FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

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Stochastic Differential Equations with Gaussian Noise and Their Applications

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Study programme: Mathematics
Study branch: Probability, Mathematical Statistics and Econometrics

Prague 2020

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I would like to thank my supervisor, RNDr. Petr Čoupek, Ph.D., for the patient guidance, support and advice he has provided.

Název práce: Stochastické diferenciální rovnice s Gaussovským šumem a jejich aplikace

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Katedra: Katedra pravděpodobnosti a matematické statistiky

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Abstrakt: V diplomové práci je studován vícerozměrný frakcionální Brownův pohyb, který může mít různé hodnoty Hurstova parametru v různých složkách, pro tento proces je dokázána věta Girsanovova typu. Dále jsou ukázány dvě různé aplikace této věty na stochastické diferenciální rovnice řízené vícerozměrným frakcionálním Brownovým pohybem. Nejprve jsou nalezeny postačující podmínky pro existenci slabého řešení rovnic s driftem, který může být napsán jako součet regulární a neregulární části, a difúzním koeficientem, který závisí na čase a splňuje jisté podmínky zajištující jeho integrabilitu vzhledem k řídícímu procesu. Tyto výsledky jsou užity k důkazu existence slabého řešení rovnice popisující stochastický harmonický oscilátor. Věta Girsanovova typu je poté využita k nalezení maximálně věrohodného skalárního parametru, který se vyskytuje v driftu rovnice s aditivním šumem.

Klíčová slova: Stochastické diferenciální rovnice, Frakcionální Brownův pohyb, Girsanovova věta, Slabé řešení

Title: Stochastic Differential Equations with Gaussian Noise and Their Applications

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Abstract: In the thesis, multivariate fractional Brownian motions with possibly different Hurst indices in different coordinates are considered and a Girsanov-type theorem for these processes is shown. Two applications of this theorem to stochastic differential equations driven by multivariate fractional Brownian motions (SDEs) are given. Firstly, the existence of a weak solution to an SDE with a drift coefficient that can be written as a sum of a regular and a singular part and a diffusion coefficient that is dependent on time and satisfies suitable conditions is shown. The results are applied for the proof of existence of a weak solution of an equation describing stochastic harmonic oscillator. Secondly, the Girsanov-type theorem is used to find the maximum likelihood scalar estimator that appears in the drift of an SDE with additive noise.

Keywords: Stochastic Differential Equations, Fractional Brownian Motion, Girsanov Theorem, Weak Solution

Contents

Introduction

In the thesis, we study multivariate fractional Brownian motions and stochastic differential equation driven by them. A multivariate fractional Brownian motion is a generalization of an n-dimensional fractional Brownian motion; it is allowed that each of its components has a different value of the Hurst parameter. This provides higher flexibility as far as the regularity properties of the processes are concerned. For instance, multivariate fractional Brownian motions are used to describe observation of a brain at low frequencies by functional Magnetic Resonance Imaging; see [3], [2]. They are also used to describe structural properties, such as deformability, stacking energy, position preference and propeller twist, of the Escherichia coli chromosome; and returns and volatility of the DAX index; see [5]. Basic properties and general context of multivariate fractional Brownian motions are presented in [4].

Let $\mathbb{H} = (H_1, H_2, \ldots, H_n)^\top \in (0,1)^n$. In the present work, we consider the multivariate fractional Brownian motion $B^{\mathbb{H}}$; that is the n-dimensional stochastic process whose components are fractional Brownian motions with Hurst parameters H_i and the components are independent. Moreover, we consider the stochastic differential equation

$$X_t = x_0 + \int_0^t b(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B^{\mathbb{H}}, \quad t \in [0,T], \tag{1}$$

where $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \to \mathcal{L}(\mathbb{R}^n)$ are Borel measurable functions.

Conditions under which the strong solution to equaution (1) exists are determined. Similar problems have been already treated in the literature. In particular, we refer to [12] or [30] (and references therein) where stochastic differential equations driven by fractional Brownian motion are considered. For the special case when the Hurst parameter is equal to one half (i.e. Brownian motion) we refer to [21].

Subsequentely, a Girsanov-type theorem for multivariate fractional Brownian motions is given. This theorem is the main tool in the thesis and it is a generalization of the Girsanov-type theorem for one-dimensional fractional Brownian motion which is discussed, e.g., in [10] or [32], and of the n-dimensional case which is discussed in [39].

Two applications of the Girsanov-type theorem are then given. Firstly, it is used to show the existence of a weak solution to the equation

$$X_t = x_0 + \int_0^t [b_1(r, X_r) + b_2(r, X_r)]\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B^{\mathbb{H}}_r, \quad t \in [0,T], \tag{2}$$

where $\sigma : [0,T] \to \mathcal{L}(\mathbb{R}^n)$ is a Borel measurable function with invertible values such that its integral with respect to $B^{\mathbb{H}}$ has a continuous version and b_1 and b_2 are two Borel measurable functions $[0,T] \times \mathbb{R}^n \to \mathbb{R}^n$. Moreover, function b_1 is assumed to be locally Lipschitz and of at most linear growth in the space variable, the map $[(t,x) \mapsto \sigma(t)^{-1} b_2(t,x)]$ is assumed to be of at most linear growth in those coordinates that correspond to the singular coordinates of $B^{\mathbb{H}}$, and Hölder continuous in both time and space variables in those components that correspond to the regular components of $B^{\mathbb{H}}$.

The topic of existence of a weak solution to stochastic differential equation driven by fractional Brownian motions has been already treated in many articles. For the case $H = \frac{1}{2}$, we refer to, e.g., [21]. Stochastic differential equations driven by a one-dimensional fractional Brownian motion are discussed, for example, in [11], [8] or [27] where σ is supposed to be equal to one, or we can refer to [24], [39], where σ is supposed to be a Borel function satisfying certain conditions. In fact, the article [39] is a main inspiration for the first part of the present thesis. Most of the results in the first part are a generalization of the article [39] and emphasis was put on the parts that are different for the $\mathbb{H}$-fractional Brownian motion. It is shown that the arguments [39] can be used even in the case of a multivariate fractional Brownian motion with different Hurst indices in different coordinates provided that the singular Hurst indices differ from the regular ones by at most one half.

The results are used to show the existence of a weak solution to the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_t + 2\gamma\frac{\mathrm{d}}{\mathrm{d}t}x_t + \omega^2 x_t = \rho(t)\frac{\mathrm{d}}{\mathrm{d}t}B^{\mathbb{H}}_t, \quad t \in [0, T], \tag{3}$$

where γ is an $n \times n$ real matrix, ω^2 is an $n \times n$ real positive semidefinite matrix and ρ is a function $\rho : [0, T] \to \mathcal{L}(\mathbb{R}^n)$ satisfying certain conditions. This equation is a generalized version of [17, Eq. 2].

Secondly, we apply the Girsanov-type theorem in the estimation of the drift parameter in equation

$$X_t = \theta \int_0^t b(r, X_r)\,\mathrm{d}r + B^{\mathbb{H}}_t, \quad t \geq 0. \tag{4}$$

We find the maximum likelihood estimate (MLE) for parameter θ based on a continuous observation of one trajectory of the solution of (4) and we give sufficient conditions for its strong consistency and asymptotic normality.

Statistical inference for stochastic diffrenetial equations driven by Brownian motion has been treated in many monographs; see, e.g., [23], [25]. On the other hand, the literature concerning inference for fractional diffusions is much more scarce. We refer, for example, to [10] and [28] where a MLE of the drift parameter of a fractional Brownian motion is studied; to [6], [9], and [20] where a MLE of the drift parameter of a fractional Ornstein-Uhlenbeck process is treated and to [40] where a MLE of θ in the general equation (4) with $b(t, x)$ not dependent on t is considered.

The thesis is divided into four sections. In the first section, we remind basic definitions and theorems from stochastic calculus, we also present fractional Brownian motion and its generalization - multivariate fractional Brownian motion with possibly different Hurst indices in different coordinates. We remind some basic concepts of fractional calculus. Then the Wiener integral with respect to multivariate fractional Brownian motion is introduced.

The second section is devoted to Girsanov theorem and the Girsanov-type theorem for multivariate fractional Brownian motions.

The third section is devoted to stochastic differential equations driven by multivariate fractional motions. Firstly, we present the conditions under which process defined as the Wiener integral with respect to multivariate fractional Brownian motion is continuous and subsequently, we find conditions that assure the existence of the strong solution to equation (1).

In the last section, two applications of the Girsanov-type theorem are given. Firstly, we show the existence of a weak solution to equation (2) and we give an example of such an equation. Lastly, we find the maximum likelihood estimate for the parameter θ in equation (4) and discuss its properties.

The thesis comprises and partially extends the following paper:

M. Camfrlová, P. Čoupek, *Applications of the Girsanov theorem for multivariate fractional Brownian motions*, submitted, 2020.

1. Preliminaries

1.1 Stochastic Process

In this section we recall some basic definitions and theorems in the field of stochastic processes.

We shortly remind the definition of a stochastic process and its measurability.

Definition 1 (Filtered Probability Space, Stochastic Process). *Let Ω be a non-empty set, $\mathcal{F}$ be a σ-algebra on Ω and P be a probability measure on $(\Omega, \mathcal{F})$ then the triplet $(\Omega, \mathcal{F}, \mathsf{P})$ is called a probability space.*

If $(\mathcal{F}_t)_{t \geq 0}$ is a filtration of $\mathcal{F}$ (that is a non-decreasing sequence of σ-algebras on Ω such that $\mathcal{F}_t \subset \mathcal{F}$ for every $t \geq 0$) then the probability space equipped with a filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is called a filtered probability space.

Let $I \neq \emptyset$, $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $(S, \mathcal{S})$ a measurable space. A family of random variables $X = \{X_t, t \in I\}$ defined on $(\Omega, \mathcal{F}, P)$ with values in S is called a stochastic process.

Definition 2 (Adapted Process). *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered space. We say that a stochastic process $\{X_t, t \geq 0\}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathsf{P})$ is $\mathcal{F}_t$-adapted if the random variable X_t is $\mathcal{F}_t$-measurable for every $t \geq 0$.*

Definition 3 (Gaussian Process). *We say that a stochastic process $X = \{X_t, t \in I\}$ is Gaussian if for every $n \in \mathbb{N}$ and for every $t_1, \ldots, t_n \in I$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ has a Gaussian distribution.*

Gaussian processes are determined by their mean and covariance functions. Conversely, we remind the theorem that ensures the existence of a stochastic process that is Gaussian with a certain mean and covariance function.

Theorem 1 (The Daniell-Kolmogorov Theorem for Gaussian Process). *Let $I \neq \emptyset$ and let $g : I^2 \to \mathbb{R}$ be a positive semidefinite function. Then there exists a probability space and a real, centered Gaussian process $X = \{X_t, t \in I\}$ defined on it such that the following equation holds:*

$$E\ X_t X_s = g(t, s), \quad t, s \in I$$

Proof. The proof can be found in e.g. [19, Theorem 2.4].

□

Next, we recall the Kolmogorov-Chentsov Theorem which ensures the existence of a continuous version of processes with certain properties.

Theorem 2 (The Kolmogorov-Chentsov Theorem). *Suppose that $(\Omega, \mathcal{F}, \mathsf{P})$ is a probability space and $\{X_t, t \in [0, \infty)\}$ is an n-dimensional stochastic process for some $n \in \mathbb{N}$ defined on $(\Omega, \mathcal{F}, \mathsf{P})$ which satisfies that for every $T \in (0, \infty)$ there exist α, β and $c > 0$ such that*

$$E\|X_t - X_s\|^\alpha \leq c|t - s|^{1+\beta}, \quad t, s \in [0, T],$$

Then there exists a continuous modification Y, which is locally Hölder continuous with exponent γ for every $\gamma \in (0, \frac{\beta}{\alpha})$, i.e., for every $T \in (0, \infty)$ it holds

$$\mathsf{P}\left[\omega : \sup_{\substack{0<t,s<T \\ 0<t-s<h_T(\omega)}} \frac{Y_t(\omega) - Y_s(\omega)}{|t-s|^\gamma} \leq \delta\right] = 1,$$

where $h_T(\omega)$ is an a.s. positive random variable and δ is an appropriate constant.

Proof. The proof can be found e.g. in [19, Theorem 2.8]

□

1.2 Stochastic Calculus for fBm

In this section, we recall the defintion of fractional Brownian motion and some of its properties. We also define $\mathbb{H}$-fractional Brownian motion which is an n-dimensional fractional Brownian motion with possibly different Hurst parameters in each component. An introduction to fractional Brownian motions can be found in, e.g., [7], [8], [26] and [1]

1.2.1 Definition and Properties of fBm

Firstly, we recall the definition of the standard Brownian motion.

Definition 4 (Brownian Motion)**.** *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered space, the process $\{W_t, t \geq 0\}$ is called an n-dimensional $\mathcal{F}_t$-Brownian motion if it is $\mathcal{F}_t$-adapted, continuous, $W_0 = 0$ a.s., and if for all $0 \leq s < t < \infty$ the increments $W_t - W_s$ are independent of $\mathcal{F}_s$ and have normal distribution with zero mean and covariance matrix $(t-s)\mathrm{Id}_n$ (Id_n denotes the identity matrix from $\mathbb{R}^n$ to $\mathbb{R}^n$).*

The existence of the Brownian motion is ensured as follows: the function defined on $[0, \infty)^2$ by

$$(t, s) \mapsto min(t, s), \quad t, s \geq 0$$

is positive-semidefinite and therefore there exist centered Gaussian processes $W^1, \ldots, W^n$ on $[0, \infty)$ with this covariance function by Theorem 1. Furthemore, these are continuous by Theorem 2. We may moreover assume that $W^1, \ldots, W^n$ are independent (by the standard argument of product probability spaces). Finally the process $W = (W^1, \ldots, W^n)^T$ satisfies the above definition with the filtration generated by W itself.

We continue with the definition of a fractional Brownian motion which is a generalization of the Brownian motion.

Definition 5 (Fractional Brownian motion)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $H \in (0, 1)$. The process $B^H = \{B_t^H,\ t \geq 0\}$ defined on this probability space that is centered Gaussian and that has the covariance function given by*

$$R^H(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad s, t \geq 0, \tag{1.1}$$

is called the fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$.

Clearly, an fBm is the Brownian motion if $H = \frac{1}{2}$. The covariance function of fBm defined by (1.1) is a positive semidefinite function and therefore, such process exists by Theorem 1. By Theorem 2 it can be assumed that the fBm has almost surely locally Hölder-continuous trajectories of order $H - \varepsilon$ for any $\varepsilon \in (0, H)$.

It can be shown that the increments of B^H are independent only if $H = \frac{1}{2}$ and that B^H is a semimartingale also for $H = \frac{1}{2}$ only. For other choices of H, the process B^H is not a semimartingale; see, e.g. [30, Section 1 and 2.1]. This leads to the need of a different definition of a stochastic integral with respect to the fractional Brownian motion than in the case of the standard Brownian motion.

Now, we generalize the concept of a fractional Brownian motion and define a multivariate $\mathbb{H}$-fractional Brownian motion.

Definition 6 ($\mathbb{H}$-fractional Brownian Motion)**.** *Let* $\mathbb{H} = (H_1, \ldots, H_n) \in (0,1)^n$ *and let* B^{H_i}, $i = 1, \ldots, n$, *be a fractional Brownian motion with Hurst parameter* H_i *and assume that* B^{H_i} *is independent of* B^{H_j} *whenever* $i \neq j$. *Let* B^{H_i}, $i = 1, \ldots, n$ *be defined on the same probability space. We define the multivariate* $\mathbb{H}$*-fractional Brownian motion (*$\mathbb{H}$*-fractional Brownian motion) as the stochastic process* $B^{\mathbb{H}} = (B^{H_1}, \ldots, B^{H_n})^T$.

Clearly, it holds that

$$\mathsf{E} B_s^{\mathbb{H}} (B_t^{\mathbb{H}})^\top = R^{\mathbb{H}}(s,t)$$

for every $(s,t) \in [0,\infty)^2$ where $R^{\mathbb{H}} : [0,\infty)^2 \to \mathcal{L}(\mathbb{R}^n)$ is defined by

$$R^{\mathbb{H}}(s,t) := \operatorname{diag}\{R^{H_i}(s,t)\}_{i=1}^n.$$

Such process surely exists by similar arguments as for Brownian motion.

1.2.2 Fractional Calculus

In this section we provide the definitons of the fractional Riemann-Liovuville integral and derivative which are useful to describe some integral operators associated with fractional Brownian motion.

This section is only devoted to the definitions and results that are to be used in the next sections. All definitions and results provided can be found in, e.g., [36].

Definition 7. *Let* $g \in L^1(a,b)$, *we define the fractional Riemann-Liovuville integrals of order* $\alpha > 0$ *for a.e.* $t \in (a,b)$ *as*

1) $I_{a+}^{\alpha} g(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s) \mathrm{d}s$,

2) $I_{b-}^{\alpha} g(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} g(s) \mathrm{d}s$.

The operators I_{a+}^{α} and I_{b-}^{α} are usually called the right-sided Riemann-Liouville fractional integral and left-sided Riemann-Liouville fractional integral respectively. The convergence of the integrals at the singularity is understood in the L^p-sense.

The operators I^{α}_{a+} and I^{α}_{b-} are extended to allow for $\alpha = 0$ by setting I^{0}_{a+} and I^{0}_{b-} to be the identity operator, this is not a standard notation but this notation will be useful in the following chapters.

We denote by $I^{\alpha}_{a+}(L^p(a,b))$ the image of $L^p(a,b),\ p \in [1,\infty)$ under the operator I^{α}_{a+} and similarly for I^{α}_{b-}.

We define the inverse operators $I^{-\alpha}_{a+}$ and $I^{-\alpha}_{b-}$.

Definition 8. *Let $-\infty < a < b < \infty$ and $g \in I^{\alpha}_{a+}(L^p(a,b))$, $p \in [1,\infty)$, $\alpha \in (0,1)$. We define the fractional Riemann-Liovuville derivatives of order α for a.e. $t \in (a,b)$ as:*

1. $I^{-\alpha}_{a+}g(t) := \frac{1}{\Gamma(1-\alpha)}\left(\frac{g(t)}{(t-a)^{\alpha}} + \alpha\int_a^t \frac{g(t)-g(s)}{(t-s)^{\alpha+1}}\mathrm{d}s\right)$,
2. $I^{-\alpha}_{b-}g(t) := \frac{1}{\Gamma(1-\alpha)}\left(\frac{g(t)}{(b-t)^{\alpha}} + \alpha\int_t^b \frac{g(t)-g(s)}{(s-t)^{\alpha+1}}\mathrm{d}s\right)$.

As in the case of fractional integrals, the operators $I^{-\alpha}_{a+}$ and $I^{-\alpha}_{b-}$ are usually called the right-sided Riemann-Liouville fractional derivative and the left-sided Riemann-Liouville fractional derivative respectively.

The operators I^{α}_{a+} and I^{α}_{b-} are extended to $\alpha = -1$ by setting I^{-1}_{a+} and I^{-1}_{b-} to be the first derivative in the L^p sense, as before this is not a standard notion.

Fractional derivatives are inverse of the fractional integrals and vice versa in the following sense.

For $\alpha \in (0,1)$ and $p \in [1,\infty)$ it holds

$$I^{-\alpha}_{a+}(I^{\alpha}_{a+}g) = g,\ g \in L^1(a,b)$$

$$I^{\alpha}_{a+}(I^{-\alpha}_{a+}g) = g,\ g \in I^{\alpha}_{a+}(L^p(a,b))$$

Analougous relation holds for the operators I^{α}_{b-} and $I^{-\alpha}_{b-}$.

It also holds that

$$I^{\alpha}_{a+}I^{\beta}_{a+} = I^{\alpha+\beta}_{a+},\ \alpha,\ \beta > 0.$$

Analogous relation holds for I^{α}_{b-}.

1.2.3 Integration for Multivariate fBms

This section is devoted to the construction of a Wiener integral with respect to the $\mathbb{H}$-fractional Brownian motion. We already mentioned that we cannot proceed as in the case of the standard Brownian motion when constructiong the integral because a fractional Brownian motion is a semimartingale only if $H = \frac{1}{2}$. For our purposes, however, it suffices to consider only deterministic integrands. The integral is constructed similarily as in [1].

Firstly, we recall the Gauss hypergeometric function ${}_2F_1(a,b,c,z)$ (for a systematic survey, see, e.g., [37, Chapter 1]). It is defined for any $a,\ b \in \mathbb{R}$, any $z \in \mathbb{R}$, $|z| < 1$, and any $c \neq 0, -1, \ldots$ by

$${}_2F_1(a,b,c,z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k$$

where $(a)_0 = 0$ and $(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)}$, where Γ is the Gamma function, i.e. $\Gamma(z) := \int_0^{\infty} x^{z-1}e^{-x}dx,\ \Re(z) > 0$. For complex arguments z with $|z| \geq 1$ it can be analytically continued along any path in the complex plane that avoids the point 1 and infinity.

The covariance function R^H can be described via a certain Volterra-type kernel. If $H = \frac{1}{2}$, define the kernel[1] $K_H : [0,T]^2 \to \mathbb{R}$ by

$$K_H(t,r) := \mathbf{1}_{(0,t)}(r).$$

If $H \neq \frac{1}{2}$, define the kernel $K_H : [0,T]^2 \to \mathbb{R}$ by

$$\begin{aligned} K_H(t,r) := & \frac{c_H}{\Gamma\left(H+\frac{1}{2}\right)}(t-r)^{H-\frac{1}{2}} \\ & \times {}_2F_1\left(H-\frac{1}{2}, \frac{1}{2}-H, H+\frac{1}{2}, 1-\frac{t}{r}\right) \end{aligned}$$

for $(t,r) \in [0,T]^2$ such that $0 < r < t \leq T$ and by $K_H(t,r) := 0$ otherwise. Here, the constant c_H is given by

$$c_H := \sqrt{\frac{\pi H(1-2H)}{\Gamma(2-2H)\cos(\pi H)}}.$$

It is well known that the following equality holds true

$$R^H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)\mathrm{d}u \tag{1.2}$$

for every $(s,t) \in [0,T]^2$; see [10, Lemma 3.1]. In [1], we can find a more convenient expression for the kernel K_H.

If $H < \frac{1}{2}$ then the kernel can be expressed as folows

$$K_H(t,s) = \tilde{c}_H\left[(\frac{t}{s})^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}\mathrm{d}u\right]\mathbf{1}_{[t>s]},$$

where the constant c_H is given by

$$\tilde{c}_H := \sqrt{\frac{2H}{(1-2H)\mathrm{B}(1-2H, H+\frac{1}{2})}}$$

with $\mathrm{B}(a,b) := \int_0^1 u^{a-1}(1-u)^{b-1}\mathrm{d}u,\ a,b > 0$, being the Beta function.

If $H < \frac{1}{2}$ then the kernel can be expressed as folows

$$K_H(t,s) = c_H^* \int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}\mathrm{d}u\mathbf{1}_{[t>s]},$$

where

$$c_H^* := \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}.$$

[1]Where $\mathbf{1}_{(0,t)}$ denotes the indicator function on the interval $(0,t)$.

Next, we define

$$K_{\mathbb{H}}(t,s) := \operatorname{diag}\left\{K_{H_i}(t,s)\right\}_{i=1}^{n} = \begin{bmatrix} K_{H_1}(t,s) & & 0 \\ & \ddots & \\ 0 & & K_{H_n}(t,s) \end{bmatrix}.$$

It is clear that the covariance matrix of an $\mathbb{H}$-fractional Brownian motion can be expressed similarly as in (1.2). In particular, it holds that

$$R^{\mathbb{H}}(t,s) = \int_0^{t\wedge s} K_{\mathbb{H}}(t,u)K_{\mathbb{H}}(s,u)\mathrm{d}u$$

for every $(s,t) \in [0,T]^2$.

Now we can proceed to the definition of the integral with respect to the $\mathbb{H}$-fractional Brownian motion for simple functions.

Let $m \in \mathbb{N}$ and denote by $\mathcal{E}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))$ (the symbol $\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$ denotes the space of $m \times n$ real matrices identified with linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$) the space of simple functions on the interval $[0,T]$ with values in the space $\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$, i.e. every $\psi \in \mathcal{E}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))$ is of the form

$$\psi = \sum_{i=0}^{N-1} A_i \mathbf{1}_{[t_i,t_{i+1})} \tag{1.3}$$

for some $N \in \mathbb{N}$, some partition $\{t_i\}_{i=0}^N$ of the interval $[0,T]$ such that $0 = t_0 < t_1 < \ldots < t_N = T$, and some set $\{A_i\}_{i=0}^{N-1} \subset \mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$ (if $m = n = 1$, we simply write $\mathcal{E}(0,T)$). For a step function ψ that is represented by (1.3), the Wiener integral with respect to the multivariate $\mathbb{H}$-fractional Brownian motion $B^{\mathbb{H}}$ is defined by

$$I_T(\psi) := \sum_{i=0}^{N-1} A_i(B^{\mathbb{H}}_{t_{i+1}} - B^{\mathbb{H}}_{t_i}). \tag{1.4}$$

In what follows, we extend I_T from the space of simple functions to a larger space of admissible integrands. For that we need to define the linear operator

$$\partial K^*_{\mathbb{H}} : \mathcal{E}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)) \to L^2(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))$$

by

$$(\partial K^*_{\mathbb{H}}\psi)(s) := \psi(s)K_{\mathbb{H}}(T,s) + \int_s^T [\psi(r) - \psi(s)](\partial_1 K_{\mathbb{H}})(r,s)\,\mathrm{d}r.$$

Here, $\partial_1 K_{\mathbb{H}} = \operatorname{diag}\left\{\partial_1 K_{H_i}\right\}_{i=1}^n$ where $\partial_1 K_{H_i}$ denotes the partial derivative of K_{H_i} in the first variable. In one-dimensional case we simply write ∂K^*_H for $H \in (0,1)$. The following theorem is crucial for the construction of the integral.

Theorem 3. *Let $\psi, \phi \in \mathcal{E}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))$. Then there is the following equality:*

$$\begin{aligned} &\mathrm{E}\langle \int_0^T \psi(t)\mathrm{d}B^{\mathbb{H}}_t, \int_0^T \phi(t)\mathrm{d}B^{\mathbb{H}}_t \rangle_{\mathbb{R}^m} \qquad (1.5)\\ &= \langle \partial K^*_{\mathbb{H}}(\psi), \partial K^*_{\mathbb{H}}(\phi) \rangle_{L^2([0,T],\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))} \\ &=: \langle \psi, \phi \rangle_{\mathcal{D}^{\mathbb{H}}(0,T,\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m))}. \end{aligned}$$

Proof. For the simplicity of the proof we will assume that $n = m$. To prove (1.5), we use the fact that

$$(\partial K_H^* \mathbf{1}_{[0,t]})(s) = K_H(t,s)\mathbf{1}_{[0,t]}(s), \tag{1.6}$$

for $0 \le s < t \le T$.

Let ψ, ϕ be of the form

$$\psi = \sum_{k=0}^{N-1} a_k \mathbf{1}_{[t_k, t_{k+1})}$$

and

$$\phi = \sum_{j=0}^{N-1} b_j \mathbf{1}_{[t_j, t_{j+1})},$$

for some $N \in \mathbb{N}$, some partition $\{t_i\}_{i=0}^N$ of the interval $[0,T]$ such that $0 = t_0 < t_1 < \ldots < t_N = T$ and some sets $\{a_i\}_{i=0}^{N-1}, \{b_j\}_{j=0}^{N-1} \subset \mathcal{L}(\mathbb{R}^n)$. Denote $A = (A^{il})_{i,l=1}^n$, where $A \in \mathcal{L}(\mathbb{R}^n)$. We may write

$$\begin{aligned}
\mathsf{E}\left\langle \int_0^T \psi \,\mathrm{d}B_t^{\mathbb{H}}, \int_0^T \phi(t)\,\mathrm{d}B_t^{\mathbb{H}} \right\rangle_{\mathbb{R}^n} \\
&= \mathsf{E}\left\langle \sum_{k=0}^{N-1} a_k (B_{t_{k+1}}^{\mathbb{H}} - B_{t_k}^{\mathbb{H}}), \sum_{j=0}^{N-1} b_j (B_{t_{j+1}}^{\mathbb{H}} - B_{t_j}^{\mathbb{H}}) \right\rangle_{\mathbb{R}^n} \\
&= \sum_{i,l,m=1}^{n} \sum_{k,j=0}^{N-1} \mathsf{E}\left[a_k^{il} (B_{t_{k+1}}^{H_l} - B_{t_k}^{H_l}) b_j^{im} (B_{t_{j+1}}^{H_m} - B_{t_j}^{H_m}) \right] \\
&= \sum_{i,l=1}^{n} \sum_{k,j=0}^{N-1} a_k^{il} b_j^{im} \Big[R^{H_l}(t_{k+1}, t_{j+1}) - R^{H_l}(t_{k+1}, t_j) \\
&\qquad\qquad - R^{H_l}(t_k, t_{j+1}) + R^{H_l}(t_k, t_j) \Big] \\
&=: (*)
\end{aligned}$$

by using the linearity of the inner product, the fact that every fractional Brownian motion is centered and uncorrelatedness of B^{H_m} and B^{H_l} whenever $l \neq m$. Moreover, by using equations (1.6) and (1.2) we obtain:

$$\begin{aligned}
(*) = \sum_{i,l=1}^{n} \sum_{k,j=0}^{N-1} \Bigg(a_k^{il} b_j^{il} \int_0^T \Big[&\partial K_{H_l}^*(\mathbf{1}_{[0,t_{k+1})})(r) \partial K_{H_l}^*(\mathbf{1}_{[0,t_{j+1})})(r) \\
&- \partial K_{H_l}^*(\mathbf{1}_{[0,t_{k+1})})(r) \partial K_{H_l}^*(\mathbf{1}_{[0,t_j)})(r) \\
&- \partial K_{H_l}^*(\mathbf{1}_{[0,t_k)})(r) \partial K_{H_l}^*(\mathbf{1}_{[0,t_{j+1})})(r) \\
&+ \partial K_{H_l}^*(\mathbf{1}_{[0,t_k)})(r) \partial K_{H_l}^*(\mathbf{1}_{[0,t_j)})(r)) \Big] \,\mathrm{d}r \Bigg)
\end{aligned}$$

$$= \sum_{i,l=1}^{n} \int_0^T \Bigg[\Bigg(\sum_{k=0}^{N-1} a_k^{il} \left[\partial K_{H_l}^*(\mathbf{1}_{[0,t_{k+1})})(r) - \partial K_{H_l}^*(\mathbf{1}_{[0,t_k)}))(r) \right] \Bigg) \times \Bigg(\sum_{j=0}^{N-1} b_j^{il} \left[\partial K_{H_l}^*(\mathbf{1}_{[0,t_{j+1})})(r) - \partial K_{H_l}^*(\mathbf{1}_{[0,t_{kj})})(r) \right] \Bigg) \Bigg] \,\mathrm{d}r$$

$$=: (**).$$

We finish the proof by using the linearity of ∂K_H^*

$$(**) = \sum_{i,l=1}^{n} \int_0^T \partial K_{H_l}^*(\psi_{il}) \partial K_{H_l}^*(\phi_{il}) \,\mathrm{d}r$$

$$= \langle \partial K_{\mathbb{H}}^*(\psi), \partial K_{\mathbb{H}}^*(\phi) \rangle_{L^2(0,T;\mathcal{L}(\mathbb{R}^n))}$$

$$=: \langle \psi, \phi \rangle_{\mathcal{D}^{\mathbb{H}}(0,T,\mathcal{L}(\mathbb{R}^n))}.$$

□

For $H \in (0,1)$, the operator $\partial K_H^* : \mathcal{E}(0,T) \to L^2(0,T)$ is injective and it can be described by the fractional operators defined in section 1.2.2. In particular, it follows that for $\psi \in \mathcal{E}(0,T)$, $\partial K_H^* \psi$ is given by

$$\partial K_H^* \psi = c_H r^{\frac{1}{2}-H} I_{T-}^{H-\frac{1}{2}} r^{H-\frac{1}{2}} \psi,$$

and for $\psi \in (\partial K_H^*)(\mathcal{E}(0,T))$, the inverse $(\partial K_H^*)^{-1}\psi$ is given by

$$(\partial K_H^*)^{-1} \psi = c_H^{-1} r^{\frac{1}{2}-H} I_{T-}^{\frac{1}{2}-H} r^{H-\frac{1}{2}} \psi,$$

cf. [7, p. 30 and p. 36] and [1, section 8]. So it follows that

$$\langle \psi, \varphi \rangle_{\mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))} := \langle \partial K_{\mathbb{H}}^* \psi, \partial K_{\mathbb{H}}^* \varphi \rangle_{L^2(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))},$$

$\psi, \varphi \in \mathcal{E}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))$, defines an inner product on $\mathcal{E}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))$ (with $\|\cdot\|_{\mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))}$ being the induced norm). We obtain from equality (1.5) that

$$\|I_T(\psi)\|_{L^2(\Omega;\mathbb{R}^m)} = \|\psi\|_{\mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))}$$

holds for every $\psi \in \mathcal{E}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))$ and therefore, the operator

$$I_T : \mathcal{E}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)) \to L^2(\Omega;\mathbb{R}^m)$$

defined by formula (1.4) is a linear isometry. Denote by $\mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))$ the completion of $\mathcal{E}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))$ with respect to the norm $\|\cdot\|_{\mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))}$.

The operator I_T admits a unique extension to a linear isometry, denoted again by I_T, from $\mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))$ to a closed linear subspace of the space $L^2(\Omega;\mathbb{R}^m)$. For $\psi \in \mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))$, the m-dimensional random variable $I_T(\psi)$ is called the Wiener integral of f with respect to the process $B^{\mathbb{H}}$. We will use the following notation:

$$\int_0^T \psi(r) \,\mathrm{d}B_r^{\mathbb{H}} := I_T(\psi).$$

We also define for $\psi \in \mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))$, $0 \le s < t \le T$ the integral

$$\int_s^t \psi(r)\,\mathrm{d}B_r^{\mathbb{H}} := \int_0^T \psi(r)\mathbf{1}_{[s,t]}(r)\,\mathrm{d}B_r^{\mathbb{H}}.$$

There are some known continuous embeddings to the space of integrands in one-dimensional case which give us a better idea of what the domain for I_T looks like. Therefore, assume that n=1.

If $H \in (0, \frac{1}{2}]$, we may mention the continuous embedding

$$\mathcal{C}^{\delta}([0,T]) \hookrightarrow \mathcal{D}^H(0,T)$$

for any $\delta \in (\frac{1}{2} - H, 1)$.

If $H \in [\frac{1}{2}, 1)$, then there is the continuous embedding

$$L^{\frac{1}{H}}(0,T) \hookrightarrow \mathcal{D}^H(0,T);$$

see, e.g., [31, section 2.1] for the proofs of these claims.

The above described procedure is a generalization of the case when the integrator is a scalar H-fractional Brownian motion and we refer, for example, to the works [1], [33], or to the monograph [7] and the many references therein.

In what follows, let us mention that there is a one-to-one correspondence between an n-dimensional Brownian motion and a multivariate fractional Brownian motion.

Theorem 4. *Let $\mathbb{H} \in (0,1)^n$. If $\{W_t, t \in [0,T]\}$ is an n-dimensional Brownian motion, then the process $\{B_t^{\mathbb{H}}, t \in [0,T]\}$ defined by*

$$B_t^{\mathbb{H}} := \int_0^t K_{\mathbb{H}}(t,r)\,\mathrm{d}W_r, \quad t \in [0,T], \tag{1.7}$$

is a multivariate $\mathbb{H}$-fractional Brownian motion. On the other hand, if $B^{\mathbb{H}}$ is a multivariate $\mathbb{H}$-fractional Brownian motion, then it follows that the process $\{W_t, t \in [0,T]\}$ defined by

$$W_t := \int_0^t (\partial K_{\mathbb{H}}^*)^{-1}(\mathbf{1}_{[0,t]}\mathrm{Id}_n)(r)\,\mathrm{d}B_r^{\mathbb{H}}, \quad t \in [0,T]. \tag{1.8}$$

is an n-dimensional Brownian motion. Moreover, in both cases, their augmented generated filtrations coincide.

Proof. Clearly, process defined by (1.8) is centered and Gaussian. We notice that for $\varphi = (\varphi_{ji})_{j,i=1}^n$ it holds that

$$\partial K_{\mathbb{H}}^*(\varphi) = (\partial K_{H_i}^*(\varphi_{ji}))_{i,j=1}^n \tag{1.9}$$

and it also holds that

$$\partial K_H^*(0) \equiv 0. \tag{1.10}$$

The covariance matrix can be written as

$$E\ W_t W_s^T = E\left[\int_0^T (\partial K_{\mathbb{H}}^*)^{-1}(\mathbf{1}_{[0,t]}\mathrm{Id}_n)(u)\mathrm{d}B_u^{\mathbb{H}}\right]\left[\int_0^T (\partial K_{\mathbb{H}}^*)^{-1}(\mathbf{1}_{[0,s]}\mathrm{Id}_n)(u)\mathrm{d}B_u^{\mathbb{H}}\right]^T,$$

$t, s \in [0, T]$. By using (1.9) and (1.10), we notice that it is sufficient to calculate

$$E[\int_0^T (\partial K_{H_i}^*)^{-1}(\mathbf{1}_{[0,t]})(u)\mathrm{d}B_u^{H_i}][\int_0^T (\partial K_{H_i}^*)^{-1}(\mathbf{1}_{[0,s]})(u)\mathrm{d}B_u^{H_i}]^T = \langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{L^2(0,T)}$$
$$= \min(t, s).$$

And from (1.5) it follows that

$$E\ W_t W_s^T = \min(s, t)\mathrm{Id}_n.$$

The rest of the proof can be found in [10, Corollary 3.1, Remark 3.2, and Theorem 4.8], [13, Theorem 1], [32, formulas (5) and (6)], and [11, formulas (3) and (4)].

□

2. Girsanov Theorem

In this chapter, a Girsanov-type theorem for multivariate $\mathbb{H}$-fractional Brownian motions is given.

Let $\mathbb{H} \in (0,1)^n$ be fixed in this subsection and let $B^{\mathbb{H}}$ be the $\mathbb{H}$-fractional Brownian motion. We recall that $\{W_t, t \in [0,T]\}$ denotes the process defined by (1.8).

Before we proceed we need to define an integral operator associated with the kernel K_H. Define $L_H : L^2(0,T) \longrightarrow I_{0+}^{H+\frac{1}{2}}(L^2(0,T))$ for $\varphi \in L^2(0,T)$ by

$$(L_H\varphi)(t) := \int_0^t K_H(t,s)\varphi(s)\mathrm{d}s$$

We show its image is indeed $I_{0+}^{H+\frac{1}{2}}(L^2(0,T))$ and it is bijective onto this space.

Lemma 5. *The operator $L_H : L^2(0,T) \longrightarrow I_{0+}^{H+\frac{1}{2}}(L^2(0,T))$ is bijective.*

Proof. Firstly, by Theorem from [36, Theorem 10.4] we have

$$L_H(L^2(0,T)) = I_{0+}^{H+\frac{1}{2}}(L^2(0,T))$$

. Hence, it remains to show the injectivity of L_H. It is sufficient to show that there exists the inverse. From [11, formulas (5),(6), and (9),(10)] we have that for $\varphi \in I_{0+}^{H+\frac{1}{2}}(L^2([0,T],\mathbb{R})$ the operator L_H^{-1} is of the form

$$L_H^{-1}\varphi = \begin{cases} c_H^{-1} r^{\frac{1}{2}-H} I_{0+}^{H-\frac{1}{2}} r^{H-\frac{1}{2}} I_{0+}^{-2H}\varphi, & H \in (0,\frac{1}{2}), \\ c_H^{-1} I_{0+}^{-1}\varphi, & H = \frac{1}{2}, \\ c_H^{-1} r^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} r^{\frac{1}{2}-H} I_{0+}^{-1}\varphi, & H \in (\frac{1}{2},1). \end{cases}$$

□

Remark. Note also that if the function φ is absolutely continuous, then it can also be compututed for $H \in (0,\frac{1}{2})$, $L_{\mathbb{H}}^{-1}\varphi$ as

$$L_H^{-1}\varphi = c_H^{-1} r^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} r^{\frac{1}{2}-H} I_{0+}^{-1}\varphi;$$

see [11, formula (11)].

From Theorem [36, Theorem 10.4] we also get another expression for the operator L_H

$$L_H\varphi = \begin{cases} c_H I_{0+}^{2H} r^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} r^{H-\frac{1}{2}}\varphi, & H \in (0,\frac{1}{2}), \\ c_H I_{0+}^{1}\varphi, & H = \frac{1}{2}, \\ c_H I_{0+}^{1} r^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} r^{\frac{1}{2}-H}\varphi, & H \in (\frac{1}{2},1). \end{cases}$$

The multivariate extension of the operator L_H is defined for $\mathbb{H} \in (0,1)^n$ and $\varphi \in L^2(0,T;\mathbb{R}^n)$ by

$$L_{\mathbb{H}}\varphi := \mathrm{diag}\,\{L_{H_i}\}_{i=1}^n\varphi. \tag{2.1}$$

In order to describe the operator we set for $\mathbb{H} \in (0,1)^n$ and $\varphi \in L^2(0,T;\mathbb{R}^n)$

$$I_{0+}^{\mathbb{H}+\frac{1}{2}}\varphi := \operatorname{diag}\{I_{0+}^{H_i+\frac{1}{2}}\}_{i=1}^n\varphi.$$

and it follows by Lemma 5 that the operator $L_{\mathbb{H}}$ defined by (2.1) is bijective from the space $L^2(0,T;\mathbb{R}^n)$ onto the space $I_{0+}^{\mathbb{H}+\frac{1}{2}}(L^2(0,T;\mathbb{R}^n))$. Moreover, its inverse is given for $\varphi \in I_{0+}^{\mathbb{H}+\frac{1}{2}}(L^2(0,T;\mathbb{R}^n))$ by

$$L_{\mathbb{H}}^{-1}\varphi = \operatorname{diag}\{L_{H_i}^{-1}\}_{i=1}^n\varphi.$$

Before we formulate the Girsanov theorem for the $\mathbb{H}$-fractional Brownian motion we recall the Girsanov theorem for the standard Brownian motion.

Theorem 6 (Girsanov theorem)**.** *Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathsf{P})$ be a filtered probability space, where $0<T<\infty$. Let $\{B_t, t\in[0,T]\}$ be an $\mathcal{F}_t$-Brownian motion under the measure P. Let $\{X_t, t\in[0,T]\}$ be an adapted process to the filtration generated by B satisfying $\int_0^T X_t^2 \mathrm{d}t < \infty$ a.s. and define*

$$\Lambda_T := \exp\left(\int_0^T X_t \mathrm{d}B_t - \frac{1}{2}\int_0^T X_t^2 \mathrm{d}t\right). \tag{2.2}$$

Suppose that $E[\Lambda_T]=1$ and define a measure Q on $(\Omega,\mathcal{F})$ by $\frac{dQ}{dP} := \Lambda_T$. Then under the measure Q the process $\{\tilde{B}_t, t\in[0,T\}]$ defined by

$$\tilde{B}_t := B_t - \int_0^T X_s \mathrm{d}s \tag{2.3}$$

is an $\mathcal{F}_t$-Brownian motion.

Proof. The proof can be found, for example, in [16, Section 2].

□

We can finally proceed to the Girsanov Theorem for $\mathbb{H}$-fractional Brownian motions.

Theorem 7. *Let $\mathbb{H} \in (0,1)^n$. Let $\{B_t^{\mathbb{H}}, t\in[0,T]\}$ be an $\mathbb{H}$-fractional Brownian motion defined on some probability space $(\Omega,\mathcal{F},\mathsf{P})$ and let $\{W_t, t\in[0,T]\}$ be the Brownian motion defined by formula (1.8). Let $\{u_t, t\in[0,T]\}$ be an n-dimensional stochastic process adapted to the filtration generated by $B^{\mathbb{H}}$ such that*

$$u \in L^1(0,T;\mathbb{R}^n) \quad \textit{and} \quad \int_0^{\cdot} u_r \,\mathrm{d}r \in I_{0+}^{\mathbb{H}+\frac{1}{2}}\left(L^2(0,T;\mathbb{R}^n)\right)$$

are satisfied P-almost surely. Define the process $\{v_t, t\in[0,T]\}$ by

$$v_t := L_{\mathbb{H}}^{-1}\left(\int_0^{\cdot} u_r \,\mathrm{d}r\right)(t), \quad t\in[0,T],$$

and the random variable $\mathcal{E}_T$ by

$$\mathcal{E}_T := \exp\left\{\int_0^T v_r^\top \,\mathrm{d}W_r - \frac{1}{2}\int_0^T \|v_r\|_{\mathbb{R}^n}^2 \,\mathrm{d}r\right\}.$$

If $\mathsf{E}\mathcal{E}_T = 1$*, then the process* $(\tilde{B}_t^{\mathbb{H}})_{t\in[0,T]}$ *defined by*

$$\tilde{B}_t^{\mathbb{H}} := B_t^{\mathbb{H}} - \int_0^t u_r \,\mathrm{d}r, \quad t \in [0,T],$$

is a multivariate $\mathbb{H}$*-fractional Brownian motion under the probability measure* $\tilde{\mathsf{P}}$ *that is defined by*

$$\frac{\mathrm{d}\tilde{\mathsf{P}}}{\mathrm{d}\mathsf{P}} := \mathcal{E}_T.$$

and it is adapted to the filtration generated by $B^{\mathbb{H}}$*.*

Proof. We use the standard Girsanov theorem 6 for the adapted, square integrable process v. We obtain that the process $\tilde{W}_t = W_t - \int_0^t v_s \mathrm{d}s$ is a Brownian motion under the probability $\tilde{P}$. The desired result follows from the following equalities (that hold for every $t \in [0,T]$ $\tilde{\mathsf{P}}$-a.s.):

$$\begin{aligned}
\tilde{B}_t^{\mathbb{H}} &= B_t^{\mathbb{H}} - \int_0^t u_r \,\mathrm{d}r \\
&= \int_0^t K_{\mathbb{H}}(t,r)\,\mathrm{d}W_r - \int_0^t K_{\mathbb{H}}(t,r) v_r \,\mathrm{d}r \\
&= \int_0^t K_{\mathbb{H}}(t,r)\,\mathrm{d}\tilde{W}_r.
\end{aligned}$$

□

3. Stochastic Differential Equations

In this chapter, we find sufficient conditions for the existence of a stochastic process $\{X_t, t \in [0,T]\}$ defined on a given probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with continuous sample paths that satisfies the equation

$$X_t = x_0 + \int_0^t b(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B_r^{\mathbb{H}} \tag{3.1}$$

for every $t \in [0,T]$ P-almost surely, where $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \to \mathcal{L}(\mathbb{R}^n)$ are Borel measurable functions and $x_0 \in \mathbb{R}^n$.

3.1 Continuity of the Wiener integral

For the existence of the solution of (3.1) we need to show that, under certain conditions, the process $\{Z_t, t \in [0,T]\}$ defined by

$$Z_t := \int_0^t \sigma(u)\,\mathrm{d}B_u^{\mathbb{H}}, \quad t \in [0,T], \tag{3.2}$$

has continuous sample paths so that we are able to use the standard Picard iteration scheme.

Let us fix the following notation. We will write $f \in S^{H+}$ with $H \in (0,1)$ if there exists $\delta > 0$ such that $f \in S^{H+\delta}$ where

$$S^{H+\delta} := \begin{cases} \mathcal{C}^{\frac{1}{2}-H+\delta}([0,T];\mathbb{R}^n), & H \in (0,\frac{1}{2}), \\ L^{\frac{1}{H}+\delta}(0,T;\mathbb{R}^n), & H \in [\frac{1}{2},1), \end{cases}$$

Similarly, we will write $f \in S^{\mathbb{H}+}$ for $\mathbb{H} = (H_1, H_2, \ldots, H_n)^\top \in (0,1)^n$ if there exists $\delta = (\delta_1, \delta_2, \ldots, \delta_n)^\top \in (0,\infty)^n$ such that $f \in \bigtimes_{i=1}^n S^{H_i+\delta_i}$.

For the purposes of the forthcoming Proposition 10 let us recall the following theorem.

Theorem 8. *Let X be an n-dimensional random variable which is normally distributed with zero mean. For every $k \in \mathbb{N}$, there exists a finite positive constant $c(k)$ such that*

$$E\|X\|^{2k} \le c(k)(E\,\|X\|^2)^k$$

holds. In particular, $c(k)$ is given by $c(k) = (2k-1)^k$.

Proof. The proof can be found in [29, Corollary 2.8.14].

□

We will also use the following lemma.

Lemma 9. *Let $s < t \in [0,T]$ and let $\sigma = (\sigma_{\cdot 1}, \sigma_{\cdot 2}, \ldots, \sigma_{\cdot n}) \in \mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n))$. The integrals $\int_s^t \sigma_{\cdot k}(r)\,\mathrm{d}B_r^{H_k}$ and $\int_s^t \sigma_{\cdot l}(r)\,\mathrm{d}B_r^{H_l}$ are uncorrelated $\mathbb{R}^n$-valued random variables whenever $k \neq l$, $k,l \in \{0,\ldots,n\}$.*

Proof. Denote $\sigma_{\cdot k} = (\sigma_{1k}, \sigma_{2k}, \dots, \sigma_{nk})^T$ for $k \in \{1, \dots, n\}$. We want to show that

$$\mathsf{E}\left(\int_s^t \sigma_{\cdot k}(r)\,\mathrm{d}B_r^{H_k}\right)\left(\int_s^t \sigma_{\cdot l}(r)\,\mathrm{d}B_r^{H_l}\right)^T = 0_{n\times n}$$

whenever $k \neq l$. For $\sigma = \mathbf{1}_{[s,t]}\mathrm{Id}_n$ the claim follows directly from the independence of B^{H_k} and B^{H_l} for $k \neq l$. For a simple function $\sigma\mathbf{1}_{[s,t]} \in \mathcal{E}(0,T;\mathcal{L}(\mathbb{R}^n))$ the claim follows from the linearity of the expected value. For $\sigma \in \mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n))$, we can find approximative functions. Take $j, m \in \{1, \dots, n\}$. Find sequences $(\sigma_{mk}^N)_{N=1}^\infty$ and $(\sigma_{jl}^N)_{N=1}^\infty \in \mathcal{E}(0,T)$ such that they approximate the functions σ_{jl} and σ_{mk}, i.e. it holds that $\|\sigma_{jl}^N - \sigma_{jl}\|_{\mathcal{D}^{H_l}(0,T)} \to 0, N \to \infty$, and $\|\sigma_{mk}^N - \sigma_{mk}\|_{\mathcal{D}^{H_k}(0,T)} \to 0, N \to \infty$. Denote by σ_{mk}^0 a zero $n \times n$ matrix with the only non-zero element σ_{mk} in the m-th line and k-th column. Then it is clear that $\|(\sigma_{mk}^N)^0 - \sigma_{mk}^0\|_{\mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n))} \to 0, N \to \infty$. Then from the already proven and from (1.5) it holds that

$$\begin{aligned}
&\mathsf{E}\left(\int_s^t \sigma_{mk}(r)\,\mathrm{d}B_r^{H_k}\right)\left(\int_s^t \sigma_{jl}(r)\,\mathrm{d}B_r^{H_l}\right)\\
&\qquad = \mathsf{E}\langle \int_s^t \sigma_{mk}^0(r)\,\mathrm{d}B_r^{\mathbb{H}}, \int_s^t \sigma_{jl}^0(r)\,\mathrm{d}B_r^{\mathbb{H}}\rangle_{\mathbb{R}^n}\\
&\qquad = \langle \sigma_{mk}^0\mathbf{1}_{[s,t]}, \sigma_{jl}^0\mathbf{1}_{[s,t]}\rangle_{\mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n))}\\
&\qquad = \lim_{N\to\infty} \langle (\sigma_{mk}^N)^0\mathbf{1}_{[s,t]}, (\sigma_{jl}^N)^0\mathbf{1}_{[s,t]}\rangle_{\mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n))}\\
&\qquad = \lim_{N\to\infty} \mathsf{E}\left\langle \int_s^t (\sigma_{mk}^N)^0(r)\,\mathrm{d}B_r^{\mathbb{H}}, \int_s^t (\sigma_{jl}^N)^0(r)\,\mathrm{d}B_r^{\mathbb{H}}\right\rangle_{\mathbb{R}^n}\\
&\qquad = 0,
\end{aligned}$$

which completes the proof. □

Theorem 10. *If $\sigma \in S^{\mathbb{H}+}$, then the integral process $\{Z_t, t \in [0,T]\}$ defined by formula* (3.2) *has a version with continuous sample paths.*

Moreover, there exists Hölder bound $\nu > 0$ such that the process $\{Z_t, t \in [0,T]\}$ has a version with γ-Hölder continuous sample paths for every $\gamma \in (0,\nu)$.

Proof. Write $\sigma = (\sigma_{\cdot 1}, \sigma_{\cdot 2}, \dots, \sigma_{\cdot n})$ where $\sigma_{\cdot k}$ denotes the k-th column of σ for $k \in \{1, \dots, n\}$.

Firstly, we verify that the integral $\int_0^t \sigma(s)\mathrm{d}B_s^{\mathbb{H}}$ is well defined. To this end, we need to show that $\sigma \in \mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n))$. Let $k \in \{1, \dots, n\}$. Firstly, suppose that $H_k < \frac{1}{2}$. In this case we have by [31, section 2.1] the inclusion

$$\mathcal{C}^\gamma([0,T];\mathbb{R}^n) \subset \mathcal{D}^{H_j}(0,T;\mathbb{R}^n)$$

for $1 > \gamma > \frac{1}{2} - H_k$. In the case $H_k \geq \frac{1}{2}$ we have by [31, section 2.1] the following continuous embeddings:

$$L^{1/H_k+\delta}([0,T];\mathbb{R}^n) \subset L^{1/H_k}([0,T];\mathbb{R}^n) \subset \mathcal{D}^{H_k}(0,T;\mathbb{R}^n). \tag{3.3}$$

Since $\sigma \in S^{\mathbb{H}+}$ the process is Z is clearly well defined.

Next, we want to verify the Kolmogorov-Chentsov condition from Theorem 2:

$$\exists \alpha > 0, \beta > 0, K > 0, \forall t, s \in [0,T]\ \mathrm{E}\|Z_t - Z_s\|^\beta \leq K|t-s|^{1+\alpha}.$$

Let $0 \le s < t \le T$ be fixed. Since the coordinates of $B^{\mathbb{H}}$ are independent fractional Brownian motions and the columns of σ are deterministic functions, the integrals $\int_s^t \sigma_{\cdot k}(r)\,\mathrm{d}B_r^{H_k}$ and $\int_s^t \sigma_{\cdot l}(r)\,\mathrm{d}B_r^{H_l}$ are uncorrelated $\mathbb{R}^n$-valued random variables whenever $k \neq l$, see Lemma 9. Using this property we compute

$$\begin{aligned}\mathrm{E}\|Z_t - Z_s\|_{\mathbb{R}^n}^2 &= \mathrm{E}\left\|\int_s^t \sigma(u)\mathrm{d}B^{\mathbb{H}}\right\|_{\mathbb{R}^n}^2 \\ &= \sum_{k=1}^n \mathsf{E}\left\|\int_s^t \sigma_{\cdot k}(r)\,\mathrm{d}B_r^{H_k}\right\|_{\mathbb{R}^n}^2\end{aligned}$$

Firstly, suppose that $H_k < \frac{1}{2}$. By using the isometry (1.5) and using the same process as in [39, Proposition 3.1] we obtain the following inequality[1]

$$\mathsf{E}\left\|\int_s^t \sigma_{\cdot k}(r)\,\mathrm{d}B_r^{H_k}\right\|_{\mathbb{R}^n}^2 = \|\partial K_{H_k}^* \sigma_{\cdot k}\|_{L^2(s,t;\mathbb{R}^n)}^2 \preceq \|\sigma_{\cdot k}\|^2_{C^{\frac{1}{2}-H_k+\delta_k}([0,T];\mathbb{R}^n)} (t-s)^{2H_k}.$$

We find a similar inequality for $H_k \ge \frac{1}{2}$. For $r + s \le T$ denote $\sigma_{\cdot k}^s(r) := \sigma_{\cdot k}(r+s)$. By using (3.3) and the Hölder inequality we obtain

$$\begin{aligned}\mathsf{E}\left\|\int_s^t \sigma_{\cdot k}(r)\,\mathrm{d}B_r^{H_k}\right\|_{\mathbb{R}^n}^2 &= \mathsf{E}\left\|\int_0^{t-s} \sigma_{\cdot k}(r+s)\,\mathrm{d}B_r^{H_k}\right\|_{\mathbb{R}^n}^2 \\ &\le \mathsf{E}\left\|\int_0^{t-s} \sigma_{\cdot k}^s(r)\,\mathrm{d}B_r^{H_k}\right\|_{\mathbb{R}^n}^2 \\ &\preceq \|\sigma_{\cdot k}^s\|^2_{L^{\frac{1}{H_k}}(0,t-s;\mathbb{R}^n)} \\ &\preceq \|\sigma_{\cdot k}\|^2_{L^{\frac{1}{H_k}}(0,T;\mathbb{R}^n)} \\ &\preceq \|\sigma_{\cdot k}\|^2_{L^{\frac{1}{H_k}+\delta_k}(0,T;\mathbb{R}^n)} (t-s)^{\frac{2\delta_k H_k^2}{1+\delta_k H_k}}.\end{aligned}$$

We conclude that

$$\mathrm{E}\|Z_t - Z_s\|^2 \preceq (t-s)^{2\nu},$$

where $\nu := min_{k\in\{1,2,\dots,n\}} G_k$ and

$$G_k := \begin{cases} H_k, & H_k \in (0, \frac{1}{2}), \\ \frac{\delta_k H_k^2}{1+\delta_k H_k}, & H_k \in [\frac{1}{2}, 1). \end{cases}$$

Now, by using Theorem 8, we have for every $j \in \mathbb{N}$

$$\mathrm{E}\|Z_t - Z_s\|^{2j} \preceq (t-s)^{2j\nu}.$$

From the Kolmogorov-Chentsov Theorem 2, the process Z has Hölder continuous modification of order $\gamma < \frac{2j\nu-1}{2j}$ for every $j \in \mathbb{N}$. The proof is completed by taking $j \longrightarrow \infty$ which gives us that Z is Hölder continuous version of order γ for every $\gamma \in (0,\nu)$. □

[1] If there exists a constant C such that $A \le CB$ and the value of this constant is not important, we simply write $A \preceq B$ throughout the thesis.

From now on, if there exists a version of the process Z that has Hölder continuous trajectories, we will identify the process Z with this version.

Note also that Theorem 10 holds with obvious modifications for $\delta_k = \infty$. For example, if σ is such that $\sigma_{\cdot k} \in L^\infty(0, T; \mathbb{R}^n)$ for every k such that $H_k \in [\frac{1}{2}, 1)$, then $\nu = \min_k H_k$.

Also, let $m \in \mathbb{N}$. If σ is such that $\sigma_{\cdot k} \in L^m(0, T; \mathbb{R}^n)$ (i.e. $\delta_k = m - \frac{1}{H_k}$) for every k such that $H_k \in [\frac{1}{2}, 1)$, then from the proof of the Theorem 10 it can be seen that the Hölder bound for Z is $\nu = min_{k \in \{1,2,\ldots,n\}} G_k$ where

$$G_k := \begin{cases} H_k, & H_k \in (0, \frac{1}{2}), \\ H_k - \frac{1}{m}, & H_k \in [\frac{1}{2}, 1). \end{cases}$$

3.2 Strong Solution

Let us fix $\mathbb{H} \in (0,1)^n$ and a multivariate $\mathbb{H}$-fractional Brownian motion $B^{\mathbb{H}}$ defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ for this subsection. Let $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, T] \to \mathcal{L}(\mathbb{R}^n)$ be Borel measurable functions and let $x_0 \in \mathbb{R}^n$. Assume that $\sigma \in \mathcal{D}^{\mathbb{H}}(0, T; \mathcal{L}(\mathbb{R}^n))$.

Consider the following equation:

$$\begin{aligned} X_t &= x_0 + \int_0^t b(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B_r^{\mathbb{H}}, \quad t \in [0, T], \\ X_0 &= x_0. \end{aligned} \tag{3.4}$$

Let us recall the definition of a strong solution to a stochastic differential equation.

Definition 9. *Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $\mathbb{H} \in (0,1)^n$. Let $\{B_t^{\mathbb{H}}, t \in [0, T]\}$ be an $\mathbb{H}$-fractional Brownian motion defined on this space. We say that a continuous process $\{X_t, t \in [0, T]\}$ defined on $(\Omega, \mathcal{F}, \mathsf{P})$ which is adapted to the filtration generated by $B^{\mathbb{H}}$ is a strong solution to the equation* (3.4) *on the interval $[0, T]$ if it satisfies the equation* (3.4) *for every $t \in [0, T]$ P-a.s.*

We say that the solution is unique if for every two solutions $\{X_t, t \in [0, T]\}$ and $\{\tilde{X}_t, t \in [0, T]\}$ it holds that $\mathsf{P}\{X_t = \tilde{X}_t$ for every $t \in [0, T]\} = 1$.

Theorem 11. *Let $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a Borel measurable function that satisfies the following two conditions:*

(I) There exists a finite positive constant K_b such that for every $t \in [0, T]$ and every $x \in \mathbb{R}^n$ it holds that

$$\|b(t, x)\|_{\mathbb{R}^n} \leq K_b(1 + \|x\|_{\mathbb{R}^n}).$$

(II) For every $N \in \mathbb{N}$ there exists a finite positive constant K_N such that for every $t \in [0, T]$ and every $x, y \in \mathbb{R}^n$ that satisfy $\|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^n} \leq N$ it holds that

$$\|b(t, x) - b(t, y)\|_{\mathbb{R}^n} \leq K_N \|x - y\|_{\mathbb{R}^n}.$$

Assume also that $\sigma : [0,T] \to \mathcal{L}(R^n)$ *is a Borel measurable function that belongs to the space* $S^{\mathbb{H}+}$. *Then there exists a unique strong solution to equation* (3.4).

Moreover, the solution has γ*-Hölder continuous sample paths for every* $\gamma \in (0,\nu)$ *where* ν *is the Hölder bound for the integral process from the proof of the Theorem 10 and the estimate*

$$\|X\|_{\mathcal{C}^\gamma(0,T;\mathbb{R}^n)} \leq C(1 + \|Z\|_{\mathcal{C}^\gamma(0,T;\mathbb{R}^n)}), \tag{3.5}$$

where C *is a finite positive constant that depends on* $T,\ K_b,\ x_0,\ \gamma$, *holds* P*-a.s.*

Proof. From Theorem 10 it follows that the process Z, where $Z_t = \int_0^t \sigma(u)\,\mathrm{d}B_u^{\mathbb{H}}$, is continuous and therefore, we can use standard Picard iteration scheme to show that there exists a unique $\mathbb{R}^n$-valued continuous stochastic process $\{X_t, t \in [0,T]\}$ that satisfies the random differential equation

$$X_t = x_0 + \int_0^t b(r, X_r)\,\mathrm{d}r + Z_t. \tag{3.6}$$

The rest of the theorem can be proved exactly as in [39, Theorem 3.6]. $\square$

4. Applications of Girsanov Theorem

4.1 Weak Solution

Let $\mathbb{H} \in (0,1)^n$ be fixed in this subsection. Let also $b_1, b_2 : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \to \mathcal{L}(\mathbb{R}^n)$ be Borel measurable functions and $x_0 \in \mathbb{R}^n$. Assume that $\sigma \in \mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n))$. Denote $b := b_1 + b_2$. In this section, we will consider the following equation:

$$\bar{X}_t = \bar{x}_0 + \int_0^t b(s,\bar{X}_s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}\bar{B}^{\mathbb{H}}_s, \quad t \in [0,T]. \tag{4.1}$$

Definition 10. *We say that a triplet* $((\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathsf{P}),\bar{B}^{\mathbb{H}},\bar{X})$ *is a weak solution to* (4.1) *if* $(\Omega,\mathcal{F},\mathcal{F}_t,\mathsf{P})$ *is a filtered probability space,* $\bar{B}^{\mathbb{H}}$ *is an* $\mathbb{H}-$ *fBm that is* $\mathcal{F}_t$*-adapted,* $\bar{X}$ *is an* $\mathcal{F}_t$*-adapted continuous stochastic process and both* $\bar{B}^{\mathbb{H}}$ *and* $\bar{X}$ *are defined on* $(\Omega,\mathcal{F},\mathsf{P})$ *and satisfy equation* (4.1) *for every* $t \in [0,T]$ $\mathsf{P}-a.s.$

In the sequel we will make use of the following. For $\lambda \in (0,1]$, consider the space

$$\tilde{\mathcal{C}}^\lambda([0,T];\mathbb{R}^n) := \left\{ f \in \mathcal{C}^\lambda([0,T];\mathbb{R}^d) \,\middle|\, \forall \varepsilon > 0\, \exists \delta > 0 \right.$$
$$\left. \forall s,t \in (0,T), 0 < |t-s| < \delta \implies \frac{|f(t)-f(s)|}{|t-s|^\lambda} < \varepsilon \right\}$$

equipped with the norm $\|\cdot\|_{\mathcal{C}^\lambda([0,T];\mathbb{R}^n)}$. It follows that $\tilde{\mathcal{C}}^\lambda([0,T];\mathbb{R}^n)$ is separable (see [22, Theorem 1.4.11]) and there are the inclusions

$$\mathcal{C}^{\kappa_2}([0,T];\mathbb{R}^n) \subset \tilde{\mathcal{C}}^{\kappa_1}([0,T];\mathbb{R}^n) \subset \mathcal{C}^{\kappa_1}([0,T];\mathbb{R}^n)$$

whenever $0 < \kappa_1 < \kappa_2 \leq 1$ (see [22, Exercise 1.2.10 (ii)]). Therefore, if the assumptions of Theorem 10 are satisfied, the integral process $\{Z_t, t \in [0,T]\}$ can be viewed as a $\tilde{\mathcal{C}}^\gamma([0,T];\mathbb{R}^n)$-valued Gaussian random variable for any $\gamma \in (0,\nu)$ where ν is the Hölder bound for Z.

We will make use of the following theorem and lemma in the sequel.

Theorem 12 (The Fernique Theorem)**.** *Let Z be a Gaussian centered random variable and let Z have values in a separable Banach space $(U,\|.\|_U)$. Then there exists a finite positive constant K, such that*

$$\mathsf{E}\exp\{K\|Z\|_U^2\} < \infty. \tag{4.2}$$

Proof. The proof can be found in [14, Théorème d' integrabilité].

□

Lemma 13. *Let $H \in (\frac{1}{2},1)$ and $s > 0$. Then there exists a finite positive constant A_H such that*

$$\int_0^s \frac{|s^{\frac{1}{2}-H} - r^{\frac{1}{2}-H}|}{(s-r)^{\frac{1}{2}+H}}\,\mathrm{d}r \leq A_H s^{1-2H}.$$

Proof. For the proof see [38, Lemma 1.3]. □

To prove the existence of the weak solution to the equation (4.1) we will use the Girsanov-type Theorem 7. Initially, its application is given.

Theorem 14. *Let $\{B_t^{\mathbb{H}}, t \in [0,T]\}$ be a multivariate $\mathbb{H}$-fractional Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Assume that the function b_1 satisfies conditions (I) and (II). Assume also that the function σ belongs to the space $S^{\mathbb{H}+}$ and that for every $t \in [0,T]$, the matrix $\sigma(t)$ is invertible. Let $\nu > 0$ be the Hölder bound for the integral process Z that is defined by (3.2) and assume additionally that the functions b_2 and σ satisfy the following condition:*

(III) If, for $k \in \{1, 2, \ldots, n\}$, the parameter H_k belongs to $(0, \frac{1}{2}]$, then there exists a constant $K_k > 0$ such that for every $t \in [0,T]$ and every $x \in \mathbb{R}^n$ it holds that

$$|[\sigma(t)^{-1} b_2(t,x)]_k| \leq K_k(1 + \|x\|_{\mathbb{R}^n});$$

and if H_k belongs to $(\frac{1}{2}, 1)$, then $H_k < \nu + \frac{1}{2}$ and there exist constants $\alpha_k \in (H_k - \frac{1}{2}, 1]$, $\beta_k \in (\frac{(2H_k - 1)}{2\nu}, 1]$, and $K_k > 0$ such that for every $s, t \in [0,T]$ and every $x, y \in \mathbb{R}^n$ it holds that

$$\begin{aligned} |[\sigma(t)^{-1} b_2(t,x)]_k - [\sigma(s)^{-1} b_2(s,y)]_k| &\leq \\ &\leq K_k(|t-s|^{\alpha_k} + \|x-y\|_{\mathbb{R}^n}^{\beta_k}). \end{aligned}$$

Here, $[z]_k$ denotes the k-th component of $z \in \mathbb{R}^n$.

Denote by $\{X_t, t \in [0,T]\}$ the strong solution[1] *to the equation*

$$X_t = x_0 + \int_0^t b_1(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B_r^{\mathbb{H}}, \quad t \in [0,T].$$

Then the process $\{\tilde{B}_t^{\mathbb{H}}), t \in [0,T]\}$ given by

$$\tilde{B}_t^{\mathbb{H}} := B_t^{\mathbb{H}} - \int_0^t \sigma(r)^{-1} b_2(r, X_r)\,\mathrm{d}r, \quad t \in [0,T],$$

is a multivariate $\mathbb{H}$-fractional Brownian motion under the probability measure $\tilde{\mathsf{P}}$ that is defined by

$$\frac{\mathrm{d}\tilde{\mathsf{P}}}{\mathrm{d}\mathsf{P}} := \mathcal{E}_T := \exp\left\{\int_0^T v_r^\top\,\mathrm{d}W_r - \frac{1}{2}\int_0^T \|v_r\|_{\mathbb{R}^n}^2\,\mathrm{d}r\right\} \tag{4.3}$$

where $\{v_t, t \in [0,T]\}$ is the stochastic process given by

$$v_t := L_{\mathbb{H}}^{-1}\left(\int_0^\cdot \sigma(r)^{-1} b_2(r, X_r)\,\mathrm{d}r\right)(t), \quad t \in [0,T].$$

Proof. Let $\{u_t, t \in [0,T]\}$ be defined by $u_t := \sigma(t)^{-1} b_2(t, X_t)$ for $t \in [0,T]$. To prove the theorem we need to verify that the assumptions of Theorem 7 are satisfied, i.e. we need to show that the following conditions

$$\int_0^\cdot u_r\,\mathrm{d}r \in I_{0+}^{H+\frac{1}{2}}\left(L^2(0,T;\mathbb{R}^n)\right) \quad \text{and} \quad \mathsf{E}\mathcal{E}_T = 1$$

[1] The existence and uniqueness of such solution is ensured by Theorem 11.

hold P-almost surely. The first condition is equivalent to $v \in L^2(0,T;\mathbb{R}^n)$ P-almost surely. To prove the second condition it is sufficient to show that there exist $\Delta > 0$ and a partition $0 = t_0 < t_1 < \ldots < t_{N(\Delta)} = T$ of the interval $[0,T]$ whose mesh size is smaller than Δ such that

$$\mathsf{E}\exp\left\{\int_{t_i}^{t_{i+1}} \|v_r\|_{\mathbb{R}^n}^2 \,\mathrm{d}r\right\} < \infty$$

for every $i \in \{0,1,\ldots,N(\Delta)-1\}$. This is because this last condition implies by [15, Lemma 7.1.3] that

$$\mathsf{E}\left[\exp\left\{\int_{t_i}^{t_{i+1}} v_r^\top \,\mathrm{d}W_r - \frac{1}{2}\int_{t_i}^{t_{i+1}} \|v_r\|_{\mathbb{R}^n}^2 \,\mathrm{d}r\right\}\middle| \mathcal{F}_{t_i}^{B^{\mathbb{H}}}\right] = 1$$

holds P-almost surely for every $i \in \{0,1,\ldots,N(\Delta)-1\}$, where $(\mathcal{F}_t^{B^{\mathbb{H}}})_{t\in[0,T]}$ is the filtration generated by $B^{\mathbb{H}}$ and using the above equality iteratively yields $\mathsf{E}\mathcal{E}_T = 1$.

Step 1 We want to show that $v \in L^2(0,T;\mathbb{R}^n)$ P-almost surely. Let $0 \le s < t \le T$. We have by the definition of L_H^{-1} from the proof of Lemma 5 that

$$\begin{aligned}\int_s^t \|v_u\|_{\mathbb{R}^n}^2 \,\mathrm{d}u &= \sum_{k=1}^n \int_s^t \left| L_{H_k}^{-1}\left(\int_0^{\cdot} [\sigma(r)^{-1} b_2(r,X_r)]_k \,\mathrm{d}r\right)(u)\right|^2 \mathrm{d}u \\ &\preceq \sum_{k=1}^n \int_s^t \left| u^{H_k-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H_k}\left(r^{\frac{1}{2}-H_k}[\sigma(r)^{-1}b_2(r,X_r)]_k\right)(u)\right|^2 \mathrm{d}u\end{aligned}$$

where I_{0+}^{α} is the fractional operator defined in subchapter 1.2.2. Now, set

$$I_k(s,t) := \int_s^t \left| u^{H_k-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H_k}\left(r^{\frac{1}{2}-H_k}[\sigma(r)^{-1}b_2(r,X_r)]_k\right)(u)\right|^2 \mathrm{d}u \tag{4.4}$$

for $k \in \{1,2,\ldots,n\}$. We will proceed similarily as in [39, Theorem 4.2 and 4.3] where the detailed computation can be found. If $H_k \in (0,\frac{1}{2})$ then from the definition of the operator $I_{0+}^{\frac{1}{2}-H_k}$ and the assumption (III), we obtain the inequality

$$I_k(s,t) \le \frac{K_k^2(1+\|X\|_{L^\infty(0,T;\mathbb{R}^n)})^2}{\Gamma^2(\frac{1}{2}-H_k)} \mathrm{B}^2\left(\frac{3}{2}-H_k, \frac{1}{2}-H_k\right)\int_s^t u^{1-2H_k}\,\mathrm{d}u.$$

Next, we use estimate (3.5) and estimate $\int_s^t u^{1-2H_k}\,\mathrm{d}u \preceq (t-s)^{\frac{1}{2}}$. We obtain the following inequality

$$I_k(s,t) \le C_k^{(1)}(1+(t-s)^{\frac{1}{2}}\|Z\|_{\mathcal{C}^\gamma(0,T;\mathbb{R}^n)}^2), \tag{4.5}$$

where $C_k^{(1)}$ is a finite positive constant that depends on $\mathbb{H},\ T,\ K_b,\ K_k, x_0$ which holds for any $s,t \in [0,T],\ s<t$, and $\gamma \in (0,\nu)$.

On the other hand, if $H_k \in (\frac{1}{2},1)$, then by using the definition of the operator $I_{0+}^{\frac{1}{2}-H_k}$ and the triangle inequality we obtain that the inequality

$$I_k(s,t) \le \frac{2}{\Gamma^2(\frac{3}{2}-H_k)}(I_{1k}+I_{2k})$$

holds P-a.s. Here, I_{1k} and I_{2k} are given by

$$I_{1k} = \int_0^{t-s} \|(v+s)^{\frac{1}{2}-H_k}[\sigma^{-1}(v+s)b_2(v+s, X_{v+s})]_k\|^2 \, \mathrm{d}v$$

and

$$I_{2k} = \int_0^{t-s} \|(H_k - \frac{1}{2})(v+s)^{H_k-\frac{1}{2}} \int_0^{v+s} \frac{[\psi(v+s)]_k - [\psi(r)]_k}{(v+s-r)^{H_k+\frac{1}{2}}} \, \mathrm{d}r\|^2 \, \mathrm{d}v$$

with

$$\psi(r) = r^{\frac{1}{2}-H_k}[\sigma^{-1}(r)b_2(r, X_r)]_k.$$

Next, we suppose without loss of generality that

$$|[\sigma^{-1}(0)b_2(0, x_0)]_k| \leq K_k, \tag{4.6}$$

where K_k is defined in (III). Then, by estimate (4.6), estimate (3.5), assumption (III) and the Hölder continuity of X we obtain following inequality which holds P-a.s.:

$$I_{1K} \leq B_{1k}(1 + (t-s)^{2(1-H_k)}\|Z\|^{2\beta_k}_{\mathcal{C}^\gamma(0,T;\mathbb{R}^n)}), \tag{4.7}$$

where B_{1k} is a finite positive constant that depends on $\mathbb{H}$, K_k, T, γ, α_k, β_k, K_b, x_0. The inequality holds for every $\gamma \in (\frac{2H_k-1}{2\beta_k}, \nu)$.

We find a similar inequality for I_{2k}. By the triangle inequality, Lemma 13, estimate 4.6, assumption (III), the Hölder continuity of X and (3.5) we obtain the following inequality which holds P-a.s.:

$$I_{2K} \leq B_{2k}(1 + (t-s)^{2(1-H_k)}\|Z\|^{2\beta_k}_{\mathcal{C}^\gamma(0,T;\mathbb{R}^n)}) \tag{4.8}$$

where B_{2k} is a finite positive constant that depends on $\mathbb{H}$, K_k, T, γ, α_k, β_k, K_b, x_0, A_{H_k}.

Combining (4.7) and (4.8) the inequality

$$I_k(s,t) \leq C_k^{(2)}(1 + (t-s)^{2(1-H_k)}\|Z\|^{2\beta_k}_{\mathcal{C}^\gamma(0,T;\mathbb{R}^n)}), \tag{4.9}$$

where $C_k^{(1)}$ is a finite positive constant is obtained. The inequality holds for any $s, t \in [0,T]$, $s < t$ and $\gamma \in (\frac{2H_k-1}{2\beta_k}, \nu)$.

If $H_k = \frac{1}{2}$ the computation is straightforward. By using conditon (III) and estimate (3.5) we obtain:

$$\int_s^t \|v_u\|^2_{\mathbb{R}^n} \, \mathrm{d}u \leq C_k^{(3)}(1 + (t-s)\|Z\|^2_{\mathcal{C}^\gamma(0,T;\mathbb{R}^n)}),$$

for some $C_k^{(3)}$ and $\gamma \in (0, \nu)$.

Set $\beta_k := 1$ for any k such that $H_k \in (0, \frac{1}{2}]$ and denote

$$H_0 := \min\{\frac{1}{2}, \min_{k: H_k \in (\frac{1}{2}, 1)} 2(1-H_k)\} \quad \text{and} \quad \nu_0 := \max_{k: H_k \in (\frac{1}{2}, 1)} \frac{2H_k - 1}{2\beta_k}.$$

We see from inequalities (4.5) and (4.9) that for every $\gamma \in (\nu_0, \nu)$ the estimate

$$\int_s^t \|v_u\|^2_{\mathbb{R}^n} \, \mathrm{d}u \leq C_0 \left(1 + (t-s)^{H_0} \sum_{k=1}^n \|Z\|^{2\beta_k}_{\mathcal{C}^\gamma([0,T];\mathbb{R}^d)}\right).$$

holds with some finite positive constant C_0. It follows from this estimate that $v \in L^2(0,T;\mathbb{R}^n)$ P-almost surely by choosing $s=0$ and $t=T$.

Step 2 Let $\gamma \in (\nu_0, \nu)$. Assumptions of Theorem 10 are satisfied and therefore, the integral process $\{Z_t, t \in [0,T]\}$ can be viewed as a $\tilde{\mathcal{C}}^\gamma([0,T];\mathbb{R}^n)$-valued Gaussian random variable. Due to the separability of this space, we can use the Fernique Theorem 12. Hence, there exists a finite constant $K_0 > 0$ such that

$$\mathsf{E}\exp\{K_0 \|Z\|^2_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)}\} < \infty. \tag{4.10}$$

Let $\Delta > 0$ be such that $nC_0\Delta^{H_0} < K_0$ and let $0 = t_0 < t_1 < \ldots < t_{N(\Delta)} = T$ be a partition of the interval $[0,T]$ whose mesh size is smaller than Δ. For $i \in \{0,1,\ldots,N(\Delta)-1\}$, we have that

$$\begin{aligned}
&\mathsf{E}\exp\left\{\int_{t_i}^{t_{i+1}} \|v_u\|^2_{\mathbb{R}^n}\,\mathrm{d}u\right\} \leq \\
&\quad \leq \mathsf{E}\exp\left\{C_0\left(1+(t_{i+1}-t_i)^{H_0}\sum_{k=1}^n \|Z\|^{2\beta_k}_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)}\right)\right\}\\
&\quad = \mathrm{e}^{C_0}\mathsf{E}\exp\left\{\left(\sum_{k=1}^n \|Z\|^{2\beta_k}_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)}\right)\frac{K_0}{n}\right\}\mathbf{1}_{[\|Z\|_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)}\leq 1]}\\
&\qquad + \mathrm{e}^{C_0}\mathsf{E}\exp\left\{\left(\sum_{k=1}^n \|Z\|^{2\beta_k}_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)}\right)\frac{K_0}{n}\right\}\mathbf{1}_{[\|Z\|_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)}> 1]}\\
&\quad \leq \mathrm{e}^{K_0+C_0}\mathsf{P}(\|Z\|_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)} \leq 1)\\
&\qquad + \mathrm{e}^{C_0}\mathsf{E}\exp\left\{K_0\|Z\|^2_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)}\right\}\mathbf{1}_{[\|Z\|_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)}> 1]}
\end{aligned}$$

which is finite by (4.10). Thus, the claim is proved.

□

Remark. Assume that $\mathbb{H}$ contains at least one element larger than $\frac{1}{2}$ and one element smaller than $\frac{1}{2}$ and let $\sigma \in S^{\mathbb{H}+}$ be such that $\sigma_{\cdot k} \in L^\infty(0,T;\mathbb{R}^n)$ whenever k is such that $H_k \in (\frac{1}{2},1)$. Then it follows that $\nu = \min_{k:H_k\in(0,\frac{1}{2}]} H_k$. On the other hand, condition (III) in Theorem 14 says that ν has to be greater than $\max_{k:H_k\in(\frac{1}{2},1)} H_k - \frac{1}{2}$. Therefore, Theorem 14 can be applied if (besides the remaining conditions) the condition

$$\min_{k:H_k\in(0,\frac{1}{2}]} H_k > \max_{k:H_k\in(\frac{1}{2},1)} H_k - \frac{1}{2}$$

is satisfied. Roughly speaking, this means that the singular values of Hurst indices cannot differ from the regular values too much.

On the other hand, if we assume that $\mathbb{H}$ contains only values larger than $\frac{1}{2}$ and let $m \in \mathbb{N}$ and let $\sigma \in L^m(0,T;\mathcal{L}(\mathbb{R}^n))$ for some $m \in \mathbb{N}$, then it follows that $\nu = \min_{k\in\{1,\ldots,n\}} H_k - \frac{1}{m}$. Then Theorem 14 can be applied if (besides the remaining conditions) the condition

$$\min_{k\in\{1,\ldots,n\}} H_k - \frac{1}{m} > \max_{k\in\{1,\ldots,n\}} H_k - \frac{1}{2}$$

is satisfied. Therefore, the regular indices cannot differ by more than $\frac{1}{2} - \frac{1}{m}$. If $\sigma \in L^\infty(0,T;\mathcal{L}(\mathbb{R}^n))$, then $\nu = \min_{k\in\{1,\ldots,n\}} H_k$ and there is no restriction on the indeces.

If $\mathbb{H}$ consists of equal elements, the result corresponds with [39, Theorem 4.2 and 4.3]. In [24, Theorem 4.1], the case $n = 1$ and $H > \frac{1}{2}$ is studied and similar assumption is made on σ and a b_2 as in assumption (III) for $H < \frac{1}{2}$.

Before we proceed to the proof of the existence of a weak solution to equation (4.1) we formulate the following lemma.

Lemma 15. *Let $\{u_t,\ t \in [0,T]\}$ be a process such that its trajectories are P-a.s. in $L^\infty([0,\ T]; \mathbb{R}^n)$. Then for any $\varphi = (\varphi_{.1}, \ldots, \varphi_{.n}) \in S^{\mathbb{H}+}$ we have*

$$\int_0^t \varphi(s)\,\mathrm{d}B_s^{\mathbb{H}} = \int_0^t \varphi(s)\,\mathrm{d}\tilde{B}_s^{\mathbb{H}} + \int_0^t \varphi(s)u(s)\,\mathrm{d}s \quad \tilde{\mathsf{P}} - a.s., t \in [0,T], \tag{4.11}$$

where $\{\tilde{B}_t^{\mathbb{H}}, t \in [0,T]\}$ is the process defined in Theorem 7, i.e.

$$\tilde{B}_t^{\mathbb{H}} := B_t^{\mathbb{H}} - \int_0^t u_r\,\mathrm{d}r, \quad t \in [0,T]$$

.

Proof. Equality (4.11) can be easily shown to hold for $\varphi = \mathbf{1}_{[a,b)}\mathrm{Id}_n$, $a, b \in [0,t]$, by using the definition of $\tilde{B}^{\mathbb{H}}$ and the fact that

$$\int_0^t \varphi(s)\,\mathrm{d}B_s^{\mathbb{H}} = B_b^{\mathbb{H}} - B_a^{\mathbb{H}}.$$

Using the linearity of the integral we can extend this result to all step functions $\varphi \in \mathcal{E}(0,T;\mathcal{L}(\mathbb{R}^n))$.

By the definition of $\mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n))$, for $\varphi \in S^{\mathbb{H}+}$, there exists a sequence $(\varphi^m)_{m=1}^\infty = (\varphi_{.1}^m, \ldots, \varphi_{.n}^m)_{m=1}^\infty \in \mathcal{E}(0,T;\mathcal{L}(\mathbb{R}^n))$ such that

$$\|\varphi^m - \varphi\|^2_{\mathcal{D}^{\mathbb{H}}(0,T;\mathcal{L}(\mathbb{R}^n))} = \sum_{i=1}^n \|\varphi_{.i}^m - \varphi_{.i}\|^2_{\mathcal{D}^{H_i}(0,T;\mathbb{R}^n)} \longrightarrow 0, m \to +\infty.$$

The rest of the proof is the same as in [39, Proposition 5.1] using the Itô-type isometry (1.5) and the fact that the probability measures P and $\tilde{\mathsf{P}}$ are equivalent. □

Theorem 16. *Assume that the function b_1 satisfies conditions (I) and (II). Assume also that the function σ belongs to the space $S^{\mathbb{H}+}$ and that for every $t \in [0,T]$, the matrix $\sigma(t)$ is invertible. Assume finally, that the functions b_2 and σ satisfy condition (III). Then the equation* (4.1) *admits a weak solution.*

Proof. Let $\{B_t^{\mathbb{H}}, t \in [0,T]\}$ be a multivariate $\mathbb{H}$-fractional Brownian motion that is defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. By Theorem 11, there exists a process $\{X_t, t \in [0,T]\}$ with continuous sample paths that is adapted to the filtration generated by $B^{\mathbb{H}}$ and that satisfies the equation

$$X_t = x_0 + \int_0^t b_1(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B_r^{\mathbb{H}},$$

for every $t \in [0,T]$ P-almost surely. On the other hand, by Theorem 14, the process $\{\tilde{B}_t^{\mathbb{H}}, t \in [0,T]\}$ defined by

$$\tilde{B}_t^{\mathbb{H}} := B_t^{\mathbb{H}} - \int_0^t \sigma(r)^{-1} b_2(r, X_r)\,\mathrm{d}r, \quad t \in [0,T]$$

is a multivariate $\mathbb{H}$-fractional Brownian motion under the probability measure $\tilde{\mathsf{P}}$ that is given by formula (4.3).

The assumptions of Lemma 15 are satisfied and hence, it follows

$$\begin{aligned} X_t &= x_0 + \int_0^t b_1(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B_r^{\mathbb{H}} \\ &= x_0 + \int_0^t b_1(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}\tilde{B}_r^{\mathbb{H}} + \int_0^t \sigma(r)\sigma^{-1}(r) b_2(r, X_r)\,\mathrm{d}r \\ &= x_0 + \int_0^t b(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}\tilde{B}_r^{\mathbb{H}}, \quad t \in [0, T], \ \tilde{\mathsf{P}} - a.s. \end{aligned}$$

This shows that the triplet $((\Omega, \mathcal{F}, (\mathcal{F}_t^{B^{\mathbb{H}}})_{t\in[0,T]}, \tilde{\mathsf{P}}), X, \tilde{B}^{\mathbb{H}})$ is a weak solution to (4.1).

□

4.1.1 Example

In this section, we use the results from the previous sections to show the existence of a weak solution to equation that models the harmonic oscillator with stochastic forcing. We generalize the formal equation from [17, Eq. (2)].

Let $\mathbb{H} \in (0, 1)^n$. Consider the formal equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x_t + 2\gamma \frac{\mathrm{d}}{\mathrm{d}t} x_t + \omega^2 x_t = \rho(t) \frac{\mathrm{d}}{\mathrm{d}t} B_t^{\mathbb{H}}, \quad t \in [0, T], \tag{4.12}$$

where $\{B_t^{\mathbb{H}}, t \in [0, T]\}$ is an $\mathbb{H}$-fractional Brownian motion, γ is an $n \times n$ real matrix, ω^2 is an $n \times n$ real positive semidefinite matrix and $\rho : [0, T] \to \mathcal{L}(\mathbb{R}^n)$. Assume that $\rho(t)$ is invertible for all $t \in [0, T]$ and that $\rho \in S^{\mathbb{H}+}$. We want to show there exists a weak solution to this equation. Equation (4.12) can be given rigorous meaning as follows

$$\begin{aligned} x_t &= x_0 + \int_0^t u_r\,\mathrm{d}r, \qquad (4.13) \\ u_t &= u_0 - \int_0^t (2\gamma u_r + \omega^2 x_r)\,\mathrm{d}r + \int_0^t \rho(r)\,\mathrm{d}B_r^{\mathbb{H}}, \quad t \in [0, T]. \end{aligned}$$

Let $\{\tilde{B}_t^{\mathbb{H}}, t \in [0, T]\}$ be an $\mathbb{H}$-fractional Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. We also consider the equation

$$\begin{aligned} z_t &= x_0 + \int_0^t v_r\,\mathrm{d}r, \qquad (4.14) \\ v_t &= u_0 + \int_0^t \rho(r)\,\mathrm{d}\tilde{B}_r^{\mathbb{H}}, \quad t \in [0, T]. \end{aligned}$$

We rewrite the equations (4.13) and (4.14) so that they agree with the notation from the previous sections. Let $z := (x^T, u^T)^T \in \mathbb{R}^{2n}$, $t \in [0, T]$, and denote $b_2^*(z) := -2\gamma u - \omega^2 x$. Denote

$$X_t := (x_t^T, u_t^T)^T, \ Z_t := (z_t^T, v_t^T)^T, \ y_0 := (x_0^T, u_0^T)^T$$

$$b_1(z) := (u^T, 0_{1\times n})^T, \ b_2(z) := (0_{1\times n}, (b_2^*(z))^T)^T,$$

$$\sigma(t) = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \rho(t) \end{bmatrix}.$$

Let $\mathbb{H}_* \in (0,1)^n$ and let $\mathbb{H}_{2n} = (\mathbb{H}_*^T, \mathbb{H}^T)^T$. Let $\tilde{B}^{\mathbb{H}_{2n}}$ be an $\mathbb{H}_{2n}$-fractional Brownian motion such that its first n components are fractional Brownian motions with Hurst parameters $\mathbb{H}_*$ defined on $(\Omega, \mathcal{F}, \mathsf{P})$ that are independent of each other and independent of $\tilde{B}^{\mathbb{H}}$ and the second n components are the components of $\tilde{B}^{\mathbb{H}}$. We can write

$$X_t = y_0 + \int_0^t (b_1(X_r) + b_2(X_r))\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B_r^{\mathbb{H}_{2n}}, \quad t \in [0,T] \tag{4.15}$$

and

$$Z_t = y_0 + \int_0^t b_1(Z_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}\tilde{B}_r^{\mathbb{H}_{2n}}, \quad t \in [0,T], \tag{4.16}$$

where $B^{\mathbb{H}_{2n}}$ is the $2n$-dimensional $\mathbb{H}_{2n}$-fractional Brownian motion.

We would like to use Theorem 16 to find a weak solution to equation (4.15). However, note that for every $t \in [0,T]$, $\sigma(t)$ is not invertible and therefore, it seems like Theorem 16 cannot be applied. Nevertheless, if we define

$$\sigma^{-1}(t) := \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \rho^{-1}(t) \end{bmatrix}, \quad t \in [0,T],$$

we can see that it is an "inverse" matrix for $\sigma(t)$ in the following sense:

$$\sigma(t)\sigma^{-1}(t) = \sigma^{-1}(t)\sigma(t) = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \mathrm{Id}_n \end{bmatrix}, \quad t \in [0,T].$$

The inverse matrix is needed in the proof of Theorem 14. However, we can see that the proof proceeds the same manner with the "inverse" matrix $\sigma^{-1}(t)$ since

$$\|L_{\mathbb{H}_{2n}}^{-1}\left(\int_0^\cdot \sigma(r)^{-1} b_2(X_r)\,\mathrm{d}r\right)(t)\|_{\mathbb{R}^{2n}} = \|L_{\mathbb{H}}^{-1}\left(\int_0^\cdot \rho(r)^{-1} b_2^*(X_r)\,\mathrm{d}r\right)(t)\|_{\mathbb{R}^n}.$$

Further, we distinguish two cases.

Case I.

Assume that $\mathbb{H} \in (0,\frac{1}{2})^n$ and $\rho_{.k} \in \mathcal{C}^{\frac{1}{2}-H_k+\delta_k}(0,T;\mathbb{R}^n)$ for some $\delta_k > 0$, $k \in \{1,\dots,n\}$. Since b_1 is Lipschitz continuous, it satisfies the conditions (I) and (II) as well as all the assumptions of Theorem 11 and there exists a strong solution to equation (4.16).

We show that there exists a weak solution to equation (4.15). We need to verify that condition (III) is satisfied. Take $z = (x^T, u^T) \in \mathbb{R}^{2n}$, $t \in [0,T]$ and $k \in \{1,\dots,n\}$, condition (III) is satisfied since using the continuity of ρ we obtain

$$\begin{aligned} \|\sigma^{-1}(t) b_2(z)\|_{\mathbb{R}^{2n}} &= \|\rho^{-1}(t)(2\gamma u + \omega^2 x)\|_{\mathbb{R}^n} \\ &\le \|\rho^{-1}(t)\|_{\mathcal{L}(\mathbb{R}^n)} \|2\gamma u + \omega^2 x\|_{\mathbb{R}^n} \\ &\le K\|z\|_{\mathbb{R}^{2n}}, \end{aligned}$$

for some $K > 0$. From Theorem 16 it follows that there is a weak solution to equation (4.15) and therefore there is a weak solution to equation (4.13).

Case II.

Assume that $\mathbb{H} \in (0,1)^n$ is such that

$$\min_{k:H_k\in(0,1)} H_k > \max_{k:H_k\in(0,1)} H_k - \frac{1}{2} \tag{4.17}$$

and ρ does not depend on t, i.e. ρ is an $n \times n$ real matrix. Note that if all $H_i,\ i \in \{1,\dots,n\} \geq \frac{1}{2}$ then condition (4.17) is satisfied. Since b_1 is Lipschitz continuous, it satisfies conditions (I) and (II) as well as all the assumptions of Theorem 11 and there exists a strong solution to (4.16). We show that there exists a weak solution to equation (4.15). Take $z = (x^T, u^T),\ q = (y^T, w^T) \in \mathbb{R}^{2n}$ and $k \in \{1,\dots,n\}$, condition (III) is satisfied since

$$|[\rho^{-1}(2\gamma u + \omega^2 x)]_k| = |\rho^{-1}_{k.}(2\gamma u + \omega^2 x)| \leq \|\rho^{-1}_{k.}\|_{\mathbb{R}^n}\|2\gamma u + \omega^2 x\|_{\mathbb{R}^n} \leq K\|z\|_{\mathbb{R}^{2n}}$$

and

$$\begin{aligned}|[\rho^{-1}(2\gamma u + \omega^2 x)]_k - |[\rho^{-1}(2\gamma w + \omega^2 y]_k| &= |\rho^{-1}_{k.}(2\gamma(u-w) + \omega^2(x-y))| \\ &\leq C\|z - q\|_{\mathbb{R}^{2n}},\end{aligned}$$

for some $K,\ C > 0$. Here, $\rho^{-1}_{k.}$ denotes k-th row of ρ^{-1}. From Theorem 16 it follows that there is a weak solution to equation (4.15) and therefore there is a weak solution to equation (4.13).

4.2 Estimation of the Drift

In this section we use the Girsanov Theorem 14 to find the maximum likelihood estimate (MLE) of an unknown parameter in a stochastic differential equation with additive noise represented by multivariate $\mathbb{H}$-fractional Brownian motion based on the observation of a trajectory of a solution.

Let $(B^{\mathbb{H}}_t)_{t\geq 0}$ be a multivariate $\mathbb{H}$-fractional Brownian motion defined on probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $b : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be a Borel function.

Consider the equation

$$X_t = \theta \int_0^t b(s, X_s)\mathrm{d}s + B^{\mathbb{H}}_t, \quad t \geq 0, \tag{4.18}$$

where $\theta \in \mathbb{R}$ is an unknown parameter.

We, moreover, suppose that the following condition holds so that the solution to the equation exists.

(IV) For every $T > 0$ there exists a finite positive constant C_T such that for every $s, t \in [0,T]$ and $x, y \in \mathbb{R}^n$ it holds that

$$\|b(t,x) - b(s,y)\|_{\mathbb{R}^n} \leq C_T\|x - y\|_{\mathbb{R}^n}.$$

This condition, obviously, ensures that the conditions (I) and (II) are satisfied. Therefore, there exists a strong solution to equation (4.18) on the interval $[0,T]$ for every $T > 0$. For every $T > 0$, denote the solution to equation (4.18) $X^{\theta,T}_t$. Then, if we define

$$X^\theta_t := X^{\theta,N+1}_t$$

for every $t \in [N, N+1]$, $N \in \mathbb{N}$, it is obvious that $\{X_t^\theta, t \geq 0\}$ is a strong solution to (4.18).[2]

Let $T > 0$. If $\theta = 0$, then notice that the equation (4.18) is of the form

$$X_t = B_t^{\mathbb{H}}, t \in [0, T],$$

which means that under the probability measure $\mathsf{P}_0 := \mathsf{P}$, the process X^0 is a multivariate $\mathbb{H}$-fractional Brownian motion.

On the other hand, if $\theta \neq 0$, then it holds that X^θ is an $\mathbb{H}$-fractional Brownian motion under the probability P_θ that is defined by the formula

$$\frac{\mathrm{d}\mathsf{P}_\theta}{\mathrm{d}\mathsf{P}_0} := \mathcal{E}_T := \exp\left\{\int_0^T v_r^\top \mathrm{d}W_r - \frac{1}{2}\int_0^T \|v_r\|_{\mathbb{R}^n}^2 \mathrm{d}r\right\}$$

where $\{W, t \in [0, T]\}$ is the Brownian motion constructed from $B^{\mathbb{H}}$ by formula (1.8) and

$$v_t := K_{\mathbb{H}}^{-1}\left(\int_0^\cdot [-\theta b(r, X_r^\theta)]\mathrm{d}r\right)(t), \quad t \in [0, T].$$

This follows from Theorem 14. All the assumptions of the theorem are satisfied and it holds that the process defined by

$$\tilde{B}_t^{\mathbb{H}} := B_t^{\mathbb{H}} + \theta \int_0^T b(t, X_r^\theta)\mathrm{d}r, \qquad t \in [0, T], \tag{4.19}$$

is an $\mathbb{H}$-fractional Brownian motion under the probability P_θ.

From (4.18), it can be seen that $X_t^\theta = \tilde{B}_t^{\mathbb{H}}$, for every $t \in [0, T]$ $\quad \mathsf{P}_0$-a.s.

Theorem 17. *The MLE of parameter θ in equation* (4.18) *based on a observation of a trajectory of its solution X^θ on* $[0, T]$ *is given by*

$$\hat{\theta}_T = -\frac{\int_0^T Q_r^\top \mathrm{d}W_r}{\int_0^T \|Q_r\|_{\mathbb{R}^n}^2 \mathrm{d}r} \tag{4.20}$$

where $\{Q_t, t \in [0, T]\}$ is given by

$$Q_t = K_{\mathbb{H}}^{-1}\left(\int_0^\cdot b(r, X_r^\theta)dr\right)(t), \quad t \in [0, T],$$

and where $\{W_t, t \in [0, T]\}$ is the Wiener process given by formula (1.8).

Proof. From theorem 14 it follows that

$$\mathsf{P}_\theta(X^{\theta,T} \in A) = \int_{\{\omega \in \Omega: X^{\theta,T}(\omega) \in A\}} \mathcal{E}_T(\omega)\mathrm{d}\mathsf{P}_0(\omega), \quad A \in \mathcal{B}(\mathcal{C}([0, T]; \mathbb{R}^n)).$$

Therefore, we can find the MLE by maximizing the function

$$F(\theta) := \log \frac{\mathrm{d}\mathsf{P}_\theta}{\mathrm{d}\mathsf{P}_0} = -\theta \int_0^T Q_r^\top \mathrm{d}W_r - \frac{\theta^2}{2}\int_0^T \|Q_r\|_{\mathbb{R}^n}^2 \mathrm{d}r. \tag{4.21}$$

This gives us the result. $\square$

[2]Definition 9 may be extended to an infinite interval in an obvious way.

From the proof of Theorem 7 equality

$$X_t^\theta = \int_0^t K_{\mathbb{H}}(t,s) d\tilde{W}_s,$$

where

$$\tilde{W}_t := W_t + \theta \int_0^t Q_s \,\mathrm{d}s \tag{4.22}$$

holds for every $t \in [0,T]$ P_θ-almost surely. Hence, the following expression holds

$$\tilde{W}_t = \int_0^t (\partial K_{\mathbb{H}}^*)^{-1} \mathbf{1}_{[0,t)} \mathrm{Id}_n(.))(s) \,\mathrm{d}X_s^\theta, \qquad t \in [0,T],\ \mathsf{P}_\theta - a.s. \tag{4.23}$$

Now, we may notice from (4.22) that the following formula holds P_θ-a.s:

$$\int_0^T Q_r^\top \,\mathrm{d}W_r = \int_0^T Q_r^\top \,\mathrm{d}\tilde{W}_r - \theta \int_0^T \|Q_r\|_{\mathbb{R}^n}^2 \,\mathrm{d}r. \tag{4.24}$$

Therefore, we have an alternative expression for function F (given by (4.21)):

$$F(\theta) = -\theta \int_0^t Q_s \,\mathrm{d}\tilde{W}_s + \frac{\theta^2}{2} \int_0^t Q_s^2 \,\mathrm{d}s$$

which follows from (4.22) and (4.21).

Hence, if we maximize this function, we obtain the alternative form of the MLE

$$\hat{\theta}_T = \frac{\int_0^T Q_r^\top \,\mathrm{d}\tilde{W}_r}{\int_0^T \|Q_r\|_{\mathbb{R}^n}^2 \,\mathrm{d}r}. \tag{4.25}$$

Note that this estimate is observable if we observe the whole trajectory of the solution X. This follows from (4.23).

In what follows, we give sufficient conditions for strong consistency and asymptotic normality of the MLE.

Theorem 18. *If*

$$\int_0^T \|Q_r\|_{\mathbb{R}^n}^2 \,\mathrm{d}r \quad \xrightarrow[T\to\infty]{\mathsf{P}-\text{a.s.}} \quad \infty, \tag{4.26}$$

then $\hat{\theta}_T$ is strongly consistent, i.e. $\hat{\theta}_T \xrightarrow[T\to\infty]{\mathsf{P}-\text{a.s.}} \theta$.

Proof. If we replace (4.24) in (4.25), we get that

$$\hat{\theta}_T - \theta = \frac{\int_0^T Q_r^\top \,\mathrm{d}W_r}{\int_0^T \|Q_r\|_{\mathbb{R}^n}^2 \,\mathrm{d}r}.$$

Now it is sufficient to apply the strong law of large numbers for martingales; see, e.g., [35, Exercise V.1.6]. □

Remark. In the case $n = 1$ and $b(t,x) \equiv b(x)$, a sufficient condition for the validity of the convergence (4.26) is given in [40, Theorem 2] for $H \in (0, \frac{1}{2})$ and in [40, Theorem 3] for $H \in (\frac{1}{2}, 1)$. Moreover, it is shown in [18, Theorem 1.3] and [40, Proposition 3] that the convergence (4.26) is satisfied for the particular case $b(t,x) \equiv x$.

Theorem 19. *If there exists $C \in \mathbb{R}$, $C > 0$, for which the convergence*

$$\frac{1}{T}\int_0^T \|Q_r\|_{\mathbb{R}^n}^2 \,\mathrm{d}r \quad \xrightarrow[T\to\infty]{\mathsf{P}-\text{a.s.}} \quad \frac{1}{C^2} \tag{4.27}$$

is satisfied, then the MLE $\hat{\theta}_T$ is an asymptotically normal estimate of the parameter θ in equation (4.18), i.e. there is the following convergence in law: $\sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow[T\to\infty]{\mathrm{D}} Z$ where $Z \sim \mathsf{N}(0, C^2)$.

Proof. It follows from the previous proof that

$$\sqrt{T}(\hat{\theta}_T - \theta) = \frac{\frac{1}{\sqrt{T}}\int_0^T Q_r^\top \,\mathrm{d}W_r}{\frac{1}{T}\int_0^T \|Q_r\|_{\mathbb{R}^n}^2 \,\mathrm{d}r}$$

holds P-a.s. The proof is finished using the central limit theorem for martingales; see, e.g., [34, Theorem 1.49]. □

Remark. In the case of a one-dimensional fractional Ornstein-Uhlenbeck process ($n = 1$ and $b(t, x) = x$), asymptotic normality of the MLE $\hat{\theta}_T$ is proved in [9, Theorem 2] where a condition analogous to condition (4.27) is shown to be valid.

5. Conclusion

The family of $\mathbb{H}$-fractional Brownian motions provides a generalization of n-dimensional fractional Brownian motions. It can be useful to consider such processes since they provide higher flexibility as far as regularity is concerned.

A Girsanov-type theorem for $\mathbb{H}$-fractional Brownian motions was given in the thesis and it was applied on two problems. Firstly, an existence theorem for a weak solution of a stochastic differential equation driven by an $\mathbb{H}$-fractional Brownian motion was provided following the article [39]. Conclusion was made that the assumptions made on drift and diffusion coefficient lead to restrictions on the Hurst parameters of the $\mathbb{H}$-fractional Brownian motion. Specifically, it was shown that if the components of the diffusion coefficient corresponding to Hurst parameters greater than one half are in L^∞, then the regular values of $\mathbb{H}$ cannot differ from the singular values by more than one half. The existence result was applied to show the existence of a weak solution to an equation modelling harmonic oscillator.

Maximum likelihood estimator of drift parameter was found by using the Girsanov-type theorem and sufficient conditions for asymptotic normality and strong consistency were given.

A possible direction of future research could be the investigation whether different assumptions on the drift and diffusion in the Girsanov-type Theorem 14 would lead to less restrictve assumptions on the distance of Hurst parameters.

It would also be desirable to find sufficient conditions for assumptions (4.27) and (4.26) to hold that would be more useful for practical applications. Inspiration for this investigation could be taken from [40, Section 4], where equation with additive one-dimensional fractional Brownian motion is studied and Malliavin calculus is used to find sufficient conditions for strong consistency of the MLE. Another subject that could be studied is finding the MLE of the drift parameter in equation where the diffusion part is not a constant but a general function instead.

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List of Abbreviations and Symbols

a.s.	...	almost surely
a.e.	...	almost everywhere
$\mathbb{R}^n$	...	n-dimensional real space
$\mathbb{N}$	...	set of natural numbers
$\mathcal{C}(F;S)$	...	space of continuous functions from F with values in S
$\mathcal{C}^\delta(F;S)$	...	space of $\delta-$Hölder continuous functions from F to S
$L^p(F;S)$	...	L^p space of functions on F with values in S
$\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$	...	space of bounded linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$
$\mathcal{L}(\mathbb{R}^n)$	...	space of bounded linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$
$\langle .,. \rangle_F$	...	dot product on space F
$\Vert . \Vert_F$	...	norm on space F
Id_n	...	identity operator from $\mathbb{R}^n$ to $\mathbb{R}^n$
$\mathrm{diag}\{a_i\}_{i=1}^n$	...	diagonal matrix with the elements a_i on the diagonal
$\mathbf{1}_A$	...	indicator function of the set A
E	...	expected value
A^T	...	transposition of the matrix A
Γ	...	Gamma function
B	...	Beta function
$\mathcal{B}$	...	Borel sigma algebra on a metric space X
$\Re(z)$	...	Real part of complex number z
$A \preceq B$	...	$A \leq CB$ for some constant C which is not important
$\bigtimes_{i=1}^n S_i$	...	Cartesian product of vector spaces S_i