# Crystal spacetime

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We construct an infinite crystal-like structure consisting of individual black holes held in stable equilibrium by the repulsion of their electric charges. This solution belongs to the Majumdar-Papapetrou family but one needs to deal with the infinite sums appearing in the metric. In addition to axial symmetry, the solution exhibits a discrete translational symmetry, while far away from the axis, it is fully cylindrically symmetric. We study the singularities and horizons of the spacetime, its asymptotics, and the behavior of charged test particles.

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## I. INTRODUCTION

The issue of symmetries is of special importance in general relativity since finding exact solutions to the Einstein-Maxwell equations is difficult. One thus often starts from assumptions on the symmetries of the spacetime in question, simplifying the underlying reality. This is certainly the case with the Schwarzschild solution but also with most of the basic cosmological models where we assume *local* homogeneity and isotropy. The reasoning is that although the universe is definitely not smooth on small scales, it becomes ever more uniform when viewed on larger scales. It is then crucial to see whether the local properties of spacetime conspire to produce the global symmetry of the standard FLRW metrics. Therefore, it would be of interest to have an exact solution that would locally be non-smooth while approximating homogeneity asymptotically.

One approach is to study the effect of a set of discrete individual masses arranged either regularly or randomly. This line of thought led to the lattice universes introduced by Lindquist and Wheeler already in 1957[1]. They considered a regular cubic spatial lattice of Schwarzschild solutions glued together at the boundaries of the neighboring cells. The resulting spacetime is non-stationary and it satisfies Einstein equations everywhere apart from the interfaces where the equations are only satisfied approximately, see[2] for a review. A similar, very simple but exact solution was studied in[3]. Another approach is based on an approximative, cellular-like solution expanding the metric in $M/L$, with $M$ mass of the cell and $L$ its typical dimension[4]. These papers investigate a space-like hypersurface at the moment of time symmetry when it is instantaneously static[5], see[6] for a recent review. The solutions are charged and contain a finite number of black holes distributed upon a 3-sphere. The model generally does not include dynamics although there are papers on the evolution of some special curves representing the evolution of the initial data[7]. The solution thus does not describe the spacetime as a whole and it is not possible, for instance, to locate the position of event horizons—one can find apparent horizons instead[8].

In this paper, we apply a different method and construct a solution involving an infinite number of sources with a discrete local symmetry and translational invariance on large scales. The simplest situation occurs in one dimension: one puts identical point sources on an infinite straight line and distributes them equidistantly. Due to the symmetry of the sources, the system is in balance and thus remains static. The basic example of this type is an infinite string of Schwarzschild black holes studied in[9]. This solution, however, has two drawbacks: firstly, its balance is only due to its reflection symmetry around any of the black holes so that an arbitrary perturbation which would not be periodic along the axis would either collapse or explode the whole system. And, secondly, there is in fact additional motivation in our search for this solution: we want to construct a source that would approximate the field of an infinitely thin string with linear mass and linear charge densities (ECS) studied in[10,11] and that would perhaps avoid the singularity along the axis of the spacetime. This has been done in[10] with a source in the form of a cylinder of dust yet this approach did not bring the discrete translational symmetry along the axis that we wanted to have. The discrete multi Schwarzschild spacetime of[9], on the other hand, has the right symmetries but also asymptotics that are very different from the charged string. We thus need to generalize this solution.

In order to keep the point sources in stable balance, we need a means to counter their gravitational attraction so it is natural to add an electromagnetic interaction by charging the sources and ensuring Coulomb repulsion between them. For a finite number of non-extremal Reissner-Nordström black holes, this has been done in[12]. This solution is of course asymptotically flat and it does not feature the discrete translational symmetry. One can combine the methods of[12] and[9] to obtain an infinite string of charged black holes along the axis of symmetry but the asymptotics do not correspond to the ECS solution, which is extreme. We thus have two options: one can take the non-extreme solution and take the extreme limit in the end, or start from a string of extremal black holes from the beginning. The latter is the path we adopted in this paper.

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The spacetime due to an infinite string of extremally charged black holes belongs to the Majumdar-Papapetrou family but it requires a special treatment due to convergence issues related to the infinite set of sources. Let us note that this issue is non trivial since one cannot apply the results of[13] as there are no accumulation points in our sequence of sources and their total mass is infinite. The sought exact solution is static and describes the entire spacetime so it is straightforward to find black-hole horizons. It can also be generalized to a dynamical solution including a cosmological constant. Moreover, it is asymptotically translationally invariant along the axis of symmetry as required, mimicking thus the large-scale homogeneity of the observed universe as an emergent phenomenon due to the local properties of the sources. In comparison, the models of[5] are locally symmetric (the 3-sphere hypersurface is) but lack any global symmetry. Our model is the exact opposite: there is only a discrete translational symmetry along the axis of symmetry, which, as we go farther away from the axis, passes into a translational symmetry with its corresponding Killing vector field and conservation of the parallel component of the linear momentum for test particles. We will further show that its asymptotics indeed correspond to the ECS spacetime of[10,11].

Due to the alignment of the sources we call these solutions "crystals," as they resemble an infinite one-dimensional crystallographic structure. To construct the crystals, we adopt two different approaches and discuss their differences, shortcomings, and merits. We study a Majumdar-Papapetrou (MP) spacetime constructed directly as an infinite sum of point sources. We look into two subcases: In Section II, all the masses are identical while in Section III, there are alternating positive and negative masses of an equal magnitude. We investigate convergence and other properties of the corresponding mathematical expressions and determine the location of singularities and other physical properties of the resulting spacetime. In Section IV, we review a construction due to[14] and based on a 5D MP solution[15] and a subsequent reduction of the number of dimensions. This results in an additional scalar field that can be viewed as representing matter contained in the spacetime. A similar situation involving a scalar field has been studied in case of bouncing cosmologies[16]. The corresponding metric has a closed form which enables us to study its properties analytically. An analogous reduction was applied in[17] albeit from 4 to 3 dimensions.

Before turning to the crystal solutions, let us first briefly review the properties of the Majumdar-Papapetrou solution in arbitrary dimension $D = n+1, n \geq 3$[?] The corresponding metric, ${}^{D}g$, reads[18]

$$ {}^{D}g = -U^{-2}\mathrm{d}t^2 + {}^{n}h_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{1} $$

where $t$ is a time-like Killing coordinate, so that the metric is static with the function $U = U(x^i)$ only depending on Cartesian-like spatial coordinates $x^i$[?] . The spatial metric ${}^{n}h_{ij}$ is conformally flat

$$ {}^{n}h = U^{\frac{2}{n-2}} \cdot {}^{n}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{2} $$

These coordinates describe well the region above the horizons. The electromagnetic potential $A$ and the electromagnetic field tensor $F$ read

$$ A = c_n \frac{\mathrm{d}t}{U},\; F = \mathrm{d}A = -c_n \sum_{i=1}^{n} \frac{U_{,i}}{U^2}\mathrm{d}x^i \wedge \mathrm{d}t \tag{3} $$

with $c_n = \sqrt{\frac{n-1}{2(n-2)}}$. One particular solution, in which we are interested, is a multi black-hole spacetime of the form

$$ U(x) = 1 + \sum_{i=1}^{N} \frac{Q_i}{r_i^{n-2}},\; r_i^2 = \sum_{a=1}^{n} (x^a - x_i^a)^2, \tag{4} $$

with the corresponding charge current[18]

$$ \sqrt{-\det{}^{D}g}\, J^0 = -\frac{c_n}{4\pi}\Delta_\delta U = = \frac{c_n \pi^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}-1\right)} \sum_{i=1}^{N} Q_i \cdot {}^{n}\delta\left(x - x_i\right). \tag{5} $$

Here $\Gamma$ is the Gamma function, $Q_i$ are constants of dimension $(\text{length})^{D-3} = (\text{length})^{n-2}$, and ${}^{n}\delta$ is the $n$-dimensional Dirac delta function. It can be shown that $Q_i$ determines the mass and also charge of each black hole and for $Q_i > 0$ the source located at $r_i = 0$ looks like a point, but in fact it represents a regular sphere $\mathbb{S}^{n-1}$ of dimension $n-1$ (and for $Q_i < 0$ the surface $r_i = 0$ corresponds to the location of a naked singularity). In $D = 4$ there exists a coordinate transformation, which regularizes the metric at a (arbitrarily chosen) horizon $r_i = 0$ and the horizon is smooth[19]. However, in $D > 4$ this holds only for a single black hole ($N = 1$). For $N = 2,3$ it was shown that the horizon is not smooth[20] while for a higher number of black holes the situation is still unclear, see also[21].

We now proceed to construct the crystal-like infinite structure consisting of charged point sources.

## II. MAJUMDAR-PAPAPETROU CRYSTAL IN 4D

In a previous paper we constructed a solution describing an extremally charged, infinite, straight string[11] and discussed its physical properties. One of interesting questions is whether this solution can be obtained as a limiting case of a spacetime, which would consist of an infinite number of extremal black holes located on the axis of symmetry, as one would expect intuitively. This can be achieved within the MP class as it allows for superposition of solutions since the Einstein equations reduce to a single flat-space Laplace's equation, which also yields the corresponding scalar potential of the Maxwell field. ECS should then be obtained in the limit of vanishing distance between the point sources while keeping the mass (and thus also the charge) per unit length of the axis constant. We are, however, interested in studying a situation when the spacing between sources is constant. Viewed by an observer located radially far away from the symmetry axis compared to the distance between two neighboring point sources, one again expects to obtain the ECS solution and recover the translational invariance along the axis. Therefore, we need the corresponding electrostatic potential of such a configuration in classical physics, describing an infinite number of point charges situated on the $z$-axis. We then construct the GR solution by simply inserting this axially symmetric potential, $U(\rho,z)$, into the general MP metric (1) with $n=3$

$$\mathrm{d}s^2 = -U^{-2}\mathrm{d}t^2 + U^2\left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\phi^2 + \mathrm{d}z^2\right), \qquad (6)$$

which we have written here in cylindrical coordinates. To this end, we shall now sum directly the fields of separate point sources located on the axis.

### A. Adding the charges

We place the individual identical charges of magnitude $Q$ at evenly spaced points along the $z$-axis (the separation between neighboring points, $k>0$, can be set to any positive value) and construct the corresponding potential as follows

$$U(\rho,z) := 1 + \frac{Q}{k}\varphi(\rho,z). \qquad (7)$$

From now on, we shall use dimensionless coordinates rescaled by $k$. The above relation defines a new structure function, $\varphi$, describing formally the distribution of the sources

$$\varphi(\rho,z) = \sum_{n=-\infty}^{\infty}\hat{\varphi}_n, \quad \hat{\varphi}_0 = \frac{1}{r_0}, \quad \hat{\varphi}_{n\neq 0} = \frac{1}{r_n} - \frac{1}{\sqrt{n^2}}. \qquad (8)$$

Here, $r_n = \sqrt{\rho^2+(z-n)^2}$ is the ($k$-rescaled) Cartesian distance from a specific point source, or puncture. Asymptotically, contribution of distant punctures goes as $1/n$ near the origin and we thus need to subtract such term to ensure convergence of the potential from distant charges here. We also include a factor of unity to yield the Minkowski spacetime if all the charges are equal to zero. It is convenient to rewrite the sum for $n\geq 1$ and take out the $\hat{\varphi}_0$ term, since it is divergent at the origin.[?] We thus obtain

$$\varphi(\rho,z) \equiv \varphi_0 + \varphi_{\neg 0}, \quad \varphi_{\neg 0} = \sum_{n=1}^{\infty}\varphi_n,$$
$$\varphi_n = \hat{\varphi}_n + \hat{\varphi}_{-n}, \quad \varphi_0 = \hat{\varphi}_0. \qquad (9)$$

The resulting potential has the mirror symmetry

$$\varphi_n(\rho,z) = \varphi_n(\rho,-z) \Rightarrow \varphi(\rho,z) = \varphi(\rho,-z) \qquad (10)$$

and it is also periodic in $z$ with $\varphi(\rho,z+1) = \varphi(\rho,z)$, see[22] for details. It is thus also mirror-symmetric with respect to $z=1/2$. On the axis with $\rho=0$, we are able to find a closed-form expression for the series, which is valid for $z\in(-1,1)$ and yields an expression for the full potential that can be extended to the real axis using the above symmetries:

$$\varphi(0,z) = \frac{1}{|z|} - H(z) - H(-z), \qquad (11)$$

with $H(z)$ the harmonic number[?] . We would like to establish uniform convergence of the series so we need a bound independent of $\rho$ and $z$. We find $\varphi_n \sim 1/n$ but this is not sufficient. Restricting to the strip $0\leq z\leq 1/2$, let us now look at the derivatives of the potential for $n\geq 1$:

$$0 \leq \frac{\partial\hat{\varphi}_n}{\partial z} \leq \frac{4}{(2n-1)^2}, \quad -\frac{8}{3\sqrt{3}(2n-1)^2} \leq \frac{\partial\hat{\varphi}_n}{\partial\rho} \leq 0$$

and

$$-\frac{1}{n^2} \leq \frac{\partial\hat{\varphi}_{-n}}{\partial z} \leq 0, \quad -\frac{2}{3\sqrt{3}n^2} \leq \frac{\partial\hat{\varphi}_{-n}}{\partial\rho} \leq 0.$$

Using definition (9) and the triangle inequality, we conclude

$$\left|\frac{\partial\varphi_n}{\partial z}\right| \leq \frac{8n^2-4n+1}{n^2(2n-1)^2}, \quad \left|\frac{\partial\varphi_n}{\partial\rho}\right| \leq \frac{2\left(8n^2-4n+1\right)}{3\sqrt{3}n^2(2n-1)^2}. \quad (12)$$

This is already sufficient as the corresponding series converge in both cases and we thus have uniform absolute-convergence for the derivatives of the series (9) (the same applies to higher derivatives of the series). We also know that the series itself converges along the axis, which implies it converges locally uniformly within the strip $z\in[0,1/2]$ and we can exchange the summation and derivatives, $\nabla_\mu\nabla_\nu\varphi_{\neg 0} = \sum_{n=1}^{\infty}\nabla_\mu\nabla_\nu\varphi_n$—we have thus shown that the above series satisfies Laplace's equation here. Adding the term $1/r_0$ and using the symmetries of the potential it then follows that the full potential is a solution of the Laplace's equation throughout the entire space save for the punctures. For details, we refer the reader to[22].

The resulting plots of $\varphi$ are shown in Figure 1a and 1b. This brings us to the question of asymptotic behavior of the potential. We notice that $\varphi_{n,\rho}$ are negative and increasing functions of $n$ so that we can use integral estimates to find

$$\frac{\partial\varphi_0}{\partial\rho} + \frac{\partial\varphi_1}{\partial\rho} + \int_1^\infty \frac{\partial\varphi_n}{\partial\rho}\mathrm{d}n \leq \frac{\partial\varphi}{\partial\rho} \leq \frac{\partial\varphi_0}{\partial\rho} + \int_1^\infty\frac{\partial\varphi_n}{\partial\rho}\mathrm{d}n. \quad (13)$$

We now evaluate the integral to obtain

$$-\frac{2}{\rho} - \frac{1}{\rho^2} + \frac{3z^2+4}{2\rho^4} + O\left(\frac{1}{\rho^5}\right) \leq \frac{\partial\varphi}{\partial\rho} \leq$$
$$\leq -\frac{2}{\rho} + \frac{1}{\rho^2} - \frac{3z^2+2}{2\rho^4} + O\left(\frac{1}{\rho^5}\right). \qquad (14)$$

It then follows that $\varphi_{,\rho}\to 0$ for $\rho\to\infty$ and that $\varphi_{,\rho}\sim -2/\rho$ and thus $\varphi\sim -2\ln\rho$. We can also see that for large $\rho$ the dependence on $z$ vanishes as expected, confirming thus the existence of an asymptotic axial Killing vector. Both these facts are consistent with the ECS spacetime[11] and the leading-order asymptotics in ECS and the present spacetime are thus identical far away from the axis. In fact, this is consistent with the result obtained by applying the approach of[9] with the seed metric from[12] and taking the limit to an extreme solution in the end, see A.

We conclude by noting that in the vicinity of the origin of (spherical) coordinates, $r\ll 1$, we can write

$$\varphi(r,\theta) = \frac{1}{r} + \frac{\bar{\zeta}(3)}{2}\left[1-3\cos(2\theta)\right]r^2 + O(r^4). \qquad (15)$$

It follows then that, locally, the situation near the individual sources is identical to the original MP solution: the punctures are not singularities but horizons and the solution can be extended through it to the other side[19]. As $\varphi$ ranges throughout $\mathbb{R}$, with $Q \neq 0$, the surface $\varphi = -k/Q$ implies $U = 0$ and is thus a location of a singularity since this is where the Maxwell invariant, $F_{\alpha\beta}F^{\alpha\beta} = -2(\nabla U)^2/U^4$, diverges. For $k/Q \lesssim -2.77$ the surface is disconnected, otherwise it is connected, see Figure 1c-1f. In fact, its surface area as well as the length of any curve contained in it vanishes and it is thus rather a point-like object than a true surface. It is then of interest that although the Majumdar-Papapetrou solution with a finite number of point sources does not contain naked singularities, it is not the case here as suggested in[13].

### B. Physical properties of the field

The electric charge producing the field (3) and enclosed in a sphere around the origin of (spherical) coordinates is

$$\frac{Q_{\rm sph}}{k} = -\frac{1}{8\pi}\int_0^{\pi}\int_0^{2\pi} r^2 U_{,r}\sin\theta \mathrm{d}\theta\mathrm{d}\phi = \frac{Q}{k} + O(r^2). \quad (16)$$

We see that $Q$ corresponds to the charge of each individual black hole and the total electric charge of any volume is given by the number of grid points on the section of the axis intersecting the volume. In fact, we can write the sources of the field as

$$\rho = -\frac{1}{4\pi}\Delta\varphi = -\frac{1}{4\pi}\left(\Delta\varphi_0 + \sum_{n=1}^{\infty}(-1)^n\Delta\varphi_n\right) = \\ = \sum_{n=-\infty}^{\infty} {}^3\delta\left(x,y,z-n\right) = \frac{1}{2\pi\rho}\text{III}_1(z)\delta(\rho), \quad (17)$$

where $\text{III}_1(z)$ is the Dirac comb distribution with support located at grid points along the axis.

We now proceed to study trajectories of test particles by first writing the two integrals of motion due to the axial symmetry and staticity of the metric

$$E = \frac{qU - \dot{t}}{U^2}, L_z = \rho^2 U^2 \dot{\phi}, \quad (18)$$

with the dot denoting a derivative with respect to the affine parameter. Additionally, the axial component of the 4-momentum, $p_z$, is also conserved asymptotically since $\dot{p}_z \sim U_{,z}$ due to the asymptotic Killing vector field mentioned above.

Test particles can remain static either anywhere if they have the same specific charge as the sources of the field, or centered between any two neighboring point sources on the axis and then their charge can be arbitrary. Radial motion is only possible at $z = 0$ or $z = 1/2$, within the mirror planes. Photons then have no turning points unlike massive particles, which can only move below or above two radii defined by

$$U^2\dot{\rho}^2 = (q - EU - 1)(q - EU + 1) = 0, \quad (19)$$

since the potential is monotonic. As there is no horizon on the axis for $z = 1/2$ a radially moving particle can oscillate below the lower radius here, while it will cross the black-hole horizon at $z = 0$. Motion parallel to the axis is only possible on the axis itself for both photons, which have no turning points, and massive particles, which can have up to 4 turning points between adjacent grid points depending on the velocity of the particles and some of them can also oscillate between the turning points.

Another interesting class of trajectories are circular geodesics, which only exist within the mirror planes again. Denoting $\dot{\phi} = \omega$, null geodesics satisfy

$$\dot{t} = \rho\omega U^2, U_{,z} = 0, U + 2\rho U_{,\rho} = 0, \quad (20)$$

so they can only occur at radii determined by the last relation—there can be two such radii, one radius or none. Their coordinate angular velocity is fixed as $1/\rho U^2$ due to the first relation. Time-like geodesics require

$$U_{,\rho}(-\gamma q U + \rho^2\omega^2 U^4 + \gamma^2) + \rho\omega^2 U^5 = 0, \quad (21)$$

$$U_{,z}(-\gamma q U + \rho^2\omega^2 U^4 + \gamma^2) = 0, \quad (22)$$

$$\rho^2\omega^2 U^2 - \frac{\gamma^2}{U^2} = -1, \quad (23)$$

with $\gamma = \dot{t}$. The middle equation implies motion within the mirror planes again. One can substitute for $\omega^2$ from the last equation into the first one, obtaining a quadratic equation for $\gamma$ and substituting the resulting two solutions back into the expression for $\omega$. Therefore, these particles can exist within a range of radii, including the circular null orbit radii, which is a special case, and they generally admit two different angular velocities for the same radius. Asymptotically, we get

$$\gamma = 1 + \frac{Q(1 - q - 2\ln\rho)}{k} + O\left(\frac{Q^2}{k^2}\right), \quad (24)$$

$$\omega^2 = \frac{2(1-q)Q}{k\rho^2} + O\left(\frac{Q^2}{k^2}\right). \quad (25)$$

Compared with ECS electrogeodesics[11], we see that the source is again compatible with linear mass and charge density of $Q/k$.

We have seen that although we are very limited due to the form of the metric components in terms of an infinite sum (7) and (8) we are still able to infer some interesting facts about the resulting spacetime, which are in line with our intuitive expectations. However, in order to achieve uniform convergence throughout the domain of the infinite sum (8), we needed to introduce a regulator function. It is then natural to ask ourselves the question: could we do without it?

## III. CRYSTAL OF ALTERNATING CHARGES

If one admits negative mass of the sources then an infinite crystal of alternating masses can be constructed the same way as in the previous section. The advantage of using point sources of opposite masses is that the sums appearing in the metric converge better and one can expect asymptotic flatness far away from the axis. On the other hand, we lose stability

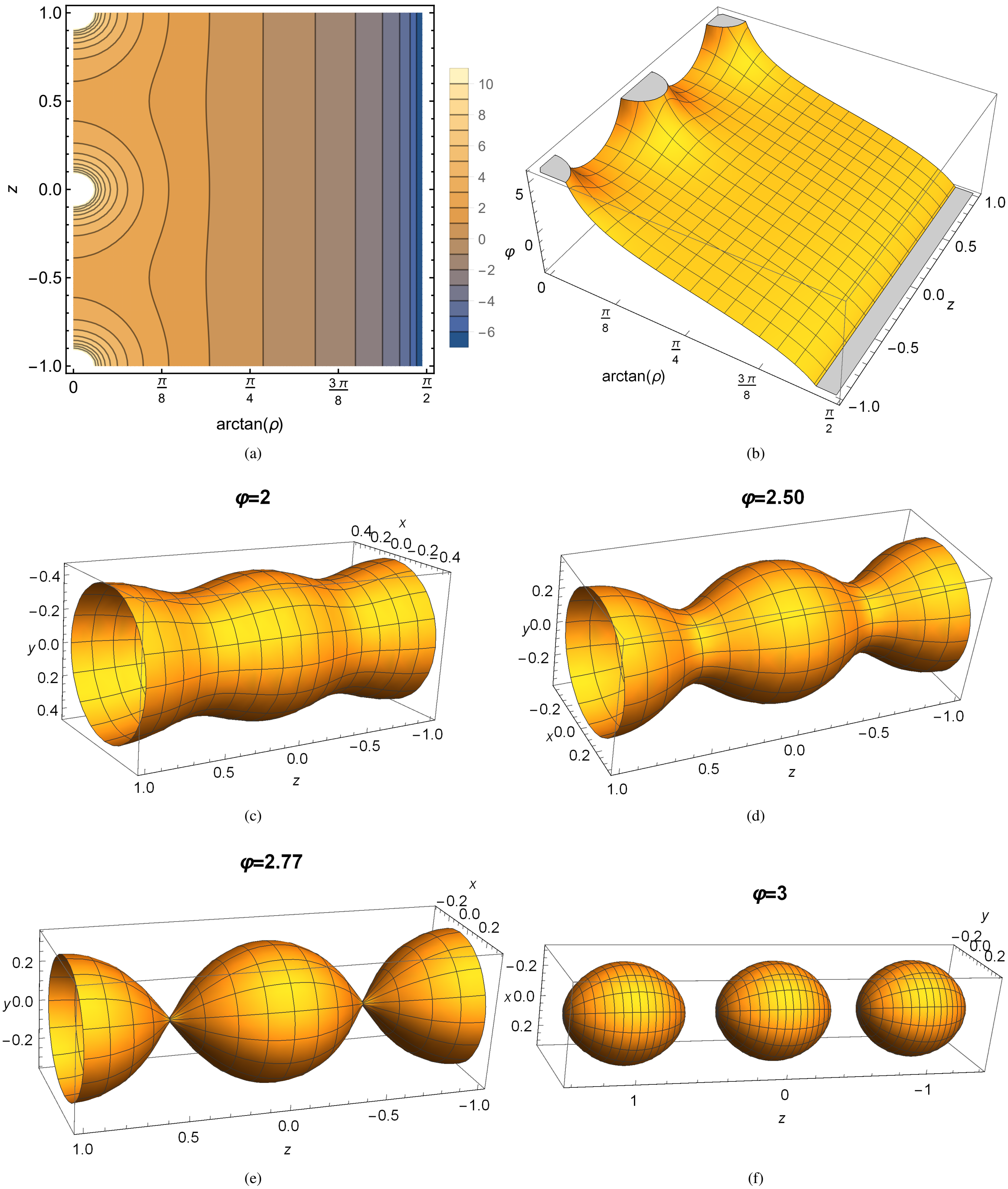



FIG. 1: (a) Conformal contour plot of $\varphi$; (b) conformal 3D plot of $\varphi$—we notice divergences in the asymptotic region $\rho \gg 1$ and around the lattice points. However, scalar invariants remain bounded at these locations. (c)-(f) 3D plots of various isopotential surfaces of $\varphi$. These define the singular surface $U = 0$ given by $\varphi = -k/Q$. For $k/Q \lesssim -2.77$ the surface is disconnected and the singularity splits. This requires black holes of a negative mass. Notice that although in this coordinate system the singularities look like surfaces, they are in fact points of zero area.

against perturbations. The master function from the general MP form (6) now reads

$$U(\rho,z) = 1 + \frac{Q}{k}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{r_n} := 1 + \frac{Q}{k}\,\chi(\rho,z), \tag{26}$$

with $r_n$ is the Cartesian distance defined below (8). Using similar arguments as above and the fact that this time we have a sum with alternating signs, one can show that the series converges uniformly on $\mathbb{R}_0^+ \times [0,1/2]$—we refer the reader to[22] for details. It may be also seen that $\chi(\rho,z) = \chi(\rho,-z) = -\chi(\rho,z+1)$, i.e., the structure function has mirror symmetry and has an antiperiod 1 and a period 2 in $z$, see Figure 2a and 2b. However, due to the presence of 1 in (26), the master function only has a period 2. On the axis, the series can be expressed in a closed form using the Lerch transcendent. There is again the outer singularity but this time it only envelopes every second grid point, see Figure 2c-2f.**?**

It can be shown that the sum (26) and its second derivatives converge uniformly throughout its domain. Uniform convergence is crucial since derivatives then commute with the sums and it enables us to prove that the resulting potential satisfies Laplace's equation. Using the symmetries of the potential, one then extends the definition of the potential for any $\rho$ and $z$, obtaining thus a full solution of the Einstein-Maxwell equations. For details of the entire proof, see[22].

Using integral estimates, we get the following bounds on $\chi$

$$-\frac{2}{\rho^2} + \frac{6z^2+17}{2\rho^4} + O\left(\frac{1}{\rho^5}\right) \le \frac{\partial\chi}{\partial\rho} \le \le \frac{2}{\rho^2} - \frac{6z^2+13}{2\rho^4} + O\left(\frac{1}{\rho^5}\right). \tag{27}$$

We thus conclude that $\chi_{,\rho} \sim \rho^{-2}$ or faster. We can also see that for large $\rho$ the dependence on $z$ vanishes again. In fact, away from the axis, we can do better using Fourier series decomposition of the sought potential, $\chi$, which is periodic and at least $C^2$. Finding the Fourier coefficients directly via integrals is hard so we choose a different approach. We assume the potential contains separated modes of the form

$$\chi \sim \sum_{n=1}^{\infty} A_n(z)B_n(\rho), \tag{28}$$

where each mode satisfies Laplace's equation, leading to

$$-\alpha_n^2 = \frac{A_n''(z)}{A_n(z)} = -\frac{1}{B_n(\rho)}\left(B_n''(\rho) + \frac{B_n'(\rho)}{\rho}\right). \tag{29}$$

Here we already assumed z-periodicity of the solution in choosing the correct sign of the separation constant. We can thus write each mode as a product of the form

$$[a_n\sin(\alpha_n z) + b_n\cos(\alpha_n z)]\,[c_n I_0(\alpha_n\rho) + d_n K_0(\alpha_n\rho)]\,.$$

The potential $\chi$ is reflection symmetric, $\chi(z) = \chi(-z)$, so that $a_n = 0$ while its anti-periodicity yields

$$A_n(z+1) = -A_n(z) \Rightarrow \cos(\pi n) = -1 \Rightarrow n = 1,3,5,\ldots \tag{30}$$

Finally, we know that $I_0$ diverges at $\rho \to \infty$, which implies $c_n = 0$. We thus arrive at the expression

$$\chi = \sum_{l=1}^{\infty} f_l\cos[\alpha_l z]\,K_0[\alpha_l\rho]\,, \alpha_l = \pi(2l-1). \tag{31}$$

Since we have

$$\|K_0(\pi n\rho)\| \le K_0(\delta)\exp[\delta(1-n)]\,, \rho \ge \delta > 0, \tag{32}$$

the sum converges absolutely uniformly for $\rho \ge \delta$. Near the axis, however, $K_0$ diverges. We thus determine coefficients $f_l$ from the electric charge only formally. The classical charge inside a small cylinder located symmetrically along the axis reads

$$4\pi Q(R,h) = -2\pi\left(\int_0^R \chi_{,z}\|_{z=-h}^{z=+h}\rho\;\mathrm{d}\rho + \int_{-h}^{h}(\chi_{,\rho}\rho)\|_{\rho=R}\mathrm{d}z\right). \tag{33}$$

The first term yields

$$2\sum_{l=1}^{\infty} f_l\sin[\alpha_l h]\left[RK_1[\alpha_l R] - \frac{1}{\alpha_l}\right], \tag{34}$$

which vanishes in the limit $R \to 0$ thanks to the Bessel function. The second term reads

$$2\pi\sum_{l=1}^{\infty}\int_{-h}^{h} f_l\cos[\alpha_l z]\,\alpha_l RK_1[\alpha_l R]\,\mathrm{d}z \tag{35}$$

and in the limit $R \to 0$ we obtain

$$2\pi\sum_{l=1}^{\infty}\int_{-h}^{h} f_l\cos[\alpha_l z]\,\mathrm{d}z. \tag{36}$$

Now, similarly to the homogeneous crystal and 17, we can write the charge density as

$$\rho = \frac{\text{Ш}_2(z) - \text{Ш}_2(z-1)}{2\pi\rho}\delta(\rho) \tag{37}$$

But we can also express the Dirac comb function as

$$\text{Ш}_2(z) - \text{Ш}_2(z-1) = \sum_{l=1}^{\infty} 2\cos[\alpha_l z]\,. \tag{38}$$

Comparing the corresponding terms in 38 and 36, we conclude $f_l = 4$. We thus have

$$\chi = 4\sum_{l=1}^{\infty}\cos[\alpha_l z]\,K_0[\alpha_l\rho]\,,\;\; \alpha_l = \pi(2l-1). \tag{39}$$

Returning now to the asymptotic behavior of the potential 27, we use the leading-order term for the Bessel function and since the dependence on $z$ vanishes asymptotically, we express the potential in the plane $z = 0$ to obtain

$$\chi = \sum_{l=1}^{\infty} 4\sqrt{\pi}e^{-\alpha_l\rho}\left(\frac{1}{\sqrt{2\alpha_l\rho}} + O\left(\frac{1}{(\alpha_l\rho)^{3/2}}\right)\right). \tag{40}$$

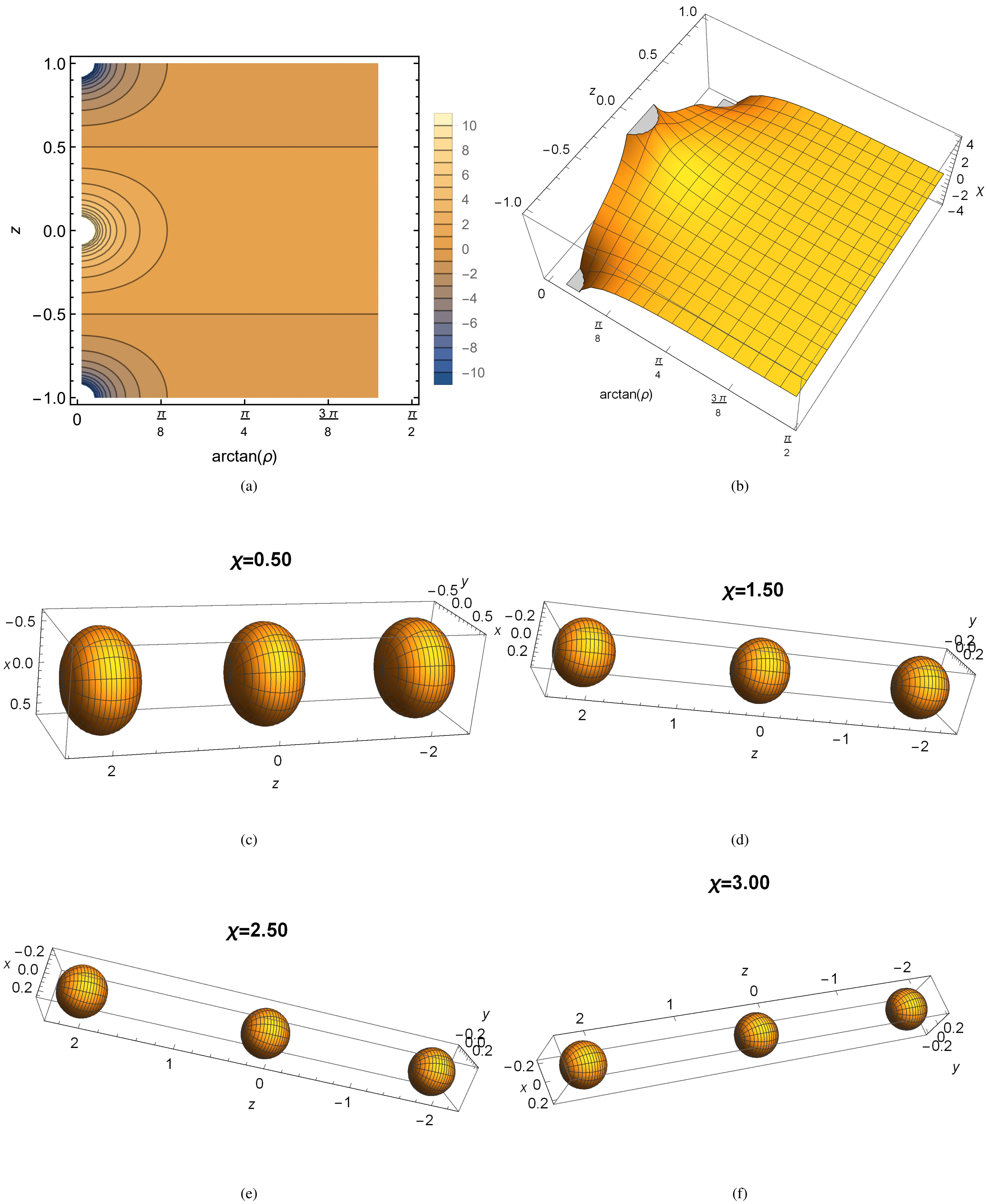



FIG. 2: (a) Conformal contour plot of $\chi$; (b) conformal 3D plot of $\chi$; (c)-(f) 3D plots of singular surfaces $\chi = -k/Q$ for various values of $Q/k$. The surface is always disconnected and envelopes every second lattice point.

The sum has no closed formula, but we can get a lower and upper integral estimates as follows, omitting the higher-order terms

$$\frac{\sqrt{2}}{\pi}\frac{e^{-\pi\rho}}{\rho^{3/2}} \leq \chi \leq 2\sqrt{2}\frac{e^{-\pi\rho}}{\rho^{1/2}}. \tag{41}$$

Numerical evaluation finally suggests $\chi \approx 2.74e^{-\pi\rho}\rho^{-1/2}$, which is close to the upper estimate. The exponential decay of the master function with radial distance from the axis means that the spacetime is radially asymptotically flat and it approaches Minkowski faster than for any isolated system. Applying the same procedure to the uniform crystal, we find $\varphi = -2\ln\rho + 4\sum_{l=1}^{\infty}\cos[\alpha_l z]K_0[\alpha_l\rho]$ with $\alpha_l = 2\pi l$. Taking now the limit $k \to 0^+$, the entire spacetime apart from the axis gets squashed to form the radial cylindrical infinity of the ECS spacetime as expected intuitively.

We have shown above that the alternating structure has certain advantages over the homogeneous crystal, particularly better convergence and asymptotic properties. Another option is to construct a smooth crystal consisting of Yukawa-like potentials of the form

$$\sum_{n=-\infty}^{+\infty}\frac{e^{-\alpha r_n}}{r_n}. \tag{42}$$

Thanks to the exponentials, the potential converges absolutely uniformly and is always positive so there is no naked singularity. This, however, comes at the expense of additional matter content in the form of charged dust. On the other hand, it would be much more comfortable to work with closed-form expressions appearing in the terms above. To this end, we now proceed to obtain a similar crystal-like 4D structure from a very different starting point.

## IV. 5D MAJUMDAR-PAPAPETROU CRYSTAL REDUCED TO 4D

As we have seen, one is able to study some of the properties of the spacetime even though its metric is in the form of an infinite series. It is however not possible to give, for instance, closed-form solutions of geodesic motion or the individual metric components. We thus tried to approach the problem in a different way by looking for a solution to the Laplace's equation with the required smooth azimuthal and discrete axial symmetries, yet the resulting series involved the same problems as the solution presented above. However, we did find a way of circumventing these issues by using a closed-form 5D Majumdar-Papapetrou solution of the required symmetry and reducing its dimension to 4. The clear advantage of this technique is that it yields the correct symmetries while the metric coefficients are analytic expressions. On the other hand, it introduces an additional scalar field as a source of the gravitational field in the resulting 4D Einstein equations.

In 5 dimensions, the corresponding general expression for the master function of an infinite crystal (4) consisting of point sources located at evenly distributed points along the axis yields the following expression in dimensionless hyper-cylindrical coordinates rescaled by $k$, the distance between the sources

$$U(\rho,z) = 1 + \frac{Q}{k^2}\sum_{n=-\infty}^{+\infty}\frac{1}{r_n^2} = 1 + \frac{Q}{k^2}\,\eta(\rho,z), \tag{43}$$

where $Q$ has the dimension of area, $r_n$ is defined below (8), and we defined a new structure function, $\eta(\rho,z)$. This can be summed to a closed form[14,15]

$$\eta(\rho,z) = \frac{\pi}{\rho}\frac{\sinh(2\pi\rho)}{\cosh(2\pi\rho)-\cos(2\pi z)}. \tag{44}$$

This situation is rather typical in higher dimensions where odd dimensions often differ profoundly from their even counterparts. The metric does not depend on one of the angular coordinates and takes the following form

$$\frac{\mathrm{d}s^2}{k^2} = U\left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\phi_2{}^2 + \rho^2\sin^2\phi_2\mathrm{d}\phi_3{}^2 + \mathrm{d}z^2\right) - U^{-2}\mathrm{d}t^2. \tag{45}$$

We now reduce to 4D using the $\phi_3$ coordinate to obtain the 4D metric and drop the index in $\phi_2$ to write

$$\frac{\mathrm{d}s^2}{k^2} = -U^{-2}\mathrm{d}t^2 + U\left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\phi^2 + \mathrm{d}z^2\right), \tag{46}$$

with an additional dilaton field

$$\Phi(\rho,\phi,z) = \sqrt{U}\rho\sin\phi. \tag{47}$$

The electromagnetic part yields

$$A = \frac{\sqrt{3}}{2}\frac{\mathrm{d}t}{U},\quad F = \frac{\sqrt{3}}{2}\frac{\mathrm{d}t}{U^2}\wedge\left(U_{,\rho}\mathrm{d}\rho + U_{,z}\mathrm{d}z\right). \tag{48}$$

For details of the reduction and the corresponding field equations, we refer the reader to[22]. The solution thus has the interesting property of being axially symmetric as regards the gravitational field while the scalar field does depend on the azimuthal coordinate. The reduced Einstein-Maxwell equations yield a single linear relation

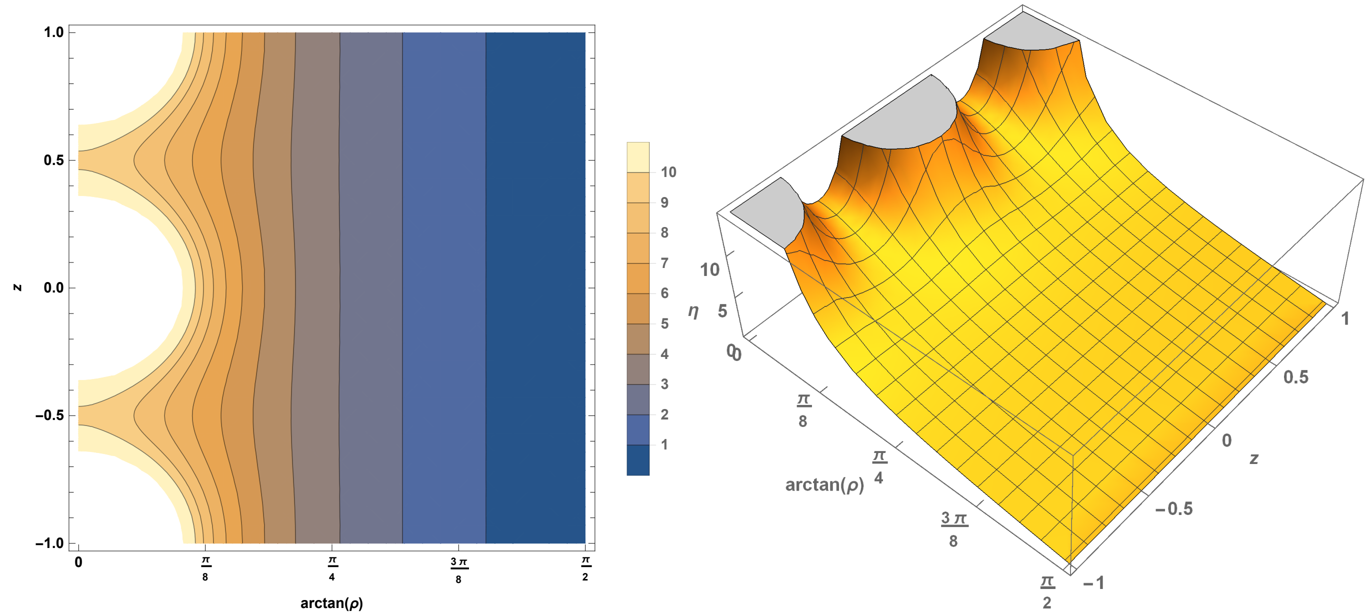
$$U_{,\rho\rho} + U_{,zz} + 2\frac{U_{,\rho}}{\rho} = 0. \tag{49}$$

We interpret the reduced electromagnetic part of the solution as a vacuum Maxwellian field. The dilaton field satisfies $3\Box\Phi = \Phi R$. Both the dilatonic and electromagnetic fields appear on the RHS of Einstein equations as sources of the resulting gravitational field.

### A. The reduced geometry

The structure function, $\eta(\rho,z)$, has the following obvious symmetries:

$$\eta(\rho,z) = \eta(\rho,-z) = \eta(\rho,z+1) = \eta(\rho,z-1). \tag{50}$$




(a) (b)

FIG. 3: (a) Conformal contour plot of $\eta$; (b) conformal 3D plot of $\eta$. Far away from the axis, $\rho \gg 1$, the structure function vanishes, producing an asymptotically flat spacetime.

On the axis and in the mirror plane, we find

$$\eta(0,z) = \frac{\pi^2}{\sin^2(\pi z)}, \quad \eta(\rho,0) = \frac{\pi}{\rho}\frac{\sinh(2\pi\rho)}{\cosh(2\pi\rho)-1}, \tag{51}$$

while far from the axis, $\rho \gg 1$, the spacetime is asymptotically flat again as expected

$$\eta = \frac{\pi}{\rho} + O(\rho^{-2}), \tag{52}$$

with the electromagnetic field vanishing as well, whereas the scalar field diverges as $\rho$. For $Q > 0$ the function $\eta$ is always positive, so there is no singularity for $\rho > 0$, see Figure 3. Just as in the previous sections, transforming to spherical coordinates centered at $\rho = z = 0$, we find for $r \ll 1$

$$\eta(r,\theta) = \frac{1}{r^2} + \frac{\pi^2}{3} + \frac{\pi^4}{45}\left[1 + 2\cos(2\theta)\right] r^2 + O(r^4). \tag{53}$$

The Ricci and Kretschmann scalars are finite at $r = 0$

$$R = -\frac{6k^2}{Q}, \quad \mathscr{K} = \frac{68k^4}{Q^2}. \tag{54}$$

Therefore, $r = 0$ is a non-singular null surface with an area $4\pi Q$. It is possible again to extend the spacetime to negative values of $r$ down to $U = 0$, which is a singular point. To see this, let us use the Ricci scalar, for instance?

$$R = -3\frac{U_{,z}^2 + U_{,\rho}^2}{2U^3}, \tag{55}$$

and this expression diverges for $U = 0$, which is a hypersurface of zero volume. The distance to the horizon is infinite and its surface gravity vanishes which is consistent with an extreme horizon as expected. We conclude that, in this respect, the solution behaves analogously to the standard Majumdar-Papapetrou multi-black-hole solution, where, however, $R = 0$ and one rather uses a Maxwell invariant. Since it contains no naked singularities and since it is asymptotically flat far away from the axis the present spacetime is more well behaved than the previous 4D MP solution obtained as a sum of infinitely many individual 4D black holes. Let us now look at other features of the solution.

### B. Physical properties of the field

The reduced Maxwell equations imply that the charge enclosed in a volume $V$ reads

$$4\pi Q_V = \int_{\partial V} \Phi F_{\mu\nu} r^\mu n^\nu \mathrm{d}\Sigma, \tag{56}$$

where $n^\nu$ is the timelike normal and $r^\mu$ is the spacelike normal to the surface $\Sigma = \partial V$. Therefore, the charge $Q_{\text{sph}}$ in a small sphere around $r = 0$ and thus the charge of each black hole in the crystal is

$$\frac{Q_{\text{sph}}}{k} = -\int_0^\pi \int_0^{2\pi} \frac{\sqrt{3} r^3 \sin^2\theta |\sin\phi| U_{,r}}{8\pi} \mathrm{d}\theta \mathrm{d}\phi = \\ = \frac{\sqrt{3}}{2}\frac{Q}{k^2} + O(r). \tag{57}$$

To interpret the spacetime physically, we investigate motion of charged test particles now, starting with static test particles. We assume the standard form of the electrogeodesic equation with the Lorentz force due to the Maxwell tensor (48). Any particle can stand still at the points of symmetry on the axis in between the neighboring punctures while particles with a specific charge

$$q^2 = \frac{4}{3} \tag{58}$$

can be located anywhere outside of horizons. In order for the test bodies to stand still the specific charge of the black holes must be $1/q^2$ in analogy with the usual 4-dimensional MP solutions—it is different from 1 since the black holes interact through electromagnetic, gravitational, and dilatonic fields, see[14].

As expected due to the symmetry, radial geodesics only exist in mirror planes and they can pierce the horizons or escape to radial infinity in case of null geodesics, while time-like radial geodesics are bounded. We can also have axial geodesics moving along the $z$-axis.

The most interesting case are circular electrogeodesics, which we calculate for large $\rho$ within the mirror plane, $z = 0$, as required by the symmetry. Writing $\dot{\phi} = \omega$, we find for null orbits

$$\begin{aligned} \dot{t} &= \rho\omega U^{3/2}, \\ 0 &= 2U + 3\rho U_{,\rho}. \end{aligned} \tag{59}$$

There is a null geodesic coinciding with the horizon, similarly to extreme Reissner-Nordström. Massive particles yield

$$\begin{aligned} &0 = 2U_{,\rho} + 2U^2\omega^2\rho + 3UU_{,\rho}\omega^2\rho^2 - \sqrt{3}U_{,\rho}q\sqrt{1+\rho^2\omega^2U}, \\ &\dot{t} = U\sqrt{1+\rho^2\omega^2U}. \end{aligned} \tag{60}$$

For large $\rho$, we obtain

$$\omega^2 = \frac{\pi Q\left(1 \pm \frac{\sqrt{3}}{2}q\right)}{k^2\rho^3} + O\left(\rho^{-4}\right). \tag{61}$$

Due to the asymptotic flatness, the leading order corresponds to the Reissner-Nordström black hole with the dimensionless angular velocity $\omega^2 = \frac{M \pm Qq}{\rho^3}$. We can thus see that far away from the axis the crystal source behaves as a point source of mass $\pi Q/k$ and charge $\frac{\sqrt{3}}{2}\pi Q/k$. The specific charge of the source thus agrees with (58).

## V. CONCLUSIONS

Our aim was to find a solution exhibiting locally a discrete translational symmetry along a symmetry axis containing the sources of the field while if looked upon from a distance, it would approach the full cylindrical symmetry of a charged line, the ECS. We approached our goal in two complementary ways—firstly, we constructed the solution as a sum of separate extremal black holes distributed evenly along an infinite line and held in equilibrium by their electric charges. And, secondly, we took a 5-dimensional solution of the required symmetry and reduced the number of dimensions by one. Both approaches have their advantages and drawbacks.

The constructive approach produces a solution in the form of an infinite sum the uniform convergence of which needs to be dealt with in order to show it is indeed a solution of Einstein-Maxwell equations. There is no closed-form relation for the metric or any subsequent terms derived from the metric, such as geodesic motion, etc. On the other hand, the physical meaning of the parameters appearing in the solution is quite as expected and the gravitational field is due to the masses and electric field of the point sources only. Despite having to work with the infinite sums, we were able to infer the asymptotic properties of the solution and show that the field diverges far away from the axis, approximating the ECS solution.

The second approach overcomes the obstacle of infinite sums since we can readily express the results of the 5D sums in a closed form, which is then simply reduced to 4D. We then work with the metric in the usual manner and it is straightforward to calculate field strengths or motion of test particles. On the other hand though, the dimensional reduction introduces an additional scalar field that becomes another source of curvature and, moreover, even the equations governing the "electromagnetic" field differ from Maxwell equations. Einstein equations retain their usual form but the source terms contain the scalar field as well now. It is of interest that the scalar field does not share the symmetry of the gravitational field it helps to produce. Asymptotically, this is not the ECS although in 5D this is the case. In fact, the dimensionally reduced solution is asymptotically flat far away from the axis while, in contrast, the scalar field diverges.

## ACKNOWLEDGMENTS

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### Appendix A: Ernst potential approach

Following[9], we now show that their approach gives asymptotics consistent with our solution. The method is based on Ernst equation because any cylindrically symmetric solution is automatically axially symmetric. The original paper superposed an infinite number of Schwarzschild solutions but we need to use a different potential—here, we add Reissner-Nordström black holes, adopting the method of[12] who, however, only had a finite number of black holes. For the metric, we can write in Weyl coordinates

$$\mathrm{d}s^2 = f^{-1}\left[Q(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\mathrm{d}\phi^2\right] - f\mathrm{d}t^2, \tag{A1}$$

with $f$ and $Q$ functions of $\rho$ and $z$. To linearize the ensuing equation for $f$, one transforms the potential

$$f = \frac{R^2 - d^2}{(R+m)^2}, \tag{A2}$$

defining a new function $R$, the mass of the black hole $m$, and a constant $d^2 := m^2 - e^2$, where $e$ is the charge of the black hole. The last step is to define

$$R = d\frac{1+\bar{f}}{1-\bar{f}}. \tag{A3}$$

The resulting equation for $\bar{f}$ is linear and we can thus add solutions. The potential corresponding to the Reissner-Nordström black hole is

$$\bar{f} = \frac{\left(d+z-\sqrt{(d+z)^2+\rho^2}\right)\left(d-z-\sqrt{(d-z)^2+\rho^2}\right)}{\rho^2}. \tag{A4}$$

We superpose these solutions, shifting them by multiples of the lattice constant $k$ along the axis and define $\exp\omega_0 := \bar{f}$ in keeping with[9]:

$$\omega(z,\rho) = \omega_0(z,\rho) + \sum_{n=1}^{\infty}\left[\omega_0(z+nk,\rho)+\omega_0(z-nk,\rho)+\frac{4d}{nk}\right]. \tag{A5}$$

The sum converges according to the same arguments as per[9] and we now explore its asymptotic behavior, taking derivative of it with respect to the cylindrical radial coordinate. The leading-order term is

$$\frac{\partial\omega(z,\rho)}{\partial\rho^2} \sim \sum_{n=-\infty}^{\infty}\frac{d}{\left((z+kn)^2+\rho^2\right)^{3/2}}. \tag{A6}$$

We estimate the sum through an integral to obtain

$$\bar{f} = \rho^{\frac{4d}{k}}. \tag{A7}$$

Finally, expressing the original potential $f = -g_{tt} = 1/U^2$ through A3 and A2, we find

$$f = \frac{4d^2\rho^{\frac{4d}{k}}}{\left(-m\rho^{\frac{4d}{k}}+d\rho^{\frac{4d}{k}}+d+m\right)^2}. \tag{A8}$$

We now take the extreme-charge limit of $d \to 0$ and keep the lowest-order term

$$f = \left(\frac{k}{2m\log(\rho)}\right)^2, \tag{A9}$$

which yields $U \sim \pm 2(m/k)\log(\rho)$ in accordance with our conclusions below 14.


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