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Consequences and applications of the Fock space representation theorem

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In Prague, 4.5.2020 Daniela Flimmel

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Abstract: In this thesis, we deal with selected applications of the Fock space representation theorem. One of the most important is the covariance identity, which can yield in an estimation of the correlation function of a point process having Papangelou conditional intensity. We used this result to generalise some asymptotic results for Gibbs particle processes. Namely, in combination with Stein's method, we derived bounds for the Wasserstein distance between the standard normal distribution and the distribution of an innovation of a Gibbs particle process. As an application, we present a central limit theorem for a functional of a Gibbs segment process with pair potential.

Keywords: Fock space, difference operator, Poisson process, point process with density, point process with conditional intensity, Stein's method, Gibbs particle process, Wasserstein distance

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Introduction

Let F be a Poisson functional, i.e. a measurable function of a Poisson process η. If F is square integrable, then the infinite sequence is of expected difference operators of F is an element of the Fock space. By the Fock space representation theorem, the second moment of F equals the squared norm in the Fock space. In this thesis, we present selected theoretical applications closely related to the Fock space representation theorem.

The first chapter covers basic definitions and results within the theory of point processes needed in the rest of the work. In particular, we define class of point processes with density with respect to the distribution of a Poisson process η. Also, point processes having Papangelou conditional intensity are introduced in order to define Gibbs particle process with pair potential.

The second chapter is devoted to the Fock space representation theorem. We introduce related definitions, e.g. the difference operator, and then show the proof of the Fock space representation theorem. As a direct application, we prove the covariance identity. This chapter is a compilation of theoretical results in Last and Penrose, 2017, Chapters 18 and 20.

In the third chapter, we study properties of the hard-core process in the real line. The first original contribution of this work is evaluating the first order correlation function of the hard-core model. In order to obtain this result, analytical form of the normalising constant in the density is derived. An application to the maximum likelihood estimation is presented.

The last chapter reviews some applications of Stein's method in stochastic geometry. First, we explain the basic principle of Stein's method and next, we derive bounds on the Wasserstein distance in four different settings, each covered in a separate section. First, we recall results covered in Last and Penrose, 2017, Chapter 21, where the Wasserstein distance is estimated for a functional of a Poisson point process using the covariance identity. In addition, we include a solution to Exercise 21.4 in Last and Penrose, 2017 as an application. In Section 4.3, we show general bounds on the Wasserstein distance between the standard normal distribution and the distribution of an innovation of a point process having Papangelou conditional intensity. One of the most important classes of point processes having Papangelou conditional intensity is formed by Gibbs point processes, that enable us to incorporate dependence among the points. In Section 4.4, the Wasserstein distance is estimated for Gibbs point processes with pair potential on $\mathbb{R}^d$. Sections 4.3 and 4.4 are covered in Torrisi, 2017. The second main contribution of this thesis is presented in Section 4.5. While in the literature, asymptotic results for Gibbs point processes are formulated exclusively in $\mathbb{R}^d$, $d \in \mathbb{N}$, here, we generalize the result of Section 4.4 for Gibbs particle processes, i.e. Gibbs point processes on the space $\mathcal{C}^{(d)}$ of all non-empty, compact subsets of $\mathbb{R}^d$ with the Hausdorff metric. Among others, we used the estimates for the correlation function based on the covariance identity to prove the result. As an application, we present a central limit theorem for a functional of a Gibbs planar segment process.

Chapter 1

Theoretical framework

Throughout this work, we will always assume the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If not specified otherwise, we will assume that we are working on a locally compact, Polish (i.e. separable, complete, metric) space $\mathbb{X}$ equipped with Borel σ-field $\mathcal{B}(\mathbb{X})$, with reference measure σ. Denote by $\mathcal{B}_0(\mathbb{X})$ the set of all bounded Borel sets in $\mathbb{X}$. The aim of this chapter is to introduce the theoretic fundamentals and notation needed for further results.

1.1 Point process as a random measure

The background presented in Sections 1.1 and 1.3 can be found in Møller and Waagepetersen, 2003 and Last and Penrose, 2017, Chapter 5.

Definition 1.1 (Locally finite measure)**.** A measure μ on $\mathbb{X}$ is said to be locally finite if

$$\mu(B) < \infty, \quad \forall B \in \mathcal{B}_0(\mathbb{X}).$$

Notation. Let $\mathbf{M}$ denote the space of all locally finite measures on $\mathbb{X}$ and $\mathbf{N}$ the subset of $\mathbf{M}$ containing only measures that take values in $\mathbb{N} \cup \{0, \infty\}$. Further, we will denote by $\mathcal{M}$ the smallest σ-field on $\mathbf{M}$ which makes all the projections $\mu \mapsto \mu(B)$ measurable for all Borel sets B. On the space $\mathbf{N}$, we define σ-field $\mathcal{N}$ by

$$\mathcal{N} = \{M \cap \mathbf{N} : M \in \mathcal{M}\}.$$

Elements of $\mathbf{M}$ (or $\mathbf{N}$) are measures. Although, we will alternatively handle them as locally finite random sets of points. Therefore, we will write $x \in \mu$ for $\mu \in \mathbf{M}$ (or $\mathbf{N}$) instead of $x \in \mathit{supp}\ \mu$.

Definition 1.2 (Random measure)**.** A random measure on $\mathbb{X}$ is a measurable mapping

$$\Psi : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbf{M}, \mathcal{M}).$$

Definition 1.3 (Point process)**.** A point process on $\mathbb{X}$ is a measurable mapping

$$\mu : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbf{N}, \mathcal{N}).$$

Remark. Point process is a special example of the random measure. Measurable space $(\mathbf{N}, \mathcal{N})$ is often called the outcome space of a point process on $\mathbb{X}$.

Definition 1.4 (Simple point process)**.** A point process μ on $\mathbb{X}$ is called simple if

$$\mathbb{P}(\mu(\{x\}) \leq 1, \ \forall x \in \mathbb{X}) = 1.$$

Definition 1.5 (Distribution of a point process)**.** Let μ be a point process on $\mathbb{X}$. By the distribution of the point process μ, we understand the probability measure P_μ on the space $(\mathbf{N}, \mathcal{N})$ given by

$$P_\mu(A) = \mathbb{P}(\mu \in A), \quad A \in \mathcal{N}.$$

Definition 1.6 (Intensity measure)**.** Let μ be a point process on $\mathbb{X}$. The measure on $\mathbb{X}$ defined by

$$\lambda(B) = \mathbb{E}\left[\mu(B)\right], \quad B \subset \mathbb{X},$$

is called the intensity measure of the point process μ.

Definition 1.7 (Intensity function)**.** Let λ be the intensity measure of a point process μ on $\mathbb{X}$ satisfying for some non-negative measurable function χ

$$\lambda(B) = \int_B \chi(x)\sigma(\mathrm{d}x), \quad B \subset \mathbb{X}.$$

Then χ is called the intensity function of point process μ. If χ is constant, we talk about the intensity of the point process μ.

Suppose that μ is a point process on $\mathbb{X}$, and we wish to give each $x \in \mu$ a random mark y with values in some locally compact, Polish space $(\mathbb{Y}, \mathcal{Y})$, called the mark space. In this way, we are able to construct a point process on the product space $\mathbb{X} \times \mathbb{Y}$. Let K be a probability kernel from $\mathbb{X}$ to $\mathbb{Y}$, i.e. a mapping $K : \mathbb{X} \times \mathcal{Y} \to [0,1]$, such that $K(x, \cdot)$ is a probability measure for each $x \in \mathbb{X}$, and $K(\cdot, C)$ is measurable for each $C \in \mathcal{Y}$. Further, for $x \in \mathbb{X}$, we denote by δ_x the Dirac measure in x.

Definition 1.8. (K-marking) Let μ be a point process on $\mathbb{X}$ and ν a random variable with values in $\mathbb{N} \cup \{0, \infty\}$ such that $\mu = \sum_{n=0}^{\nu} \delta_{x_n}$. Let $y_1, y_2, \ldots$ be random elements in $\mathbb{Y}$, such that the conditional distribution of $(y_n)_{n \leq m}$ given $\nu = m$ and $(x_n)_{n \leq m}$ is that of independent random variables with distribution $K(x_n, \cdot), n \leq m$. Then the point process

$$\xi := \sum_{n=1}^{\nu} \delta_{(x_n, y_n)}$$

is called a K-marking on μ.

Definition 1.9. (p-thinning) Let $p : \mathbb{X} \to [0,1]$ be measurable and consider the probability kernel K from $\mathbb{X}$ to $[0,1]$ defined by

$$K_p(x, \cdot) := (1 - p(x))\delta_0 + p(x)\delta_1, \quad x \in \mathbb{X}.$$

If ξ is a K_p-marking of a point process μ, then $\xi(\cdot \times \{1\})$ is called a p-thinning of μ.

Remark. If a point process μ has an intensity function $\lambda(x)$, then its p-thinning has the intensity function $p(x)\lambda(x)$.

Definition 1.10 (Poisson point process)**.** Let λ be a σ-finite measure on $\mathbb{X}$ such that for all points $x \in \mathbb{X}$ it holds $\lambda(\{x\}) = 0$. The Poisson point process on $\mathbb{X}$ with intensity measure λ is a point process η on $\mathbb{X}$ satisfying the following conditions

1. for every compact set $B \subset \mathbb{X}$, $\eta(B)$ is a Poisson distributed random variable with parameter $\lambda(B)$;
2. if $B_1, \dots, B_n,\ n \in \mathbb{N}$, are pairwise disjoint compact subsets of $\mathbb{X}$, then $\eta(B_1), \dots, \eta(B_n)$ are independent random variables.

Theorem 1.1. *(Thinning theorem) Let $p : \mathbb{X} \to [0, 1]$ be measurable and let η_p be a p-thinning of a Poisson process η. Then η_p and $\eta - \eta_p$ are independent Poisson processes.*

Theorem 1.2. *(Superposition theorem) Let $\eta_i,\ i \in \mathbb{N}$, be a sequence of independent Poisson processes on $\mathbb{X}$ with intensity measures λ_i. Then*

$$\eta := \sum_{i=1}^{\infty} \eta_i$$

is a Poisson process with intensity measure $\lambda = \lambda_1 + \lambda_2 + \cdots$.

1.2 Finite point processes

Theoretical background related to this section is covered in Baddeley, 2007.

Definition 1.11 (Finite point process)**.** A point process μ on $\mathbb{X}$ is called finite if

$$\mathbb{P}(\mu(\mathbb{X}) < \infty) = 1.$$

Let η be a finite Poisson point process on $\mathbb{X}$ with intensity measure λ and distribution P_η. In this section, we will introduce the class of point processes having density with respect to P_η.

Definition 1.12 (Finite point process with density with respect to the distribution of the Poisson point process)**.** Consider a measurable mapping $p : \mathbf{N} \to \mathbb{R}_+$ satisfying

$$\int_{\mathbf{N}} p(\mathbf{x}) dP_\eta(\mathbf{x}) = 1.$$

A point process μ with distribution P_μ, such that

$$P_\mu(A) = \int_A p(\mathbf{x}) dP_\eta(\mathbf{x}), \quad A \in \mathcal{N},$$

is called the point process with density p with respect to the distribution of the Poisson point process η.

Theorem 1.3 (Baddeley, 2007, page 62-63)**.** *For a point process μ with density p with respect to the distribution of a Poisson point process η with intensity measure λ on a bounded set $B \subset \mathbb{R}^d$ we have*

$$\mathbb{P}(\mu \in A) = e^{-\lambda(B)} \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_B \dots \int_B \mathbf{1}\{(x_1, \dots, x_n) \in A\} \right. \\ \left. p(x_1, \dots, x_n) \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_n) \right)$$

for any $A \in \mathcal{N}$, and

$$\mathbb{E}\left[g(\eta)\right] = e^{-\lambda(B)} \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_B \dots \int_B g(x_1, \dots, x_n) \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_n) \right), \quad (1.1)$$

$$\mathbb{E}\left[g(\mu)\right] = e^{-\lambda(B)} \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_B \dots \int_B g(x_1, \dots, x_n) \right. \\ \left. p(x_1, \dots, x_n) \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_n) \right)$$

for any integrable function $g : \mathbf{N} \to \mathbb{R}_+$.

Remark. The identities for the point process μ in Theorem 1.3 can be shortly rewritten as follows

$$\mathbb{P}(\mu \in A) = \mathbb{E}\left[p(\eta) 1_A(\eta)\right],$$
$$\mathbb{E}\left[g(\mu)\right] = \mathbb{E}\left[g(\eta) p(\eta)\right].$$

Example 1.1 (Strauss process)**.** The Strauss process is a standard example of a finite point process with density with respect to the distribution of a Poisson point process. A special case of the Strauss process is the hard-core process. We will study this particular example in more detail in Chapter 3.

Let $B \subset \mathbb{R}^d,\ d \in \mathbb{N}$, be bounded set and $\sigma = Leb$ on B, where Leb denotes the Lebesgue measure of the corresponding dimension. Denote by $n(\mathbf{x})$ the number of points in a realization $\mathbf{x} \in \mathbf{N}$ in B of some point process. Let for real parameters $\beta > 0,\ 0 \le \gamma \le 1$ and $r > 0$ define the number of points in the realization $\mathbf{x}$ in B being close to u as a function

$$t(u, \mathbf{x}) = \sum_{x \in \mathbf{x}} \mathbf{1}\{\|u - x\| < r\},$$

and furthermore, define the number of pairs of points in a realization $\mathbf{x}$ in B being at most r units apart from each other as a function

$$s(\mathbf{x}) = \sum_{x, y \in \mathbf{x}} \mathbf{1}\{\|x - y\| < r\}.$$

The Strauss point process is constructed as a finite point process having density p with respect to the distribution of the Poisson point process η with intensity measure $\lambda = Leb$,

$$p(\mathbf{x}) = \alpha \beta^{n(\mathbf{x})} \gamma^{s(\mathbf{x})},$$

where α is the normalising constant.

Remark. The normalising constant α is not in full generality available in a closed form as a function of variables β, γ and r. For special choice of $\gamma = 0$, we get the hard-core process κ_r with parameter $r > 0$, for which we will derive the normalising constant in case of $\mathbb{R}^1$ later on in Chapter 3.

If we set $\gamma = 1$, we will get the Poisson point process with the intensity measure λ.

1.3 Point processes with Papangelou conditional intensity

Definition 1.13 (Papangelou conditional intensity)**.** Let μ be a point process on $\mathbb{X}$ with distribution P_μ. Suppose $C^!$ is a measure on $\mathbb{X} \times \mathbf{N}$ that is absolutely continuous with respect to $\sigma \otimes P_\mu$ and satisfies for every $B \subset \mathbb{X}$ Borel set and $A \in \mathcal{N}$

$$C^![B \times A] = \mathbb{E}\left[\sum_{x \in \mu} \mathbf{1}\{x \in B\} \mathbf{1}\{\mu - \delta_x \in A\}\right].$$

Then the Radon-Nikodým derivative $\lambda^* : \mathbb{X} \times \mathcal{N} \to \mathbb{R}_+$ of $C^!$ with respect to the product measure $\sigma \otimes P_\mu$ is called the Papangelou conditional intensity (or simply conditional intensity) of μ.

Remark. In fact, conditional intensity λ^* is defined to satisfy for all $B \subset \mathbb{X}$ Borel and $A \in \mathcal{N}$ the condition

$$C^![B \times A] = \int_B \mathbb{E}\left[\lambda^*(u, \mu) \mathbf{1}\{\mu \in A\}\right] \sigma(\mathrm{d}u).$$

We can interpret the conditional intensity of the point process μ as the conditional probability that in an infinitesimal neighbourhood of some fixed point $x \in \mathbb{X}$, there will be a point of μ, given we know the location of all points of μ outside this neighbourhood.

Theorem 1.4 (Georgii-Nguyen-Zessin formula)**.** *Let μ be a point process on $\mathbb{X}$ with Papangelou conditional intensity λ^* and intensity measure λ. Then*

$$\mathbb{E} \int_{\mathbb{X}} f(x, \mu - \delta_x) \mu(\mathrm{d}x) = \mathbb{E} \int_{\mathbb{X}} f(x, \mu) \lambda^*(x, \mu) \lambda(\mathrm{d}x)$$

for all non-negative functions $f : \mathbb{X} \times \boldsymbol{N} \to \mathbb{R}$.

Definition 1.14 (Innovation of a point process)**.** Let μ be a point process on $\mathbb{X}$ with Papangelou conditional intensity λ^* and intensity measure λ. Then we define the innovation of the point process μ as a random variable

$$I_\mu(\varphi) := \sum_{x \in \mu} \varphi(x, \mu - \delta_x) - \int_{\mathbb{X}} \varphi(x, \mu) \lambda^*(x, \mu) \lambda(\mathrm{d}x)$$

for any measurable $\varphi : \mathbb{X} \times \mathbf{N} \to \mathbb{R}$, for which $|I_{\mathbf{x}}(\varphi)| < \infty$ for μ-a.a. $\mathbf{x} \in \mathbf{N}$.

Remark. By Theorem 1.4, we have immediately $\mathbb{E}[I_\mu(\varphi)] = 0$ for any innovation defined above.

Definition 1.15. [Repulsive conditional intensity] The Papangelou conditional intensity λ^* of a point process μ on $\mathbb{X}$ is said to be repulsive if

$$\lambda^*(x, \mathbf{x}) \geq \lambda^*(x, \mathbf{y}), \quad \text{whenever } \mathbf{x}, \mathbf{y} \in \mathbf{N}, \mathbf{x} \subseteq \mathbf{y}, x \in \mathbb{X}.$$

Definition 1.16 (Conditional intensity of p-th order)**.** Let μ be a point process on $\mathbb{X}$. For $p \in \mathbb{N}$, we define the conditional intensity of the p-th order as a measurable function $\lambda_p^* : \mathbb{X}^p \times \mathbf{N} \to [0, \infty)$ defined by

$$\lambda_p^*(u_1, \dots, u_p, \mu) := \lambda^*(u_1, \mu)\lambda^*(u_2, \mu + \delta_{u_1}) \cdots \lambda^*(u_p, \mu + \delta_{u_1} + \cdots + \delta_{u_{p-1}}),$$

$u_1, \dots, u_p \in \mathbb{X}$.

Remark. Theorem 1.4 can be iterated, i.e. for $p \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E} \int_{\mathbb{X}^p} & f(u_1, \dots, u_p, \mu - \delta_{u_1} - \cdots - \delta_{u_p})\mu^{(p)}(\mathrm{d}(u_1, \dots, u_p)) \\ &= \mathbb{E} \int_{\mathbb{X}^p} f(u_1, \dots, u_p, \mu)\lambda_p^*(u_1, \dots, u_p, \mu)\lambda^p(\mathrm{d}(u_1, \dots, u_p)) \end{aligned}$$

for each measurable function $f : \mathbb{X}^p \times \mathbf{N} \to [0, \infty)$, where $\mu^{(p)}$ is the p-th factorial measure of μ.

Definition 1.17 (Correlation function)**.** Let μ be a point process with conditional intensity of the p-th order λ_p^*, $p \in \mathbb{N}$. Then the expectation of λ_p^*,

$$\rho_p(u_1, \dots, u_n) = \mathbb{E}\ \lambda_p^*(u_1, \dots, u_p, \mu), \quad u_1, \dots, u_p \in \mathbb{X},$$

is called the correlation function of the p-th order of the point process μ.

Notation. We will further use also the notation ρ for the first order correlation function ρ_1.

For point processes with density with respect to the distribution of Poisson point process defined in Section 1.2, the following representation of the conditional intensity holds true.

Theorem 1.5 (Baddeley, 2007, page 65)**.** *Let μ be a finite point process on a bounded set $B \subset \mathbb{R}^d$ with density p. Assume that*

$$p(\mathbf{x}) > 0 \Rightarrow p(\mathbf{y}) > 0, \quad \mathbf{x}, \mathbf{y} \in \mathbf{N}, \mathbf{y} \subset \mathbf{x}.$$

Then the conditional intensity of the point process μ exists and equals

$$\lambda^*(u, \mathbf{x}) = \frac{p(\mathbf{x} \cup \{u\})}{p(\mathbf{x})}, \quad \mathbf{x} \in \mathbf{N}, u \in B, u \notin \mathbf{x}, \mathbb{P}(u \in \mu) = 0.$$

If we have $p(\mathbf{x}) = 0$, we set $\lambda^*(u, \mathbf{x}) = 0$.

Remark. In a similar way as in Theorem 1.5, we can see for $p \in \mathbb{N}, p > 1$, that

$$\lambda_p^*(u_1, \dots, u_p, \mathbf{x}) = \frac{p(\mathbf{x} \cup \{u_1, \dots, u_p\})}{p(\mathbf{x})},$$

where $u_1, \dots, u_p \in B$ are pairwise different. Note that λ_p^* is a symmetric function in variables $u_1, \dots, u_p$.

1.4 Gibbs point processes with pair potential on $\mathbb{R}^d$

In this section, we will introduce Gibbs point processes with pair potential on the Euclidean space $\mathbb{R}^d$ defined by the Papangelou conditional intensity. Presented theory is based on Torrisi, 2017. A generalisation of this definition will follow in the next section.

Assume the space $\mathbb{R}^d, d \in \mathbb{N}$, equipped with the Lebesgue measure *Leb* of the corresponding dimension.

Definition 1.18 (Pair potential)**.** We call a function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ the pair potential if it is Borel measurable and $\phi(x) = \phi(-x)$ for any $x \in \mathbb{R}^d$.

Definition 1.19 (Relative energy)**.** For $x \in \mathbf{N}$ and $u \in \mathbb{R}^d$, we define the relative energy of interaction between the point u and the configuration $\mathbf{x}$ by

$$E(u, \mathbf{x}) = \begin{cases} \sum\limits_{y \in \mathbf{x}} \phi(u - y), & \text{if } \sum\limits_{y \in \mathbf{x}} |\phi(u - y)| < \infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

Definition 1.20 (Gibbs point process with pair potential)**.** Take $\tau > 0$ and a pair potential ϕ. A point process μ on $\mathbb{R}^d$ is called the Gibbs point process with activity τ and pair potential ϕ if its Papangelou conditional intensity takes form

$$\lambda^*(u, \mathbf{x}) = \tau \exp\{-E(u, \mathbf{x})\}, \quad u \in \mathbb{R}^d, \ \mathbf{x} \in \mathbf{N}.$$

Remark. Let $T = \{\tau_x, \ x \in \mathbb{R}^d\}$ be the shift group, where $\tau_x : \mathbf{N} \to \mathbf{N}$ is the translation by the vector $-x \in \mathbb{R}^d$. A point process μ on $\mathbb{R}^d$ is said to be stationary if μ is invariant with respect to T. Conditions for the existence of a stationary Gibbs point process with pair potential are given in Ruelle, 1970. We can not say anything about the uniqueness, so we always consider one such stationary process.

Definition 1.21 (inhibitory Gibbs point process)**.** We call a Gibbs point process μ with pair potential ϕ inhibitory if $\phi \geq 0$ on $\mathbb{R}^d$.

Definition 1.22 (Gibbs point process with finite range)**.** A Gibbs point process μ with pair potential ϕ is said to have finite range if $1 - e^{-\phi}$ has compact support.

In Section 1.2, we have defined the hard-core point process on $\mathbb{R}^d$ by its probability density. Since the idea of the hard-core process is based on repulsive interactions between each two points, it is also possible to define it as a Gibbs point process with pair potential.

Example 1.2 (Hard-core point process with pair potential)**.** Take $r > 0$ fixed and let $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be a real function defined by

$$\phi(y) = \begin{cases} 0, & \text{if } ||x|| > r, \\ +\infty, & \text{if } ||x|| \leq r. \end{cases} \tag{1.2}$$

Set the relative energy between point $u \in \mathbb{R}^d$ and system of points $\mathbf{x} \in \mathbf{N}$ as

$$E(u, \mathbf{x}) = \begin{cases} \sum_{y \in \mathbf{x}} \phi(u-y), & \text{if } \sum_{y \in \mathbf{x}} |\phi(u-y)| < \infty, \\ +\infty, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0, & \text{if } ||u-y|| > r, \ \forall y \in \mathbf{x}. \\ +, \infty & \text{otherwise.} \end{cases}.$$

Then we define the hard-core point process with pair potential by its Papangelou conditional intensity

$$\lambda^*(u, \mathbf{x}) = \tau \exp\{-E(u, \mathbf{x})\} = \tau \mathbf{1}_{[||u-y||>r, \ \forall y \in \mathbf{x}]}, \quad u \in \mathbb{R}^d, \mathbf{x} \in \mathbf{N}.$$

Lemma 1.6. *Let μ be a hard-core point process on $\mathbb{R}^d$ with pair potential ϕ and activity $\tau > 0$. Then μ is inhibitory and has finite range.*

Proof. Point process μ is inhibitory straightforwardly from the definition of its pair potential. It is also not difficult to see, that μ has finite range, i.e. function $g(x) := 1 - e^{-\phi(x)}$ has compact support. The function g is nonzero if and only if $x \in b(0, r)$, which is a compact subset of $\mathbb{R}^d$. □

1.5 Particle processes

Particle processes are point processes in the space of all nonempty compact subsets of $\mathbb{R}^d$, $d \in \mathbb{N}$. For further details, see Schneider and Weil, 2008, Section 4.1.

Notation. Denote by $\mathcal{C}^d$ the system of all compact subsets in $\mathbb{R}^d$ and set

$$\mathcal{C}^{(d)} := \mathcal{C}^d \setminus \{\emptyset\}.$$

Let $K \in \mathcal{C}^{(d)}$. Then we denote by $B(K)$ the circumscribed ball of K and by $z(K)$ the centre of $B(K)$. Further, we denote

$$\mathcal{C}_0^{(d)} := \{K \in \mathcal{C}^{(d)} : z(K) = 0\},$$

where 0 is the origin in $\mathbb{R}^d$.

Take $K, \ L \in \mathcal{C}^{(d)}$. By the Minkowski sum of K and L, we mean

$$K + L := \{x + y : x \in K, \ y \in L\}.$$

Moreover, we denote $\check{K} := \{-x : x \in K\}$.

Definition 1.23 (Hausdorff metric)**.** Let $||\cdot||$ denote the Euclidean distance. On $\mathcal{C}^{(d)}$, the Hausdorff metric is defined by

$$\rho_H(K, L) := \max\left\{\max_{x \in K} \inf_{y \in L} ||x-y||, \max_{y \in L} \inf_{x \in K} ||x-y||\right\}, \quad K, L \in \mathcal{C}^{(d)}.$$

We define particle processes as point processes on the space $\mathcal{C}^{(d)}$ equipped with Hausdorff metric. Such space is a Polish, locally compact space (see Rataj, 2006).

Definition 1.24 (Particle process)**.** Let $\mathcal{C}^{(d)}$ be equipped with the Hausdorff metric, $\mathcal{B}(\mathcal{C}^{(d)})$ be the Borel σ-field on $\mathcal{C}^{(d)}$ and let $\mathbf{N}^d$ denote the space of all measures μ on $\mathcal{C}^{(d)}$ with values in $\mathbb{N}_0 \cup \{\infty\}$, such that $\mu(\{L \in \mathcal{C}^{(d)} : L \cap K \neq \emptyset\}) < \infty$ for all $K \in \mathcal{C}^{(d)}$. We equip $\mathbf{N}^d$ with the smallest σ-field such that the mappings $\mu \mapsto \mu(A)$ are measurable for all measurable $A \subset \mathcal{C}^{(d)}$. Then a point process on $\mathcal{C}^{(d)}$ is called the particle process.

Definition 1.25 (Stationary particle process)**.** For a measure ξ on $\mathcal{C}^{(d)}$ set $t_x\xi(A) := \int \mathbf{1}\{K + x \in A\}\xi(\mathrm{d}K)$, $A \in \mathcal{C}^{(d)}$, and $K + x := \{y + x : x \in K\}$. We call a particle process μ stationary, if $t_x\mu \overset{d}{=} \mu$ for each $x \in \mathbb{R}^d$.

Theorem 1.7. *The intensity measure of a stationary particle process μ can be decomposed as*

$$\theta(A) = \gamma \int_{\mathcal{C}_0^{(d)}} \int_{\mathbb{R}^d} \mathbf{1}\{K + x \in A\}\mathrm{d}x \; \mathbb{Q}(\mathrm{d}K), \quad A \in \mathcal{B}(\mathcal{C}^{(d)}), \tag{1.3}$$

where $\gamma > 0$ and $\mathbb{Q}$ is a probability measure on $\mathcal{B}(\mathcal{C}_0^{(d)})$.

1.5.1 Gibbs particle processes

In the context of Gibbs particle processes, we often use the reference intensity measure $\lambda = \theta/\gamma$. Also, we will make the assumption that there exists $R > 0$ such that

$$\mathbb{Q}(\{K \in \mathcal{C}_0^{(d)} : B(K) \subset b(0, R)\}) = 1. \tag{1.4}$$

Definition 1.26 (Gibbs particle process)**.** We call a particle process μ the Gibbs particle process with Papangelou conditional intensity λ^* and activity $\tau > 0$ if

$$\mathbb{E} \int_{\mathcal{C}^{(d)}} \varphi(K, \mu - \delta_K)\mu(\mathrm{d}K) = \tau\mathbb{E} \int_{\mathcal{C}^{(d)}} \varphi(K, \mu)\lambda^*(K, \mu)\lambda(\mathrm{d}K)$$

holds or all measurable $\varphi : \mathcal{C}^{(d)} \times \mathbf{N}^d \to [0, \infty)$.

Remark. In this thesis, we will work with the conditional intensity of the form

$$\lambda^*(K, \mu) := \tau \exp\left\{- \int g(K \cap L)\mu(\mathrm{d}L)\right\}, \quad K \in \mathcal{C}^{(d)}, \tag{1.5}$$

where $g : \mathcal{C}^d \to [0, \infty)$ is a measurable, translation invariant function with property $g(\emptyset) = 0$, called the pair potential.

Remark. The existence of a stationary marked Gibbs point process is discussed in Mase, 2000.

Definition 1.14 can be rewritten for an innovation of a Gibbs particle process μ as

$$I_\mu(\varphi) = \sum_{K \in \mu} \varphi(K, \mu - \delta_K) - \int_{\mathcal{C}^{(d)}} \varphi(K, \mu)\lambda^*(K, \mu)\lambda(\mathrm{d}K).$$

The following lemma gives a lower and an upper bound for the p-th order correlation function of a Gibbs particle process. These bounds are similar to those in Stucki and Schuhmacher, 2014 for Gibbs point processes on $\mathbb{R}^d$. Although, both are derived using different methods. Bounds in Stucki and Schuhmacher, 2014 are built based on Stein's method for Poisson approximation (see Xia, 2005), meanwhile bounds that we will use can be obtained using the Fortuin-Kasteleyn-Ginibre inequality and the covariance identity (both in Last and Penrose, 2017).

Take $p \in \mathbb{N}$ and denote by $U_p, V_p : (\mathcal{C}^{(d)})^p \to [0, \infty)$ a measurable functions defined by

$$U_p(K_1, \dots, K_p) := \sum_{1 \le i < j \le p} g(K_i \cap K_j),$$

$$V_p(K_1, \dots, K_p) := \int\limits_{\mathcal{C}^{(d)}} \left(1 - e^{-\sum_{i=1}^{p} g(K \cap K_i)}\right) \lambda(\mathrm{d}K),$$

$K_1, \dots, K_p \in \mathcal{C}^{(d)}$.

Lemma 1.8 (Last, 2017)**.** *Let μ be a stationary Gibbs particle process with activity $\tau > 0$ and particle distribution $\mathbb{Q}$. Take $p \in \mathbb{N}$, then*

$$\rho_p(K_1, \dots, K_p) \ge \tau^p \exp\{-U_p(K_1, \dots, K_p)\} \exp\{-\tau V_p(K_1, \dots, K_p)\}$$

for λ^p-a.a. $K_1, \dots, K_p \in \mathcal{C}^{(d)}$.

If, moreover, there exists $b \in [0, \infty)$ such that $V_1(K) \le b$ for λ-a.a. $K \in \mathcal{C}^{(d)}$, then

$$\rho_p(K_1, ..., K_p) \le \tau^p \exp\{-U_p(K_1, ..., K_p)\} \left(1 - e^{-\tau b}(1 - \exp\{-\tau V_p(K_1, ..., K_p)\})\right)$$

for λ^p-a.a. $K_1, \dots, K_p \in \mathcal{C}^{(d)}$.

Example 1.3 (Planar segment process)**.** Denote by $S \subset \mathcal{C}^{(2)}$ the space of all planar segments with fixed length $r > 0$. Any segment $K \in S$ can be parametrized as $K = (x, \varphi)$, where $x \in \mathbb{R}^2$ is the centre and $\varphi \in \mathbb{S}^1$ is the axial orientation.

Take $a > 0$. On $\mathcal{C}^{(2)}$, we define a pair potential

$$g(K) := a\mathbf{1}\{K \ne \emptyset\}, \quad K \in \mathcal{C}^{(2)}. \tag{1.6}$$

We define the segment process ξ in $\mathbb{R}^2$ as a Gibbs particle process with intensity measure λ concentrated on S, activity $\tau > 0$ and the Papangelou conditional intensity defined as

$$\lambda^*(K, \mathbf{x}) = \tau \exp\left\{-a \int\limits_S \mathbf{1}\{K \cap L \ne \emptyset\} \mathrm{d}\mathbf{x}(L)\right\} \quad K \in S, \mathbf{x} \in \mathbf{N}^2.$$

In fact, $\lambda^*(K, \mathbf{x}) = \tau e^{-aN_{\mathbf{x}}(K)}$, where $N_{\mathbf{x}}(K)$ denotes the number of intersections of K with the segments in $\mathbf{x}$. The directional distribution $\mathbb{Q}$ can be interpreted as an even probability measure on the unit sphere $\mathbb{S}^1$.

Chapter 2

Fock space representation

This chapter is devoted to the Fock space representation theorem. We will introduce the definition of the Fock space, show the proof of this theorem and then cover some applications. The most important application will follow in a separate Chapter 4. Results presented in this chapter can be found in Last and Penrose, 2017, Chapters 18 and 20. Suppose, we have a Poisson point process η on $\mathbb{X}$ with $\sigma-$finite intensity measure λ and distribution P_η. First, recall some results that will be needed.

Theorem 2.1 (Bogachev, 2007, page 146)**.** *Let $\boldsymbol{W}$ be a vector space of $\mathbb{R}$-valued, bounded functions on $\mathbb{X}$ that contains the constant functions. Further, suppose that for every increasing, uniformly bounded sequence of non-negative functions $f_n \in \boldsymbol{W}, n \in \mathbb{N}$, the function $f = lim_{n\to\infty} f_n$ belongs to $\boldsymbol{W}$. Let $\boldsymbol{G}$ be a subset of $\boldsymbol{W}$ that is closed with respect to multiplication. Then $\boldsymbol{W}$ contains all bounded $\sigma(\boldsymbol{G})$-measurable functions on $\mathbb{X}$.*

Theorem 2.2. *(Multivariate Mecke Equation) Let $m \in \mathbb{N}$. Then, for every $f : \mathbb{X}^m \times \boldsymbol{N} \to \mathbb{R}_+ \cup \{\infty\}$ measurable function*

$$\mathbb{E}\left[\int_{\mathbb{X}^m} f(x_1,\dots,x_m,\eta)\eta^{(m)}(\mathrm{d}(x_1,\dots,x_n))\right]$$
$$= \int_{\mathbb{X}^m} \mathbb{E}\left[f(x_1,\dots,x_m,\eta+\delta_{x_1}+\dots+\delta_{x_m})\right]\lambda^m(\mathrm{d}(x_1,\dots,x_m)), \quad (2.1)$$

as long as the right-hand side is finite.

2.1 Difference operator

Definition 2.1 (Difference operator)**.** Let $F : \mathbf{N} \to \mathbb{R}$ be a measurable function. For $y \in \mathbb{X}$, we define the difference operator as a functional $D_y F : \mathbf{N} \to \mathbb{R}$ satisfying

$$D_y F(\mu) = F(\mu + \delta_y) - F(\mu).$$

Iterating this definition, we get for $n \in \mathbb{N},\ n \geq 2$, the difference operator of the $n-$th order defined as a functional $D^n_{y_1,\dots,y_n} F : \mathbf{N} \to \mathbb{R}$

$$D^n_{y_1,\dots,y_n} F = D^1_{y_1} D^{n-1}_{y_2,\dots,y_n} F,$$

where $D^1 = D$ and $D^0 F = F$.

Remark. We can see that for the $n-$th order difference operator, it holds

$$D^n_{y_1,\dots,y_n} F(\mu) = \sum_{J \subset \{1,\dots,n\}} (-1)^{n-|J|} F\left(\mu + \sum_{j \in J} \delta_{y_j}\right). \tag{2.2}$$

Also, it will be useful to notice that $D^n_{y_1,\dots,y_n} F$ is symmetric mapping in $y_1 \dots, y_n \in \mathbb{X}$ and $(\mu, y_1, \dots, y_n) \mapsto D^n_{y_1,\dots,y_n} F(\mu)$ is measurable.

Notation. For purposes of the following section, we will denote for $F : \mathbf{N} \to \mathbb{R}$ and $n \in \mathbb{N}$

$$T_n F(y_1, \dots, y_n) = \mathbb{E}[D^n_{y_1,\dots,y_n} F(\eta)]$$

and set $T_0 F = \mathbb{E}[F(\eta)]$, whenever these expectations exist. Set $T_n F(y_1, \dots, y_n) = 0$ otherwise.

Note that the mapping $T_n F : \mathbb{X}^n \to \mathbb{R}$ is again symmetric and measurable.

2.2 Fock space representation theorem

To obtain the Fock space representation theorem, we need to introduce some further notation.

Notation. The scalar product of $f,\ g \in L^2(\mathbb{X}^n, \lambda^n)$ is denoted by

$$\langle f,\ g \rangle_n = \int_{\mathbb{X}^n} f g \ \mathrm{d}\lambda^n.$$

The associated norm is then denoted by

$$||f||_n = \sqrt{\langle f, f \rangle_n}.$$

Definition 2.2 (Fock space)**.** Let H_n denote the space of all symmetric functions $f \in L^2(\mathbb{X}^n, \lambda^n)$ equipped with norm $||f||_n$. Then we define the Fock space H as the space of all sequences $f = (f_n)_{n \geq 0}$, $f_n \in H_n$, i.e. as the product space

$$H = \bigtimes_{n=0}^{\infty} H_n.$$

with a scalar product defined by

$$\langle f, g \rangle_H = \sum_{i=0}^{\infty} \frac{1}{n!} \langle f_n, g_n \rangle_n, \quad f, g \in H.$$

Remark. H is a Hilbert space.

Theorem 2.3 (Fock space representation)**.** *Let $f, g \in L^2(\boldsymbol{N}, P_\eta)$. Then*

$$\mathbb{E}\ [f(\eta) g(\eta)] = (\mathbb{E}\ [f(\eta)])\,(\mathbb{E}\ [g(\eta)]) + \sum_{n=1}^{\infty} \frac{1}{n!} \langle T_n f, T_n g \rangle_n. \tag{2.3}$$

In particular,

$$\mathbb{E}\ [f(\eta)^2] = (\mathbb{E}\ [f(\eta)])^2 + \sum_{n=1}^{\infty} \frac{1}{n!} ||T_n f||_n^2. \tag{2.4}$$

Remark. Let us for $f \in L^2(\mathbf{N}, P_\eta)$ define the sequence $Tf = (T_n f)_{n\geq 0}$. Theorem 2.3 gives us that $Tf \in H$ and

$$\mathbb{E}[f(\eta)g(\eta)] = \langle Tf, Tg\rangle_H.$$

Proof. We will prove the theorem in four steps. In prior to show that the equality holds for arbitrary $f, g \in L^2(\mathbf{N}, P_\eta)$, we will prove it for special space of bounded and measurable functions, which will be proved to be dense in $L^2(\mathbf{N}, P_\eta)$. Than we apply some approximation arguments to prove the theorem.

Step 1 Let $\mathcal{X}_0$ be the system of all measurable sets $B \in \mathcal{B}(\mathbb{X})$ for which $\lambda(B) < \infty$. Denote by $\mathbb{R}_0(\mathbb{X})$ the space of all bounded functions $v : \mathbb{X} \to \mathbb{R}_+$ vanishing outside some $B \in \mathcal{X}_0$. Furthermore, denote by $\mathbf{G}$ the space of all (bounded and measurable) functions $g : \mathbf{N} \to \mathbb{R}$ of the form

$$g(\mu) = a_1 e^{-\mu(v_1)} + \ldots + a_n e^{-\mu(v_n)},$$

where $n \in \mathbb{N}, a_1, \ldots, a_n \in \mathbb{R}$ and $v_1, \ldots, v_n \in \mathbb{R}_0(\mathbb{X})$. Let us show that equality (2.3) holds for $f, g \in \mathbf{G}$.

By linearity, it is sufficient to consider functions f and g of the form

$$f(\mu) = \exp[-\mu(v)], \quad g(\mu) = \exp[-\mu(w)]$$

for $v, w \in \mathbb{R}_0(\mathbb{X})$. First, we will calculate $T_n f$ and $T_n g$ for $n \in \mathbb{N}$. For each $\mu \in \mathbf{N}$ and $x \in \mathbb{X}$, we have

$$f(\mu + \delta_x) = \exp\left[-\int_{\mathbb{X}} v(y)(\mu + \delta_x)(\mathrm{d}y)\right] = \exp[-\mu(v)]\exp[-v(x)],$$

and therefore,

$$D_x f(\mu) = \exp[-\mu(v)](\exp[-v(x)] - 1).$$

Iterating this identity, we can get for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in \mathbb{X}$ that

$$D^n_{x_1,\ldots,x_n} f(\mu) = \exp[-\mu(v)] \prod_{i=1}^{n} (\exp[-v(x_i)] - 1). \tag{2.5}$$

Recall that for the Poisson point process η with intensity measure λ, the Laplace functional takes form

$$L_\eta(u) = \exp\left[-\lambda(1 - e^{-u})\right], \quad u : \mathbb{X} \to \mathbb{R}_+. \tag{2.6}$$

From (2.5) and (2.6), we obtain that

$$T_n f = \exp[-\lambda(1 - e^{-v})] \prod_{i=1}^{n} (\exp[-v(x_i)] - 1). \tag{2.7}$$

Analogously for g. Since $v, w \in \mathbb{R}_0(\mathbb{X})$ it follows that $T_n f, T_n g \in H_n, n \geq 0$. Using again equality (2.6), we obtain that

$$E[f(\eta)g(\eta)] = \exp[-\lambda(1 - e^{-(v+w)})].$$

Now, we can compute the right hand side of (2.3)

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n!}\langle T_n f, T_n g\rangle_n \\
&\quad = \exp[-\lambda(1-e^{-v})]\exp[-\lambda(1-e^{-w})]\sum_{n=1}^{\infty}\frac{1}{n!}\lambda^n(((e^{-v}-1)(e^{-w}-1))^{\otimes n}) \\
&\quad = \exp[-\lambda(2-e^{-v}-e^{-w})]\exp[\lambda((e^{-v}-1)(e^{-w}-1))] \\
&\quad = \exp[-\lambda(1-e^{-(v+w)})]
\end{aligned}$$

and hence the assertion holds true for $f, g \in \mathbf{G}$.

Step 2 We need to prove that the set $\mathbf{G}$ is dense in $L^2(\mathbf{N}, P_\eta)$. Let $\mathbf{W}$ be the space of all bounded measurable $g : \mathbf{N} \to \mathbb{R}$ that can be approximated in $L^2(\mathbf{N}, P_\eta)$ by functions in $\mathbf{G}$. We want to use the functional version of the monotone class theorem, i.e. Theorem 2.1. We can see that space $\mathbf{G}$ is closed under uniformly bounded convergence. It also contains the constant functions and it is closed under multiplication. If we denote by $\mathcal{N}'$ the smallest σ-field on $\mathbf{N}$ such that $\mu \mapsto h(\mu)$ is measurable for all $h \in \mathbf{G}$, then according the Theorem 2.1 $\mathbf{W}$ contains any bounded $\mathcal{N}'$-measurable g.

On the other hand we can write for every $C \in \mathcal{X}_0$ that

$$\mu(C) = \lim_{t\to 0_+} t^{-1}(1-e^{-t\mu(C)}), \quad \mu \in \mathbf{N},$$

such that $\mu \mapsto \mu(C)$ is $\mathcal{N}'$-measurable. Since λ is σ-finite, for any $C \in \mathcal{X}$ there exists a monotone sequence $C_k \in \mathcal{X}_0, k \in \mathbb{N}$ such that $C = \cup C_k$, so that $\mu \mapsto \mu(C)$ is $\mathcal{N}'$-measurable. Thus, $\mathcal{N}' = \mathcal{N}$ and it follows that $\mathbf{W}$ contains all bounded measurable functions. Hence $\mathbf{W}$ is dense in $L^2(\mathbf{N}, P_\eta)$.

Step 3 For further purposes we would like to show that $f, f^1, f^2, \ldots \in L^2(\mathbf{N}, P_\eta)$ satisfying $f^k \to f$ in $L^2(\mathbf{N}, P_\eta)$ as $k \to \infty$ implies

$$\lim_{k\to\infty}\int_{C^n}\mathbb{E}\left[|D^n_{x_1,\ldots,x_n}f(\eta) - D^n_{x_1\ldots,x_n}f^k(\eta)|\right]\lambda^n(\mathrm{d}(x_1,\ldots,x_n)) = 0 \qquad (2.8)$$

for all $n \in \mathbb{N}$ and $C \in \mathcal{X}_0$. According to (2.2), it is sufficient to prove

$$\lim_{k\to\infty}\int_{C^n}\mathbb{E}\left[\left|f\left(\eta+\sum_{i=1}^m\delta_{x_i}\right) - f^k\left(\eta+\sum_{i=1}^m\delta_{x_i}\right)\right|\right]\lambda^n(\mathrm{d}(x_1,\ldots,x_n)) = 0 \qquad (2.9)$$

for all $m \in \{0, \ldots, n\}$. The case of $m = 0$ is obvious. Assuming $m \in \{0, \ldots, n\}$, we apply on the integral inside the limit (2.8) the multivariate Mecke equation, i.e Theorem 2.2.

$$\begin{aligned}
&\int_{C^n}\mathbb{E}\left[\left|f\left(\eta+\sum_{i=1}^m\delta_{x_i}\right) - f^k\left(\eta+\sum_{i=1}^m\delta_{x_i}\right)\right|\right]\lambda^n(\mathrm{d}(x_1,\ldots,x_n)) \\
&= \lambda(C)^{n-m}\int_{C^m}\mathbb{E}\left[\left|f\left(\eta+\sum_{i=1}^m\delta_{x_i}\right) - f^k\left(\eta+\sum_{i=1}^m\delta_{x_i}\right)\right|\right]\lambda^m(\mathrm{d}(x_1,\ldots,x_m))
\end{aligned}$$

$$= \lambda(C)^{n-m}\mathbb{E}\left[\int_{C^m} \left|f(\eta) - f^k(\eta)\right| \eta^m(\mathrm{d}(x_1,\ldots,x_m))\right]$$
$$\leq \lambda(C)^{n-m}\mathbb{E}\left[\left|f(\eta) - f^k(\eta)\right| \eta^{(m)}(C^m)\right]$$
$$\leq \lambda(C)^{n-m}(\mathbb{E}\left[(f(\eta) - f^k(\eta))^2\right])^{\frac{1}{2}}(\mathbb{E}\left[(\eta^{(m)}(C^m))^2\right])^{\frac{1}{2}}.$$

Last estimate follows from the Cauchy-Schwartz inequality. Since all moments of the Poisson distribution exists, we obtain (2.9) and hence (2.8).

Step 4 Recall the polarization identity of the scalar product

$$4\langle f, g\rangle_{\mathbf{H}} = \langle f+g, f+g\rangle_{\mathbf{H}} - \langle f-g, f-g\rangle_{\mathbf{H}}.$$

Because of the linearity of the scalar product, it is sufficient to show that (2.4) holds to prove the theorem.

Since the system $\mathbf{G}$ is dense in $L^2(\mathbf{N}, P_\eta)$, for every $f \in L^2(\mathbf{N}, P_\eta)$ there is a sequence $f^k \in \mathbf{G}$ such that $f^k \to f$ in $L^2(\mathbf{N}, P_\eta)$ as $k \to \infty$. In step 3, we proved that $Tf^k, k \in \mathbb{N}$, is a Cauchy sequence in $\mathbf{H}$, hence has a limit $\tilde{f} = (\tilde{f}_n) \in \mathbf{H}$, meaning that

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} \frac{1}{n!} ||T_n f^k - \tilde{f}_n||_n^2 = 0. \tag{2.10}$$

Taking the limit in the identity $\mathbb{E}[f^k(\eta)^2] = \langle Tf^k, Tf^k\rangle_{\mathbf{H}}$ yields

$$\mathbb{E}[f(\eta)^2] = \langle \tilde{f}, \tilde{f}\rangle_{\mathbf{H}}.$$

Equation (2.10) immediately implies that $\tilde{f}_0 = \mathbb{E}[f(\eta)] = T_0 f$. It remains to show that for any $n \geq 1$, we have

$$\tilde{f}_n = T_n f, \quad \lambda^n\text{-a.e.} \tag{2.11}$$

Let $C \in \mathcal{X}_0$ and let $B := C^n$. Denote by $(\lambda^n)_B$ the restriction of the measure λ^n to B. By (2.10) $T_n f^k$ converges to f in $L^2(B, (\lambda^n)_B)$ and hence also in $L^1(B, (\lambda^n)_B)$. Meanwhile, by the definition of T_n and the equality (2.8), $T_n f^k$ converges in $L^1(B, (\lambda^n)_B)$ to $T_n f$. Hence the uniqueness of these limits yields $\tilde{f}_n = T_n f \lambda^n$- a. e. on B. Since λ is assumed to be $\sigma-$finite, this implies (2.11) and hence the theorem. □

2.3 Covariance identity

Covariance identity is a direct consequence of the Fock space representation theorem. It can be further used, for instance, to obtain bounds on the Wasserstein distance between the standard normal distribution and distribution of a Poisson functional (cf. Section 4.2).

Notation. Denote

$$L_\eta^0 = \{F; F = f(\eta)\ \mathbb{P}\text{-a.s. for some measurable } f: \mathbf{N} \to \mathbb{R}\}.$$

If $F \in L_\eta^0$, $F = f(\eta)$, then f is called a representative of the functional F. Further, denote for $q > 0$

$$L_\eta^q = \{F \in L_\eta^0;\ \mathbb{E}\left[|F|^q\right] < \infty\}.$$

Assume, we have a square integrable Poisson functional F (i.e. $F \in L^2_\eta$) and $t \in [0,1]$. To obtain the covariance identity, we need to introduce Poisson functional P_tF defined by a combination of t-thinning and independent superposition. Then we will be able to rewrite the Fock space series representation as an integral equation involving only the first order difference operator and the operator P_t.

Definition 2.3. (Operator P_t) Let for $F \in L^1_\eta$ with a representative f define

$$P_tF = \mathbb{E}\left[\int_{\mathbf{N}} f(\eta_t + \mu)\Pi_{(1-t)\lambda}(\mathrm{d}\mu)\middle|\eta\right], \quad t \in [0,1],$$

where η_t is a t-thinning of η and $\Pi_{\lambda'}$ denotes the distribution of a Poisson process with intensity measure λ'.

Lemma 2.4. *[Last and Penrose, 2017, page 265] Let $f : \Omega \times \mathbb{X} \to \bar{\mathbb{R}}_+$ be a measurable function, $f \in L^1(\Omega \times \mathbb{X}, \mathbb{P} \otimes \lambda)$. Let $\mathcal{G} \subset \mathcal{A}$ be a $\sigma-$field. Then there is a measurable version of $\mathbb{E}[f(x)|\mathcal{G}]$ satisfying*

$$\mathbb{E}\left[\int_{\mathbb{X}} f(x)\lambda(\mathrm{d}x)\middle|\mathcal{G}\right] = \int_{\mathbb{X}} \mathbb{E}\left[f(x)|\mathcal{G}\right]\lambda(\mathrm{d}x), \quad \mathbb{P}\text{-}a.s.. \tag{2.12}$$

Remark. According to Theorems 1.1 and 1.2, we get that

$$\Pi_\lambda = \mathbb{E}\left[\int_{\mathbf{N}} \mathbf{1}\{\eta_t + \mu \in \cdot\}\Pi_{(1-t)\lambda}(\mathrm{d}\mu)\right]. \tag{2.13}$$

It follows that the definition of P_tF does not depend on the chosen representative of F up to almost sure equality. Moreover, using Lemma 2.4 gives us equality

$$P_tF = \int_{\mathbf{N}} \mathbb{E}\left[f(\eta_t + \mu)|\eta\right]\Pi_{(1-t)\lambda}(\mathrm{d}\mu), \quad t \in [0,1].$$

We can also see that

$$P_t = \mathbb{E}\left[f(\eta_t + \eta'_{1-t})|\eta\right],$$

where η'_{1-t} is a Poisson process with intensity measure $(1-t)\lambda$, independent of the pair (η, η_t).

It follows from (2.13) that

$$\mathbb{E}\left[P_tF\right] = \mathbb{E}\left[F\right], \quad F \in L^1_\eta \tag{2.14}$$

and $P_tF \in L^1_\eta$, whenever $F \in L^1_\eta$.

Using the conditional version of the Jensen inequality and equality (2.14), we can determine an estimate for the p-th absolute moment of P_tF.

Lemma 2.5. *(Contractivity property) For any $p \geq 1$, $F \in L^p_\eta$ and $t \in [0,1]$, we have*

$$\mathbb{E}\left[|P_tF|^p\right] \leq \mathbb{E}\left[|F|^p\right].$$

Proof. Let f be a representative of F and denote $g = |f|^p, G = g(\eta)$. Then we can estimate

$$\begin{aligned}\mathbb{E}\,[|P_tF|^p] &= \mathbb{E}\,\left|\mathbb{E}\,[f(\eta_t+\eta'_{1-t})|\eta]\right|^p \leq \mathbb{E}\,\left[\mathbb{E}\,[|f(\eta_t+\eta'_{1-t})|^p|\eta]\right]\\ &= \mathbb{E}\,\left[\mathbb{E}\,[g(\eta_t+\eta'_{1-t})|\eta]\right] = \mathbb{E}\,[P_tG] = \mathbb{E}\,[G] = \mathbb{E}\,[|F|^p].\end{aligned}$$

□

Lemma 2.6. *(Mehler's formula) Let $F \in L^2_\eta$, $n \in \mathbb{N}$ and $t \in [0,1]$. Then*

$$D^n_{x_1,\dots,x_n}(P_tF) = t^n P_t D^n_{x_1,\dots,x_n}F, \quad \lambda^n\text{-a.a. } (x_1,\dots,x_n) \in \mathbb{X}^n,\ \mathbb{P}-a.s.$$

In particular,

$$\mathbb{E}\,[D^n_{x_1,\dots,x_n}P_tF] = t^n\mathbb{E}\,[D^n_{x_1,\dots,x_n}F], \quad \lambda^n\text{-a.a. } (x_1,\dots,x_n) \in \mathbb{X}^n.$$

Notation. Let for $F \in L^2_\eta$ denote by DF the mapping $(\omega, x) \mapsto (D_xF)(\omega)$. The next theorem will additionally require $DF \in L^2(\Omega \times \mathbb{X}, \mathbb{P} \otimes \lambda)$, i.e.

$$\mathbb{E}\left[\int_{\mathbb{X}}(D_xF)^2\lambda(\mathrm{d}x)\right] < \infty.$$

Theorem 2.7 (Covariance Identity)**.** *For any $F, G \in L^2_\eta$ such that $DF, DG \in L^2(\Omega \times \mathbb{X}, \mathbb{P} \otimes \lambda)$, we have*

$$\mathsf{E}[FG] - \mathsf{E}[F]\,\mathsf{E}[G] = \mathsf{E}\left[\int_{\mathbb{X}}\int_0^1 (D_xF)(P_tD_xG)\mathrm{d}t\lambda(\mathrm{d}x)\right]. \tag{2.15}$$

Proof. Using first the Cauchy-Schwartz inequality and then the contractivity property (Lemma 2.5) we can estimate

$$\begin{aligned}&\left(\mathbb{E}\left[\int_{\mathbb{X}}\int_0^1 |D_xF||P_tD_xG|\mathrm{d}t\lambda(\mathrm{d}x)\right]\right)^2\\ &\qquad\leq \mathbb{E}\left[\int_{\mathbb{X}}(D_xF)^2\lambda(\mathrm{d}x)\right]\mathbb{E}\left[\int_{\mathbb{X}}\int_0^1(P_tD_xG)^2\mathrm{d}t\lambda(\mathrm{d}x)\right]\\ &\qquad\leq \mathbb{E}\left[\int_{\mathbb{X}}(D_xF)^2\lambda(\mathrm{d}x)\right]\mathbb{E}\left[\int_{\mathbb{X}}(D_xG)^2\lambda(\mathrm{d}x)\right],\end{aligned}$$

which is finite by the assumption. Therefore, using Fubini's theorem and Mehler's formula (Lemma 2.6), we obtain that the right-hand side of (2.15) equals

$$\int_{\mathbb{X}}\int_0^1 t^{-1}\,\mathsf{E}\,[(D_xF)(P_tD_xG)]\mathrm{d}t\lambda(\mathrm{d}x). \tag{2.16}$$

We can now apply the Fock space representation (Theorem 2.3) to the expectation inside the integral. For $t \in [0,1]$ and taking into account also Lemma 2.6 we obtain

$$\begin{aligned}\mathsf{E}\,[(D_xF)(D_xP_tG)] = t\,\mathsf{E}\,[D_xF]\,\mathsf{E}\,[D_xG] \\ + \sum_{i=1}^{\infty} \frac{t^{n+1}}{n!} \int_{\mathbb{X}^n} \mathsf{E}\,[D^{n+1}_{x_1,\dots,x_n,x}F]\,\mathsf{E}\,[D^{n+1}_{x_1,\dots,x_n,x}G]\lambda^n(\mathrm{d}(x_1,\dots,x_n)).\end{aligned}$$

We want to insert this expression into formula (2.16) and use Fubini's theorem (to be justified below). Compute (2.16) as

$$\begin{aligned}&\int_{\mathbb{X}} \int_0^1 \mathsf{E}\,[D_xF]\,\mathsf{E}\,[D_xG]\mathrm{d}t\lambda(\mathrm{d}x) \\ &\quad + \sum_{n=1}^{\infty} \int_0^1 \frac{t^n}{n!}\mathrm{d}t \int_{\mathbb{X}} \int_{\mathbb{X}^n} \mathsf{E}\,[D^{n+1}_{x_1,\dots,x_n,x}F]\,\mathsf{E}\,[D^{n+1}_{x_1,\dots,x_n,x}G]\lambda^n(\mathrm{d}(x_1,\dots,x_n))\lambda(\mathrm{d}x) \\ &= \int_{\mathbb{X}} \mathsf{E}\,[D_xF]\,\mathsf{E}\,[D_xG]\lambda(\mathrm{d}x) \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \int_{\mathbb{X}} \int_{\mathbb{X}^n} \mathsf{E}\,[D^{n+1}_{x_1,\dots,x_n,x}F]\,\mathsf{E}\,[D^{n+1}_{x_1,\dots,x_n,x}G]\lambda^n(\mathrm{d}(x_1,\dots,x_n))\lambda(\mathrm{d}x) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} \mathsf{E}\,[D^n_{x_1,\dots,x_n}F]\,\mathsf{E}\,[D^n_{x_1,\dots,x_n}G]\lambda^n(\mathrm{d}(x_1,\dots,x_n)).\end{aligned}$$

Eventually, by Theorem 2.3, this equals to $\mathsf{E}\,[FG] - \mathsf{E}\,[F]\,\mathsf{E}\,[G]$, which yields the asserted formula (2.15). The use of Fubini's theorem is justified by identity (2.4) and the Cauchy-Schwartz inequality. □

To conclude this section, we can point out the consequences of the Fock space representation theorem (Theorem 2.3) occurring in this work. First of all, the Fock space representation theorem plays a crucial role while proving the covariance identity (Theorem 2.7).

In section 4.2, we obtain bounds on the Wasserstein distance between the standard normal distribution and the distribution of a Poisson functional by combining the covariance identity with Stein's method.

Furthermore, using the covariance identity, it is possible to prove Lemma 1.8, which is needed in Section 4.5 to derive bounds on the Wasserstein distance between the standard normal distribution and the distribution of an innovation of a Gibbs particle process.

Chapter 3

Hard-core process in the real line

It is generally a very non-trivial or impossible task to evaluate terms $T_n f$, $f \in L^2(\mathbf{N}, P_\eta)$ in the Fock space representation theorem because of the expectation. We will show that it is non-trivial even for case of $n = 0$. The Poisson functionals will be defined as indicator functions of a Poisson point process being included in the domain of the hard-core process on the real line. We will determine some characteristics such as the normalising constant in the density or the correlation function of the first order.

We will restrict ourselves to the space $\mathbb{R}^1$. Recall first the general definition of hard-core process κ_r on a bounded set $B \subset \mathbb{R}^d$ with density with respect to the distribution of a Poisson point process η with intensity measure $\lambda = Leb$.

Definition 3.1 (Hard-core proces)**.** For $r > 0$ denote

$$S_n^r = \Big\{(x_1, \dots, x_n) \in B^{!n} : \| x_i - x_j \| \geq r, \text{ for all } i \neq j\Big\}, \quad n \geq 1,$$

and

$$S^r = \bigcup_{n=0}^{\infty} I_n(S_n^r),$$

where $I_k : B^{!k} \to \mathbf{N}_k$ is defined by $I_k(x_1, \dots, x_k) = \delta_{x_1} + \ldots + \delta_{x_k}$ and $B^{!k}$ is the set of all ordered k-tuples of pairwise different points of the set B, i.e.

$$B^{!k} = \{(x_1, \dots, x_k) : x_i \in B, x_i \neq x_j, \text{ for all } i \neq j\}.$$

$I_0(S_0^r)$ can by understood as the one point set containing only null measure.

Then, we define the hard-core process κ_r as a point process with density p with respect to distribution of a Poisson point process η with intensity measure $\lambda = Leb$, where

$$p(\mathbf{x}) = \alpha_r \mathbf{1}\{\mathbf{x} \in S^r\}, \quad \mathbf{x} \in \mathbf{N}.$$

Remark. For the normalising constant α_r from Definition 3.1, it holds

$$\alpha_r = \frac{1}{\mathbb{P}(\eta \in S^r)}.$$

Notation. While working on the real line, we denote for $n \geq 1$

$$S^r_{n,u_1,\dots,u_k}[a,b] = \Bigg\{(x_1,\dots,x_n) \in [a,b]^n : \ \|x_i - x_j\| \geq r, \underset{i,j\in\{1,\dots,n\}}{\underset{i\neq j}{\forall}} ; \|x_i - u_j\| \geq r,$$

$$i = 1,\dots,n, \ j = 1,\dots,k\Bigg\}, \text{ if } (u_1,\dots,u_k) \in S^r_k[a,b]$$

$$= \emptyset, \quad \text{otherwise}$$

and

$$S^r_n[a,b] = \Bigg\{ (x_1,\dots,x_n) \in [a,b]^n : \| x_i - x_j \| \geq r, \underset{i,j\in\{1,\dots,n\}}{\underset{i\neq j}{\forall}} \Bigg\}.$$

Now, we will derive the analytical form of the normalising constant and the correlation function $\rho^{(r)}$ for the hard-core process κ_r on an interval $[a,b], a < b$. Note that since we are working on a closed interval, hard-core property allows the process κ_r to have only finitely many points. The maximum number of points depends on length of the interval $[a,b], a < b$ and on the parameter r.

Theorem 3.1. *Set $r > 0$ and denote by $m := \lfloor \frac{b-a}{r} \rfloor + 1$ the maximal possible number of points of the point process κ_r in the interval $[a,b]$. Then the normalising constant α_r of the hard-core point process κ_r on $[a,b], a < b$, can be expressed as following*

$$\alpha_r = \frac{1}{e^{-(b-a)}\left(1 + \sum\limits_{n=1}^{m} \frac{1}{n!}(b-a-(n-1)r)^n\right)}.$$

Proof. Using Theorem 1.3 we can see for the Poisson point process η that

$$\begin{aligned}
\mathbb{P}\,(\eta \in S^r) &= \mathbb{E}\,[\mathbf{1}\{\eta \in S^r\}] \\
&= e^{-(b-a)}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_a^b \cdots \int_a^b \mathbf{1}\{(x_1,\dots,x_n) \in S^r_n[a,b]\}\ \mathrm{d}x_1 \dots \mathrm{d}x_n\right) \\
&= e^{-(b-a)}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} Leb(S^r_n[a,b])\right).
\end{aligned}$$

Further for $n = 1,\dots,m$ we compute $Leb(S^r_n[a,b])$. If $n > m$, then $Leb(S^r_n[a,b]) = 0$. Denote

$$I(i_1,\dots,i_n) = Leb(\{(x_1,\dots,x_n) \in [a,b]^n : x_{i_1} < \ldots < x_{i_n};\ \| x_{i_k} - x_{i_l} \| \geq r,\ \forall k \neq l\}).$$

Then,

$$Leb(S^r_n[a,b]) = \sum_{(i_1,\dots,i_n)\in P_n} I(i_1,\dots,i_n), \quad n \geq 1,$$

where P_n denotes the set of all permutations on the set $\{1, \dots, n\}$. It is not difficult to see that $I : P_n \to \mathbb{R}_+$ is symmetric in its variables, thus

$$Leb(S_n^r[a,b]) = n! I(1, \dots, n), \quad n \geq 1.$$

It is sufficient to compute $I(1, \dots, n)$.

$$\begin{aligned}
I(1,\dots,n) &= \\
&= \int_{a}^{b-(n-1)r} \int_{x_1+r}^{b-(n-2)r} \int_{x_2+r}^{b-(n-3)r} \cdots \int_{x_{n-3}+r}^{b-2r} \int_{x_{n-2}+r}^{b-r} \int_{x_{n-1}+r}^{b} 1 \, \mathrm{d}x_n \mathrm{d}x_{n-1} \dots \mathrm{d}x_1 \\
&= \int_{a}^{b-(n-1)r} \int_{x_1+r}^{b-(n-2)r} \int_{x_2+r}^{b-(n-3)r} \cdots \int_{x_{n-3}+r}^{b-2r} \int_{x_{n-2}+r}^{b-r} (b - r - x_{n-1}) \, \mathrm{d}x_{n-1} \dots \mathrm{d}x_1 \\
&= \int_{a}^{b-(n-1)r} \int_{x_1+r}^{b-(n-2)r} \int_{x_2+r}^{b-(n-3)r} \cdots \int_{x_{n-3}+r}^{b-2r} \int_{0}^{b-2r-x_{n-2}} y \, \mathrm{d}y \mathrm{d}x_{n-2} \dots \mathrm{d}x_1 \\
&= \frac{1}{2} \int_{a}^{b-(n-1)r} \int_{x_1+r}^{b-(n-2)r} \int_{x_2+r}^{b-(n-3)r} \cdots \int_{x_{n-3}+r}^{b-2r} (b - 2r - x_{n-2})^2 \, \mathrm{d}x_{n-2} \dots \mathrm{d}x_1 \\
&= \frac{1}{2} \int_{a}^{b-(n-1)r} \int_{x_1+r}^{b-(n-2)r} \int_{x_2+r}^{b-(n-3)r} \cdots \int_{0}^{b-3r-x_{n-3}} y^2 \, \mathrm{d}y \mathrm{d}x_{n-3} \dots \mathrm{d}x_1 \\
&= \frac{1}{(n-1)!} \int_{a}^{b-(n-1)r} (b - (n-1)r - x_1)^{n-1} \, \mathrm{d}x_1 \\
&= \frac{1}{(n-1)!} \int_{0}^{b-a-(n-1)r} y^{n-1} \, \mathrm{d}y \\
&= \frac{(b - a - (n-1)r)^n}{n!}.
\end{aligned}$$

For each integral of the form

$$\int_{x_{n-j-1}+r}^{b-jr} (b - jr - x_{n-j})^j \, \mathrm{d}x_{n-j}, \quad j = 0, \dots, n-2,$$

we used the substitution $z = b - jr - x_{n-j}$. Thus, for $n = 1, \dots, m$, we get

$$Leb(S_n^r[a,b]) = n! \cdot \frac{(b - a - (n-1)r)^n}{n!} = (b - a - (n-1)r)^n.$$

Finally, by Theorem 1.3 we prove the assertion by the following argument

$$\alpha_r = \frac{1}{\mathbb{P}(\eta \in S^r)} = \frac{1}{e^{-(b-a)}\left(1 + \sum\limits_{n=1}^{\infty} \frac{1}{n!} Leb(S_n^r[a,b])\right)}$$
$$= \frac{1}{e^{-(b-a)}\left(1 + \sum\limits_{n=1}^{m} \frac{1}{n!}(b-a-(n-1)r)^n\right)}.$$

□

Example 3.1 (Maximum likelihood estimation of the parameter r)**.** Since we determined the analytical form of the normalising constant α_r, and hence the density of the hardcore process from the class $\kappa_r, r > 0$, we can quite easily construct an estimate of the parameter r. For given realisation $\mathbf{x} \in \mathbf{N}$ of the point process κ_r on $[a,b]$, we maximize the likelihood function

$$l(r,\mathbf{x}) = \alpha_r \mathbf{1}\{\mathbf{x} \in S^r\} = \frac{\mathbf{1}\left\{\min\limits_{(x_1,x_2)\in \mathbf{x}^2_{\neq}} |x_1 - x_2| \geq r\right\}}{e^{-(b-a)}\left(1 + \sum\limits_{n=1}^{\lfloor \frac{b-a}{r} \rfloor + 1} \frac{1}{n!}(b-a-(n-1)r)^n\right)}.$$

Function $\sum_{n=0}^{\lfloor \frac{b-a}{r} \rfloor + 1} \frac{1}{n!}(b-a-(n-1)r)^n$ is decreasing for $0 < r \leq \min\limits_{(x_1,x_2)\in \mathbf{x}^2_{\neq}} |x_1 - x_2|$, and so the maximum of likelihood function $l(r,\mathbf{x})$ lies in point

$$\hat{r} = \min_{(x_1,x_2)\in \mathbf{x}^2_{\neq}} |x_1 - x_2|,$$

which is the estimate of the parameter r using the maximum likelihood method.

Theorem 3.2. *Let $u \in [a,b]$ and $r > 0$. Denote by $m_1 = \lfloor \frac{u-a}{r} \rfloor$, resp. $m_2 = \lfloor \frac{b-u}{r} \rfloor$ the maximal possible number of points of point process κ_r on $[a, u-r]$, resp. $[u+r, b]$. Then, for the correlation function $\rho^{(r)}$ of the hard-core process κ_r on $[a,b], a < b$, it holds*

$$\rho^{(r)}(u) = \frac{\sum\limits_{n=0}^{m_1+m_2} \sum\limits_{p=\max(0,n-m_2)}^{\min(n,m_1)} \frac{1}{p!(n-p)!}(u-a-pr)^p(b-u-(n-p)r)^{n-p}}{1 + \sum\limits_{n=1}^{m} \frac{1}{n!}(b-a-(n-1)r)^n}.$$

Proof. Before we go any further, it is useful to became aware of the following relations,

$$\begin{aligned}\rho^{(r)}(u) &= \mathbb{E}[p(\eta \cup \{u\})] \\ &= \mathbb{E}[\alpha_r \mathbf{1}\{\eta \cup \{u\} \in S^r\}] \\ &= \frac{\mathbb{P}(\eta \cup \{u\} \in S^r)}{\mathbb{P}(\eta \in S^r)} \\ &= \frac{1 + \sum_{n=1}^{\infty} \frac{1}{n!} Leb(S_{n,u}^r[a,b])}{1 + \sum_{n=1}^{\infty} \frac{1}{n!} Leb(S_n^r[a,b])}.\end{aligned} \tag{3.1}$$

In the last equality, we used again formula (1.1) from Theorem 1.3.

We can easily evaluate the denominator of the last fraction according to Theorem 3.1. We will now focus on the corresponding nominator, i.e. on the computation of the measure of the set $S^r_{n,u}[a,b]$. For $n = 1, \ldots, m_1 + m_2$, we will compute the measure of $S^r_{n,u}[a,b]$. If $n > m_1 + m_2$, then $Leb(S^r_{n,u}[a,b]) = 0$.

The main idea of the proof is very similar to the one in the proof of Theorem 3.1. We will again decompose the set $S^r_{n,u}[a,b]$ into $n!$ disjoint parts, that are determined by the ordering $x_{i_1} < \ldots < x_{i_n}, (i_1, \ldots, i_n) \in P_n$. Furthermore, we denote the number of elements in n-tuple $(x_1, \ldots, x_n)$ being smaller than u by

$$p := \sharp\{x \in \{x_1, \ldots, x_n\}; x < u\}$$

For technical purposes, set $x_0 = a$, $x_{n+1} = b$ and denote

$$I(p, i_1, \ldots, i_n) = Leb\Big(\Big\{ (x_1, \ldots, x_n) \in [a,b]^n :$$
$$x_{i_1} < \ldots < x_{i_p} \le u - r < u + r \le x_{i_{p+1}} < \ldots < x_{i_n};\ \ \| x_{i_k} - x_{i_l} \| \ge r, \quad \forall k \ne l\Big\}\Big).$$

Again, we need to realized that the mapping $I : \mathbb{N} \times P_n \to \mathbb{R}_+$ is symmetric for given $p \in \mathbb{N}$ on the set P_n. It is not difficult to show that $I(p, \cdot, \ldots, \cdot)$ is non-zero only for $p = \max(0, n - m_2), \ldots, \min(n, m_1)$. Thus, for $n \ge 1$

$$Leb(S^r_{n,u}[a,b]) = n! \sum_{p=\max(0,n-m_2)}^{\min(n,m_1)} I(p, 1, \ldots, n).$$

Similar steps as in the proof of Theorem 3.1 are used here while computing $I(p, 1, \ldots, n)$. Let p be fixed. Then,

$$\begin{aligned}
&I(p, 1, ..., n) = \\
&= \int_a^{u-pr} \int_{x_1+r}^{u-(p-1)r} \cdots \int_{x_{p-2}+r}^{u-2r} \int_{x_{p-1}+r}^{u-r} \int_{u+r}^{b-((n-p)-1)r} \cdots \int_{x_{n-2}+r}^{b-r} \int_{x_{n-1}+r}^{b} 1 \; \mathrm{d}x_n \ldots \mathrm{d}x_1 \\
&= \int_a^{u-pr} \int_{x_1+r}^{u-(p-1)r} \cdots \int_{x_{p-1}+r}^{u-r} \int_{u+r}^{b-((n-p)-1)r} \cdots \int_{x_{n-2}+r}^{b-r} (b - r - x_{n-1}) \; \mathrm{d}x_{n-1} \ldots \mathrm{d}x_1 \\
&= \int_a^{u-pr} \int_{x_1+r}^{u-(p-1)r} \cdots \int_{u+r}^{b-((n-p)-1)r} \cdots \int_{x_{n-3}+r}^{b-2r} \frac{(b - 2r - x_{n-1})^2}{2} \; \mathrm{d}x_{n-2} \ldots \mathrm{d}x_1 \\
&= \int_a^{u-pr} \int_{x_1+r}^{u-(p-1)r} \cdots \int_{x_{p-2}+r}^{u-2r} \int_{x_{p-1}+r}^{u-r} \frac{(b - u - (n-p)r)^{(n-p)}}{(n-p)!} \; \mathrm{d}x_p \ldots \mathrm{d}x_1 \\
&= \frac{(b - u - (n-p)r)^{(n-p)}}{(n-p)!} \int_a^{u-pr} \int_{x_1+r}^{u-(p-1)r} \cdots \int_{x_{p-2}+r}^{u-2r} \int_{x_{p-1}+r}^{u-r} 1 \; \mathrm{d}x_p \ldots \mathrm{d}x_1
\end{aligned}$$

$$= \frac{(b-u-(n-p)r)^{(n-p)}}{(n-p)!} \int\limits_{a}^{u-pr} \int\limits_{x_1+r}^{u-(p-1)r} \cdots \int\limits_{x_{p-2}+r}^{u-2r} (u-2r-x_{p-1}) \, \mathrm{d}x_{p-1} \ldots \mathrm{d}x_1$$

$$= \frac{(b-u-(n-p)r)^{(n-p)}}{(n-p)!} \int\limits_{a}^{u-pr} \int\limits_{x_1+r}^{u-(p-1)r} \cdots \int\limits_{x_{p-3}+r}^{u-3r} \frac{(u-3r-x_{p-2})^2}{2!} \, \mathrm{d}x_{p-2} \ldots \mathrm{d}x_1$$

$$= \frac{(b-u-(n-p)r)^{(n-p)}}{(n-p)!} \int\limits_{a}^{u-pr} \frac{(u-pr-x_1)^{(p-1)}}{(p-1)!} \, \mathrm{d}x_1$$

$$= \frac{(b-u-(n-p)r)^{(n-p)}}{(n-p)!} \frac{(u-a-pr)^p}{p!}.$$

We derived the Lebesgue measure of the set $S^r_{n,u}[a,b]$ for $n \geq 1$ as

$$Leb(S^r_{n,u}[a,b]) = n! \sum_{p=\max(0,n-m_2)}^{\min(n,m_1)} \frac{1}{p!(n-p)!}(u-a-pr)^p(b-u-(n-p)r)^{n-p}. \quad (3.2)$$

The right hand side of (3.2) is applicable in case of $n = 0$ as well. By plugging the formula (3.2) into the last equality in (3.1), we prove the assertion. □

Remark. It is also possible to express the correlation function $\rho^{(r)}$ as

$$\rho^{(r)}(u) = \frac{\sum\limits_{n=0}^{m_1+m_2} \sum\limits_{p=\max(0,n-m_2)}^{\min(n,m_1)} \frac{1}{p!(n-p)!} Leb(S^r_p[a,u-r]) Leb(S^r_{n-p}[u+r,b])}{1 + \sum\limits_{n=1}^{m} \frac{1}{n!} Leb(S^r_n[a,b])}.$$

The following picture demonstrates the behaviour of the correlation function $\rho^{(r)}$ in case of $[a,b] = [0,2]$ depending on the point $u \in [0,2]$ for different choices of the parameter $r \in (0,2]$.

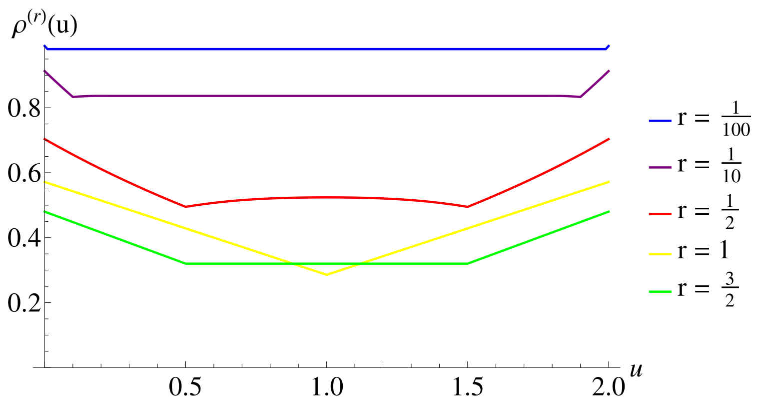

Figure 3.1:

Figure 3.1 : Plot showing the dependence of the correlation function $\rho^{(r)}$ of the point process κ_r on the point $u \in [0,2]$ for different choices of parameter $r \in (0,2]$. We can observe that for parameter r being close to zero, the hard-core process behaves similarly to the Poisson point process with constant intensity equal to

1. On the other hand with increasing parameter r realisations with points being close to each other are eliminated and consequently, the intensity decreases.

When trying to evaluate the correlation function of the hard-core model in $\mathbb{R}^2$ or higher dimensions, we are not able to get explicit formulas. It is generally known that moment formulas are not available for a large class of point processes with known conditional intensity. Therefore, it is useful to derive stochastic approximations of these quantities, cf. Stucki and Schuhmacher, 2014 or Lemma 1.8 in this thesis.

Chapter 4

Stein's method

Stein's method is a general method used to obtain bounds on distance between two probability distributions. Typical aim in case of stochastic geometry is to approximate behaviour of certain functionals of point processes with standard normal distribution or with Poisson distribution. In this work, we will focus on the normal approximation and deriving bounds for Wasserstein distance, even though other choices of probability distances are applicable. For example bound on total variation distance using Stein's method is discussed in Schuhmacher and Stucki, 2014.

In Last and Penrose, 2017, Chapter 21, Stein's method is combined with the covariance identity presented in Section 2.3. This combination yields an upper bound on the Wasserstein distance between the standard normal distribution and distribution of a Poisson functional.

In Torrisi, 2017, bounds on the Wasserstein distance betwen the standard normal distribution and distribution of an innovation are proved using Malliavin-Stein calculus for point processes having Papangelou conditional intensity. Consequently, these bounds are derived for Gibbs point processes in $\mathbb{R}^d, d \in \mathbb{N}$.

The main contribution of this chapter is combining a consequence of the covariance identity and the general bound for innovations from Torrisi, 2017 to generalize bounds for Gibbs particle processes.

Each method will be discussed and examples demonstrating its usage will be incorporated.

4.1 Principle of Stein's method

Results in Sections 4.1 and 4.2 can be seen in Last and Penrose, 2017, Chapter 21.

Notation. Let $\mathbf{Lip}(1)$ denote the space of all Lipschitz functions $h : \mathbb{R} \to \mathbb{R}$ with a Lipschitz constant less then or equal to one.

Moreover, denote by $\mathbf{AC}_{1,2}$ the set of all differentiable functions $g : \mathbb{R} \to \mathbb{R}$ such that g' is absolutely continuous and

$$sup\{|g'(x)| : x \in \mathbb{R}\} \leq \sqrt{2/\pi}, \qquad sup\{|g''(x)| : x \in \mathbb{R}\} \leq 2$$

for almost all $x \in \mathbb{R}$, where g'' is a Radon-Nikodým derivative of g'.

Proposition 4.1 (Stein's equation)**.** *Given fixed function $h \in \boldsymbol{Lip}(1)$, there exists a function $g \in \boldsymbol{AC}_{1,2}$ solving so called Stein's equation*

$$h(x) - \mathsf{E}[h(Z)] = g'(x) - xg(x), \tag{4.1}$$

where Z denotes a standard normal distributed random variable.

Definition 4.1. (Wasserstein distance) Let X, Y be two real random variables. The Wasserstein distance between X and Y is defined by

$$d_W(X,Y) = \sup_{h \in \mathbf{Lip}(1)} |\,\mathsf{E}\,[h(X)] - \mathsf{E}\,[h(Y)]|.$$

Remark. It can be shown, that if a sequence of random variables $X_n,\ n \in \mathbb{N}$ converges to a random variable X with respect to d_W, then X_n converges to X in distribution.

Theorem 4.2. *(Stein's method) Let $F : \boldsymbol{N} \to \mathbb{R}$ be a measurable function such that $\mathbb{E}[|F|] < \infty$. Then*

$$d_W(F,Z) \le \sup_{g \in \boldsymbol{AC}_{1,2}} |\,\mathsf{E}\,[g'(F) - Fg(F)]|.$$

Proof. Theorem 4.2 is a simple consequence of Proposition 4.1. For every function $h \in \mathbf{Lip}(1)$, we can find a function $g \in \mathbf{AC}_{1,2}$ solving the Stein's equation (4.1). From that it follows

$$|\,\mathsf{E}\,[h(F)] - \mathsf{E}\,[h(Z)]| = |\,\mathsf{E}\,[g'(F) - Fg(F)]|.$$

Plugging this expression into the deffinition of the Wasserstein distance and taking the supremum yields the assertion. □

4.2 Bounds for Poisson functionals

In Chapter 2, we have proved the covariance identity for two Poisson functionals using the Fock space representation theorem (Theorem 2.3). By combining the covariance identity with Stein's method, we can obtain an upper bound on the Wasserstein distance between the standard normal distribution and distribution of a Poisson functional.

Suppose that η is a Poisson point process with intensity measure λ and distribution P_η.

Theorem 4.3. *Let $F \in L^2_\eta$ satisfies $DF \in L^2(\Omega \times \mathbb{X}, \mathsf{P} \otimes \lambda)$ and $\mathsf{E}[F] = 0$. Then*

$$\begin{aligned} d_W(F,Z) \le & \mathbb{E}\left[\left|1 - \int_{\mathbb{X}} \int_0^1 (P_t D_x F)(D_x F) \mathrm{d}t \lambda(\mathrm{d}x)\right|\right] \\ & + \mathbb{E}\left[\int_{\mathbb{X}} \int_0^1 |P_t D_x F|(D_x F)^2 \mathrm{d}t \lambda(\mathrm{d}x)\right]. \end{aligned}$$

Since the bound in Theorem 4.3 involves the operator P_t, it is often difficult to apply. However, by using again the covariance identity (Theorem 2.7) and the contractivity property (Lemma 2.5), it is possible to determine bound depending only on the first and the second order difference operators, which could be evaluated for some simple choices of the Poisson functionals (see Example 4.1).

Theorem 4.4 (Second order Poincaré inequality)**.** *Suppose that $F \in L^2_\eta$ satisfies $DF \in L^2(\Omega \times \mathbb{X}, \mathsf{P} \otimes \lambda)$, $\mathsf{E}[F] = 0$ and $Var[F] = 1$. Denote*

$$\alpha_{F,1} := 2 \left[\int_{\mathbb{X}^3} (\mathsf{E}[(D_{x_1}F)^2(D_{x_2}F)^2])^{1/2} (\mathsf{E}[\Delta_{x_1,x_2,x_3}(F)])^{1/2} \lambda^3(\mathrm{d}(x_1,x_2,x_3)) \right]^{1/2},$$

$$\alpha_{F,2} := \left[\int_{\mathbb{X}^3} \mathsf{E}[\Delta_{x_1,x_2,x_3}(F)] \lambda^3(\mathrm{d}(x_1,x_2,x_3)) \right]^{1/2},$$

$$\alpha_{F,3} := \int_{\mathbb{X}} \mathsf{E}[|D_x F|^3] \lambda(\mathrm{d}x),$$

where $\Delta_{x_1,x_2,x_3}(F) = (D^2_{x_1,x_3}F)^2(D^2_{x_2,x_3}F)^2$. Then the upper bound on the Wasserstein distance can be expressed in terms of the constants $\alpha_{F,1}, \alpha_{F,2}, \alpha_{F,3}$ as

$$d_W(F,Z) \le \alpha_{F,1} + \alpha_{F,2} + \alpha_{F,3}.$$

Example 4.1 (CLT for non-homogeneous Poisson processes)**.** Let η be a Poisson point process on $\mathbb{R}_+$, whose intensity measure λ satisfies $0 < \lambda([0,t]) < \infty$ for all sufficiently large t and $\lambda[0,\infty) = \infty$.

We will define Poisson functionals $F_t,\ t > 0$ as the normalized difference between the actual number of points of point process η in the interval $[0,t]$ and the expected number of points in this interval, i.e.

$$F_t(\eta) = \frac{\eta([0,t]) - \lambda([0,t])}{\sqrt{\lambda([0,t])}}.$$

We want to use Theorem 4.4 to induce the central limit theorem. First, we have to verify its assumptions. We can observe that all moments of F_t exists, since Poisson distribution has all moments finite. Furthermore, since

$$\mathsf{E}\,[\eta([0,t])] = Var[\eta([0,t])] = \lambda([0,t]),$$

the assumptions on the variance and the expectation are evidently satisfied.

Take an arbitrary point $x \in \mathbb{R}_+$. Then for the difference operator of the functional F_t, it holds from the definition that

$$D_x F_t(\eta) = \frac{(\eta + \delta_x)([0,t]) - \eta([0,t])}{\sqrt{\lambda([0,t])}} = \frac{\mathbf{1}[x \in [0,t]]}{\sqrt{\lambda([0,t])}}.$$

The difference operator of F_t is no longer random, which implies that the assumption of square integrability of DF holds and moreover, the difference operators of the higher orders are zero.

It remains to plug the difference operator of F_t into the formulae for the constants $\alpha_{F,1}, \alpha_{F,2}, \alpha_{F,3}$ in Theorem 4.4, i.e.

$$\begin{aligned}
&\alpha_{F,1} = 0,\\
&\alpha_{F,2} = 0,\\
&\alpha_{F,3} = \int \mathbb{E}[|D_x F|^3]\lambda(dx) = \frac{1}{(\lambda([0,t]))^{\frac{3}{2}}} \int \mathbf{1}[x \in [0,t]]\lambda(dx) = \frac{1}{\sqrt{\lambda([0,t])}}.
\end{aligned}$$

Thus,

$$d_W(F(t), Z) \leq \frac{1}{\sqrt{\lambda([0,t])}}. \tag{4.2}$$

The right hand side of (4.2) tends to infinity as t goes to infinity, hence we have derived the central limit theorem.

4.3 Bounds for innovations of point processes with Papangelou conditional intensity

We are interested in bounds on the Wasserstein distance between the standard normal distribution and distribution of so called innovation introduced in the first chapter by Definition 1.14.

Suppose that these innovation are induced by a point process μ having Papangelou conditional intensity λ^* and intensity measure λ. Using theory of Malliavin-Stein calculus, the following very general bound is proved in Torrisi, 2017.

Theorem 4.5 (Torrisi, 2017, page 6). *Let $\varphi : \mathbb{X} \times \boldsymbol{N} \to \mathbb{R}$ be a measurable function satisfying*

$$\mathbb{E}\left[\int_{\mathbb{X}} |\varphi(x,\mu)|\lambda^*(x,\mu)\lambda(\mathrm{d}x)\right] < \infty \text{ and } \mathbb{E}\left[\int_{\mathbb{X}} |\varphi(x,\mu)|^2\lambda^*(x,\mu)\lambda(\mathrm{d}x)\right] < \infty.$$

Then,

$$\begin{aligned}
d_W(I_\mu(\varphi), Z) \leq \sqrt{\frac{2}{\pi}}\mathbb{E}&\left[\left|1 - \int_{\mathbb{X}} \varphi_x D_x I_\mu(\varphi)\lambda^*(x,\mu)\lambda(\mathrm{d}x)\right|\right]\\
&+\mathbb{E}\left[\int_{\mathbb{X}} |\varphi(x,\mu)||D_x I_\mu(\varphi)|^2\lambda^*(x,\mu)\lambda(\mathrm{d}x)\right].
\end{aligned}$$

Remark. Big advantage of Theorem 4.5 is that it allows the innovation φ to depend also on a given realisation of point process. That gives us opportunity to study important functionals as the volume of intersections between particles in this realisation, etc.. Unfortunately, in most cases this bound is not applicable, since it is impossible to evaluate for most point processes. There is one modification in Torrisi, 2017 enabling us to express the bound above in terms of difference operators of φ instead of $I_\mu(\varphi)$. This modification also solves this general case but with the same difficulty to be applied.

The following result simplifies considerably the bound in Theorem 4.5, but with the price that the function φ no longer depends on a given realisation, hence it is only function on $\mathbb{X}$.

Notation. For brevity, define functions $\alpha_2 : \mathbb{X}^2 \to \mathbb{R},\ \alpha_3 : \mathbb{X}^3 \to \mathbb{R}$ by

$$\alpha_2(x, y, \mu) := \mathbb{E}[\lambda^*(x, \mu)\lambda^*(y, \mu)],$$
$$\alpha_3(x, y, z, \mu) := \mathbb{E}[\lambda^*(x, \mu)\lambda^*(y, \mu)\lambda^*(z, \mu)],$$

for $x, y, z \in \mathbb{X}$ and the point process μ on $\mathbb{X}$.

Theorem 4.6 (Torrisi, 2017, page 10)**.** *Let $\varphi : \mathbb{X} \to \mathbb{R}$ be a measurable function such that*

$$\int_{\mathbb{X}} |\varphi(x)|\, E\left[\lambda^*(x, \mu)\right] \lambda(\mathrm{d}x) < \infty \text{ and } \int_{\mathbb{X}} |\varphi(x)|^2\, E\left[\lambda^*(x, \mu)\right] \lambda(\mathrm{d}x) < \infty. \tag{4.3}$$

Then,

$$\begin{aligned}
&d_W(I_\mu(\varphi), Z) \leq \\
&\sqrt{\frac{2}{\pi}}\sqrt{1 - 2\int_{\mathbb{X}} |\varphi(x)|^2\, E[\lambda^*(x, \mu)]\lambda(\mathrm{d}x) + \int_{\mathbb{X}^2} |\varphi(x)\varphi(y)|^2 \alpha_2(x, y, \mu)\lambda(\mathrm{d}x)\lambda(\mathrm{d}y)} \\
&+ \int_{\mathbb{X}} |\varphi(x)|^3\, E[\lambda^*(x, \mu)]\lambda(\mathrm{d}x) \\
&+ \sqrt{\frac{2}{\pi}} \int_{\mathbb{X}^2} |\varphi(x)\varphi(y)|\, E[|D_x\lambda^*(y, \mu)|\lambda^*(x, \mu)]\lambda(\mathrm{d}x)\lambda(\mathrm{d}y) \\
&+ 2\int_{\mathbb{X}^2} |\varphi(x)|^2\varphi(y)|\, E[|D_x\lambda^*(y, \mu)|\lambda^*(x, \mu)]\lambda(\mathrm{d}x)\lambda(\mathrm{d}y) \\
&+ \int_{\mathbb{X}^3} |\varphi(x)\varphi(y)\varphi(z)|\, E[|D_x\lambda^*(y, \mu)D_x\lambda^*(z, \mu)|\lambda^*(x, \mu)]\lambda(\mathrm{d}x)\lambda(\mathrm{d}y)\lambda(\mathrm{d}z).
\end{aligned}$$

Moreover, if we add the assumption of repulsivity (see Definition 1.15), we can express bound of Theorem 4.6 using the correlation function up to the third order.

Corollary 1 (Torrisi, 2017, page 12)**.** Let $\varphi : \mathbb{X} \to \mathbb{R}$ be a measurable function such that

$$\int_{\mathbb{X}} |\varphi(x)|\, \mathsf{E}\left[\lambda^*(x, \mu)\right] \lambda(\mathrm{d}x) < \infty \text{ and } \int_{\mathbb{X}} |\varphi(x)|^2\, \mathsf{E}\left[\lambda^*(x, \mu)\right] \lambda(\mathrm{d}x) < \infty.$$

Then

$$
\begin{aligned}
&d_W(I_\mu(\varphi), Z) \leq \\
&\sqrt{\frac{2}{\pi}}\sqrt{1 - 2\int\limits_{\mathbb{X}} |\varphi(x)|^2 \rho_1(x)\lambda(\mathrm{d}x) + \int\limits_{\mathbb{X}^2} |\varphi(x)\varphi(y)|^2 \alpha_2(x, y, \mu)\lambda(\mathrm{d}x)\lambda(\mathrm{d}y)} \\
&+ \int\limits_{\mathbb{X}} |\varphi(x)|^3 \rho_1(x)\lambda(\mathrm{d}x) + \sqrt{\frac{2}{\pi}}\int\limits_{\mathbb{X}^2} |\varphi(x)\varphi(y)|(\alpha_2(x, y, \mu) - \rho_2(x, y))\lambda(\mathrm{d}x)\lambda(\mathrm{d}y) \\
&+ 2\int\limits_{\mathbb{X}^2} |\varphi(x)|^2 \varphi(y)|(\alpha_2(x, y, \mu) - \rho_2(x, y))\lambda(\mathrm{d}x)\lambda(\mathrm{d}y) \\
&+ \int\limits_{\mathbb{X}^3} |\varphi(x)\varphi(y)\varphi(z)|(\alpha_3(x, y, z, \mu) - \rho_3(x, y, z))\lambda(\mathrm{d}x)\lambda(\mathrm{d}y)\lambda(\mathrm{d}z).
\end{aligned}
$$

Remark. For some cases, the bound in Corollary 1 can be actually used to induce a central limit theorem if we know the analytical form of the correlation functions up to the third order. We saw in Chapter 3 that the first order correlation function can be computed analytically for the hard-core on the real line.

4.4 Bounds for innovations of Gibbs point processes on $\mathbb{R}^d$

In Torrisi, 2017, bounds on the Wasserstein distance between the distribution of an innovation of a Gibbs point process with pair potential and the standard normal distribution are derived. We will recall this result and show its application on the hard-core process, which can be defined also using its Papangelou conditional intensity. Generalisation of this result for Gibbs particle processes will follow in Section 4.5.

For purposes of this section, assume the space $\mathbb{R}^d$, $d \in \mathbb{N}$, equipped with the Lebesgue measure *Leb* of the corresponding dimension.

Another simplification when deriving bounds on the Wasserstein distance arises if we know the exact form of the conditional intensity. This is the case of the Gibbs point processes with pair potential given by Definition 1.20.

Theorem 4.7 (Torrisi, 2017, page 20)**.** *Let μ be a stationary Gibbs point process with activity $\tau > 0$ and pair potential $\phi : \mathbb{R}^d \to [0, +\infty]$, and suppose*

$$\varphi \in L^1(\mathbb{R}^d, Leb) \cap L^2(\mathbb{R}^d, Leb).$$

If, moreover, μ has finite range, then for any $p,\ q,\ p',\ q' > 1$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{p'} + \frac{1}{q'} = 1$,

$$d_W(I_\mu(\varphi), Z) \leq \sqrt{\frac{2}{\pi}}\sqrt{1 - 2c_1||\varphi||^2_{L^2(\mathbb{R}^d, Leb)} + \tau c_2||\varphi||^4_{L^2(\mathbb{R}^d, Leb)}} + c_2 A,$$

where

$$
\begin{aligned}
A :=&||\varphi||^3_{L^3(\mathbb{R}^d,Leb)} + \sqrt{\frac{2}{\pi}}\tau||\varphi||^2_{L^2(\mathbb{R}^d,Leb)}||1-e^{-\phi}||_{L^1(\mathbb{R}^d,Leb)} \\
&+ 2\tau||\varphi||^2_{L^q(\mathbb{R}^d,Leb)}||1-e^{-\phi}||_{L^1(\mathbb{R}^d,Leb)} \\
&+ \tau^2||\varphi||_{L^{pp'}(\mathbb{R}^d,Leb)}||\varphi||_{L^{pq'}(\mathbb{R}^d,Leb)}||\varphi||_{L^q(\mathbb{R}^d,Leb)}||1-e^{-\phi}||^2_{L^1(\mathbb{R}^d,Leb)}
\end{aligned}
$$

and

$$
c_1 := \frac{\tau}{1+\tau||1-e^{-\phi}||_{L^1(\mathbb{R}^d,Leb)}}, \qquad c_2 := \frac{\tau}{2-\exp\{-\tau||1-e^{-\phi}||_{L^1(\mathbb{R}^d,Leb)}\}}.
$$

Example 4.2 (Central limit theorem for an innovation of a hard-core process in $\mathbb{R}^d$)**.** Application of Theorem 4.7 can be illustrated on the hard-core process introduced in Example 1.3. We can induce a central limit theorem for the normalized number of points of the hard-core process.

Theorem 4.8. *Consider for each $n \in \mathbb{N}$ a stationary hard-core point process $\mu^{(n)}$ in $\mathbb{R}^d$ with activity $\tau_n > 0$ such that $\tau_n \to \tau$ as $n \to \infty$, and with pair potential*

$$
\phi_n(y) = \begin{cases} 0, & \text{if } ||x|| > r_n, \\ +\infty, & \text{if } ||x|| \le r_n, \end{cases} \tag{4.4}
$$

where $r_n \geq 0$, $a_n \to 0$ as $n \to \infty$. Let K_n, $n \in \mathbb{N}$, be bounded borel sets in $\mathbb{R}^d$ such that $Leb(K_n) \to \infty$ as $n \to \infty$. Define functions

$$
\varphi_n(x) = \frac{1}{\sqrt{\tau_n Leb(K_n)}} \cdot \mathbf{1}_{K_n}(x), \quad n \in \mathbb{N}, x \in \mathbb{R}^d.
$$

Then,

$$
d_W(I_{\mu^{(n)}}(\varphi_n), Z) \to 0
$$

as $n \to \infty$.

Proof. According to Lemma 1.6, the hard-core point processes $\mu^{(n)}$ have finite ranges. Also, for every $n \in \mathbb{N}$

$$
\int_{\mathbb{R}^d} |\varphi_n(x)| \mathrm{d}x = \int_{\mathbb{R}^d} \left| \frac{1}{\sqrt{\tau_n Leb(K_n)}} \cdot \mathbf{1}_{K_n}(x) \right| \mathrm{d}x = \sqrt{\frac{Leb(K_n)}{\tau_n}} < \infty
$$

and

$$
\int_{\mathbb{R}^d} |\varphi_n(x)|^2 \mathrm{d}x = \int_{\mathbb{R}^d} \left| \frac{1}{\sqrt{\tau_n Leb(K_n)}} \cdot \mathbf{1}_{K_n}(x) \right|^2 \mathrm{d}x = \frac{1}{\tau_n} < \infty.
$$

Hence, the assumptions of Theorem 4.7 are satisfied and so we can compute bounds on the Wasserstein distance between the standard normal distribution Z and the innovation $I_{\mu^{(n)}}(\varphi_n)$ for each $n \in \mathbb{N}$.

First, we need to compute the L^1 norm of the function $1 - e^{-\phi_n}$.

$$
||1-e^{-\phi_n}||_{L^1(\mathbb{R},Leb)} = \int_{\mathbb{R}^d} |1-e^{-\phi_n(x)}| \mathrm{d}x = \int_{b(0,r_n)} 1 \mathrm{d}x = \pi r_n^2.
$$

Set $p = q = p' = q' = 2$ and compute for given $n \in \mathbb{N}$ the constants $A^{(n)}, c_1^{(n)}$ and $c_2^{(n)}$ from Theorem 4.7.

$$c_1^{(n)} = \frac{\tau_n}{1 + \tau_n ||1 - e^{-\phi_n}||_{L^1(\mathbb{R}^d, Leb)}} = \frac{\tau_n}{1 + \tau_n \pi r_n^2},$$

$$c_2^{(n)} = \frac{\tau_n}{2 - \exp\{-\tau_n ||1 - e^{-\phi_n}||_{L^1(\mathbb{R}^d, Leb)}\}} = \frac{\tau_n}{2 - \exp\{-\tau_n \pi r_n^2\}}$$

and

$$\begin{aligned} A^{(n)} &= ||\varphi_n||^3_{L^3(\mathbb{R}^d, Leb)} + \sqrt{\frac{2}{\pi}} \tau_n ||\varphi_n||^2_{L^2(\mathbb{R}^d, Leb)} ||1 - e^{-\phi_n}||_{L^1(\mathbb{R}^d, Leb)} \\ &+ 2\tau_n ||\varphi_n||^2_{L^2(\mathbb{R}^d, Leb)} ||1 - e^{-\phi_n}||_{L^1(\mathbb{R}^d, Leb)} \\ &+ \tau_n^2 ||\varphi_n||^2_{L^4(\mathbb{R}^d, Leb)} ||\varphi_n||_{L^2(\mathbb{R}^d, Leb)} ||1 - e^{-\phi_n}||^2_{L^1(\mathbb{R}^d, Leb)} \\ &= \frac{1}{\tau_n^{3/2} \sqrt{Leb(K_n)}} + \sqrt{\frac{2}{\pi}} \pi r_n^2 + 2\pi r_n^2 + \frac{\sqrt{\tau_n}}{\sqrt{Leb(K_n)}} (\pi r_n^2)^2. \end{aligned}$$

We can see that

$$c_1^{(n)} \to \tau, \qquad c_2^{(n)} \to \tau, \qquad A^{(n)} \to 0$$

as $n \to \infty$. All together, using bound from Theorem 4.7, we obtain

$$\begin{aligned} d_W(I_{\mu^{(n)}}(\varphi_n), Z) &\leq \sqrt{\frac{2}{\pi}} \sqrt{1 - 2c_1^{(n)} ||\varphi_n||^2_{L^2(\mathbb{R}^d, Leb)} + \tau_n c_2^{(n)} ||\varphi_n||^4_{L^2(\mathbb{R}^d, Leb)}} + c_2^{(n)} A^{(n)} \\ &= \sqrt{\frac{2}{\pi}} \sqrt{1 - 2c_1^{(n)} \frac{1}{\tau_n} + c_2^{(n)} \frac{1}{\tau_n}} + c_2^{(n)} A^{(n)}, \end{aligned}$$

which tends to 0 as n approaches $+\infty$. □

4.5 Bounds for innovations of Gibbs particle processes

Let $\mathbb{X} = \mathcal{C}^{(d)},\ d \in \mathbb{N}$, be the space of all compact subsets of $\mathbb{R}^d$ with the Hausdorff metric and with a σ-finite reference measure σ. We know that the space $\mathcal{C}^{(d)}$ is a Polish, locally compact space, and hence we can use theorems from Section 4.3.

Consider a stationary Gibbs particle process with activity $\tau > 0$ and particle distribution $\mathbb{Q}$. Recall, that the conditional intensity of μ takes form

$$\lambda^*(K, \mu) = \tau \exp\left\{-\int g(K \cap L)\mu(\mathrm{d}L)\right\}, \quad K \in \mathcal{C}^{(d)}, \tag{4.5}$$

where g is the pair potential. Since $g \geq 0$ we have that λ^* is repulsive.

By computing the difference operator of λ^* and plugging the result into the bound presented in Theorem 4.6, we can obtain the following result.

Theorem 4.9. *Let μ be a stationary Gibbs particle process with activity $\tau > 0$ and particle distribution $\mathbb{Q}$ as in Definition 1.20. In addition, we suppose that $\varphi : \mathcal{C}^{(d)} \to \mathbb{R}$ satisfies conditions of Theorem 4.6. Then*

$$
\begin{aligned}
& d_W(I_\mu(\varphi), Z) \leq \sqrt{\frac{2}{\pi}} \cdot \\
& \cdot \sqrt{1 - 2 \int\limits_{\mathcal{C}^{(d)}} |\varphi(K)|^2 \mathbb{E}[\lambda^*(K,\mu)] \lambda(\mathrm{d}K) + \int\limits_{(\mathcal{C}^{(d)})^2} |\varphi(K)\varphi(L)|^2 \alpha_2(K,L,\mu) \lambda(\mathrm{d}K)\lambda(\mathrm{d}L)} \\
& + \int\limits_{(\mathcal{C}^{(d)})^2} |\varphi(K)|^3 \mathbb{E}[\lambda^*(K,\mu)] \lambda(\mathrm{d}K) \\
& + \sqrt{\frac{2}{\pi}} \int\limits_{(\mathcal{C}^{(d)})^2} |\varphi(K)\varphi(L)| |1 - e^{-g(K \cap L)}| \alpha_2(K,L,\mu) \lambda(\mathrm{d}K)\lambda(\mathrm{d}L) \\
& + 2 \int\limits_{(\mathcal{C}^{(d)})^2} |\varphi(K)|^2 |\varphi(L)| |1 - e^{-g(K \cap L)}| \alpha_2(K,L,\mu) \lambda(\mathrm{d}K)\lambda(\mathrm{d}L) \\
& + \int\limits_{(\mathcal{C}^{(d)})^3} |\varphi(K)\varphi(L)\varphi(M)| |1 - e^{-g(L \cap K)}| |1 - e^{-g(L \cap M)}| \cdot \\
& \qquad \cdot \alpha_3(K,L,M,\mu) \lambda(\mathrm{d}K)\lambda(\mathrm{d}L)\lambda(\mathrm{d}M).
\end{aligned}
$$

Proof. Idea of the proof is similar to the proof of Theorem 5.1 in Torrisi, 2017 for Gibbs point processes with pair potential on $\mathbb{R}^d$.

Take $K \in \mathcal{C}^{(d)}$. We will use formula (4.5) to compute the difference operator at point K of $\lambda^*(L, \mathbf{x}), L \in \mathcal{C}^{(d)}$.

$$
\begin{aligned}
& D_K \lambda^*(L, \mathbf{x}) \\
& = \tau \exp\left\{ - \int\limits_{\mathcal{C}^{(d)}} g(L \cap M)(\mathbf{x} + \delta_K)(\mathrm{d}M) \right\} - \tau \exp\left\{ - \int\limits_{\mathcal{C}^{(d)}} g(L \cap M)\mathbf{x}(\mathrm{d}M) \right\} \\
& = \tau \exp\left\{ - \int\limits_{\mathcal{C}^{(d)}} g(L \cap M)\mathbf{x}(\mathrm{d}M) \right\} \left(e^{-g(L \cap K)} - 1 \right) \\
& = \lambda^*(L, \mathbf{x}) \left(e^{-g(L \cap K)} - 1 \right).
\end{aligned}
$$

By plugging this result into the bound of Theorem 4.6, we obtain the assertion immediately. □

Corollary 2. Let μ be a stationary Gibbs particle satisfying conditions of Lemma 1.8. Then,

$$
\tau e^{-\tau b} \leq \mathbb{E}\lambda^*(K, \mu) \leq \tau \tag{4.6}
$$

for λ-a.a. $K \in \mathcal{C}^{(d)}$.

Proof. For the lower bound, we set $p = 1$ in Lemma 1.8 and recall the definition of the first order correlation function

$$
\rho(K) = \mathbb{E}\lambda^*(K, \mu), \quad K \in \mathcal{C}^{(d)}.
$$

Further, we use that $e^{-\tau b} \leq e^{-\tau V_1(K)}$ for λ-a.a. $K \in \mathcal{C}^{(d)}$. The upper bound is a direct consequence of the definition of λ^* (formula 4.5). □

Corollary 2 gives us uniform bounds on $\mathbb{E}\lambda^*(K,\mu)$. We will use them to prove the main result of this chapter.

Theorem 4.10. *Let μ be a stationary Gibbs particle process with activity $\tau > 0$ and particle distribution $\mathbb{Q}$. Suppose, there exists $b \in [0,\infty)$ such that $V_1(K) \leq b$ for λ-a.a. $K \in \mathcal{C}^{(d)}$. Moreover, assume that $\varphi : \mathcal{C}^{(d)} \to \mathbb{R}$ satisfies*

$$\varphi \in L^1(\mathcal{C}^{(d)},\lambda) \cap L^2(\mathcal{C}^{(d)},\lambda).$$

Then for any $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$d_W(I_\mu(\varphi), Z) \leq \sqrt{\frac{2}{\pi}}\sqrt{1 - 2\tau e^{-\tau b}||\varphi||^2_{L^2(\mathcal{C}^{(d)},\lambda)} + \tau^2||\varphi||^4_{L^2(\mathcal{C}^{(d)},\lambda)}} + \tau A,$$

where

$$\begin{aligned} A :=& ||\varphi||^3_{L^3(\mathcal{C}^{(d)},\lambda)} + \sqrt{\frac{2}{\pi}}\tau||\varphi||^2_{L^p(\mathcal{C}^{(d)},\lambda)} D_q + 2\tau||\varphi||^2_{L^{2p}(\mathcal{C}^{(d)},\lambda)}||\varphi||_{L^p(\mathcal{C}^{(d)},\lambda)} D_q \\ &+ \tau^2||\varphi||^3_{L^p(\mathcal{C}^{(d)},\lambda)} D'_q \end{aligned}$$

and

$$D_q := \left(\int_{(\mathcal{C}^{(d)})^2} |1 - e^{-g(K\cap L)}|^q \lambda(\mathrm{d}K)\lambda(\mathrm{d}L) \right)^{1/q},$$

$$D'_q := \left(\int_{(\mathcal{C}^{(d)})^3} |1 - e^{-g(K\cap L)}|^q |1 - e^{-g(K\cap M)}|^q \lambda(\mathrm{d}K)\lambda(\mathrm{d}L)\lambda(\mathrm{d}M) \right)^{1/q}.$$

Proof. We would like to estimate terms of the bound in Theorem 4.9 individually. First of all, we need to verify the assumptions. We use the trivial estimate following straightforwardly from the definition of the conditional intensity λ^* of μ, i.e.

$$\lambda^*(K,\mathbf{x}) = \tau \exp\left\{ -\int_{\mathcal{C}^{(d)}} g(K \cap L)\mathbf{x}(\mathrm{d}L) \right\} \leq \tau, \tag{4.7}$$

for any $K \in \mathcal{C}^{(d)}$ and $\mathbf{x} \in \mathbf{N}^d$.

Therefore, from the integrability assumptions

$$\int_{\mathcal{C}^{(d)}} |\varphi(K)|\mathbb{E}[\lambda^*(K,\mu)]\lambda(\mathrm{d}K) < \tau||\varphi||_{L^1(\mathcal{C}^{(d)},\lambda)} < \infty$$

and

$$\int_{\mathcal{C}^{(d)}} |\varphi(K)|^2\mathbb{E}\left[\lambda^*(K,\mu)\right]\lambda(\mathrm{d}K) < \tau||\varphi||_{L^2(\mathcal{C}^{(d)},\lambda)} < \infty$$

and hence, the assumptions are verified.

Using the notation of Theorem 4.9, we can obtain the following bounds on α_2 and α_3 based on the corollary of Lemma 1.8.

$$\begin{aligned} \alpha_2(K,L,\mu) &\leq \tau^2, \\ \alpha_3(K,L,M,\mu) &\leq \tau^3, \end{aligned} \tag{4.8}$$

whenever $K, L, M \in \mathcal{C}^{(d)}$.

Finally, we can estimate the terms in Theorem 4.9 individually. Take $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and suppose, that

$$\varphi \in L^3(\mathcal{C}^{(d)}, \lambda) \cap L^p(\mathcal{C}^{(d)}, \lambda) \cap L^{2p}(\mathcal{C}^{(d)}, \lambda).$$

Otherwise, there is nothing to prove.

In the first term, we will use estimates (4.6) and (4.8) to obtain the bound

$$\begin{aligned} &\sqrt{1 - 2\int\limits_{\mathcal{C}^{(d)}} |\varphi(K)|^2 \mathbb{E}[\lambda^*(K,\mu)]\lambda(\mathrm{d}K) + \int\limits_{(\mathcal{C}^{(d)})^2} |\varphi(K)\varphi(L)|^2 \alpha_2(K,L,\mu)\lambda^2(\mathrm{d}K)\lambda(\mathrm{d}L)} \\ &\leq \sqrt{1 - 2\tau e^{-\tau b}\int\limits_{\mathcal{C}^{(d)}} |\varphi(K)|^2 \lambda(\mathrm{d}K) + \tau^2 \int\limits_{(\mathcal{C}^{(d)})^2} |\varphi(K)\varphi(L)|^2 \lambda(\mathrm{d}K)\lambda(\mathrm{d}L)} \\ &\leq \sqrt{1 - 2\tau e^{-\tau b}||\varphi||^2_{L^2(\mathcal{C}^{(d)},\lambda)} + \tau^2 ||\varphi||^4_{L^2(\mathcal{C}^{(d)},\lambda)}}. \end{aligned}$$

The second term can be estimated analogically.

$$\int\limits_{\mathcal{C}^{(d)}} |\varphi(K)|^3 \mathbb{E}[\lambda^*(K,\mu)]\lambda(\mathrm{d}K) \leq \tau \int\limits_{\mathcal{C}^{(d)}} |\varphi(K)|^3 \lambda(\mathrm{d}K) \leq \tau ||\varphi||^3_{L^3(\mathcal{C}^{(d)},\lambda)}.$$

In the following two terms, we will use additionally Hölder's inequality on the product space $((\mathcal{C}^{(d)})^2, \lambda^{\otimes 2})$ for functions

$$f_1(K,L) := |\varphi(K)\varphi(L)| \quad \text{and} \quad {f'}_1(K,L) := |1 - e^{-g(K\cap L)}|, \quad (K,L) \in (\mathcal{C}^{(d)})^2$$

in the first case and

$$f_2(K,L) := |\varphi(K)|^2|\varphi(L)| \quad \text{and} \quad {f'}_2(K,L) := |1 - e^{-g(K\cap L)}|, \quad (K,L) \in (\mathcal{C}^{(d)})^2$$

in the second one. Term $\alpha_2(x,y)$ will be estimated similarly as in the previous case. Thus,

$$\begin{aligned} &\sqrt{\frac{2}{\pi}} \int\limits_{(\mathcal{C}^{(d)})^2} |\varphi(K)\varphi(L)||1 - e^{-g(K\cap L)}|\alpha_2(K,L,\mu)\lambda(\mathrm{d}K)\lambda(\mathrm{d}L) \\ &\leq \tau^2 \sqrt{\frac{2}{\pi}} \left(\int\limits_{(\mathcal{C}^{(d)})^2} |\varphi(K)\varphi(L)|^p \lambda(\mathrm{d}K)\lambda(\mathrm{d}L) \right)^{\frac{1}{p}} \left(\int\limits_{(\mathcal{C}^{(d)})^2} |1 - e^{-g(K\cap L)}|^q \lambda(\mathrm{d}K)\lambda(\mathrm{d}L) \right)^{\frac{1}{q}} \\ &= \tau^2 D_q \sqrt{\frac{2}{\pi}} \left(\int\limits_{\mathcal{C}^{(d)}} |\varphi(K)|^p \lambda(\mathrm{d}K) \right)^{\frac{1}{p}} \left(\int\limits_{\mathcal{C}^{(d)}} |\varphi(L)|^p \lambda(\mathrm{d}L) \right)^{\frac{1}{p}} \\ &= \tau^2 D_q \sqrt{\frac{2}{\pi}} ||\varphi||^2_{L^p(\mathcal{C}^{(d)},\lambda)} \end{aligned}$$

and

$$2 \int\limits_{(\mathcal{C}^{(d)})^2} |\varphi(K)|^2 |\varphi(L)| |1 - e^{-g(K\cap L)}| \alpha_2(K, L, \mu) \lambda(\mathrm{d}K) \lambda(\mathrm{d}L)$$

$$\leq 2\tau^2 \left(\int\limits_{\mathcal{C}^{(d)2}} |\varphi(K)|^{2p} |\varphi(L)|^p \lambda(\mathrm{d}K) \lambda(\mathrm{d}L) \right)^{\frac{1}{p}} \left(\int\limits_{\mathcal{C}^{(d)2}} |1 - e^{-g(K\cap L)}|^q \lambda(\mathrm{d}K) \lambda(\mathrm{d}L) \right)^{\frac{1}{q}}$$

$$\leq 2\tau^2 D_q \left(\int\limits_{\mathcal{C}^{(d)}} |\varphi(K)|^{2p} \lambda(\mathrm{d}K) \right)^{\frac{1}{p}} \left(\int\limits_{\mathcal{C}^{(d)}} |\varphi(L)|^p \lambda(\mathrm{d}L) \right)^{\frac{1}{p}}$$

$$\leq 2\tau^2 D_q ||\varphi||^2_{L^{2p}(\mathcal{C}^{(d)},\lambda)} ||\varphi||_{L^p(\mathcal{C}^{(d)},\lambda)}.$$

In the last term, we will use Hölder's inequality on the product space $((\mathcal{C}^{(d)})^3, \lambda^{\otimes 3})$ for functions

$f(K, L, M) := |\varphi(K)\varphi(L)\varphi(M)|$ and $f'(K, L, M) := |1 - e^{-g(K\cap L)}||1 - e^{-g(K\cap M)}|$, where $(K, L, M) \in (\mathcal{C}^{(d)})^3$.

$$\int\limits_{(\mathcal{C}^{(d)})^3} |\varphi(K)\varphi(L)\varphi(M)||1 - e^{-g(K\cap L)}||1 - e^{-g(K\cap M)}| \alpha_3(K, L, M, \mu) \lambda(\mathrm{d}K) \lambda(\mathrm{d}L) \lambda(\mathrm{d}M)$$

$$\leq \tau^3 \left(\int\limits_{(\mathcal{C}^{(d)})^3} |\varphi(K)\varphi(L)\varphi(M)|^p \lambda(\mathrm{d}K) \lambda(\mathrm{d}L) \lambda(\mathrm{d}M) \right)^{1/p}$$

$$\cdot \left(\int\limits_{(\mathcal{C}^{(d)})^3} |1 - e^{-g(K\cap L)}|^q |1 - e^{-g(K\cap M)}|^q \lambda(\mathrm{d}K) \lambda(\mathrm{d}L) \lambda(\mathrm{d}M) \right)^{1/q}$$

$$= \tau^3 D'_q \left(\int\limits_{\mathcal{C}^{(d)}} |\varphi(K)|^p \lambda(\mathrm{d}K) \right)^{1/p} \left(\int\limits_{\mathcal{C}^{(d)}} |\varphi(L)|^p \lambda(\mathrm{d}L) \right)^{1/p} \left(\int\limits_{\mathcal{C}^{(d)}} |\varphi(M)|^p \lambda(\mathrm{d}M) \right)^{1/p}$$

$$= \tau^3 D'_q ||\varphi||^3_{L^p(\mathcal{C}^{(d)},\lambda)}.$$

Adding these estimates together yields the theorem. □

4.5.1 Central limit theorem for an innovation of a Gibbs planar segment process

As an example of possible application of Theorem 4.10, we will derive a central limit theorem for an innovation of a Gibbs planar segment process defined in Example 1.3. First, it is useful to realize the following property of planar segment processes.

Lemma 4.11. *Let ξ be a stationary Gibbs planar segment process with activity $\tau > 0$, particle distribution $\mathbb{Q}$ concentrated on S and pair potential g defined by formula* (1.6)*. Then*

$$V_1(K) \le (1 - e^{-a})4\pi r^2$$

for λ-a.a. $K \in S$.

Proof. From the definition of V_1 and using Theorem 1.7, we obtain

$$\begin{aligned} V_1(K) &= \int_{\mathcal{C}^{(2)}} \left(1 - e^{-g(K\cap L)}\right) \lambda(\mathrm{d}L) = (1 - e^{-a}) \int_{\mathcal{C}^{(2)}} \mathbf{1}\{K \cap L \neq \emptyset\}\lambda(\mathrm{d}L) \\ &= (1 - e^{-a}) \int_{S_0} \int_{\mathbb{R}^2} \mathbf{1}\{K + (L + x) \neq \emptyset\}\mathrm{d}x\mathbb{Q}(\mathrm{d}L) \\ &= (1 - e^{-a}) \int_{S_0} Leb(K + \check{L})\mathbb{Q}(\mathrm{d}L), \end{aligned}$$

where $K + \check{L}$ is the Minkowski sum of K and $\check{L}$ and S_0 is the system of segments centred in the origin.

We assumed (1.4), i.e.

$$\mathbb{Q}(\{K \in S : B(K) \subset b(0, r)\}) = 1.$$

Thus, we can estimate $Leb(K + \check{L})$ by $Leb(b(0, 2r)) = \pi 4r^2$ and hence the assertion. □

We present a central limit theorem for a functional of a Gibbs planar segment process, where the functional is the normalized number of segments in a window. We take windows from a van Hove sequence (Ruelle, 1970), i.e. monotone increasing sequence of bounded Borel sets converging to $\mathbb{R}^2$.

Theorem 4.12. *Consider for each $n \in \mathbb{N}$ a stationary Gibbs planar segment process $\xi^{(n)}$ with activity $\tau_n > 0$ and uniform directional distribution $\mathbb{Q}$. Assume that $\tau_n \to \tau, \tau > 0$, as $n \to \infty$. Define the pair potential of $\xi^{(n)}$ by*

$$g_n(K) = a_n \mathbf{1}\{K \neq \emptyset\}, \quad K \in \mathcal{C}^{(2)},$$

where $a_n \ge 0, a_n \to 0$, as $n \to \infty$. Let $W_n, n \in \mathbb{N}$ be a van Hove sequence of convex sets in $\mathbb{R}^2$. For each $n \in \mathbb{N}$ define functions

$$\varphi_n(K) = \frac{1}{\sqrt{\tau_n Leb(W_n)}} \cdot \mathbf{1}\{K \cap W_n \neq \emptyset\}, \quad K \in S.$$

Then

$$d_W(I_{\xi^{(n)}}(\varphi_n), Z) \to 0$$

as $n \to \infty$.

Proof. We want to use Theorem 4.10 for $S \subset \mathcal{C}^{(2)}$. The reference measure is $\lambda = \theta/\gamma$.

First, we have to verify the assumptions of Theorem 4.10. By Lemma 4.11, we can set $b_n = (1 - e^{-a_n})\pi 4r^2$. Evidently, $b_n \to 0$ as $n \to \infty$.

Further, for every $n \in \mathbb{N}$,

$$\begin{aligned}
\int\limits_{\mathcal{C}^{(2)}} |\varphi_n(x)| \lambda(\mathrm{d}K) &= \int\limits_{\mathcal{C}^{(2)}} \frac{\mathbf{1}\{K \cap W_n \neq \emptyset\}}{\sqrt{\tau_n Leb(W_n)}} \lambda(\mathrm{d}K) \\
&= \frac{1}{\sqrt{\tau_n Leb(W_n)}} \int\limits_{S_0} \int\limits_{\mathbb{R}^2} \mathbf{1}\{(K+x) \cap W_n \neq \emptyset\} \mathrm{d}x \mathbb{Q}(\mathrm{d}K) \\
&= \frac{1}{\sqrt{\tau_n Leb(W_n)}} \int\limits_{S_0} Leb(\check{K} + W_n) \mathbb{Q}(\mathrm{d}K) < \infty,
\end{aligned}$$

since W_n is bounded and K is the segment of the length r. Integration is over directions only. Similarly,

$$\begin{aligned}
\int\limits_{\mathcal{C}^{(2)}} |\varphi_n(x)|^2 \lambda(\mathrm{d}K) &= \int\limits_{\mathcal{C}^{(2)}} \frac{\mathbf{1}\{K \cap W_n \neq \emptyset\}}{\tau_n Leb(W_n)} \lambda(\mathrm{d}K) \\
&= \frac{1}{\tau_n Leb(W_n)} \int\limits_{S_0} \int\limits_{\mathbb{R}^2} \mathbf{1}\{(K+x) \cap W_n \neq \emptyset\} \mathrm{d}x \mathbb{Q}(\mathrm{d}K) \\
&= \frac{1}{\tau_n Leb(W_n)} \int\limits_{S_0} Leb(\check{K} + W_n) \mathbb{Q}(\mathrm{d}K) < \infty.
\end{aligned}$$

Hence, the assumptions of Theorem 4.10 are satisfied and so we can compute the explicit bounds on the Wasserstein distance between the standard normal distribution Z and the distribution of the innovation $I_{\xi^{(n)}}(\varphi_n)$ for each $n \in \mathbb{N}$.

Take $p = q = 2$ and focus on the constants $D_2^{(n)}, D_2'^{(n)}, n \in \mathbb{N}$, defined in Theorem 4.10. For $D_2^{(n)}$, we have

$$\begin{aligned}
D_2^{(n)} &= \left(\int\limits_{(\mathcal{C}^{(2)})^2} |1 - e^{-g_n(K \cap L)}|^2 \lambda(\mathrm{d}K) \lambda(\mathrm{d}L) \right)^{1/2} \\
&= \left(\int\limits_{S^2} |1 - e^{-a_n}|^2 \mathbf{1}\{K \cap L \neq \emptyset\} \lambda(\mathrm{d}K) \lambda(\mathrm{d}L) \right)^{1/2}.
\end{aligned}$$

Functions $f_n(x,y) := |1 - e^{-a_n}|^2 \mathbf{1}\{K \cap L \neq \emptyset\}, K, L \in S$ converge uniformly to 0 as $n \to \infty$ and therefore,

$$\begin{aligned}
\lim_{n\to\infty} D_2^{(n)} &= \lim_{n\to\infty} \left(\int\limits_{S^2} f_n(K,L) \lambda(\mathrm{d}K) \lambda(\mathrm{d}L) \right)^{1/2} \\
&= \left(\int\limits_{S^2} \lim_{n\to\infty} f_n(K,L) \lambda(\mathrm{d}K) \lambda(\mathrm{d}L) \right)^{1/2} \\
&= 0.
\end{aligned}$$

Further, we use Lemma 4.11 to estimate the constant $D_2'^{(n)}$ for fixed $n \in \mathbb{N}$. We have that

$$
\begin{aligned}
D_2'^{(n)} &= \left(\int\limits_{(\mathcal{C}^{(2)})^3} |1 - e^{-g(K\cap L)}|^2 |1 - e^{-g(K\cap M)}|^2 \lambda(\mathrm{d}K)\lambda(\mathrm{d}L)\lambda(\mathrm{d}M) \right)^{1/2} \\
&= \left(\int\limits_{(\mathcal{C}^{(2)})^2} |1 - e^{-g(K\cap L)}|^2 \left(\int\limits_{\mathcal{C}^{(2)}} |1 - e^{-g(K\cap M)}|^\lambda(\mathrm{d}M) \right) \lambda(\mathrm{d}K)\lambda(\mathrm{d}L) \right)^{1/2} \\
&\leq (1 - e^{-a_n})^2 4\pi r^2 D_2^{(n)}.
\end{aligned}
$$

Hence, also $D_2'^{(n)}$ converges to 0 as $n \to \infty$.

Take some fixed $n \in \mathbb{N}$ and $\alpha > 1$. Using Theorem 1.7 and Steiner theorem (Schneider and Weil, 2008), we obtain

$$
\begin{aligned}
||\varphi_n||_{L^\alpha(\mathcal{C}^{(d)},Leb)} &= \left(\int\limits_{\mathcal{C}^{(2)}} \left| \frac{\mathbf{1}\{K \cap W_n \neq \emptyset\}}{\sqrt{\tau_n Leb(W_n)}} \right|^\alpha \lambda(\mathrm{d}K) \right)^{\frac{1}{\alpha}} \\
&= \frac{1}{\sqrt{\tau_n Leb(W_n)}} \left(\int\limits_{S_0} Leb(\check{K} + W_n) \mathbb{Q}(\mathrm{d}K) \right)^{\frac{1}{\alpha}} \\
&= \frac{1}{\sqrt{\tau_n Leb(W_n)}} \left(Leb(W_n) + \frac{r}{\pi} U(W_n) \right)^{\frac{1}{\alpha}},
\end{aligned}
$$

where $U(W_n)$ denotes the perimeter of the set W_n.

Therefore, the constant $A^{(n)}$ can be evaluated as

$$
\begin{aligned}
A^{(n)} &= ||\varphi_n||^3_{L^3(\mathcal{C}^{(2)},\lambda)} + \sqrt{\frac{2}{\pi}} \tau_n ||\varphi_n||^2_{L^2(\mathcal{C}^{(2)},\lambda)} D_2^{(n)} + 2\tau_n ||\varphi_n||^2_{L^4(\mathcal{C}^{(2)},\lambda)} ||\varphi_n||_{L^2(\mathcal{C}^{(2)},\lambda)} D_2^{(n)} \\
&\quad + \tau_n^2 ||\varphi_n||^3_{L^2(\mathcal{C}^{(2)},\lambda)} D_2'^{(n)} \\
&= \frac{1}{\tau_n^{3/2}} \left(\frac{1}{\sqrt{Leb(W_n)}} + \frac{r}{\pi} \frac{U(W_n)}{Leb(W_n)^{3/2}} \right) + \sqrt{\frac{2}{\pi}} \left(1 + \frac{r}{\pi} \frac{U(W_n)}{Leb(W_n)} \right) D_2^{(n)} \\
&\quad + \frac{2}{\sqrt{\tau_n}} \left(\frac{1}{\sqrt{Leb(W_n)}} + \frac{r}{\pi} \frac{U(W_n)}{Leb(W_n)^{3/2}} \right) D_2^{(n)} + \sqrt{\tau_n} \left(1 + \frac{r}{\pi} \frac{U(W_n)}{Leb(W_n)} \right)^{3/2} D_2'^{(n)}.
\end{aligned}
$$

We can see, that $A^{(n)} \to 0$ as $n \to \infty$.

Finally, by using the bound from Theorem 4.10, we obtain

$$
\begin{aligned}
d_W(I_{\xi^{(n)}}(\varphi_n), Z) &\leq \sqrt{\frac{2}{\pi}} \sqrt{1 - 2\tau_n e^{-\tau_n b_n} ||\varphi_n||^2_{L^2(\mathcal{C}^{(2)},\lambda)} + \tau_n^2 ||\varphi_n||^4_{L^2(\mathcal{C}^{(2)},\lambda)}} + \tau_n A^{(n)} \\
&= \sqrt{\frac{2}{\pi}} \sqrt{1 - 2e^{-\tau_n b_n} \left(1 + \frac{r}{\pi} \frac{U(W_n)}{Leb(W_n)} \right) + \left(1 + \frac{r}{\pi} \frac{U(W_n)}{Leb(W_n)} \right)^2} \\
&\quad + \tau_n A^{(n)},
\end{aligned}
$$

which tends to 0 as n approaches $+\infty$. □

The assumption of $a_n \to 0$ as $n \to \infty$ in Theorem 4.12 is somewhat limiting (analogously to the assumption of $r = 1/n$ in Example 5.9 in Torrisi, 2017, where r was the hard-core distance). The presented methodology does not enable to relax the assumption $a_n \to 0$ and it is an open problem for our further research how to do it.

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List of Abbreviations

$\mathbb{R}$ .... real numbers
$\mathbb{R}_+$ .... non-negative real numbers
$\mathbb{N}$ .... integral numbers
$\mathbb{N}_0$ .... integral numbers and zero
$\mathbb{P}(A)$ .... probability of event A
$\mathbb{E}(X)$ .... expected value of a random variable X
$(\Omega, \mathcal{A}, \mathbb{P})$ .... probability space
$\lambda_1 \otimes \lambda_2$ .... product of measures λ_1 and λ_2
$\mathbb{X} \times \mathbb{Y}$ .... product space generated by spaces $\mathbb{X}$ and $\mathbb{Y}$
$\mathbf{1}\{B\}$ .... indicator function of a set B
δ_y .... Dirac measure concentrated in a point y
$(\mathbb{X}, \sigma)$ .... space $\mathbb{X}$ with reference measure σ
$||f||_{L_p(\mathbb{X},\sigma)} = (\int_{\mathbb{X}} |f|^p \mathrm{d}\sigma)^{\frac{1}{p}}$ .... L_p norm in the space $(\mathbb{X}, \sigma)$
$L_p(\mathbb{X}, \sigma)$ .... $\Big\{f : ||f||_{L_p(\mathbb{X},\sigma)} < \infty, f$ measurable function on $(\mathbb{X}, \sigma)\Big\}$
$\mathcal{C}^d$ .... the space of all compact subsets in $\mathbb{R}^d$
$\mathcal{C}^{(d)}$ .... the space of all nonempty compact subsets in $\mathbb{R}^d$
$\mathcal{C}_0^{(d)}$ .... $B \in \mathcal{C}^{(d)}$ with the centre of the circumscribed ball in the origin
$b(x, r)$ .... ball in $\mathbb{R}^d$ with the radius $r > 0$ and the centre in $x \in \mathbb{R}^d$
$\mathbb{S}^1$ .... unit sphere in $\mathbb{R}^2$
Leb .... Lebesgue measure