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On representations of Chekanov-Eliashberg algebras

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Abstract: In this thesis, we study modern invariants of Legendrian knots on $\mathbb{R}^3$ with a standard contact structure. We introduce the notion of Chekanov-Eliashberg algebra (DGA) and Legendrian contact homology. Then we consider representations of DGA as a way how to derive some computable invariants of Legendrian knots. Finally, we will find equivalence classes of graded 2-dimensional irreducible representations for a certain Legendrian knot.

Keywords: Legendrian knot, Chekanov-Eliashberg algebra, representation

Contents

Introduction

0.1 Motivation

“Contact geometry is all geometry.”

– V. I. Arnold

Contact geometry can be seen as an odd dimensional sibling of symplectic geometry. Pursuing the first ideas of contact geometry leads us to Christian Huygens [Gei05], who studied optics in 17th century. But first explicit use of contact geometry dates to early 20th century, when Sophus Lie's work on differential equations[Lie12] was written.

Legendrian knots are natural objects on contact 3-manifolds. Their importance appears in Eliashberg's work on tight versus overtwisted dichotomy of contact 3-manifolds [Eli89]. Also, Ding and Geiges [DG04] described that every closed connected 3-manifold can be obtained by surgery along a Legendrian link on $\mathbb{S}^3$ with a standard contact structure.

To understand Legendrian knots, we can study their classification on given contact manifold up to Legendrian isotopy. This leads to classical invariants of Legendrian knots. Those are: the underlying topological knot type, rotation number, and Thurston-Bennequin invariant. Many knot types are completely determined by classical invariants. Given an example, this is the case of unknot, as showed Eliashberg and Fraser [EF09], or torus knots, as proven by Etnyre and Honda [EH01].

Unfortunately, Chekanov [Che02] described two Legendrian knots of topological type $m(5_2)$ that have the same classical invariants and are not Legendrian isotopic. This was done by studying Chekanov-Eliashberg differential graded algebra (DGA) for Legendrian knots. Chekanov-Eliashberg DGA was found by Eliashberg, Givental, and Hofer in their work on symplectic field theory [EGH00] and independently combinatorially defined by Chekanov for Legendrian knots in $\mathbb{R}^3$ with a standard contact structure [Che02]. The homology of Chekanov-Eliashberg DGA is called Legendrian contact homology (LCH) and it is a modern invariant of Legendrian knots. However, LCH is often difficult to compute.

Luckily, Chekanov also introduced the notion of augmentations as a way how to linearize Chekanov-Eliashberg DGA and obtain some computable invariants. Which is, for example, linearized LCH. Augmentations form two important Fukaya-type A_∞ categories. The first one, Aug_+, was described by the work of Ng, Rutherford, Shende, Sivek, Zaslow [NRS$^+$15] and the second one, Aug_-, was originally defined by Bourgeois and Chantraine [BC$^+$14].

It is possible to generalize the notion of augmentations to representations of Chekanov-Eliashberg DGA, where augmentations can be seen as a 1-dimensional case of representations. Representations can be used for the linearization of Chekanov-Eliashberg DGA. Also, Chantraine, Dimitroglou Rizell, Ghiggini, and Golovko [CRGG16] constructed A_∞ categories Rep_+ and Rep_- that are analogues of Aug_+ and Aug_-. From this perspective, studying representations of Chekanov-Eliashberg DGA is an important task allowing us to understand powerful Fukaya-type invariants of Legendrian submanifolds.

Sivek [Siv13] gave an example of a Legendrian knot which admit higher dimensional representations, but does not admit any augmentations. Hence, the study of higher-dimensional representations is meaningful. Note that the representations discussed by Sivek were ungraded. The goal of this thesis is to describe 2-dimensional irreducible representations (and hence the representations coming only from higher dimensions) of certain knot that are graded.

0.2 Outline

The thesis is organized as follows.

In the first chapter, we introduce contact 3-manifolds and study the behavior of the contact structure. Namely, we will be interested in the global/local character of contact 1-forms in relation to the contact structure. Also, we explain using Frobenius theorem the notion of "maximal non-integrability" of the contact structure. We finish the chapter with Darboux's theorem.

In the second chapter, we are concerned with projections of Legendrian knots on standard $(\mathbb{R}^3, \xi_{std})$. We start with a front projection and then continue with Lagrangian projection. Moreover, we outline a resolution as a way to switch the knot diagrams between these projections.

In the third chapter, we define classical invariants of Legendrian knots. Also, we introduce an operation on Legendrian knots called stabilization. Here, we study the stabilization for better understanding of classical invariants.

In the fourth chapter, we conclude to the main terms of the thesis that are Chekanov-Eliashberg DGA and its homology (LCH). After we properly define Chekanov-Eliashberg DGA with $\mathbb{Z}_2$ coefficients, we study invariance of stable tame isomorphism class of Chekanov-Eliashberg DGA. And hence we obtain the invariance of LCH up to Legendrian isotopy. Then we lift Chekanov-Eliashberg DGA to $\mathbb{Z}[t, t^{-1}]$. We finish this chapter with a note on LCH of stabilized Legendrian knots.

In the fifth chapter, we inspect representations of Chekanov-Eliashberg DGA as a tool for obtaining some useful invariants of Legendrian knots. We also briefly study exact Lagrangian cobordisms and derive some further properties of representations.

In the sixth chapter, we conclude to the computation of irreducible graded 2-dimensional representations of Chekanov-Eliashberg algebra for a Legendrian knot of type $m(5_2)$.

1. Contact manifold

In this chapter, we will follow [Han08].

From now on, let M be a 3-dimensional manifold.

Definition 1. *A **plane field** ξ on M is a smooth choice of the 2-dimensional subspace $\xi_p \subset T_pM$ for every point $p \in M$. It means that for each $p \in M$ there exists neighborhood $U_p \subseteq M$ and set $S = \{X_1, X_2\}$ of two pointwise linearly independent vector fields on U_p such that S is a spanning set of ξ_q for each $q \in U_p$.*

*Next, a smooth choice of the 1-dimensional subspace T_pM at each point $p \in M$ is called a **line field**.*

Definition 2. *A 1-form α on M is called a **contact form** if*

$$\alpha \wedge d\alpha \neq 0, \tag{1.1}$$

i.e. for every point $p \in M$ and every basis $\{X_1, X_2, X_3\}$ of T_pM holds

$$\alpha_p \wedge d\alpha_p(X_1, X_2, X_3) \neq 0.$$

*A plane field ξ on M is called a **contact structure** if for every point $p \in M$ there exists a neighborhood $U_p \subseteq M$ and contact form α defined on U_p such that $\xi = \ker \alpha \subset TU_p$.*

*The pair (M, ξ) is called a **contact manifold**.*

Remark. At first, we note that any plane field ξ on M can be locally written as a kernel of some 1-form. We choose an arbitrary Riemannian metric g and construct a normal bundle $\xi^\perp$ with respect to this metric. Then for any $p \in M$ exists by the definition of vector bundle a neighborhood $U_p \subseteq M$ such that $\xi^\perp$ is on U_p diffeomorphic to $U_p \times \mathbb{R}^1$, i.e., $\xi^\perp$ is trivial on U_p. So, we can take a nonzero section v on U_p. Then we construct 1-form α_{U_p} on U_p as $\alpha_{U_p} := i_v g$. Clearly, $\xi_{U_p} = \ker \alpha_{U_p}$.

Now, we will see that the coorientability of ξ is sufficient and necessary condition to define ξ globally as a kernel of some 1-form. Coorientabilty means that quotient bundle $TM/\xi \cong \xi^\perp$ is orientable. By the definition, orientability of real vector bundle guarantees covering by trivializations such that transition functions are vector space orientation-preserving. But $\xi^\perp$ is a line bundle, so it has to be a trivial bundle. Hence, if ξ is coorientable, then we can take a global nonzero section v of $\xi^\perp$. Thus analogously defined 1-form will be also global. Conversely, let $\xi = \ker \alpha$ on M, where α is a contact form. Then it is possible to find a nonvanishing global section of the line bundle $\xi^\perp$ using α. Hence $\xi^\perp$ is orientable and ξ is coorientable.

From now on, we will be interested only in studying coorientable contact structures.

Remark. Observe that there are many contact forms for given (M, ξ). More precisely, if α is a contact form and f is a non-vanishing function on M, then also $\alpha' = f\alpha$ is a contact form. It quickly follows from the computation: $\alpha' \wedge d\alpha' = f\alpha \wedge d(f\alpha) = f\alpha \wedge (df \wedge \alpha + f d\alpha) = f^2 \alpha \wedge d\alpha$, where in the last equality we used the anticommutativity of the exterior product.

Remark. If α is a defining contact form on the contact manifold (M, ξ), then $\alpha \wedge d\alpha$ is a non-vanishing top-form on M, so M must be orientable.

Let's concentrate on a condition (1.1) in the definition of the contact form. From this condition follows that contact structure is *"maximally non-integrable"*. Now, we are going to explain the meaning of this statement.

Definition 3. *Let ξ be a plane field on M. An* ***integral submanifold*** *N for ξ is a submanifold of M such that for every point $p \in N$ holds $T_pN \subset \xi$. Observe that* $\dim N \leq \operatorname{rank} \xi$.

A plane field ξ is called ***integrable*** *if M can be covered by coordinate charts $\left\{U_a, \varphi_a \equiv (x_a^1, x_a^2, x_a^3)\right\}_{a \in \mathcal{A}}$ such that $\xi = \operatorname{span}\left\{\frac{\partial}{\partial x_a^1}, \frac{\partial}{\partial x_a^2}\right\}$ on U_a for each $a \in \mathcal{A}$.*

Remark. We see that plane field ξ is integrable if and only if for every point $p \in M$ there exists an integral submanifold $N \subset M$ for ξ.

Theorem 1 (Frobenius theorem)**.** *A plane field ξ on the manifold M is integrable if and only if for all X, $Y \in \Gamma(\xi)$ holds $[X, Y] \in \Gamma(\xi)$.*

For the proof of Frobenius theorem, we refer the reader to [War10].

Lemma 2. *Let ξ is a plane field on M defined as a kernel of the* 1*-form α. Then the following are equivalent:*

(*i.*) *α is a contact form,*

(*ii.*) $d\alpha|_\xi \neq 0$,

(*iii.*) $[X, Y]_p \notin \xi$*, at every $p \in M$, for all pointwise linearly independent $X, Y \in \Gamma(\xi)$.*

Proof. We choose an arbitrary point $p \in M$.
$(i.) \Rightarrow (ii.)$: Let $d\alpha_p|_{D_p} = 0$. By the definition, there exists a basis v_1, v_2 of ξ_p such that $d\alpha_p(v_1, v_2) = 0$. Next, we complete v_1, v_2 with a non-zero vector v_0 of $\xi_p^\perp$ to the basis v_0, v_1, v_2 of T_pM. Clearly, $\alpha_p \wedge d\alpha_p(v_0, v_1, v_2) = 0$.
$(ii.) \Rightarrow (i.)$: Let $d\alpha_p|_{\xi_p} \neq 0$. Then there exist nonzero vectors $v_1, v_2 \in \xi_p$ such that $d\alpha_p(v_1, v_2) \neq 0$. Take a nonzero vector v_0 of $\xi_p^\perp$, then evaluating (v_0, v_1, v_2) on $\alpha_p \wedge d\alpha_p$ gives $\alpha \wedge d\alpha \neq 0$.
$(ii.) \Leftrightarrow (iii.)$: Let α is a 1-form on M. Now, we prove, using the Cartan's magic formula and properties of the Lie derivative on tensors, the following identity:

$$\begin{aligned} d\alpha(X, Y) = i_X d\alpha(Y) &= (L_X\alpha)(Y) - (di_X\alpha)(Y) \\ &= L_X(\alpha(Y)) - \alpha(L_XY) - (di_X\alpha)(Y) \\ &= X\alpha(Y) - \alpha(L_XY) - (di_X\alpha)(Y) \\ &= X\alpha(Y) - \alpha([X, Y]) - d(\alpha(X))(Y) \\ &= X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]). \end{aligned}$$

Since this identity holds also for the vector fields $X, Y \in \Gamma(\xi = \ker \alpha)$, we obtain:

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) = -\alpha([X, Y]).$$

Thus $d\alpha(X, Y)$ vanishes if and only if $[X, Y] \in \Gamma(\xi)$. □

Remark. By Lemma 2 and Frobenius theorem, we see that the contact structure is out of being integrable as far as possible. Hence, we say that the contact structure is *"maximally non-integrable"*.

Definition 4. *Let (M,ξ) be a contact manifold with the contact form α. R is called a* ***Reeb vector field*** *associated with α if it is a vector field on (M,ξ) that is defined by the following system of equations:*

(i.) $\alpha(R) = 1$,

(ii.) $i_R(d\alpha) = 0$.

Remark. From the following consideration, we find out that on the contact manifold (M,ξ) exists one and only one Reeb vector field associated with the contact form α.

We know from Lemma 2 that at each point $p \in M$ is $d\alpha_p|_{\xi_p} \neq 0$. So, $d\alpha_p$ has 1-dimensional kernel. From $(i.)$ in the definition of the Reeb vector field, we obtain the existence of the unique line field on M. Then we get from $(ii.)$ a unique section of this line field.

Example. Consider $\mathbb{R}^3$ with Cartesian coordinates (x,y,z) and 1-form

$$\alpha_{std} = dz - ydx.$$

Observe that α_{std} is a contact form, since $\alpha_{std} \wedge d\alpha_{std} = (dz - ydx) \wedge (dx \wedge dy) = dx \wedge dy \wedge dz \neq 0$. Then contact structure $\ker \alpha_{std}$ is called a **standard contact structure** and denoted by ξ_{std} (see Figure 1.1). Note that ξ_{std} is spanned by $\{\frac{\partial}{\partial y}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial x}\}$. And also, $\frac{\partial}{\partial z}$ is a Reeb vector field associated with α_{std}.

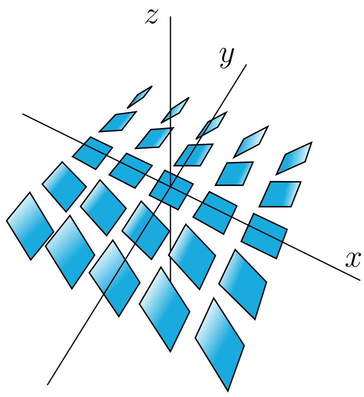


Figure 1.1: Standard contact structure ξ_{std}.

Definition 5. *Two contact manifolds (M_0,ξ_0) and (M_1,ξ_1) are called* ***contactomorphic*** *if there exists diffeomorphism $f : M_0 \to M_1$ such that $f_*(\xi_0) = \xi_1$.*

Let $\xi_0 = \ker \alpha_0$ and $\xi_1 = \ker \alpha_1$. Then we can equivalently say that (M_0,ξ_0) and (M_1,ξ_1) are called contactomorphic if there exists nonvanishing smooth function g on M_0 and diffeomorphism $f : M_0 \to M_1$ such that $f^*(\alpha_1) = g\alpha_0$.

We finish this section with Darboux's theorem, which says that all contact manifolds are locally contactomorphic. In other words, there are no local invariants in contact geometry. This property distinguishes contact geometry from Riemannian geometry, where, for example, curvature is a local invariant.

Theorem 3 (Darboux [Hon]). *Let (M, ξ) be a contact manifold, where $\xi = \ker \alpha$. Then for any point $p \in M$ exists a neighborhood U_p and diffeomorphism $\phi : U_p \to V \subset \mathbb{R}^3$ such that $\phi(p) = 0$ and $\phi^*(\alpha_{std}) = \alpha$.*

Proof. There is a coordinate chart $(\varphi, (x, y, z))$ such that $\varphi(p) = 0$ and induced contact structure $\xi_1 := \varphi_*(\xi)$ satisfies $\xi_1(0) = \ker dz_0$. Hence, we can take a contact form

$$\alpha = dz + f dx + g dy$$

on some small neighborhood of 0.

Contact structure ξ_1, restricted to xz-plane, is given by the span of vector field $X = \frac{\partial}{\partial x} - f(x, 0, z)\frac{\partial}{\partial z}$. By Fundamental theorem of ODEs, there exists a flow ϕ of the vector field X starting at $(0, 0, z)$ with time coordinate x. Since z-axis and X are transverse, ϕ is a diffeomorphism on some neighborhood of 0 in xz-plane by Inverse function theorem. Hence, we can take new coordinates (x, z) such that $X = \frac{\partial}{\partial x}$. Since $X \in \ker \alpha$, we obtain that $f(x, 0, z) = 0$.

Next, ξ_1 restricted to $x = const$ is given by the span of vector field $\widetilde{X} = \frac{\partial}{\partial y} - g\frac{\partial}{\partial z}$. Note that $\widetilde{X}$ is well defined on some neighborhood of 0 in $\mathbb{R}^3$ and is transverse to xz-plane. By Fundamental theorem of ODE's, we can similarly construct a flow $\widetilde{\phi}$ of the vector field $\widetilde{X}$, which starts from xz-plane and has time coordinate y. Also, by Inverse function theorem, we obtain new coordinates (x, y, z) such that $\widetilde{X} = \frac{\partial}{\partial y}$. Since $\widetilde{X} \in \ker \alpha$, we can write near 0:

$$\alpha = dz + f(x, y, z)dx,$$

where $f(0, 0, 0) = 0$.

Moreover, we see from contact condition that $\alpha \wedge d\alpha = -\frac{\partial f}{\partial y} dx \wedge dy \wedge dz \neq 0$, so $\frac{\partial f}{\partial y} \neq 0$. Then the map $(x, y, z) \mapsto (x, -f(x, y, z), z)$ is invertible in some neighborhood of 0.

Thus we can make a change of coordinates around p such that ξ is a standard contact structure around 0 in these coordinates and we are done. $\square$

2. Legendrian knots

We assume that reader is familiar with some basic concepts of knot theory, like smooth knots, equivalence of knots and knot diagrams. For further details, we refer the reader to [Rol04]. In this chapter, we will follow [Etn05] and [Han08].

Definition 6. *A **Legendrian knot** K on the contact manifold (M, ξ) is an embedding of $\mathbb{S}^1$ into (M, ξ) such that $T_xK \subset \xi_x$ for any $x \in K$.*

Definition 7. *Let K_0 and K_1 be Legendrian knots on the contact manifold (M, ξ). We say that K_0 and K_1 are **Legendrian isotopic** if there exists a map $f : \mathbb{S}^1 \times [0,1] \to M$ such that:*

(i.) $f(\mathbb{S}^1 \times \{t\})$ is a Legendrian knot for any $t \in [0,1]$,

(ii.) $f(\mathbb{S}^1 \times \{t\}) = K_t$ for $t \in \{0,1\}$.

Remark. We will distinguish Legendrian knots up to Legendrian isotopy.

From now on, we will be studying Legendrian knots on $(\mathbb{R}^3, \xi_{std})$.

Remark. By Darboux's Theorem 3, we will be also studying Legendrian knots locally on arbitrary contact 3-manifold.

Remark. Let us suggest a regular parametrization of Legendrian knot K, which is given by a map

$$\phi : \mathbb{S}^1 \to \mathbb{R}^3 : \theta \mapsto (x(\theta), y(\theta), z(\theta)). \tag{2.1}$$

Since $\phi'(\theta) = (x'(\theta), y'(\theta), z'(\theta)) \in \xi_{\phi(\theta)}$, we obtain the following identity:

$$z'(\theta) - y(\theta)x'(\theta) = 0, \tag{2.2}$$

by evaluating $(dz - ydx)_{\phi(\theta)}$ on vector $\phi'(\theta)$.

There are two natural projections of Legendrian knots in $(\mathbb{R}^3, \xi_{std})$. We will discuss them in the following two sections.

2.1 Front projection

Definition 8. *A **front projection** is a canonical map $\pi_F : (\mathbb{R}^3, \xi_{std}) \to \mathbb{R}^2_{x,z} : (x, y, z) \mapsto (x, z)$.*

Definition 9. *Let K be a Legendrian knot with a front projection parametrized by $\phi_F : \mathbb{S}^1 \to \mathbb{R}^2 : \theta \mapsto (x(\theta), z(\theta))$. And let $\theta_0 \in \mathbb{S}^1$, then $\phi_F(\theta_0)$ is called a **cusp point** if $x'(\theta_0) = 0$.*

Definition 10. *Let $f, \widetilde{f}$ be maps from the manifold N_1 to the manifold N_2. We say that f is C^0**-approximated** by $\widetilde{f}$ if for any $\varepsilon > 0$ locally (in coordinates) we have $||f(x) - \widetilde{f}(x)|| < \varepsilon$. Moreover, f is C^k**-approximated** by $\widetilde{f}$ if the approximation condition holds for any derivative up to order k.*

Theorem 4. *Let K be a Legendrian knot. Then front projection $\pi_F(K)$ has no vertical tangents.*
K can be C^0-approximated by a Legendrian knot $\widetilde{K}$ Legendrian isotopic to K such that $\pi_F(\widetilde{K})$ is paramatrized by immersion except of finite number of semi-cubical cusp points and all self-intersections of this parametrization are transversal.
Moreover, given a parametrization $\widetilde{\phi}_F(\theta) = (x(\theta), z(\theta))$ of $\pi_F(\widetilde{K})$, we can recover any y-coordinate of $\widetilde{K}$ in the following way:

$$y(\theta_0) = \begin{cases} \frac{z'(\theta_0)}{x'(\theta_0)} & \text{if } \widetilde{\phi}_F(\theta_0) \text{ is not a cusp point,} \\ \lim_{\theta \to \theta_0} \frac{z'(\theta)}{x'(\theta)} & \text{if } \widetilde{\phi}_F(\theta_0) \text{ is a cusp point.} \end{cases} \tag{2.3}$$

Proof. Let K be parametrized by ϕ as in (2.1). Then $\phi_F(\theta) = (x(\theta), z(\theta))$ will denote its front projection. From equation (2.2), we see that if $x'(\theta)$ vanishes, then also $z'(\theta)$ vanishes. Thus it is not possible to obtain θ_0 such that a tangent space of $\phi_F(\theta_0)$ is spanned by $\frac{\partial}{\partial z}$. Hence, there are no vertical tangents in front projection.

Because any immersion of $\mathbb{S}^1$ into $\mathbb{R}^2$ must have vertical tangents, we see from the above that ϕ_F can not be an immersion. Since it is possible that K has $x'(\theta) = 0$ on open intervals, we C^0-approximate K by a Legendrian isotopic Legendrian knot $\widehat{K}$ with a finite number of cusp points and there $x(\theta)'' \neq 0$. This can be done by "sharpening" the cusp points, see Figure 2.1. Let $\widehat{\phi}$ be a regular parametrization of $\widehat{K}$, then clearly $\widehat{\phi}_F$ is an immersion everywhere except cusp points.

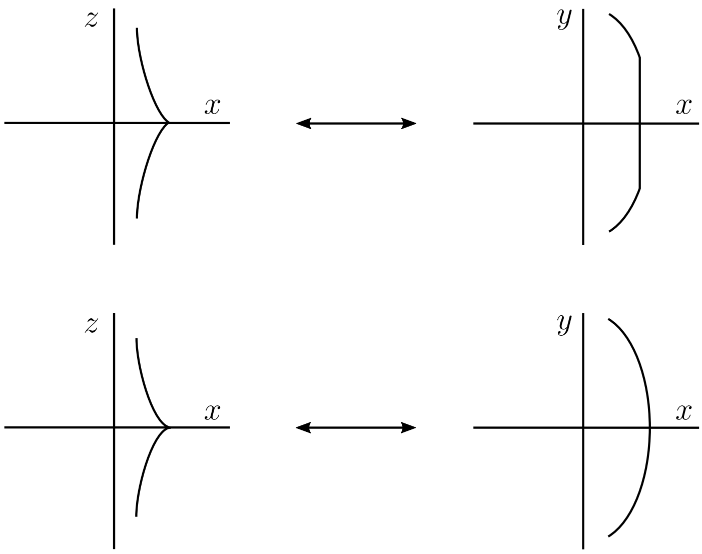


Figure 2.1: At each row, there are front and Lagrangian projections of Legendrian arcs. At the first row, we see Legendrian arc with $x'(\theta) = 0$ on open interval and, in the second row, there is a C^0-approximating Legendrian arc with $x'(\theta) = 0$ in the single cusp point.

From equation (2.2), we observe that $x'(\theta)$ vanishes maximally up to order of $z'(\theta)$. Thus y-coordinate can be recovered as in relation (2.3). However, obtained y-coordinates are not necessary smooth functions, since, from relation (2.3), we only have guaranteed continuity of recovered y-coordinates. To obtain a smoothness, we need to apply the following procedure.

Let $\widehat{\phi}_F(\theta_0)$ be a cusp point. Then $x'(\theta_0) = z'(\theta_0) = 0$. Because $\widehat{\phi}$ is an embedding, thus an immersion, we get that $y'(\theta_0) \neq 0$. So we can reparametrize $\widehat{\phi}$ by a parameter s such that our cusp point is attended for $s = 0$ and locally around this point

$$y(s) = s + a,$$

for some constant a. Note that locally around this cusp point is $x(s)$ a Morse function, where 0 is the only critical point. Then by Morse lemma [Mil63], x can be locally written as

$$x(s) = s^2 g(s) + x(0),$$

where $g(s)$ is a smooth function such that $g(0) \neq 0$. Now, we make the following C^0-approximation of $g(s)$ around 0 by $h(s)$. Put

$$h(s) = \begin{cases} g(0), & |s| < \varepsilon, \text{ where } \varepsilon \text{ is sufficiently small,} \\ g(s), & \text{otherwise.} \end{cases}$$

Then $x'(s) = 2sg(0)$ for $|s| < \varepsilon$. Using the equation (2.2) on the small neighborhood of 0 we obtain $z'(s)$ and by integrating also $z(s)$. Hence, around the cusp point at $s = 0$, we get the semi-cubic parametrization $\widetilde{\phi}$ of Legedrian knot $\widetilde{K}$:

$$\widetilde{\phi}(s) = \Big(g(0)s^2 + b, s + a, -g(0)(2s^3/3 + as^2) + c\Big),$$

where a, b, c are some real numbers and $g(0) \neq 0$. Clearly, $\widehat{K}$ is C^2-approximated by a Legendrian isotopic knot $\widetilde{K}$.

If we have some non-transverse intersection in the front projection, then we can apply Elementary transversality theorem [GG74] on the x-coordinate. Leaving y-coordinate unchanged, we can recover z-coordinate by integrating from equation (2.2). This leads to C^0-approximation by a Legendrian isotopic knot with only transverse self-intersections in the front projection. ☐

Corollary 5. *Any knot diagram that is satisfies the following properties:*

(*i.*) *there are no vertical tangents,*

(*ii.*) *the only non-smooth points are semi-cubic cusp points,*

(*iii.*) *at each crossing is the slope of the overcrossing smaller than the slope of undercrossing (here we suggest a standard orientation of $\mathbb{R}^3$ with y-axis having a direction "into the page"),*

represents an unique Legendrian knot K. Such a diagram is called ***front diagram*** *of Legendrian knot K.*

Examples:

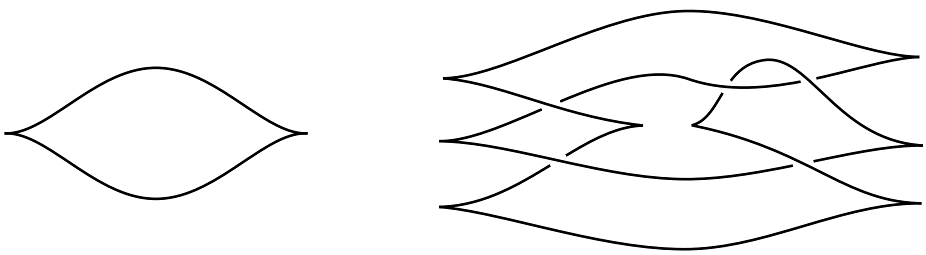


Figure 2.2: Front diagrams of Legendrian knots: unknot (left), $m(5_2)$ (right).

Lemma 6 (Reidemeister moves in front projection [Swi92])**.** *Let K_1 and K_2 be Legendrian knots and D_1, D_2 be front diagrams of K_1 and K_2, respectively. Then K_1 and K_2 are Legendrian isotopic if and only if we can pass from D_1 to D_2 by a regular homotopy and a finite sequence of moves of the following type:*

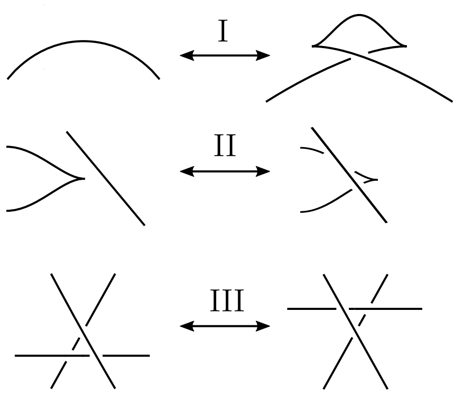


Figure 2.3: Reidemeister moves I, II, III (it is also necessary to consider these moves rotated by 180° around all three coordinate axes).

Lemma 7. *Any topological knot in $(\mathbb{R}^3, \xi_{std})$ can be C^0-approximated by a Legendrian knot. In particular, for any topological knot type, there exists a Legendrian knot of this type.*

Proof. For a given topological knot T in $(\mathbb{R}^3, \xi_{std})$, we take its knot diagram D_T. We would like to approximate D_T by some knot diagram satisfying conditions of Corollary 5. Then we can lift this new knot diagram to the desired Legendrian knot. Let us do the following observation.

We can C^0-approximate any arc A that can be immersed into $\mathbb{R}_{xz}$ by a Legendrian curve isotopic to this arc relative to the endpoints. This can be done by adding small locally non-intersecting zigzags (see Figure 2.4) to C^0-approximate the curve $\pi_F(A)$ in particular way. The slope of the zigzag-curve will be such that the lifted Legendrian curve also C^0-approximates y-coordinate of A.

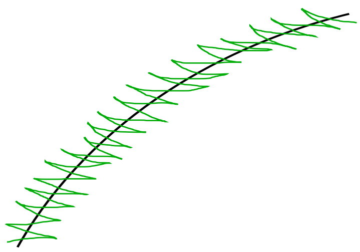


Figure 2.4: Diagrams in xz-plane of arc (black) and its C^0-approximation (green zigzags), respectively.

Applying this idea on a knot diagram D_T, we finish the proof. □

Remark. By Darboux's Theorem 3, Lemma 7 can be stated for an arbitraly contact 3-manifold. For details see [Han08].

Remark. In practice, we use the following moves to obtain Legendrian knot of a given topological knot type:

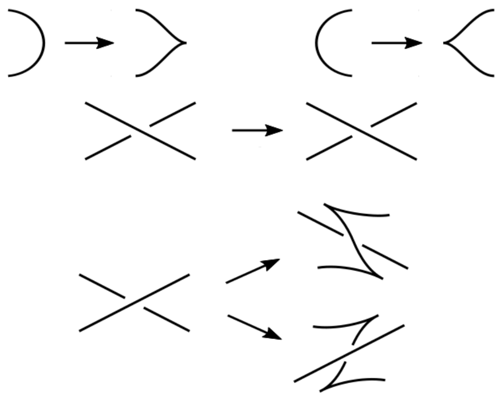


Figure 2.5: local changes of topological knot to obtain a Legendrian knot.

As we will see later, making a zigzag in the knot diagram has some unwanted consequences for the Legendrian isotopy class of the resulted Legendrian knot. And hence, it is better to avoid zigzags if possible.

2.2 Lagrangian projection

Definition 11. *A* ***Lagrangian projection*** *is a canonical map* $\pi_L : (\mathbb{R}^3, \xi_{std}) \to \mathbb{R}^2_{x,y} : (x, y, z) \mapsto (x, y)$.

Theorem 8. *Let* $\phi(\theta) = (x(\theta), y(\theta), z(\theta))$ *be a regular parametrization of Legendrian knot* K*, then its Lagrangian projection* ϕ_L *is an immersion. Moreover, we can recover* K *from* ϕ_L *by:*

$$z(\theta) = z(0) + \int_0^\theta y(\theta)x'(\theta)d\theta. \tag{2.4}$$

Any immersion $g : \mathbb{S}^1 \to \mathbb{R}^2 : \theta \mapsto (x(\theta), y(\theta))$ *can be lifted uniquely, up to translation in* z*-direction, to the Legendrian knot if:*

(i.) $\int_0^{2\pi} y(\theta)x'(\theta)d\theta = 0$,

(ii.) $\int_{\theta_0}^{\theta_1} y(\theta)x'(\theta)d\theta \neq 0$ *for all* $\theta_0 \neq \theta_1$ *with* $g(\theta_0) = g(\theta_1)$.

Proof. First, if we assume that ϕ_F is not an immersion, then $x'(\theta) = y'(\theta) = 0$ for some $\theta \in [0, 2\pi]$. And from identity (2.2) we obtain that $z'(\theta) = 0$, but ϕ is an immersion. Contradiction.

Next, since $z(\theta) = z(0) + \int_0^\theta z'(\theta)d\theta$, by identity (2.2) relation (2.4) holds .

We would like to lift g to Legendrian knot using relation (2.4). In order to do that we need to have correctly defined function $z : [0, 2\pi] \to \mathbb{R}$. Hence, we require that $z(0) = z(2\pi)$, which is equivalent to (*i.*) in statement of the theorem.

Since Legendrian knots are embeddings, we need to lift each double point to two points with different z-coordinate. And this condition is equivalent to (*ii.*) in the statement of the theorem. $\square$

Remark. By Stokes theorem, we see that the condition (*i.*) from the previous theorem is equivalent to the zero oriented area of the immersed closed curve given by g.

In general, diagrammatical description of Lagrangian projection is much more difficult than description of the front diagram. However, there are some advantages of Lagrangian projection as we will see later.

Lemma 9 (Reidemeister moves in Lagrangian projection [Kal05])**.** *Let K_1 and K_2 be Legendrian knots and D_1, D_2 knot diagrams representing in Lagrangian projection K_1 and K_2, respectively. Then K_1 and K_2 are Legendrian isotopic only if we can pass from D_1 to D_2 by a finite sequence of moves of the following type:*

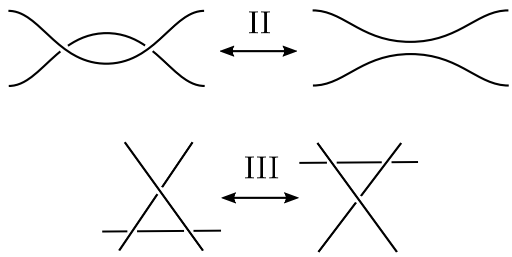


Figure 2.6: Reidemeister moves II, III (it is also necessary to consider these moves rotated by 180° around all three coordinate axes).

From the front diagram of given Legendrian knot, it is possible to obtain a Lagrangian projection, which represents the same knot. This leads to the following definition.

Definition 12. *Let D be a front diagram of arbitrary Legendrian knot. Then knot diagram obtained from D by regular homotopy and replacing all cusps as in the Figure 2.7 is called the **resolution of** D.*

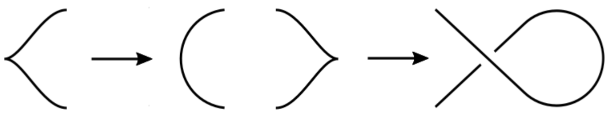


Figure 2.7: Local changes of the front diagram D to the resolution.

Lemma 10 ([Ng03])**.** *Let D_1 be a front diagram of Legendrian knot K_1. By the resolution D_2 of D_1, it is possible to produce a Lagrangian projection of knot K_2 that is Legendrian isotopic to K_1.*

Proof. We would like to change D_1 by a regular homotopy to the front diagram D_1' such that we obtain a corresponding diagram D_2 in Lagrangian projection after applying moves as in Figure 2.7 to D_1'. For illustration see Figure 2.8 on the next page.

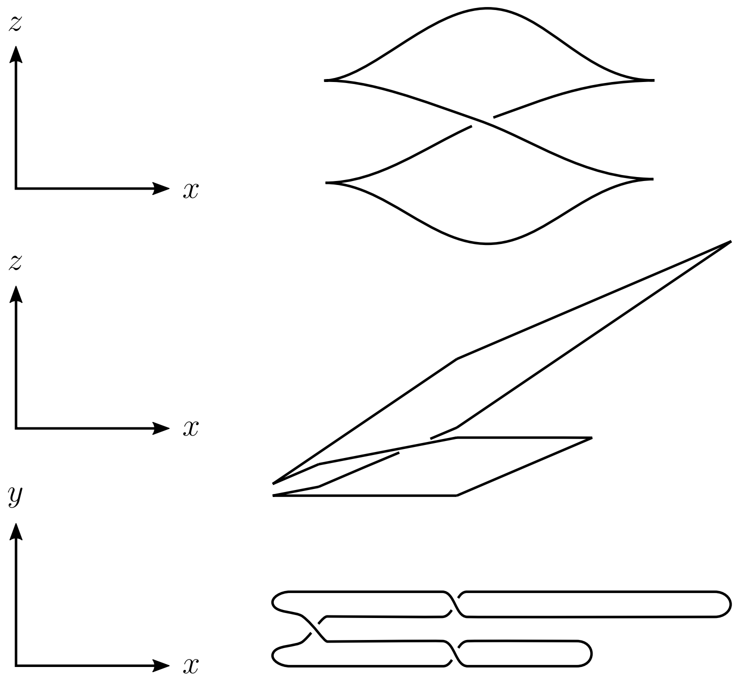


Figure 2.8: Example of resolution for the unknot. On the top, we see the front diagram D_1. In the middle, we see the distorted diagram D_1', containing only line segments and exceptional segments (viewed as corners of D_1'). On the bottom, we see the resolving diagram D_2 in xy-plane.

Let D_1 have k left cusps and any of them has different z-coordinate from others. We would like to obtain D_1' such that D_1' consists only line segments that are glued together by small smooth "exceptional segments". Each line segment has some slope of $\{0, 1, \ldots, 2k-1\}$ and for each x-coordinate corresponding line segments have different slopes.

Construction of D_1' will be following. We will start with left cusps. Each left cusp will be simply two line segments with slopes $j, j+1$ for some j that are smoothly joined by an exceptional segment. If we need to have a crossing of two segments, we will interchange their slopes by smoothly adding two exceptional segments and glue each segment with the slope of the opposite segment. Thus we obtain a crossing. Construction of the right cusp is similar as in the case of the crossing. Only before the crossing of two segments, we will glue them smoothly by the exceptional segment. Such a construction always exists.

By relation (2.3), we recover y-coordinate and obtain a Lagrangian projection corresponding to D_1'. Line segments will be transformed to line segments parallel to x-axis and after transformation, no pair of line segments will intersect. Also, each interchange of slope will produce a crossing in xy-plane. Thus, each right cusp will produce a loop and there is an identification of each crossing in xz-plane with some of the remaining crossings in xy-plane. So Lagrangian projection corresponding to D_1' is clearly a diagram D_2 obtained by moves as in Figure 2.7. $\square$

3. Classical invariants

In this chapter, we will follow [Etn05].

From now on, all Legendrian knots will be oriented.

Remark. Since Legendrian isotopy is also a topological isotopy of the underlying topological knot, the most obvious classical invariant of Legendrian knots is the underlying **topological knot type**.

Now, we conclude to the definition of the remaining two classical invariants:

Definition 13. *Let K be a Legendrian knot.*

*We denote **rotation number** of K by $r(K)$, which is defined as*

$$r(K) = \frac{1}{2}(\#D - \#U),$$

where $\#U$ is a number of upward cusps and $\#D$ is a number of downward cusps, when we follow the orientation of K in $\pi_F(K)$.

*We denote **Thurston-Bennequin invariant** of K by $tb(K)$, which is defined in terms of front projection as*

$$tb(K) = \#P - \#N - \#\textit{right cusps},$$

where $\#P$ and $\#N$ are numbers of positive and negative crossings in $\pi_F(K)$, respectively (see Figure 3.1).

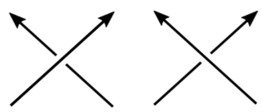


Figure 3.1: positive crossing (left) and negative crossing (right).

Remark. It is straightforward to verify that tb and r are not changing after Reidemeister moves in front projection, hence they are invariants of Legendrian knots by Lemma 6.

Remark. Using the same construction as in Lemma 10 we can also compute tb and r in Lagrangian projection of Legendrian knot K.

Then the rotation number can be viewed as a winding number of tangent vectors to $\pi_L(K)$, so

$$r(K) = \text{winding}(\pi_L(K)).$$

Next, Thurston-Bennequin invariant can be computed as

$$tb(K) = \#P - \#N,$$

where $\#P$ and $\#N$ are numbers of positive and negative crossings in $\pi_L(K)$, respectively.

Remark. Note that Thurston-Bennequin invariant does not depend on the orientation of given Legendrian knot.

Definition 14. *Let K be a Legendrian knot.* ***Stabilization*** *is an operation on K, which changes an arbitrary strand in $\pi_F(K)$ by one of the following zigzags (see Figure 3.2). If we added downward cusps, then the stabilization of K is called* ***positive*** *and denoted by $S_+(K)$. Otherwise stabilization is of K is called* ***negative*** *and denoted by $S_-(K)$. Legendrian knot $S_\pm(K)$ is called* ***stabilized****.*

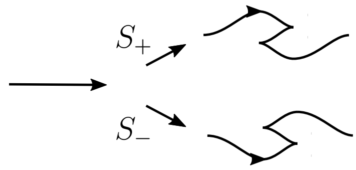


Figure 3.2: stabilization in front projection.

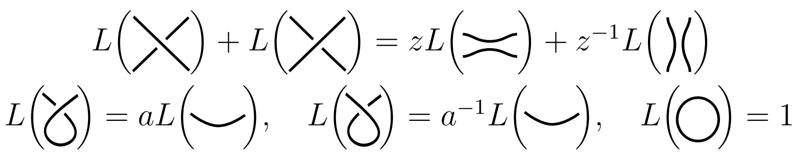
Remark. Stabilization is a well-defined operation, i.e. it does not depend on which strand of the given Legendrian knot is chosen for a stabilization. To prove this, it is necessary to check that we can move with a zigzag along cusps and crossing without changing a Legendrian isotopy knot type of the given Legendrian knot (see [FT97]).

Remark. Observe that stabilization does not change the topological knot type. Stabilization changes the remaining two classical invariants of Legendrian knot K in the following way:

$$r(S_\pm(K)) = r(K) \pm 1 \text{ and } tb(S_\pm(K)) = tb(K) - 1.$$

Thus we can obtain a Legendrian knot of an arbitrary negative tb for given topological type. Conversely, there is an upper bound on tb for a given topological class of Legendrian knots. But, first, we introduce the notion of Kauffman polynomial for Legendrian knots.

Take a front diagram of Legendrian knot K and make all cusps smooth. For resulting diagram D of the smooth knot, we state polynomial $L_D(a, z)$ as a polynomial that satisfies four skein relations



and is an invariant under Reidemeister moves II and III.

Definition 15. ***Kauffman polynomial*** *F_K of Legendrian knot K is defined as*

$$F_K(a, z) = a^{\#N - \#P} L_D(a, z).$$

Remark. Kauffman polynomial always exists and is an invariant of topological knot [Kau90].

Theorem 11 ([Rud90])**.** *Let K be a Legendrian knot. Then K satisfies* ***Kauffman bound****, i. e.*

$$tb(K) \leq \textit{min-deg}_a F_K(a, z) - 1.$$

Theorem 12 ([FT97])**.** *Two Legendrian knots with the same topological type become Legendrian isotopic after a finite number of stabilizations.*

Remark. Finally, note that using the construction as in Lemma 10 we can alternatively define a stabilization in terms of Lagrangian projection as in Figure 3.3.

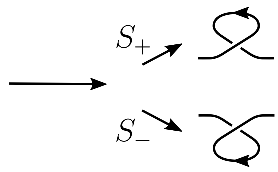


Figure 3.3: stabilization in Lagrangian projection.

4. Legendrian contact homology

4.1 Differential graded algebra

In this short section, we would like to recall some definitions from general algebra. S will denote an unital commutative ring in these definitions.

Definition 16. *An S-**algebra** $\mathcal{A}$ is an S-module with an associative bilinear map:*

$$m : \mathcal{A} \otimes_S \mathcal{A} \to \mathcal{A}.$$

For simplicity, we will rather write $x \cdot y$ instead of $m(x \otimes_S y)$ for all $x, y \in \mathcal{A}$.

Definition 17. *Let $\mathcal{A}$ be a S-algebra and G a cyclic group. We say that $\mathcal{A}$ is (G-)**graded** if $\mathcal{A}$ as an S-module can be decomposed as a direct sum of S-modules:*

$$\mathcal{A} = \bigoplus_{i \in \mathbb{G}} \mathcal{A}^i$$

such that for any $x \in \mathcal{A}^n$ and $y \in \mathcal{A}^m$ it holds that $x \cdot y \in \mathcal{A}^{n+m}$.

*Moreover, if $x \in \mathcal{A}^n$, we say that x is a **homogeneous element** of grading n and write $|x| = n$.*

*An S-linear map between graded S-algebras is called **graded morphism of degree** n if it maps homogeneous elements of degree m to homogeneous elements of degree $n + m$. If $n = 0$, we call this map simply a **graded morphism**.*

Definition 18. *$(\mathcal{A}, \partial)$ is called a **differential graded S-algebra (DGA)** if it is a graded S-algebra $\mathcal{A}$ with an S-linear map $\partial : \mathcal{A} \to \mathcal{A}$ satisfying:*

(*i.*) $\partial \circ \partial = 0$,

(*ii.*) $\partial(\mathcal{A}^n) \subset \mathcal{A}^{n-1}$ (*i.e.* ∂ *has grading* -1),

(*iii.*) *Leibniz formula: $\partial(x \cdot y) = \partial(x) \cdot y + (-1)^{|x|} x \cdot \partial(y)$ for any homogeneous elements $x, y \in \mathcal{A}$.*

*Such a map ∂ is called **differential**.*

*Moreover, $(\mathcal{A}, \partial)$ is called **semi-free** if the underlying graded S-algebra $\mathcal{A}$ is free. Let $\{a_1, \ldots, a_n\}$ be a set of generators of $(\mathcal{A}, \partial)$. To emphasize them, we sometimes use the notation $(\mathcal{A}(a_1, \ldots, a_n), \partial)$.*

*Let $(\mathcal{A}, \partial)$ and $(\mathcal{A}', \partial')$ be two DGAs and $\phi : \mathcal{A} \to \mathcal{A}'$ a graded morphism. Then ϕ is called a **chain map** between $(\mathcal{A}, \partial)$ and $(\mathcal{A}', \partial')$ if $\phi \circ \partial = \partial' \circ \phi$.*

$\mathfrak{dg}$ *will denote a category, whose objects are DGAs and morphisms are chain maps between them.*

From now on, all DGAs are assumed to be semi-free.

4.2 Chekanov-Eliashberg DGA

In this section, we follow [ENS$^+$02] and [Che02].

Definition 19. *Let K be a Legendrian knot. An integral curve for the Reeb vector field ($\frac{\partial}{\partial z}$) with endpoints on K is called a **Reeb chord**.*

From now on, unless otherwise specified, all the singularities of Legendrian knots in Lagrangian projection will be a finite number of double points, where we also have orthogonal crossings.

Remark. Note that the last convention is always satisfiable, since any Legendrian knot can be C^0-approximated by a Legendrian isotopic Legendrian knot satisfying this convention.

Remark. Observe that Reeb chords for given Legendrian knot K correspond to double points of $\pi_L(K)$. We denote the set of double points of $\pi_L(K)$ by $Q(K)$.

Now, we can conclude to the combinatorial definition of Chekanov-Eliashberg DGA with $\mathbb{Z}_2$ coefficients.

Let K be a Legendrian knot. Then the definition of **Chekanov-Eliashberg DGA** ($\mathcal{A}_K, \partial_K$) **of** K will be divided into three parts: algebra, grading and differential.

The algebra: $\mathcal{A}_K$ will be a free unital commutative $\mathbb{Z}_2$-algebra generated by $Q(K)$.

The grading: $\mathcal{A}_K$ will be graded over $\mathbb{Z}_{2r(K)}$ as follows. To any generator a of $\mathcal{A}_K$, we associate a path γ_a along $\pi_L(K)$ starting from ovecrossing in a (denote it a^+) and going to undercrossing in a (denote it a^-). Then $r(\gamma_a)$ will be a (fractional) winding number of a tangent vectors to the curve γ_a. The grading of a is defined by

$$|a| = 2r(\gamma_a) - 1/2 \bmod 2r(K).$$

Note that $r(\gamma_a)$ is an odd multiple of $1/4$. Finally, expand the grading on $\mathcal{A}_K$ by $|ab| = |a| + |b|$. Thus $|1| = |1 \cdot 1| = |1| + |1| = 0$.

The differential: We would like to define the differential by some appropriate count of immersed polygons, but first it is necessary to state the following term.

Neighborhood of any double point in $\pi_L(K)$ is divided by crossings into four quadrants. To each of the quadrants, we would like to associate a **Reeb sign** defined as in Figure 4.1.

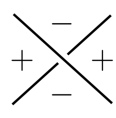


Figure 4.1: Reeb sign.

Let P_{k+1} denote a (curved) convex $(k+1)$-sided polygon in $\mathbb{R}^2$ with vertices $v_0, \dots, v_k$ appearing in counterclockwise order around the polygon. Each of $b_0, \dots, b_k$ take a value in $Q(K)$. Then we define the set

$$\mathcal{M}_{b_0,\dots,b_k} = \{u : (P_{k+1}, \partial P_{k+1}) \to (\mathbb{R}^2_{xy}, \pi_L(K))\}/\sim,$$

where $\sim$ is reparametrization of the domain and u is an orientation-preserving immersion sending v_i to b_i for $i \in \{0, \dots, k\}$.

Moreover, for $u \in \mathcal{M}_{b_0,\dots,b_k}$, we call vertex v_i of P_{k+1} **positive** (with respect to u) if the neighborhood of v_i is mapped to the quadrant of b_i with + Reeb sign (i.e. positive quadrant). Otherwise, the vertex v_i is called **negative**.

Then by $\mathcal{M}^{b_0}_{b_1,\dots,b_k}$ we will denote the subset of $\mathcal{M}_{b_0,\dots,b_k}$ containing immersions, where b_0 is the only positive vertex. And $\mathcal{M}^{b_0}_{b_1,\dots,b_k}$ is called the **set of admissible immersions**.

Now, we can conclude to the definition of differential $\partial_K : \mathcal{A}_K \to \mathcal{A}_K$. Let a be a generator of $\mathcal{A}_K$. The we put

$$\partial_K(a) = \sum_{\substack{k \geq 0 \\ b_1,\dots,b_k \text{ generators of } \mathcal{A}_K}} (\#_2 \mathcal{M}^a_{b_1,\dots,b_k}) b_1 \dots b_k, \tag{4.1}$$

where $\#_2 \mathcal{M}^a_{b_1,\dots,b_k}$ means modulo 2 count of elements of $\mathcal{M}^a_{b_1,\dots,b_k}$. Finally, we extend ∂_K on $\mathcal{A}_K$ by $\mathbb{Z}_2$-linearity and Leibniz formula.

Note that $\partial_K(1) = \partial_K(1 \cdot 1) = \partial_K(1) \cdot 1 + (-1)^{|1|} 1 \cdot \partial_K(1) = 2 \cdot \partial_K(1)$, hence $\partial_K(1) = 0$. Of course, here the sign in Leibniz formula is redundant, however in definitions of Chekanov-Eliashberg algebras with more sophisticated coefficients the sign plays a substancial role.

When it is clear from the context, sometimes we will write simply DGA instead of Chekanov-Eliashberg DGA.

Next, we are going to show in the following three lemmata that the differential ∂_K is well defined.

Lemma 13. *For any Legendrian knot K, there is only finite number of admissible immersed polygons.*

Lemma 14. *∂_K has degree -1.*

Lemma 15. *$\partial_K^2 = 0$.*

Remark. It is possible to define Chekanov-Eliashberg DGA also in terms of front projection. It will lead to DGA generated not only by the set of double points, but also by the set of right cusps. There will be also other differences and analogues of Lemmata 13, 14, 15 will be a little bit more complicated. For details, see [Ng03].

Remark. We would like to derive an important relation between Reeb signs of vertices of the immersed polygons. By H we will denote a height function on double points in $\pi_L(K)$, i.e. let b be a double point in $\pi_L(K)$, then $H(b)$ will be equal to the difference between z-coordinate of preimages of b^+ and b^- in K. Let each of $b_0, \dots, b_k$ take a value in $Q(K)$ and $u \in \mathcal{M}_{b_0,\dots,b_k}$ is an immersion with the domain P_{k+1}.

Then we lift $u(\partial P_{k+1})$ to $\tilde{u}(\partial P_{k+1})$ on K in a way that $\pi_L(\tilde{u}(\partial P_{k+1})) = u(\partial P_{k+1})$. Note that $\tilde{u}(\partial P_{k+1})$ together with Reeb chords, corresponding to $b_0, \dots, b_k$, make a piecewise smooth loop. Thus

$$\sum_{v \in Q^+} H(u(v)) - \sum_{v \in Q^-} H(u(v)) + \int_{\partial P_{k+1}} \tilde{u}^*(dz) = 0,$$

where Q^+ and Q^- are sets of positive and negative vertices of P_{k+1}, respectively. By the relation $dz = ydx$ and Stokes theorem, we obtain

$$\sum_{v \in Q^+} H(u(v)) - \sum_{v \in Q^-} H(u(v)) = \int_{P_{k+1}} u^*(dx \wedge dy) > 0. \tag{4.2}$$

Now, we make a few useful observations about this inequality. Since H takes values only in $\mathbb{R}^+$, there is only finite number of sets $\mathcal{M}_{b_0,\ldots,b_k}$. Also, note that at least one of vertices of P_{k+1} is positive. If we suggest an admissible immersion, it is not possible to have positive and negative vertex mapped to the same double point.

Proof of Lemma 13. Due to the previous remark, it remains to show that for given double points $a, b_1, \ldots, b_k$ in $\pi_L(K)$, the set $\mathcal{M}^a_{b_1,\ldots,b_k}$ is finite.

Let $\{C_1, \ldots, C_m\}$ be a set of components of $\mathbb{R}^2 \setminus \pi_L(K)$ and $\{S_1, \ldots, S_m\}$ denotes the set of areas of corresponding components. Now, we suppose that $\mathcal{M}^a_{b_1,\ldots,b_k}$ is nonempty and we take $u \in \mathcal{M}^a_{b_1,\ldots,b_k}$. Then

$$\int_{P_{k+1}} u^*(dx \wedge dy) = \sum_{i=1}^{m} n_i S_i,$$

where n_i are non-negative integers that are equal to the cardinality of $u^{-1}(p)$ for any $p \in C_i$. From relation (4.2), we see that there is only finite number of possibilities, how to choose n_i. Since any immersion in $\mathcal{M}^a_{b_1,\ldots,b_k}$ is uniquely given by its cardinalities n_i on components of $\mathbb{R}^2 \setminus \pi_L(K)$, the set $\mathcal{M}^a_{b_1,\ldots,b_k}$ is also finite. $\square$

Proof of Lemma 14. Let $u \in \mathcal{M}^a_{b_1,\ldots,b_k}$ and P_{k+1} be a corresponding polygon with vertices $v_0, \ldots, v_k$. Note that $u(\partial P_{k+1})$ can be seen as an union of paths (arcs) $u(\overline{v_0, v_1}), \ldots, u(\overline{v_{k-1}, v_k}), u(\overline{v_k, v_0})$ in $\pi_L(K)$. We denote them by $\eta_0, \ldots, \eta_n$. From the diagram

$$a^- \xrightarrow{-\gamma_a} a^+ \xrightarrow{\eta_0} b_1^+ \xrightarrow{\gamma_{b_1}} b_1^- \xrightarrow{\eta_1} b_2^+ \cdots b_k^- \xrightarrow{\eta_n} a^-,$$

we observe that the union $u(\partial P_{k+1}) \cup -\gamma_a \cup \gamma_{b_1} \cup \cdots \cup \gamma_{b_k}$ corresponds to the unique oriented loop Γ_u in K. Since $K \cong \mathbb{S}^1$, Γ_u coincides with an unique element of $H_1(K, \mathbb{Z}_{2r(K)}) = \mathbb{Z}_{2r(K)}$. And we denote this element by $n(u)$.

Now, we would like to count the rotation number of the loop Γ_u in two ways. Recall that the Γ_u corresponds to the union $u(\partial P_{k+1}) \cup -\gamma_a \cup \gamma_{b_1} \cup \cdots \cup \gamma_{b_k}$. Hence, it is equal to $n(u)r(K)$. But, we can also count the rotation number of Γ_u by parts, i.e. as the sum of contributions of smooth pieces of the union. Then smooth pieces of $u(\partial P_{k+1})$ contributes by $1 - (k+1)/4$, since we substract $1/4$ for each corner of immersed polygon. And next, by the definition, $r(\gamma_a)$ is equal to $(2|a|+1)/4$ and $r(\gamma_{b_i})$ is equal to $(2|b_i|+1)/4$ for $i \in \{1, \ldots, k\}$.

Thus

$$n(u)r(K) = 1 - \frac{k+1}{4} - \frac{2|a|+1}{4} + \sum_{i=1}^{k} \frac{2|b_i|+1}{4}.$$

Which can be rewritten as

$$|a| = \sum_{i=1}^{k} |b_i| - 2n(u)r(K) - 1.$$

Next, we obtain that

$$|a| = \sum_{i=1}^{k} |b_i| - 1, \tag{4.3}$$

because we count grading over $\mathbb{Z}_{2r(K)}$. Note that for the same reason relation (4.3) is independent on the choice of paths γ_i. Hence, the grading is well defined. Finally, formula 4.1 and Leibniz formula finishes the proof. □

Proof of Lemma 15. Strategy of the proof will be following: by Leibniz formula, it is enough to check $\partial_K^2 = 0$ only for generators of $\mathcal{A}_K$. In more detail, we will show that each word of ∂_K^2 corresponds to some immersed polygon with one non-convex vertex (we call him an obtuse polygon). But, such an obtuse polygon can be constructed in two ways by gluing a pair of admissible immersed polygons. Thus, if any word appears in ∂_K^2, then it appears even number of times and they cancel, since we count modulo 2.

1. *Construction of the obtuse polygon*
   Let a be a generator of $\mathcal{A}_K$ and W be an arbitrary word in $\partial_K^2(a)$. Then W can be written in the form

   $$b_1 \dots b_{i-1} c_1 \dots c_{k'} b_{i+1} \dots b_k,$$

   for $i, k, k' \in \mathbb{N}$ and $i < k$ and where W is corresponding to some elements $u \in \mathcal{M}^{a}_{b_1,\dots,b_k}$ and $u' \in \mathcal{M}^{b_i}_{c_1,\dots,c_{k'}}$. We use also the following notation; P_{k+1} will be an associated polygon to u with vertices $\{v_0, \dots, v_k\}$ and $P'_{k'+1}$ will be an associated polygon to u' with vertices $\{w_0, \dots, w_{k'}\}$.

   Now, we would like to glue polygons $P'_{k'+1}$ and P_{k+1} to an obtuse polygon $\mathcal{P}$ such that u' and u will be glued to the orientation-preserving immersion $U : (\mathcal{P}, \partial\mathcal{P}) \to (\mathbb{R}^2_{xy}, \pi_L(K))$ and immersed polygon $U(\mathcal{P})$ will have letters from the word aW appearing along its boundary. We put

   $$\mathcal{P} := P'_{k'+1} \cup_g P_{k+1},$$

   where g is a (smooth) embedding such that $u \circ g = u'$ on $\operatorname{Dom} g$, $g(w_0) = v_i$ and $\operatorname{Dom} g \subseteq \partial P'_{k'+1}$ and $\operatorname{Im} g \subseteq \partial P_{k+1}$ are connected and largest possible. If S' is a neighborhood of w_0 in $P'_{k'+1}$ and S is a neighborhood of v_i in P_{k+1}, we can WLOG assume that w_0 and v_i are glued as in the Figure 4.2. There are several cases of gluing.

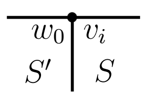
   

   Figure 4.2: Gluing of neighborhoods of vertices w_0 and v_i.

   If $k' > 0$, then see Figure 4.3.

   (a) $v_{i+1} \notin \operatorname{Im} g$ and $w_{k'-1} \in \operatorname{Dom} g$,

   (b) $v_{i+1} \in \operatorname{Im} g$ and $w_{k'-1} \notin \operatorname{Dom} g$,

(c) $v_{i+1} \in \operatorname{Im} g$ and $w_{k'-1} \in \operatorname{Dom} g$; in this case, $\mathcal{P}$ will have only negative vertices, but this contradicts relation (4.2). Thus this case is not possible.

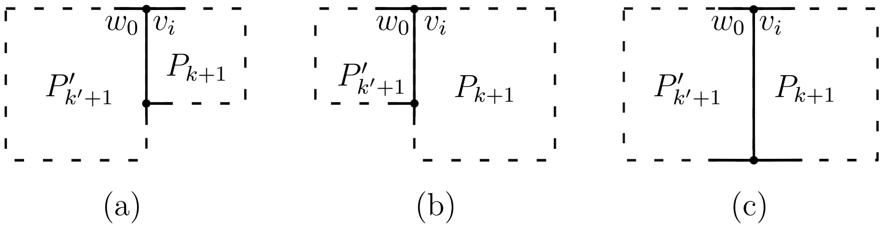


Figure 4.3: Gluing polygons $P'_{k'+1}$ and P_{k+1} to obtain non-convex polygon $\mathcal{P}$, and where the dashed lines denote unspecified part of boundaries of glued polygons.

Note that it is not possible to have simultaneously $v_{i+1} \notin \operatorname{Im} g$ and $w_{k'-1} \notin \operatorname{Dom} g$, because then $\operatorname{Im} g$ or $\operatorname{Dom} g$ will be not largest possible.

If $k' = 0$, then see Figure 4.4.

(d) $v_{i+1} \in \operatorname{Im} g$ and $\operatorname{Dom} g \neq P'_1$; this case is displayed by the same figure as (a),

(e) $v_{i+1} \in \operatorname{Im} g$ and $\operatorname{Dom} g = P'_1$; observe that in this case $g(v_{i+1}) \sim w_0 \sim g(v_i)$, where $\sim$ is equivalence relation induced by g.

Otherwise, if $v_{i+1} \notin \operatorname{Im} g$ and $\operatorname{Dom} g = P'_1$, then there exists $x \in \partial P_{k+1}$ such that $x \sim v_i$ but $x \neq v_i$. Moreover, we need to glue some pieces of the boundary of $\mathcal{P}$ to obtain the required obtuse polygon. We put

$$\mathcal{P} := \mathcal{P} \cup_h \mathcal{P},$$

where h is a (smooth) embedding such that $U \circ h = U$ on $\operatorname{Dom} h$, $h(v_i) = x$ and $\operatorname{Dom} h \subseteq \partial\mathcal{P}$ and $\operatorname{Im} h \subseteq \partial\mathcal{P}$ are connected and largest possible. So, now, we need to distinguish the following cases:

(f) $v_{i+1} \notin \operatorname{Im} h$ and $v_{i-1} \in \operatorname{Dom} h$,

(g) $v_{i+1} \in \operatorname{Im} h$ and $v_{i-1} \notin \operatorname{Dom} h$,

(h) $v_{i+1} \in \operatorname{Im} h$ and $v_{i-1} \in \operatorname{Dom} h$; in this case $\mathcal{P}$ will have only negative vertices, but this contradicts relation (4.2). Hence, this case is not possible.

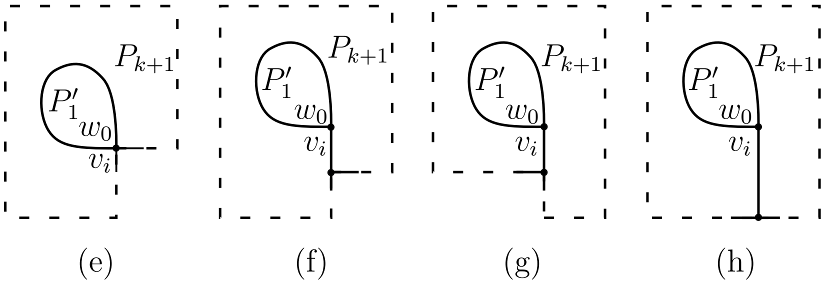


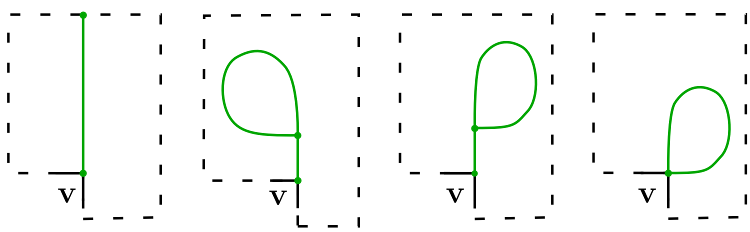
Figure 4.4: Gluing polygons P'_1 and P_{k+1} to obtain non-convex polygon $\mathcal{P}$, and where the dashed lines denote unspecified parts of boundaries of glued polygons.

Observe that such a construction of $\mathcal{P}$ and U always exists and is unique up to reparametrization of U. Let $\mathbf{v}$ denote the vertex of $\mathcal{P}$ with an obtuse angle. Note that $\mathcal{P}$ has either one positive vertex or the neighborhood of $\mathbf{v}$ is mapped by U to two quadrants with $+$ Reeb sign and one with $-$ Reeb sign.

2. *Cutting* $\mathcal{P}$

At $\mathbf{v}$, there are two segments pointing into $\mathcal{P}$. We would like to extend each of these segments by tracking curve γ until this curve does not self-intersect or reach the boundary of $\mathcal{P}$ and moreover γ will satisfy $U(\gamma) \subseteq \pi_L(K)$. Note that endpoint of tracking γ will correspond to some double point of $\pi_L(K)$.

There are four possibilities (A)–(D) of γ, see Figure 4.5. Also, γ in each case divides the polygon $\mathcal{P}$ into two polygons P and P' and also divide U into two admissible immersions u and u' that can be glued as in the step 1. to $(\mathcal{P}, U)$ and correspond to the word W. Here it is necessary to say that each of u and u' is admissible. This follows from the construction and relation (4.2).



(A) (B) (C) (D)

Figure 4.5: Cutting non-convex polygon $\mathcal{P}$ by γ into two polygons P, P', where γ is denoted by green line.

Observe that in a case (D) we obtain two pairs of different admissible immersions (i.e. each pair corresponds to another word in $\partial_K^2(a)$). This is true since $\mathbf{v}$ appears as a vertex glued from vertices v_{i+1} and v_i and each of them contributes by one admissible immersed polygon. Other cases also produce pairs of two different admissible immersions. And we are done.

$\square$

4.3 Modern invariants

Note that it is possible "to change" Chekanov-Eliashberg DGA by Reidemeister moves in Lagrangian projection. For example, Reidemeister move II increases (decreases) the number of generators of underlying algebra by two. Thus, the DGA itself is not an invariant under Legendrian isotopy. But, if we look closer, we find out that Reidemeister moves preserve certain equivalence class of Chekanov-Elisashberg DGAs. This leads to the notion of modern invariants of Legendrian knots (i.e. non-classical invariants). In this section, we follow [ENS+02] and [Che02].

Definition 20. *A chain isomorphism of DGAs*

$$\phi : (\mathcal{A}(a_1, \dots, a_n), \partial) \to (\mathcal{A}'(a'_1, \dots, a'_n), \partial')$$

is called ***elementary*** *if there is some* $j \in \{1, \dots, n\}$ *such that*

$$\phi(a_i) = \begin{cases} a'_i, & i \neq j, \\ a'_i + u, & i = j, \text{ for some } u \in \mathcal{A}'(a'_1, \dots, a'_j, a'_{j+1}, \dots, a'_n). \end{cases}$$

The composition of some number of elementary isomorphisms is called ***tame isomorphism****.*

Definition 21. *Let* $(E_k(e_1, e_2), \partial_{E_k})$ *be a differential* $\mathbb{Z}$*-graded algebra that is given by* $|e_1| = k, |e_2| = k - 1$ *and* $\partial_{E_k}(e_1) = e_2$, $\partial_{E_k}(e_2) = 0$*.*

Then the degree k ***stabilization*** *of a DGA* $\mathcal{A}(a_1, \dots, a_n), \partial)$ *graded by* G *is a DGA*

$$(S_k(\mathcal{A}), \partial_{S_k}) := (\mathcal{A}, \partial) \coprod (E_k(e_1, e_2), \partial_{E_k})$$

graded by $G \otimes \mathbb{Z} \cong G$*, where* ∂_{S_k} *is given by* $\partial_{S_k}(a_i) = \partial(a_i)$, $\partial_{S_k}(e_j) = \partial_{E_K}(e_j)$ *and the degrees of generators are images of the initial degrees under the natural homomorphisms* $G, \mathbb{Z} \to G \otimes \mathbb{Z}$*.*

Two DGAs $(\mathcal{A}, \partial)$ *and* $(\mathcal{A}', \partial')$ *are called* ***stable tame isomorphic*** *if there exist sequences of stabilizations* $S_{i_1}, \dots, S_{i_n}$ *and* $S_{j_1}, \dots, S_{j_m}$ *and a tame isomorphism*

$$\phi : (S_{i_n}(\dots(S_{i_1}(\mathcal{A}), \partial_{S_{i_1}}) \dots \partial_{S_{j_n}}) \to (S_{j_m}(\dots(S_{i_1}(\mathcal{A}'), \partial_{S_{j_1}}) \dots \partial_{S_{j_m}}).$$

Lemma 16. *Stable tame isomorphic DGAs have same homology groups.*

Proof. It is sufficient to prove the lemma for $(\mathcal{A}, \partial)$ and $(S(\mathcal{A}), \partial_S)$. Consider the natural projection τ and inclusion i

$$\mathcal{A} \xrightarrow{\ i\ } S(\mathcal{A}) \xrightarrow{\ \tau\ } \mathcal{A} .$$

We would like to show that τ and i are homotopy equivalent, since then they will induce desired isomorphism of homology groups.

Observe that $\tau \circ i = \mathrm{Id}_{\mathcal{A}}$. On the other hand, we would like to show that $i \circ \tau$ and $\mathrm{Id}_{\mathcal{A}}$ are chain homotopic, i.e. there is a graded linear map $P : S(\mathcal{A}) \to S(\mathcal{A})$ of degree 1 such that

$$i \circ \tau + \mathrm{Id}_{S(\mathcal{A})} = \partial_S \circ P + P \circ \partial_S. \tag{4.4}$$

Let $x, b \in S(\mathcal{A})$ and $a \in \mathcal{A}$.

- If $x = a$, then put $P(x) := 0$. Relation (4.4) is satisfied, since

$$\begin{aligned} (\tau \circ i + \mathrm{Id}_{S(\mathcal{A})})(a) &= 0 \\ &= (\partial_S \circ P + P \circ \partial_S)(a). \end{aligned}$$

- If $x = ae_2b$, then put $P(x) := ae_1b$. Relation (4.4) is satisfied, since

$$\begin{aligned}(\tau \circ i + \mathrm{Id}_{S(\mathcal{A})})(ae_2b) &= ae_2b \\ &= (\partial_S a)e_1b + ae_2b + ae_1(\partial_S b) \\ &\quad + P\Big((\partial_S a)e_2b + ae_2(\partial_S b)\Big) \\ &= \partial_S(ae_1b) + P\Big((\partial_S a)e_2b + a(\partial_S e_2)b + ae_2(\partial_S b)\Big) \\ &= (\partial_S \circ P + P \circ \partial_S)(ae_2b).\end{aligned}$$

- If $x = ae_1b$, then put $P(x) := 0$. Relation (4.4) is satisfied, since

$$\begin{aligned}(\tau \circ i + \mathrm{Id}_{S(\mathcal{A})})(ae_1b) &= ae_1b \\ &= P\Big((\partial_S a)e_1b + ae_2b + ae_1(\partial_S b)\Big) \\ &= P\Big((\partial_S a)e_1b + a(\partial_S e_1)b + ae_1(\partial_S b)\Big) \\ &= (\partial_S \circ P + P \circ \partial_S)(ae_1b).\end{aligned}$$

Thus, τ and i are homotopy equivalent and we are done. $\square$

Theorem 17. *Stable tame isomorphism class of $(\mathcal{A}_K, \partial_K)$ is an invariant of Legendrian knot K.*

Proof. Let K be a Legendrian knot with DGA $(\mathcal{A}_K, \partial_K)$. We would like to prove that if we change K by one of the following the Reidemeister moves for Lagrangian projection (see Figure 4.6) to the Legendrian knot K', then we obtain stable tame isomorphic DGA $(\mathcal{A}_{K'}, \partial_{K'})$. We look on the Reidemeister move as a smooth path $\{K_t\}_{t\in[-\epsilon/2,\epsilon/2]}$ in the space of Legendrian knots, where $K = K_{-\epsilon/2}$, $K' = K_{\epsilon/2}$ and the only non-generic Legendrian knot appears at $t = 0$.

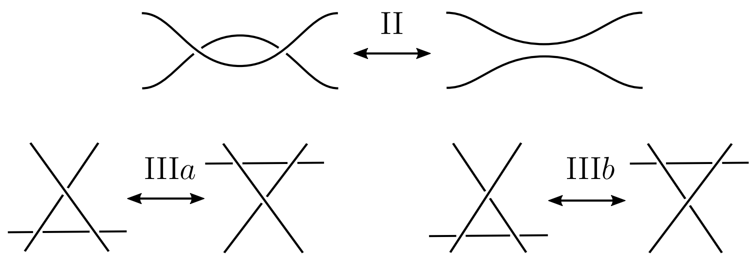


Figure 4.6: Reidemeister moves in Lagrangian projection that will be considered. For each move, the part of diagram that represents K will be on the left and the part of diagram that represents K' will be on the right.

1. *Move IIIa.*
   First, note that graded algebras $\mathcal{A}_K$ and $\mathcal{A}_{K'}$ are isomorphic. Hence, we can label their generators as $a, b, c, a_1, \ldots, a_n$, where a, b, c will label double points as in the Figure 4.7. It remains to show that $\partial_K = \partial_{K'}$.

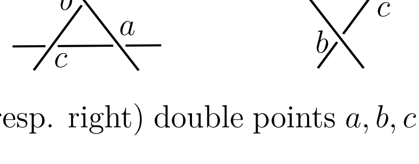


Figure 4.7: On the left (resp. right) double points a, b, c of $\pi_L(K)$ (resp. $\pi_L(K')$).

By Leibniz formula, it will be enough to construct a bijection B between admissible immersed polygons for K and K'. B will map admissible immersed triangle T with vertices a, b, c to admissible immersed triangle T' with vertices a, b, c. If we apply relation (4.2) on T, then we obtain that

$$H(b) + H(c) < H(a). \tag{4.5}$$

Besides T and T', it is not possible to have admissible immersed polygon with side $[a, b]$, $[a, c]$ or $[b, c]$. Indeed, if $[a, b]$ or $[a, c]$ is such a side, then a will be an image of the negative vertex and b or c, respectively, will be an image of the positive vertex, which is in contradiction with relation (4.5). Also, $[b, c]$ can not be such a side, since admissible immersed polygon can not have two positive vertices by relation (4.2).

Thus, B will be bijection between admissible immersed polygons with the same covered quadrants and double points as vertices. Moreover, except cases T and T', this mapping will be continuous deformation in t, see Figure 4.8.

Figure 4.8: Examples of mapping admissible immersed polygons by B.

2. *Move IIIb.*
Similarly as in the previous case, graded algebras $\mathcal{A}_K$ and $\mathcal{A}_{K'}$ are isomorphic. Also, $a, b, c, a_1, \ldots, a_n$ will denote their generators, where a, b, c will label double points as in the Figure 4.9. We would like to show that tame isomorphism $\phi : (\mathcal{A}_K, \partial_K) \to (\mathcal{A}_{K'}, \partial_{K'})$, defined on generators by

$$\phi(w) = \begin{cases} w + bc, & w = a, \\ w, & \text{else}, \end{cases}$$

is well defined. In another words, we would like to verify that $\phi \circ \partial_K = \partial_{K'} \circ \phi$.

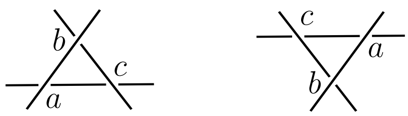


Figure 4.9: On the left (resp. right) double points a, b, c of $\pi_L(K)$ (resp. $\pi_L(K')$)

At first, we make the following *discussion* about admissible immersed polygons with side $[a, b]$, $[a, c]$ or $[b, c]$.

Immersed triangles T and T' with vertices a, b, c are not admissible. Thus, admissible immersed polygons with side $[a, b]$ or $[a, c]$ can not have a as an image of the positive vertex.

Next, we inspect admissible immersed polygons with the side $[b, c]$. Denote them $P_{[b,c]}$ and $P'_{[b,c]}$. We will show that a_i will be a positive vertex of immersed $P_{[b,c]}$ (or $P'_{[b,c]}$). By relation (4.2), b and c can not be images of positive vertex, since they are images of the negative vertex. Now, assume that a is an image of positive vertex. By relation (4.2), if we compare the signed sum of heights, then we see that the area of immersed $P_{[b,c]}$ (or $P'_{[b,c]}$) is less than the area of immersed triangle T (or T'). Also, as ϵ goes to zero, areas of triangles vanish. But, areas of immersed $P_{[b,c]}$ (or $P'_{[b,c]}$) can not vanish. Hence, we get a contradiction.

Also, admissible immersed polygon with side $[a, c]$ or $[a, b]$, where a is an image of the positive vertex, can not contain vertex b or c, respectively. This follows directly from the signed count of heights in this polygon and in T (or T').

It is enough to check that ϕ is a chain morphism only for a and $g \in Q(K)$ such that $\partial_K(g)$ contains a letter a. Indeed, by the discussion, we see that for all other generators the differential remains the same. Also, note that by relation (4.2) we have $g \neq a$.

We would like to group admissible immersed polygons of $\partial_K(g)$ and $\partial_{K'}(g)$ into three groups:

(i) immersed polygons of $\partial_{K'}(g)$ that locally look as in Figure 4.10(a) and the corresponding immersed polygons of $\partial_K(g)$ that locally look as in Figure 4.10(b) and Figure 4.10(c). By the "corresponding" we mean that (b) and (c) contribute to $\partial_K(g)$ with the same letters as (a) contribute to $\partial_{K'}(g)$, only (c) contributes with bc instead of a. Also, by the discussion is this grouping correct, i.e. it is unique defined and all immersed polygons of (a), (b), (c) will be contained in some triple.

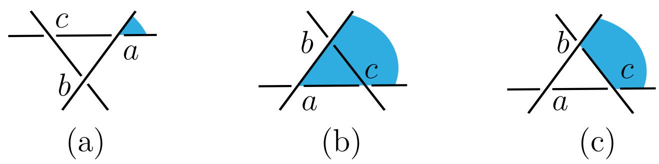


Figure 4.10

(ii) immersed polygons of $\partial_K(g)$ that locally look as in Figure 4.11(a) and the corresponding immersed polygons of $\partial_{K'}(g)$ that locally looks as in Figure 4.11(b) and Figure 4.11(c). By the "corresponding", we mean the same as above, only switch K and K'. Also, by the discussion this grouping is correct.

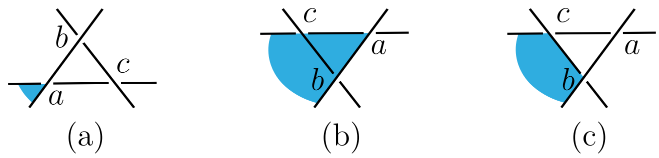


Figure 4.11

(iii) remaining immersed polygons of $\partial_K(g)$ and $\partial_{K'}(g)$. Clearly, there is a bijection between them, which pairs those with exactly same covered quadrants and double points as vertices.

For $j = \text{i, ii, iii}$, we denote by $\partial^j_K(g)$ or $\partial^j_{K'}(g)$ contributions of elements of these three groups to $\partial_K(g)$ or $\partial_{K'}(g)$, respectively. Clearly, $\partial^{\text{i}}_K(g) = \phi \circ \partial^{\text{i}}_{K'}(g)$, $\phi \circ \partial^{\text{ii}}_K(g) = \partial^{\text{ii}}_{K'}(g)$, $\partial^{\text{iii}}_K(g) = \partial^{\text{iii}}_{K'}(g)$. And since $\phi^2 = \mathrm{Id}_{\mathcal{A}_K}$ and $\phi(g) = g$, ϕ is a chain morphism in this case.

Now, we would like to compare $\partial_K a$ to $\partial_{K'} a$. Note that for small ϵ, we can assume that $H(a) > H(b)$ and $H(a) > H(c)$. Hence, by relation (4.2), neither of $\partial_K(a), \partial_K(b), \partial_K(c)$ contain a letter a. Thus, $\phi \circ \partial_K \circ \phi^{-1}(a) = \phi \circ \partial_K(a + bc) = \partial_K(a + bc)$. It remains to verify that

$$\partial_K(a) + \partial_{K'}(a) = b(\partial_K(c)) + (\partial_K(b))c. \tag{4.6}$$

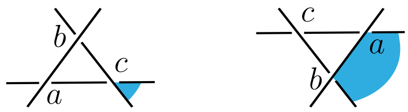
To prove relation (4.6), we will construct the following groups. We start with groups containing immersed polygons of $\partial_K(c)$:

(i) immersed polygons of $\partial_K(c)$ that locally look as in Figure 4.12(a) and the corresponding immersed polygons of $\partial_K(a)$ that locally looks as in Figure 4.12(b). By the "corresponding", we mean that (b) contribute to $\partial_K(a)$ with the same letters that (a) contribute to $\partial_K(c)$, only (b) contribute moreover with b at the begining. Also, by the discussion the correspondence is bijective.

(a) (b)

Figure 4.12

(ii) immersed polygons of $\partial_K(c)$ that locally look as in Figure 4.13(a) and the corresponding immersed polygons of $\partial_{K'}(a)$ that locally looks as in Figure 4.13(b). By the "corresponding", we mean the same as above. Also, by the discussion the correspondence is bijective.

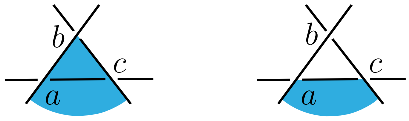


(a) (b)

Figure 4.13

Now, we will make analogue groups for $\partial_K(b)$:

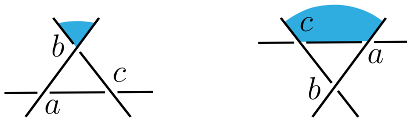
(iii) immersed polygons of $\partial_K(b)$ that locally look as in Figure 4.14(a) and the corresponding immersed polygons of $\partial_K(a)$ that locally looks as in Figure 4.14(b). By the "corresponding", we mean that (b) contributes to $\partial_K(a)$ with the same letters that (a) contribute to $\partial_K(b)$, only (b) contribute moreover with c at the end. Also, by the discussion the correspondence is bijection.



(a) (b)

Figure 4.14

(iv) immersed polygons of $\partial_K(b)$ that locally look as in Figure 4.15(a) and the corresponding immersed polygons of $\partial_{K'}(a)$ that locally looks as in Figure 4.15(b). By the "corresponding", we mean the same as above. Also, by the discussion the correspondence is bijection.



(a) (b)

Figure 4.15

Still, there is just one group of admissible immersed polygons to inspect:

(v) remaining immersed polygons of $\partial_K a$ and $\partial_{K'} a$, clearly there is a bijection between them pairing those with exactly same covered quadrants and doublepoints as vertices.

If we look back at relation (4.6), we see that terms contributed by the elements of the group (v) cancel each other (since we count mod 2). Now, we inspect remaining terms in relation (4.6). But, each term on the left side of relation (4.6) is also on the right side and vice versa. This is true, since correspondences in the groups (i)–(iv) are bijections. Hence, we find out that relation (4.6) holds.

3. *Move II.*
Let $a, b, a_1, \ldots, a_n, b_1, \ldots, b_m$ be generators of graded algebra $\mathcal{A}_K$, where a, b are double points of K as in the Figure 4.16. Let X be a two-sided ideal in $\mathcal{A}_K$ generated by a, b. Since algebras $\mathcal{A}_K/X$ and $\mathcal{A}_{K'}$ are isomorphic, we can label corresponding generators of $\mathcal{A}_{K'}$ by $a_1, \ldots, a_n, b_1, \ldots, b_m$. Moreover, by relation (4.2), we can assume that:

$$H(a_n) \geq \cdots \geq H(a_1) \geq H(a) > H(b) \geq H(b_1) \geq \cdots \geq H(b_m).$$

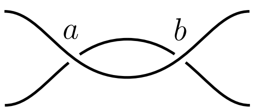


Figure 4.16: Double points a, b of $\pi_L(K)$.

Next, we compare the area of the admissible immersed 2-gon with vertices a, b to areas of other admissible immersed polygons with vertex a covered by a positive quadrant. If we use the similar argument as in the discussion of the move IIIb., then we see that $\partial_K(a)$ contains only admissible immersed 2-gon as an immersed polygon with vertex b. Hence, we can write, $\partial_K(a) = b + v$, where v contains only terms in b_i.

Let $(S_{|a|}(\mathcal{A}_{K'}), \partial_S)$ be a degree $|a|$ stabilization of $(\mathcal{A}_{K'}, \partial_{K'})$. Next, ϕ_0 will be a graded isomorphism $\phi_0 : \mathcal{A}_K \to S_{|a|}(\mathcal{A}_{K'})$ which is on generators defined as

$$\phi_0(w) = \begin{cases} e_1, & w = a, \\ e_2 + v, & w = b, \\ w, & \text{else.} \end{cases}$$

Also, $\mathcal{A}_i$ will denote graded algebra $\mathcal{A}_K(a, b, a_1, \ldots, a_i, b_1, \ldots, b_m)$ and τ and i will be natural projection and inclusion, respectively:

$$\mathcal{A}_{K'} \xrightarrow{i} S_{|a|}(\mathcal{A}_{K'}) \xrightarrow{\tau} \mathcal{A}_{K'} .$$

Now, we would like to prove following two claims.

Claim 1. $\phi_0|_{\mathcal{A}_0}$ is a chain map.

The claim is true for b_i, since, by relation (4.2), we see that $\partial_K(b_i)$ contains only terms of b_j for $j > i$. For a, b, we may check that:

$$\phi_0 \circ \partial_K(a) = \phi_0(b + v) = e_2 = \partial_S(e_1) = \partial_S \circ \phi_0(a)$$

and

$$\phi_0 \circ \partial_K(b) = \phi_0 \circ \partial_K(v) = \partial_S(v) = \partial_S(v + e_2) = \partial_S \circ \phi_0(b),$$

where we used that $\partial_K^2 = 0$ and $\partial_K(a) = b + v$. Hence, the claim holds.

Claim 2. $\tau \circ \partial_S \circ \phi_0 = \tau \circ \phi_0 \circ \partial_K$.

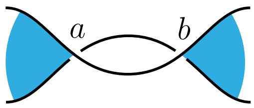
Using Claim 1, it remains to verify the relation for generators a_i. We will denote by W_1 the sum of terms that occur in both $\partial_K(a_i)$ and $\partial_S(a_i)$. W_2 will denote the sum of terms of $\partial_K(a_i)$ that involve b. Then we can write:

$$\partial_K(a_i) = W_1 + W_2 + W_3 \text{ and } \partial_S(a_i) = W_1 + W_4.$$

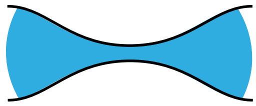
Since terms in W_1 contain neither a nor b, we get $\phi_0(W_1) = W_1$. Terms in W_3 need to contain a, but $\phi_0(a) = e_1$, thus $\tau \circ \phi_0(W_3) = 0$. Terms in W_4 correspond to admissible immersed polygons that locally look as in Figure 4.17(b). There is a bijective correspondence between these admissible immersed polygons and pairs of admissible immersed polygons in $\pi_L(K)$: one with vertex a covered with a positive quadrant and second with b covered with a negative quadrant. These quadrants look locally as in Figure 4.17(a). Hence W_4 is the same as W_2, only every appearance of b is replaced by v, i.e. $W_4 = \tau \circ \phi_0(W_2)$. Now, we can conclude to the computation:

$$\begin{aligned} \tau \circ \partial_S \circ \phi_0(a_i) &= \tau \circ \partial_S(a_i) \\ &= \tau(W_1 + W_4) \\ &= \tau(\phi_0(W_1) + W_4) \\ &= \tau \circ \phi_0(W_1 + W_2) \\ &= \tau \circ \phi_0(W_1 + W_2 + W_3) \\ &= \tau \circ \phi_0 \circ \partial_S(a_i). \end{aligned}$$

Thus, the claim holds.



(a)



(b)

Figure 4.17

Now, we would like to construct desired tame isomorphism $\phi_n : (\mathcal{A}_K, \partial_K) \to (S_{|a|}(\mathcal{A}_{K'}), \partial_S)$. This will be done by induction starting from ϕ_0 and which for each $i \leq n$ will give an isomorphism $\phi_i : \mathcal{A}_K \to S_{|a|}(\mathcal{A}_{K'})$ such that $\phi_i|_{\mathcal{A}_i}$ is a chain map.

Let $P : S_{|a|}(\mathcal{A}_{K'}) \to S_{|a|}(\mathcal{A}_{K'})$ be an analogue chain homotopy between $i \circ \tau$ and $\mathrm{Id}_{S_{|a|}(\mathcal{A}_{K'})}$ (for simplicity, we will write Id) as in the proof of Lemma 16. Recall that this map satisfies:

$$i \circ \tau + \mathrm{Id} = \partial_S \circ P + P \circ \partial_S.$$

Next, we define graded automorphism $g_i : S_{|a|}(\mathcal{A}_{K'}) \to S_{|a|}(\mathcal{A}_{K'})$ as

$$g_i(w) = \begin{cases} w + P\Big(\partial_S(w) + \phi_{i-1} \circ \partial_K(w)\Big), & w = a_i, \\ w, & \text{otherwise.} \end{cases}$$

And finally, put $\phi_i = g_i \circ \phi_{i-1}$.

Since step with ϕ_0 is already done, we can conclude to the induction step. By induction hypothesis, we can assume that $\phi_{i-1}|_{\mathcal{A}_{i-1}}$ is a chain map. Then $\phi_i|_{\mathcal{A}_{i-1}}$ is also a chain map. It remains to show that $\phi_i \circ \partial_K(a_i) = \partial_S \circ \phi_i(a_i)$. Since $\tau \circ P = 0$, for all i it holds that $\tau \circ g_i = \tau$ and consequently $\tau \circ \phi_i = \tau \circ \phi_0$. Also, observe that $\partial_K(a_i) \in \mathcal{A}_{i-1}$. Using Claim 2 we compute that:

$$\begin{aligned}
\phi_{i-1} \circ \partial_K(a_i) &= i \circ \tau \circ \phi_{i-1} \circ \partial_K(a_i) + \partial_S \circ P \circ \phi_{i-1} \circ \partial_K(a_i) \\
&\quad + P \circ \partial_S \circ \phi_{i-1} \circ \partial_K(a_i) \\
&= i \circ \tau \circ \phi_0 \circ \partial_K(a_i) + \partial_S \circ P \circ \phi_{i-1} \circ \partial_K(a_i) \\
&\quad + P \circ \phi_{i-1} \circ \partial_K^2(a_i) \\
&= i \circ \tau \circ \partial_S \circ \phi_0(a_i) + \partial_S \circ P \circ \phi_{i-1} \circ \partial_K(a_i) \\
&= (\mathrm{Id} + \partial_S \circ P + P \circ \partial_S) \circ \partial_S(a_i) + \partial_S \circ P \circ \phi_{i-1} \circ \partial_K(a_i) \\
&= \partial_S(a_i + P \circ \partial_S(a_i) + P \circ \phi_{i-1} \circ \partial_K(a_i)) \\
&= \partial_S \circ g_i(a_i).
\end{aligned}$$

Since $\phi_{i-1} \circ \partial_K(a_i) \in \mathcal{A}_{i-1}$, we conclude that

$$\phi_i \circ \partial_K(a_i) = g_i \circ \phi_{i-1} \circ \partial_K(a_i) = \phi_{i-1} \circ \partial_K(a_i) = \partial_S \circ g_i(a_i) = \partial_S \circ \phi_i(a_i).$$

Hence, DGAs $(\mathcal{A}_K, \partial_K)$ and $(\mathcal{A}_{K'}, \partial_{K'})$ are stable tame isomorphic and we are done.

$\square$

From Theorem 17 and Lemma 16, we immediately obtain the following important corollary.

Corollary 18. *Homology $H_*(\mathcal{A}_K, \partial_K)$ is an invariant of a Legendrian knot K.*

Definition 22. *Let K be a Legendrian knot. Homology of $(\mathcal{A}_K, \partial_K)$ is called* ***Legendrian contact homology*** *of K and denoted by $LCH_*(K)$.*

4.4 Chekanov-Eliashberg DGA over $\mathbb{Z}[t, t^{-1}]$

Observe that invariants coming from Chekanov-Eliashberg DGA over $\mathbb{Z}_2$ are a bit limited. For example, established DGA does not detect the orientation of Legendrian knots, and hence induced LCH does not detect the orientation too. Thus, we would like to introduce the following Chekanov-Eliashberg DGA over $\mathbb{Z}[t, t^{-1}]$, which will produce more powerful invariants of Legendrian knots. We will follow [EN18].

First, we will fix a base point $*$ in given Legendrian knot K, which is in $\pi_L(K)$ different from double points. Similarly as before, we would like to define DGA $(\mathcal{A}_K, \partial_K)$ in terms of algebra, grading, and differential.

The algebra: $\mathcal{A}_K$ will be a free unital noncommutative $\mathbb{Z}$-algebra generated by $Q(K)$ and letters t, t^{-1} satisfying relations $t \cdot t^{-1} = t^{-1} \cdot t = 1$.

The grading: $\mathcal{A}_K$ will be graded over $\mathbb{Z}$ as follows. For t and t^{-1}, we put $|t| = 2r(K)$ and $|t^{-1}| = -2r(K)$. To any double point a we associate a path γ_a along $\pi_L(K)$ starting from a^+, going to a^- and missing a base point $*$. Then, we put

$$|a| = 2r(\gamma_a) - 1/2$$

and expand the grading onto whole $\mathcal{A}_K$.

The differential: first, we need to introduce a definition of **orientation sign**, which will be associated to each quadrant of crossing in $\pi_L(K)$ as in Figure 4.18.

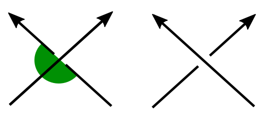


Figure 4.18: Orientation sign for green coloured quadrants is equal to -1 and for remaining quadrants is equal to 1.

Let each of $a, b_1, \ldots, b_k$ take a value in $Q(K)$. Similarly as in the proof of Lemma 14, we associate for $u \in \mathcal{M}^a_{b_1,\ldots,b_k}$ a set of paths $\eta_0, \ldots, \eta_k$ such that their union is $u(\partial P_{k+1})$. Moreover, we now require that η_i follows the orientation of K. Then, $t(\eta_i)$ will denote $t^{n_i(u)}$, where $n_i(u)$ is sign count of passing η_i through the basepoint $*$.

We put

$$w(u) = t(\eta_0)b_0t(\eta_1)b_1 \ldots t(\eta_k)b_k$$

and define

$$\delta(u) = \delta(a) \prod_{i=1}^{k} \delta(b_i)$$

as a product of orientations signs of quadrant that covers vertices of $u(\partial P_{k+1})$.

Then, the definition of differential $\partial_K : \mathcal{A}_K \to \mathcal{A}_K$ will be the following. For a generator a of $\mathcal{A}_K$, we put

$$\partial_K(a) = \sum_{\substack{k \geq 0 \\ b_1,\dots,b_k \in Q(K)}} \sum_{u \in \mathcal{M}^a_{b_1,\dots,b_k}} \delta(u) w(u)$$

and $\partial_K(t) = \partial_K(t^{-1}) = 0$. Finally, extend ∂_K on $\mathcal{A}_K$ by Leibniz formula.

Analogues of Lemmata 13–16 and Theorem 17 can be proven for this version of DGA, see [EN18].

From now on, we will use this new version of DGA.

4.5 A note on Chekanov-Eliashberg DGA

Even though Chekanov-Eliashberg DGA produces, as we will see further, powerful invariants of Legendrian knots, there is a large class of Legendrian knots that are not Legendrian isotopic, but have stable tame isomorphic DGAs.

Lemma 19 ([Che02]). *Stabilized Legendrian knots have stable tame isomorphic DGAs. In particular, their Legendrian contact homology vanishes.*

Proof. Let K be a Legendrian knot. By the definition of stabilization in Lagrangian projection, we add a double point a with a small loop to K. Using relation (4.2) we see that the area of the lifted loop is equal to $H(a)$. Hence, we can add a double point a such that its height is smaller than any other heights of the double points in $\pi_L(S(K))$. The only possible contributions of admissible immersed polygons to $\partial_{S(K)}(a)$ will be admissible immersed monogones. If there were two contribution by monogones to $\partial_{S(K)}(a)$, then the knot would be an unstabilized unknot (see Figure 2.2), which leads to the contradiction. And so, $\partial_{S(K)}(a) = t^n$ for some integer n.

Now, we inspect DGA $(\mathcal{A}_{S(K)}, \partial_{S(K)})$. Let w be its cycle. Since

$$\partial_{S(K)}(at^{-n}w) = \partial_{S(K)}(a)t^{-n}w + a\partial_{S(K)}(t^{-n}w) = w,$$

w is also a boundary. Thus, $(\mathcal{A}_{S(K)}, \partial_{S(K)})$ is stable tame isomorphic to the trivial algebra, i.e. DGA with only generator a such that $\partial(a) = 1$. In particular, $LCH_*(S(K)) = 0$. □

5. Representations of Chekanov-Eliashberg DGA

5.1 Augmentations

Since Chekanov-Eliashberg DGA is non-commutative and often not of finite rank, even in a fixed degree, it is often difficult to compute LCH. In this section, we would like to introduce augmentations of DGA and linearized LCH, which will produce some useful invariants of Legendrian knots. We will follow [EN18] and [Ghi]. Also, $\mathbb{F}$ will denote a finite field.

Definition 23. *An* ***ungraded augmentation*** ϵ *of DGA* $(\mathcal{A}_K, \partial_K)$ *into* $\mathbb{F}$ *is an algebra homomorphism*

$$\epsilon : \mathcal{A}_K \to \mathbb{F},$$

such that $\epsilon(1) = 1$ *and* $\epsilon \circ \partial_K = 0$.
Moreover, if $\epsilon(a) = 0$ *for* $|a| \neq 0$*, then* ϵ *is called a* ***(graded) augmentation****.*

Remark. We immediately obtain from the definition above that $\epsilon(t)$ is invertible, hence a nonzero element of $\mathbb{F}$.

Remark. Note that (graded or ungraded) augmentations do not exist for all Legendrian knots, because the existence implies $LCH_* \neq 0$. Hence, by Lemma 19, we see that it is not possible to construct them for stabilized Legendrian knots.

To have some sufficient condition for the existence of augmentations, we introduce the following theorem.

Theorem 20 ([Rut06])**.** *Let K be a Legendrian knot. Then Kauffman bound of K is an equality if and only if DGA $(\mathcal{A}_K, \partial_K)$ admits an ungraded augmentation into $\mathbb{Z}_2$.*

Remark. Note that it is possible to extend augmentation ϵ over stabilization of DGA. This can be done simply by sending both e_1, e_2 to $\epsilon(e_1) = \epsilon(e_2) = 0$.
Hence, the existence of (graded or ungraded) augmentation is an invariant of Legendrian knot.

However, more important property of augmentations is that they can be used for linearization of DGA. And thus they consequently produce linearized LCH, which will have nicer properties to handle than LCH.

Now, we would like to linearize DGA and hence obtain linearized LCH. Let us assume that ϵ is an augmentation of $(\mathcal{A}_K, \partial_K)$ into $\mathbb{F}$. Next, define DGA $(\mathcal{A}_K^\epsilon, \partial)$, where

$$\mathcal{A}_K^\epsilon = \frac{\mathcal{A}_K \otimes \mathbb{F}}{\langle t = \epsilon(t) \rangle}.$$

It is a free $\mathbb{F}$-algebra generated by $Q(K)$. Since t is invertible, $\epsilon(t) \neq 0$, and hence $|t| = 0$. Thus, grading on $\mathcal{A}_K^\epsilon$ can be simply inherited from grading of $\mathcal{A}_K$. And also, ∂ is obtained from the action of ∂_K on $\mathcal{A}_K$ such that for any $a \in \mathcal{A}_K^\epsilon$, each occurrence of $t^{\pm 1}$ in $\partial_K(a)$ is replaced by $\epsilon(t^{\pm 1}) \in \mathbb{F}^\times$.

We introduce word length filtration

$$\mathcal{A}_K^\epsilon = (\mathcal{A}_K^\epsilon)^0 \supset (\mathcal{A}_K^\epsilon)^1 \supset \cdots \supset (\mathcal{A}_K^\epsilon)^n \supset \dots,$$

where $(\mathcal{A}_K^\epsilon)^n$ is a subalgebra of $\mathcal{A}_K^\epsilon$, which is generated as a vector space by words in $Q(K)$ of length at least n. We would like to have differential on $\mathcal{A}_K^\epsilon$ that preserves word length filtration. Observe that ∂ preserves word length filtration on $\mathcal{A}_K^\epsilon$ if and only if $\partial(a)$ does not contain a constant term for any $a \in Q(K)$. But this is never satisfied.

Let ϕ^ϵ be an elementary automorphism on $\mathcal{A}_K^\epsilon$, which is given by

$$\phi^\epsilon(a) = a + \epsilon(a)$$

for $a \in Q(K)$. Then we define a new differential ∂_K^ϵ on $\mathcal{A}_K^\epsilon$ as

$$\partial_K^\epsilon = \phi^\epsilon \circ \partial \circ (\phi^\epsilon)^{-1}$$

Clearly, $\partial_K^\epsilon \circ \partial_K^\epsilon = 0$ and ∂_K^ϵ has degree -1.

Lemma 21. *∂_K^ϵ preserves word length filtration on $\mathcal{A}_K^\epsilon$.*

Proof. Let $a \in Q(K)$ and $\partial(a) = c + \sum b_1 \dots b_k$ for some $c \in \mathbb{F}$. Note that $(\phi^\epsilon)^{-1}(a) = a - \epsilon(a)$, so

$$\begin{aligned}
\partial_K^\epsilon &= \phi^\epsilon \circ \partial(a - \epsilon(a)) \\
&= \phi^\epsilon(c + \textstyle\sum b_1 \dots b_k) \\
&= \phi^\epsilon(c) + \textstyle\sum \phi^\epsilon(b_1) \dots \phi^\epsilon(b_k) \\
&= c + \textstyle\sum (b_1 + \epsilon(b_1)) \dots (b_k + \epsilon(b_k)) \\
&= c + \textstyle\sum \epsilon(b_1) \dots \epsilon(b_k) + \text{terms in } (\mathcal{A}_K^\epsilon)^1.
\end{aligned}$$

But $c + \sum \epsilon(b_1) \dots \epsilon(b_k) = \epsilon \circ \partial(a) = 0$, hence $\partial_K^\epsilon\Big((\mathcal{A}_K^\epsilon)^1\Big) \subseteq (\mathcal{A}_K^\epsilon)^1$. Trivially, $\partial_K^\epsilon\Big((\mathcal{A}_K^\epsilon)^0\Big) \subseteq (\mathcal{A}_K^\epsilon)^0$ and since ∂_K^ϵ satisfies Leibniz formula, the proof for $(\mathcal{A}_K^\epsilon)^{n>1}$ follows. $\square$

Next, $(\partial_K^\epsilon)_1$ will denote differential induced from ∂_K^ϵ as a map

$$(\partial_K^\epsilon)_1 : \left(\frac{(\mathcal{A}_K^\epsilon)^1}{(\mathcal{A}_K^\epsilon)^2}\right) \to \left(\frac{(\mathcal{A}_K^\epsilon)^1}{(\mathcal{A}_K^\epsilon)^2}\right).$$

Since $(\partial_K^\epsilon)_1 \circ (\partial_K^\epsilon)_1 = 0$ and $(\partial_K^\epsilon)_1$ has degree -1, we can put the following definition.

Definition 24. *For given augmentation ϵ of $(\mathcal{A}_K, \partial_K)$ into $\mathbb{F}$, we define* ***linearized Legendrian contact homology*** *as*

$$LCH_*^\epsilon(K) = H_*\left(\frac{(\mathcal{A}_K^\epsilon)^1}{(\mathcal{A}_K^\epsilon)^2}, (\partial_K^\epsilon)_1\right).$$

Remark. It is also possible to define equivalently $LCH_*^\epsilon(K)$ as $H_*(A, (\partial_K^\epsilon)_1)$, where A is $\mathbb{F}$-vector space generated by $Q(K)$ and differential $(\partial_K^\epsilon)_1$ is defined as follows. Let $a \in Q(K)$, then

$$(\partial_K^\epsilon)_1(a) = \sum_{\substack{k \geq 1 \\ b_1, \dots, b_k \in Q(K)}} \sum_{u \in \mathcal{M}^a_{b_1,\dots,b_k}} \sum_{j=1}^{k} \delta(u) \epsilon(t)^{\sum_{i=0}^k n_i(u)} \epsilon(b_1) \dots \epsilon(b_{j-1}) \epsilon(b_{j+1}) \dots \epsilon(b_k) b_j.$$

Now, we are going to state the following famous theorem. This theorem was originally stated by Chekanov [Che02] for augmentations from DGA with $\mathbb{Z}_2$ coefficients into $\mathbb{Z}_2$. But, the result can be naturally extended for (graded or ungraded) augmentations from DGA over $\mathbb{Z}[t, t^{-1}]$ into $\mathbb{F}$.

Theorem 22. *The set*

$$\left\{ \text{Isomorphism classes of } LCH_*^{\epsilon}(K) \;\middle|\; \begin{array}{c} \epsilon \text{ is a (graded or ungraded)} \\ \text{augmentation of } (\mathcal{A}_K, \partial_K) \text{ into } \mathbb{F} \end{array} \right\}$$

is an invariant of Legendrian knot K.
Alternatively, the set of Poincaré polynomials

$$P^{\epsilon}(t) = \sum_{i=-\infty}^{\infty} \dim(LCH_i^{\epsilon}(K))t^i$$

over all augmentations ϵ of $(\mathcal{A}_K, \partial_K)$ into $\mathbb{F}$ is an invariant of K.

Remark. As a consequence of the previous theorem, Chekanov [Che02] gave an example of two Legendrian knots K_1, K_2 (see Figure 5.1).

They are both of the topological knot type $m(5_2)$. Also, rotation numbers are both equal to 0. And Thurston-Bennequin invariant is for both knots equal to 1. Hence, they have same classical invariants.

Also, the cardinality of the set of Poincaré polynomials is in both cases equal to 1. However, the polynomials are different. In the case of K_1, we obtain a polynomial $2+x$. But, in the case of K_2, we get a polynomial $x^{-2}+x+x^2$. Thus, K_1 and K_2 are not Legendrian isotopic.

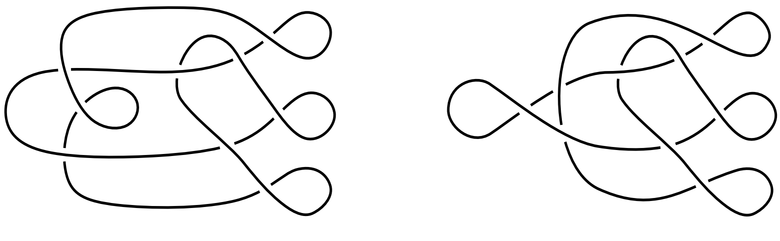


Figure 5.1: Chekanov's knot K_1 (left) and K_2 (right).

5.2 Representations

In this section, we generalize the notion of augmentations into n-dimensional representations, where augmentations can be seen as 1-dimensional representations.

Definition 25. *An **ungraded n-dimensional representation** ϵ of $(\mathcal{A}_K, \partial_K)$ into $\mathbb{F}$ is an algebra homomorphism*

$$\epsilon : \mathcal{A}_K \to Mat(n, \mathbb{F}),$$

such that $\epsilon(1) = 1$ and $\epsilon \circ \partial_K = 0$.
*Moreover, if $\epsilon(a) = 0$ for $|a| \neq 0$, then ϵ is called a **(graded) n-dimensional representation**.*

Remark. Analogously, the existence of representation implies nonvanishing LCH. And hence, it is not possible to construct (graded or ungraded) representations of stabilized knots.

Remark. Also, nonvanishing LCH is not a sufficient condition for the existence of representation. Sivek gave an example of $m(10_{132})$ knot (see Figure 5.2) that has nonvanishing LCH, but do not admit any representation into $\mathbb{Z}_2$. Note that the result can be generalized for representations into $\mathbb{F}$, see [DRG15].

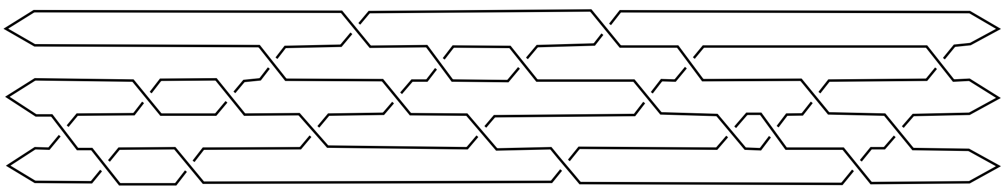


Figure 5.2: Front diagram of Legendrian knot $m(10_{132})$.

We continue with an important remark.

Remark. Sivek [Siv13] described that $(p, -q)$-torus knots with $p \geq 3$ odd and $q > p$ (see 5.3) admit ungraded 2-dimensional representations into $\mathbb{Z}_2$, but do not admit ungraded 1-dimensional representations into $\mathbb{Z}_2$. From this perspective, studying higher dimensional representations is meaningful.

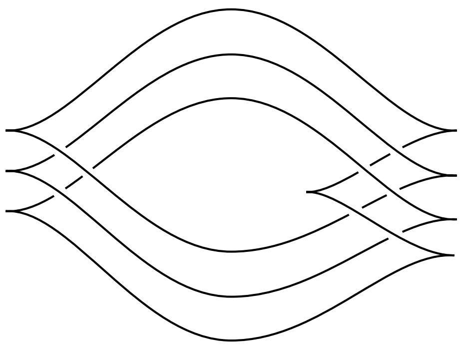


Figure 5.3: Front diagram of $(3, -4)$-torus knot.

Remark. The definition of Chekanov-Eliashberg DGA can be extended for Legendrian links with a base point for each component. Also, for Legendrian links, we can extend a definition of (graded or ungraded) n-dimensional representation. For details, we refer the reader to [Ng03].

Remark. Now, we introduce an interesting connection between augmentations and higher dimensional representations. But first, we make the following construction.

Let K be a Legendrian knot and n is a positive integer. In front projection, we construct a Legendrian link K' consisting n-copies of K, where each of them is different from all others by a small shift in z direction. Then take a small segment of K' that is oriented from left to right and does not contain any singularities. And replace such a segment with a positive full twist, see Figure 5.4. Finally, denote the resulting link by $K^{(n)}$. This construction is a well-defined operation on Legendrian knots, see [NT04].

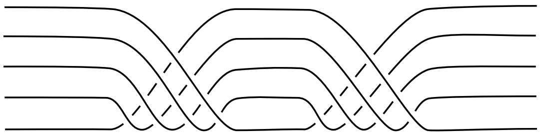


Figure 5.4: Positive full twist of 5 parallel straight lines.

Theorem 23 ([NR13]). *Let K be a Legendrian knot. Then DGA $(\mathcal{A}_K, \partial_K)$ admits ungraded n-dimensional representation into $\mathbb{Z}_2$ if and only if DGA $(\mathcal{A}_{K^{(n)}}, \partial_{K^{(n)}})$ admits ungraded augmentation into $\mathbb{Z}_2$.*

Remark. Linearized LCH can be stated also for n-dimensional representations, where $n > 1$. For details, we refer the reader to [CRGG16].

Definition 26. *Two representations $\epsilon_1, \epsilon_2 : \mathcal{A}_K \to Mat(n, \mathbb{F})$ are called **equivalent**, if there is a vector space isomorphism $f : \mathbb{F}^n \to \mathbb{F}^n$ such that $f \circ \epsilon_1 = \epsilon_2 \circ f$ for each $a \in \mathcal{A}_K$.*

Definition 27. *Representation $\epsilon : \mathcal{A}_K \to Mat(n, \mathbb{F})$ is called **irreducible** if trivial subspaces are the only subspaces $V \subseteq \mathbb{F}^n$ such that $\epsilon(a)V \subseteq V$ for each $a \in \mathcal{A}_K$.*

5.3 Exact Lagrangian cobordisms

Here, we give a brief introduction to exact Lagrangian cobordisms and show an interesting consequence to representations of Chekanov-Eliashberg DGA.

Definition 28. *Let L_1, L_2 be Legendrian links. We say that surface Σ in $\Big(\mathbb{R}_t \times \mathbb{R}^3_{x,y,z}, d(e^t \alpha_{std})\Big)$ is an **exact Lagrangian cobordism** from L_1 to L_2 (see Figure 5.5) and write $L_1 \prec_\Sigma L_2$ if there exists $T > 0$ such that the following holds:*

(i.) $\Sigma \cap \Big((T, \infty) \times \mathbb{R}^3\Big) = (T, \infty) \times L_2,$

(ii.) $\Sigma \cap \Big((-\infty, -T) \times \mathbb{R}^3\Big) = (-\infty, -T) \times L_1,$

(iii.) there exists a smooth function $f : \Sigma \to \mathbb{R}$ such that $e^t \alpha_{std}|_{T\Sigma} = df$ and f is a constant when $t \leq -T$ and $t \geq T$.

*Also, $(T, \infty) \times L_2 \subset \Sigma$ and $(-\infty, -T) \times L_1 \subset \Sigma$ are called **positive** and **negative end** of Σ, respectively.*

*If Σ is diffeomorphic to a cylinder, then we call such an exact Lagrangian cobordism a **concordance**.*

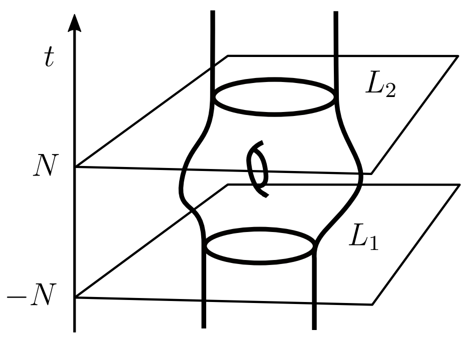


Figure 5.5: Exact Lagrangian cobordism Σ.

Definition 29. *Let L_1, L_2, L_3 be Legendrian links such that $L_1 \prec_{\Sigma_{1,2}} L_2$ and $L_2 \prec_{\Sigma_{2,3}} L_3$.* ***Concatenation*** *of $\Sigma_{1,2}$ and $\Sigma_{2,3}$ is an exact Lagrangian cobordism from L_1 to L_3 obtained by gluing the positive end of $\Sigma_{1,2}$ with the negative end of $\Sigma_{2,3}$. We denote such an exact Lagrangian cobordism by $\Sigma_{1,2} \odot \Sigma_{2,3}$.*

Remark. It is possible to establish the category $\mathfrak{Cob}$ of exact Lagrangian cobordisms. Objects will be Lagrangian links and morphisms will be exact Lagrangian cobordisms between them. Then, the composition of morphisms in $\mathfrak{Cob}$ will be concatenation and identity morphism for Legendrian link L is given as an exact Lagrangian cobordism $\mathbb{R} \times L$.

As is shown in the following theorem, there is a contravariant functor Φ from $\mathfrak{Cob}$ to $\mathfrak{dg}$.

Theorem 24 ([EHK16], [Kar19])**.** *If $L_1 \prec_\Sigma L_2$, then there is an induced chain map*

$$\Phi_\Sigma : (\mathcal{A}_{L_2}, \partial_{L_2}) \to (\mathcal{A}_{L_1}, \partial_{L_1})$$

such that

(*i.*) *if $L_1 \prec_{\Sigma_{1,2}} L_2$ and $L_2 \prec_{\Sigma_{2,3}} L_3$, then $\Phi_{\Sigma_{1,2} \odot \Sigma_{2,3}} = \Phi_{\Sigma_{1,2}} \circ \Phi_{\Sigma_{2,3}}$,*

(*ii.*) *if $L_1 \prec_\Sigma L_2$ and $L_2 \prec_{\widehat{\Sigma}} L_1$ and Σ is isotopic through exact Lagrangian cobordisms with $\widehat{\Sigma}$, then Φ_Σ and $\Phi_{\widehat{\Sigma}}$ are chain homotopic,*

(*iii.*) $\Phi_{\mathbb{R}\times L} = Id_{\mathcal{A}_L}$.

Theorem 25 ([EHK16])**.** *Let L_1, L_2 be Legendrian links. There is an exact Lagrangian cobordism Σ from L_1 to L_2 if L_1 is obtained from L_2 by one of the following:*

(*i.*) *Legendrian isotopy; Σ is in this case a concordance,*

(*ii.*) *deleting an unstabilized unknot component of L_2 that is contractible in the complement of the remainder of L_2,*

(*iii.*) *a saddle move in front projection as in Figure 5.6.*

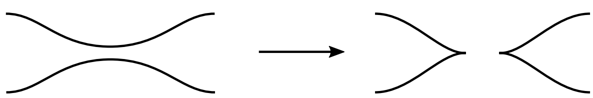


Figure 5.6: Saddle move in front projection between L_2 (left) and L_1 (right).

Definition 30. *Exact Lagrangian cobordisms from the previous theorem are called* ***elementary*** *and exact Lagrangian cobordism obtained by their concatenation is called* ***decomposable****.*

Theorem 26 ([CRGG15])**.** *If there is a concordance $L_1 \prec_\Sigma L_2$, then the chain map Φ_Σ induces an inclusion*

$$\frac{\left\{\begin{array}{c} n\text{-}dim\ representation \\ of\ (\mathcal{A}_{L_1}, \partial_{L_1}) \end{array}\right\}}{isomorphism} \hookrightarrow \frac{\left\{\begin{array}{c} n\text{-}dim\ representation \\ of\ (\mathcal{A}_{L_2}, \partial_{L_2}) \end{array}\right\}}{isomorphism}.$$

Corollary 27. *The number of equivalence classes of n-dimensional representations is an invariant of Legendrian knot.*

Proof. Let K_1 be a Legendrian knot Legendrian isotopic to K_2. For $i = 1, 2$, I_i will denote the set of equivalence classes of n-dimensional representations from $(\mathcal{A}_{K_i}, \partial_{K_i})$ into $\mathbb{F}$. Since F is finite, I_i are also finite.

By Theorem 25, it follows that K_1 is concordant to K_2. And hence, Theorem 26 induces the inclusion from I_1 to I_2. However, K_2 is also Legendrian isotopic to K_1. Thus, we analogously obtain an inclusion from I_2 to I_1. Then, the corollary follows. $\square$

6. Computations

Recall that Sivek's higher dimensional representations for torus knots were all ungraded [Siv13]. In this chapter, we would like to describe an example of higher dimensional representations that are graded. More precisely, we will study irreducible higher dimensional representations, since these are representations not coming from augmentations.

And hence, we are going to find, how many different 2-dimensional irreducible 2-graded representations over $\mathbb{Z}_2$ we can get for Legendrian knot K, see Figure 6.1. Observe that K is really a Legendrian knot, since $\pi_L(K)$ can be obtained by Lemma 10 from $\pi_F(m(5_2))$, which is defined on Figure 2.2.

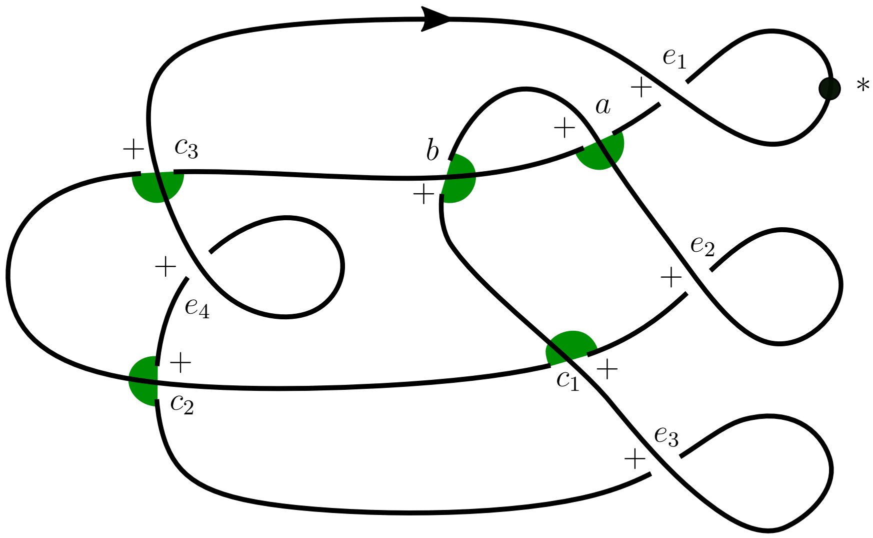


Figure 6.1: Legendrian knot K in Lagrangian projection.

Chekanov-Eliashberg DGA algebra with a basepoint $(\mathcal{A}_K, \partial_K)$ is generated by 9 double points $a, b, c_1, c_2, c_3, e_1, e_2, e_3, e_4$. Now, we compute the grading of generators. Since the rotation number $r(K)$ is equal to 0, the grading is given by

$$|a| = |b| = |c_1| = |c_2| = |c_3| = |t| = |t^{-1}| = 0 \text{ and } |e_1| = |e_2| = |e_3| = |e_4| = 1.$$

Next, we compute the differential ∂_K on generators. Clearly,

$$\partial_K a = \partial_K b = \partial_K c_1 = \partial_K c_2 = \partial_K c_3 = 0.$$

There are 3 disks contributing to the count of $\partial_K e_1$ (Figure 6.2), thus

$$\partial_K e_1 = t + c_3 + c_3 ba.$$

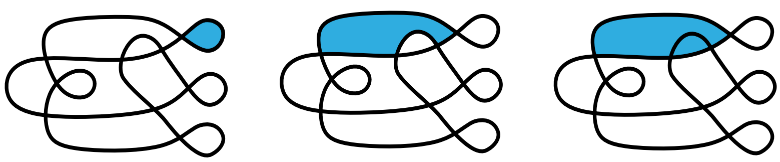


Figure 6.2: 3 disks contributing to $\partial_K e_1$.

For $\partial_K e_2$, there are 5 disks (Figure 6.3), thus

$$\partial_K e_2 = 1 - abc_1 - c_1 - a - ac_2c_3.$$

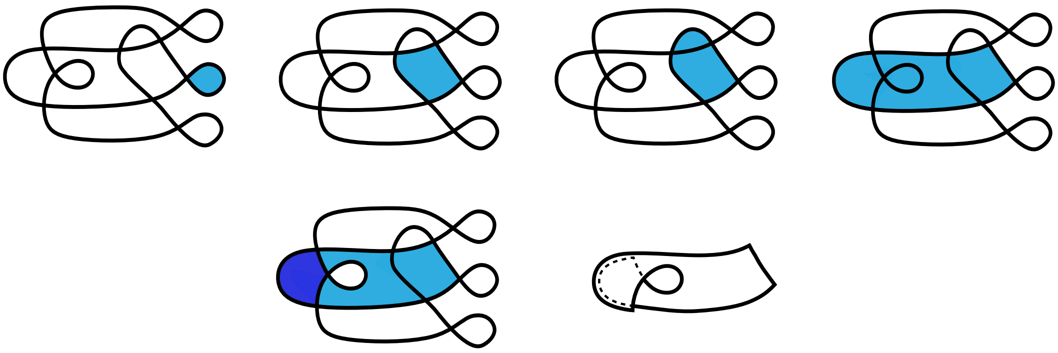


Figure 6.3: 5 disks contributing to $\partial_K e_2$. In the second row left, there is an admissible immersed disk, where the dark part describes immersion 2 to 1. On the right, we visualize the same disk with two overlapping parts that correspond to the dark blue area on the left.

For $\partial_K e_3$, there are 2 disks (Figure 6.4), thus

$$\partial_K e_3 = 1 + c_1c_2.$$

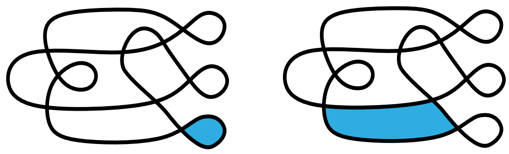


Figure 6.4: 2 disks contributing to $\partial_K e_3$.

And finally, for $\partial_K e_4$, there are also 2 disks (Figure 6.5), thus

$$\partial_K e_4 = 1 + c_3c_2.$$

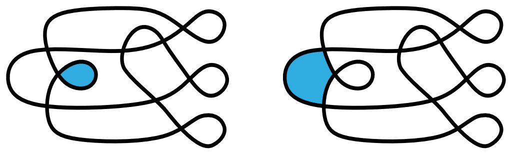


Figure 6.5: 2 disks contributing to $\partial_K e_4$.

Now, we are going to state the following proposition, which is the analog to the Proposition 3.15. in [LR18].

Proposition 28. *There is a bijection*

$$Rep_{grad}(K, \mathbb{Z}_2^2) \leftrightarrow \{(A, B) \in Mat(2, \mathbb{Z}_2) \times Mat(2, \mathbb{Z}_2) | M_1(AB) = \{\emptyset\}\},$$

where $Rep_{grad}(K, \mathbb{Z}_2^2)$ denotes the set of graded 2-dimensional representations of $(\mathcal{A}_K, \partial_K)$ into $\mathbb{Z}_2$. Also, $M_1(AB)$ denotes the set of 1-eigenvectors of the matrix AB.

Proof. We would like to show that the map $f \in \mathrm{Rep}_{grad}(K, \mathbb{Z}_2^2) \mapsto (A, B) = (f(a), f(b))$ is our required bijection.

Take $f \in \mathrm{Rep}_{grad}(K, \mathbb{Z}_2^2)$. From the fact that $f \circ \partial_K(e_1) = 0$, it follows that $f(c_3)(I_2 + f(b)f(a)) = f(t)$. Since t is invertible, $f(t)$ must be an invertible element of $Mat(2, \mathbb{Z}_2)$. Hence, $I_2 + f(b)f(a)$ is invertible and BA has not 1 as an eigenvalue. Because AB and BA have same eigenvalues, $M_1(AB) = \{0\}$ as required. (The last statement follows from the fact that if $\lambda \in \mathbb{Z}_2$ and $v \in M_\lambda(BA)$ is a nonzero vector, then $A(v) \in M_\lambda(AB)$. And analogously for opposite direction.)

On the other hand, there is an inverse map mapping a pair (A, B), such that $M_1(AB) = \{0\}$, to the element $f \in \mathrm{Rep}_{grad}(K, \mathbb{Z}_2^2)$, which is given by

$$\begin{aligned}
&f(a) = A, f(b) = B, f(c_1) = (I_2 + AB)^{-1},\\
&f(c_2) = I_2 + AB, f(c_3) = (I_2 + AB)^{-1}, f(t) = (I_2 + AB)^{-1}(I_2 + BA)
\end{aligned}$$

and $f(e_i) = 0$ for $i \in \{1, \ldots, 4\}$, because we need graded representation. Also, since (I_2+BA) is invertible, $f(t^{-1})$ can be correctly defined as $(f(t))^{-1}$. Next, f is as an (unital) algebra homomorphism, thus the corresponding element to (A, B) is uniquely defined on the whole $\mathcal{A}_K$. By the linearity and Leibniz formula, we see that $f \circ \partial_K = 0$, so we are done. □

To find irreducible representations, we will often use the following lemma.

Lemma 29. *Let f be an element of $Rep_{grad}(K, \mathbb{Z}_2^2)$ and (A, B) corresponding matrices as in the previous lemma. Then, f is irreducible if and only if A and B have no common eigenvector.*

Proof. Let v be a nonzero element of $\mathbb{Z}_2^2$ such that $f(a)v = \lambda_a v$ and $f(b)v = \lambda_b v$ for some $\lambda_a, \lambda_b \in \mathbb{Z}_2$. We would like to show that $f(x)v \in span\{v\}$ for any $x \in \mathcal{A}_K$. Since f is algebra homomorphism, it is sufficient to check this condition only for generators of $\mathcal{A}_K$:

$$\begin{aligned}
f(e_i)v &= 0 \quad \text{for } i \in \{1, \ldots, 4\},\\
f(c_1)v &= (1 + \lambda_a\lambda_b)^{-1}v = v,\\
f(c_2)v &= (1 + \lambda_a\lambda_b)v = v,\\
f(c_3)v &= (1 + \lambda_a\lambda_b)^{-1}v = v,\\
f(t)v &= (1 + \lambda_a\lambda_b)^{-1}(1 + \lambda_b\lambda_a)v = v,\\
f(t^{-1})v &= (f(t)^{-1})v = v,
\end{aligned}$$

where we used the fact that $M_1(AB) = \{0\}$, and the knowledge from linear algebra that inverse matrix have same eigenvectors with inverse eigenvalues.

On the other hand, if f is reducible, then by the definition there exists a nontrivial subspace V of $\mathbb{Z}_2^2$ such that $f(x)V \subseteq V$ for any $x \in \mathcal{A}_K$. Since V is 1-dimensional, $V = span\{v\}$ for some $v \in V$ and v is an eigenvector of A and B. And hence f is reducible. □

Now, we conclude to the computation of the irreducible representations. With Proposition 28 in mind, we are going to find pairs (A, B) that will correspond to irreducible representations.

We will denote the entries of A, B and AB by $(a_{ij})_{i,j=1,2}$, $(b_{ij})_{i,j=1,2}$ and $(x_{ij})_{i,j=1,2}$, respectively.

Observe that in $\mathbb{Z}_2^2$ there are 3 nontrivial subspaces. They are given by the *span* of $(0,1)^T, (1,0)^T$ and $(1,1)^T$.

Since AB can not have 1 as an eigenvalue, there are only two possibilities for a characteristic polynomial $P_{AB}(\lambda)$:

$$P_{AB}(\lambda) = \begin{cases} \lambda^2 & \text{case } 1, \\ \lambda^2 + \lambda + 1 & \text{case } 2. \end{cases}$$

Case 1: From the characteristic polynomial, we see that $x_{11}x_{22} + x_{12}x_{21} = 0$ and $x_{11} + x_{22} = 0$. Thus, there are only 4 possibilities for the matrix AB:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

a) $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$**:**

Note that AB has 2-dimensional kernel. If at least one of A and B has also 2-dimensional kernel, then by Lemma 29 the second matrix has no eigenvector. And there are only 2 matrices with this property in $\mathrm{Mat}(2, \mathbb{Z}_2)$. So, we obtain the following four irreducible representations:

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right],$$
$$\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right],$$
$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right],$$
$$\left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right].$$

Otherwise, assume that both A and B have 1-dimensional kernel. Let v_a be a 0-eigenvector of A, i.e. *span* of v_a is equal to $\ker(A)$. Then, the matrix B maps v_a to some vector v_b. Since AB is a zero matrix, $v_b \in \ker(A)$. Because $\ker(A)$ is 1-dimensional, v_b is equal to 0 or v_a. Hence, v_a is an eigenvector of B. But, it is a contradiction to Lemma 29.

b) $AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$**:**

We have that $\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{11}a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, which is a matrix with the only eigenvector $(1,1)^T$. Hence, by Lemma 29, a pair (A, B) will represent an irreducible representation if and only if $(1,1)^T$ is not an eigenvector of A or B.

Since $(1,1)^T$ is 0-eigenvector of AB, there are several possibilities of nonzero vectors v_B such that $B(1,1)^T = v_B$ and $Av_B = 0$. If $B(1,1)^T = 0$, there are also few possibilities of vectors v_A such that $A(1,1)^T = v_A$. And for each v_B (respective v_A), we will find matrices A, B that correspond to irreducible representations:

- $\begin{cases} B(1,1)^T = (1,0)^T & (*), \\ A(1,0)^T = (0,0)^T & (**). \end{cases}$

  (In this case, $v_B = (1,0)^T$.) We obtain from $(**)$ that $a_{11} = a_{21} = 0$. Thus, $\begin{pmatrix} a_{12}b_{21} & a_{12}b_{22} \\ a_{22}b_{21} & a_{11}a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. And from $(*) : b_{11} + b_{12} = 0$, we get the following two irreducible representations:

$$\left[\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right],$$

$$\left[\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right].$$

- $\begin{cases} B(1,1)^T = (0,1)^T & (*), \\ A(0,1)^T = (0,0)^T & (**). \end{cases}$

  We obtain from $(**)$ that $a_{12} = a_{22} = 0$. Thus, $\begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{21}b_{11} & a_{21}b_{12} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. And from $(*) : b_{21} + b_{22} = 0$, we get the following two irreducible representations:

$$\left[\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right],$$

$$\left[\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right].$$

- $\begin{cases} B(1,1)^T = (1,1)^T, \\ A(1,1)^T = (0,0)^T. \end{cases}$

  This case will not add any irreducible representation by Lemma 29.

- $\begin{cases} B(1,1)^T = (0,0)^T & (*), \\ A(1,1)^T = (0,1)^T & (**). \end{cases}$

  (In this case, $v_B = (0,0)^T$ and $v_A = (0,1)^T$.)From $(**)$, since $x_{11} = 1$, we get that $a_{11} = a_{12} = 1$. By $(*)$ and $x_{12} = x_{21} = 1$, we obtain the following two irreducible representations:

$$\left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right],$$

$$\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right].$$

- $\begin{cases} B(1,1)^T = (0,0)^T & (*), \\ A(1,1)^T = (1,0)^T & (**). \end{cases}$

  From $(**)$, since $x_{22} = 1$, we get that $a_{21} = a_{22} = 1$. By $(*)$ and $x_{12} = x_{21} = 1$, we obtain the following two irreducible representations:

$$\left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right],$$

$$\left[\begin{pmatrix}0&1\\1&1\end{pmatrix},\begin{pmatrix}0&0\\1&1\end{pmatrix}\right].$$

- $\begin{cases}B(1,1)^T=(0,0)^T,\\ A(1,1)^T=(1,1)^T \text{ or } (0,0)^T.\end{cases}$

  Neither of both cases will contribute with an irreducible representation by Lemma 29.

c) $AB=\begin{pmatrix}0&1\\0&0\end{pmatrix}$**:**

We have that $\begin{pmatrix}a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22}\\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{11}a_{22}\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, which is the matrix with the only eigenvector $(1,0)^T$. So, we will be looking for a pair (A,B) such that $(1,0)^T$ is not an eigenvector of A or B.

Similarly, as in **b)**, there are the following cases:

- $\begin{cases}B(1,0)^T=(1,0)^T,\\ A(1,0)^T=(0,0)^T.\end{cases}$

  This case will not add any irreducible representation by Lemma 29.

- $\begin{cases}B(1,0)^T=(0,1)^T & (*),\\ A(0,1)^T=(0,0)^T & (**).\end{cases}$

  From $(*)$ and $(**)$, we obtain that $a_{12}=a_{22}=b_{11}=0$ and $b_{21}=1$. Next, since $1=x_{12}=a_{11}b_{12}+0$, we get $a_{11}=b_{12}=1$. Finally, from x_{22} we obtain the following two irreducible representations:

$$\left[\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\1&1\end{pmatrix}\right],$$

$$\left[\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix}\right].$$

- $\begin{cases}B(1,0)^T=(1,1)^T & (*),\\ A(1,1)^T=(0,0)^T & (**).\end{cases}$

  We obtain from $(*)$ that $b_{11}=b_{21}=1$. Next, it follows from x_{12} and $(**)$ that $a_{11}=a_{12}=1$. x_{12} gives us two possibilities for a pair $(b_{12},b_{22})=(0,1)$ or $(1,0)$. Then, from x_{22} and $(**)$ we conclude the following two irreducible representations:

$$\left[\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&1\\1&0\end{pmatrix}\right],$$

$$\left[\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix}\right].$$

- $\begin{cases}B(1,0)^T=(0,0)^T,\\ A(1,0)^T=(1,0)^T \text{ or } (0,0)^T.\end{cases}$

  Both cases will not contribute by any irreducible representations by Lemma 29.

- $\begin{cases} B(1,0)^T = (0,0)^T & (*), \\ A(1,0)^T = (0,1)^T & (**). \end{cases}$

  Immediately, we obtain from $(*)$ and $(**)$ that $a_{11} = b_{11} = b_{21} = 0$ and $a_{21} = 1$. Next, from x_{12} we get that $a_{12} = b_{22} = 1$. Finally, from x_{22} we conclude the following two irreducible representations:

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right],$$

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

- $\begin{cases} B(1,0)^T = (0,0)^T & (*), \\ A(1,0)^T = (1,1)^T & (**). \end{cases}$

  Immediately, we obtain from $(*)$ and $(**)$ that $b_{11} = b_{21} = 0$ and $a_{11} = a_{21} = 1$. If $a_{12} = 0$, then we obtain the following irreducible representation:

$$\left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right].$$

  Otherwise, if $a_{12} = 1$, then we conclude from x_{12} and x_{22} that $b_{12} = 0$ and get the following irreducible representation:

$$\left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

d) $AB = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$**:**

Lemma 30. *For any irreducible representation (A, B) such that $AB = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, there is an irreducible representation (B^T, A^T) such that $B^T A^T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Moreover, this correspondence is bijection between irreducible representations with product AB equal to $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.*

Proof. Fist, we make the following observation. Let V be a finite-dimensional vector space. Also, f will denote a linear map $f : V \to V$ and f^* will be a dual map $f^* : V^* \to V^*$. Note that if v is λ-eigenvector of f, then also $f^*(v^*) = \lambda v^*$.

It is well known that for any matrix a dual map can be seen as a transpose matrix. Hence, it follows that if two matrices have a common eigenvector, then also their transpose matrices have a common eigenvector. By Lemma 29 and the fact that 2-times transpose matrix is the same matrix, we obtain our required bijective correspondence. □

By the previous lemma and the case **c)**, we immediately obtain all irreducible representations for **d)**:

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right],$$

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right],$$

$$\left[\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right],$$
$$\left[\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\right],$$
$$\left[\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right],$$
$$\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right],$$
$$\left[\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\right],$$
$$\left[\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\right].$$

Case 2: Characteristic polynomial has in this case no root. Hence, AB has no eigenvector and representation, which is given by (A, B), is automatically irreducible. There are two possibilities for the product AB:

$$\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}.$$

e) $AB = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$**:**

We have the the following system of equations $\begin{pmatrix}a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22}\\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{11}a_{22}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$. Now, if we look at x_{22}, we would like to distinguish several cases:

- $(a_{21}, b_{12}, a_{11}, a_{22}) = (0, 0, 1, 1)$: it follows from x_{12} and x_{21} that $a_{12} = b_{21} = 1$. Then, from x_{11}, we get $a_{11} = b_{11} = 1$. So, the concluded irreducible representation:
$$\left[\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right].$$
- $(a_{21}, b_{12}, a_{11}, a_{22}) = (1, 0, 1, 1)$: we obtain from x_{12} that $a_{12} = 1$. Then, from x_{11} and x_{21}, we conclude to the following irreducible representation:
$$\left[\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right].$$
- $(a_{21}, b_{12}, a_{11}, a_{22}) = (0, 1, 1, 1)$: we obtain from x_{21} that $b_{21} = 1$. Then, from x_{11} and x_{12}, we conclude to the following irreducible representation:
$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\right].$$
- $(a_{21}, b_{12}, a_{11}, a_{22}) = (1, 1, 0, 0)$: it follows from x_{12} and x_{21} that $a_{11} = b_{11} = 1$. Then, from x_{11}, we get $a_{12} = b_{21} = 1$. So, the concluded irreducible representation:
$$\left[\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\right].$$

- $(a_{21}, b_{12}, a_{11}, a_{22}) = (1, 1, 1, 0)$: we obtain from x_{12} that $a_{11} = 1$. Then, from x_{11} and x_{21}, we conclude to the following irreducible representation:

$$\left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right].$$

- $(a_{21}, b_{12}, a_{11}, a_{22}) = (1, 1, 0, 1)$: we obtain from x_{21} that $b_{11} = 1$. Then, from x_{11} and x_{12}, we conclude the following irreducible representation:

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right].$$

f) $AB = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$**:**

Lemma 31. *For any irreducible representation (A, B) such that $AB = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, there is a irreducible representation $(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)A, B\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$ such that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)AB\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Moreover, this correspondence is a bijection between irreducible representations with product AB equal to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.*

Proof. It immediately follows from the fact that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = I_2$, so the correspondence is a bijection. □

By the previous lemma and the case **e)**, we immediately obtain all irreducible representations for **f)**:

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right],$$
$$\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right],$$
$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right],$$
$$\left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right],$$
$$\left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right],$$
$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right].$$

Now, to distinguish irreducible representations, we are going to sort them into equivalence classes.

Note that there are only 5 nontrivial vector space isomorphisms on $\mathbb{Z}_2^2$:

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \delta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since $\beta \circ \gamma = \delta$ and $\gamma \circ \beta = \epsilon$, we will use only α, β, γ to find equivalence classes. Observe, how these isomorphisms will change by conjugating some arbitrary element of $\mathrm{Mat}(2, \mathbb{Z}_2)$:

$$\begin{aligned}
\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha^{-1} &= \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \\
\beta \begin{pmatrix} a & b \\ c & d \end{pmatrix} \beta^{-1} &= \begin{pmatrix} a+c & a+b+c+d \\ c & c+d \end{pmatrix}, \\
\gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} \gamma^{-1} &= \begin{pmatrix} a+b & b \\ a+b+c+d & b+d \end{pmatrix}.
\end{aligned}$$

With this in mind, I found (by hand) 10 equivalence classes of irreducible representations:

1. $\left\{ \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right] \right\}$

2. $\left\{ \left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right] \right\}$

3. $\left\{ \left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right] \right\}$

4. $\left\{ \left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \right\}$

5. $\left\{ \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right] \right\}$

6. $\left\{ \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right], \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right], \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right] \right\}$

7. $\left\{ \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right] \right\}$

8. $$\left\{ \left[\begin{pmatrix}0&1\\1&1\end{pmatrix},\begin{pmatrix}0&0\\1&1\end{pmatrix}\right],\left[\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}1&1\\0&0\end{pmatrix}\right],\left[\begin{pmatrix}0&1\\1&1\end{pmatrix},\begin{pmatrix}0&1\\0&1\end{pmatrix}\right],\right.$$
$$\left.\left[\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right],\left[\begin{pmatrix}0&1\\1&1\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}\right],\left[\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\1&0\end{pmatrix}\right]\right\}$$

9. $$\left\{ \left[\begin{pmatrix}0&1\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&1\end{pmatrix}\right],\left[\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}1&1\\1&0\end{pmatrix}\right],\left[\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\1&1\end{pmatrix}\right],\right.$$
$$\left.\left[\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&1\\1&0\end{pmatrix}\right],\left[\begin{pmatrix}0&0\\1&1\end{pmatrix},\begin{pmatrix}0&1\\1&1\end{pmatrix}\right],\left[\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}1&1\\1&0\end{pmatrix}\right]\right\}$$

10. $$\left\{ \left[\begin{pmatrix}0&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix}\right],\left[\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix}\right],\left[\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix}\right],\right.$$
$$\left.\left[\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix}\right],\left[\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix}\right],\left[\begin{pmatrix}0&0\\1&1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix}\right]\right\}$$

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List of Abbreviations

S	unital commutative ring
$\mathbb{F}, \mathbb{F}^{\times}$	finite field, set of nonzero elements of $\mathbb{F}$
G	cyclic group
TM	tangent bundle of M
$\Gamma(\xi)$	space of sections of the plane field ξ
α_{std}	standard contact form
ξ_{std}	standard contact structure
i_X	contraction by a vector field X (interior product)
α_{std}	standard contact form
L_X	Lie derivative with respect to the vector field X
R	Reeb vector field
K	Legendrian knot
L	Legendrian link
T	topological knot
D	knot diagram
π_F, π_L	front and Lagrangian projections, respectively
$\|\| \cdot \|\|$	Euclidean norm
$r(K)$	rotation number of K
$tb(K)$	Thurston-Bennequin invariant of K
$\#P, \#N$	number of positive and negative crossings, respectively
$S_{\pm}(K)$	positive and negative stabilization of K, respectively
$F_K(a, z)$	Kauffman polynomial of K
$Q(K)$	set of double-points in $\pi_L(K)$
P_{k+1}	$(k+1)$-sided convex polygon
$\mathcal{M}_{b_0,\ldots,b_k}$	set of immersed polygons with vertices $b_0, \ldots, b_k$
$\mathcal{M}^a_{b_1,\ldots,b_k}$	set of admissible immersed polygons with vertices $a, b_1, \ldots, b_k$
DGA	(Chekanov-Eliashberg) differential graded algebra
$(\mathcal{A}_K, \partial_K)$	Chekanov-Eliashberg DGA of K
$\|a\|$	degree of $a \in \mathcal{A}_K$
LCH	Legendrian contact homology
H	height function on double-points in $\pi_L(K)$
γ, η	paths along $\pi_L(K)$
e_1, e_2	"new" generators of the stabilized DGA
$\delta(u)$	product of orientation signs of vertices in admissible immersed polygon u
$n_i(u)$	sign count of passing η_i through the basepoint $*$
ϵ	augmentation or more generally representation of DGA (only in the proof of Theorem 17 "ϵ" denotes small positive real number)
LCH^{ϵ}	linerize LCH by the augmentation ϵ
Σ	exact Lagrangian cobordism
$L_1 \prec_{\Sigma} L_2$	cobordance Σ from L_1 to L_2
$\Sigma_1 \odot \Sigma_2$	concatenation of Σ_1 and Σ_2
$\mathfrak{Cob}, \mathfrak{dg}$	categories of exact Lagrangian cobordisms and DGAs, resp.
Φ	contravariant functor from $\mathfrak{Cob}$ to $\mathfrak{dg}$
$\mathrm{Rep}_{grad}(K, \mathbb{Z}_2^2)$	set of graded 2-dimensional representations of $(\mathcal{A}_K, \partial_K)$ into $\mathbb{Z}_2$
$M_1(A)$	set of 1-eigenvectors of the matrix A