounds on middle binomial coefficient of N. D. Kazarinoff [42]. Thus, for every $m \in \mathbb{N}$,

$$\binom{2m}{m}^2 \geq \frac{16^m}{4m} \quad \text{and} \quad 0.418 \cdot 1.466^m > G_m.$$
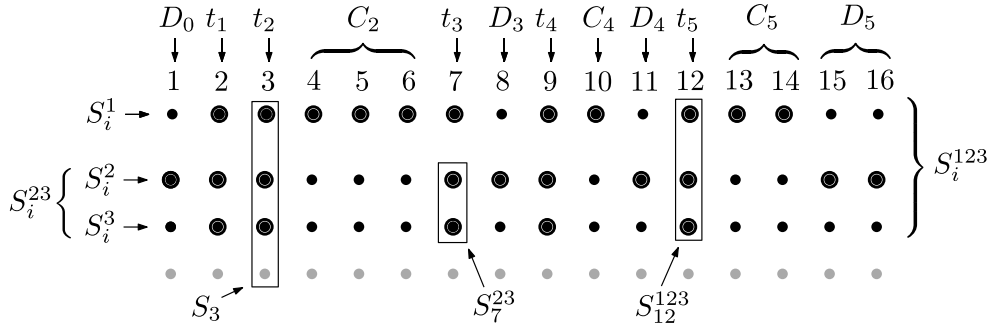
Figure 2.5: Step 4. An example of containment $K_{A,B} = (26, \chi) \preceq K_r = (64, \chi)$ where the set S is marked by bigger dots. Here $n = 26$, $r = 16$, $A = \{1, 2, 6, 7, 9\}$, and $B = \{2, 3, 4, 6, 8\}$.

Since $\frac{1}{4 \cdot 0.418 \cdot 1.466^4} > 0.129$ and $\frac{16}{1.466^5} > 2.362$, we need an m_0 with $m \geq m_0$ implying $0.129 \cdot 2.362^m \geq m$. It is easy to check that this holds with $m_0 = 4$. Thus in Step 4, $|X_n| \geq G_n$ holds for every $n \geq 5 \cdot 4 = 20$.

To find m_1 we proceed similarly with

$$|X_n| \geq \binom{2m}{m}^2 \geq \frac{16^m}{4m} > 3.064 \cdot 11.090^m \geq F_{5m+4} \geq F_n$$

for any $m \geq 15$, thus $n \geq 75$. That concludes the second part of the statement and the proof of Proposition 2.44 is complete. $\qquad\square$

2.4.2 Crossing matrices

In this section we establish the necessary groundwork to prove Theorem 2.12. First we review some results on two-dimensional matrices. Then we introduce three-dimensional "crossing" matrices with entries in $\{0, 1, *\}$ derived from colorings, and reduce by Lemmas 2.71, 2.72 and 2.74 problems on three-dimensional matrices to two-dimensional situation. However, we must deal with many new situations that may happen in three dimensions.

We recall the two-dimensional results and call two- or three-dimensional matrices simply matrices when their dimension is clear from the context. To simplify that, we denote two-dimensional matrices by N and three-dimensional matrices by M.

Results on two-dimensional matrices

We review some results in the article [44] in order that we can use them later; they also inspired this article. Then we give and prove one new result on two-dimensional matrices that we also use later. For $r, s \in \mathbb{N}$ let

$$N : [r] \times [s] \to \{0, 1\}$$

be an $r \times s$ binary matrix.

Definition 2.45 (Number al(N) of binary changes)**.** *Every row and column of an $r \times s$ binary matrix N consists of alternating intervals of zeros and ones. We*

denote by al(N) *the maximum number of these intervals in a row or a column of* N.

Equivalently, al(N) is one plus the maximum number of subwords 01 and 10 in a row or a column of N.

Definition 2.46 (Positions $R(N, j)$, $C(N, i)$, $R(N)$ and $C(N)$). *For $j \in [s]$ we let $C(N, j) \subset [r]$ denote the set of the row indices of the largest entries of these intervals in column j, with r omitted: $i \in C(N, j)$ if and only if $N(i, j) \neq N(i + 1, j)$. We set $C(N) = \bigcup_{j=1}^{s} C(N, j)$. Similarly, for $i \in [r]$ we let $R(N, i) \subset [s]$ be the column indices j such that $N(i, j) \neq N(i, j + 1)$ and $R(N) = \bigcup_{i=1}^{r} R(N, i)$.*

Unlike $|C(\cdot)|$, the quantity $|R(\cdot)|$ is in [44] defined only implicitly as the number a in [44, Lemma 3.12], which is the next lemma. Below in Proposition 2.53 and in the proof of Proposition 2.90 we consider more general matrices N with entries 0, 1 and $*$. Then, as before, al(N) is one plus the maximum number of subwords 01 and 10 in a row or a column of N, $R(N, i)$ is the set of j such that $\{N(i, j), N(i, j + 1)\} = \{0, 1\}$, $R(N) = \bigcup_{i=1}^{r} R(N, i)$ and also $C(N, j)$ and $C(N)$ are defined as before. The following paraphrases a lemma in [44].

Lemma 2.47 (Klazar [44], 2008). *For every binary matrix N,*

$$|R(N)| \leq (\mathrm{al}(N) - 1)(2|C(N)| + 1).$$

By transposing the matrix we get the same bound with exchanged $R(N)$ and $C(N)$. Thus if a sequence $(\mathrm{al}(N_n))_{n \geq 1}$, where N_n are binary matrices, is bounded then either both sequences $(R(N_n)|)_{n \geq 1}$ and $(|C(N_n)|)_{n \geq 1}$ are bounded, or both are unbounded.

Let I_r be the $r \times r$ identity matrix with 1's on the main diagonal and 0's elsewhere and U_r be the $r \times r$ upper diagonal matrix with 1's on the main diagonal and above it and 0's below it.

Definition 2.48 (Similar and strongly similar matrices). *Two square matrices N' and N are similar if $N = N'$ or one arises from the other by vertical and/or horizontal flip and/or by exchanging 0 and 1. If the horizontal flip is not allowed, we say that N' and N are strongly similar.*

Remark 2.49. *In [44, bottom of p. 15] strong similarity is called similarity. Our relation of similarity is not used in [44].*

The notion of similar matrices result from the symmetries of colorings appearing in Section 2.4.1 in the definitions of colorings of types W_2 and W_3. Strong similarity implies similarity. There are four matrices that are both similar and strongly similar to I_r, four matrices strongly similar to U_r and eight matrices similar to U_r.

Definition 2.50 (Containment of matrices). *For two matrices N and N' we say that N' is contained in N, and write $N' \preceq N$, if N' arises from N by deleting some rows and some columns. In other words, if*

$$N' : [r'] \times [s'] \to \{0, 1, *\} \quad and \quad N : [r] \times [s] \to \{0, 1, *\}$$

then for some increasing injections $f : [r'] \to [r]$ *and* $g : [s'] \to [s]$ *we have for every* $i \in [r']$ *and* $j \in [s']$ *that*

$$N'(i, j) = N(f(i), g(j)).$$

We then also say that N' *is a submatrix of* N.

Another result from [44] used in the end of the proof of Theorem 2.12 is the next one.

Lemma 2.51 (Klazar [44], 2008). *Let* $(N_n)_{n \geq 1}$ *be an infinite sequence of binary matrices such that the sequence* $(\mathrm{al}(N_n))_{n \geq 1}$ *is bounded but the sequence* $(|C(N_n)|)_{n \geq 1}$ *is unbounded. Then one of the following holds.*

(i) *For every* $r \in \mathbb{N}$ *there is an* n *such that* N_n *contains a matrix* I'_r *strongly similar to* I_r.

(ii) *For every* $r \in \mathbb{N}$ *there is an* n *such that* N_n *contains a matrix* U'_r *strongly similar to* U_r.

By transposing the matrices N_n we deduce that this lemma holds also when $(|C(N_n)|)_{n \geq 1}$ is replaced with $(|R(N_n)|)_{n \geq 1}$. In Lemma 2.71 we lift Lemma 2.51 to three-dimensional matrices.

In the end of the proof of Theorem 2.12 we need the next result which does not appear in [44] and which we prove here. We recall further terminology from [44] that we need in next proposition.

Definition 2.52 (Wealthy colorings of type 2). *A coloring* $\chi = ([3r], \chi)$, $r \in \mathbb{N}$, *of pairs (i.e.* $\chi : \binom{[3r]}{2} \to \{0, 1\}$) *is* r-*wealthy of type 2 if none of the* r *triples* $\{3i - 2, 3i - 1, 3i\}$, $i \in [r]$, *is* χ-*monochromatic.*

Proposition 2.53. *Let* $(N_n)_{n \geq 1}$ *be a sequence of symmetric square matrices with* $*s$ *on the main diagonal and* $0s$ *and* $1s$ *outside it and such that the sequence* $(\mathrm{al}(N_n))_{n \geq 1}$ *is bounded but the sequence*

$$(|R(N_n)|)_{n \geq 1} = (|C(N_n)|)_{n \geq 1}$$

is unbounded. Each matrix N_n *defines a binary coloring* $K_n = ([r(n)], \psi_n)$ *of pairs, where* $r(n)$ *is the number of rows or columns in* N_n, *by*

$$\psi_n(\{i, j\}) = N_n(i, j) = N_n(j, i).$$

Then (α) *or* (β) *holds.*

(α) *For every* $r \in \mathbb{N}$ *there is an* n *such that* K_n *contains an* r-*wealthy coloring of type* 2.

(β) *For every* $r \in \mathbb{N}$ *there is an* n *such that* N_n *contains a binary matrix* I'_r *strongly similar to* I_r, *or for every* $r \in \mathbb{N}$ *there is an* n *such that* N_n *contains a binary matrix* U'_r *strongly similar to* U_r. *Moreover, for every* r *the matrices* I'_r *and* U'_r *lie above the diagonal of the matrix* $N_{n(r)}$.

Proof. For a given K_n we consider (as in [44]) the *interval decomposition of K_n* which is the interval partition $I_{n,1} < I_{n,2} < \cdots < I_{n,k(n)}$ of $[r(n)]$ such that $I_{n,1}$ is the longest initial interval in $[r(n)]$ with all pairs in $\binom{I_{n,1}}{2}$ having the same color in ψ_n, $I_{n,2}$ is the longest monochromatic interval following after $I_{n,1}$, and so on. In Section 2.4.3 we use the same decomposition (called there nuclear decomposition) for colorings of triples. Clearly, for every $i < k(n)$ we have $|I_{n,i}| \geq 2$ and every interval $I_{n,i}$ with $i < k(n)$ contains elements $a = a_{n,i}, b = b_{n,i}$ such that $a \neq b$ and $\psi_n(\{a,b\}) \neq \psi_n(\{b, \min(I_{n,i+1})\})$. There are two cases, either the sequence $(k(n))_{n \geq 1}$ of lengths of the interval decompositions is unbounded or it is bounded.

In the former case, when the quantity $k(n)$ attains arbitrarily large values, we consider for $i = 1, 2, \ldots, \lfloor k(n)/2 \rfloor$ the triples $T_{n,i} = \{a_{n,2i-1}, b_{n,2i-1}, \min(I_{n,2i})\}$. These triples are disjoint, in fact

$$T_{n,1} < T_{n,2} < \cdots < T_{n,\lfloor k(n)/2 \rfloor},$$

and each triple $T_{n,i}$ is not monochromatic in ψ_n. Thus we get case (α) of the present proposition.

The latter case is that the quantity $k(n)$ as a function of n is bounded, say $k(n) \leq c$ for every n and a constant $c \in \mathbb{N}$. Then we consider for every pair of intervals $I = I_{n,i}, J = I_{n,j}$ with $1 \leq i, j \leq k(n)$ the submatrix $M_{I,J}$ of N_n formed by the positions in the rows with indices in I and in the columns with indices in J. Note that every diagonal matrix $M_{I,I}$ consists of $*$s on the main diagonal and of only 0s or only 1s elsewhere because every interval $I_{n,i}$ is ψ_n-monochromatic. Also, $M_{I,J} = M_{J,I}^T$ and if $i < j$ then $M_{I,J}$ lies above the diagonal of N.

We claim that for any $n \in \mathbb{N}$ there exist indices $i_n, j_n \in [k(n)]$ with $i_n < j_n$ and such that for $I_n = I_{n,i_n}, J_n = I_{n,j_n}$ the sequence $(|R(M_{I_n,J_n})|)_{n \geq 1}$ is unbounded. To see it, recall that $R(N_n)$ is the set of the column indices $j \in [k(n)]$ such that $\{N_n(i,j), N_n(i, j+1)\} = \{0, 1\}$ for some row index $i = i_j \in [k(n)]$. The column indices $j \in R(N_n)$ are of two kinds. The first js are such that the two positions (i_j, j) and $(i_j, j+1)$ lie in one matrix $M_{I,J}$; both j and $j+1$ lie in one interval J of the interval decomposition of K_n. The second js are the remaining ones when j and $j+1$ lie in two consecutive intervals of the interval decomposition of K_n. So if a j is of the second kind then $j = \max(J)$ for an interval J of the interval decomposition of K_n and there are at most c of them. Also, we already noted that for no j of the first kind the two positions (i_j, j) and $(i_j, j+1)$ lie in a diagonal matrix $M_{I,I}$ and we know there are at most c^2 matrices $M_{I,J}$. Thus there exist intervals $I_n < J_n$ such that

$$|R(M_{I_n,J_n})| \geq \frac{|R(N_n)| - c}{c^2}$$

and the sequence $(|R(M_{I_n,J_n})|)_{n \geq 1}$ is unbounded. Since the sequence $(\mathrm{al}(N_n))_{n \geq 1}$ is bounded, $(\mathrm{al}(M_{I_n,J_n}))_{n \geq 1}$ is bounded too. We may therefore apply Lemma 2.51 to the sequence of binary matrices $(M_{I_n,J_n})_{n \geq 1}$ and get case (β) of the present proposition. $\square$

Three-dimensional matrices

We extend the notion of binary matrices in two directions. First, we consider three-dimensional matrices, where any position has three coordinates. Then we

allow three types of entries; binary colors $0, 1$ and new star symbol $*$. This new symbol means "position not defined", as one may observe from following definitions.

Definition 2.54 ($*$-binary matrices). *A (three-dimensional) matrix M is a map $M : [r] \times [s] \times [t] \to \{0, 1, *\}$, $r, s, t \in \mathbb{N}$, we also say that M is $*$-binary.*

We visualize three-dimensional matrices M as cubes in $\mathbb{R}^3$, with edges parallel to the coordinate axes and such that the origin in M is the front top left corner, the first coordinate increases in the left-to-right direction, the second one in the top-to-bottom direction, and the third one in the front-to-back direction, as shown in Figure 2.6. We will work with special $*$-binary matrices derived from colorings which have entry $*$ when two of the three elements in a triple in the binary coloring coincide, as given in the next formal definition.

Definition 2.55 (Crossing matrix and its base sets). *Let $H = (n, \chi)$ be a coloring of triples and*

$$X = \{x_1 < \cdots < x_r\}, \ Y = \{y_1 < \cdots < y_s\} \ and \ Z = \{z_1 < \cdots < z_t\}$$

be nonempty subsets of $[n]$. We call the $r \times s \times t$ matrix $M : [r] \times [s] \times [t] \to \{0, 1, *\}$, given as

$$M(i, j, k) = \begin{cases} \chi(\{x_i, y_j, z_k\}) & \text{if } x_i, \ y_j \ and \ z_k \ are \ three \ distinct \ elements \ and \\ * & \text{else (when } |\{x_i, y_j, z_k\}| \leq 2) \, , \end{cases}$$

the crossing matrix of H and denote it $M_{X,Y,Z}$. The sets X, Y and Z are the base sets of the crossing matrix M.

Clearly, a crossing matrix is binary if and only if its base sets are disjoint. Now we dive deeply to the structure of crossing matrices. Thus, we need notation for particular subsets of crossing matrices.

Definition 2.56 (Rows, columns, shafts and lines of M). *Let M be an $r \times s \times t$ $*$-binary matrix. For fixed $J \in [s]$ and $K \in [t]$, the finite sequence*

$$\overline{r}(J, K) := (M(i, J, K))_{i=1}^{r} \in \{0, 1, *\}^r$$

is a row of M. In some situations we understand under a row also the set

$$\overline{r}(J, K) = \{(i, J, K) : \ i \in [r]\}$$

of its positions in M. We define columns $\overline{c}(I, K)$ and shafts $\overline{s}(I, J)$ in M similarly by fixing the first and third, respectively the first and second, coordinate of M. Rows, columns, and shafts are the lines of M.

A row in a two-dimensional matrix has the first coordinate fixed and the second one variable. However, a row in a three-dimensional matrix here has the first coordinate variable and the second and third one fixed. Similarly for columns in which case the second coordinate is variable, while the first and third one are fixed. This discrepancy in terminology for two- and three-dimensional matrices forces us later to exchange coordinates in layer and cross-matrices (defined below). A three-dimensional $r \times s \times t$ matrix has st rows, rt columns and rs shafts, but they cannot be ordered in a natural way as rows and columns in two-dimensional matrices.

Definition 2.57 (Number al(M) for $*$-binary matrices). *Let M be an $r \times s \times t$ $*$-binary matrix. Let al(M) be one plus the maximum number of the pairs $a_i a_{i+1} \in \{01, 10\}$ in a line of M, taken over all $st + rt + rs$ lines of M.*

Definition 2.58 (Positions of binary changes). *For any row given by the coordinates (J, K) we define $R(M; J, K) \subset [r]$ to be the indices i such that we have $M(i, J, K)M(i + 1, J, K) \in \{01, 10\}$. We set*

$$R(M) = \bigcup_{J, K} R(M; J, K),$$

with the union over all rows of M. Analogously, we define $C(M; I, K) \subset [s]$, $S(M; I, J) \subset [t]$,

$$C(M) = \bigcup_{I, K} C(M; I, K) \qquad and \qquad S(M) = \bigcup_{I, J} S(M; I, J).$$

Note that $R(M) \subset [r]$ contains exactly the indices i such that the two (j, k)-entries in the two-dimensional matrices $M(i, j, k)$ and $M(i + 1, j, k)$ are 0 and 1.

Example. Let M be the three-dimensional $5 \times 3 \times 2$ $*$-binary matrix given by two two-dimensional matrices $N_1(a, b) = M(b, a, 1)$ and $N_2(a, b) = M(b, a, 2)$, where in both cases $a \in [5]$ and $b \in [3]$ that may bee seen as layers of M, where

$$N_1 = \begin{pmatrix} * & 0 & 1 & 1 & 1 \\ 0 & 0 & * & 0 & 0 \\ 1 & 1 & * & * & * \end{pmatrix} \qquad and \qquad N_2 = \begin{pmatrix} 1 & 0 & * & * & 1 \\ 1 & 0 & * & 1 & 0 \\ 0 & * & 1 & * & 0 \end{pmatrix}.$$

One may observe the coordinate swap in the definition of N_1 and N_2 that is caused by the different definition of rows in two- and three-dimensional matrices. Now one may see that

$$\begin{array}{lll} R(M; 1, 1) = \{2\}, & R(M; 1, 2) = \{1\} & R(M) = \{1, 2, 4\} \\ R(M; 2, 1) = \emptyset, & R(M; 2, 2) = \{1, 4\} \quad and & C(M) = \{1, 2\} \\ R(M; 3, 1) = \emptyset, & R(M; 3, 2) = \emptyset & S(M) = \{1\} \end{array}$$

$\triangle$

Note that if M is a crossing matrix of a coloring K then any line of M either contains at most two stars or consists only of stars. Indeed, for any line two coordinates are fixed and one varies. Therefore, if the fixed coordinates represent the same vertex, we end with a line of stars. Otherwise, there are at most two values for the changing coordinate that produces star, namely values of the fixed coordinates. One can prove following analogue of Lemma 2.47

Lemma 2.59. *For every $r \times s \times t$ binary matrix M, if al(M) $\leq l$ and both $|R(M)|$ and $|C(M)|$ are bounded by m then*

$$|S(M)| \leq (l - 1)(m + 1)^2.$$

Moreover, two symmetric variants hold: $|S(M)|$ may be switched with $|R(M)|$ or with $|C(M)|$.

Proof. We assume that $\mathrm{al}(M) \leq l$, $|R(M)| \leq m$ and $|C(M)| \leq m$. The other two symmetric variants are treated in the same way and we omit their proof. We first bound the cardinality of the set D consisting of the pairs $(x, y) \in [r] \times [s]$ such that $M(x, y, k) \neq M(x, y, k+1)$ for some $k \in [t-1]$ but $M(x', y', k) = M(x', y', k+1)$ whenever $x' + y' < x + y$. Let $(x, y) \in D$ with $x, y > 1$. Then

$$M(x,\, y,\, k) \neq M(x,\, y,\, k+1) \quad \text{but} \quad M(x-1,\, y,\, k) = M(x-1,\, y,\, k+1)$$

and $x - 1 \in R(M)$. Also $y - 1 \in C(M)$, by a symmetric argument. We have $x, y > 1$ and the map

$$D \to R(M) \times C(M) \quad \text{such that} \quad (x,\, y) \mapsto (x-1,\, y-1)$$

is an injection, hence D has at most m^2 such elements (x, y). By a similar argument D has at most $2m$ elements of the form $(1, y)$ and $(x, 1)$ with $x, y > 1$. Thus $|D| \leq m^2 + 2m + 1 = (m+1)^2$. We consider the map

$$S(M) \to D \quad \text{such that} \quad k \mapsto (x,\, y) \quad (\text{with } M(x,\, y,\, k) \neq M(x,\, y,\, k+1)).$$

If $|S(M)| > (l-1)(m+1)^2$ then there exists a pair $(x, y) \in D$ and l shaft indices k_i with $1 \leq k_1 < k_2 < \cdots < k_l < t$ such that $M(x, y, k_i) \neq M(x, y, k_i + 1)$ for every $i \in [l]$. The shaft $\bar{s}(x, y)$ has at least $l+1$ alternating intervals of zeros and ones, in contradiction with the bound $\mathrm{al}(M) \leq l$. $\qquad \square$

Thus in the definition of p-tame colorings in Section 2.4.3 it does not matter which two of the three quantities $R(M)$, $C(M)$ and $S(M)$ we chose; we select $R(M)$ and $C(M)$.

Definition 2.60 (Containment of $*$-binary matrices)**.** *For two $*$-binary matrices*

$$M: \; [r] \times [s] \times [t] \to \{0,\, 1,\, *\} \quad \textit{and} \quad M': \; [r'] \times [s'] \times [t'] \to \{0,\, 1,\, *\}$$

we say that M' is a submatrix of M, and write $M' \preceq M$, if there are increasing injections $f: \; [r'] \to [r]$, $g: \; [s'] \to [s]$ and $h: \; [t'] \to [t]$ such that for every $(I, J, K) \in [r'] \times [s'] \times [t']$ one has

$$M'(I, J, K) = M(f(I), g(J), h(K)).$$

We also say that M' is contained in M.

Easily, $M'' \preceq M'$ and $M' \preceq M$ imply $M'' \preceq M$. It is clear that if $M' \preceq M$ and $M = M_{X,Y,Z}$ is a crossing matrix of a coloring, then there are sets $X' \subset X$, $Y' \subset Y$ and $Z' \subset Z$ such that $M' = M_{X',Y',Z'}$ is a crossing matrix of the same coloring. More precisely, the sets X', Y' and Z' are given by

$$X' = X(f([r'])), \; Y' = Y(g([s'])) \; \text{ and } \; Z' = Z(h([t']))$$

where X, Y, Z, f, g, h, r', s' and t' are as above.

Definition 2.61 (Base sets for submatrices)**.** *We call the above sets X', Y' and Z' the base sets of the submatrix M' of matrix M.*

For a three-dimensional $r \times s \times t$ $*$-binary matrix M we define in a moment two-dimensional $*$-binary layer and cross- matrices N. They are motivated by colorings of type W_2 and W_3, respectively. We fix the order of kinds of lines in two- and three-dimensional matrices as: rows first, then columns and shafts last. We define the N derived from M so that this order is preserved when we go from the lines in N to their counterparts in M. Therefore we define a layer matrix N of M with exchanged coordinates; we already mentioned this terminological peculiarity. We do the same for cross-matrices (and illustrate it by the examples below) even if for them some lines in N do not have corresponding lines in M.

Definition 2.62 (Layer matrices). *For $i \in [3]$, N is an i-layer matrix of M if N arises from M by fixing the i-th coordinate and swapping the remaining ones.*

1. *N is a 1-layer matrix of M if $N(a,b) = M(z,b,a)$ with a fixed z.*

2. *N is a 2-layer matrix of M if $N(a,b) = M(b,z,a)$ with a fixed z.*

3. *N is a 3-layer matrix of M if $N(a,b) = M(b,a,z)$ with a fixed z.*

We call N a layer matrix of M if it is a i-layer matrix of M for some $i \in [3]$.

For example, in part 1 a row in N determines a column in M, a column in N determines a shaft in M, and the orders 'row' < 'column' and 'column' < 'shaft' agree. We need the following definition for the notion of base sets of layer matrices and cross-matrices (Definition 2.68).

Definition 2.63 (Index sets). *Let $A = \{a_1 < a_2 < \cdots < a_r\} \subset \mathbb{N}$, $A \neq \emptyset$, and $B \subset [r]$ be two sets. By index set $A(B)$ we mean the set*

$$A(B) := \{a_b : \; b \in B\} \; .$$

Definition 2.64 (Base sets for layer matrices). *If $M = M_{X,Y,Z}$ is a crossing matrix of a coloring with $r = |X|$, $s = |Y|$ and $t = |Z|$, and if N is the 1-layer matrix of M given by the fixed coordinate $z \in [r]$, then we call the three sets*

$$(X', Y', Z') := (X(\{z\}), Y, Z)$$

the base sets of the 1-layer matrix N of M. If N is a 2-layer (resp. a 3-layer) matrix of M, we define its base sets in the analogous way.

Definition 2.65 ((i,j)-d-cross-matrices). *For $\{i < j\} \subset [3]$, the (i,j)-d-cross-matrix N of M is defined by equating the i-th and j-th coordinate in M and swapping them with the remaining coordinate.*

1. *N is the $(1,2)$–d-cross-matrix of M if $N(a,b) = M(b,b,a)$ (we assume that $r = s$).*

2. *N is the $(1,3)$–d-cross-matrix of M if $N(a,b) = M(b,a,b)$ (we assume that $r = t$).*

3. *N is the $(2,3)$–d-cross-matrix of M if $N(a,b) = M(b,a,a)$ (we assume that $s = t$).*

For example, in part 2 a row in N determines variable coordinates $\{1,3\}$ in M, a column in N determines variable coordinate $\{2\}$ in M, and the orders 'row' < 'column' and $\{1,3\} < \{2\}$ (by the minimal elements) agree.

Definition 2.66 ((i,j)-ad-cross-matrices). *Similarly, for $\{i < j\} \subset [3]$ we define the (i,j)-ad-cross-matrix N of M as follows.*

1. *N is the $(1,2)$–ad-cross-matrix of M if $N(a,b) = M(b, s - b + 1, a)$ (we assume that $r = s$).*

2. *N is the $(1,3)$–ad-cross-matrix of M if $N(a,b) = M(b, a, t - b + 1)$ (we assume that $r = t$).*

3. *N is the $(2,3)$–ad-cross-matrix of M if $N(a,b) = M(b, a, t - a + 1)$ (we assume that $s = t$).*

For example, in part 3 a row in N determines variable coordinate $\{1\}$ in M, a column in N determines variable coordinates $\{2,3\}$ in M, and the orders 'row' < 'column' and $\{1\} < \{2,3\}$ (by the minimal elements) agree.

Definition 2.67 (Cross-matrices). *We say that N is an (i,j)-cross-matrix of M if it is an (i,j)-d-cross-matrix or an (i,j)-ad-cross-matrix of M. We say that N is a cross-matrix of M if it is an (i,j)-cross-matrix of M for some $\{i < j\} \subset [3]$.*

Definition 2.68 (Base sets for cross-matrices). *If $M = M_{X,Y,Z}$ is a crossing matrix of a coloring $K = (n, \chi)$ with $r = |X|$, $s = |Y|$ and $t = |Z|$, and if N is a cross-matrix of M, then we call the three sets X, Y, Z the base sets of the cross-matrix N of M.*

Thus a cross-matrix N of $M = M_{X,Y,Z}$ has the same base sets as M. Unlike for submatrices and layer matrices, now N cannot be viewed as a crossing matrix of the coloring K but is of course still determined by K. For example, if N is the $(1,3)$-ad-cross-matrix of $M = M_{X,Y,Z}$ then

$$N(i, j) = \chi(\ \{X(\{j\}), Y(\{i\}), Z(\{t - j + 1\})\}\ )$$

or $= *$ if two of the three elements in $\{\cdot, \cdot, \cdot\}$ coincide.

The above defined two-dimensional matrices N related to M are illustrated in Figure 2.6. In the following lemma whose easy proof we omit we review some relations between the operations of taking a submatrix, taking a layer matrix and taking a cross-matrix.

Lemma 2.69. *Let M be a three-dimensional $*$-binary matrix. Then for any two-dimensional $*$-binary matrix N' the following holds.*

1. *There is a layer matrix N of M such that $N' \preceq N$ if and only if there is a submatrix M' of matrix M such that N' is a layer matrix of M'.*

2. *If there is a cross-matrix N of M such that $N' \preceq N$ then there is a submatrix M' of matrix M such that N' is a cross-matrix of M'. The opposite implication in general does not hold.*
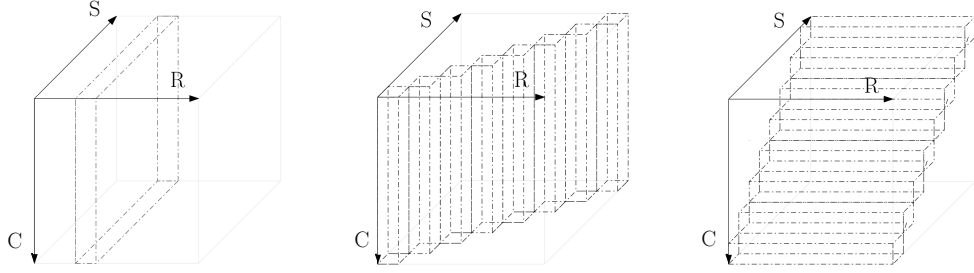
Figure 2.6: The dashed part of the matrix M displays a 1-layer matrix, the $(1,3)$-d-cross-matrix and the $(2,3)$-ad-cross-matrix.

In the direction $\Longleftarrow$ in part 1 matrices N' and M' determine N uniquely, and similarly in the direction $\Longrightarrow$ in part 2. In the direction $\Longrightarrow$ in part 1 the matrix M' is not determined uniquely by N' and N, and to make it unique we may take for example the smallest M'.

In Section 2.4.5 we work with matrices N' obtained from matrices M, N and M' in the ways described in the left and right sides in parts 1 and 2, and the base sets of N' will be important. It is straightforward to determine them if $M = M_{X,Y,Z}$ is a crossing matrix of a coloring, if we know the fixed coordinate for the layer matrix N, and if we know how M' and N' embed in their supermatrices. For illustration we describe it in detail in one of the four situations, the left side in part 1. Suppose that $M = M_{X,Y,Z}$ is a crossing matrix of a coloring and with $|X| = r$, $|Y| = s$ and $|Z| = t$, that N is the $t \times r$ 2-layer matrix of M given by the fixed coordinate $z \in [s]$, and that N' is the $t' \times r'$ submatrix of N given by the increasing injections $f : [t'] \to [t]$ and $g : [r'] \to [r]$. Then M has the base sets X, Y and Z, N has the base sets $X, Y(\{z\})$ and Z, and N' has the base sets $X(g([r'])), Y(\{z\})$ and $Z(f([t']))$.

Definition 2.70 (R-full and C-full matrices). *Let* $N : [r] \times [s] \to \{0, 1, *\}$ *be a two-dimensional $*$-binary matrix. We say that N is R-full if there exist r distinct column indices $s_i \in [s-1]$ such that*

$$\{N(i, s_i), N(i, s_i + 1)\} = \{0, 1\} \ \ for \ \ i = 1, 2, \ldots, r \,.$$

We define similarly that N is C-full if for some s distinct row indices r_i, for $i \in [s]$ the values $N(r_i, i)$ and $N(r_i + 1, i)$ are 0 and 1 or vice versa.

The following crucial lemma is a three-dimensional version of Lemma 2.51 and reduces three-dimensional matrices to two-dimensional ones.

Lemma 2.71. *Let $(M_n)_{n \geq 1}$ be an infinite sequence of three-dimensional $*$-binary matrices with the property that the sequence $(\mathrm{al}(M_n))_{n \geq 1}$ is bounded but the sequence $(|R(M_n)|)_{n \geq 1}$ is unbounded. Then (i) or (ii) holds.*

(i) *Every matrix M_n either has a 2-layer matrix N_n such that the sequence $(|R(N_n)|)_{n \geq 1}$ is unbounded, or every matrix M_n has a 3-layer matrix N_n such that the sequence $(|R(N_n)|)_{n \geq 1}$ is unbounded.*

(ii) *Every matrix M_n has a submatrix M'_n which has a $(2,3)$-cross-matrix N_n that is R-full and such that the sequence $(|R(N_n)|)_{n \geq 1}$ is unbounded.*

Proof. Let the sequences $(\mathrm{al}(M_n))_{n\geq1}$ and $(|R(M_n)|)_{n\geq1}$ be bounded and unbounded, respectively. Hence for every $n \in \mathbb{N}$ there is a finite sequence

$$\Big((J_{n,1},\, K_{n,1}),\, (J_{n,2},\, K_{n,2}),\, \ldots,\, (J_{n,m(n)},\, K_{n,m(n)})\Big)$$

of mutually distinct coordinates $(J_{n,j}, K_{n,j}) \in \mathbb{N}^2$ of rows in M_n such that the sequence $(m(n))_{n\geq1}$ is unbounded and for every $j \in [m(n)]$ we have that

$$R\left(M_n;\, J_{n,j},\, K_{n,j}\right) \setminus \bigcup_{i=1}^{j-1} R(M_n;\, J_{n,i},\, K_{n,i}) \neq \emptyset\,.$$

First we suppose that there is a sequence $(J_n)_{n\geq1} \subset \mathbb{N}$ such that the sequence

$$\left(|\{j \in [m(n)] \mid J_{n,j} = J_n\}|\right)_{n\geq1}$$

is unbounded, i.e. some second coordinate J has unbounded multiplicity. Then by the above displayed non-equality the 2-layer matrices

$$N_n(a,\, b) = M_n(b,\, J_n,\, a)$$

of M_n are such that the sequence $(|R(N_n)|)_{n\geq1}$ is unbounded and we have case (i). Similarly, if there is a sequence $(K_n)_{n\geq1} \subset \mathbb{N}$ such that the numbers of indices $j \in [m(n)]$ with $K_{n,j} = K_n$ form an unbounded sequence, then the 3-layer matrices $N_n(a,b) = M_n(b,a,K_n)$ of M_n give an unbounded sequence $(|R(N_n)|)_{n\geq1}$.

Suppose that the maximum multiplicity of both the J and the K coordinate is bounded and therefore the numbers of distinct J coordinates and of distinct K coordinates are unbounded. Using twice the Erdős–Szekeres lemma, which says that every sequence of $(k-1)^2+1$ numbers has a k-term monotone subsequence, we deduce that there is a sequence $(Z_n)_{n\geq1}$ of sets of indices

$$Z_n = \{j_{n,1} < j_{n,2} < \cdots < j_{n,|Z_n|}\} \subset [m(n)]$$

such that the sequence $(|Z_n|)_{n\geq1}$ is unbounded and for every n both sequences

$$\overline{J_n} = \left(J_{n,j_{n,i}}\right)_{1\leq i\leq|Z_n|} \quad \text{and} \quad \overline{K_n} = \left(K_{n,j_{n,i}}\right)_{1\leq i\leq|Z_n|}$$

are monotonic, each strictly increases or strictly decreases. We define

$$M_n^0(a,\, b,\, c) = M_n(a,\, J_{n,j_{n,b}},\, K_{n,j_{n,c}})$$

and consider the submatrix M_n' of M_n with the positions $(a, J_{n,j_{n,b}}, K_{n,j_{n,c}})$ where a runs in the first dimension of M_n and b,c run in $[|Z_n|]$. When the sequences $\overline{J_n}$ and $\overline{K_n}$ are monotonic in the same sense, both increase or both decrease, we take the $(2,3)$-d-cross-matrix

$$N_n(a,\, b) = M_n^0(b,\, a,\, a) \quad \text{or} \quad N_n(a,\, b) = M_n^0(b,\, |Z_n| - a + 1,\, |Z_n| - a + 1)$$

of M_n', respectively. When $\overline{J_n}$ increases and $\overline{K_n}$ decreases or vice versa, we take the $(2,3)$-ad-cross-matrix

$$N_n(a,\, b) = M_n^0(b,\, a,\, |Z_n| - a + 1) \quad \text{or} \quad N_n(a,\, b) = M_n^0(b,\, |Z_n| - a + 1,\, a)$$

of M_n', respectively. By the above displayed non-equality every matrix N_n is R-full. In particular, since $(|Z_n|)_{n\geq1}$ is unbounded, the sequence $(|R(N_n)|)_{n\geq1}$ is unbounded and we have case (ii). $\qquad\square$

When we replace rows with columns, we obtain the following symmetric result. It has an analogous (or symmetric) proof which we omit. But note that because of the interchange of coordinates in the definitions of layer and cross-matrices, we cannot just replace in Lemma 2.71 'R' with 'C'.

Lemma 2.72. *Let* $(M_n)_{n\geq 1}$ *be an infinite sequence of three-dimensional* $*$*-binary matrices with the property that the sequence* $(\mathrm{al}(M_n))_{n\geq 1}$ *is bounded but the sequence* $(|C(M_n)|)_{n\geq 1}$ *is unbounded. Then (i) or (ii) holds.*

(i) *Every matrix* M_n *either has a 1-layer matrix* N_n *such that the sequence* $(|R(N_n)|)_{n\geq 1}$ *is unbounded, or every matrix* M_n *has a 3-layer matrix* N_n *such that the sequence* $(|C(N_n)|)_{n\geq 1}$ *is unbounded.*

(ii) *Every matrix* M_n *has a submatrix* M'_n *which has a* $(1,3)$*-cross-matrix* N_n *that is* C*-full and such that the sequence* $(|C(N_n)|)_{n\geq 1}$ *is unbounded.*

Remark 2.73. *Note that in the case (ii) of Lemma 2.71 each row of the matrix of* N_n *is also a row of the matrix* M_n*, in the sense that if* $M_n = M_{X_n,Y_n,Z_n}$ *is a crossing matrix of a coloring then the three first base sets of the matrices* N_n*,* M'_n *and* M_n *are all equal to* X_n*. Similarly, in the case (ii) of Lemma 2.72 each column of* N_n *is in the analogous sense a column of the matrix* M_n*.*

Lemma 2.74. *Suppose that* $(M_n)_{n\geq 1}$ *is a sequence of three-dimensional* $*$*-binary matrices, that every matrix* M_n *has a binary submatrix* M'_n *such that the sequence* $(\mathrm{al}(M'_n))_{n\geq 1}$ *is bounded, and that every matrix* M'_n *has a cross-matrix* N_n *such that the sequence* $(\mathrm{al}(N_n))_{n\geq 1}$ *is unbounded. Then every matrix* M'_n *has a (binary) layer matrix* P_n *such that the sequence* $(\mathrm{al}(P_n))_{m\geq 1}$ *is bounded and both sequences* $(|R(P_n)|)_{n\geq 1}$ *and* $(|C(P_n)|)_{n\geq 1}$ *are unbounded.*

Proof. By passing to a subsequence of $n = 1, 2, \ldots$ we may assume that the type of cross-matrix N_n of M'_n is constant in n. For concreteness we assume that each N_n is a $(2,3)$-d-cross-matrix of M'_n, the other five cases for other cross-matrices are treated by similar arguments. Thus, by the above definition of cross-matrices,

$$N_n(a,\, b) = M'_n(b,\, a,\, a)\,.$$

Let L be a general three-dimensional $r \times s \times s$ binary matrix. We define a $(2,3)$-*diagonal of* L as $L(x,y,y)$ where $x \in [r]$ is fixed and y runs in $[s]$ and set

$$\mathrm{al}_{23d}(L) = 1 + \max_{x\in[r]} |\{i \in [s-1]:\ \{L(x,\, i,\, i),\, L(x,\, i+1,\, i+1)\} = \{0,\, 1\}\}|\,.$$

Unboundedness of the sequence $(\mathrm{al}(N_n))_{n\geq 1}$ implies unboundedness of the sequence $(\mathrm{al}_{23d}(M'_n))_{n\geq 1}$ because rows in N_n are rows in M'_n and thus contribute to $\mathrm{al}(N_n)$ by a bounded amount, but columns in N_n are $(2,3)$-diagonals in M'_n to which boundedness of $\mathrm{al}(M'_n)$ does not apply. Thus we see that every matrix M'_n has a 1-layer matrix P_n (see Figure 2.8 in Section 2.4.5 for a similar situation) such that, if $s(P_n)$ denotes the number of subwords 01 and 10 on the main diagonal of P_n, the sequence $(s(P_n))_{n\geq 1}$ is unbounded. Now let N be any two-dimensional binary $r \times r$ matrix with $s(N) = k \geq 1$ and $\mathrm{al}(N) = l$. Clearly, $l \geq 2$. Let $i \in [r-1]$ be such that

$$\{N(i,\, i),\, N(i+1,\, i+1)\} = \{0,\, 1\}\,,$$

that is, the position (i, i) on the diagonal of N contributes 1 to $s(N)$. Considering the position $(i + 1, i)$ in N we see that either $i \in R(N)$ or $i \in C(N)$. Thus either $|R(N)| \geq k/2$ or $|C(N)| \geq k/2$. If the latter inequality holds then by Lemma 2.47 we have that

$$|R(N)| \geq \frac{1}{2} \left(\frac{|C(N)|}{l - 1} - 1 \right) \geq \frac{1}{2} \left(\frac{k}{2(l - 1)} - 1 \right).$$

As this is less than $k/2$, the displayed lower bound on $|R(N)|$ is true also when the former inequality holds. Since the sequence $(\mathrm{al}(P_n))_{n \geq 1}$ is bounded (P_n is a layer matrix of M_n') and the sequence $(s(P_n))_{n \geq 1}$ is unbounded, by the displayed lower bound on $|R(N)|$ we see that the sequence $(|R(P_n)|)_{n \geq 1}$ is unbounded. By symmetry, this argument shows that also the sequence $(|C(P_n)|)_{n \geq 1}$ is unbounded. $\square$

To be precise, we will use this lemma in the situation when not $\mathrm{al}(M_n')$ but only the numbers $\mathrm{al}(M_n)$ are bounded, but M_n will be such that this still implies boundedness of $\mathrm{al}(M_n')$ for any (not necessarily binary) submatrix M_n' of M_n. This is explained in the remark just before Proposition 2.89.

2.4.3 Tame colorings

In this section we present special type of colorings called p-tame colorings. We prove that p-tame colorings contain only polynomially many subcolorings. Thus, in the number of subcolorings, they are kind of opposite to wealthy colorings.

Definition 2.75 (Nuclear decomposition). *For a coloring $H = (n, \chi)$ we define its nuclear decomposition $\mathrm{nu}(H)$ to be the interval partition $I_1 < I_2 < \cdots < I_s$, $s \in \mathbb{N}$, of $[n]$ such that I_1 is the longest initial χ-monochromatic interval, I_2 is the longest χ-monochromatic interval following after I_1, and so on.*

Each interval I_i in the nuclear decomposition of H is χ-monochromatic and for every $i \in [s - 1]$ we have that $|I_i| \geq 3$ and the set $I_i \cup \{\min(I_{i+1})\}$ is not χ-monochromatic. We note that if the nuclear decomposition of H has length $s \geq 2r$ then $H \succeq K$ where K is an r-wealthy coloring of type $W_{4,1}$. Indeed, it is easy to find triples $a_i, b_i, c_i \in I_{2i-1}$, $i \in [r]$, of mutually distinct elements such that none of the r quadruples

$$\{ a_i, b_i, c_i, \min(I_{2i}) \} \subset I_{2i-1} \cup I_{2i}, \ i \in [r],$$

is χ-monochromatic. Nuclear decompositions are analogous to the decompositions in the proof of Proposition 2.53.

Definition 2.76 (p-tame colorings). *Let $p \in \mathbb{N}$. We say that a coloring $H = (n, \chi)$ is p-tame if its nuclear decomposition*

$$\mathrm{nu}(H) = \{I_1 < I_2 < \cdots < I_s\}$$

satisfies the following five conditions (cf. the remark after Lemma 2.59).

    1. The length of the nuclear decomposition $s \leq p$.

2. For all integers u, v, w with $1 \leq u < v < w \leq s$ one has $\mathrm{al}(M_{I_u, I_v, I_w}) \leq p$ (see Section 2.4.2 for the definition of $\mathrm{al}(\cdot)$).

3. For all integers u, v, w with $1 \leq u < v < w \leq s$ one has $|R(M_{I_u, I_v, I_w})| \leq p$ and $|C(M_{I_u, I_v, I_w})| \leq p$ (see Section 2.4.2 for the definition of $R(\cdot)$ and $C(\cdot)$).

4. For all integers u, v with $1 \leq u < v \leq s$ one has $\mathrm{al}(M_{I_u, I_u, I_v}) \leq p$ and $\mathrm{al}(M_{I_u, I_v, I_v}) \leq p$.

5. For all integers u, v with $1 \leq u < v \leq s$ one has $|R(M_{I_u, I_u, I_v})| \leq p$, $|C(M_{I_u, I_u, I_v})| \leq p$, $|R(M_{I_u, I_v, I_v})| \leq p$ and $|C(M_{I_u, I_v, I_v})| \leq p$.

Proposition 2.77. *For every $n \geq 2$ and $p \geq 3$ there are at most n^{10p^6} p-tame colorings (n, χ).*

Proof. Let $p, n \in \mathbb{N}$ with $n \geq 2$ and $p \geq 3$ and $H = (n, \chi)$ be a p-tame coloring. We upper-bound the number of different colorings H. Recall the well known formula $\binom{n-1}{k-1}$ for the number of partitions of $[n]$ into k nonempty intervals $I_1 < I_2 < \cdots < I_k$. The number of possibilities for the interval partitions $I_1 < I_2 < \cdots < I_s$ of $[n]$ satisfying condition 1 and for the colors $\chi \,|\, \binom{I_i}{3} \equiv 0, 1$ as $i = 1, 2, \ldots, s$ is at most

$$c_1 = \sum_{s=1}^{p} \binom{n-1}{s-1} 2^s \leq n^{p-1}(2 + 2^2 + \cdots + 2^p) \leq (2n)^p.$$

The colors of the edges outside $\binom{I_i}{3}$ are determined by crossing matrices of three types:

(i) M_{I_u, I_v, I_w} for u, v and w with $1 \leq u < v < w \leq s$,

(ii) M_{I_u, I_u, I_v} for u and v with $1 \leq u < v \leq s$, and

(iii) M_{I_u, I_v, I_v} for u and v with $1 \leq u < v \leq s$.

First we bound, for fixed u, v and w, the number of matrices $M = M_{I_u, I_v, I_w}$ of type (i) satisfying conditions 2 and 3. Let $p_1 = |I_w| \leq n$. Using $\mathrm{al}(M) \leq p$ we bound the number of all possible types of shafts (as binary strings) in M by

$$c_2 = 2 \sum_{i=1}^{p} \binom{p_1 - 1}{i - 1} \leq c_1 \leq (2n)^p.$$

We set $p_2 = |I_v| \leq n$ and bound the number of two-dimensional 1-layer matrices of M obtained by fixing the first (row) coordinate. Each of these layer matrices consists of p_2 shafts, parametrized by the second coordinate, which form at most $|C(M)| + 1$ different intervals of shafts because two consecutive shafts may differ only when the first shaft has its (second) coordinate in $C(M)$. Each of these intervals of shafts can be selected in at most c_2 ways (by the type of shafts in the interval) and positions of these intervals (the second coordinate of the first shaft in the interval) in at most $\binom{p_2 - 1}{|C(M)|}$ ways. Hence we have an upper bound on the number of those two-dimensional 1-layer matrices

$$c_3 = \sum_{i=0}^{p} \binom{p_2 - 1}{i} c_2^{i+1} \leq (p+1)(n-1)^p (2n)^{p^2 + p} \leq (2n)^{2p^2}.$$

Finally, we bound the number of whole matrices M. Let $p_3 = |I_u| \leq n$. Like before, M consists of p_3 1-layer matrices (parametrized by the row coordinate) that form exactly $|R(M)| + 1$ different intervals. Thus the number of matrices M of type (i) is bounded by

$$c_4 = \sum_{i=0}^{p} \binom{p_3 - 1}{i} c_3^{i+1} \leq (p+1)(n-1)^p (2n)^{2p^3 + 2p^2} \leq (2n)^{4p^3}$$

(as before we use that $(p+1)(n-1)^p \leq (2n)^p$).

To bound, for fixed u and v, the number of crossing matrices of types (ii) and (iii) satisfying conditions 4 and 5 we note two things. First, these numbers for types (ii) and (iii) are equal (consider reversals). Second, the number of matrices of type (ii) has the same upper bound $(2n)^{4p^3}$ as that of matrices of type (i). This is because any matrix M of type (ii) is symmetric in the first two coordinates ($M(i, i, k) = *$ and $M(i, j, k) = M(j, i, k)$ for $i \neq j$) and therefore in every 1-layer matrix of M obtained by fixing the first coordinate to i it suffices to consider the triangle with the second coordinate j satisfying $j < i$. Now the bounds used in the previous paragraph apply also here and bound the number of these triangles.

Thus the number of interval partitions $I_1 < I_2 < \cdots < I_s$ of $[n]$, $s \leq p$, with monochromatic intervals is at most c_1, the triples $\{a < b < c\}$ with elements from three distinct intervals can be colored in at most $c_4^{\binom{s}{3}}$ ways, and those from two distinct intervals (either a, b or b, c lie in the same interval) in at most $c_4^{\binom{s}{2}}$ ways. Together there are at most

$$c_1 c_4^{2\binom{s}{2}} c_4^{\binom{s}{3}} \leq c_1 c_4^{p^3} \leq (2n)^{4p^6 + p} \leq n^{10p^6}$$

p-tame different colorings H. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.4.4 Six auxiliary lemmas on matrices and colorings

In this subsection we first state and prove six lemmas ensuring presence of r-wealthy colorings of type W_i in the colorings whose crossing matrices satisfy certain conditions, usually involving quantities $\mathrm{al}(\cdot)$, $|R(\cdot)|$ and $|C(\cdot)|$. The lemmas are given here in the order in which they are employed in the proofs of Propositions 2.89 and 2.90 given in Section 2.4.5.

Throughout the proof of the main theorem, we make use of the fact that base sets of crossing matrices do not "overlap". In that case, it turns out, that the argumentation is easier. We formalize that in next definition.

Definition 2.78 (Non-intertwined sets and matrices)**.** *Let $A_1, A_2, \ldots, A_k \subset \mathbb{N}$ be finite sets of positive integers. We say that $A_1, A_2, \ldots, A_k$ are non-intertwined if for some permutation $\{i_1, i_2, \ldots, i_k\} = \{1, 2, \ldots, k\}$ one has*

$$A_{i_1} < A_{i_2} < \cdots < A_{i_k}.$$

A three-dimensional crossing matrix M or its submatrix M' or its layer matrix N or its cross-matrix N' is non-intertwined if the base sets of the respective matrix are non-intertwined.

Clearly if M is non-intertwined then so are the matrices M', N and N'. It may happen that M is not non-intertwined but M' or N is (but not N' that has the same base sets as M). Non-intertwined base sets are motivated by enforcing colorings of type W_2 and type W_3 in the next three lemmas and their corollaries.

Recall that I_r denotes the two-dimensional $r \times r$ identity matrix and U_r denotes the two-dimensional $r \times r$ upper-triangular matrix. Every non-intertwined crossing matrix is binary, has no $*$ entry.

Lemma 2.79. *Let $K = (n, \chi)$ be a coloring, $M = M_{X,Y,Z}$ be a crossing matrix of K, $M' = M_{X',Y',Z'}$ be a submatrix of M and N be a non-intertwined layer matrix of M'. If N is similar to the matrix I_r or to the matrix U_r, then $K \succeq K'$ where K' an r-wealthy coloring of type W_2. Moreover, each base set of the coloring K' is mapped in the containment $K' \preceq K$ in one of the sets X, Y and Z.*

Proof. We assume that the matrix N is similar to I_r (the case of U_r is analogous and leads to colorings of type $W_{2,2}$) and that N is a 3-layer matrix of M' (the cases of 1-layer and 2-layer matrices are analogous). Thus M' has dimensions $r \times r \times t$ with $r = |X'| = |Y'|$ and $t = |Z'|$. We see by the definition of crossing matrices, layer matrices, base sets and of similarity to I_r that if

$$X' = \{a_1 < \cdots < a_r\},\ Y' = \{b_1 < \cdots < b_r\}\ \text{and}\ Z' = \{c_1 < \cdots < c_t\}$$

(these are subsets of $[n]$) then for some $k \in [t]$ the sets X', Y' and $\{c_k\}$ are non-intertwined and for every $i, j \in [r]$ we have

$$N(i,\, j) = M'(j,\, i,\, k) = \chi(\{a_j,\, b_i,\, c_k\}) = \begin{cases} 1 & \text{if } i = j \text{ and} \\ 0 & \text{if } i \neq j, \end{cases}$$

or 1 and 0 are switched and i and/or j may be in $\chi(\cdots)$ replaced with $r - i + 1$ and/or $r - j + 1$, respectively. Then the coloring $K' \preceq K$ obtained by normalization of the restriction of K to the set $X' \cup Y' \cup \{c_k\} \subset [n]$ is an r-wealthy coloring of type $W_{2,1}$. It is clear that each base set of K' is mapped in the containment in one of the sets X, Y and Z. $\qquad\square$

Corollary 2.80. *Suppose that $(K_m)_{m \geq 1}$ is a sequence of colorings. We assume for every m that M_m is a crossing matrix of K_m, $M'_m \preceq M_m$, and that N_m is a non-intertwined layer matrix of M'_m such that the sequence $(\mathrm{al}(N_m))_{m \geq 1}$ is bounded but the sequence $(|R(N_m)|)_{m \geq 1}$ or the sequence $(|C(N_m)|)_{m \geq 1}$ is unbounded. Then for every $r \in \mathbb{N}$ there is an m such that $K_m \succeq K'_m$ where K'_m is an r-wealthy coloring of type W_2. Moreover, each base set of the coloring K'_m is mapped in the containment $K'_m \preceq K_m$ in one of the base sets of M_m.*

Proof. By Lemma 2.51 and the remark after it, for every $r \in \mathbb{N}$ there is an $m = m(r)$ such that N_m has a submatrix I'_r strongly similar to I_r or a submatrix U'_r strongly similar to U_r. We apply Lemma 2.79 to the non-intertwined matrices I'_r and U'_r (each is also a layer matrix of a submatrix of $M'_{m(r)}$, see part 1 of Lemma 2.69) and get that for every $r \in \mathbb{N}$ there is an m such that the coloring K_m contains an r-wealthy coloring K'_m of type W_2. It is clear that each base set of K'_m is mapped in the containment in one of the base sets of M_m. $\qquad\square$

Lemma 2.81. *Let $K = (n, \chi)$ be a coloring, $M = M_{X,Y,Z}$ be a crossing matrix of K, $M' = M_{X',Y',Z'}$ be a submatrix of M and N be a non-intertwined cross-matrix of M'. If N is similar to the matrix I_r or to the matrix U_r, then $K \succeq K'$ where K' an r-wealthy coloring of type $W_{3,1}$ or type $W_{3,2}$. Moreover, each base set of the coloring K' is mapped in the containment $K' \preceq K$ in one of the sets X, Y and Z.*

Proof. Unlike in Lemma 2.79 here the matrices N and M' have the same base sets. We assume that the matrix N is similar to I_r (the case of U_r is analogous and leads to colorings of type $W_{3,2}$) and that N is a $(1,2)$-d-cross-matrix of M' (the other five cases of cross-matrices are analogous). Thus M' has dimensions $r \times r \times r$ with $r = |X'| = |Y'| = |Z'|$. We see by the definition of crossing matrices, cross-matrices, base sets and of similarity to I_r that the sets

$$X' = \{a_1 < \cdots < a_r\}, \ Y' = \{b_1 < \cdots < b_r\} \ \text{and} \ Z' = \{c_1 < \cdots < c_r\}$$

are non-intertwined subsets of $[n]$. Thus for every $i, j \in [r]$ we have

$$N(i,\, j) = M'(j,\, j,\, i) = \chi(\{a_j,\, b_j,\, c_i\}) = \begin{cases} 1 & \text{if } i = j \text{ and} \\ 0 & \text{if } i \neq j\,, \end{cases}$$

or 1 and 0 are switched and i and/or j may be in $\chi(\cdots)$ replaced with $r - i + 1$ and/or $r - j + 1$, respectively. Then the coloring $K' \preceq K$ obtained by normalization of the restriction of K to the set $X' \cup Y' \cup Z' \subset [n]$ is an r-wealthy coloring of type $W_{3,1}$. It is clear that each base set of K' is mapped in the containment in one of the sets X, Y and Z. $\square$

Corollary 2.82. *Suppose that $(K_m)_{m \geq 1}$ is a sequence of colorings. We assume for every m that M_m is a crossing matrix of K_m, $M'_m \preceq M_m$, and that N_m is a non-intertwined cross-matrix of M'_m such that the sequence $(\mathrm{al}(N_m))_{m \geq 1}$ is bounded but the sequence $(|R(N_m)|)_{m \geq 1}$ or the sequence $(|C(N_m)|)_{m \geq 1}$ is unbounded. Then for every $r \in \mathbb{N}$ there is an m such that $K_m \succeq K'_m$ where K'_m is an r-wealthy coloring of type $W_{3,1}$ or type $W_{3,2}$. Moreover, each base set of the coloring K'_m is mapped in the containment $K'_m \preceq K_m$ in one of the base sets of M_m.*

Proof. Again, unlike in Corollary 2.80 here the matrices N_m and M'_m have the same base sets. By Lemma 2.51 and the remark after it, for every $r \in \mathbb{N}$ there is an $m = m(r)$ such that N_m has a submatrix I'_r strongly similar to I_r or a submatrix U'_r strongly similar to U_r. We apply Lemma 2.81 to the non-intertwined matrices I'_r and U'_r (each is also a cross-matrix of a submatrix of $M'_{m(r)}$, see part 2 Lemma 2.69) and get that for every $r \in \mathbb{N}$ for an m the coloring K_m contains an r-wealthy coloring K'_m of type $W_{3,1}$ or type $W_{3,2}$. It is clear that each base set of K'_m is mapped in the containment in one of the base sets of M_m. $\square$

Lemma 2.83. *Suppose that $(K_m)_{m \geq 1}$ is a sequence of colorings. We assume for every m that M_m is a crossing matrix of K_m, that $M'_m \preceq M_m$ and the sequence $(\mathrm{al}(M'_m))_{m \geq 1}$ is bounded, and that N_m is a non-intertwined cross-matrix of M'_m such that the sequence $(|R(N_m)|)_{m \geq 1}$ or the sequence $(|C(N_m)|)_{m \geq 1}$ is unbounded. Then for every $r \in \mathbb{N}$ there is an m such that $K_m \succeq K'_m$ where K'_m is an r-wealthy coloring of one and the same type W_2 or type $W_{3,1}$ or type $W_{3,2}$. Moreover, each base set of K'_m is mapped in the containment $K'_m \preceq K_m$ in one of the base sets of M_m.*

Proof. We assume that the sequence $(|R(N_m)|)_{m\geq 1}$ is unbounded, the other case when the sequence $(|C(N_m)|)_{m\geq 1}$ is unbounded is very similar. We consider two cases depending on whether the sequence $(\mathrm{al}(N_m))_{m\geq 1}$ is bounded or not. If it is bounded then we apply Corollary 2.82 to the matrices N_m and conclude that for every $r \in \mathbb{N}$ there is an m such that the coloring K_m contains an r-wealthy coloring of type $W_{3,1}$ or type $W_{3,2}$ and that the condition on bases sets holds. If the sequence $(\mathrm{al}(N_m))_{m\geq 1}$ is unbounded then by Lemma 2.74 every matrix M'_m has a (square) layer matrix P_m such that the sequence $(\mathrm{al}(P_m))_{m\geq 1}$ is bounded but the sequence $(|R(P_m)|)_{m\geq 1}$ is unbounded. Note that P_m is non-intertwined because N_m and M'_m are non-intertwined. We apply Corollary 2.80 to the matrices P_m and conclude that for every $r \in \mathbb{N}$ there is an m such that the coloring K_m contains an r-wealthy coloring of type W_2. It is clear that the condition on base sets is again satisfied. $\qquad\square$

Lemma 2.84. *Let $r \in \mathbb{N}$, $K = (n, \chi)$ be a coloring, $M = M_{X,X,Y}$ with $X < Y$ be a crossing matrix of K, and M' be a submatrix of M whose $(2,3)$-d-cross-matrix N has dimensions $r \times 3r$ and satisfies 1 or 2.*

    1. For $i \in [r]$, $N(i, 3i - 2) = *$ and $\{N(i, 3i - 1), N(i, 3i)\} = \{0, 1\}$.

    2. For $i \in [r]$, $\{N(i, 3i - 2), N(i, 3i - 1)\} = \{0, 1\}$ and $N(i, 3i) = *$.

Then K contains an r-wealthy coloring K' of type $W_{4,2}$. Moreover, each base set of K' is mapped in the containment in one of the sets X and Y.

Proof. We assume that 1 holds, assumption 2 is treated similarly. We have that $N(a, b) = M'(b, a, a)$, $M' = M_{X',X'',Y'}$ for sets $X', X'' \subset X$ and $Y' \subset Y$ with

$$X' = \{x_1 < x_2 < \cdots < x_{3r}\}, \ X'' = \{x_1 < x_4 < x_7 < \cdots < x_{3r-2}\}$$

and $Y' = \{y_1 < y_2 < \cdots < y_r\}$, and that for every $i \in [r]$,

$$\{\chi(\{x_{3i-2} < x_{3i-1} < y_i\}), \ \chi(\{x_{3i-2} < x_{3i} < y_i\})\} = \{0, 1\}.$$

Normalization of the restriction of K to the set $X' \cup Y' \subset [n]$ is an r-wealthy coloring K' of type $W_{4,2}$. It is clear that the first base set of K' is in the containment mapped in X and the second one in Y. $\qquad\square$

Definition 2.85 (Above-diagonal and below-diagonal cross-matrix)**.** *Let K be a coloring, $M = M_{X,Y,Y}$ with $X < Y$ be a three-dimensional $*$-binary crossing matrix of K and M' be a submatrix of M with the base sets $\{r_1 < \cdots < r_s\} \subset X$ and $\{c_1 < \cdots < c_r\}, \{s_1 < \cdots < s_r\} \subset Y$. If $c_i < s_i$ (resp. $c_i > s_i$) for every $i \in [r]$, we say that the $r \times s$ $(2,3)$-d-cross-matrix N of M', $N = N(i, j) = M(r_j, c_i, s_i)$, is an above-diagonal (resp. a below-diagonal) cross-matrix.*

    Clearly, each such cross-matrix N is binary.

Lemma 2.86. *Let $(K_m)_{m\geq 1}$ be a sequence of colorings and $M_m = M_{X_m,Y_m,Y_m}$ with $X_m < Y_m$ be $*$-binary crossing matrices of K_m. Let each matrix M_m have a submatrix M'_m whose (two-dimensional) $(2,3)$-d-cross-matrix N'_m is above-diagonal, resp. below-diagonal. Suppose that the sequence $(\mathrm{al}(M_m))_{m\geq 1}$ is bounded but*

$$\textit{both sequences } (|R(N'_m)|)_{m\geq 1} \textit{ and } (\mathrm{al}(N'_m))_{m\geq 1} \textit{ are unbounded}.$$

Then for every $r \in \mathbb{N}$ there is an m such that K_m contains an r-wealthy coloring of one and the same type $W_{3,3}$ or type W_2.

Proof. We assume that all matrices N'_m are above-diagonal and treat the other case when they are below-diagonal at the end. As we know, every N'_m is binary. Since the numbers of subwords 01 and 10 in the rows of N'_m are bounded (each row of N'_m is a row of M'_m), for every m there exists a column of N'_m with index c_m such that the sequence $(\mathrm{al}(N'_m(\cdot, c_m)))_{m \geq 1}$ is unbounded, where $\mathrm{al}(N'_m(\cdot, c_m))$ is defined as one plus the number of subwords 01 and 10 in the c_m-th column of N'_m. Let P_m be the 1-layer matrix of M_m containing the positions of the c_m-th column of N'_m (see Figure 2.8 at the end of the proof of Proposition 2.90). Clearly, P_m is a square symmetric matrix with $*$s on the main diagonal and 0s and 1s elsewhere. Let $r(m)$ be the number of rows in N'_m and

$$(c_m, \, x_{m,i}, \, y_{m,i}), \ i = 1, \, 2, \, \ldots, \, r(m) \, ,$$

with

$$x_{m,1} < x_{m,2} < \cdots < x_{m,r(m)} \ \text{ and } \ y_{m,1} < y_{m,2} < \cdots < y_{m,r(m)}$$

be the positions in M_m of the c_m-th column of N'_m. Hence

$$N'_m(a, \, c_m) = M_m(c_m, \, x_{m,a}, \, y_{m,a}) = P_m(y_{m,a}, \, x_{m,a}).$$

The sequence $(\mathrm{al}(N'_m(\cdot, c_m)))_{m \geq 1}$ is unbounded and thus so is $(|R(P_m)|)_{m \geq 1} = (|C(P_m)|)_{m \geq 1}$. Indeed, consider in P_m the zig-zag path

$$Z_m = \left(u_1, \, u'_1, \, u_2, \, u'_2, \, u_3, \, \ldots, u'_{r(m)-1}, \, u_{r(m)} \right) \, ,$$

where $u_i = (y_{m,i}, x_{m,i})$ and $u'_i = (y_{m,i+1}, x_{m,i})$, which has turns in the u'_is and the u_is with $1 < i < r(m)$. Since columns and shafts of M correspond to rows and columns of N, the path Z_m lies in P_m below the main diagonal of $*$s and contains therefore only 0s and 1s. Let $e_m = \mathrm{al}(N'_m(\cdot, c_m)) - 1$ be the number of subwords 01 and 10 in the binary sequence

$$\left(P_m(u_1), \, P_m(u_2), \, \ldots, \, P_m(u_{r(m)}) \right)$$

where $P_m(u_1)$ means $P(y_{m,1}, x_{m,1})$ etc. It follows that at least $e_m/2$ of the vertical segments (u_i, u'_i) of Z_m contain a subword 01 or 10, or this holds for the horizontal segments (u'_i, u_{i+1}). Thus $(|R(P_m)|)_{m \geq 1} = (|C(P_m)|)_{m \geq 1}$ is unbounded.

Hence we can apply Proposition 2.53 to the sequence $(P_m)_{m \geq 1}$ and its case (α) or (β) holds. It is not hard to see that in case (α) because of the type 2 colorings of pairs produced by the matrices P_m, for every $r \in \mathbb{N}$ there is an m such that the coloring K_m contains an r-wealthy coloring of type $W_{3,3}$ (determined by the fixed vertex corresponding to the fixed first coordinate and by an r-wealthy coloring of pairs of type 2 that follows after it). In case (β) we consider submatrices $P'_m \preceq P_m$ strongly similar to the identity matrix I_r or to the upper triangular matrix U_r. Each P'_m is non-intertwined as it lies above the diagonal of P_m. By part 1 of Lemma 2.69 each P'_m is also a layer matrix of a submatrix of M_m. We apply Lemma 2.79 to the matrices P'_m and conclude that for every $r \in \mathbb{N}$ there is an m such that the coloring K_m contains an r-wealthy coloring of type W_2.

If all matrices N'_m are below-diagonal, the same argument works for the zig-zag path

$$Z''_m = \left(u_1, \, u''_1, \, u_2, \, u''_2, \, u_3, \, \ldots, u''_{r(m)-1}, \, u_{r(m)} \right) \, ,$$

where $u_i = (y_{m,i}, x_{m,i})$ and $u''_i = (y_{m,i}, x_{m,i+1})$, which lies above the diagonal of $*$s. $\qquad\square$

Lemma 2.87. *Let $(K_m)_{m \geq 1}$ be a sequence of colorings and $M_m = M_{X_m, Y_m, Y_m}$ with $X_m < Y_m$ be $*$-binary crossing matrices of K_m. Let each matrix M_m have a submatrix M_m' whose (two-dimensional) $(2,3)$-d-cross-matrix N_m' is above-diagonal, resp. below-diagonal. Suppose that*

$$(|R(N_m')|)_{m \geq 1} \text{ is unbounded but } (\mathrm{al}(N_m'))_{m \geq 1} \text{ is bounded.}$$

Then for every $r \in \mathbb{N}$ there is an m such that K_m contains an r-wealthy coloring of one and the same type $W_{3,1}$ or type $W_{3,2}$ or type $W_{4,2}$.

Proof. Let $K_m = (n_m, \chi_m)$. We assume that all matrices N_m' are above-diagonal and treat the other case when they are below-diagonal at the end. As we know, every N_m' is binary. Using Lemma 2.51 to the sequence of matrices $(N_m')_{m \geq 1}$ and passing to a subsequence of $m = 1, 2, \dots$ and to submatrices of the corresponding matrices N_m', we may suppose that every matrix M_m has a submatrix M_m' whose $(2,3)$-d-cross-matrix N_m' is above-diagonal and strongly similar to the $m \times m$ identity matrix I_m, or to the $m \times m$ upper triangular matrix U_m. We first assume that for every m one has $N_m' = I_m$ and then we consider other matrices N_m' strongly similar to I_m. The case of N_m' strongly similar to U_m is deferred to the end. Let

$$\{r_{m,1} < \cdots < r_{m,m}\} \subset X_m \text{ and } \{c_{m,1} < \cdots < c_{m,m}\}, \{s_{m,1} < \cdots < s_{m,m}\} \subset Y_m$$

be the base sets of N_m' and M_m', so for any $i, j \in [m]$ we have $N_m'(i,j) = M_m'(r_{m,j}, c_{m,i}, s_{m,i})$. Moreover, $c_{m,i} < s_{m,i}$, since N_m' is above diagonal. By the assumption, $\chi_m(\{r_{m,j}, c_{m,i}, s_{m,i}\}) = 1$ if $i = j$ and $= 0$ if $i \neq j$. We consider the numbers

$$T_m = \max\left\{|L| : \; L \subset [m], \; \bigcap_{l \in L}(c_{m,l},\, s_{m,l}) \neq \emptyset\right\},$$

where

$$(c_{m,l},\, s_{m,l}) = \{x \in \mathbb{N} : \; c_{m,l} < x < s_{m,l}\}.$$

We distinguish the cases of unbounded and bounded sequence $(T_m)_{m \geq 1}$, respectively. In the former case we see that for every $r \in \mathbb{N}$ there is an m such that K_m contains a coloring $K_r' = (3r, \psi_r)$ satisfying for every $i, j \in [r]$ that

$$\psi_r(\{j,\, r + i,\, 2r + i\}) = \begin{cases} 1 & \cdots \quad i = j, \\ 0 & \cdots \quad i \neq j. \end{cases}$$

This is an r-wealthy coloring of type $W_{3,1}$. In the latter case $(T_m)_{m \geq 1}$ is bounded. Then for every $r \in \mathbb{N}$ there is an m such that K_m contains the coloring $K_r'' = (6r, \theta_r)$ satisfying for every $i, j \in [2r]$ that

$$\theta_r(\{j,\, 2r + 2i - 1,\, 2r + 2i\}) = \begin{cases} 1 & \cdots \quad i = j, \\ 0 & \cdots \quad i \neq j. \end{cases}$$

We partition $[2r + 1, 6r]$ in intervals $S_1 < S_2 < \cdots < S_r$ of length four each,

$$S_i = \{2r + 4i - 3,\, 2r + 4i - 2,\, 2r + 4i - 1,\, 2r + 4i\}.$$

For each $j \in [r]$ we take the three triples

$$U_{j,k} = \{2j - 1 < 2r + 4j - 3 + k < 2r + 4j - 2 + k\}, \ k = 0, 1, 2 \,.$$

Their respective θ_r-colors are $1, ?, 0$. We take the two triples for $k = 0, 1$ or $k = 1, 2$ with different colors. Thus for every $j \in [r]$ there exist three distinct elements $a_j, b_j, c_j \in S_j$ such that

$$\theta_r(\{2j - 1, \, a_j, \, b_j\}) \neq \theta_r(\{2j - 1, \, a_j, \, c_j\}) \,.$$

We see that K_r'' and K_m contain an r-wealthy coloring of type $W_{4,2}$.

If $N_m' \neq I_m$ is strongly similar to the identity matrix I_m, it arises from I_m by exchanging 0 and 1 and/or replacing the main diagonal with the antidiagonal. The exchange of 0 and 1 has no effect on the resulting type $W_{4,2}$ coloring and we assume that $N_m = I_m'$ is the antidiagonal unit matrix. Suppose that $(T_m)_{m \geq 1}$ is unbounded. In the colorings K_r' the order of elements in the interval $[r]$ is reversed but this leads again to colorings of type $W_{3,1}$. If $(T_m)_{m \geq 1}$ is bounded, in the colorings K_r'' the order of elements in the interval $[2r]$ is reversed. However, when we replace "$2j - 1$" with "$2r - 2j + 2$" in the argument of the triples $U_{j,k}$, this leads again to colorings of type $W_{4,2}$.

Suppose that for every m the matrix N_m' is strongly similar to the upper triangular matrix U_m. We first assume that $N_m' = U_m$ for every m and modify the previous argument by replacing the phrases "$i = j$" and "$i \neq j$" with "$i \leq j$" and "$i > j$", respectively. If the sequence $(T_m)_{m \geq 1}$ is unbounded, we obtain by an argument analogous to the previous one colorings of type $W_{3,2}$. If the sequence $(T_m)_{m \geq 1}$ is bounded, for every $r \in \mathbb{N}$ there is an m such that K_m contains the coloring $L_r = (6r, \phi_r)$ with the property that for every $i, j \in [2r]$, $\phi_r(\{j, 2r + 2i - 1, 2r + 2i\}) = 1$ if $i \leq j$ and $= 0$ if $i > j$. Using the same argument with the triples $U_{j,k}$ as in the case with $N_m' = I_m$, we get r-wealthy colorings of type $W_{4,2}$.

Suppose that $N_m' \neq U_m$ but N_m' is strongly similar to U_m. The effect of exchange of 0 and 1 is again clear and therefore we may assume that for every m we have $N_m' = U_m'$, the matrix with 1's on the antidiagonal and above it and 0s below it. We modify the argument in the first part by replacing the phrases "$i = j$" and "$i \neq j$" with "$i \leq d + 1 - j$" and "$i > d + 1 - j$", respectively, where d is the dimension of the square matrix in question, $d = m, r$ and $2r$, respectively. If the sequence $(T_m)_{m \geq 1}$ is unbounded, we obtain as before r-wealthy colorings of type $W_{3,2}$. If the sequence $(T_m)_{m \geq 1}$ is bounded, for every $r \in \mathbb{N}$ there is an m such that K_m contains the coloring $L_r' = (6r, \phi_r')$ with the property that for every $i, j \in [2r]$, $\phi_r'(\{j, 2r + 2i - 1, 2r + 2i\}) = 1$ if $i \leq 2r - j + 1$ and $= 0$ if $i > 2r - j + 1$. For each $j \in [r]$ the three triples

$$U_{j,k}' = \{2j - 1 < 6r - 4j + 3 + k < 6r - 4j + 4 + k\}, \ k = 0, 1, 2 \,,$$

with ϕ_r'-colors $1, ?, 0$ show as before that for every $r \in \mathbb{N}$ for some m the colorings L_r' and K_m contain an r-wealthy coloring of type $W_{4,2}$.

The case when all matrices N_m' are below-diagonal is handled by an argument almost identical to the previous one, only the phrases "$c_{m,i} < s_{m,i}$", "$(c_{m,l}, s_{m,l})$" and "$c_{m,l} < x < s_{m,l}$" have to be replaced with "$s_{m,i} < c_{m,i}$", "$(s_{m,l}, c_{m,l})$" and "$s_{m,l} < x < c_{m,l}$", respectively. □

2.4.5 Proof of Theorem 2.12

We have arrived in the last part of the proof of Theorem 2.12. This subsection combines the six lemmas with the previous results and finish in Propositions 2.89 and 2.90 the proof of Theorem 2.12.

First we generalize p-tameness for the ideals of colorings.

Definition 2.88 (p-tame sets)**.** *For $p \in \mathbb{N}$ a set X of colorings is p-tame if every coloring $H \in X$ is p-tame.*

See Section 2.4.3 for the definition of tame colorings. If an ideal $X \subset \mathcal{C}_3$ is not p-tame for any p, then for every $p \in \mathbb{N}$ there is a coloring $H \in X$ such that its nuclear decomposition $\mathrm{nu}(H)$ violates one of the conditions 1–5 of p-tameness in the definition in Section 2.4.3 for this p. This clearly implies that there is a $c \in [5]$ such that for every $p \in \mathbb{N}$ there is a coloring $H \in X$ such that $\mathrm{nu}(H)$ violates condition c of p-tameness for p. We first look at violation of conditions 1–4 and then at the more complicated situation when condition 5 is violated. As Propositions 2.89 and 2.90 show, violation of one of the conditions 1–5 produces coloring of one of the four types (with subtypes) W_1–W_4. For better readability of the proofs we emphasize each obtained coloring of **type W_i** by the bold type. We also emphasize by the bold type the beginning of discussion of each **first** or **second** or **third** ... case of the argument in either proof of the two propositions. For example, the proof of Proposition 2.89 has the following cases.

       Condition 1 is violated.

       Condition 2 or **condition 4** is violated.

$$\textbf{Condition 3} \begin{cases} |\mathbf{R}(\cdot)| \textbf{ unbounded} \begin{cases} \textbf{conclusion (i)} \text{ holds} \\ \textbf{conclusion (ii)} \text{ holds} \end{cases} \\ |\mathbf{C}(\cdot)| \textbf{ unbounded} \end{cases}$$

The proof of Proposition 2.90 has more complicated structure of cases.

It is true that if M is a binary three-dimensional matrix and $M' \preceq M$ then $\mathrm{al}(M') \leq \mathrm{al}(M)$. For $*$-binary matrices this inequality in general does not hold, but fortunately a modified version which we use does hold: if $M = M_{I,J,K}$ is a crossing matrix of a coloring with the base sets $I = J < K$ or $I < J = K$ and $M' \preceq M$, then $\mathrm{al}(M') \leq \mathrm{al}(M) + 1$. This holds because every line of M contains at most one $*$ or consists only of $*$s.

Proposition 2.89. *Let $c \in [4]$ and let $X \subset \mathcal{C}_3$ be an ideal of colorings such that for every $p \in \mathbb{N}$ there is a coloring $H \in X$ whose nuclear decomposition $\mathrm{nu}(H)$ violates condition c of p-tameness for p. Then there is an $i \in [4]$ such that for every $r \in \mathbb{N}$ the ideal X contains an r-wealthy coloring of type W_i.*

Proof. We consider three cases.

    Condition 1 is violated. Then for every $r \in \mathbb{N}$ there is a coloring $H \in X$ such that the length of its nuclear decomposition is at least $2r$. As we noted in Section 2.4.3, $H \succeq K$ where K is an r-wealthy coloring of type $W_{4,1}$. Therefore for every $r \in \mathbb{N}$ the ideal X contains an r-wealthy coloring of **type $W_{4,1}$**.

Condition 2 is violated or **condition 4** is violated. If condition 2 is violated then for every $r \in \mathbb{N}$ there is a coloring $H \in X$ that has a binary crossing matrix $M = M_{I,J,K}$, where $I < J < K$ are three intervals in $\mathrm{nu}(H)$, with $\mathrm{al}(M) \geq r$. It is easy to see that then $H \succeq K$ where K is an $(r+2)$-wealthy coloring of type W_1. Hence for every $r \in \mathbb{N}$ the ideal X contains an r-wealthy coloring of **type W_1**. If condition 4 is violated, the only difference is that the line in the crossing matrix M witnessing $\mathrm{al}(M) \geq r$ may contain one $*$. Again, for every $r \in \mathbb{N}$ the ideal X contains an $(r+2)$-wealthy coloring of **type W_1**.

Condition 3 is violated. We assume that condition 2 is not violated, which means that there is a constant p_0 such that for every coloring $H \in X$ and every three intervals $I < J < K$ in $\mathrm{nu}(H)$ we have $\mathrm{al}(M_{I,J,K}) \leq p_0$. At the same time there exists a sequence $(H_m)_{m \geq 1}$ of coloring $H_m = (n_m, \chi_m)$ in X with three intervals $I_m < J_m < K_m$ in $\mathrm{nu}(H_m)$ such that, if we denote by $M_m = M_{I_m, J_m, K_m}$ the corresponding crossing matrix, the sequence $(\mathrm{al}(M_m))_{m \geq 1}$ is bounded by p_0 but one of the sequences $R = (|R(M_m)|)_{m \geq 1}$ and $C = (|C(M_m)|)_{m \geq 1}$ is unbounded. Each matrix $M_m = M_{I_m, J_m, K_m}$ is non-intertwined and so are the matrices below derived from M_m and we may apply to them Corollary 2.80 and Lemma 2.83. First we suppose that the **sequence R is unbounded** and defer the case when C is unbounded to the end. Thus the hypothesis of Lemma 2.71 is satisfied and its conclusion (i) or (ii) holds.

If the **conclusion (i) of Lemma 2.71** holds then every matrix M_m has a layer matrix N_m such that the sequence $(|R(N_m)|)_{m \geq 1}$ is unbounded. Note that the sequence $(\mathrm{al}(N_m))_{m \geq 1}$ is bounded. Therefore by Corollary 2.80 (with $M'_m = M_m$) for every $r \in \mathbb{N}$ there is an m such that the coloring H_m contains an r-wealthy coloring of **type W_2**.

If the **conclusion (ii) of Lemma 2.71** holds then every matrix M_m has a submatrix M'_m that has a cross-matrix N_m such that the sequence $(|R(N_m)|)_{m \geq 1}$ is unbounded. Also, the sequences $(\mathrm{al}(M_m))_{m \geq 1}$ and $(\mathrm{al}(M'_m))_{m \geq 1}$ are bounded. We apply Lemma 2.83 to the matrices N_m and conclude that for every $r \in \mathbb{N}$ there is an m such that the coloring H_m contains an r-wealthy coloring of one and the same type **type W_2** or **type $W_{3,1}$** or **type $W_{3,2}$**.

If the **sequence C is unbounded,** we make use of Lemma 2.72 instead of Lemma 2.71. Now the only difference is that in conclusion (i) of Lemma 2.72 we have unbounded sequence $(|C(N_m)|)_{m \geq 1}$ instead of $(|R(N_m)|)_{m \geq 1}$. But Corollary 2.80 applies in this case too. If conclusion (ii) holds then we again use Lemma 2.83. In total we get again that for every $r \in \mathbb{N}$ there is an m such that the coloring H_m contains an r-wealthy coloring of one and the same **type W_2** or **type $W_{3,1}$** or **type $W_{3,2}$**. $\square$

The last case is when condition 5 is violated and the previous proposition does not apply.

Proposition 2.90. *Let $X \subset \mathcal{C}_3$ be an ideal of colorings such that for some $p_0 \in \mathbb{N}$ for every coloring $H \in X$ it holds that $\mathrm{nu}(H)$ satisfies each of the conditions 1–4 of p-tameness for $p = p_0$, but for every $p \in \mathbb{N}$ there is a coloring $H \in X$ such that $\mathrm{nu}(H)$ violates condition 5 of p-tameness for p. Then there is an $i \in [4]$ such that for every $r \in \mathbb{N}$ the ideal X contains an r-wealthy coloring of type W_i.*

Proof. Let X be as stated. It follows that there is a sequence of colorings $H_m = (n_m, \chi_m) \in X$, $m \in \mathbb{N}$, and intervals $I_m < J_m$ in $\mathrm{nu}(H_m)$ such that one of the

cases (a)–(d) holds.

(a) $M_m := M_{I_m, I_m, J_m}$, the sequence $(\mathrm{al}(M_m))_{m \geq 1}$ is bounded and the sequence $(|R(M_m)|)_{m \geq 1}$ is unbounded.

(b) $M_m := M_{I_m, I_m, J_m}$, the sequence $(\mathrm{al}(M_m))_{m \geq 1}$ is bounded and the sequence $(|C(M_m)|)_{m \geq 1}$ is unbounded.

(c) $M_m := M_{I_m, J_m, J_m}$, the sequence $(\mathrm{al}(M_m))_{m \geq 1}$ is bounded and the sequence $(|R(M_m)|)_{m \geq 1}$ is unbounded.

(d) $M_m := M_{I_m, J_m, J_m}$, the sequence $(\mathrm{al}(M_m))_{m \geq 1}$ is bounded and the sequence $(|C(M_m)|)_{m \geq 1}$ is unbounded.

It is in fact unimportant that $I_m < J_m$ are two intervals in $\mathrm{nu}(H_m)$, it suffices to assume that they are just two subsets in $[n_m]$. We prove then that in each of the cases (a)–(d) there is an $i \in [4]$ such that for every $r \in \mathbb{N}$ there is an m such that $H_m \succeq K_m$ for an r-wealthy coloring K_m of type W_i.

Cases (b) and (d) reduce to case (a). Before we begin with case (a) we show that cases (b) and (d) reduce to it. Indeed, the matrices $M_m = M_{I_m, I_m, J_m}$ are symmetric in the first two coordinates, which means that for every m every 3-layer matrix of M_m is a symmetric (two-dimensional) matrix and $C(M_m) = R(M_m)$. So cases (a) and (b) are the same. Suppose that the colorings H_m and sets $I_m < J_m$ satisfy conditions of case (d). The reversing map $r_m : [n_m] \to [n_m]$ given by $r_m(x) = n_m - x + 1$ turns each coloring $H_m = (n_m, \chi_m)$ in its reversal $H'_m = (n_m, \chi'_m)$, $\chi' = \chi \circ r_m$, given by

$$\chi'_m(\{a,\, b,\, c\}) = \chi_m(\{n_m - a + 1,\, n_m - b + 1,\, n_m - c + 1\}).$$

Each matrix $M_m = M_{I_m, J_m, J_m}$ then turns in the matrix $M'_m = M_{I'_m, I'_m, J'_m}$ where $I'_m = r_m(J_m)$ and $J'_m = r_m(I_m)$. Then

$$\mathrm{al}(M'_m) = \mathrm{al}(M_m) \ \text{ and } \ |C(M'_m)| = |R(M'_m)| = |C(M_m)|\,.$$

Thus the colorings H'_m and sets $I'_m < J'_m$ produce matrices M'_m satisfying conditions of case (a). Assuming that we have solved it, we have an $i \in [4]$ such that for every $r \in \mathbb{N}$ there is an m such that $H'_m \succeq K_m$ for an r-wealthy coloring K_m of type W_i. Since the family of wealthy colorings of type W_i is closed to reversals and H_m arises from H'_m as its reversal (and $\succeq$ is preserved by reversals), $H_m \succeq K_m$ too. We see that an r-wealthy coloring of the same **type W_i** is contained in H_m as well and case (d) is solved.

Case (a). Recall that $M_m = M_{I_m, I_m, J_m}$ for two subsets $I_m < J_m$ in $[n_m]$, the sequence $(\mathrm{al}(M_m))_{m \geq 1}$ is bounded and the sequence $(|R(M_m)|)_{m \geq 1}$ is unbounded. The hypothesis of Lemma 2.71 is satisfied and its conclusion (i) or (ii) holds.

Suppose that **conclusion (i) of Lemma 2.71** holds. Now, unlike in the previous proposition, we have to treat its two subcases differently because the second base set of M_m coincides with the first one but the third base set lies after the first one. The **first subcase** is that every matrix M_m has a 2-layer $*$-binary matrix N_m, where N_m is a matrix that has one column of $*$s and elsewhere only 0s and 1s, such that the sequence $(\mathrm{al}(N_m))_{m \geq 1}$ is bounded but the sequence $(|R(N_m)|)_{m \geq 1}$ is unbounded. The column of $*$s splits N_m in two submatrices

N'_m and N''_m which are non-intertwined because each lies on only one side of the column of $*$s. For one of them, say N'_m, we have that $|R(N'_m)| \geq \frac{1}{2}|R(N_m)|$. The other case with N''_m is similar. Thus the sequence $(\mathrm{al}(N'_m))_{m \geq 1}$ is bounded but the sequence $(|R(N'_m)|)_{m \geq 1}$ is unbounded. By part 1 of Lemma 2.69, each matrix N'_m is also a layer matrix of a submatrix of M_m. We apply Corollary 2.80 to the matrices N'_m and get that for every $r \in \mathbb{N}$ there is an m such that the coloring H_m contains an r-wealthy coloring of **type $\mathbf{W_2}$**.

The **second subcase** in conclusion (i) is that every matrix M_m has a 3-layer $*$-binary matrix N_m such that the sequence $(\mathrm{al}(N_m))_{m \geq 1}$ is bounded but the sequence

$$(|R(N_m)|)_{m \geq 1} = (|C(N_m)|)_{m \geq 1}$$

is unbounded. Here even $R(N_m) = C(N_m)$ because each N_m is a square symmetric matrix with $*$s on the diagonal and 0s and 1s elsewhere (the first two base sets of M_m and N_m are the same). We apply Proposition 2.53 to the matrices $(N_m)_{m \geq 1}$ and get that its case (α) or (β) holds. **Case (α)** implies that for every $r \in \mathbb{N}$ there is an m such that H_m contains an r-wealthy coloring of **type $\mathbf{W_{3,3}}$** (determined by the fixed vertex corresponding to the fixed third coordinate and by an r-wealthy coloring of pairs of type 2 that precedes it). In **case (β)** we can use Lemma 2.79 because the submatrices I'_r and U'_r of N_m produced by case (β) are non-intertwined (they lie above the diagonal of N_m) and each is also a layer matrix of a submatrix of M_m, by part 1 of Lemma 2.69. We get that for every $r \in \mathbb{N}$ there is an m such that the coloring H_m contains an r-wealthy coloring of **type $\mathbf{W_2}$**.

Suppose that **conclusion (ii) of Lemma 2.71** holds. Then every matrix $M_m = M_{I_m,I_m,J_m}$ has a submatrix M'_m that has a $*$-binary R-full $(2,3)$-cross-matrix N_m such that the sequence $(|R(N_m)|)_{m \geq 1}$ is unbounded. For concreteness we suppose that every matrix $\mathbf{N_m}$ **is a $\mathbf{(2,3)}$-d-cross-matrix** of M'_m, which means that $N_m(i,j) = M'_m(j,i,i)$, and postpone the other case with $(2,3)$-ad-cross-matrix to the end of case (a). R-fullness of the matrices N_m implies that if N_m has $r(m)$ rows then there are $r(m)$ distinct column indices $p_{m,i}$ such that for each $i = 1, 2, \ldots, r(m)$ we have

$$\{N_m(i, p_{m,i}), N_m(i, p_{m,i} + 1)\} = \{0, 1\}.$$

In the sense of Remark 2.73, each row of N_m is a row in M_m. Thus each row of N_m, as a word over $\{0, 1, *\}$, has exactly one $*$. Also, the numbers of subwords 01 and 10 in the rows of N_m are bounded as a function of m. The last fact implies, since the sequence $(|R(N_m)|)_{m \geq 1}$ is unbounded, that the sequence $(r(m))_{m \geq 1}$ of numbers of rows in the matrices N_m is unbounded. Let the column index of the $*$ in the row i of N_m be $q_{m,i}$. We note that $q_{m,1} < q_{m,2} < \cdots < q_{m,r(m)}$. Either $p_{m,i} > q_{m,i}$ or $p_{m,i} < q_{m,i}$ holds for at least half of the indices $i \in [r(m)]$. Note that $p_{m,i} = q_{m,i}$ is impossible because the left side is a column index of a 0 or a 1 and the right side is a column index of a $*$.

We assume that the **former inequality $\mathbf{p_{m,i} > q_{m,i}}$** holds for at least half of the rows and later indicate how to deal with the latter inequality. Using again the Erdős–Szekeres lemma we may suppose, by passing to a subsequence of the sequence $m = 1, 2, \ldots$ and to submatrices of the corresponding matrices N_m, that for every $m \in \mathbb{N}$ the sequence

$$P(m) = \left(p_{m,1}, p_{m,2}, \ldots, p_{m,r(m)} \right)$$

$$
\begin{array}{c}
t_{m,1} \\
\\
\\
t_{m,2} \\
\\
\\
t_{m,3}
\end{array}
\begin{pmatrix}
* & \mathbf{0} & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & * & \mathbf{1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & * & \mathbf{1} & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & * & 0 & \mathbf{0} & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & * & 0 & \mathbf{0} & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & * & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & * & \mathbf{0} & 1
\end{pmatrix}
$$

Figure 2.7: An example of a matrix N_m and row indices $t_{m,i}$ defined in the proof of Proposition 2.90. The positions $(i, p_{m,i})$ are indicated by the bold type.

is decreasing, or that for every $m \in \mathbb{N}$ it increases.

If the **sequences P(m) decreases**, $p_{m,1} > p_{m,2} > \cdots > p_{m,r(m)}$, we consider the submatrix N'_m of N_m consisting of the whole columns with indices $\geq p_{m,r(m)}$. The matrix N'_m is non-intertwined because it lies to the right of all $*$s in N_m. By part 2 of Lemma 2.69, N'_m is also a cross-matrix of a submatrix M''_m of M'_m. Clearly, the sequence $(\mathrm{al}(M''_m))_{m \geq 1}$ is bounded (because $(\mathrm{al}(M_m))_{m \geq 1}$ is bounded) and the sequence $(|R(N'_m)|)_{m \geq 1}$ is unbounded. We apply Lemma 2.83 to the matrices N'_m and conclude that for every $r \in \mathbb{N}$ there is an m such that the coloring H_m contains an r-wealthy coloring of one and the same **type $\mathbf{W_2}$** or **type $\mathbf{W_{3,1}}$** or **type $\mathbf{W_{3,2}}$**.

If the **sequences P(m) increases**, $p_{m,1} < p_{m,2} < \cdots < p_{m,r(m)}$, we define for each m an increasing sequence

$$
1 = t_{m,1} < t_{m,2} < \cdots < t_{m,s(m)}
$$

of row indices as follows. Recall that in the row i, $p_{m,i}$ is the column index of the first letter of a subword 01 or 10 and $q_{m,i} < p_{m,i}$ and is the column index of the unique $*$. For $i > 1$ we let $t_{m,i}$ be the minimum index j such that $j > t_{m,i-1}$ and $q_{m,j} > p_{m,t_{m,i-1}} + 1$ if such j exists, and set $t_{m,i}$ to be the last row index in N_m otherwise; thus $t_{m,s(m)} = r(m)$ is the number of rows in N_m. The indices $t_{m,i}$ are illustrated by Figure 2.7. We consider two cases, of bounded and unbounded sequence $(s(m))_{m \geq 1}$ of the numbers of the indices $t_{m,i}$, respectively.

We assume that the **sequence $(\mathbf{s(m)})_{\mathbf{m} \geq \mathbf{1}}$ is bounded**. Then for every m there is an index $l(m) \in [r(m) - 1]$ such that the sequence

$$
(u_m)_{m \geq 1} := \left(t_{m,l(m)+1} - t_{m,l(m)} - 2 \right)_{m \geq 1}
$$

is unbounded, where the numbers u_m counts rows in N_m between the $(t_{m,l(m)}+1)$-th and $(t_{m,l(m)+1})$-th row. We denote again by N'_m the R-full submatrix of N_m formed by the intersection of these u_m rows and the columns lying to the right of the $(p_{m,l(m)} + 1)$-th column. Because of the definition of the indices $t_{m,i}$ and since $P(m)$ increases, we see as for the above matrices N'_m that each of the matrices N'_m here is non-intertwined. Also, the sequence $(|R(N'_m)|)_{m \geq 1}$ is unbounded because $|R(N'_m)| \geq u_m$. We argue as for the above matrices N'_m and conclude by part 2 of Lemma 2.69 and by Lemma 2.83 that for every $r \in \mathbb{N}$ there is an m such that the coloring H_m contains an r-wealthy coloring of one and the same **type $\mathbf{W_2}$** or **type $\mathbf{W_{3,1}}$** or **type $\mathbf{W_{3,2}}$**.

If the **sequence $(s(m))_{m \geq 1}$ is unbounded** we consider for each m the submatrix of N_m denoted again by N'_m and formed by the intersection of

the rows $t_{m,j}$ and the columns $q_{m,t_{m,j}}$, $p_{m,t_{m,j}}$ and $p_{m,t_{m,j}} + 1$ for $j \in [s(m) - 1]$.

Let $r' = r'(m) = s(m) - 1$. The $r' \times 3r'$ matrix N'_m has for $i \in [r']$ the entries $N'_m(i, 3i-2) = *$ and $\{N'_m(i, 3i-1), N'_m(i, 3i)\} = \{0, 1\}$. By part 1 of Lemma 2.84, for every $r' \in \mathbb{N}$ there is an m such that the coloring H_m contains an r'-wealthy coloring of **type $W_{4,2}$**.

We consider the **case when $p_{m,i} < q_{m,i}$** holds for at least half of the row indices i of N_m. We again use the Erdős–Szekeres lemma, pass to a subsequence of indices m and to submatrices of the corresponding matrices N_m and again distinguish two cases depending on whether the sequence

$$P(m) = \left(p_{m,1}, \, p_{m,2}, \, \ldots, \, p_{m,r(m)}\right)$$

decreases or increases. In the **case when $P(m)$ decreases**, when $p_{m,1} > p_{m,2} > \cdots > p_{m,r(m)}$, we consider the submatrix N'_m of N_m formed by the columns of N_m with indices $\leq p_{m,1}$. The matrix N'_m is non-intertwined because it lies to the left of all $*$s in N_m. We argue as before and by means of part 2 of Lemma 2.69 and Lemma 2.83 conclude that for every $r \in \mathbb{N}$ there is an m such that the coloring H_m contains an r-wealthy coloring of one and the same **type W_2** or **type $W_{3,1}$** or **type $W_{3,2}$**.

In the **case when $P(m)$ increases**, when $p_{m,1} < p_{m,2} < \cdots < p_{m,r(m)}$, we again define certain row indices of N_m denoted again $t_{m,i}$,

$$1 = t_{m,1} < t_{m,2} < \cdots < t_{m,s(m)} \, ,$$

as follows. For $i > 1$ we set $t_{m,i}$ to be the minimum j such that $j > t_{m,i-1}$ and $p_{m,j} + 1 > q_{m,t_{m,i-1}}$ if such j exists, and set $t_{m,i}$ to be the last row index in N_m else. Thus again $t_{m,s(m)} = r(m)$ is the number of rows of N_m. As before we consider two cases depending on whether the sequence $(s(m))_{m \geq 1}$ is bounded. In the **case when $(s(m))_{m \geq 1}$ is bounded**, for some indices $l(m) \in [r(m) - 1]$ the difference

$$t_{m,l(m)+1} - t_{m,l(m)} - 1$$

is unbounded in m and like before we consider the R-full submatrix N'_m of N_m formed by the intersection of this many rows lying between the $(t_{m,l(m)} - 1)$-th and $(t_{m,l(m)+1} - 1)$-th row and the columns lying to the left of the $q_{m,t_{m,l(m)}}$-th column. Then as before the matrix N'_m is non-intertwined because it lies to the left of all $*$s in N_m and $|R(N'_m)|$ is at least the number of rows in N'_m and hence is unbounded in m. We again conclude by means of part 2 of Lemma 2.69 and Lemma 2.83 that for every $r \in \mathbb{N}$ there is an m such that the coloring H_m contains an r-wealthy coloring of one and the same **type W_2** or **type $W_{3,1}$** or **type $W_{3,2}$**.

If the **sequence $(s(m))_{m \geq 1}$ is unbounded** we again consider the matrix N'_m formed by the intersection of

the rows $t_{m,j}$ and the columns $p_{m,t_{m,j}}$, $p_{m,t_{m,j}} + 1$ and $q_{m,t_{m,j}}$ for $j \in [s(m) - 1]$.

This is an $r' \times 3r'$ matrix, where $r' = r'(m) = s(m) - 1$, and for every $i \in [r']$ we have that $\{N'_m(i, 3i - 2), N'_m(i, 3i - 1)\} = \{0, 1\}$ and $N'_m(i, 3i) = *$. By part 2 of

Lemma 2.84, for every $r' \in \mathbb{N}$ there is an m such that the coloring H_m contains an r'-wealthy coloring of **type $\mathbf{W_{4,2}}$**.

The last case of case (a) to consider is when conclusion (ii) of Lemma 2.71 holds and each matrix $\mathbf{N_m}$ **is a $(2,3)$-ad-cross-matrix** of a submatrix M'_m of the matrix $M_m = M_{I_m, I_m, J_m}$ which is a crossing matrix of the coloring H_m; we know that the sequence $(\mathrm{al}(M_m))_{m \geq 1}$ is bounded and the sequence $(|R(N_m)|)_{m \geq 1}$ is unbounded. We use a finer version of the argument by which we above reduced case (d) to case (a). We reduce the case of $(2,3)$-ad-cross-matrix N_m to the just resolved case of $(2,3)$-d-cross-matrix N_m. Instead of global reversal of colorings we will use local reversals. Namely, we replace matrices N_m, M'_m and M_m and colorings H_m with the matrices $F(N_m)$, $F(M'_m)$ and $F(M_m)$ and coloring H'_m, respectively, as follows. F is the mapping transforming a general three-dimensional $*$-binary matrix $M \colon [r] \times [s] \times [t] \to \{0, 1, *\}$ to the matrix $F(M) \colon [r] \times [s] \times [t] \to \{0, 1, *\}$ given by

$$F(M)(a,\, b,\, c) = M(a,\, b,\, t - c + 1)\,.$$

If $M = M_{I,I,J}$ is a crossing matrix of a coloring $H = (n, \chi)$, where $I, J \subset [n]$ with $I < J$, then $F(M) = M_{I,I,J}$ is the crossing matrix of the coloring $H' = (n, \chi')$ (for the same sets I and J) that arises from H by reversing the order of elements in the subset J. In more details, if $J = \{y_1 < y_2 < \cdots < y_t\} \subset [n]$ and for an $x \in [n]$ we denote $\overline{x} = x$ if $x \in [n] \setminus J$ and $\overline{x} = \overline{y_j} = y_{t-j+1}$ if $x = y_j \in J$ then for every $\{x, y, z\} \in \binom{[n]}{3}$ also $\{\overline{x}, \overline{y}, \overline{z}\} \in \binom{[n]}{3}$, so no new $*$ is created, and

$$\chi'(\{x,\, y,\, z\}) := \chi(\{\overline{x},\, \overline{y},\, \overline{z}\})\,.$$

The mapping F reverses orders of positions in shafts in M but preserves orders of positions in rows and columns and only changes their places. F also transforms the $(2,3)$-ad-cross-matrix N of a submatrix M' of M to the $(2,3)$-d-cross-matrix $F(N)$ of the submatrix $F(M')$ of $F(M)$ (in the definitions of $F(N)$, $F(M')$ and $F(M)$ a position (a, b, c) in $F(M)$ becomes position $(a, b, t-c+1)$ in M). We have $\mathrm{al}(F(M)) = \mathrm{al}(M)$ and even $R(F(N)) = R(N)$. We can apply to the matrices $F(N_m)$, $F(M'_m)$ and $F(M_m)$ and colorings H'_m the previously resolved case of $(2,3)$-d-cross-matrix. We deduce that there is a symbol i_0 which is '2' or '3, 1' or '3, 2' or '4, 2' such that for every $r \in \mathbb{N}$ there is an m such that $H'_m \succeq K_m$ where K_m is an r-wealthy coloring of type W_{i_0}.

We claim that then also $H_m \succeq K_m$. We prove it by verifying that each time $K_m \preceq H'_m$ in the previous argument, the transformation $F^{-1} = F$ yields $K_m \preceq H_m$. We have $K_m \preceq H'_m$ either by four applications of Lemma 2.83 or by two applications of Lemma 2.84. By the statements of Lemmas 2.83 and 2.84, each base set of K_m is mapped in the containment $K_m \preceq H'_m$ in one of the base sets of $F(M_m)$ which are I_m, I_m and J_m. Since H_m is obtained back from H'_m by reversing the order of elements in the interval J_m and each family of type W_{i_0} colorings is closed to reversing the order of elements in any base set, we finally see that $H_m \succeq K_m$ too and H_m contains an r-wealthy coloring of **type $\mathbf{W_{io}}$**.

Case (c). Recall that $M_m = M_{I_m, J_m, J_m}$ for two subsets $I_m < J_m$ in $[n_m]$, the sequence $(\mathrm{al}(M_m))_{m \geq 1}$ is bounded and the sequence $(|R(M_m)|)_{m \geq 1}$ is unbounded. Thus, again, the hypothesis of Lemma 2.71 is satisfied and its conclusion (i) or (ii) holds.

Suppose that **conclusion (i) of Lemma 2.71** holds. Thus every matrix M_m has a 2-layer or a 3-layer matrix N_m, where N_m is a two-dimensional $*$-binary matrix with one row of $*$s and 0s and 1s elsewhere, such that the sequence $(\mathrm{al}(N_m))_{m \geq 1}$ is bounded but the sequence $(|R(N_m)|)_{m \geq 1}$ is unbounded. Since the second and third base set of M_m are the same, we may assume that N_m is a 3-layer matrix of M_m. The row of $*$s splits N_m in binary matrices N'_m and N''_m and for one of them, say N'_m, we have $|R(N'_m)| \geq \frac{1}{2}|R(N_m)|$, thus $(N'_m)_{\geq 1}$ is unbounded. In general, the matrix N_m may not be non-intertwined. However, the splitting of N_m enforces, as in case (a), that both N'_m and N''_m are non-intertwined. Thus we have a sequence $(N'_m)_{\geq 1}$ of two-dimensional binary matrices such that the sequence $(\mathrm{al}(N'_m))_{m \geq 1}$ is bounded but the sequence $(|R(N'_m)|)_{m \geq 1}$ is unbounded. We conclude as before by part 1 of Lemma 2.69 and Corollary 2.80 that for every $r \in \mathbb{N}$ there is an m such that the coloring H_m contains an r-wealthy coloring of **type $\mathbf{W_2}$**. The argument is the same when $|R(N''_m)| \geq \frac{1}{2}|R(N_m)|$.

Suppose that **conclusion (ii) of Lemma 2.71** holds. This means that every matrix M_m has a submatrix M'_m that has an R-full $(2,3)$-cross-matrix N_m such that the sequence $(|R(N_m)|)_{m \geq 1}$ is unbounded. As before we suppose for concreteness that every matrix $\mathbf{N_m}$ **is a $(\mathbf{2,3})$-d-cross-matrix** of M'_m and defer $(2,3)$-ad-cross-matrices to the end of case (c). We partition N_m by its rows in two submatrices N'_m and N''_m (see Figure 2.8). By Remark 2.73 the rows in matrices N_m, M'_m and M_m are the same. If

$$\{c_{m,1} < c_{m,2} < \cdots < c_{m,p(m)}\} \subset J_m \quad \text{and} \quad \{s_{m,1} < s_{m,2} < \cdots < s_{m,p(m)}\} \subset J_m$$

is the second and third base set of M'_m, respectively, with $p(m)$ being the common second and third dimension of M'_m, then the i-th row of N_m is

$$N_m(i,\, j) = M_m(j,\, c_{m,i},\, s_{m,i})\, .$$

We define the submatrix N'_m (resp. N''_m) as consisting of the rows i of N_m for which $c_{m,i} < s_{m,i}$ (resp. $c_{m,i} > s_{m,i}$). We cannot have $c_{m,i} = s_{m,i}$, then the i-th row of N_m would contain only $*$s and this contradicts R-fullness of N_m. In the sense of the definition preceding Lemma 2.86 the matrix N'_m is above-diagonal and the matrix N''_m below-diagonal. Clearly, N_m, N'_m and N''_m are binary matrices but in general are not non-intertwined. However, one of the sequences $(|R(N'_m)|)_{m \geq 1}$ and $(|R(N''_m)|)_{m \geq 1}$ is unbounded.

We first suppose that the **sequence $(|\mathbf{R(N'_m)}|)_{\mathbf{m} \geq \mathbf{1}}$ is unbounded** and discuss the other case of unbounded sequence $(|R(N''_m)|)_{m \geq 1}$ later. We consider two subcases. If the **sequence $(\mathrm{al}(\mathbf{N'_m}))_{\mathbf{m} \geq \mathbf{1}}$ is unbounded**, by part 2 of Lemma 2.69 we can apply to the matrices N'_m Lemma 2.86 and conclude that for every $r \in \mathbb{N}$ there is an m such that the coloring H_m contains an r-wealthy coloring of one and the same **type $\mathbf{W_{3,3}}$** or **type $\mathbf{W_2}$**. If the **sequence $(\mathrm{al}(\mathbf{N'_m}))_{\mathbf{m} \geq \mathbf{1}}$ is bounded** then by part 2 of Lemma 2.69 we can apply to the matrices N'_m Lemma 2.87 and conclude that for every $r \in \mathbb{N}$ there is an m such that the coloring H_m contains an r-wealthy coloring of one and the same **type $\mathbf{W_{3,1}}$** or **type $\mathbf{W_{3,2}}$** or **type $\mathbf{W_{4,2}}$**.

The case when the **sequence $(|\mathbf{R(N''_m)}|)_{\mathbf{m} \geq \mathbf{1}}$ is unbounded** is almost identical to the previous case. We can use Lemmas 2.86 and 2.87 as before since they apply to below-diagonal d-cross-matrices N''_m too. We get the same conclusions,
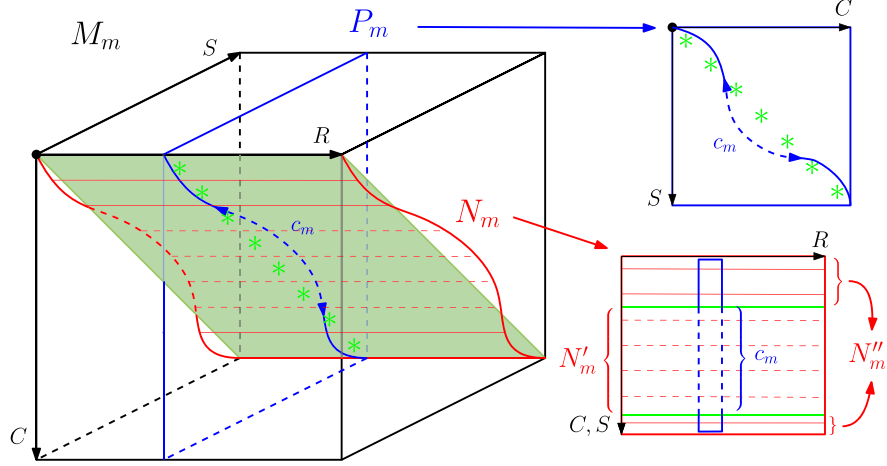
Figure 2.8: Proof of Proposition 2.90, case (c). The $(2,3)$-d-cross-matrix N_m of M'_m is the red wavy surface in the matrix $M_m = M_{I_m, J_m, J_m}$ $(I_m < J_m)$ and is disjoint to the green plane that contains only $*$s.

for every $r \in \mathbb{N}$ there is an m such that H_m contains an r-wealthy coloring of one and the same **type W_2** or **type $W_{3,1}$** or **type $W_{3,2}$** or **type $W_{3,3}$** or **type $W_{4,2}$**.

The last case of case (c) to consider is when conclusion (ii) of Lemma 2.71 holds and for every m the matrix **N_m is a $(2,3)$-ad-cross-matrix** of the submatrix M'_m of the matrix $M_m = M_{I_m, J_m, J_m}$ which is a crossing matrix of the coloring H_m; we know that the sequence $(\mathrm{al}(M_m))_{m \geq 1}$ is bounded and the sequence $(|R(N_m)|)_{m \geq 1}$ is unbounded. Now the i-th row of N_m is

$$N_m(i,\, j) = M_m(j,\, c_{m,i},\, s_{m,p(m)-i+1})\ .$$

Like before we partition N_m by its rows in two submatrices and define N'_m (resp. N''_m) as consisting of the rows i of N_m such that

$$c_{m,i} < s_{m,p(m)-i+1}\ \left(\text{resp.}\ c_{m,i} > s_{m,p(m)-i+1}\right).$$

Equality here again does not occur because of R-fullness of N_m. From these inequalities it follows, since (see above) both the $c_{m,i}$ and the $s_{m,i}$ increase in i, that for the rows i in N'_m all $c_{m,i}$s precede all $s_{m,p(m)-i+1}$s, and the other way around for N''_m. Thus both matrices N'_m and N''_m are non-intertwined. Clearly, one of the sequences $(|R(N'_m)|)_{m \geq 1}$ and $(|R(N''_m)|)_{m \geq 1}$, say the first one, is unbounded (the other case is identical). By part 2 of Lemma 2.69 we know that each matrix N'_m is also a cross-matrix of a submatrix of the matrix M'_m. We apply to the matrices N'_m Lemma 2.83 and conclude that for every $r \in \mathbb{N}$ there is an m such that the coloring H_m contains an r-wealthy coloring of one and the same **type W_2** or **type $W_{3,1}$** or **type $W_{3,2}$**. □

Proof. (Proof of Theorem 2.12.) Let $X \subset \mathcal{C}_3$ be an ideal of colorings. If X is p-tame for some $p \in \mathbb{N}$ then by Proposition 2.77 we have that

$$|X_n| \leq n^{10p^6}\ \text{ for every } n \in \mathbb{N}\,.$$

Else by Propositions 2.89 and 2.90 there is an $i \in [4]$ such that for every $r \in \mathbb{N}$ the ideal X contains an r-wealthy coloring of type W_i. For $i = 1$ Lemma 2.25 gives

$|X_n| \geq 2^{n-2}$ for every $n \in \mathbb{N}$ (W_1 colorings). For $i = 2$ Proposition 2.31 gives $|X_n| \geq F_n$ ($\approx 1.618^n$) for every $n \in \mathbb{N}$ (W_2 colorings). For $i = 3$ Proposition 2.39 gives

$$|X_n| > \frac{0.28 \cdot 1.587^n}{\sqrt{n}} \geq G_n \ (\approx 1.466^n) \ \text{ for every } n \geq 23$$

($W_{3,1}$ and $W_{3,2}$ colorings) and Lemma 2.40 gives $|X_n| \geq F_n$ for every $n \in \mathbb{N}$ ($W_{3,3}$ colorings). For $i = 4$ Proposition 2.43 gives $|X_n| \geq G_n$ for every $n \in \mathbb{N}$ ($W_{4,1}$ colorings) and Proposition 2.44 gives

$$|X_n| \geq \left(\frac{\lfloor \frac{2(n-4)}{5} \rfloor}{\lfloor \frac{n-4}{5} \rfloor} \right)^2 (\approx 1.751^n) \geq G_n \ \text{ for every } n \geq 20$$

($W_{4,2}$ colorings). In the second displayed bound, the range $n \geq 23$ applies to the second inequality, the first one holds for every $n \geq 1$. In the third displayed bound, the range $n \geq 20$ applies again to the second inequality, the first one holds already for $n \geq 9$. So in all cases (see also part 3 of Lemma 2.29) we have that $|X_n| \geq G_n$ for every $n \geq 23$. $\qquad \square$

2.5 Concluding remarks

Recall that in the following discussion we have $l = 2$ colors. If we look at the last proof of Theorem 2.12, we see that the smallest lower bound is for type $W_{4,1}$ colorings, and the next smallest one is for type $W_{3,1}$ and $W_{3,2}$ colorings in Proposition 2.39. Therefore we have the following result.

Corollary 2.91. *Let $X \subset \mathcal{C}_3$ be an ideal of colorings that for infinitely many $r \in \mathbb{N}$ does not contain an r-wealthy coloring of type $W_{4,1}$. Then either $|X_n|$ has at most polynomial growth or $|X_n| > 2^{2(n-2)/3}/\sqrt{2n}$ for any large enough $n \in \mathbb{N}$.*

In the remark after the proof of Lemma 2.25 we showed that the lower bound $|X_n| \geq 2^{n-2}$ for type W_1 colorings is tight. Similarly, we noted after the proof of Proposition 2.43 that its lower bound $|X_n| \geq G_n$ for type $W_{4,1}$ colorings is tight too. We do not know if the lower bounds for the other colorings of types W_2, W_3 and $W_{4,2}$ are tight. We suspect that some of them are not but could not improve them. If one could improve the lower bound in Proposition 2.39 for type $W_{3,1}$ and $W_{3,2}$ colorings to $|X_n| \geq F_n$, the next strengthening of the corollary would follow.

Hypothesis 2.92. *Let $X \subset \mathcal{C}_3$ be an ideal of colorings that for infinitely many $r \in \mathbb{N}$ does not contain an r-wealthy coloring of type $W_{4,1}$. Then either $|X_n|$ grows at most polynomially or $|X_n| \geq F_n$ for every large enough $n \in \mathbb{N}$.*

For any fixed $k \geq 1$ we define the sequence $(G_n^k)_{n \geq 1} = (G_1^k, G_2^k, \dots)$ by the recurrence

$$G_1^k = G_2^k = \dots = G_{k-1}^k = 1, \ G_k^k = 2 \ \text{ and } \ G_n^k = G_{n-1}^k + G_{n-k}^k \ \text{ for } n > k.$$

Thus $F_{n-1} = G_n^2$ and $G_n = G_n^3$. The bounds in Theorems 2.11 and 2.12 suggest the following conjecture.

Hypothesis 2.93. *Let $X \subset \mathcal{C}_k$ be an ideal of colorings. Then there is a constant $c > 0$ such that either $|X_n| \leq n^c$ for every $n \in \mathbb{N}$ or $|X_n| \geq G_n^k$ for every large enough n.*

The lower bound G_n^k is tight since, as for $k = 3$, the ideal $S(k) \subset \mathcal{C}_k$ of colorings (n, χ), where $\chi : \binom{[n]}{k} \to \{0, 1\}$, such that for some disjoint k-intervals $I_1 < I_2 < \cdots < I_r$ in $[n]$ one has $\chi(I_j) = 0$ for every $j \in [r]$ but $\chi(E) = 1$ for all other edges, satisfies $|S(k)_n| = G_n^k$. We briefly considered the case $k = 4$ and found all potential wealthy colorings leading to violation of one of the tameness conditions. Hopefully this approach can be generalized for any $k > 2$. However, the analogy of type $W_{4,2}$ colorings produces in the general version sufficiently many colorings only for $k \leq 34$. For larger k one has to find a better way to generate sufficiently many different colorings, but this should not be a problem since the bound for $W_{4,2}$ colorings given in Proposition 2.44 is larger than G_n.

Growth functions of ideals of ordered 3-uniform hypergraphs should be investigated in more detail. We only determined the jump from constant to linear growth and the jump from polynomial to exponential growth. Finer polynomial jumps and exponential jumps for ordered graphs are described in [12]. To obtain similar results for ordered k-uniform hypergraph for general $k > 2$ may be the goal of a future work.

3. Wilf Equivalence

> These results, which are partly combinatorial and partly real mathematics.
>
> *A. Joseph, Oxford, 1997*

Previous chapters were devoted to the ideals of integer partitions and ordered graphs and hypergraphs. In this chapter we extend this terminology to any partial ordered set with a size function. However, we have to pay a price for the general theory. Given characterizations are almost impossible to put under one universal result because of the different variability they provide. For example, the characterization of partitions ideals, unlike that for ideals of graphs, enables high jumps of the growth function.

While it is hard to describe the full range of growth functions in general, we focus on describing ideals with equal growth function. For that reason we introduce the notion of Wilf equivalence. We say that two families of structures are Wilf equivalent, if the number of structures that avoid the former family of patterns is alike the number of structures that avoid the latter family of patterns.

Wilf equivalence, in this terminology, originated in the area of counting permutations with forbidden patterns. It remains so far mostly restricted to it, with some forays in the enumeration of both integer and set partitions, words, and compositions. The origin of the term Wilf equivalence dates back to Herbert S. Wilf's survey article [70] where he described the equivalence of pattern-free permutation classes in the way we define it in Section 3.1.

From that point, many authors undertook the notion of Wilf equivalent sets. Some of the recent works are Backelin, West and Xin [9], Bloom and Saracino [17], Burstein and Pantone [22], Chamberlain, Cochran, Ginsburg, Micelli, Riehl and Zhang [24], Chen and Narayanan [25], Cohen [26], Dwyer and Elizalde [28], Jelínek and Mansour [52], Kitaev, Liese, Remmel and Sagan [43], Lee and Sah [46], Simion and Schmidt [63], Stankova [65], Wilf [69]. Our main inspiration was the paper of Remmel [56] that introduces the method of Subsection 3.3.

In Section 3.1 we introduce ideals of posets and define Wilf equivalence. Section 3.2 provides an easy way to generate pairs that are Wilf equivalent via automorphism method. Section 3.3 deals with the Cohen–Remmel method that is mainly used for integer partitions. The final Section 3.4 summarises given results and includes couple of open problems that arises from the results of this chapter.

3.1 Introduction

In this section we formally define Wilf equivalence for two, possibly infinite, sets A and B of patterns. Before that, we make clear the notation in use. A poset is a pair $(P, \preceq)$, where P is a set and $\preceq$ is a partial order of P. We refer to $\preceq$ as the containment relation or the substructure relation.

We recall basic notation. A set $A \subset P$ is an antichain in $(P, \preceq)$ if no two elements of A are comparable by $\preceq$. That is, for any distinct $a, b \in A$ we have both $a \not\preceq b$ and $b \not\preceq a$. The minimal element in a poset $(P, \preceq)$ is any element

$a \in P$ such that no element $b \neq a$ of P satisfies $b \preceq a$. We define set B to be the set of minimal elements of A with respect to $\preceq$ and write $B = \min_{\preceq}(A)$.

Definition 3.1 (Ideal in a poset). *Let $(P, \preceq)$ be a poset. The set $X \subset P$ is an ideal if for any pair $a, b \in P$ where $a \preceq b$ and $b \in X$ we have $a \in X$. That is, X is downward closed to $\preceq$.*

Definition 3.2 (Poset with size function). *Let $P = (P, \preceq)$ be a countable poset. We consider a size function*

$$| \cdot | : P \to \mathbb{N}_0 = \{0, 1, \dots\},$$

such that $a \preceq b$ implies $|a| \leq |b|$. We call $(P, \preceq, | \cdot |)$ a sized poset. Further, let X be an ideal and $n \in \mathbb{N}$. We set

$$X_n = \{a \in X : |a| = n\}$$

to be the subset of X that contains all elements of size n.

Note that in any poset the size function respects the containment. In the following definition we give an important example of an ideal defined via set of forbidden elements that generalizes our examples.

Definition 3.3 (Forbidden set). *For any $B \subset P$ we define the set*

$$F(B) = \{a \in P : \ \forall b \in B : b \not\preceq a\}$$

of all elements of P that do not contain any element of B.

Easily, $F(B)$ is an ideal since $b \not\preceq a$ and $a \preceq c$ means that $b \not\preceq c$. Finally, we proceed to the definition of Wilf equivalent sets.

Definition 3.4 (Wilf equivalence). *Let $A, B \subset P$. We say that sets A and B (of forbidden patters) are Wilf equivalent if for any $n \in \mathbb{N}_0$ we have*

$$|F(A)_n| = |F(B)_n|.$$

We write that $A \sim_W B$.

Clearly, $\sim_W$ is an equivalence relation. So $A \sim_W B$ means that for every non-negative integer n, there are as many elements in P with size n and not containing any $a \in A$, as those not containing any $b \in B$.

One can easily check that $A \sim_W \min_{\preceq}(A)$ for every $A \subset P$. Hence sets $A, B \subset P$ for which $A \sim_W B$ is in question may be assumed to be antichains. Because of the bijective correspondence between ideals in P and antichains, a question on Wilf equivalence of two sets A and B (resp. their minimal elements) is a question on the equality of the growth functions

$$n \mapsto |X_n| \ \text{ and } \ n \mapsto |Y_n|$$

of two ideals $X = F(A)$ and $Y = F(B)$.

One of the best known examples of the Wilf equivalence of two sets A, B is the theorem of Euler [31] which gives the equivalence between two sets of (integer) partitions with different forbidden sets. Note that

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$$

is a partition of size $|\lambda| = n = \sum_{i=1}^{k} \lambda_i$ and a containment relation for partitions, $\mu \succeq \lambda$, means that μ contains λ as a (possibly sparse) subsequence. For example,

$$(5, 3, 2, 2, 1, 1, 1) \succeq (5, 2, 1, 1) \text{ but } (5, 3, 2, 2, 1, 1, 1) \not\succeq (3, 2, 2, 2, 1) .$$

Equivalently, for every $n \in \mathbb{N}$, the multiplicity of n in μ is at least the multiplicity of n in λ.

Theorem 3.5 (The Euler identity [30], 1741). *For every $n \in \mathbb{N}_0$, there are as many partitions λ of size n with all parts λ_i odd ($\lambda_i \equiv 1$ (mod 2) for every i) as partitions μ of size n with all parts μ_i mutually distinct ($\mu_i > \mu_{i+1}$ for every i).*

The two ideals of partitions are not defined by forbidden patterns A and B, but one can easily observe that in fact

$$A = \{(2), (4), (6), \ldots\} \text{ and } B = \{(1,1), (2,2), (3,3), \ldots\}.$$

Hence, the Euler identity says that $A \sim_W B$.

There are many other examples of Wilf equivalent pairs given by Andrews [5, Corollary 1.2], Remmel [56, Corollary 2.1], Huang [38, Theorem 1.1], the Catalan identities for permutations by Simion and Schmidt [63], Bóna [20, Chapter 4.2].

One can establish a Wilf equivalence of two sets in many ways. In the next sections we describe several methods that produce sets that are Wilf equivalent. We describe two of them more thoroughly: the automorphism method and the Cohen–Remmel method.

3.2 Automorphism method

The simplest method for generating Wilf equivalent pairs uses the fact that any size-preserving automorphism of a poset sends any set of forbidden patterns to a Wilf equivalent set.

Theorem 3.6 (Automorphism method). *Suppose that*

$$P = (P, \preceq, |\cdot|) \text{ and } \Phi \colon P \to P$$

is a countable poset P with a size function $|\cdot|$, and a size-preserving automorphism Φ. That is; Φ is a bijection such that for every $a, b \in P$ one has $|a| = |\Phi(a)|$ and $a \preceq b$ if and only if $\Phi(a) \preceq \Phi(b)$. Then for every set $A \subset P$ we have

$$A \sim_W B = \Phi(A).$$

Note that a countable poset may have a monomorphism that is not an automorphism. But every size-preserving monomorphism is an automorphism since any injection of finite set is a surjection.

Proof. For every $n \in \mathbb{N}_0$ we have the restriction

$$\Phi \colon F(A)_n \to F(B)_n$$

because if $c \in P$ does not contain any $a \in A$, then $\Phi(c)$ has the same size and does not contain any $b \in B$; otherwise $b \preceq \Phi(c)$ would imply $a = \Phi^{-1}(b) \preceq c$. Similar argument shows that we have the restriction $\Phi^{-1} \colon F(B)_n \to F(A)_n$. The two restrictions form a bijection between $F(A)_n$ and $F(B)_n$. $\qquad\square$

Example. There are as many binary sequences with length n and no consecutive pattern 10011, as those avoiding 00110, due to the two automorphisms of the corresponding poset of binary words: one reverses the sequences and the other swaps 0 and 1. $\qquad\triangle$

We present general examples of Theorem 3.6 for several combinatorial structures. The most famous one is for permutations and one can find it in the following subsection 3.2.1. Next subsection 3.2.2 deals with the automorphisms of number partitions and compositions. Finally, in subsection 3.2.3 we present our results on the subword and sparse subword poset. The results for permutations was proved by Smith [64]. Results for partitions, compositions and words are our own work.

3.2.1 Automorphisms for permutations

Let P consists of all permutations π of $[n]$. These are finite sets of pairs

$$\pi \subset [n] \times [n], \ n \in \mathbb{N}_0,$$

such that for every $i \in [n]$ there is exactly one $j \in [n]$ with $(i, j) \in \pi$ and exactly one $j \in [n]$ with $(j, i) \in \pi$. They may also be written as bijections $\pi \colon [n] \to [n]$ or as sequences $\pi = a_1 a_2 \ldots a_n$, $a_i = j$ corresponds to $(i, j) \in \pi$. The size of π is defined by $|\pi| = n$. Containment $\rho \preceq \pi$ means that there are two increasing mappings $f, g \colon [|\rho|] \to [|\pi|]$,

$$(i, j) \in \rho \iff (f(i), g(j)) \in \pi.$$

In other words, $\pi = a_1 a_2 \ldots a_n$ has a subsequence that is order-isomorphic to the sequence $\rho = b_1 b_2 \ldots b_m$. Seven Catalan identities provide an example of applying automorphism method to obtain Wilf equivalent pairs.

Lemma 3.7 (Catalan identities)**.** *Consider the set partition of permutations of size* 3*:*
$$P_3 = U \cup V := \{123, 321\} \cup \{132, 231, 312, 213\}.$$

Then for every $n \in \mathbb{N}_0$ and any of the seven pairs $\{\pi, \rho\} \in \binom{U}{2}$ and $\{\pi, \rho\} \in \binom{V}{2}$ we have the Wilf equivalence
$$\{\pi\} \sim_W \{\rho\}.$$

Proof. The poset $(P, \preceq, |\cdot|)$ of permutations has eight size-preserving automorphisms. They are defined by

$$\pi \mapsto \pi' \in P_n, \quad (i, j) \mapsto (i', j')',$$

where the last three prime operations, each with two options, mean: for every n either do nothing, $i' = i$, or for every n set $i' = n - i + 1$, do the same with j', and for every n either do nothing, $(i, j)' = (i, j)$, or for every n switch the coordinates by $(i, j)' = (j, i)$. These eight automorphisms partition P_3 in the blocks U and V and give Lemma 3.7 as an instance of Theorem 3.6. More generally, if Φ is any of the eight automorphisms and $A \subset P$ is any set of permutations, then we have $A \sim_W \Phi(A)$. $\qquad\square$

Example. Consider the reversal automorphism $\Phi_r(a_1 a_2 \ldots a_n) = a_n a_{n-1} \ldots a_1$ that may be represented as a correspondence $(i, j) \mapsto (n + 1 - i, j)$. Then Theorem 3.7 implies that

$$\{213\} \sim_W \{312\}, \qquad \{132\} \sim_W \{231\}, \qquad \{123\} \sim_W \{321\}.$$

$\triangle$

A well known result in enumerative combinatorics (Bóna [20, Chapter 4.2]) says that $\{\pi\} \sim_W \{\rho\}$ also for $\pi \in U$ and $\rho \in V$. The adjective 'Catalan' comes from the fact that the common growth function equals to the n-th Catalan number C_n. Hence for every $\pi \in P_3$ and every $n \in \mathbb{N}_0$,

$$|F(\{\pi\})|_n = \frac{1}{n+1}\binom{2n}{n}.$$

The reason that we presented Theorem 3.7 is that one can prove it more easily than this stronger but singular claim, and moreover as an instance of a general principle. Simion and Schmidt [63, Lemma 1] hint rather implicitly to the eight size-preserving automorphisms of the permutation poset.

First proof that there is no other automorphism of poset of permutations was given by Smith [64, Theorem 2.4]. Since for every permutation π its size $|\pi|$ is one less than the cardinality of the maximum chain ending in π, the qualification 'size-preserving' and the size function may be omitted.

Theorem 3.8 (Smith [64], 2006). *The poset of permutations $(P, \preceq)$ has no other automorphism besides the eight described after Theorem 3.6. They are compositions of following three automorphisms $\Phi : P \to P$:*

(A) Vertical flip $\Phi(i, j) = (n + 1 - i, j)$ that coincides with the reversal automorphism.

(B) Horizontal flip $\Phi(i, j) = (i, n + 1 - j)$.

(C) Diagonal flip $\Phi(i, j) = (j, i)$.

The eight automorphisms of the poset of permutations are sometimes called the symmetries. Interestingly and unlike for partitions which we discuss later, restricting to an ideal X of permutations may give new automorphisms besides the symmetries. Albert, Atkinson and Claesson [1] give the following example of an automorphism ξ on a particular ideal X.

Example (Automorphism of an ideal X). Let $X = F(\{132, 312\})$ be an ideal of permutations with two forbidden patterns (=permutations) 132 and 312. It is easy to observe that X consists of permutations $\pi \in P_n$ that are the union of an
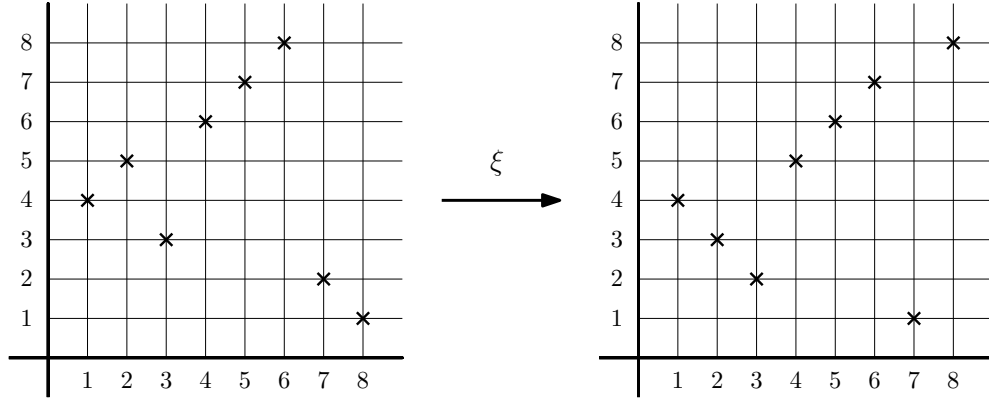
Figure 3.1: An example of automorphism ξ of ideal $X = F(\{132, 312\})$ that cannot be generalised to an automorphism (symmetry) of permutations. Here, $\xi(45367821) = 43256718$.

increasing sequence with a decreasing sequence, where the increasing terms are all greater than the decreasing terms, as shown on both pictures of Figure 3.1

To any permutation $\pi \in X_n$ we assign a word $\omega(\pi) = c_2 c_3 \ldots c_n$ such that for any $i \in [2, n]$ we define

$$c_i = \begin{cases} a & \text{if } \pi(i) > \pi(1), \\ b & \text{if } \pi(i) < \pi(1). \end{cases}$$

The correspondence $\omega : X_n \to \{a, b\}^{n-1}$ is a bijection since one can easily reconstruct the permutation $\omega^{-1}(w)$ from the word $w \in \{a, b\}^{n-1}$: start with $(\omega^{-1}(w))(1) = 1$ and prolong the increasing, resp. decreasing, sequence for any letter a, resp. b, from w. Let $r : \{a, b\}^{n-1} \to \{a, b\}^{n-1}$ be the mapping that reverses the order of letters in a word. Now we may finally define mapping $\xi = \omega^{-1} r \omega$, see the example

$$\xi(45367821) = \omega^{-1} r \omega(45367821) = \omega^{-1} r(abaaabb) = \omega^{-1}(bbaaaba) = 43256718.$$

The latter result [1] states that ξ is an automorphism of X. One may observe the transformation in the example in Figure 3.1.

$\triangle$

3.2.2 Automorphisms for partitions and compositions

We move to the partition poset and show that it has no nontrivial size-preserving automorphism. We are not aware of any literature on the topic, except for Starovojt [67] whose ordering of partitions is different from ours. We deduce our result from a more general property of the partition poset.

Recall that a partition λ is a k-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of non-increasingly ordered parts $\lambda_1, \lambda_2, \ldots, \lambda_k$. The size function $|\lambda|$ is the sum $\sum_{i=1}^{k} \lambda_k$ of the parts. And the partial order $\lambda \preceq \gamma$ means that the multiplicity of any part of λ is at most the multiplicity of the same part in γ. Note that we used the same relation $\preceq$ in Chapter 2, Definition 1.9.

Definition 3.9 (Downsets and upsets)**.** *For the poset* $(P, \preceq)$ *of partitions and* $\lambda \in P$ *we set*

$$\downarrow\!\lambda = \{\kappa \in P : \ \kappa \preceq \lambda\}, \qquad \uparrow\!\lambda = \{\kappa \in P : \ \kappa \succeq \lambda\} \ .$$

to be the downset of λ *and the upset of* λ. *More generally, we may define upset and downset in the same way for any partially ordered set.*

Now we may conclude with the main theorem of this subsection. It shows that there is no nontrivial automorphism of $P = (P, \preceq, |\cdot|)$.

Theorem 3.10 (Isomorphisms of downsets)**.** *Let* $P = (P, \preceq, |\cdot|)$ *be the sized poset of all partitions and* $\lambda \neq \mu \in P$ *be two different partitions. Then*

$$(\downarrow\!\lambda, \preceq, |\cdot|) \not\cong (\downarrow\!\mu, \preceq, |\cdot|).$$

Equivalently, there is no size-preserving isomorphism between the two downsets of P. *Thus for every ideal* X *in* P *the subposet* $(X, \preceq, |\cdot|)$ *has no nontrivial size-preserving automorphism.*

We note that the "size-preserving" condition is essential. Otherwise, we may take the ideal $X = \{(3,1), (3), (1), \emptyset\}$ which has a nontrivial automorphism that is not size-preserving. Indeed, we set $\Phi(3,1) = (3,1)$, $\Phi(3) = 1$ and $\Phi(1) = 3$ and obtain an automorphism of X that is not size-preserving.

Proof of Theorem 3.10. We first show that any partition $\mu \in P$ can be reconstructed just from the isomorphism class of the sized subposet $Q = (\downarrow\!\mu, \preceq, |\cdot|)$. We use the sized graph $G = (\downarrow\!\mu, E, |\cdot|)$ in which $\{\mu, \nu\} \in E$ if and only if $\mu \neq \nu$, $\mu \succeq \nu$, and $\mu \succeq \kappa \succeq \nu$ implies $\mu = \kappa$ or $\nu = \kappa$.

Let $n \in \mathbb{N}$ and $m_n(\mu)$ be the maximal length l of a path $v_0, v_1, \dots, v_l$ in G such that $v_0 = \mu$ is the maximum element of Q and for any $i \in [l]$ we have $|v_i| = |v_0| - in$. It is not hard to see that $n \in \mathbb{N}$ has in μ multiplicity $m \in \mathbb{N}_0$ if and only if $m_n(v_0) = m$.

For any size-preserving automorphism $\Phi : X \to X$ and any $\lambda \in X$ the restriction of Φ on the subposet $\downarrow\!\lambda$ shows that

$$(\downarrow\!\lambda, \preceq, |\cdot|) \cong (\downarrow\!\Phi(\lambda), \preceq, \cdot|).$$

Since Φ is nontrivial, there is a $\lambda \in X$ with $\lambda \neq \Phi(\lambda)$ and we get a contradiction with the first part. $\square$

Corollary 3.11 (Automorphisms of partitions)**.** *The sized partition poset* $(P, \preceq, |\cdot|)$ *has no nontrivial size-preserving automorphism.*

Proof. It is an immediate consequence of Theorem 3.10 for $X = P$. $\square$

Therefore for partitions the automorphism method may not be applied. Despite this, interestingly, large parts of the partition poset are nontrivially isomorphic.

Theorem 3.12 (Isomorphisms of upsets)**.** *Let* $P = (P, \preceq, |\cdot|)$ *be the sized partition poset and* $\lambda, \mu \in P$ *be two partitions. Then*

$$(\uparrow\!\lambda, \preceq, |\cdot|) \cong (\uparrow\!\mu, \preceq, |\cdot|) \iff |\lambda| = |\mu|.$$

Proof. The implication $\implies$ is clear when the size of the smallest element is considered. For the reverse implication we suppose that $|\lambda| = |\mu|$ and define the isomorphism $\Phi : \uparrow\lambda \to \uparrow\mu$ as follows. For any $\kappa \in \uparrow\lambda$ we take all parts (with multiplicities) of κ that witnesses the inclusion $\lambda \preceq \kappa$ and substitute them with all parts (with multiplicities) of μ. That mapping is trivially an isomorphism. $\square$

The situation is a bit different in the poset of compositions $(C, \preceq, |\cdot|)$. By compositions we mean the partitions with prescribed order of the parts, hence compositions are in a bijective correspondence with finite sequences of positive integers. Thus the set of partitions is a subposet of the set of compositions.

The containment relation respects the order of parts; that is subcomposition is a (possibly sparse) subsequence. The size function does not change, it still sums all parts of the composition. For any composition $\lambda \in C$ we also define upsets $\uparrow\lambda = \{\kappa \in C : \kappa \succeq \lambda\}$ and downsets $\downarrow\lambda = \{\kappa \in C : \kappa \preceq \lambda\}$ of λ. In contrast to partitions, one can find two different compositions with isomorphic downsets, for example

$$(\downarrow(1,2,3,4), \preceq, |\cdot|) \cong (\downarrow(1,3,2,4), \preceq, |\cdot|)$$

and basically, any permutation of $[4]$ would work there. Despite this fact the number of automorphisms for compositions does not change much. There are two trivial automorphisms, identity and reversal composition:

$$\Phi_{id}(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n),$$
$$\Phi_r(a_1, a_2, \ldots, a_n) = (a_n, \ldots, a_2, a_1).$$

Note that in latter definitions and further proofs we use $\Phi(a_1, \ldots, a_n)$ instead of formal expression $\Phi((a_1, \ldots, a_n))$.

Theorem 3.13 (Automorphisms of compositions). *The poset $P = (C, \preceq, |\cdot|)$ of number compositions has no nontrivial size-preserving automorphism. There are only two trivial automorphisms: the identity Φ_{id} and the reversal Φ_r.*

Proof. Suppose that there is a nontrivial automorphism Φ. Easily, the size-preserving condition guarantees that $\Phi(a) = (a)$ for all positive integers a.

Moreover, Φ preserves multiplicities of parts. That is, for any $\lambda \in C$ and $a \in \mathbb{N}$, the multiplicity of a in λ equals the multiplicity of a in $\Phi(\lambda)$. To prove that, we choose any $\lambda \in C$ and construct a sized graph $G_\lambda = (\downarrow\lambda, E, |\cdot|)$ in which $\{\mu, \nu\} \in E$ if and only if $\mu \neq \nu$, $\mu \succeq \nu$, and $\mu \succeq \kappa \succeq \nu$ implies $\mu = \kappa$ or $\nu = \kappa$. Observe that graphs G_λ and $G_{\Phi(\lambda)}$ are isomorphic. Now, any part x of λ has multiplicity $m \in \mathbb{N}$ if and only if the maximal length l of a path $v_0, v_1, \ldots, v_l$ in G_λ such that $v_0 = \lambda$ and for any $i \in [l]$ we have $|v_i| = |v_0| - ix$ equals $l = m$. Hence Φ only reorders parts of any given composition.

Therefore, a composition that is formed by deleting any part x of λ is mapped to a composition where x is deleted from $\Phi(\lambda)$. It follows that $\{\Phi(a,b), \Phi(b,a)\} = \{(a,b), (b,a)\}$ for all couples $(a,b) \in \mathbb{N}^2$. Thus $\Phi(a,a) = (a,a)$. We show that either

$$\Phi(t) = \Phi_{id}(t) \text{ for all } t \in \mathbb{N}^2 \quad \text{or} \quad \Phi(t) = \Phi_r(t) \text{ for all } t \in \mathbb{N}^2. \qquad (3.1)$$

Indeed, let us, for contrary, suppose that $\Phi(a', b') = (b', a')$ and $\Phi(c', d') = (c', d')$ for mutually different $a', b', c', d' \in \mathbb{N}$. We note that one may always ensure that

latter parts are mutually different. Indeed, we consider sufficiently large pair of integers and compare the type of the image to one of the given pairs. Next we claim that there are mutually different positive integers $a, b, c \in \mathbb{N}$ such that

$$\Phi(a, b) = (a, b) \quad \text{and} \quad \Phi(a, c) = (c, a). \tag{3.2}$$

We prove (3.2) by inspecting the image of (a', d'). If $\Phi(a', d') = (a', d')$ then we set $a = a'$, $b = d'$ and $c = b'$. On the other hand, if $\Phi(a', d') = (d', a')$ then $\Phi(d', a') = (a', d')$ and $\Phi(d', c') = (d', c')$ and we set $a = d'$, $b = c'$ and $c = a'$. That establishes (3.2).

Supposing (3.2), we inspect the image of (a, b, c) and (a, c, b) under Φ. Since both (a, b) and (a, c) are subcompositions of (a, b, c) and (a, c, b) thus both images $\Phi(a, b, c)$ and $\Phi(a, c, b)$ must contain $\Phi(a, b) = (a, b)$ and $\Phi(a, c) = (c, a)$. Since Φ preserves multiplicities of parts and (c, a, b) is the only composition that contains both (a, b) and (c, a) (and parts a, b and c), we have $\Phi(a, b, c) = (c, a, b) = \Phi(a, c, b)$, contradiction with the injectivity of Φ. Hence (3.1) is true. If the latter of (3.1) is true then we consider $\Phi \circ \Phi_r$ instead of Φ. Thus we may further suppose that for all $t \in \mathbb{N}^2$ we have

$$\Phi(t) = \Phi_{id}(t) = t.$$

Let A be a composition such that $A \neq B = \Phi(A)$ and $|A|$ is minimal. Let A starts with part a. Let $t > 0$ be the number of a's at the beginning of A, and $s \geq 0$ be the number of a's at the beginning of B. Hence

$$A = (\overbrace{a, a, \dots, a}^{t}, b, \dots) \quad \text{and} \quad B = (\overbrace{a, a, \dots, a}^{s}, c, \dots),$$

where $b, c \in \mathbb{N} \setminus \{a\}$ are two (not necessary different) positive integers. Let for $s > 0$

$$A' = (\overbrace{a, \dots, a}^{t-1}, b, \dots) \quad \text{and} \quad B' = (\overbrace{a, \dots, a}^{s-1}, c, \dots),$$

are subcomposition of A and B, respectively. In terms of t and s, at least one of the following happens.

- If $t > s \geq 1$ then by the minimality of A we have $\Phi(B') = B'$. Thus B' must be also contained in A. But that is not possible since any subcomposition of A either has at least $t - 1$ a's at the beginning or has the size smaller then B'.

- If $t \geq 2$ and $s = 0$ then we argue similarly with A'. We have $\Phi(A') = A'$, A' is contained in A, but is A' is not contained in B. Indeed, Φ preserves multiplicities and therefore A' can be obtained from B only by deleting some a, which is not possible.

- If $t = 1$ and $s = 0$, we denote by k, resp. l, the multiplicity of a in A, resp. the sum of multiplicities of non-a parts in A. Easily, $k, l \geq 1$ since otherwise $A = B$. Also, first part of (3.1) implies $k + l > 2$.

  If $l = 1$ then $k \geq 2$. Let A'' be the subcomposition of A obtained by deleting the last part a from A. Easily A'' is contained in A and by minimality, $\Phi(A'') = A''$. Thus A'' must be contained also in B, which is not possible.

On the other hand we suppose $l > 1$. We argue similarly with B'' that is obtained by deleting the last non-a part of B. One may check that B'' is not a subcomposition of A.

- If $t < s$ then $s > 2$ and we apply first case for inverse automorphism Φ^{-1}.

- If $t = s$ then $t = s > 0$ we conclude that $A = B$. Indeed, A' is contained in A, $\Phi(A') = A'$ and A' can be obtained from B only by deleting first a, thus $A' = B'$. However, then $A = B$, contradiction.

In all cases either B without one part cannot be reconstructed from A deleting only one part or vice versa. So there is no trivial automorphism. $\square$

3.2.3 Automorphisms for words

In this section we give similar results to the Theorem 3.8 and Corollary 3.11 for the universe of words. Especially the result of Theorem 3.18, where all automorphisms for a sparse word poset are enlisted, are similar to the composition case. It is so because the containment relation differs only in the values of the size function, which does not enforce new structural properties.

We first introduce two possibilities to define containment relation in the poset of words and in both cases prove that there are only trivial automorphisms of a poset $(P, \preceq, |\cdot|)$. Note that the adjective "trivial" contains not only identity and reversal as before but also the permutation of the words.

Definition 3.14 (Set of words). *Let $k \in \mathbb{N}$. We set*

$$[k]^* = \{u = a_1 a_2 \ldots a_m : \ a_i \in [k], \ m \in \mathbb{N}_0\}$$

to be the base set of all (finite) words over alphabet $[k]$.

Here $|u| = m$ is the size (usually called length) of the word $u = a_1 a_2 \ldots a_m$.

Definition 3.15 (Subword poset). *Let $u = a_1 a_2 \ldots a_m \in P$ and $v = b_1 b_2 \ldots b_M \in P$ be two words over alphabet $[k]$. We define (tight) containment relation*

$$u \preceq_t v \iff \exists i \in [M - m + 1]: \ b_{i+1} = a_1, b_{i+2} = a_2, \ldots, b_{i+m} = a_m$$

and call $(P, \preceq_t, |\cdot|)$ the (tight) k-subword poset.

Definition 3.16 (Sparse subword poset). *Let $u = a_1 a_2 \ldots a_m$ and $v = b_1 b_2 \ldots b_M$ be two words over alphabet $[k]$. We define sparse containment by*

$$u \preceq_s v \iff \exists 1 \le i_1 < i_2 < \cdots < i_m \le M: \ b_{i_1} = a_1, b_{i_2} = a_2, \ldots, b_{i_m} = a_m$$

and call $(P, \preceq_s, |\cdot|)$ the sparse k-subword poset.

Note that $\preceq_s$ is structurally same as the containment for partitions and compositions. The difference is the base set and the size function. We use that fact in the proof of Theorem 3.18.

For $k = 1$ both posets of words are isomorphic to the simple poset $(\mathbb{N}_0, \le, |\cdot|)$ with $|n| = n$, but already for $k = 2$ things get interesting. We gave in Section 3.2 an example of two sets of forbidden patterns $\{10011\} \sim_W \{00110\}$. That is an

example in the isomorphic copy of the 2-subword poset which has the alphabet [2] replaced with $\{0, 1\}$.

There are three types of trivial automorphisms for both posets $(P, \preceq_t, |\cdot|)$ and $(P, \preceq_s, |\cdot|)$. They either do nothing, reverse the words or permute the alphabet. Thus we define three trivial automorphisms

$$\Phi_{id}(a_1 a_2 \ldots a_n) = (a_1 a_2 \ldots a_n),$$
$$\Phi_r(a_1 a_2 \ldots a_n) = (a_n \ldots a_2 a_1),$$
$$\Phi_\pi(a_1 a_2 \ldots a_n) = (\pi_{a_1} \pi_{a_2} \ldots \pi_{a_n}),$$

where $\pi : [n] \to [n]$ is a permutation of $[n]$. The obvious $2 \cdot k!$ automorphisms of both posets are compositions of Φ_{id}, Φ_r and Φ_π for different permutations π. One can easily check that all these mappings are size-preserving automorphisms, and we call them trivial automorphisms. We prove that there are no other automorphisms in the subword poset.

Theorem 3.17 (Automorphism of subword poset)**.** *For every integer $k \geq 2$, the k-subword poset $(P, \preceq_t, |\cdot|)$ has only $2 \cdot k!$ size-preserving automorphisms. They are of the form $\Phi_{ir} \circ \Phi_\pi$, where $\Phi_{ir} \in \{\Phi_{id}, \Phi_r\}$ is either identity or reverse, and Φ_π is a permutation of the alphabet $[k]$.*

Proof. For contradiction we suppose that $\Phi : [k]^* \to [k]^*$ is an automorphism that cannot be written as a composition of identity Φ_{id}, reversal Φ_r and permutation automorphism Φ_π. We may suppose that

(i) $\Phi(a) = a$ for any $a \in [k]$ and

(ii) $\Phi(ab) = ab$ for any $a \neq b \in [k]$.

Indeed, there is a permutation π of $[k]$ such that $\Phi \circ \Phi_\pi$ satisfies condition (i), so we may consider $\Phi \circ \Phi_\pi$ instead of Φ. To ensure (ii) we prove that any automorphism Φ satisfy either $\Phi(ab) = \Phi_{id}(ab) = ab$ for all pairs $a, b \in [k]$ or satisfy $\Phi(ab) = \Phi_r(ab) = ba$ for all pairs $a, b \in [k]$. Thus either Φ or $\Phi \circ \Phi_r$ satisfy condition (ii).

For any couple $(a, b) \in [k]^2$ of distinct letters we have $\{\Phi(ab), \Phi(ba)\} = \{ab, ba\}$ since condition (i) implies $\Phi(a) = a$ and $\Phi(b) = b$, and both a and b are subwords of ab. We call a couple (a, b) reversed if $\Phi(ab) = ba$. Note that the order is not important there since (b, a) is reversed if and only if (a, b) is reversed. Clearly $\Phi(aa) = aa$.

We describe the situation in the terms of reversed couples. Our claim is that either all couples $(a, b) \in [k]^*$ of distinct letters are reversed or all such couples are not reversed. Let (a, b) be a reversed couple of distinct letters. Since (ii) is clear for $k = 2$ we suppose that there is $c \in [k] \setminus \{a, b\}$. We focus on the possible values of $\Phi(abc)$ and $\Phi(bc)$. Since $\Phi(abc)$ have to contain both c and ba then $\Phi(abc) \in \{cba, bac\}$. However, because $\Phi(bc) \in \{bc, cb\}$, and only cb is a subword of an element of $\{cba, bac\}$ there is only one option for $\Phi(abc)$; that is $\Phi(abc) = cba$. That implies $\Phi(bc) = cb$ and (b, c) is a reversed couple. Using similar argument we prove that any couple of distinct letters of $[k]^*$ is reversed. Thus we may suppose that (ii) holds.

Hence we assume that our nonidentical automorphism Φ satisfy (i) and (ii). Let $A \in [k]^*$ be a minimal witness of $\Phi(A) \neq A$; that is $\Phi(A) \neq A$ and

$$\Phi(A') = A' \text{ for any } A' \text{ such that } |A'| < |A|. \tag{3.3}$$

By what we have already proved, $|A| \geq 3$. We focus on the structure of A and $B = \Phi(A)$. Let $A = aQc$ and $B = bRd$, where $a, b, c, d \in [k]$ and $|Q| = |R| = |A| - 2 > 0$. From (3.3) we have $\{aQ, Qc\} = \{bR, Rd\}$. We analyze this equality of sets. If $aQ = Qc = bR = Rd$ then $A = B$, contradiction. If $aQ = bR \neq Qc = Rd$ then $a = b$, $c = d$, $Q = R$, and again $A = B$. So the last case remains, that is

$$aQ = Rd \neq Qc = bR. \tag{3.4}$$

Since $|Q| = |R| > 0$, the first letter of Q is b, and the first letter of R is a. Hence $Q = bQ'$ and $R = aR'$. If $|Q'| = |R'| = 0$ then $A = abc$ and $B = bad$ and (3.4) implies $A = aba$ and $B = bab$. On the other hand, $|Q'| = |R'| > 0$ together with (3.4) implies that $Q = baQ''$ and $R = abR''$. Again, for $|Q''| = 0$ we get $A = abab$ and $B = baba$, otherwise (3.4) reveals first letters of Q'' and R''. We continue revealing letters of Q and R, resp. A and B, in a similar manner. Finally we conclude that

$$A = abab \dots a/b, \quad B = baba \dots b/a,$$

where the last letter of A, resp. B, is either a or b, depending on the parity of the length of A, resp. B. Note that $a \neq b$ since otherwise $A = B$, which is in contradiction with the definition of A. We examine both parities separately.

(I) Let $|A| = |B| = 2n + 1$, where $n \geq 1$. We focus on the image of aA. There are several possibilities for $\Phi(aA)$, namely

$$\Phi(aA) = \Phi(aaba \dots a) \in \bigcup_{x \in [k]} \{xbab \dots b, bab \dots bx\}. \tag{3.5}$$

Moreover, because of $n \geq 1$, we have $aa \preceq_t aA$ and $\Phi(aa) = aa$. That means $aa \preceq_t \Phi(aA)$. However, inspection of the right side of (3.5) reveals no room for pattern aa. Hence such a word A with odd length does not exist.

(II) Let $|A| = |B| = 2n$, where $n \geq 2$. Again, the image of $aA = aaba \dots b$ is either of the form $xbab \dots a$ or $bab \dots ax$, for some $x \in [k]$. Since $aa \preceq_t aA$ and $\Phi(aa) = aa$, word aa must be present in either $xbab \dots a$ or $bab \dots ax$ for some $x \in [k]$. The only possibility is $x = a$, where aa is at the end of $bab \dots ax$. That means

$$\Phi(aA) = \Phi(aab \dots abab) = baba \dots aa.$$

Now in the same way we look for the image of $aab \dots aba$ and $aab \dots ab$, where we removed last (two) letter(s) from aA. We have to preserve the pattern aa in both images, hence

$$\Phi(aab \dots aba) = aba \dots aa,$$
$$\Phi(aab \dots ab) = ba \dots aa.$$

However, $|aab \dots ab| = |A| - 1 > 1$, but $\Phi(aab \dots ab) \neq aab \dots ab$. That is a contradiction with (3.3).

In all cases we obtained a contradiction. Therefore, there is no automorphism Φ that is not a composition of identity Φ_{id}, reversal Φ_r and permutation automorphism Φ_π. $\qquad\square$

We may conclude with a similar theorem for the sparse subword poset. The proof is a combination of two ideas. First one is finding possible outputs for all small words, as described at the beginning of the latter proof. Second idea is taken from the composition case since the containment relation is very similar for compositions and sparse subword poset.

Theorem 3.18 (Automorphism of sparse subword poset)**.** *Let $k \geq 2$. The sparse k-subword poset $(P, \preceq_s, |\cdot|)$ has no other automorphism than the $2 \cdot k!$ ones that reverse the words in $[k]^*$ and/or permute the alphabet $[k]$.*

Proof. Let $\Phi : [k]^* \to [k]^*$ be an automorphism of $(P, \preceq_s, |\cdot|)$ that is not a composition of identity Φ_{id}, reversal Φ_r and permutation automorphism Φ_π. In the first part, as in the latter proof, we justify additional assumptions (i) and (ii), given below. In the second part of the proof, as in the proof of Theorem 3.13, we reveal the beginning of a minimal word A witnessing $\Phi(A) \neq A$.

Now we justify that we may suppose that

(i) $\Phi(a) = a$ for any $a \in [k]$ and

(ii) $\Phi(ab) = ab$ for any $a \neq b \in [k]$.

First assumption comes from the fact that there is a permutation automorphism Φ_π such that $a = \Phi \circ \Phi_\pi(a)$ for any letter $a \in [k]$. Hence we may use $\Phi_\pi \circ \Phi$ instead of Φ. Second condition (ii) is more technical to obtain. Observe that for any $a, b \in [k]$ we have $\Phi(ab) \in \{ab, ba\}$, hence either

$$\Phi(ab) = ab \text{ and } \Phi(ba) = ba \qquad \text{or} \qquad \Phi(ab) = ba \text{ and } \Phi(ba) = ab.$$

Condition (ii) trivially holds for $k = 2$. Let us suppose that $k \geq 3$ and there are mutually different $a, b, c \in [k]$ such that $\Phi(ab) = ba$ and $\Phi(ac) = ac$. Then $\Phi(abc) = bac$ since both subwords ba and ac must appear in $\Phi(abc)$. Moreover, $\Phi(acb) = bac$ for the same reason. That is in contradiction with the definition of automorphism. Therefore, for any triple a, b, c and $A \subset \{a, b, c\}$ with $|A| = 2$ either $\Phi(A) = \Phi_{id}(A)$ or $\Phi(A) = \Phi_r(A)$. Applying this for all triples of $[k]$ we obtain that for any $A \subset [k]^*$ with $|A| = 2$ either $\Phi(A) = \Phi_{id}(A)$ or $\Phi(A) = \Phi_r(A)$. That establishes (ii), since either Φ or $\Phi \circ \Phi_r$ satisfy (ii).

Thus we may assume that Φ is an automorphism of $[k]^*$ that satisfies (i) and (ii) and is not a composition of Φ_{id}, Φ_r and Φ_π for any permutation π on $[k]$. Let $A \in [k]^*$ be a word such that $\Phi(A) \neq A$ and

$$\Phi(A') = A' \text{ for any } A' \text{ such that } |A'| < |A|. \tag{3.6}$$

We first show that Φ preserves the multiplicity of letters, thus A and $B = \Phi(A)$ differ only by the order of letters. Indeed, if the former is not true, there is a letter x that has more occurrences in A than in B. However, the word A' obtained by deleting (any) x from A can not be obtained from B by deleting only one letter, since the number of occurrences do not match. And the original claim easily follows.

Now we analyze the initial parts of words A and $B = \Phi(A)$, while bear in mind that $|A| \geq 3$. Let us suppose that A starts with $a \in [k]$ and t, resp. s, be the number of initial a-terms of A, resp. B. Therefore

$$A = \overbrace{aa\ldots a}^{t} A' \quad \text{and} \quad B = \overbrace{aa\ldots a}^{s} B'.$$

Note that $t > 0$ and $s \geq 0$. We inspect the relation of t and s in a similar way as in the proof of Theorem 3.13. Since the containment relation is the same as for compositions, the only difference is the size function. However, we proved that images of subwords of A can be obtained only by deleting same letters, hence the reasoning is the same as in Theorem 3.13 and no such automorphism Φ exists. $\square$

3.3   The Cohen–Remmel method

The enumeration of structures specified by several forbidden patterns evokes the inclusion-exclusion principle, which leads to another method for generating Wilf equivalent pairs.

Originally, the principle of inclusion and exclusion was first used for the counting of partitions by Cohen [26] and independently by Remmel [56], who studied the topic deeper. We already presented their results in Theorem 1.17 and gave applications that established the asymptotics of particular ideals of partitions in Theorem 1.19 and Theorem 1.20.

Here we inspect the idea of Cohen and Remmel thoroughly and generalize it beyond integer partitions. Our idea is to use the principle of inclusion and exclusion to generate Wilf equivalent pairs in wider range of structures. And this method applies to posets.

Theorem 3.19 (The generalized Cohen–Remmel method)**.** *Suppose that* $P = (P, \preceq, |\cdot|)$ *is a countable poset with a size function and* $A, B \subset P$*. Let* $f : A \to B$ *be a bijection such that for every* $n \in \mathbb{N}_0$ *and every finite subset* $I \subset A$,

$$|\{c \in P_n : \ \forall \, a \in I \ c \succeq a\}| = |\{c \in P_n : \ \forall \, a \in I \ c \succeq f(a)\}|. \qquad (3.7)$$

Then $A \sim_W B$.

Proof. We denote the two above displayed cardinalities as $a(n, I)$ and $b(n, I)$, respectively. For example, $a(n, \emptyset) = b(n, \emptyset) = |P_n|$. The inclusion-exclusion formula gives, for every $n \in \mathbb{N}_0$,

$$|F(A)_n| = \sum_{I \subset A} (-1)^{|I|} a(n, I) \ \text{ and } \ |F(B)_n| = \sum_{I \subset A} (-1)^{|I|} b(n, I).$$

The sums are over finite subsets I and are easily seen to be finite (almost all summands are 0). By the assumption, we have $a(n, I) = b(n, I)$ for all $n \in \mathbb{N}_0$. Thus $|F(A)_n| = |F(B)_n|$. $\square$

Nice illustration of Theorem 3.19 is the Euler identity (Theorem 3.5). However, in that case, the sets A, B are infinite and we need to explain more carefully why (3.7) is true. Hence we stay with even simpler example and prove the Euler identity after Theorem 3.23.

Example (Wilf equivalent sets of partitions)*.* We have the Wilf equivalence

$$A = \{(3, 2, 1), (12)\} \sim_W \{(6), (7, 5)\} = B \qquad (3.8)$$

of two sets of partitions. Indeed, Theorem 3.19 can be applied with the mapping f that to any partition λ such that

- $\lambda \succeq (3,2,1)$ and $\lambda \not\succeq (12)$ assigns a partition $\lambda \setminus (3,2,1) \cup (6) \succeq (6)$,

- $\lambda \succeq (12)$ and $\lambda \not\succeq (3,2,1)$ assigns a partition $\lambda \setminus (12) \cup (7,5) \succeq (7,5)$,

- $\lambda \succeq (12,3,2,1)$ assigns a partition $\lambda \setminus (12,3,2,1) \cup (7,6,5) \succeq (7,6,5)$,

and vice versa. Easily, f is a bijection and condition (3.7) is clearly true. Hence Theorem 3.19 produces equivalence (3.8). $\triangle$

3.3.1 The Cohen–Remmel method for partitions

To expand and study the full effect of Theorem 3.19 in theory of number partitions, e. g. to prove the Euler identity, we need a family of simple but fundamental partition identities.

Definition 3.20 (Union of set of partitions)**.** *For a finite set A of partitions we define the union of A to be the union of all $\lambda \in A$ when taken as multisets. We denote that by $\mu = \bigcup A$.*

Equivalently, for every $n \in \mathbb{N}$, the multiplicity of n in μ equals to the maximal multiplicity of n in λ, taken over all $\lambda \in A$. It is not hard to see that μ is the $\preceq$-smallest element in $(P, \preceq)$ of all partitions containing every $\lambda \in A$. We use this definition in next identities. Thereafter, the assumptions of Theorem 3.19 are easier to obtain via Theorem 3.21.

Theorem 3.21 (Fundamental partition identities)**.** *Suppose that $A, B \subset P$ are two finite sets of partitions.*

1. For every partition κ,

$$\left(\forall \lambda \in A : \; \kappa \succeq \lambda \right) \iff \kappa \succeq \bigcup A.$$

2. For every $n \in \mathbb{N}_0$,

$$|\{\kappa \in P_n : \; \forall \lambda \in A \; \kappa \succeq \lambda\}| = |\{\kappa \in P_n : \; \forall \lambda \in B \; \kappa \succeq \lambda\}|$$

if and only if

$$\left| \bigcup A \right| = \left| \bigcup B \right|.$$

Proof. 1. If $\kappa \succeq \bigcup A = \mu$ and $\lambda \in A$ then, by the definition of μ, every $n \in \mathbb{N}$ has in κ multiplicity at least as big as its multiplicity in λ and $\kappa \succeq \lambda$. If $\kappa \succeq \lambda$ for every $\lambda \in A$, then for every $n \in \mathbb{N}$ its multiplicity in κ is at least the maximal multiplicity in λ over all $\lambda \in A$ and $\kappa \succeq \mu$.

2. Let $\mu = \bigcup A$, $\nu = \bigcup B$ and $|\mu| = |\nu|$. Consider a mapping $f(\kappa)$ from the former of the displayed sets of partitions to the latter that replaces μ in κ with ν, and a similar mapping g going in the opposite way and replacing ν in κ with μ. In more details, $f(\kappa)$ arises from κ by subtracting for every $n \in \mathbb{N}$ from its multiplicity in κ the multiplicity in μ and adding the multiplicity in ν, and doing it the other way around for g. By the assumption $|\mu| = |\nu|$ the mappings f and g go from P_n to P_n and, by part 1, are correctly defined and range in the stated

sets. Also, f and g are clearly inverse one to another. Thus the cardinalities of two displayed sets of partitions in P_n are the same.

If $|\mu| < |\nu|$ then for $n = |\mu|$ the former cardinality is positive but the latter is still zero. Case $|\mu| > |\nu|$ is treated symmetrically. Therefore the statement of the theorem is true. $\square$

Note that for any partition μ there is a bijection

$$\{\lambda \in P : \ \lambda \succeq \mu\} \to P$$

that decreases sizes by $|\mu|$; for every $n \in \mathbb{N}$ it subtracts from the multiplicity in λ the multiplicity in μ. Thus for any $n \geq |\mu|$, the set $P_n \backslash F(\{\mu\})$ is in bijection with $P_{n-|\mu|}$, and the cardinality is given by

$$|P_n \backslash F(\{\mu\})| = p(n - |\mu|),$$

where $p(n) = |P_n|$ is the standard partition function (see definition of $p(n)$ in Section 1.2). Such values of $p(n)$ appear in the 'PIE-sums' of Cohen [26].

Definition 3.22. *(Remmel's condition) Let $A, B \subset P$ be two sets of partitions and $f : \ A \to B$ is a bijection such that for every finite subset $I \subset A$ we have*

$$\left| \bigcup I \right| = \left| \bigcup f(I) \right|.$$

We call this condition on A and B, respectively on f, Remmel's condition.

Now we prove the main result of Remmel [56, Theorem 2], which in our terminology amounts to a simple criterion for validity of the assumption in Theorem 3.19, in the case of the partition poset.

Theorem 3.23 (Remmel [56], 1982)**.** *If $A, B \subset P$ and $f : \ A \to B$ satisfy Remmel's condition. Then $A \sim_W B$.*

Proof. The proof is immediate by combining Theorems 3.19 and 3.21. $\square$

Theorem 3.23 is implicit in the work of Cohen [26, Theorem 7] and has a precursor already in the work of Andrews [5, Theorem 8.4]. The proof in [56] does not (explicitly) use inclusion-exclusion and instead constructs by the involution principle bijections between $F(A)_n$ and $F(B)_n$. This, in our opinion, makes it less clear and simple than it can be.

Now we present two examples. First is the proof of the Euler identity (Theorem 3.5) and second is an identity of I. Schur.

Proof of Theorem 3.5 (The Euler identity). Easily, the sets A and B of forbidden partitions are

$$A = \{(2), (4), (6), \dots\} \ \text{ and } \ B = \{(1, 1), (2, 2), (3, 3), \dots\}.$$

We define the function $f : \ A \to B$ by $(2k) \mapsto (k, k)$. Let $I \subset A$ be a finite set of partitions (l) with even part l, then

$$\left| \bigcup I \right| = \sum_{(l) \in I} l = \sum_{(l/2, l/2) \in f(I)} 2(l/2) = \left| \bigcup f(I) \right|$$

and $A \sim_W B$ by Theorem 3.23. $\square$

Example (Schur's identity). Applying Theorem 3.23 to the sets

$$A = \{(1,1),(3),(2,2),(6),(4,4,),(9),(5,5),(12),\dots\},$$
$$B = \{(2),(3),(4),(6),(8),(9),(10),(12),\dots\}$$

gives the equality between the number of partitions of n with mutually distinct parts that are congruent to ± 1 (mod 3), and the number of partitions of n with parts congruent to ± 1 (mod 6). Easily, one may verify Remmel's condition. $\triangle$

Many other examples illustrating Theorem 3.23 can be found in [56]. We give one more example of Franklin [35], also mentioned without relation to the theorem of Remmel, as an identity by Huang [38, Theorem 1.3] in an equivalent form.

Corollary 3.24 (Franklin [35], 1883). *For any $k, m, n \in \mathbb{N}$, there are as many partitions λ of n for which less than m numbers $j \in \mathbb{N}$ divisible by k have positive multiplicity in λ, as those for which less than m numbers $j \in \mathbb{N}$ have multiplicity in λ at least k.*

Proof. Let $Y = \binom{\mathbb{N}}{m}$ be the set of all subsets of $\mathbb{N}$ of the size m. For any set $X = \{x_1 > x_2 > \dots > x_m\} \in Y$ we consider the partitions

$$\kappa_X = (kx_1, kx_2, \dots, kx_m) \quad \text{and} \quad \lambda_X = (x_1^k, x_2^k, \dots, x_m^k)$$

where $x^k = x, x, \dots, x$ abbreviates the constant sequence of length k. Let

$$A = \{\kappa_X : X \in Y\}, \ B = \{\lambda_X : X \in Y\} \quad \text{and} \quad f : A \to B, \ \kappa_X \mapsto \lambda_X.$$

Clearly, the former partitions of n in the statement are $F(A)_n$, the latter are $F(B)_n$, and f is a bijection. It remains to check Remmel's condition. Let $Z \subset Y$ be a finite subset, $l \in \mathbb{N}$ an arbitrary number and $m_1 = m_1(l) \in \mathbb{N}_0$, resp. $m_2 = m_2(l) \in \mathbb{N}_0$, be the multiplicity

$$\text{of } kl \text{ in } \ \mu = \bigcup_{X \in Z} \kappa_X, \qquad \text{resp. of } l \text{ in } \ \nu = \bigcup_{X \in Z} \lambda_X.$$

Let $V = \bigcup Z \subset \mathbb{N}$ be the vertices l such that $l \in X$ for at least one $X \in Z$. It is immediate that

$$m_1 = \chi_V(l) \quad \text{and} \quad m_2 = k\chi_V(l)$$

where $\chi_V : \mathbb{N} \to \{0, 1\}$ is the characteristic function of V. Thus

$$|\mu| = \sum_{l \in \mathbb{N}} m_1(l)kl = \sum_{l \in \mathbb{N}} k\chi_V(l)l = \sum_{l \in \mathbb{N}} m_2(l)l = |\nu|$$

and Remmel's condition is verified. $\square$

Theorem 3.23 has a converse for a special type of partitions; that is disjoint partitions, as proved Remmel [56]. Recall that κ and λ are *disjoint* if for every $n \in \mathbb{N}$ the minimum of its two multiplicities in κ and λ equals 0. Remmel [56, Theorem 3] says that given two sequences of partitions $A = \{A_i\}_{i \geq 1}$ and $B = \{B_i\}_{i \geq 1}$ such that for all i the partitions A_i and B_i are disjoint, we have the equivalence $A \sim_W B$ if and only if the sequences $\{|A_i|\}_{i \geq 1}$ and $\{|B_i|\}_{i \geq 1}$ are rearrangements of each other. We noticed and remark that the statement of the

theorem, [56, Theorem 3 on p. 277], has by a typographical error incorrectly 'distinct' in place of 'disjoint'.

But in general the converse to Theorem 3.23 fails. Of course, it fails trivially, for example, for $A' = A \cup \{(2,2,2)\}$ and B where A and B are as in Theorem 3.5 and there are even more trivial failures. Therefore, we further assume that A and B are antichains. Remmel gives an example in a remark after [56, Corollary 2.3]. Let

$$A = \{(j) : \ j \in \mathbb{N}, \ j \equiv 0, 2, 3, 4 \ (\mathrm{mod} \ 6)\}$$

and

$$B = \{(j,j), (j+1,j), (j+2,j), (3j+3,3j) : \ j \in \mathbb{N}\}.$$

Then $A \sim_W B$, as proven by Schur [62]. But A and B do not satisfy Remmel's condition because, for example, $(3,2) \in B$ but no element of A has size 5.

3.3.2 Replacement method

Second application of Theorem 3.19 is based on the bijection of forbidden sets that is formed by replacing forbidden patters of the former set with forbidden patterns of the latter set. Hence the name replacement method. In contrast to the latter Cohen–Remmel method, we use this bijection for general poset.

We need to define the patterns that we replace. On one hand, when particular poset is chosen, it is evident what to replace. On the other hand, it is quite challenging to define the replacement correctly for a general poset. Therefore, we consider this subsection more inspirational than precise.

Definition 3.25 (Copies in structures). *For $a, b \in P$, an a-copy in b is the position of all places or all substructures in b isomorphic to a that realize the containment $b \succeq a$.*

For example, for a poset of sparse words we have $10 \preceq 010010$, but the 10-copy may be chosen in four ways:

$$0\mathbf{10}010, 0\mathbf{1}00\mathbf{1}0, 0\mathbf{1}001\mathbf{0}, 010\mathbf{10}.$$

Hence, to avoid all 10-copies in 010010 one have to delete at least one position of any 10-copy. For example, one might delete second and sixth position and obtain 10-free word 0001. On the other hand, only the first position of 010010 is not the position of any 10-copy. We proceed with the main theorem of this subsection.

Theorem 3.26 (Replacement method). *Suppose that $P = (P, \preceq, |\cdot|)$ is a countable poset with a size function and $A, B \subset P$ are the sets of forbidden patterns that satisfy following conditions.*

(i) *For every $c \in P$, all a-copies and all b-copies in c, for all $a \in A$ and all $b \in B$, are together pairwise disjoint.*

(ii) *There is a bijection $f : A \to B$, identical on $A \cap B$, such that for every $c \in P$, any $a \in A$, and any $b \in B$, replacement of any a-copy in c with $f(a) \in B$ produces a $c' \in P$ with $|c'| = |c|$, and replacement of any b-copy in c with $f^{-1}(b) \in A$ produces a $c'' \in P$ with $|c''| = |c|$.*

Then $A \sim_W B$.

Proof. We use the latter result of Theorem 3.19 with the same notion of sets $A, B \subset P$ and bijection $f : A \to B$ described in assumption (ii). We only need to verify (3.7).

Let $I \subset A$ and $n \in \mathbb{N}_0$. We take any $c \in P_n$ such that for any $a \in I$ we have $c \succeq a$. Since all a-copies in c are pairwise disjoint for all $a \in I$ because of both (i) and (ii), we may replace all a-copies with $f(a)$ simultaneously and obtain $c' \in P_n$ with $|c'| = |c|$. Because all elements of I had its pairwise disjoint copies in c, so do all elements of $f(I)$ and the bijectiveness of f implies that for all $a \in I$ we have $c' \succeq f(a)$. On the other hand, we may proceed in the same manner with any $c \in P_n$ such that for any $b \in f(I)$ it is true that $c \succeq b$ and by replacing $f^{-1}(b)$ with b we obtain $c'' \in P_n$ with $|c''| = |c|$. Finally, since f is bijection, for mutually different elements c_1, c_2 we obtain mutually different elements c_1' and c_2' and vice versa. Hence (3.7) is true and therefore $A \sim_W B$ holds. $\square$

It is also possible to prove Theorem 3.26 straightforwardly without using Theorem 3.19, but we want to point out the relation to the Cohen–Remmel method.

Example (Wilf equivalent pairs)*.* We consider the 2-subword poset and claim that

$$\{a\} = \{11212\} \sim_W \{11222\} = \{b\}.$$

This Wilf equivalence follows by Proposition 3.26. Indeed, neither a nor b overlaps with itself and a and b do not overlap, so assumption (i) in Proposition 3.26 is satisfied. Assumption (ii) is satisfied too because $|a| = |b|$. $\triangle$

The example $\{11212\} \sim_W \{11222\}$ does not follow from Proposition 3.6 because none of the four length-preserving automorphisms sends a to b since b ends with three 2s but a does not have constant three-term sequence on either of its ends.

We may generalize this example; that is, the set of 2-words, by presenting the concept of the poset of block ordered graphs. Let P be the set of all ordered complete edge 2-colored graphs $G = (n, \chi)$ with loops. Here n denotes the number of vertices of G, the ordering of vertices is represented by the bijection between V and $[n]$, and $\chi : [n]^2 \to [2]$ is the coloring of edges. The containment relation is a standard induced subgraph relation that respects the vertex ordering and, moreover, vertices that witness the containment of a smaller graph form a block in the former ordering. More formally, $G' = (m, \chi') \prec G = (n, \chi)$ if and only if there is a non-negative integer $l \leq n - m$ such that for all $a, b \in [m]$ (possibly $a = b$) we have
$$\chi'(\{a, b\}) = \chi(\{l + a, l + b\}).$$

Example (Wilf equivalent pairs for block ordered graphs)*.* We present an example on Figure 3.2, where two sets $A \sim_W B$, each with two graphs, are drawn. One can check that both overlapping conditions (i) and (ii) of Theorem 3.26 are satisfied. Moreover, one can obtain that all four graphs G_1, G_2, H_1 and H_2 on Figure 3.2 are Wilf equivalent
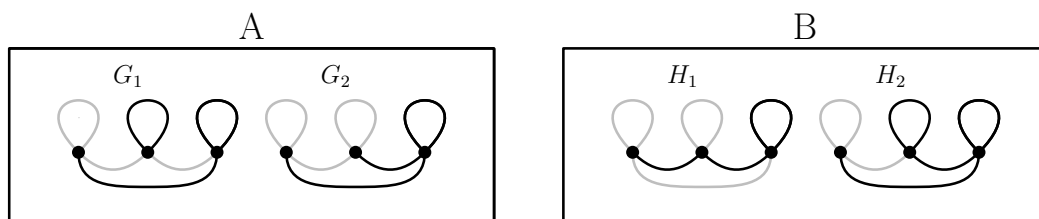$$G_1 \sim_W G_2 \sim_W H_1 \sim_W H_2.$$

$\triangle$

Figure 3.2: Two sets $A = \{G_1, G_2\}$, $B = \{H_1, H_2\}$, each consists of two ordered colored graphs with loops. Vertices of graphs are ordered from left to right, edges are colored by two colors: black and grey.

3.4 Conclusions

In this chapter we presented two main methods for generating Wilf equivalent pairs of sets. First one, the automorphism method, was based on the fact that any automorphism of a poset preserves the structure of the poset. Therefore, it maps one forbidden set to another, generating a Wilf equivalent pair. Then we investigated the Cohen–Remmel method where, under particular assumptions, we replaced forbidden structures of first type with forbidden structures of second type.

However, the example of Rogers–Ramanujan identities shows that there are other methods to find Wilf equivalent pairs. Even more complicated question is to find all Wilf equivalent pairs. That is possible in the posets with elementary structure, as mentioned Stanley [66]. He studied the poset $P = (\mathbb{N}_0^k, \preceq, |\cdot|)$ with base set $\mathbb{N}_0^k$ of all k-tuples of non-negative integers, ordering

$$(a_1, a_2, \ldots, a_k) \preceq (b_1, b_2, \ldots, b_k) \iff \forall\, i \in [k]\ a_i \leq b_i,$$

and the size function $|u| = |(a_1, a_2, \ldots, a_k)| = a_1 + a_2 + \cdots + a_k$. We call this poset Stanley's poset. In 1976 Stanley [66] proved that for every ideal X in P the counting function $n \mapsto |X_n|$ is eventually a polynomial. As an instance of a more general result on well quasiorderings, or easily directly by induction on k, it follows that in every Stanley's poset any antichain is finite. Thus in these posets we have only at most countably many antichains and ideals.

There are $n!$ trivial automorphisms of Stanley's poset. Indeed, any monomorphism that permutes the components of $(a_1, a_2, \ldots, a_k) \in P$ preserves the size and ordering of P. However, we do not know if there are other trivial automorphisms.

Hypothesis 3.27. *Let $k \in \mathbb{N}$. There are exactly $n!$ trivial automorphisms of Stanley's poset $P = (\mathbb{N}_0^k, \preceq, |\cdot|)$ defined above.*

The fact that there are countably many antichains and ideals in Stanley's poset suggests following claim.

Hypothesis 3.28. *Let $k \in \mathbb{N}$ and $P = (\mathbb{N}_0^k, \preceq, |\cdot|)$ be the Stanley's poset defined above. Then for all (finite) antichains $A, B \subset \mathbb{N}_0^k$,*

$$A \sim_W B \iff \exists\ automorphism\ \Phi\ of\ P\ such\ that\ B = \Phi(A).$$

Based on the results for permutations and ordered graphs, where ideals of permutations and colorings may have been of quite different kind, latter hypothesis

seems very improbable. However, the countability of number of ideals indicates that the case of Stanley's poset might be different.

Similarly, the finiteness argument on antichains may work for the sparse subword poset defined in Subsection 3.2.3.

Hypothesis 3.29 (On sparse subwords)**.** *Let $k \geq 2$ be a positive integer and $P = ([k]^*, \preceq_s, |\cdot|)$ be the sparse k-subword poset. Then for all (finite) antichains $A, B \subset [k]^*$,*

$$A \sim_W B \iff \exists\, \Phi: \ B = \Phi(A),$$

for some of the $2 \cdot k!$ automorphisms Φ of P.

However, one cannot generate all Wilf equivalent pairs via automorphism for more complex posets. Straightforward example is the poset of integer partitions where only two automorphisms exists, but the number of Wilf equivalent pairs is unbounded.

Moreover, one cannot generate all Wilf equivalent pairs even with the Cohen–Remmel method as demonstrates the example of Rogers–Ramaujan identities (Rogers [57], Hardy [36, Chapters 6.8–6.17], Andrews [5]). There the number of partitions of n with parts differing by at least 2 are equal to the number of partitions of n with possibly repeating parts $\equiv 1, 4$ modulo 5. That makes it an outstanding counterexample of the converse of Theorem 3.23. But a much more elementary counterexample with antichains A and B may be given.

Example (Counterexample to the converse of Theorem 3.23)**.** Let

$$A = \{(2), (3), (4), \dots\} \quad \text{and} \quad B = \{(j, i): \ j, i \in \mathbb{N}, \ j \geq i\}.$$

be the sets of partitions. Then easily $|F(A)_n| = |F(B)_n| = 1$, hence $A \sim_W B$. But Remmel's condition is not satisfied. For example, A has just one element of size 4 but B has two. $\triangle$

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List of Figures

List of publications

Conference proceedings

[P1] P. Angelini, M. A. Bekos, T. Bruckdorfer, J. Hančl Jr., M. Kaufmann, S. Kobourov, A. Symvonis, P. Valtr. *Low ply drawings of trees*. Graph drawing and network visualization, 236–248, Lecture Notes in Comput. Sci., 9801, Springer, Cham, 2016.

[P2] P. Angelini, S. Chaplick, F. De Luca, J. Fiala, J. Hančl Jr., N. Heinsohn, M. Kaufmann, S. Kobourov, J. Kratochvíl. Symvonis, P. Valtr. *On vertex- and empty-ply proximity drawings*. Graph drawing and network visualization, 24—37, Lecture Notes in Comput. Sci., 10692, Springer, Cham, 2018.

Submitted papers

[S1] J. Hančl Jr. *Complementary Schur Asymptotics for Partitions*. Submitted to Discrete Mathematics, second revision.

[S2] J. Hančl Jr., M. Klazar. *On Growth Rates of Ordered Hypergraphs*. Submitted to European Journal of Combinatorics.