FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Jan Krejčí

Optimality of function spaces for a weighted integral operator

Department of Mathematical Analysis

Supervisor of the master thesis: prof. RNDr. Luboš Pick, CSc., DSc.
Study programme: Mathematics
Study branch: Mathematical Analysis

Prague 2020

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In Prague on July 20, 2020 Jan Krejčí

First and foremost, I would like to thank my supervisor, Prof. Luboš Pick. I discovered some really nice parts of analysis thanks to his suggestions and it has been an honor collaborating on them with him. I would like to express gratitude to my friend Bára, who willingly read the entirety of this thesis, and whose comments helped to make it even better.

It would be shameful not to mention my mother, who, at certain times, seemed more concerned with my thesis than myself and who offered me invaluable support during the entirety of my studies. I wish to extend my thanks to my friends, who accompanied me during these years. Even to those who claimed that I won't be able to finish my thesis on time. Last but not least I would like to mention all the chocolate without which it would be virtually impossible to hand in the thesis on time.

Title: Optimality of function spaces for a weighted integral operator

Author: Jan Krejčí

Department: Department of Mathematical Analysis

Supervisor: prof. RNDr. Luboš Pick, CSc., DSc., Department of Mathematical Analysis

Abstract: This thesis studies questions related to the boundedness of an integral operator $\int_t^1 wf^*$, where w is a given non-increasing function and f^* is a non-increasing rearrangement of a function f. The main goal is to characterize the optimal range for the operator and a given domain and conversely optimal domain for a given range. These results are then illustrated on particular examples. Lastly, some necessary conditions for the existence of optimal space are given.

Keywords: rearrangement-invariant function space, weighted integral operator, Hardy operator, associate space, optimal range, optimal domain

Contents

1. Introduction

In various branches of research, most notably in study of PDEs and theory of function spaces, we come up with the question of embeddings between different function spaces. As a pivotal examples we can mention Sobolev embeddings of both Euclidean and Gaussian types, higher-order embeddings (as in [1]), embeddings on bad domains (as in [2]) etc.

Our interest in the topic is driven by the development of theory behind the first two embeddings. Extensive research of Gaussian-Sobolev embeddings came after they proved to be a successful attempt at generalizing (Euclidean-)Sobolev embeddings to infinite dimension bypassing several problems including the fact that Lebesgue measure is meaningless in this case on the way.

Since then, it was shown that using reduction principles (cf. [3]), the question of many embeddings can be reformulated as whether certain 1-dimensional operator, usually of Hardy type (as defined in [4]), is bounded. This operator often attains form

$$f \mapsto Tf, \text{ where } Tf(t) = T_w f(t) = \int_t^1 f^* w,$$

where w is a given decreasing non-negative function and f^* is the decreasing rearrangement of a function f (precise definitions will be given below). Mentioned examples correspond to functions $w(t)$ equal to $t^{\frac{1}{n}-1}$ in case of Euclidean-Sobolev embeddings,

$$\frac{1}{t\sqrt{1+\log\frac{1}{t}}}$$

for Gaussian-Sobolev embeddings, $\dfrac{I(t)}{t}$ for bad domains, where $I(t)$ is its isoperimetric profile (cf. [5]) and lastly $t^{\frac{m}{n}-1}$ for the higher-order embeddings. With such a broad topic and so many possible applications (cf. [6, 7, 8, 9, 10]), development of general theory is in order.

Phenomena we are mainly interested in are the following, in certain cases (e.g. Euclidean-Sobolev embeddings) we observe a certain "gain" in integrability for any given domain, where "gain" means that the range of the operator T_w and this given domain is a proper subset of this domain. Similarly, a "loss" for a given domain and operator occurs when the range is a proper superset of the domain.

What is more interesting, in case of Gaussian-Sobolev embeddings and

$$w(t) = \frac{1}{t\sqrt{1+\log\left(\frac{1}{t}\right)}}$$

both the "gain" and "loss" occur – the former happens near L^1 and the latter near L^∞ (cf. [3]). If both occur simultaneously, the question arises, where on the scale of L^1 to L^∞ is the edge – the optimal space? Which space satisfies the condition that it is optimal range for itself as a domain and simultaneously the optimal domain for itself as a range? This is the fundamental question we are trying to answer and in this thesis we lay groundwork to do it.

First, we need to characterize when there is "gain" and "loss" simultaneously. It will be shown that the Lebesgue space L^∞ is an optimal range for some domain and operator T_w if and only if $w \in L^1$. As a result, there is a "loss" near L^∞ for T_w if and only if $w \notin L^1$. Conversely, it will be shown that the optimal domain for the Lebesgue space L^1 is the Lorentz space $\Lambda^1\left(\left(\frac{1}{w}\right)'\right)$ and that implies that there is a „gain" near L^1 if and only if $tw(t) \in L^\infty$.

Under these assumptions, a characterization of the optimal range for a given domain and the operator T_w is given as well as a characterization of the optimal domain for a given range. Moreover, we give some particular examples for standard spaces as domains and in the last chapter we present results related to the question of the optimal space.

The text is organized as follows. In Chapter 2, we collect background material and fix notation. In Chapter 3, we characterize optimal range space for a given operator and domain space and, conversely, the optimal domain space for a given operator and range space. In Chapter 4, we illustrate the general results on several non-trivial examples involving mostly Lorentz-type spaces and their modifications. Finally, we give some results connected to the question of the optimal space in Chapter 5.

2. Preliminaries

In the next three sections, we would like to introduce readers who were not exposed to the sufficiently advanced theory of (primarily) function spaces to the tools and knowledge needed to understand the workframe and results of this thesis.

2.1 General settings of the thesis

There are certain (un)written rules, which will be in effect throughout this thesis. First of all, there's a fair bit of integrating throughout the thesis and if it is not specified otherwise it is done with respect to the Lebesgue measure on the interval $(0,1)$. If there is no threat of misinterpretation the integrating variable will be omitted. The Lebesgue measure is denoted μ.

Unless stated otherwise, $|U|$ denotes the Lebesgue measure of a given measurable set U. If we speak of a property being satisfied a.e. it means μ-almost everywhere in the appropriate set.

Due to this being a meager diploma thesis, we will be only working with measurable functions, which are a.e. finite. The set of all such functions will be denoted $\mathcal{M}(S)$, where S is the domain of the functions. We will also use the set of all measurable positive a.e. finite functions, which will be denoted $\mathcal{M}^+(S)$. In cases when we need to establish existence of a constant, but we are not particularly interested in it's value, we shall use a generic constant C.

Definition 2.1. (Equivalence of functions) We say that functions f and g are *equivalent* on the interval (a,b) if there exist constants C_1, $C_2 > 0$ independent of the appropriate quantities such that

$$C_1 f(t) \leq g(t) \leq C_2 f(t),\ t \in (a,b).$$

We denote the fact by writing $f \approx g$ on (a,b). Unless the interval over which functions are equivalent is specified, assume $(a,b) = (0,1)$.

2.2 Rearrangements and maximal operator

The first thing we need to do is somehow simplify the objects we are working with. To this end, we will introduce their decreasing rearrangements, which will allow us to work with a big family of functions while retaining some useful properties which we will use in our endeavor.

Definition 2.2. (Decreasing rearrangement) For any function $f \in \mathcal{M}(0,1)$ define its *decreasing rearrangement*

$$f^* : (0,1) \to [0,\infty)$$

by

$$f^*(t) = \sup\{s \geq 0 : |\{x \in (0,1) : |f(x)| > s\}| > t\}.$$

Moreover, we say that functions f and g are *equimeasurable* if their respective decreasing rearrangements coincide. It is denoted by $f \sim g$.

A very important result regarding the relationship between decreasing rearrangements and the original functions is presented in the following theorem.

Theorem 2.3. *(Hardy-Littlewood inequality) [11, Chapter 2, Theorem 2.2]*

1) If functions f and g belong to $\mathcal{M}(R, \lambda)$, then

$$\int_R |fg| d\lambda \leq \int_0^1 f^*(s) g^*(s) ds.$$

2) For $(R, \lambda) = ((0,1), \mu)$ and a measurable non-negative function g, we have that

$$\int_0^t g d\mu \leq \int_0^t g^* d\mu, \ t \in (0,1).$$

Proposition 2.4. *[11, Chapter 2, Proposition 1.7] Functions f and f^* are equimeasurable.*

Definition 2.5. (Operator f^{**}) For any function $f \in \mathcal{M}(0,1)$ define the *maximal function* or *maximal operator* of f by the formula

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds, \ t \in (0,1).$$

Proposition 2.6. *(Properties of f^{**}) [11, Chapter 2, Proposition 3.2, Property (3.10)] Suppose that f, g and f_n, $n \in \mathbb{N}$, are measurable functions and let $a \in \mathbb{R}$. Then f^{**} is a non-negative, non-increasing and continuous function on $(0,1)$. Moreover,*

1) $f^{**} = 0$ if and only if $f = 0$ a.e.,

2) $f^* \leq f^{**}$,

3) $|g| \leq |f|$ a.e. implies $g^{**} \leq f^{**}$,

4) $(af)^{**} = |a| f^{**}$,

5) $|f_n| \nearrow |f|$ a.e. implies $f_n^{**} \nearrow f^{**}$,

6) $(f+g)^{**} \leq f^{**} + g^{**}$.

While the decreasing rearrangement does not posses some of the properties we would like (e.g. it is not subadditive), the maximal operator f^{**} fares far better in being subadditive and having some nice convergence implications.

2.3 Rearrangement invariant spaces

Armed with the knowledge of a decreasing rearrangement, we want to define function spaces we will be working with, formulate some operator related definitions and then finish off with some fundamental results we will use later on.

Definition 2.7. (Rearrangement invariant space) A Banach space $X(0,1)$ of measurable functions equipped with the norm $\|\cdot\|_{X(0,1)}$ is called a *rearrangement invariant* (*r.i.*) space if

(P1) $0 \leq f \leq g$ a.e. implies $\|f\|_{X(0,1)} \leq \|g\|_{X(0,1)}$,

(P2) $0 \leq f_k \nearrow f$ implies $\|f_k\|_{X(0,1)} \nearrow \|f\|_{X(0,1)}$,

(P3) $\|1\|_{X(0,1)} < \infty$,

(P4) there exists a constant C s.t. $\int_0^1 |f| d\lambda \leq C \|f\|_{X(0,1)}$ for any $f \in X(0,1)$,

(P5) $\|f\|_{X(0,1)} = \|g\|_{X(0,1)}$ whenever $f^* = g^*$.

A norm satisfying the above properties is called an *r.i. norm.*

Let us now define another important space tied to an r.i. space which is an analogy of a dual space.

Definition 2.8. (Associate space) For a given r.i. space $X(0,1)$, define its *associate space* $(X'(0,1), \|\cdot\|_{X'(0,1)})$ as follows,

$$X'(0,1) = \{f \in \mathcal{M}(0,1) : \int_0^1 |fg| < \infty,\ \forall g \in X(0,1)\}$$

and

$$\|f\|_{X'(0,1)} = \sup_{\|g\|_{X(0,1)} \leq 1} \int_0^1 |fg|.$$

With an associate space in hand we can formulate a generalization of Hölder inequality, which will later turn out very useful.

Proposition 2.9. *(Hölder inequality for r.i. spaces) [11, Chapter 1, Theorem 2.4]*
Let $X(0,1)$ be an r.i. space, then the space $X'(0,1)$ with the norm defined above is an r.i. space and furthermore the inequality

$$\int_0^1 |fg| \leq \|f\|_{X(0,1)} \|g\|_{X'(0,1)}$$

holds for any $f \in X(0,1)$ and $g \in X'(0,1)$.

We shall mention one last object, which to some degree characterizes an r.i. space and that is its fundamental function.

Definition 2.10. (Fundamental function) Let $X(0,1)$ be an r.i. space. We define the *fundamental function* of X in the following manner,

$$\varphi_X(t) = \left\|\chi_{(0,t)}\right\|_{X(0,1)},\ t \in [0,1).$$

It can be shown (and seen in [12, Theorem 7.9.6]) that the fundamental function of any r.i. space is quasiconcave[1] and the following equality holds for any $t \in [0,1)$,

$$\varphi_X(t)\varphi_{X'}(t) = t.$$

With all of this in mind, it is now a good time to include some operator related definitions and examples of r.i. spaces.

[1] A function f on $[0,1)$ is quasiconcave if it is non-decreasing, vanishes at 0 and $\frac{f}{t}$ is non-increasing.

Definition 2.11. Assume that there exists a constant $C > 0$ such that for any $f \in X$ the operator T satisfies

$$\|Tf\|_{Y(0,1)} \leq C \|f\|_{X(0,1)}$$

for some r.i. spaces X and Y. We say that the operator $T : X \to Y$ is *bounded.* The fact will be denoted simply by

$$T : X \to Y.$$

An r.i. space Y is *continuously embedded* into an r.i. space X if the identity operator $I : X \to Y$ is bounded. We denote the fact that X is continuously embedded into Y by

$$X \hookrightarrow Y.$$

Moreover, for $T : X \to Y$ we say that Y is the *optimal range* (of the operator T and the domain X), if Y is continuously embedded into any r.i. space Z for which

$$T : X \to Z.$$

is bounded.

Similarly, X is the *optimal domain* (of the operator T and the range Y), if any r.i. space Z such that the operator

$$T : Z \to Y$$

is bounded is continuously embedded into X.

If the pair (X, X) is an optimal domain and range for an operator T, then the r.i. space X is called a *self-optimal space* (with respect to the operator T).

As stated in [3, page 3592],

$$X \hookrightarrow Y \Longleftrightarrow Y' \hookrightarrow X'$$

and

$$X \hookrightarrow Y \Longleftrightarrow X \subset Y.$$

The family of r.i. spaces is quite large. To give some examples, it contains Lebesgue spaces, Orlicz spaces and under some conditions also Lorentz or Lorentz-Zygmund spaces and, more importantly, classical Lorentz spaces $\Gamma^p(w)(0,1)$ and $\Lambda^p(w)(0,1)$.

Proposition 2.12. *[11, Chapter 2, Proposition 1.8] For any $p \in [1, \infty]$ the Lebesgue space $L^p(0,1)$ is an r.i. space.*

Definition 2.13. (Two-parameter Lorentz spaces) Let p, $q \in [1, \infty]$. Define the two parameter *Lorentz spaces* $L^{p,q} = L^{p,q}(0,1)$ and $L^{(p,q)} = L^{(p,q)}(0,1)$ as the sets of functions for which the corresponding norm (defined below) is finite,

$$\|f\|_{p,q} = \|f\|_{L^{p,q}} = \begin{cases} \left\| t^{\frac{1}{p} - \frac{1}{q}} f^*(t) \right\|_{L^q} & 0 < q < \infty, \\ \operatorname{ess\,sup}_{0<s<1} t^{\frac{1}{p}} f^*(t) & q = \infty, \end{cases}$$

$$\|f\|_{(p,q)} = \|f\|_{L^{(p,q)}} = \begin{cases} \left\| t^{\frac{1}{p} - \frac{1}{q}} f^{**}(t) \right\|_{L^q} & 0 < q < \infty, \\ \operatorname{ess\,sup}_{0<s<1} t^{\frac{1}{p}} f^{**}(t) & q = \infty. \end{cases}$$

In the definitions, we use the convention that $\frac{1}{\infty} = 0$. If we add logarithmic weights into the norm, we get Lorentz-Zygmund spaces as defined in either [12] or [3].

Definition 2.14. (Lorentz-Zygmund spaces) Let p, $q \in (0, \infty]$ and let $\alpha \in \mathbb{R}$. Define the *Lorentz-Zygmund spaces* $L^{p,q,\alpha} = L^{p,q,\alpha}(0,1)$ and $L^{(p,q,\alpha)} = L^{(p,q,\alpha)}(0,1)$ as the set of functions for which the corresponding norm (defined below) is finite,

$$\|f\|_{p,q,\alpha} = \|f\|_{L^{p,q,\alpha}} = \begin{cases} \left\| t^{\frac{1}{p}-\frac{1}{q}} \log^{\alpha}\left(\frac{e}{t}\right) f^*(t) \right\|_{L^q} & 0 < q < \infty, \\ \operatorname{ess\,sup}_{0<s<1} t^{\frac{1}{p}} \log^{\alpha}\left(\frac{e}{t}\right) f^*(t) & q = \infty, \end{cases}$$

$$\|f\|_{(p,q,\alpha)} = \|f\|_{L^{(p,q,\alpha)}} = \begin{cases} \left\| t^{\frac{1}{p}-\frac{1}{q}} \log^{\alpha}\left(\frac{e}{t}\right) f^{**}(t) \right\|_{L^q} & 0 < q < \infty, \\ \operatorname{ess\,sup}_{0<s<1} t^{\frac{1}{p}} \log^{\alpha}\left(\frac{e}{t}\right) f^{**}(t) & q = \infty. \end{cases}$$

Notice that $L^{p,q} = L^{p,q,0}$ and $L^{(p,q)} = L^{(p,q,0)}$. Moreover, $L^{p,q,\alpha} = L^{(p,q,\alpha)}$ if and only if $p > 1$.

The next theorem, cited from [3, page 3595], describes the conditions under which $L^{p,q,\alpha}$ is a rearrangement invariant space and the one after it defines the associate space of $L^{p,q,\alpha}$.

Theorem 2.15. *The Lorentz-Zygmund space $L^{p,q,\alpha}$ is an r.i. space, up to equivalent norms, if and only if one of the following conditions is satisfied:*

- $p = q = 1,\ \alpha \geq 0,$
- $1 < p < \infty,\ 1 \leq q \leq \infty,\ \alpha \in \mathbb{R},$
- $p = \infty,\ 1 \leq q < \infty,\ \alpha + \frac{1}{q} < 0,$
- $p = q = \infty,\ \alpha \leq 0.$

Theorem 2.16. *Let $L^{p,q,\alpha}$ be a given r.i. space. Then, up to equivalent norms,*

$$(L^{p,q,\alpha})' = \begin{cases} L^{\infty,\infty,-\alpha} & \text{if } p = q = 1,\ \alpha \geq 0, \\ L^{p',q',-\alpha} & \text{if } 1 < p < \infty,\ 1 \leq q \leq \infty, \alpha \in \mathbb{R}, \\ L^{(1,q',-\alpha-1)} & \text{if } p = \infty,\ 1 \leq q < \infty,\ \alpha + \frac{1}{q} < 0, \\ L^{1,1,-\alpha} & \text{if } p = q = \infty,\ \alpha \leq 0, \end{cases}$$

where

$$p' = \begin{cases} 1 & \text{for } p = \infty, \\ \infty & \text{for } p = 1, \\ \frac{p}{p-1} & \text{for } 1 < p < \infty. \end{cases}$$

With Lebesgue, Lorentz and Lorentz-Zygmund spaces covered, let us turn our attention to the last Lorentz spaces left – classical Lorentz spaces, which are a natural generalization of all of the mentioned spaces.

Definition 2.17. (Classical Lorentz spaces) Let $w \in \mathcal{M}_+(0,1)$. The *classical Lorentz spaces* $\Lambda^p(w) = \Lambda^p(w)(0,1)$ and $\Gamma^p(w) = \Gamma^p(w)(0,1)$ are defined as the set of functions for which the corresponding norm (defined below) is finite,

$$\|f\|_{\Lambda^p(w)(0,1)} = \begin{cases} (\int_0^1 (f^*(s))^p w(s) ds)^{\frac{1}{p}} & 0 < p < \infty, \\ \operatorname{ess\,sup}_{0<s<1} f^*(s) w(s) & p = \infty, \end{cases}$$

$$\|f\|_{\Gamma^p(w)(0,1)} = \begin{cases} (\int_0^1 (f^{**}(s))^p w(s) ds)^{\frac{1}{p}} & 0 < p < \infty, \\ \operatorname{ess\,sup}_{0<s<1} f^{**}(s) w(s) & p = \infty. \end{cases}$$

The spaces $\Lambda^p(w)$ above are not necessarily r.i. spaces. In fact as stated in [3], $\Lambda^p(w)$ is an r.i. space if and only if either $p = 1$ and

$$\frac{1}{s}\int_0^s w \le \frac{C}{t}\int_0^t w, \ 0 < t \le s \le 1, \tag{2.1}$$

or $1 < p < \infty$ and

$$s^p \int_s^1 r^{-p} w(r) dr \le C \int_0^s w, \ s \in (0,1),$$

or $p = \infty$ and

$$\frac{1}{s}\int_0^s \frac{1}{\overline{w}} \le \frac{C}{\overline{w}(s)}, \ s \in (0,1),$$

for some constant $C > 0$, where

$$\overline{w}(s) = \operatorname*{ess\,sup}_{0<r<s} w(r).$$

On the other hand, the spaces $\Gamma^p(w)$ satisfy most of the properties of r.i. spaces. The maximal operator is subadditive and positively homogeneous, as stated in Proposition 2.6. Moreover w is positive a.e., so

$$\|f\|_{\Gamma^p(w)} = 0$$

if and only if

$$f^{**} = 0$$

and this in turn is true if and only if $f = 0$ a.e. These properties then translate to the functional $\|f\|_{\Gamma^p(w)}$ and so it is always a norm. Similarly, properties (P1), (P2) and (P5) follow from the corresponding properties of the maximal operator.

Lastly, we want to show that $\|\cdot\|_{\Gamma^p(w)}$ satisfies the property (P4). Observe that

$$\frac{w(t)^{\frac{1}{p}}}{t}\int_0^t f^* \ge \chi_{[\frac{1}{2},1]} \frac{w(t)^{\frac{1}{p}}}{t}\int_0^t f^* \ge \chi_{[\frac{1}{2},1]} \frac{w(t)^{\frac{1}{p}}}{t}\int_0^{\frac{1}{2}} f^*.$$

Using (P1), we can formally write

$$\|f\|_{\Gamma^p(w)} = \left\| \frac{w(t)^{\frac{1}{p}}}{t}\int_0^t f^* \right\|_{L^p} \ge \left\| \chi_{[\frac{1}{2},1]} \frac{w(t)^{\frac{1}{p}}}{t}\int_0^t f^* \right\|_{L^p} \ge \frac{1}{2}\left\| \chi_{[\frac{1}{2},1]} \frac{w(t)^{\frac{1}{p}}}{t} \right\|_{L^p} \|f\|_{L^1},$$

where the last inequality owes its existence to the monotonicity of the function f^*. Now, the quantity

$$\left\| \frac{w(t)}{t} \right\|_{L^p}$$

can be infinity, but then no characteristic function $\chi_{[0,T]}$ belongs to the space, because its norm is infinity thanks to the string of inequalities above. This in turn means that the space does not contain any non-increasing function other than 0 and in particular it does not contain 1.

For $p = \infty$, the expression

$$\frac{w(x)}{x}$$

must be finite for some point $x \in (0, 1)$, otherwise

$$\|f\|_{\Gamma^\infty(w)} = \operatorname*{ess\,sup}_{t\in(0,1)} \frac{w(t)}{t} \int_0^t f^*$$

is finite if and only if $f = 0$. Consider such point x, then

$$\operatorname*{ess\,sup}_{t\in(0,1)} \frac{w(t)}{t} \int_0^t f^* \geq \frac{w(x)}{x} \int_0^x f^* \geq w(x) \|f\|_{L^1},$$

where the last inequality is again due to the monotonicity of f^*. As a result, we only need to verify non-triviality of the space $\Gamma^p(w)$.

For the special cases of $L^{(p,q,\alpha)}$, which are equal to $\Gamma^q(w)$ for a suitable choice of w, it is easy to verify that $1 \in L^{(p,q,\alpha)}$ for all p, $q > 0$ and $\alpha \in \mathbb{R}$ except the case when $p = q = \infty$ (then we need $\alpha \leq 0$).

In addition, according to [3, page 3596], any quasiconcave weakly differentiable function

$$\varphi : [0, 1) \to [0, \infty)$$

and a function φ', which is its derivative, give rise to r.i. spaces $\Lambda^1(\varphi')$ and $\Gamma^\infty(\varphi)$. Moreover these spaces are the smallest and the largest r.i. spaces with the fundamental function equivalent to the function φ, i.e.

$$\Lambda^1(\varphi') \hookrightarrow X(0, 1) \hookrightarrow \Gamma^\infty(\varphi)$$

for any r.i. space $X(0, 1)$ such that $\varphi_X \approx \varphi$. Another important property is that if

$$\overline{\varphi}(t) = \frac{t}{\varphi(t)}, \; t \in (0, 1)$$

satisfies

$$\lim_{t\to 0^+} \overline{\varphi}(t) = 0,$$

then

$$(\Lambda^1(\varphi'))' = \Gamma^\infty(\overline{\varphi}) \text{ and } (\Gamma^\infty(\varphi))' = \Lambda^1(\overline{\varphi}'). \tag{2.2}$$

Even though the next two theorems look somehow disconnected from the rest of this chapter, they will be substantial to prove some results in the chapters to come. The following theorem is a special case of [5, Corollary 9.8].

Theorem 2.18. *Let X and Y be r.i. spaces on $(0,1)$ and let $I : [0,\infty) \to [0,\infty)$ be a non-decreasing function. Then*

$$\left\| \int_t^1 \frac{f(s)}{I(s)} ds \right\|_{Y(0,1)} \leq C_1 \|f\|_{X(0,1)} \text{ for every } f \in \mathcal{M}^+(0,\infty)$$

holds true if and only if

$$\left\| \int_t^1 \frac{g(s)}{I(s)} ds \right\|_{Y(0,1)} \leq C_2 \|g\|_{X(0,1)} \text{ for every non-increasing } g \in \mathcal{M}^+(0,\infty).$$

Next result will be used in Chapter 3. It is rather complicated so to make reader's life easier it is included in the thesis instead of merely cited. It is a modification of Theorem 3.5 from [13].

Theorem 2.19. *Let $0 < p, q < \infty$. Let b be a weight function*[2] *such that*

$$0 < B(t) < \infty, \ t \in (0,1),$$

where

$$B(t) = \int_0^t b.$$

Let u be a continuous weight and let v, w be weights such that

$$0 < \int_0^x v < \infty \text{ and } 0 < \int_0^x w < \infty, \ x \in (0,1)$$

and assume that

$$\sup_{0<t<1} \frac{u(t)}{B(t)} \int_0^t \frac{b(s)}{\overline{u}(s)} < \infty,$$

where

$$\overline{u}(t) = B(t) \sup_{t \leq \tau \leq 1} \frac{u(\tau)}{B(\tau)}.$$

Finally, let $1 < p \leq q < \infty$. Then there exists a constant $C > 0$ such that the following holds for any non-increasing non-negative measurable function φ:

$$\left(\int_0^1 [(T_{u,b}\varphi)(t)]^q w(t) dt \right)^{\frac{1}{q}} \leq C \left(\int_0^1 [\varphi(t)]^p v(t) dt \right)^{\frac{1}{p}},$$

where

$$(T_{u,b}\varphi) = sup_{t \leq \tau < 1} \frac{u(\tau)}{B(\tau)} \int_0^\tau \varphi(t) b(t) dt,$$

if and only if there exists a constant $C > 0$ such that

$$\left(\int_0^x [\sup_{t \leq \tau \leq x} \overline{u}(\tau)]^q w(t) dt \right)^{\frac{1}{q}} \leq C \left(\int_0^x v(t) dt \right)^{\frac{1}{p}} \text{ for all } x \in (0,1) \tag{2.3}$$

and

$$\sup_{x \in (0,1)} \left(\int_x^1 \left(\frac{\overline{u}(t)}{B(t)} \right)^q w(t) dt \right)^{\frac{1}{q}} \left(\int_0^x \left(\frac{B(t)}{V(t)} \right)^{p'} v(t) dt \right)^{\frac{1}{p'}} < \infty, \tag{2.4}$$

where

$$V(t) = \int_0^t v.$$

[2] In the context of this theorem, weight function is measurable non-negative a.e. finite function. Later on in the thesis we shall use a different definition, which is more strict than this one.

3. Optimal range and optimal domain for the integral operator

After introducing the background we shall get to work. To outline our endeavour, we will define an integral operator T dependent on a weight function w. Then we shall prove that a norm of the optimal range of this operator and a given domain is linked to the function w and the norm of the space associated with X. After that we will prove similar assertion with respect to the given range and optimal domain. Note that in [3] similar results are proven, but only for

$$w(t) = \frac{1}{t\sqrt{1 + \log \frac{1}{t}}}.$$

We shall obtain these results as a particular consequence of the theory we develop.

Definition 3.1. (Weight function) If a non-increasing function w belongs to $\mathcal{M}^+(0,1)$, we shall call it a *weight function*.

Definition 3.2. (Operator T) For a given weight function w, define

$$T : \mathcal{M}(0,1) \to \mathcal{M}(0,1)$$

such that for any $f \in \mathcal{M}(0,1)$

$$Tf(t) = T_w f(t) = \int_t^1 w f^*.$$

With the operator defined we also define a norm on the associate space Y' of a soon-to-be-proven optimal range. The range itself is then defined somewhat implicitly as an associate space to Y'.

Definition 3.3. (Norm on Y') Given a rearrangement invariant space $X(0,1)$ and a weight w, define a functional $\|\cdot\|_{Y'(0,1)}$ as follows,

$$\|g\|_{Y'(0,1)} = \left\| w(t) \int_0^t g^* \right\|_{X'(0,1)}$$

for any $g \in \mathcal{M}$. Moreover, define $Y'(0,1) = \{f \in \mathcal{M}, \|f\|_{Y'(0,1)} < \infty\}$.

Please note that when no confusion can arise, we will be omitting sets over which the function spaces are defined (it is almost always $(0,1)$). After this definition, one might not be surprised by the following theorem.

Theorem 3.4. *Let X be an r.i. space and let w be a weight function. Then the space Y', as defined in Definition 3.3, equipped with the norm $\|\cdot\|_{Y'(0,1)}$ is an r.i. space if and only if*

$$tw(t) \in X'.$$

Proof. First we shall prove that $\|\cdot\|_{Y'(0,1)}$ is a norm and then the remaining five properties from Definition 2.7. Non-negativity and positive homogeneity follow directly from the same properties of $\|\cdot\|_{X'(0,1)}$ and in the latter case of a decreasing rearrangement. From the fact that $\|\cdot\|_{X'(0,1)}$ is a norm,

$$\|g\|_{Y'(0,1)} = 0$$

if and only if

$$w(t)\int_0^t g^* = 0 \text{ a.e.}$$

The integral $\int_0^t g^*$ is non-zero for any $t > 0$ if and only if g is non-zero a.e. and since the function $w \in \mathcal{M}^+(0,1)$ is non-increasing,

$$w(t)\int_0^t g^* = 0$$

if and only if $g = 0$ a.e.

To prove the triangle inequality, take measurable functions f and g, then

$$\begin{aligned}
\|f+g\|_{Y'(0,1)} &= \left\| w(t)\int_0^t (f+g)^* \right\|_{X'(0,1)} \\
&= \|tw(t)(f+g)^{**}(t)\|_{X'(0,1)} \\
&\leq \|tw(t)(f^{**}+g^{**})(t)\|_{X'(0,1)} \\
&\leq \left\| w(t)\int_0^t f^* \right\|_{X'(0,1)} + \left\| w(t)\int_0^t g^* \right\|_{X'(0,1)} \\
&= \|f\|_{Y'(0,1)} + \|g\|_{Y'(0,1)} .
\end{aligned}$$

The first inequality follows from the subadditivity of the maximal operator and the property (P5) of the norm $\|\cdot\|_{X'(0,1)}$, the second one then from the triangle inequality for the norm $\|\cdot\|_{X'(0,1)}$. We have proven that $\|\cdot\|_{Y'(0,1)}$ is indeed a norm.

The property (P5) follows directly from the fact that f and f^* are equimeasurable. To prove the property (P1), assume

$$0 \leq f \leq g \text{ a.e.}$$

Consequently, for any $t > 0$,

$$|\{x \in (0,1), |f(x)| > t\}| \leq |\{x \in (0,1), |g(x)| > t\}|.$$

This translates to

$$f^*(s) \leq g^*(s) \text{ for any } s \in (0,1)$$

and so

$$w(t)\int_0^t f^*(s)ds \leq w(t)\int_0^t g^*(s)ds,\ t > 0.$$

Using the property (P1) of $\|\cdot\|_{X'(0,1)}$, we get

$$\|f\|_{Y'(0,1)} = \left\| w(t)\int_0^t f^*(s)ds \right\|_{X'(0,1)} \leq \left\| w(t)\int_0^t g^*(s)ds \right\|_{X'(0,1)} = \|g\|_{Y'(0,1)} .$$

To show that the norm $\|\cdot\|_{Y'(0,1)}$ possesses the property (P2), take

$$0 \le f_k \nearrow f.$$

Using the property *5)* from Proposition 2.6,

$$\int_0^t f_n^* = t f_n^{**} \nearrow t f^{**} = \int_0^t f^*$$

and so, owing to the fact that w is non-negative,

$$0 \le w(\cdot) \int_0^{\cdot} f_n \nearrow w(\cdot) \int_0^{\cdot} f.$$

Next, using the property (P2) of the norm $\|\cdot\|_{X'(0,1)}$,

$$\|f_n\|_{Y'(0,1)} = \left\| w(t) \int_0^t f_n^* \right\|_{X'(0,1)} \nearrow \left\| w(t) \int_0^t f^* \right\|_{X'(0,1)} = \|f\|_{Y'(0,1)} .$$

With (P1), (P2) and (P5) proven, we want to show that

$$tw(t) \notin X'$$

implies that

$$\chi_{[0,T]} \notin Y' \text{ for any } T \in (0,1),$$

which means that the space is empty and so in particular

$$1 = \chi_{[0,1]} \notin Y'$$

that is Y' is not rearrangement invariant. Using monotonicity of the maximal operator and positive homogeneity, we obtain

$$\begin{aligned} \|f\|_{Y'(0,1)} &= \left\| w(t) \int_0^t f^* \right\|_{X'(0,1)} \\ &\ge \|w(t) t f^{**}(1)\|_{X'(0,1)} \\ &= f^{**}(1) \|tw(t)\|_{X'(0,1)} \\ &= \|tw(t)\|_{X'(0,1)} \|f\|_{L^1} \end{aligned} \tag{3.1}$$

for any $f \in L^1$, because

$$\|f\|_{L^1} = \|f^*\|_{L^1} = f^{**}(1). \tag{3.2}$$

To complete the argument it suffices to notice that

$$\chi_{[0,T]} \in L^1, \ T \in [0,1].$$

As a result, we know that

$$tw(t) \in X'$$

is a necessary condition for Y' to be an r.i. space. For sufficiency, we need to prove the remaining two properties. The property (P3) follows easily from the following

$$\|1\|_{Y'(0,1)} = \left\| w(t) \int_0^t 1 \right\|_{X'(0,1)} = \|tw(t)\|_{X'(0,1)} < \infty.$$

Finally we have to deal with (P4). Consider again (3.1), now assume that $f \in Y'$ instead of $f \in L^1$ and note that

$$\|tw(t)\|_{X'(0,1)} \neq 0,$$

because w is non-zero a.e. It follows that

$$\|f\|_{L_1} \leq \frac{1}{\|tw(t)\|_{X'(0,1)}} \|f\|_{Y'(0,1)}.$$

This last step completes the proof. $\square$

With this theorem in hand, we can define the space Y as follows

$$Y = \{f \in \mathcal{M}(0,1), \int_0^1 |fg| < \infty \ \forall g \in Y'\}, \tag{3.3}$$

where Y' is as in Theorem 3.4, and equip it with the dual norm

$$\|f\|_{Y(0,1)} = \sup_{\|g\|_{Y'(0,1)} \leq 1} \int_0^1 |fg|.$$

As mentioned in Proposition 2.9, $(Y, \|\cdot\|_{Y(0,1)})$ is an r.i. space.

Theorem 3.5. *(Optimal range) Let a function w and a space X be as in Theorem 3.4 and a space Y as in* (3.3)*. Then the operator*

$$T = T_w : X \to Y$$

is bounded and the space Y is the optimal range for the space X and the operator T.

Proof. Using the definition of norm on Y in the second equality, the definition of T and Fubini's theorem in the third and the fourth one respectively, we get

$$\begin{aligned}
\|T\|_{X \to Y} &= \sup_{\|f\|_{X(0,1)} \leq 1} \|Tf\|_Y \\
&= \sup_{\|f\|_{X(0,1)} \leq 1} \sup_{\|g\|_{Y'(0,1)} \leq 1} \int_0^1 |g(Tf)| \\
&= \sup_{\|f\|_{X(0,1)} \leq 1} \sup_{\|g\|_{Y'(0,1)} \leq 1} \int_0^1 |g(t)| \int_t^1 w(s) f^*(s) ds dt \\
&= \sup_{\|f\|_{X(0,1)} \leq 1} \sup_{\|g\|_{Y'(0,1)} \leq 1} \int_0^1 f^*(s) w(s) \int_0^s |g(t)| dt ds.
\end{aligned}$$

Next, we utilize Hardy-Littlewood inequality (Theorem 2.3, part *2)*), fact that $f^*(t)w(t)$ is a non-negative function and then Hölder inequality (Proposition 2.9)

$$\begin{aligned}
\|T\|_{X \to Y} &\leq \sup_{\|f\|_{X(0,1)} \leq 1} \sup_{\|g\|_{Y'(0,1)} \leq 1} \int_0^1 f^*(s) w(s) \int_0^s g^*(t) dt ds \\
&\leq \sup_{\|f\|_{X(0,1)} \leq 1} \sup_{\|g\|_{Y'(0,1)} \leq 1} \|f^*\|_{X(0,1)} \left\| w(s) \int_0^s g^* \right\|_{X'(0,1)} \\
&\leq 1.
\end{aligned}$$

The last inequality holds true, because X is an r.i. space and because of the identity

$$\left\| w(s) \int_0^s g^* \right\|_{X'(0,1)} = \|g\|_{Y'(0,1)} .$$

For the second part, we will show that whenever

$$T : X \to Z$$

for an r.i. space Z then

$$Y \hookrightarrow Z \iff Z' \hookrightarrow Y'.$$

For the second embedding, assume that

$$T : X \to Z$$

and we get, using definitions of norms on Y', X', Fubini's theorem and Hölder inequality (in this order), that

$$\begin{aligned}
\sup_{\|g\|_{Z'}\le 1} \|g\|_{Y'(0,1)} &= \sup_{\|g\|_{Z'}\le 1} \left\| w(t) \int_0^t g^* \right\|_{X'(0,1)} \\
&= \sup_{\|g\|_{Z'}\le 1} \sup_{\|f\|_{X(0,1)}\le 1} \int_0^1 |f(t)| w(t) \int_0^t g^*(s) ds dt \\
&= \sup_{\|g\|_{Z'}\le 1} \sup_{\|f\|_{X(0,1)}\le 1} \int_0^1 g^*(t) \int_t^1 w(s)|f(s)| ds dt \\
&\le \sup_{\|g\|_{Z'}\le 1} \sup_{\|f\|_{X(0,1)}\le 1} \|g^*\|_{Z'} \left\| \int_t^1 w|f| \right\|_Z ,
\end{aligned}$$

where

$$\sup_{\|g\|_{Z'}\le 1} \|g^*\|_{Z'} = \sup_{\|g\|_{Z'}\le 1} \|g\|_{Z'} \le 1,$$

because Z' is an r.i. space. The second expression is a bit more complicated. Using Theorem 2.18, on spaces X and Z and $I = \frac{1}{w}$, there exists a constant $C > 0$ such that

$$\left\| \int_t^1 w|f| \right\|_Z \le C \, \| |f| \|_{X(0,1)} , \ f \in \mathcal{M}(0,1)$$

if and only if this inequality holds for non-increasing non-negative functions. For a non-increasing function $f \in \mathcal{M}^+(0,1)$, we have

$$\left\| \int_t^1 wf \right\|_Z = \left\| \int_t^1 wf^* \right\|_Z = \|Tf\|_Z \le C \, \|f\|_{X(0,1)}$$

thanks to the boundedness of

$$T : X \to Z.$$

Note that

$$\| |f| \|_{X(0,1)} = \|f\|_{X(0,1)} ,$$

because f and $|f|$ are equimeasurable. Put together, one has that

$$\sup_{\|g\|_{Z'}\le 1} \|g\|_{Y'(0,1)} \le C$$

that is

$$Z' \hookrightarrow Y'.$$

□

Now, we proceed to the question of the optimal domain, which turns out to be more challenging than that of the optimal range. We start by finding (borderline) optimal ranges for the smallest and the biggest r.i. spaces, so that we restrict ourselves only to spaces which can be reasonably assumed to be optimal range of the operator T and some domain space.

Proposition 3.6. *(Optimal ranges for L^1 and L^∞ and $w \notin L^1$) Let $w \notin L^1$ be a differentiable weight function, such that $tw(t)$ is a non-decreasing function. Then*

- *for $tw(t) \in L^\infty$, the operator $T_w : L^1 \to Y_1 = \Lambda^1\left(\left(\frac{1}{w}\right)'\right)$ is bounded and the range Y_1 is optimal,*
- *for $tw(t) \in L^1$, the operator $T : L^\infty \to Y_\infty = \Gamma^\infty(\psi)$, where*

$$\psi(t) = \frac{t}{\int_0^t \int_s^1 w(y)dyds},$$

is bounded and the range Y_∞ is optimal.

Spaces Y_1 and Y_∞ are rearrangement invariant.

Note that the assumptions for $tw(t)$ to belong to L^∞ and L^1 are necessary conditions for the norm on Y_1' and Y_∞' to be rearrangement invariant (recall Theorem 3.4).

Proof. Utilizing Theorem 3.5, we know that the norm on the associate space satisfies

$$\begin{aligned}
\|f\|_{Y'(0,1)} &= \left\| w(t) \int_0^t f^* \right\|_{X'(0,1)} \\
&= \left\| w(t) \int_0^t f^* \right\|_\infty \\
&= \sup_{t\in(0,1)} w(t) \int_0^t f^* \\
&= \|f\|_{\Gamma^\infty(tw(t))} .
\end{aligned}$$

Space $\Gamma^\infty(tw(t))$ must be an r.i. space, because

$$tw(t) \in L^\infty$$

implies $\|\cdot\|_{Y_1'}$ is an r.i. space via Theorem 3.4. Moreover, the function $tw(t)$ is non-decreasing, $w(t)$ is non-increasing and due to the function $w \notin L^1$ being non-increasing, we have

$$\lim_{t\to 0^+} w(t) = \infty,$$

which in turn implies

$$\lim_{t\to 0^+} \frac{t}{tw(t)} = 0. \tag{3.4}$$

Consequently (via (2.2))

$$Y_1 = (Y_1')' = (\Gamma^\infty(tw(t)))' = \Lambda^1\left(\left(\frac{1}{w}\right)'\right). \tag{3.5}$$

Note that to verify that $tw(t)$ is quasiconcave, we need to show that the function vanishes at 0. However if it does not hold, then

$$\lim_{t\to 0^+} tw(t) = C > 0$$

due to monotonicity of $tw(t)$ and so

$$\Gamma^\infty(tw(t)) = \Gamma^\infty(1) = L^\infty.$$

The first equality holds, because

$$tw(t) \approx C$$

thanks to the assumption of the theorem. Moreover, space $\Gamma^\infty(1)$ is an r.i. space, because it contains constant functions, and it is continuously embedded into L^∞ due to the property *2)* of Proposition 2.6, whereas L^∞ is the smallest r.i. space. As a result, the conclusion (3.5) holds without $tw(t)$ vanishing at 0.

Regarding properties of $\Lambda^1\left(\left(\frac{1}{w}\right)'\right)$, due to $tw(t)$ being a non-decreasing function, there exists a $C > 0$ such that

$$sw(s) \leq Ctw(t),\ 0 < s \leq t \leq 1.$$

Put together with (3.4), we obtain

$$\begin{aligned}
\frac{1}{tw(t)} &\leq \frac{C}{sw(s)},\\
\frac{1}{t}\left[\frac{1}{w(x)}\right]_0^t &\leq \frac{C}{s}\left[\frac{1}{w(x)}\right]_0^s,\\
\frac{1}{t}\int_0^t \left(\frac{1}{w}\right)' &\leq \frac{C}{s}\int_0^s \left(\frac{1}{w}\right)',
\end{aligned}$$

for any $0 < s \leq t \leq 1$. As mentioned in (2.1), this condition is necessary and sufficient for the space $\Lambda^1\left(\left(\frac{1}{w}\right)'\right)$ to be rearrangement invariant.

Similarly, for the case of L^∞, we have that

$$\begin{aligned}
\|f\|_{Y'(0,1)} &= \left\|w(t)\int_0^t f^*\right\|_{X'(0,1)}\\
&= \left\|w(t)\int_0^t f^*(s)ds\right\|_{L^1}\\
&= \int_0^1 w(t)\int_0^t f^*(s)dsdt\\
&= \|f\|_{\Gamma^1(tw(t))}\,.
\end{aligned}$$

In the string of equalities above, we only used definitions of various norms. Moreover, due to the equality of norms, aforementioned $\Gamma^1(tw(t))$ must be an r.i. space, because for

$$tw(t) \in L^1$$

the space Y'_∞ is rearrangement invariant. We want to use modified characterization of associate spaces from [12, Theorem 10.4.1]. First we need to show that the function $tw(t)$ satisfies non-degenerate conditions, which in our case are equivalent to

$$\int_0^1 tw(t)dt < \infty \quad \text{and} \quad \int_0^1 w(t)dt = \infty.$$

Both of them hold, because $tw(t) \in L^\infty$ and $w(t) \notin L^\infty$. Consequently, Theorem 10.4.1 states that

$$\begin{aligned}
\|f\|_Y &= \|f\|_{(Y')'} \\
&\approx \sup_{t\in(0,1)} \frac{t}{\int_0^t sw(s)ds + t\int_t^1 w(s)ds} f^{**}(t) && (3.6)\\
&= \sup_{t\in(0,1)} \frac{f^{**}(t)}{\frac{1}{t}\int_0^t\int_s^1 w(y)dyds} \\
&= \|f\|_{\Gamma^\infty\left(\frac{t}{\int_0^t\int_s^1 w(y)dyds}\right)}. && (3.7)
\end{aligned}$$

The second equality makes use of Fubini's theorem. What remains to show is non-triviality (cf. the discussion under Definition 2.17). Denote

$$\overline{\varphi}(t) = \frac{1}{\frac{1}{t}\int_0^t\int_s^1 w(y)dyds}$$

and notice that it has an integral mean of non-increasing function

$$\int_s^1 w(y)dy$$

in the denominator. This mean attains infimum at $t = 1$ and so the supremum of $\overline{\varphi}$ is also attained at 1. Using monotonicity of w and non-negativity of w and $tw(t)$, it can be estimated in the following manner

$$\overline{\varphi}(1) = \left(\int_0^1 sw(s)ds\right)^{-1} \le \left(\int_{\frac{1}{4}}^{\frac{3}{4}} sw(s)ds\right)^{-1} \le \left(\frac{1}{8}w\left(\frac{3}{4}\right)\right)^{-1} < \infty.$$

Consequently,

$$\|1\|_{\Gamma^\infty(\overline{\varphi})} = \sup_{t\in(0,1)} \overline{\varphi}(t) = \overline{\varphi}(1) < \infty$$

and we have verified the only condition needed to show that $\Gamma^\infty(\overline{\varphi})$ is rearrangement invariant (cf. the discussion under Definition 2.17). $\square$

As we have just shown, it is natural for the (optimal) range Y to satisfy

$$\Gamma^\infty\left(\frac{t}{\int_0^t\int_s^1 w(y)dyds}\right) \hookrightarrow Y \hookrightarrow \Lambda^1\left(\left(\frac{1}{w}\right)'\right). \tag{3.8}$$

Armed with this knowledge we can answer the question of the optimal domain.

Theorem 3.7. *(Optimal domain) Let $w \notin L^1$ be a differentiable weight function, such that $tw(t)$ is a non-decreasing function and $tw(t) \in L^\infty$. Moreover, assume that $Y(0,1)$ is an r.i. space satisfying* (3.8)*. Then*

$$\|f\|_{X(0,1)} = \sup_{0 \le g \sim f} \left\| \int_t^1 gw \right\|_{Y(0,1)}, \ f \in \mathcal{M}(0,1),$$

defines an r.i. norm on the space

$$X = \{f \in \mathcal{M}(0,1), \ \|f\|_{X(0,1)} < \infty\}.$$

Moreover, $(X, \|\cdot\|_{X(0,1)})$ is the optimal domain with respect to the operator T and range Y.

Proof. There are three parts to prove – first that $\|\cdot\|_{X(0,1)}$ is a norm and that it satisfies properties (P1) to (P5) from Definition 2.7 and then that the domain is optimal.

For the first part, it is obvious that the functional $\|\cdot\|_{X(0,1)}$ is non-negative. Next, we want to show that

$$\|f\|_{X(0,1)} = 0 \Longleftrightarrow f = 0.$$

Assume that

$$\|f\|_{X(0,1)} = 0.$$

The fact that $\|\cdot\|_{Y(0,1)}$ is a norm implies

$$\int_t^1 gw = 0 \quad \forall g \text{ s.t. } 0 \le g \sim f.$$

This, paired with fact that $w \in \mathcal{M}^+(0,1)$, in turn means that

$$f \sim g = 0$$

and

$$f^* = 0 \Longleftrightarrow f = 0. \tag{3.9}$$

Conversely it is easy to see via (3.9) that if $f = 0$, then

$$g \sim f = 0$$

and so

$$\|0\|_{X(0,1)} = 0.$$

Positive homogeneity follows from the following string of equalities, for $a \neq 0$

$$\begin{aligned}
(ag)^*(s) &= \sup\{t \ge 0, |\{x \in (0,1), |ag(x)| > t\}| > s\} \\
&= \sup\left\{t \ge 0, \left|\left\{x \in (0,1), |g(x)| > \frac{t}{|a|}\right\}\right| > s\right\} \\
&= \sup\{|a|t \ge 0, |\{x \in (0,1), |g(x)| > t\}| > s\} \\
&= |a| \sup\{t \ge 0, |\{x \in (0,1), |g(x)| > t\}| > s\} \\
&= |a| g^*(s).
\end{aligned}$$

Equality also trivially holds for $a = 0$. As a result,

$$f \sim g \Longleftrightarrow af \sim ag$$

and

$$\begin{aligned}
\|af\|_{X(0,1)} &= \sup_{0 \le g \sim af} \left\| \int_t^1 gw \right\|_{X(0,1)} \\
&= \sup_{0 \le \frac{g}{a} \sim f} \left\| \int_t^1 a \left(\frac{g}{a} \right) w \right\|_{X(0,1)} \\
&= |a| \sup_{0 \le g \sim f} \left\| \int_t^1 gw \right\| \\
&= |a| \, \|f\|_{X(0,1)} \text{ for any } a \in \mathbb{R} \text{ and } f \in X.
\end{aligned}$$

Let us now prove the property (P1) and use it when proving triangle inequality. We want to show that for f, $g \in \mathcal{M}^+(0,1)$

$$f \le g \text{ a.e. } \Longrightarrow \|f\|_{X(0,1)} \le \|g\|_{X(0,1)} .$$

Using [11, Chapter 2, Corollary 7.5], for any non-negative function $h \sim f$ there is a measure preserving map

$$\sigma_h : (0,1) \to (0,1)$$

satisfying

$$h = h^* \circ \sigma_h = f^* \circ \sigma_h.$$

Assume that $f^* \le g^*$, then

$$h \le g^* \circ \sigma_h \sim g,$$

where the equimeasurability holds due to [11, Chapter 2, Proposition 7.2]. As a result, for $f \le g$ a.e., we have

$$\begin{aligned}
\|f\|_{X(0,1)} &= \sup_{0 \le h \sim f} \left\| \int_t^1 hw \right\|_{Y(0,1)} \\
&\le \sup_{0 \le h \sim f} \left\| \int_t^1 (g^* \circ \sigma_h) w \right\|_{Y(0,1)} \\
&\le \sup_{0 \le u \sim g} \left\| \int_t^1 uw \right\|_{Y(0,1)} \\
&= \|g\|_{X(0,1)} .
\end{aligned}$$

Returning to the triangle inequality, it is quite straightforward that for any pair of simple functions f, g and a (simple) function h such that

$$h \sim f + g,$$

there exists a pair of simple functions h_f and h_g such that

$$h_f \sim f, \quad h_g \sim g, \quad h = h_f + h_g. \tag{3.10}$$

With this property in hand, assume f, $g \in \mathcal{M}(0,1)$. Then there exist two sequences of non-negative simple functions $\{f_n\}$, $\{g_n\} \subset \mathcal{M}^+(0,1)$ satisfying

$$f_n \nearrow |f|, \; n \to \infty \tag{3.11}$$

and

$$g_n \nearrow |g|,\ n \to \infty.$$

Consequently,

$$\lim_{n\to\infty}(f_n+g_n)^* = (|f|+|g|)^*. \tag{3.12}$$

Assume $h \in \mathcal{M}^+(0,1)$ such that

$$h \sim |f|+|g|,$$

then again there exists a measure preserving map σ such that

$$h = h^* \circ \sigma = (|f|+|g|)^* \circ \sigma.$$

Now, define

$$h_n = (f_n+g_n)^* \circ \sigma,\ n \in \mathbb{N}.$$

The aim is to show that for any $h \sim |f|+|g|$

$$\left\| \int_t^1 hw \right\|_{Y(0,1)} \le \liminf_{n\to\infty} \left\| \int_t^1 h_n w \right\|_{Y(0,1)} \le \|f\|_{X(0,1)} + \|g\|_{X(0,1)}. \tag{3.13}$$

Certainly,

$$h_n \sim f_n + g_n,\ n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} h_n = h.$$

Moreover,

$$h_n^{**}(t) = (f_n+g_n)^{**}(t) \le f_n^{**}(t) + g_n^{**}(t) \le f^{**}(t) + g^{**}(t), \tag{3.14}$$

for any $n \in \mathbb{N}$ and $t \in (0,1)$, where the last two inequalities follow from properties *6)* and *5)* of Proposition 2.6. It shows that the functions h_n are equiintegrable. This paired with the fact that w is bounded on any interval

$$(t,1),\ t \in (0,1)$$

implies that the expression $h_n w$ is bounded as well on any such interval $(t,1)$. Moreover,

$$\lim_{n\to\infty} \int_t^1 h_n w = \int_t^1 hw,\ t \in (0,1).$$

Using Fatou's lemma ([11, Chapter 1, Theorem 1.7]) for the r.i. norm $\|\cdot\|_{Y(0,1)}$, one gets

$$\left\| \int_t^1 hw \right\|_{Y(0,1)} \le \liminf_{n\to\infty} \left\| \int_t^1 h_n w \right\|_{Y(0,1)}. \tag{3.15}$$

First inequality of (3.13) is proven. For the second inequality, due to (3.10), for any f_n and g_n, there exist simple functions h_{f_n} and h_{g_n} such that

$$h_{f_n} \sim f_n, \quad h_{g_n} \sim g_n \quad \text{and} \quad h_n = h_{f_n} + h_{g_n}.$$

As shown earlier in the proof, for every $F_n \in \{f_n,\ g_n\}$, there exists a measure preserving map σ_{F_n} such that

$$h_{F_n} = (h_{F_n})^* \circ \sigma_{F_n} = f_n^* \circ \sigma_{F_n} \leq f^* \circ \sigma_{F_n} \sim f^*,$$

where the inequality is due to (3.11) and properties of decreasing rearrangement. Finally,

$$\begin{aligned}
\left\| \int_t^1 h_n w \right\|_{Y(0,1)} &\leq \liminf_{n\to\infty} \left\| \int_t^1 h_n w \right\|_{Y(0,1)} \\
&\leq \left\| \int_t^1 h_{f_n} w \right\|_{Y(0,1)} + \left\| \int_t^1 h_{g_n} w \right\|_{Y(0,1)} \\
&\leq \left\| \int_t^1 (f^* \circ \sigma_{f_n}) w \right\|_{Y(0,1)} + \left\| \int_t^1 (g^* \circ \sigma_{g_n}) w \right\|_{Y(0,1)} \\
&\leq \sup_{0\leq u \sim f} \left\| \int_t^1 uw \right\|_{Y(0,1)} + \sup_{0\leq u\sim g} \left\| \int_t^1 uw \right\|_{Y(0,1)} \\
&= \|f\|_{X(0,1)} + \|g\|_{X(0,1)}\,.
\end{aligned}$$

We have proven (3.13). To obtain triangle inequality, it suffices to notice that

$$f + g \leq |f| + |g|$$

and so, using the already-proven property (P1), we get

$$\|f+g\|_{X(0,1)} \leq \|f\|_{X(0,1)} + \|g\|_{X(0,1)}\,.$$

The property (P5) obviously holds so it remains to show that (P3) and (P4) hold as well. Here is where the embedding chain (3.8) comes into play. Next string of inequalities follows from choosing

$$h = |f| \sim f,$$

the definition of the r.i. norm on X and finally using the first embedding in (3.8)

$$\|f\|_{X(0,1)} \geq \left\| \int_t^1 |f|w \right\|_{Y(0,1)} \geq C \left\| \int_t^1 |f|w \right\|_{\Lambda^1\left(\left(\frac{1}{w}\right)'\right)}.$$

Using Fubini's theorem, we obtain that the last term is equal to $\|f\|_1$. For the last property of an r.i. norm, note that the only non-negative functions equimeasurable with 1 are those which are 1 a.e., so

$$\begin{aligned}
\|1\|_{X(0,1)} &= \left\| \int_x^1 w \right\|_{Y(0,1)} \\
&\leq C \left\| \int_x^1 w \right\|_{\Gamma^\infty\left(\frac{t}{\int_0^t \int_s^1 w(y)dyds}\right)} \\
&= \sup_{t\in(0,1)} \frac{t}{\int_0^t \int_s^1 w(y)dyds} \int_t^1 w(t)dt \\
&\leq 1,
\end{aligned}$$

because

$$\frac{t}{\int_0^t \int_s^1 w(y)dyds} \le \frac{1}{\int_t^1 w},$$

which we get by using Fubini's theorem, (3.7) and then simply estimating from above.

Let us now turn our attention to the optimality of the space X. Assume that operator T is bounded from an r.i. space Z to Y. Then there exists a constant $C > 0$ such that for any non-increasing non-negative function $f \in Z$, one has

$$\left\| \int_t^1 wf \right\|_{Y(0,1)} = \left\| \int_t^1 wf^* \right\|_{Y(0,1)} = \|Tf\|_{Y(0,1)} \le C \|f\|_{Z(0,1)} .$$

Consequently, Theorem 2.18 implies that

$$\left\| \int_t^1 wf \right\|_{Y(0,1)} \le C \|f\|_{Z(0,1)} \tag{3.16}$$

for any non-negative $f \in Z$.

Assume $f \in Z$ is given and take

$$0 \le g \sim f,$$

then firstly $g \in Z$ due to the property (P5) of r.i. spaces and secondly, utilizing (3.16),

$$\|f\|_{X(0,1)} = \sup_{0 \le g \sim f} \left\| \int_t^1 wg \right\|_{Y(0,1)} \le C \sup_{0 \le g \sim f} \|g\|_{Z(0,1)} = C \|f\|_{Z(0,1)} , \tag{3.17}$$

where the last equality holds again because of the property (P5). Finally, (3.17) is equivalent to

$$Z \hookrightarrow X,$$

which is what we wanted to prove. □

So far, we have imposed rather natural conditions onto function w. Firstly, we assumed $tw(t) \in L^\infty$, which is actually a necessary condition for Y_1' to be an r.i. space, where Y_1 is as in Proposition 3.6, and differentiability and monotonicity of w together with $w \notin L^1$, to make sure that the optimal ranges of L^1 and L^∞ are (defined and) rearrangement invariant.

Under these assumptions we obtained a reasonable characterization of the optimal range and a bit inconvenient and impractical characterization of a norm on the optimal domain. We shall try to improve the latter one under some stronger assumptions so that it is easier to use. We will utilize the following operator

$$Uf(s) = \frac{1}{sw(s)} \sup_{s \le r \le 1} rw(r) f^*(r), \ f \in \mathcal{M}(0,1), \ s \in (0,1).$$

Lemma 3.8. *Let $Y(0,1)$ and $Z(0,1)$ be r.i. spaces. Assume that w is a continuous weight function, which satisfies*

$$\frac{1}{w(t)} \approx \int_0^t \frac{1}{sw(s)}, \ t \in (0,1), \tag{3.18}$$

and, finally, let

$$U : Y'(0,1) \to Z'(0,1).$$

Then, there exists a constant $C > 0$ such that

$$\left\| \int_t^1 fw \right\|_{Y(0,1)} \le C \left\| \int_t^1 f^* w \right\|_{Z(0,1)}, \quad f \in \mathcal{M}_+(0,1).$$

Before proving the lemma, let us discuss few functions for which the property (3.18) holds or does not hold. It is clearly not satisfied for $w(t) = 1$, because

$$1 \not\approx t \text{ on } (0,1).$$

The condition also does not hold for $\log^\alpha \frac{1}{t}$ for any $\alpha \in \mathbb{R}$. The situation is more interesting for $w(t) = t^\alpha$, because

$$t^{-\alpha} \approx \int_0^t s^{-\alpha-1} \text{ on } (0,1)$$

if and only if $\alpha < 0$.

Proof. The proof consists of a string of inequalities. First observe the following

$$\begin{aligned} w(t) \int_0^t Ug(s)ds &\approx \frac{\int_0^t Ug(s)ds}{\int_0^t (sw(s))^{-1} ds} \\ &\approx \frac{\int_0^t (\sup_{s \le u \le 1} uw(u))(sw(s))^{-1} ds}{\int_0^t (sw(s))^{-1} ds}. \end{aligned}$$

And the latter function is an integral mean of a non-increasing function

$$\sup_{s \le u \le 1} uw(u)$$

with respect to the measure

$$(sw(s))^{-1} ds \text{ on } (0,t)$$

and so it is itself non-increasing in t. Now, assume $f \in \mathcal{M}_+(0,1)$, then

$$\begin{aligned} \left\| \int_s^1 fw \right\|_{Y(0,1)} &= \sup_{\|g\|_{Y'(0,1)} \le 1} \int_0^1 |g(s)| \int_s^1 f(r)w(r)drds \\ &= \sup_{\|g\|_{Y'(0,1)} \le 1} \int_0^1 f(r)w(r) \int_0^r |g(s)|dsdr \\ &\le \sup_{\|g\|_{Y'(0,1)} \le 1} \int_0^1 f(r)w(r) \int_0^r g^*(s)dsdr \\ &\le \sup_{\|g\|_{Y'(0,1)} \le 1} \int_0^1 f(r)w(r) \int_0^r Ug(s)dsdr. \end{aligned}$$

Equalities owe their existence either to the definitions of norm or Fubini's theorem. The first inequality is a consequence of Hardy-Littlewood inequality (Theorem 2.3, part 2) and non-negativity of the function $f(t)w(t)$. The next one follows easily from the fact that

$$g^*(s) = \frac{sw(s)}{sw(s)} g^*(s) \le \frac{1}{sw(s)} \sup_{s \le r \le 1} rw(r)g^*(r) = Ug(s).$$

Utilizing the observation we made in the beginning of the proof, we see that

$$w(r)\int_0^r Ug(s)ds$$

is equivalent to a non-increasing function and so the Hardy-Littlewood inequality yields

$$\left\|\int_s^1 fw\right\|_{Y(0,1)} \leq C \sup_{\|g\|_{Y'(0,1)}\leq 1} \int_0^1 f^*(t)w(t)\int_0^t Ug(s)dsdt.$$

Using Fubini's theorem again, passing from suprema over functions g to suprema over Ug and thanks to boundedness of $U : Y' \to Z'$, we get

$$\begin{aligned}
\left\|\int_s^1 fw\right\|_{Y(0,1)} &\leq C \sup_{\|g\|_{Y'(0,1)}\leq 1} \int_0^1 f^*(t)w(t)\int_0^t Ug(s)dsdt \\
&\leq C \sup_{\|Ug\|_{Z'(0,1)}\leq 1} \int_0^1 f^*(t)w(t)\int_0^t Ug(s)dsdt \\
&\leq C \sup_{\|h\|_{Z'(0,1)}\leq 1} \int_0^1 f^*(t)w(t)\int_0^t h(s)dsdt \\
&\leq C \sup_{\|h\|_{Z'(0,1)}\leq 1} \int_0^1 |h(t)|\int_t^1 f^*(s)w(s)dsdt \\
&= C\left\|\int_t^1 f^*w\right\|_{Z(0,1)}.
\end{aligned}$$

□

Note that Property (3.18) implicitly requires the function $\frac{1}{tw(t)}$ to be integrable on any interval

$$[0,t),\ t \in (0,1).$$

With this lemma in hand we shall prove that the norm on the associate space can be expressed in a nicer form – one might call this a star-form.

Proposition 3.9. *Let $w \notin L^1$ be a differentiable weight function such that it satisfies Property* (3.18) *from Lemma 3.8, $tw(t)$ is a non-decreasing function and $tw(t) \in L^\infty$. Moreover, assume that $Y(0,1)$ is an r.i. space such that*

$$\Gamma^\infty\left(\frac{t}{\int_0^t\int_s^1 w(y)dyds}\right) \hookrightarrow Y(0,1)$$

and the operator U is bounded on $Y'(0,1)$. Then (3.8) *holds and the optimal domain $X(0,1)$ of the operator T and range Y fulfils*

$$\|f\|_{X(0,1)} \approx \left\|\int_s^1 f^*w\right\|_{Y(0,1)},\ f \in \mathcal{M}(0,1).$$

Proof. Firstly, we show that boundedness of U on Y' implies the embeddings (3.8). Due to Y' being an r.i. space, $1 \in Y'$ and so

$$U(1)(s) = \frac{K}{sw(s)}$$

as an image of 1 as well, because the operator U is bounded on Y'. So, if any function satisfies that

$$\|f\|_{\Gamma^\infty(tw(t))} = \sup_{t\in(0,1)} tw(t)f^{**}(t) < \infty,$$

then $f \in Y'$ due to *2)* of Proposition 2.6 and properties (P1) and (P5) of $\|\cdot\|_{Y'(0,1)}$. Consequently,

$$\Gamma^\infty(tw(t)) \hookrightarrow Y'$$

and

$$Y \hookrightarrow (\Gamma^\infty(tw(t)))' = \Lambda^1\left(\left(\frac{1}{w}\right)'\right).$$

As shown in the proof of Proposition 3.6, the assumptions on w are sufficient for the last equality to hold.

For the equivalence part, using Theorem 3.7, one easily gets

$$\left\|\int_s^1 f^*w\right\|_{Y(0,1)} \le \sup_{0\le g\sim f}\left\|\int_s^1 gw\right\|_{Y(0,1)} = \|f\|_{X(0,1)}$$

for any $f \in \mathcal{M}(0,1)$. Conversely, utilizing Lemma 3.8, where roles of Y and Z in the lemma are fulfilled by Y, we obtain a constant $C > 0$ such that

$$\|f\|_{X(0,1)} \le C\left\|\int_s^1 f^*w\right\|_{Y(0,1)}.$$

□

4. Examples

In this section, we will discuss optimal domains and ranges for typical spaces mentioned in Chapter 2. We start off by considering a simple weight

$$w(t) = w_{\alpha,\beta}(t) = t^{\alpha} \log^{\beta}\left(\frac{e}{t}\right)$$

for a chosen set of parameters α and β, then we shall pass under some restricting conditions to a general weight function w.

Example 4.1. Let

$$w(t) = w_{\alpha,\beta}(t) = t^{\alpha} \log^{\beta}\left(\frac{e}{t}\right)$$

such that the parameters α, $\beta \in \mathbb{R}$ satisfy

$$-1 \geq \alpha \quad \text{and} \quad \beta \geq \alpha + 1$$

and let $T : X \to Y$, where Y is the optimal range for X.

(a) If $X = L^{1,1,\xi}$, $\xi \geq 0$, then Y' is an r.i. space if and only if

$$\alpha = -1 \quad \text{and} \quad \beta \leq \xi.$$

Moreover, if the condition holds, then

$$Y' = L^{(\infty,\infty,\beta-\xi)},$$

(b) if $X = L^{\infty,\infty,\xi}$, $\xi \leq 0$, then Y' is an r.i. space if and only if

$$\alpha \in (-2, -1].$$

Moreover, if the condition holds, then

$$Y' = L^{\left(\frac{1}{\alpha+2},1,\beta-\xi\right)},$$

(c) if $X = L^{p,1,\xi}$, $p \in (1,\infty)$, $\xi \in \mathbb{R}$, then Y' is an r.i. space if and only if either of the following conditions holds

(I) $-2 + \frac{1}{p} < \alpha$.[1] If the condition holds, then

$$Y' = L^{\left(\frac{p'}{1+p'(1+\alpha)},\infty,\beta-\xi\right)},$$

(II) $-2 + \frac{1}{p} = \alpha$ and $\beta \leq \xi$. If the condition holds, then

$$Y' = L^{(\infty,\infty,\beta-\xi)},$$

(d) if $X = L^{p,q,\xi}$, $p \in (1,\infty)$ and $q > 1$, then Y' is an r.i. space if and only if either of the following conditions holds

[1] where p' is the standard notation for the number which satisfies $\frac{1}{p} + \frac{1}{p'} = 1$. Same notation is used for q'.

(I) $-2+\frac{1}{p}<\alpha$. If the condition holds, then

$$Y' = L^{\left(\frac{p'}{p'(\alpha+1)+1}, q', \beta-\xi\right)},$$

(II) $-2+\frac{1}{p}=\alpha$ and $\beta-\xi<-1+\frac{1}{q}$. If the condition holds, then

$$Y' = L^{(\infty, q', \beta-\xi)}.$$

Proof. It is obvious that

$$w(t)\int_0^t f^* = tw(t)f^{**}(t),$$

where f^{**} is non-increasing. Moreover,

$$\begin{aligned}(tw(t))' &= \left(t^{\alpha+1}\log^{\beta}\left(\frac{e}{t}\right)\right)' \\ &= (\alpha+1)t^{\alpha}\log^{\beta}\left(\frac{e}{t}\right) - \beta t^{\alpha}\log^{\beta-1}\left(\frac{e}{t}\right) \\ &= t^{\alpha}\log^{\beta-1}\left(\frac{e}{t}\right)\left((\alpha+1)\log\left(\frac{e}{t}\right)-\beta\right). \end{aligned} \tag{4.1}$$

Note that for any value of parameters α and β

$$t^{\alpha}\log^{\beta-1}\left(\frac{e}{t}\right) \geq 0, \ t \in (0,1).$$

On the other hand, $\log\left(\frac{e}{t}\right)$ is monotone and so is

$$(\alpha+1)\log\left(\frac{e}{t}\right)-\beta.$$

Finally, observe that for the assumed range of parameters

$$\sup_{t\in(0,1)} (\alpha+1)\log\left(\frac{e}{t}\right)-\beta = \alpha+1-\beta \leq 0.$$

Consequently,

$$(tw(t))' \leq 0$$

and $w(t)\int_0^t f^*$ is a non-increasing non-negative function. As a result,

$$\left(w(t)\int_0^t f^*\right)^* = w(t)\int_0^t f^*.$$

By very similar reasoning, we can check whether or not $w_{\alpha,\beta}$ is a weight function. Taking the derivative and the follow-up discussion give us the condition that $w_{\alpha,\beta}$ is a weight if and only if

$$\alpha \leq 0 \text{ and } \beta \geq \alpha.$$

These conditions are trivially satisfied under our assumptions.

Ad (a), as proven in Theorem 3.4, the space Y' is an r.i. space if and only if $tw(t)$ belongs to $X' = L^{\infty,\infty,-\xi}$. So, it must be true that

$$\left\|t^{\alpha+1}\log^{\beta}\left(\frac{e}{t}\right)\right\|_{\infty,\infty,-\xi} = \sup_{t\in(0,1)} t^{\alpha+1}\log^{\beta-\xi}\left(\frac{e}{t}\right) < \infty.$$

Taking into consideration the assumptions on α and β, the norm is finite if and only if

$$\alpha = -1 \quad \text{and} \quad \beta \leq \xi.$$

Using definitions of various norms, we obtain

$$\begin{aligned}
\|f\|_{Y'(0,1)} &= \left\| w(t) \int_0^t f^* \right\|_{X'(0,1)} \\
&= \left\| w(t) \int_0^t f^* \right\|_{\infty,\infty,-\xi} \\
&= \sup_{t\in(0,1)} \log^{\beta-\xi}\left(\frac{e}{t}\right) f^{**}(t) \\
&= \|f\|_{(\infty,\infty,\beta-\xi)}.
\end{aligned}$$

As noted in the discussion under Definition 2.17, spaces $L^{(a,b,\eta)}$ are special cases of spaces $\Gamma^p(w)$ and as such are rearrangement invariant for any a, $b > 0$, $\eta \in \mathbb{R}$, except the case when $a = b = \infty$ in which we need to assume $\eta \leq 0$.

Ad (b), similarly as in part (a), we need

$$tw(t) \in X' = L^{1,1,-\xi},$$

which is true when

$$\left\| t^{\alpha+1} \log^{\beta}\left(\frac{e}{t}\right) \right\|_{1,1,-\xi} = \int_0^1 t^{\alpha+1} \log^{\beta-\xi}\left(\frac{e}{t}\right) dt < \infty$$

and this holds true if either

$$\alpha = -2 \quad \text{and} \quad \beta - \xi < -1$$

or

$$\alpha > -2.$$

The first case cannot occur, because

$$\beta \geq \alpha + 1 = -1 \quad \text{and} \quad -\xi \geq 0.$$

In the second case,

$$\begin{aligned}
\|f\|_{Y'(0,1)} &= \left\| w(t) \int_0^t f^* \right\|_{X'(0,1)} \\
&= \left\| w(t) \int_0^t f^* \right\|_{1,1,-\xi} \\
&= \int_0^1 t^{\alpha+1} \log^{\beta-\xi}\left(\frac{e}{t}\right) f^{**}(t) dt \\
&= \|f\|_{\left(\frac{1}{\alpha+2},1,\beta-\xi\right)}.
\end{aligned}$$

Ad (c), firstly,

$$t^{\alpha+1} \log^{\beta}\left(\frac{e}{t}\right) \in X'$$

if and only if

$$\left\| t^{\alpha+1} \log^{\beta}\left(\frac{e}{t}\right) \right\|_{p',\infty,-\xi} = \sup_{t\in(0,1)} t^{\frac{1}{p'}+\alpha+1} \log^{\beta-\xi}\left(\frac{e}{t}\right) < \infty$$

and the conditions on α and β coincide with the conditions I) and II) from part c) of this theorem (it is necessary and sufficient for either of them to hold). Using the definition of the norm of Y',

$$\begin{aligned}
\|f\|_{Y'(0,1)} &= \left\| w(t) \int_0^t f^* \right\|_{p',\infty,-\xi} \\
&= \sup_{t\in(0,1)} t^{\frac{1}{p'}+\alpha+1} \log^{\beta-\xi}\left(\frac{e}{t}\right) f^{**}(t) \\
&= \left\| t^{\frac{1+p'(\alpha+1)}{p'}} \log^{\beta-\xi}\left(\frac{e}{t}\right) f^{**} \right\|_\infty \\
&= \|f\|_{\left(\frac{p'}{1+p'(\alpha+1)},\infty,\beta-\xi\right)}.
\end{aligned}$$

Note that for $\alpha = -1 - \frac{1}{p'}$,

$$\frac{p'}{1+p'(\alpha+1)} = \infty.$$

Ad (d), assume $p \in (1,\infty)$, $q > 1$, then

$$t^{\alpha+1} \log^{\beta}\left(\frac{e}{t}\right) \in X'$$

if and only if either

$$\left(\alpha > -2 + \frac{1}{p}\right) \quad \text{or} \quad \left(\alpha = -2 + \frac{1}{p} \quad \wedge \quad \beta - \xi < -1 + \frac{1}{q}\right).$$

Moreover,

$$\begin{aligned}
\|f\|^{q'}_{Y'(0,1)} &= \left\| w(t) \int_0^t f^* \right\|^{q'}_{p',q',-\xi} \\
&= \left\| t^{\frac{1}{p'}-\frac{1}{q'}+\alpha+1} \log^{\beta-\xi}\left(\frac{e}{t}\right) f^{**} \right\|^{q'}_{q'} \\
&= \left\| t^{\frac{p'(\alpha+1)+1}{p'}-\frac{1}{q'}} \log^{\beta-\xi}\left(\frac{e}{t}\right) f^{**} \right\|^{q'}_{q'} \\
&= \|f\|^{q'}_{\left(\frac{p'}{p'(\alpha+1)+1},q',\beta-\xi\right)}.
\end{aligned}$$

In the second case, $\frac{p'}{p'(\alpha+1)+1} = \infty$ □

Corollary 4.2. *Let*

$$w(t) = w_{\alpha,\beta}(t) = t^{\alpha} \log^{\beta}\left(\frac{e}{t}\right)$$

such that the parameters α, $\beta \in \mathbb{R}$ satisfy

$$-1 \geq \alpha \quad \textit{and} \quad \beta \geq \alpha + 1$$

and let $T : X \to Y$, where Y is the optimal range for X.

(a) *If* $X = L^{1,1,\xi}$, $\xi \geq 0$, *then for*

$$\alpha = -1 \quad \text{and} \quad \beta \leq \xi \leq \beta + 1$$

the space Y *is rearrangement invariant and*

$$Y = \Lambda^1\left(\log^{\xi-\beta-1}\left(\frac{e}{t}\right)\left(\log\left(\frac{e}{t}\right) + \beta - \xi\right)\right).$$

(b) *if* $X = L^{\infty,\infty,\xi}$, $\xi \leq 0$, *then* Y *is an r.i. space if and only if either*

$$(\alpha \in (-2,-1)) \quad \text{or} \quad (\alpha = -1 \ \wedge \ \beta - \xi \geq -1).$$

If one of the conditions is satisfied, then

$$Y = \Gamma^\infty\left(\frac{t}{\int_0^t \int_s^1 x^\alpha \log^{\beta-\xi}\left(\frac{e}{x}\right) dxds}\right),$$

(c) *if* $X = L^{p,1,\xi}$, $p \in (1,\infty)$, $\xi \in \mathbb{R}$, *then* Y' *is an r.i. space if and only if either of the following conditions holds*

(I) $-2+\frac{1}{p} < \alpha$ *and* $\frac{1}{p'}+\alpha \leq \beta-\xi \leq \frac{1}{p'}+\alpha+1$.[2] *Moreover, if the condition holds, then*

$$Y = \Lambda^1\left(t^{-\frac{1}{p'}-\alpha-1}\log^{\xi-\beta}\left(\frac{e}{t}\right)\left(\left(-\frac{1}{p'}-\alpha\right) + \frac{\beta-\xi}{\log\frac{e}{t}}\right)\right),$$

(II) $-2+\frac{1}{p} = \alpha$ *and* $\beta \leq \xi \leq \beta+1$. *Moreover, if the condition holds, then*

$$Y = \Lambda^1\left(\log^{\xi-\beta-1}\left(\frac{e}{t}\right)\left(\log\left(\frac{e}{t}\right) + \beta - \xi\right)\right),$$

(d) *if* $X = L^{p,q,\xi}$, $p \in (1,\infty)$, $q > 1$ *and* $\alpha > -2 + \frac{1}{p}$ *and*

(I) $q = \infty$ *and the following condition holds*

$$\left(-1 < \frac{1}{p'} + \alpha < 0\right) \quad \text{or} \quad \left(\frac{1}{p'} + \alpha = -1 \quad \wedge \quad \beta - \xi < -1\right),$$

then

$$Y = \Gamma^\infty\left(\frac{t}{\int_0^t \int_s^1 y^{\frac{1}{p'}+\alpha-1}\log^{\beta-\xi}\left(\frac{e}{y}\right) dyds}\right),$$

(II) $q \in (1,\infty)$, *then*

$$Y = \Gamma^q\left(\frac{t^{q+q'-1}\int_0^t \varphi_{d2}(s)ds \int_t^1 s^{-q'}\varphi_{d2}(s)ds}{(\int_0^t \varphi_{d2}(s)ds + t^{q'}\int_t^1 s^{-q'}\varphi_{d2}(s)ds)^{q+1}}\right),$$

where

$$\varphi_{d2}(t) = t^{\frac{q'}{p'}-1+q'(\alpha+1)}\log^{q'(\beta-\xi)}\left(\frac{e}{t}\right).$$

[2] where p' is the standard notation for the number which satisfies $\frac{1}{p}+\frac{1}{p'} = 1$. Same notation is used for q'.

All of the spaces above are rearrangement invariant except the last one, for which it will not be shown whether or not it is non-trivial.

Proof. Ad a), we have shown that

$$Y' = L^{(\infty,\infty,\beta-\xi)} = \Gamma^{\infty}\left(\log^{\beta-\xi}\left(\frac{e}{t}\right)\right).$$

Denote

$$\varphi_a(t) = \log^{\beta-\xi}\left(\frac{e}{t}\right).$$

Obviously for $\beta \le \xi$, the function φ_a is non-decreasing and vanishes at 0. As discussed in the beginning of Example 4.1, the function

$$t^a \log^b\left(\frac{e}{t}\right)$$

is non-increasing if and only if

$$a \le 0 \quad \text{and} \quad b \ge a, \tag{4.2}$$

which in our case means that the function

$$\frac{1}{t}\log^{\beta-\xi}\left(\frac{e}{t}\right)$$

is non-increasing if and only if

$$\beta - \xi \ge -1,$$

which is one of the assumptions of the theorem. Finally,

$$\lim_{t\to 0^+} \overline{\varphi_a}(t) = \lim_{t\to 0^+} t\log^{\xi-\beta}\left(\frac{e}{t}\right) = 0.$$

Using (2.2), one gets

$$Y = (Y')' = (\Gamma^{\infty}(\varphi_a))' = \Lambda^1(\overline{\varphi}') = \Lambda^1\left(\log^{\xi-\beta}\left(\frac{e}{t}\right)\left(1+\frac{\beta-\xi}{\log\frac{e}{t}}\right)\right).$$

As mentioned in (2.1), Y is an r.i. space if and only if

$$\frac{1}{t}\int_0^t \log^{\xi-\beta}\left(\frac{e}{x}\right)\left(1+\frac{\beta-\xi}{\log\frac{e}{x}}\right)dx \le \frac{C}{s}\int_0^s \log^{\xi-\beta}\left(\frac{e}{x}\right)\left(1+\frac{\beta-\xi}{\log\frac{e}{x}}\right)dx$$

for any $0 < s \le t \le 1$. It trivially holds true with $C = 1$ if

$$\log^{\xi-\beta}\left(\frac{e}{x}\right)\left(1+\frac{\beta-\xi}{\log\frac{e}{x}}\right)$$

is a non-increasing function, because then both the left-hand and right-hand sides of the inequality are integral means of the same non-increasing function and $s \le t$. From the assumption of the theorem, we know that

$$0 \le \xi - \beta$$

and this implies $\log^{\xi-\beta}\left(\frac{e}{x}\right)$ is non-increasing and so is

$$1+\frac{\beta-\xi}{\log\frac{e}{t}},$$

because the nominator of the fraction is at most 0 and the denominator is positive and non-increasing. Adding a constant does not change monotonicity. As a result the function is indeed non-increasing and so the assertion holds. It is not mentioned explicitly, but the result, rather trivially, holds even in case when $\xi=\beta$.

Ad (b), notice that the associate space found in Example 4.1 can be rewritten as follows

$$L^{\left(\frac{1}{\alpha+2},1,\beta-\xi\right)}=\Gamma^1\left(t^{\alpha+1}\log^{\beta-\xi}\left(\frac{e}{t}\right)\right).$$

We want to use [12, Theorem 10.4.1]. Denote

$$v(t)=t^{\alpha+1}\log^{\beta-\xi}\left(\frac{e}{t}\right).$$

We need to verify that the function satisfies two conditions,

$$\int_0^1\frac{v(s)}{s+1}ds<\infty\quad\text{and}\quad\int_0^1\frac{v(s)}{s}ds=\infty.$$

The first condition is due to boundedness of $s+1$ on $(0,1)$ equivalent to

$$\int_0^1 t^{\alpha+1}\log^{\beta-\xi}\left(\frac{e}{t}\right)dt<\infty$$

and it is true, in the scope of admissible parameters, if and only if $\alpha>-2$. The second one reads

$$\int_0^1 t^{\alpha}\log^{\beta-\xi}\left(\frac{e}{t}\right)=\infty.$$

It is satisfied if and only if

$$(\alpha\in(-2,-1))\quad\text{or}\quad(\alpha=-1\ \wedge\ \beta-\xi\geq-1).$$

Theorem 10.4.1 then gives us that

$$\begin{aligned}\|f\|_{\left(\Gamma^1\left(t^{\alpha+1}\log^{\beta-\xi}\left(\frac{e}{t}\right)\right)\right)'}&\approx\sup_{t\in(0,1)}\frac{t}{\int_0^t s^{\alpha+1}\log^{\beta-\xi}\left(\frac{e}{s}\right)ds+t\int_0^1 s^{\alpha}\log^{\beta-\xi}\left(\frac{e}{s}\right)ds}f^{**}(t)\\&=\sup_{t\in(0,1)}\frac{t}{\int_0^t\int_s^1 x^{\alpha}\log^{\beta-\xi}\left(\frac{e}{x}\right)dxds}f^{**}(t)\\&=\|f\|_{\Gamma^\infty\left(\frac{t}{\int_0^t\int_s^1 x^{\alpha}\log^{\beta-\xi}\left(\frac{e}{x}\right)dxds}\right)}.\end{aligned}$$

Up to equivalent norm, we have

$$Y=(Y')'=\left(\Gamma^1\left(t^{\alpha+1}\log^{\beta-\xi}\left(\frac{e}{t}\right)\right)\right)'=\Gamma^\infty\left(\frac{t}{\int_0^t\int_s^1 x^{\alpha}\log^{\beta-\xi}\left(\frac{e}{x}\right)dxds}\right).$$

Based on the discussion in Preliminaries, under Definition 2.17, we need to show that the space satisfies the property (P3) from the definition of an r.i. space. Denote

$$F(t) = \int_s^1 x^\alpha \log^{\beta-\xi}\left(\frac{e}{x}\right) dx$$

and

$$MF(t) = \frac{1}{t}\int_0^t F(s)ds.$$

Then certainly

$$\|1\|_{\Gamma^\infty\left(\frac{t}{\int_0^t\int_s^1 x^\alpha \log^{\beta-\xi}\left(\frac{e}{x}\right)dxds}\right)} = \sup_{t\in(0,1)} \frac{t}{\int_0^t\int_s^1 x^\alpha \log^{\beta-\xi}\left(\frac{e}{x}\right)dxds} = \sup_{t\in(0,1)} \frac{1}{MF(t)},$$

where $MF(t)$ is the integral mean of a non-increasing function, because

$$t^\alpha \log^{\beta-\xi}\left(\frac{e}{t}\right) \geq 0,\ t\in(0,1),$$

and it is itself non-increasing. This means that $\frac{1}{MF(t)}$ attains maximum at 1, where, using Fubini's theorem and non-negativity of

$$t^{\alpha+1}\log^{\beta-\xi}\left(\frac{e}{t}\right),$$

one gets

$$\begin{aligned}
\frac{1}{MF(1)} &= \frac{1}{\int_0^1 t^{\alpha+1}\log^{\beta-\xi}\left(\frac{e}{t}\right)dt}\\
&\leq \frac{1}{\int_{\frac14}^{\frac34} t^{\alpha+1}\log^{\beta-\xi}\left(\frac{e}{t}\right)dt}\\
&\leq \left(\min\left\{\left(\frac14\right)^{\alpha+1}, \left(\frac34\right)^{\alpha+1}\right\}\int_{\frac14}^{\frac34}\log^{\beta-\xi}\left(\frac{e}{t}\right)dt\right)^{-1}\\
&< \infty.
\end{aligned}$$

The last inequality obviously holds, because the function $\log^{\beta-\xi}\left(\frac{e}{t}\right)$ is bounded on $\left(\frac14, \frac34\right)$.

Ad (c), the associate space Y' in the second case is the same as the one in part (a) and the assumptions on parameters are identical as well, so

$$Y = \Lambda^1\left(\log^{\xi-\beta-1}\left(\frac{e}{t}\right)\left(\log\left(\frac{e}{t}\right)+\beta-\xi\right)\right).$$

Now, assume that

$$-2+\frac1p < \alpha$$

and denote

$$\varphi_c(t) = t^{\frac{1}{p'}+\alpha+1}\log^{\beta-\xi}\left(\frac{e}{t}\right).$$

The assumption implies

$$\frac{1}{p'}+\alpha+1 > 0$$

and so φ_c vanishes at 0. Next, we need to show that

$$\frac{\varphi_c(t)}{t} = t^{\frac{1}{p'}+\alpha} \log^{\beta-\xi}\left(\frac{e}{t}\right)$$

is non-increasing. In our case, using properties (4.2) the function is non-increasing if and only if

$$\alpha \leq -\frac{1}{p'} \quad \text{and} \quad \beta - \xi \geq \frac{1}{p'} + \alpha.$$

The first condition is satisfied trivially because

$$\alpha \leq -1 \leq -\frac{1}{p'}.$$

Lastly, the function φ_c needs to be non-decreasing. A discussion similar to the one in the first part of Example 4.1 yields that the function

$$t^a \log^b\left(\frac{e}{t}\right)$$

is non-decreasing if and only if

$$a \geq -1 \quad \text{and} \quad a \geq b. \tag{4.3}$$

Applied to our case, it implies

$$\frac{1}{p'} + \alpha + 1 \geq -1 \quad \text{and} \quad \beta - \xi \leq \frac{1}{p'} + \alpha + 1.$$

Put together, we need the following

$$-1 - \frac{1}{p'} < \alpha \leq -1 \quad \text{and} \quad \frac{1}{p'} + \alpha \leq \beta - \xi \leq \frac{1}{p'} + \alpha + 1. \tag{4.4}$$

Note that the first condition implies

$$\overline{\varphi_c}(t) = t^{-\frac{1}{p'}-\alpha} \log^{\beta-\xi}\left(\frac{e}{t}\right) \xrightarrow{t\to 0^+} 0, \tag{4.5}$$

and the second one

$$\xi - \beta \geq -\frac{1}{p'} - \alpha - 1 \quad \text{and} \quad \beta - \xi - \frac{1}{p'} - \alpha > 0. \tag{4.6}$$

Under the assumptions above and thanks to Property (4.5), the function φ_c is quasiconcave and

$$Y = (Y')' = (\Gamma^\infty(\varphi_c))' = \Lambda^1(\overline{\varphi_c}').$$

Moreover,

$$\overline{\varphi_c}'(t) = t^{-\frac{1}{p'}-\alpha-1} \log^{\xi-\beta}\left(\frac{e}{t}\right)\left(\left(-\frac{1}{p'} - \alpha\right) + \frac{\beta - \xi}{\log \frac{e}{t}}\right).$$

Note that thanks to the inequalities (4.6), the function $\overline{\varphi_c}'(t)$ is non-negative. Paired with assumptions (4.4) it follows that the function is also non-increasing (cf. (4.2)). Finally, for a non-increasing function the space $\Lambda^1(\overline{\varphi_c})$ is an r.i. space. The ideas are identical to the ones used in part (a) of this proof.

To deal with part (d), it was shown in Example 4.1 that

$$Y' = L^{\left(\frac{p'}{1+p'(\alpha+1)},q',\beta-\xi\right)} = \Gamma^{q'}\left(t^{\frac{q'}{p'}-1+q'(\alpha+1)}\log^{q'(\beta-\xi)}\left(\frac{e}{t}\right)\right).$$

First, assume that $q'=1$, in which case denote

$$\varphi_{d1}(t) = t^{\frac{1}{p'}+\alpha}\log^{\beta-\xi}\left(\frac{e}{t}\right).$$

We want to use [12, Theorem 10.4.1] and so we need to verify the non-triviality conditions, which are equivalent to the following ones

$$\int_0^1 t^{\frac{1}{p'}+\alpha}\log^{\beta-\xi}\left(\frac{e}{t}\right)dt<\infty \quad \text{and} \quad \int_0^1 t^{\frac{1}{p'}+\alpha-1}\log^{\beta-\xi}\left(\frac{e}{t}\right)=\infty.$$

The first condition is satisfied if and only if

$$\left(\frac{1}{p'}+\alpha>-1\right) \quad \text{or} \quad \left(\frac{1}{p'}+\alpha=-1 \quad \wedge \quad \beta-\xi<-1\right)$$

and the second one if and only if

$$\left(\frac{1}{p'}+\alpha<0\right) \quad \text{or} \quad \left(\alpha=-\frac{1}{p'} \quad \wedge \quad \beta-\xi\geq-1\right).$$

Note that second option of the latter condition does not hold for any choice of parameters, because

$$\alpha\leq-1 \quad \text{and} \quad p'\in(1,\infty).$$

Now, both of the conditions must be met so

$$\left(-1<\frac{1}{p'}+\alpha<0\right) \quad \text{or} \quad \left(\frac{1}{p'}+\alpha=-1 \quad \wedge \quad \beta-\xi<-1\right).$$

Then Theorem 10.4.1 yields with some help of Fubini's theorem (part (b) of this proof uses similar techniques)

$$Y=\left(\Gamma^{q'}\left(t^{\frac{q'}{p'}-1+q'(\alpha+1)}\log^{q'(\beta-\xi)}\left(\frac{e}{t}\right)\right)\right)'=\Gamma^{\infty}\left(\frac{t}{\int_0^t\int_s^1 y^{\frac{1}{p'}+\alpha-1}\log^{\beta-\xi}\left(\frac{e}{y}\right)dyds}\right).$$

It is also true that

$$1\in\Gamma^{\infty}\left(\frac{t}{\int_0^t\int_s^1 y^{\frac{1}{p'}+\alpha-1}\log^{\beta-\xi}\left(\frac{e}{y}\right)dyds}\right).$$

The argument follows the reasoning of non-triviality in part (b) of this proof, only the powers of t and $\log\frac{e}{t}$ are different, yet the conclusion is the same.

Assume that $q'\in(1,\infty)$ and denote

$$\varphi_{d2}(t)=t^{\frac{q'}{p'}-1+q'(\alpha+1)}\log^{q'(\beta-\xi)}\left(\frac{e}{t}\right).$$

From Example 4.1, we have that

$$Y' = L^{\left(\frac{p'}{p'(\alpha+1)+1}, q', \beta-\xi\right)} = \Gamma^{q'}\left(\varphi_{d2}\right).$$

This time, using [14, Theorem 6.2], we get

$$Y = \Gamma^{q}\left(\frac{t^{q+q'-1}\int_0^t \varphi_{d2}(s)ds \int_t^1 s^{-q'}\varphi_{d2}(s)ds}{(\int_0^t \varphi_{d2}(s)ds + t^{q'}\int_t^1 s^{-q'}\varphi_{d2}(s)ds)^{q+1}}\right).$$

$\square$

Let us present a different example. We will drop a particular form of function w and use different methods this time. In particular, we will make a clever use of the property (P1) of r.i. spaces together with an operator which assigns any function a non-increasing one. Yet, this generalization does not come up without a price. We will to assume that

$$tw(t) \in L^{\infty}.$$

Before we start, define the operator mentioned above in the following way,

$$S(f^*)(t) = S_w(f^*)(t) = \sup_{t\leq s\leq 1} w(s)\int_0^s f^*(\tau)d\tau, \; f \in \mathcal{M}(0,1).$$

Notice that due to properties of the suprema $S(f^*)$ is a non-increasing function and trivially, for any $t \in (0,1)$ and $f \in \mathcal{M}(0,1)$,

$$w(t)\int_0^t f^* \leq \sup_{t\leq s\leq 1} w(s)\int_0^s f^*(\tau)d\tau. \tag{4.7}$$

Theorem 4.3. *Let w be a continuous weight function, such that it satisfies Property* (3.18) *and*

$$tw(t) \in L^{\infty}.$$

Moreover, let

$$1 < p \leq q < \infty,$$

let the operator T be as in Definition 3.2 and $X = L^{p,q}$, then the norm of the space associated to the optimal range of X and T satisfies

$$\|f\|_{Y'(0,1)} \approx \left(\int_0^1 t^{\frac{q'}{p'}-1+q'} w^{q'}(t)\left(f^{**}(t)\right)^{q'}dt\right)^{\frac{1}{q'}}.$$

Regarding the parameters p and q, we need $q > 1$ for our argument to work and for such q the space $L^{p,q}$ is rearrangement invariant only for $p > 1$. Let us also mention the fact that Example 4.1 and this theorem intersect, but besides that, each of them cover a different family of functions.

In the example, we take

$$w_{\alpha,\beta}(t) = t^{\alpha}\log^{\beta}\left(\frac{e}{t}\right)$$

for a given range of parameters. It is obvious that for $\alpha < -1$,

$$tw_{\alpha,\beta}(t) \notin L^{\infty}.$$

Conversely, a function

$$w(t) = \frac{1}{t\sqrt{1+\log\left(\frac{1}{t}\right)}}$$

does not satisfy the assumptions of Example 4.1, but it satisfies the assumptions of Theorem 4.3. However, the proof that this function satisfies Property (3.18) is omitted.

Proof. First, observe that

$$1 < p \leq q < \infty \quad \Longrightarrow \quad \frac{q'}{p'} - 1 \leq 0.$$

Then, by means of Hardy-Littlewood inequality (Theorem 2.3) and the characterization of the norm on the space associated to the optimal range, one gets

$$\begin{aligned}
\int_0^1 t^{\frac{q'}{p'}-1}\left(w(t)\int_0^t f^*(s)ds\right)^{q'} dt &\leq \int_0^1 \left(t^{\frac{q'}{p'}-1}\right)^* \left[\left(w(t)\int_0^t f^*(s)ds\right)^*\right]^{q'} dt \\
&= \int_0^1 t^{\frac{q'}{p'}-1} \left[\left(w(t)\int_0^t f^*(s)ds\right)^*\right]^{q'} dt \\
&= \left\| w(t)\int_0^t f^* \right\|_{p',q'}^{q'} \\
&= \left\| w(t)\int_0^t f^* \right\|_{X'(0,1)}^{q'} \\
&= \|f\|_{Y'(0,1)}^{q'}.
\end{aligned}$$

Conversely, by the property (P1) of the r.i. space $L^{p',q'}$ and (4.7),

$$\begin{aligned}
\|f\|_{Y'(0,1)} &= \left\| w(t)\int_0^t f^* \right\|_{p',q'} \\
&\leq \|S(f^*)\|_{p',q'} \\
&= \left(\int_0^1 t^{\frac{q'}{p'}-1}\left[\left(\sup_{t\leq s\leq 1} w(s)\int_0^s f^*(\tau)d\tau\right)^*\right]^{q'} dt\right)^{\frac{1}{q'}} \\
&= \left(\int_0^1 t^{\frac{q'}{p'}-1}\left[\sup_{t\leq s\leq 1}\left(w(s)\int_0^s f^*(\tau)d\tau\right)\right]^{q'} dt\right)^{\frac{1}{q'}}.
\end{aligned}$$

The last equality holds due to the fact that $S(f^*)$ is a non-increasing function. As the last step, certainly the hardest one, we need to prove that under the assumptions above, there is a constant $C > 0$ such that

$$\begin{aligned}
&\left(\int_0^1 t^{\frac{q'}{p'}-1}\left[\sup_{t\leq s\leq 1}\left(w(s)\int_0^s f^*(\tau)d\tau\right)\right]^{q'} dt\right)^{\frac{1}{q'}} \\
&\leq C\left(\int_0^1 t^{\frac{q'}{p'}-1}\left[w(t)\int_0^t f^*(\tau)d\tau\right]^{q'} dt\right)^{\frac{1}{q'}}.
\end{aligned}$$

We shall use Theorem 2.19. First, we need

$$T_{u,b}(f^*)(t) = \sup_{t\le\tau\le 1}\frac{u(\tau)}{B(\tau)}\int_0^\tau f^*(\xi)b(\xi)d\xi = \sup_{t\le s\le 1}(w(s)\int_0^s f^*(\tau)d\tau) = S(f^*)(t).$$

Consequently $b = 1$,[3] and so

$$B(t) = \int_0^t 1 d\tau = t$$

and

$$u(\tau) = \tau w(\tau).$$

Finally, functions $w(t)$ and $v(t)$ from Theorem 2.19 are equal to $t^{\frac{q'}{p'}-1}$ and parameters p and q from the theorem are equal to q'.

Now, we need to verify the assumptions of Theorem 2.19. Certainly,

$$0 < B(t) = t < \infty,\ t \in (0,1)$$

and

$$u(t) = tw(t)$$

is a continuous (and so measurable and a.e. finite) non-negative function by the assumption of this theorem. Moreover,

$$0 < \int_0^t x^{\frac{q'}{p'}-1}dx = \left[x^{\frac{q'}{p'}}\right]_0^t = t^{\frac{q'}{p'}} < \infty,\ t \in (0,1),$$

due to

$$\frac{q'}{p'} > 0$$

which holds true simply because

$$0 < p', q' < \infty.$$

We know that

$$\int_0^t v = \int_0^t w = \int_0^t s^{\frac{q'}{p'}-1}ds$$

and so the assumptions on w and v hold as well. Next, observe that

$$\begin{aligned}\overline{u}(t) &= B(t)\sup_{t\le\tau\le 1}\frac{u(\tau)}{B(\tau)}\\ &= t\sup_{t\le\tau\le 1} w(\tau)\\ &= tw(t), \end{aligned}\tag{4.8}$$

where the last equality uses the fact that w is non-increasing. By utilizing Property (3.18), one has

$$\sup_{t\in(0,1)}\frac{u(t)}{B(t)}\int_0^t\frac{b(\tau)}{\overline{u}(\tau)}d\tau = \sup_{t\in(0,1)} w(t)\int_0^t\frac{1}{\tau w(\tau)}d\tau \approx 1 < \infty.$$

[3]This holds up to a multiplicative constant, which cancels out in the equality above with the same multiplicative constant coming from $B(t)$.

Now, for the first actual condition of Theorem 2.19, using (4.8) again paired with the fact that

$$tw(t) \in L^\infty(0,1),$$

one gets

$$\begin{aligned}\left(\int_0^x \left[\sup_{t\le\tau\le x} \overline{u}(\tau)\right]^{q'} t^{\frac{q'}{p'}-1}dt\right)^{\frac{1}{q'}} &= \left(\int_0^x \left[\sup_{t\le\tau\le x} \tau w(\tau)\right]^{q'} t^{\frac{q'}{p'}-1}dt\right)^{\frac{1}{q'}} \\ &\le C\left(\int_0^x t^{\frac{q'}{p'}-1}dt\right)^{\frac{1}{q'}}.\end{aligned}$$

Consequently, Condition (2.3) holds. For Condition (2.4), observe that

$$\left(\int_x^1 \left(\frac{\overline{u}(t)}{B(t)}\right)^{q'} t^{\frac{q'}{p'}-1}dt\right)^{\frac{1}{q'}} = \left(\int_x^1 w(t)^{q'} t^{\frac{q'}{p'}-1}dt\right)^{\frac{1}{q'}} =: I_1(x) \tag{4.9}$$

and

$$\begin{aligned}\left(\int_0^x \left(\frac{B(t)}{V(t)}\right)^{q} t^{\frac{q'}{p'}-1}dt\right)^{\frac{1}{q}} &= \left(\int_0^x \left(\frac{t}{\int_0^t s^{\frac{q'}{p'}-1}ds}\right)^{q} t^{\frac{q'}{p'}-1}dt\right)^{\frac{1}{q}} \\ &= \left(\int_0^x \left(t^{1-\frac{q'}{p'}}\right)^{q} t^{\frac{q'}{p'}-1}dt\right)^{\frac{1}{q}} \\ &= \left(\int_0^x t^{\frac{p'-q'}{p'(q'-1)}}dt\right)^{\frac{1}{q}} \\ &= K\left(x^{\frac{p'-q'}{p'(q'-1)}+1}\right)^{\frac{1}{q}} \\ &= Kx^{\frac{1}{q}\cdot\frac{q'(p'-1)}{p'(q'-1)}} \\ &= Kx^{\frac{p'-1}{p'}} \\ &= Kx^{\frac{1}{p}} =: I_2(x),\end{aligned}$$

where

$$K = \left(\frac{p'-q'}{p'(q'-1)}+1\right)^{-\frac{1}{q}} = \left(\frac{q'(p'-1)}{p'(q'-1)}\right)^{-\frac{1}{q}} < \infty.$$

The fourth equality in the chain above holds, because

$$\frac{p'-q'}{p'(q'-1)} > -1,$$

which, in turn, is true thanks to

$$q' > 0 \quad \text{and} \quad p' > 1.$$

Finally,

$$
\begin{aligned}
S(I_1, I_2) &:= \sup_{x\in(0,1)} I_1(x)I_2(x) \\
&= K \sup_{x\in(0,1)} \left(\int_x^1 w(t)^{q'} t^{\frac{q'}{p'}-1} dt\right)^{\frac{1}{q'}} x^{\frac{1}{p}} \\
&\leq C \sup_{x\in(0,1)} \left(\int_x^1 t^{\frac{q'}{p'}-1-q'}\right)^{\frac{1}{q'}} x^{\frac{1}{p}}. \\
&= C \sup_{x\in(0,1)} \left(x^{\frac{q'}{p'}-q'} - 1\right)^{\frac{1}{q'}} x^{\frac{1}{p}}.
\end{aligned}
$$

The first inequality holds by virtue of $tw(t) \in L^\infty$ and the constant from the integration got included in the generic constant C. Notice that

$$
\frac{q'}{p'} - q' = q'\left(\frac{1}{p'} - 1\right) < 0
$$

for any admissible choice of p and q and it means that the expression in the brackets under the supremum tends to infinity for $x \to 0^+$.

It is obvious that the expression under the supremum is bounded everywhere outside the neighbourhood of 0, due to continuity of functions involved. So it suffices to focus our attention only on this neighbourhood. There, it is true that

$$
\left(x^{\frac{q'}{p'}-q'} - 1\right)^{\frac{1}{q'}} \approx x^{\frac{1}{p'}-1}.
$$

Consequently, on the neighbourhood of 0,

$$
\left(x^{\frac{q'}{p'}-q'} - 1\right)^{\frac{1}{q'}} x^{\frac{1}{p}} \approx x^{\frac{1}{p'}+\frac{1}{p}-1} = 1
$$

and after heroic effort we have shown that Condition (2.4) holds. Using Theorem 2.19 together with the already proven inequality, one gets

$$
\begin{aligned}
\|f\|_{Y'(0,1)} &\approx \left(\int_0^1 t^{\frac{q'}{p'}-1} \left(w(t) \int_0^t f^*(s)ds\right)^{q'} dt\right)^{\frac{1}{q'}} \\
&= \left(\int_0^1 t^{\frac{q'}{p'}+q'-1} w^{q'}(t)(f^{**}(t))^{q'}\right)^{\frac{1}{q'}}.
\end{aligned}
$$

$\square$

5. Threshold results

There are two distinct directions we want to explore. First, we shall prove the result mentioned in Chapter 1. That is, there exists an r.i. space X for which L^∞ is the optimal domain (with respect to the operator T_w) if and only if

$$w \in L^1.$$

Secondly, we explore the question which is tied to finding an optimal space – under which conditions are the fundamental functions of the spaces X and Y, where Y is the optimal range of X, equivalent?

Obviously, if X is an optimal space with respect to T_w, then it is the optimal range for itself and so the fundamental functions of the domain and range must be equivalent. Basically, we are looking for necessary conditions for the space X to be optimal.

Proposition 5.1. *Given an r.i. space X and a weight function w such that*

$$tw(t) \in X',$$

then

$$T : X \to L^\infty,$$

where the operator T is as in Definition 3.2, if and only if

$$w(t) \in X'.$$

Proof. For sufficiency, assume that $w \in X'$, then for any function $f \in X$,

$$\begin{aligned}
\|Tf\|_\infty &= \sup_{t\in[0,1]} \int_t^1 wf^* \\
&= \int_0^1 wf^* \\
&\le \|f^*\|_{X(0,1)} \|w\|_{X'(0,1)} \\
&= \|f\|_{X(0,1)} \|w\|_{X'(0,1)} \\
&< \infty.
\end{aligned}$$

The second equality holds because wf^* is a non-negative function, the following inequality is of course by virtue of Hölder inequality and the equality after that is due to the properties of an r.i. space.

For necessity, assume $T : X \to Y$, where Y is the optimal range as in Theorem 3.5. In fact, as a consequence of the discussion in Preliminaries, under Definition 2.11,

$$Y \hookrightarrow L^\infty \iff L^1 \hookrightarrow Y'.$$

The second embedding is then equivalent to existence of a constant $C > 0$ such that for any integrable function f,

$$\|f\|_{Y'(0,1)} = \left\| w(t) \int_0^t f^* \right\|_{X'(0,1)} \le C \|f\|_{L^1}.$$

Certainly,
$$\chi_{[0,T]} \in L^1, \quad T \in (0,1).$$
This combined with the use of the property (P1) of the r.i. space X' implies
$$\begin{aligned}
\left\| w(t) T \chi_{(T,1]}(t) \right\|_{X'(0,1)} &\leq \left\| w(t) t \chi_{[0,T]}(t) + w(t) T \chi_{(T,1]}(t) \right\|_{X'(0,1)} \\
&= \left\| w(t) \int_0^t \chi_{[0,T]} \right\|_{X'(0,1)} \\
&= \left\| \chi_{[0,T]} \right\|_{Y'(0,1)} \\
&\leq CT
\end{aligned}$$
As a result, if
$$Y \hookrightarrow L^\infty,$$
then necessarily
$$\left\| w(t) \chi_{(T,1]}(t) \right\|_{X'(0,1)} \leq C, \quad T \in (0,1)$$
and so
$$w \in X'.$$
□

As an obvious consequence, we can characterize all the weight functions w for which
$$T : X \to L^\infty.$$

Corollary 5.2. *Let w be a weight function. There exists an r.i. space such that*
$$T = T_w : X \to L^\infty,$$
where the operator T is as in Definition 3.2, if and only if
$$w(t) \in L^1.$$

Proof. If $w(t) \in L^1$, then we can take $X = L^\infty$. Moreover $w \in X'$ and due to the property (P1) of the r.i. space L^1, we know that
$$tw(t) \in L^1 = X'.$$
Consequently we can use Proposition 5.1 and as a result
$$T : X \to L^\infty.$$
On the other hand, in order for the optimal range of T and X to be sensibly defined, one needs
$$tw(t) \in X'.$$
Pair it with the assumption that
$$w(t) \notin L^1$$
and Proposition 5.1 yields that there is no r.i. space X such that
$$T : X \to L^\infty.$$
□

Turning back to the examples of functions w given in Introduction, namely

$$w_1(t) = \frac{1}{t\sqrt{1+\log\frac{1}{t}}} \quad \text{and} \quad w_2(t) = t^{-1+\frac{m}{n}},$$

we conclude that $w_1 \notin L^1$ and so there is no r.i. space X such that

$$T_{w_1} : X \to L^\infty.$$

On the other hand, $w_2 \in L^1$ for any $m < n$ and so there exists an r.i. space X such that

$$T_{w_2} : X \to L^\infty.$$

For the second part, we will first assume that

$$w \notin L^1.$$

We will show that if the fundamental function of the optimal range is majorizing (up to an additive constant) the fundamental function of X and $tw(t)$ does have a one-sided limit at 0, then this limit must be finite, effectively implying

$$tw(t) \in L^\infty.$$

Lemma 5.3. *Let X be an r.i. space, let w be a weight function such that*

$$tw(t) \in X'$$

and

$$T = T_w : X \to Y,$$

where Y is the optimal range with respect to X and T. Assume there exists a constant $C > 0$ such that

$$\frac{\varphi_X}{\varphi_Y} \leq C,$$

where φ_X and φ_Y are the fundamental functions of the spaces X and Y, respectively, and lastly assume that $\lim_{t\to 0^+} tw(t)$ exists. Then

$$\lim_{t\to 0^+} tw(t) < \infty.$$

Proof. Let us first show an auxiliary observation. For any r.i. space X the following holds

$$\begin{aligned} \left\| tw(t)\chi_{[0,a]}(t) \right\|_{X(0,1)} &+ a \left\| w(t)\chi_{(a,1]}(t) \right\|_{X(0,1)} \\ &\approx \left\| w(t)(t\chi_{[0,a]}(t) + a\chi_{(a,1]}(t)) \right\|_{X(0,1)} \end{aligned} \tag{5.1}$$

for $t \in (0,1)$. Obviously, thanks to the triangle inequality for $\|\cdot\|_{X(0,1)}$

$$\begin{aligned} \left\| tw(t)\chi_{[0,a]}(t) \right\|_{X(0,1)} &+ a \left\| w(t)\chi_{(a,1]}(t) \right\|_{X(0,1)} \\ &\geq \left\| w(t)(t\chi_{[0,a]}(t) + a\chi_{(a,1]}(t)) \right\|_{X(0,1)}. \end{aligned}$$

On the other hand, by properties of an r.i. space,

$$\max\left\{\left\|tw(t)\chi_{[0,a]}(t)\right\|_{X'(0,1)}, a\left\|w(t)\chi_{(a,1]}(t)\right\|_{X'(0,1)}\right\}$$
$$\leq \left\|w(t)(t\chi_{[0,a]}+a\chi_{(a,1]})\right\|_{X'(0,1)},$$

because

$$\max\{tw(t)\chi_{[0,a]}(t), aw(t)\chi_{(a,1]}(t)\} \leq w(t)(t\chi_{[0,a]}(t)+a\chi_{(a,1]}(t)),\ t\in(0,1)$$

and so

$$\frac{1}{2}\left(\left\|tw(t)\chi_{[0,a]}(t)\right\|_{X'(0,1)}+a\left\|w\chi_{(a,1]}(t)\right\|_{X'(0,1)}\right)$$
$$\leq \left\|w(t)(t\chi_{[0,a]}(t)+a\chi_{(a,1]}(t))\right\|_{X'(0,1)}.$$

Now, back to the proof. If the conclusion is not true, we can find a non-increasing positive sequence $\{t_n\}$ such that

$$tw(t)\geq n \quad \text{for} \quad t\in(0,t_n).$$

Consequently,

$$\frac{\left\|tw(t)\chi_{[0,t_n]}(t)\right\|_{X'(0,1)}}{\left\|\chi_{[0,t_n]}(t)\right\|_{X'(0,1)}}\geq n.$$

Using the identity

$$\left\|\chi_{[0,t_n]}\right\|_{X(0,1)}\left\|\chi_{[0,t_n]}\right\|_{X'(0,1)}=t_n, \quad \text{for any } t_n\in(0,1), \tag{5.2}$$

we obtain

$$\frac{\left\|tw(t)\chi_{[0,t_n]}(t)\right\|_{X'(0,1)}\left\|\chi_{[0,t_n]}(t)\right\|_{X(0,1)}}{t_n}\geq n.$$

Using the definition of the norm on the space associated to the optimal range (cf. Theorem 3.4) paired with the identity (5.2) in the second equality and then the equivalence (5.1), we get

$$\begin{aligned}
\frac{\varphi_X}{\varphi_Y}(t_n) &= \frac{\left\|\chi_{[0,t_n]}\right\|_{X(0,1)}}{\left\|\chi_{[0,t_n]}\right\|_{Y(0,1)}}\\
&= \frac{\left\|\chi_{[0,t_n]}(t)\right\|_{X(0,1)}\left\|w(t)\left(t\chi_{[0,t_n]}(t)+t_n\chi_{(t_n,1]}(t)\right)\right\|_{X'(0,1)}}{t_n}\\
&\approx \frac{\left\|\chi_{[0,t_n]}(t)\right\|_{X(0,1)}\left(\left\|tw(t)\chi_{[0,t_n]}(t)\right\|_{X'(0,1)}+t_n\left\|w(t)\chi_{(t_n,1]}(t)\right\|_{X'(0,1)}\right)}{t_n}\\
&\geq \frac{\left\|\chi_{[0,t_n]}(t)\right\|_{X(0,1)}\left\|tw(t)\chi_{[0,t_n]}(t)\right\|_{X'(0,1)}}{t_n}\\
&\geq n.
\end{aligned} \tag{5.3}$$

This is clearly a contradiction and so

$$\lim_{t\to0^+} tw(t)<\infty.$$

□

Lastly, we want to give conditions which guarantee that

$$\varphi_X \approx \varphi_Y.$$

Before we do that, we shall state one more technical lemma. The point of this lemma is simple. Whether or not the fundamental functions are equivalent depends solely on their behaviour near 0. This lemma proves the part that $\varphi_X \approx \varphi_Y$ for t away from 0.

Lemma 5.4. *Let X be an r.i. space, let w be a weight function such that*

$$tw(t) \in X',$$

and

$$T : X \to Y,$$

where Y is the optimal range with respect to X and T. For any $T \in (0,1)$ and $a \in (T,1)$ the expression

$$\frac{\varphi_X}{\varphi_Y}(a)$$

is bounded from above and from below by a constant depending on T, X and the function w.

Proof. Let $T \in (0,1)$ be given and $a \in (T,1)$. As shown in (5.3)

$$\frac{\varphi_X}{\varphi_Y}(a) \approx \frac{\left\|\chi_{[0,a]}\right\|_{X(0,1)} \left(\left\|tw\chi_{[0,a]}\right\|_{X'(0,1)} + a\left\|w\chi_{(a,1]}\right\|_{X'(0,1)}\right)}{a}.$$

Due to the non-decreasing nature (via the property (P1) of r.i. spaces) of the norms involved in the expression, we can take $a = 1$ in the nominator, and underestimate a in the denominator by T, then

$$\frac{\varphi_X}{\varphi_Y}(a) \leq \frac{1}{T}\left\|1\right\|_{X(0,1)}\left(\left\|tw\right\|_{X'(0,1)} + \left\|w\chi_{(T,1]}\right\|_{X'(0,1)}\right).$$

The only thing, which is not obvious, is boundedness of $\left\|w\chi_{(T,1]}\right\|_{X'(0,1)}$. Well, for $t \geq T$,

$$\left\|w\chi_{(T,1]}\right\|_{X'(0,1)} \leq \frac{1}{T}\left\|tw\chi_{(T,1]}\right\|_{X'(0,1)} < \infty.$$

Consequently,

$$\frac{\varphi_X}{\varphi_Y}(a) \leq C(X,T,w) < \infty, \ a \in [T,1].$$

For the estimate from below, notice that due to monotonicity and non-negativity of w and t, we have that for any $a \geq T$

$$\int_0^a tw(t)dt \geq \int_{\frac{T}{2}}^T tw(t)dt \geq \frac{T^2}{4}w(T) > 0, \ a \in [T,1].$$

Utilizing this observation, one gets

$$\begin{aligned}
\frac{\varphi_X}{\varphi_Y}(a) &\approx \frac{\left\|\chi_{[0,a]}\right\|_{X(0,1)}\left(\left\|tw\chi_{[0,a]}\right\|_{X'(0,1)} + a\left\|w\chi_{(a,1]}\right\|_{X'(0,1)}\right)}{a} \\
&\geq \frac{\left\|\chi_{[0,a]}\right\|_{X(0,1)}\left\|tw\chi_{[0,a]}\right\|_{X'(0,1)}}{a} \\
&\geq C\frac{\left\|\chi_{[0,a]}\right\|_{1}\left\|tw\chi_{[0,a]}\right\|_{1}}{a} \\
&\geq C\frac{T^2}{4}w(T) = C(X,T,w) \\
&> 0
\end{aligned}$$

Note that the constants of equivalence in (5.3) are $\frac{1}{2}$ and 1 and they are not dependent on any other quantity. □

Analyzing the expression

$$\frac{\left\|\chi_{[0,a]}\right\|_{X(0,1)}(\left\|tw\chi_{[0,a]}\right\|_{X'(0,1)} + a\left\|w\chi_{(a,1]}\right\|_{X'(0,1)})}{a}$$

further and using (5.2), we arrive at conditions which have to be met for φ_X and φ_Y to be equivalent. First, let us consider an upper bound. That is equivalent to existence of a constant $C > 0$ such that on some neighbourhood of 0

$$\frac{\left\|tw\chi_{[0,a]}\right\|_{X'(0,1)}}{\left\|\chi_{[0,a]}\right\|_{X'(0,1)}} \leq C, \tag{5.4}$$

$$\frac{\left\|w\chi_{(a,1]}\right\|_{X'(0,1)}}{\left\|\chi_{[0,a]}\right\|_{X'(0,1)}} \leq \frac{C}{a}. \tag{5.5}$$

To obtain the lower bound, one of the following two conditions must be satisfied, again on some neighbourhood of 0,

$$\frac{\left\|tw\chi_{[0,a]}\right\|_{X'(0,1)}}{\left\|\chi_{[0,a]}\right\|_{X'(0,1)}} \geq C, \tag{5.6}$$

$$\frac{\left\|w\chi_{(a,1]}\right\|_{X'(0,1)}}{\left\|\chi_{[0,a]}\right\|_{X'(0,1)}} \geq \frac{C}{a}. \tag{5.7}$$

Theorem 5.5. *Let X be a given r.i. space, w a weight function such that*

$$tw(t) \in X'$$

and let Y be the corresponding optimal range. Then

$$\varphi_X \approx \varphi_Y$$

if and only if either one of the triplets of the conditions (5.4), (5.5) *and* (5.6) *or* (5.4), (5.5) *and* (5.7) *is satisfied.*

Proof. It is obvious via Lemma 5.4 that boundedness of $\frac{\varphi_X}{\varphi_Y}$ depends only on behaviour near 0 and there

$$\frac{\varphi_X}{\varphi_Y}(a) \approx \frac{\left\|\chi_{[0,a]}\right\|_{X(0,1)} \left(\left\|tw\chi_{[0,a]}\right\|_{X'(0,1)} + a\left\|w\chi_{(a,1]}\right\|_{X'(0,1)}\right)}{a}.$$

As discussed before, the conditions are equivalent to boundedness of expression on the right-hand side from either above or below. □

Corollary 5.6. *Under the assumptions of Theorem 5.5:*

(I) Assume that $w(t) \approx \frac{1}{t}$ *near 0, then*

$$\varphi_X \approx \varphi_Y$$

if and only if there is a constant $C > 0$ *such that*

$$\left\|\frac{1}{t}\chi_{(a,1]}\right\|_{X'(0,1)} \leq C\left\|\frac{1}{a}\chi_{[0,a]}\right\|_{X'(0,1)}$$

for a *close to 0.*

(II) Assume that $\lim_{t\to 0^+} tw(t) = 0$*, then*

$$\varphi_X \approx \varphi_Y$$

if and only if

$$\left\|w\chi_{(a,1]}\right\|_{X'(0,1)} \approx \frac{1}{\left\|\chi_{[0,a]}\right\|_{X(0,1)}}$$

for a *close to 0.*

Proof. Apply Theorem 5.5 to the assumptions given. □

Bibliography

[1] H. Federer and W. H. Fleming. Normal and Integral Currents. *Annals of Mathematics*, 72(3):458–520, 1960.

[2] V. G. Maz'ya and S. V. Poborchi. *Differentiable functions on bad domains.* World Scientific Publishing Co., Inc., River Edge, NJ, 1997.

[3] A. Cianchi and L. Pick. Optimal Gaussian Sobolev embeddings. *Journal of Functional Analysis*, 256(11):3588–3642, 2009.

[4] D. E. Edmunds and J. Lang. Operators of Hardy type. *Journal of Computational and Applied Mathematics*, 208(1):20–28, 2007.

[5] A. Cianchi, L. Pick, and L. Slavíková. Higher-order Sobolev embeddings and isoperimetric inequalities. *Advances in Mathematics*, 273:568–650, 2015.

[6] O. S. Rothaus. Analytic inequalities, isoperimetric inequalities and logarithmic Sobolev inequalities. *Journal of Functional Analysis*, 64:296–313, 1985.

[7] G. F. Feissner. Hypercontractive semigroups and Sobolev's inequality. *Transactions of the American Mathematical Society*, 210:51–62, 1975.

[8] J. Moser. A Sharp Form of an Inequality by Trudinger. *Indiana University Mathematics Journal*, 20:1077–1092, 1971.

[9] E. Sawyer. Boundedness of classical operators on classical Lorentz spaces. *Studia Mathematica*, 96(2):145–158, 1990.

[10] S. G. Bobkov and F. Götze. Exponential Integrability and Transportation Cost Related to Logarithmic Sobolev Inequalities. *Journal of Functional Analysis*, 163(1):1–28, 1999.

[11] C. Bennett and R. Sharpley. *Interpolation of Operators.* Academic Press Professional, Inc., USA, 1987.

[12] L. Pick, A. Kufner, O. John, and S. Fučík. *Function Spaces.* De Gruyter series in nonlinear analysis and applications. De Gruyter, 2012.

[13] A. Gogatishvili, B. Opic, and L. Pick. Weighted inequalities for Hardy-type operators involving suprema. *Collectanea Mathematica*, 57:227–255, 2006.

[14] A. Gogatishvili and L. Pick. Discretization and anti-discretization of rearrangement-invariant norms. *Publicacions Matematiques*, 47:311–358, 06 2003.