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Convex hull properties for parabolic systems of partial differential equations

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Title: Convex hull properties for parabolic systems of partial differential equations

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Abstract: The topic of this thesis is the convex hull property for systems of partial differential equations, which is a natural generalisation of the maximum principle for scalar equations. The main result of this thesis is a theorem asserting the convex hull property for the solutions of a certain class of parabolic systems of nonlinear partial differential equations. It also investigates the coefficients of linear systems. The respective results are sharp which is demonstrated by counterexamples to the convex hull property for solutions of linear elliptic and parabolic systems. The general theme is that the coupling of the system is what breaks the convex hull property, not necessarily the non-linearity.

Keywords: partial differential equations, non-linear systems, maximum principles, p-Laplacian

Contents

Notation

Throughout the thesis, $\Omega \subset \mathbb{R}^n$ will always denote a Lipschitz domain. Vector-valued functions will usually have N components and depend on n variables called x, that is $u\colon \Omega \to \mathbb{R}^N$, in the case of parabolic equations $u\colon [0,T)\times\Omega \to \mathbb{R}^N$ and the first variable is called t. Variables under the integral sign are often omitted, that is we use $\int_\Omega u := \int_\Omega u(x)\,\mathrm{d}x$.

- $x \cdot y = \displaystyle\sum_{i=1}^{n} x_i y_i$ for $x, y \in \mathbb{R}^n$
- $|x| = \sqrt{x \cdot x}$ for $x \in \mathbb{R}^n$
- $A : B = \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{N} A_{ij}B_{ij}$ for matrices $A, B \in \mathbb{R}^{n\times N}$
- $|A| = \sqrt{A : A}$ for a matrix $A \in \mathbb{R}^{n\times N}$
- $a^+ = \max\{a, 0\}$, $a^- = \max\{-a, 0\}$ for $a \in \mathbb{R} \cup \{\infty, -\infty\}$
- $|\Omega|$ the Lebesgue measure of Ω
- $\operatorname{conv} M$ the closed convex hull of the set $M \subset \mathbb{R}^N$
- $(\nabla u)_{ij} = \dfrac{\partial u^j}{\partial x_i}$, $i = 1, \dots, n$, $j = 1, \dots, N$ the gradient of $u\colon \Omega \to \mathbb{R}^N$
  In the case of functions also depending on t, we only take the gradient with respect to x.
- $(\Delta u)_j = \Delta u_j = \displaystyle\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$, $j = 1, \dots, N$ the Laplace operator for $u\colon \Omega \to \mathbb{R}^N$
- $(\operatorname{div} F)_j = \operatorname{div} F_j = \displaystyle\sum_{i=1}^{n} \frac{\partial F_{ij}}{\partial x_i}$, $j = 1, \dots, N$ the divergence of $F\colon \Omega \to \mathbb{R}^{n\times N}$
- $L^p(\Omega, \mathbb{R}^N)$ Lebesgue space of functions with values in $\mathbb{R}^N$ with the norm
$$\|u\|_p = \|u\|_{L^p(\Omega,\mathbb{R}^N)} = \left(\int_\Omega |u|^p\right)^{\frac{1}{p}}, p \le 1 < \infty, \quad \|u\|_\infty = \operatorname*{ess\,sup}_\Omega |u|$$
- $W^{1,p}(\Omega, \mathbb{R}^N)$ Sobolev space of functions with values in $\mathbb{R}^N$, with the norm
$$\|u\|_{1,p} = \|u\|_{W^{1,p}(\Omega,\mathbb{R}^N)} = \left(\|u\|_p + \|\nabla u\|_p\right)^{\frac{1}{p}}$$
- $W_0^{1,p}(\Omega, \mathbb{R}^N) \subset W^{1,p}(\Omega, \mathbb{R}^N)$ the subspace of functions with zero trace
- $W^{-1,p}(\Omega, \mathbb{R}^N) = \left(W_0^{1,p}(\Omega, \mathbb{R}^N)\right)^*$ the dual space
- $\mathcal{C}(\Omega, \mathbb{R}^N)$ continuous functions with values in $\mathbb{R}^N$
- $\mathcal{C}^k(\Omega, \mathbb{R}^N)$ the space of functions with continuous derivatives up to order k
- $L^p([0,T), X)$, $W^{1,p}([0,T), X)$, $\mathcal{C}^k([0,T), X)$ corresponding Bochner spaces
- In all the function spaces, in the scalar case $N = 1$ we omit the $\mathbb{R}^N$ symbol.

0. Preface

This thesis deals with the convex hull property for systems of partial differential equations. It is the natural generalisation of maximum principles for scalar equations. It turns out that the validity of the convex hull property for systems requires some rather strict conditions on the structure of the coupling of the system. On the other hand, rather surprisingly, (some) non-linearities are admissible for the convex hull property.

The novelties in this thesis are the following. The convex hull property holds for the parabolic p-Laplace equation. Here we prove even more: We include lower-order terms in the equation and the non-linearity may be more general than the one of the p-Laplacian. We also investigate the conditions for general linear elliptic systems. Finally, we provide counterexamples to the convex hull property for elliptic and parabolic linear systems. It is remarkable that the elliptic counterexample needs variable coefficients in order to fail the convex hull property (as we will demonstrate) while the parabolic counterexample has constant coefficients. This is in line with the observations made in Kresin and Maz'ya [2012] on elliptic and parabolic systems.

The structure of the text is as follows. Chapter 1 contains a discussion of known results. It starts by recalling the maximum principles for scalar equations, then it proceeds to survey known results on the convex hull property. Moreover a new proof for Laplace equation in $\mathbb{R}^N$ is demonstrated as an illustration. Chapter 2 then contains the necessary theory for projections to convex sets, which will serve as the essential tool for proving our main results on the convex hull property in Chapter 3. The final Chapter 4 contains the counterexamples mentioned above.

The notation used is rather standard, but for the sake of completeness, a list is included at the beginning. The Appendix at the end contains a list of results from analysis that are used throughout the text.

1. Introduction

Recall from Notation that throughout the thesis $\Omega \subset \mathbb{R}^n$ will always be a Lispchitz domain.

1.1 Maximum principles for scalar equations

In this section we recall known maximum principles for elliptic partial differential equations, both for classical and weak solutions. The topic has been studied extensively, and the results for classical solutions can be found for instance in Evans [2010], and for both classical and weak solutions in Gilbarg and Trudinger [2001]. Our main interest will be in the *weak maximum principle*, since that is the one we will attempt at generalising to the convex hull property in systems of equations.

Theorem 1.1 (Weak maximum principle for elliptic equations, classical solutions)**.** *Let us consider the differential operator L in the nondivergence form*

$$Lu = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu, \tag{1.1}$$

where $a_{ij}, b_i, c \in \mathcal{C}(\Omega)$ and the matrix $A = (a_{ij})_{i,j=1}^n$ is everywhere symmetric and uniformly elliptic, that is, there exists $\alpha > 0$ such that for every $x \in \Omega$

$$\forall \xi \in \mathbb{R}^n : A\xi \cdot \xi \geq \alpha \left|\xi\right|^2.$$

Assume that $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ solves

$$Lu = 0 \quad \text{in } \Omega. \tag{1.2}$$

(i) If $c \equiv 0$ in Ω, then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u \quad \text{and} \quad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

(ii) If $c \geq 0$ in Ω, then

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

Sketch of proof. The proof is based on the fact that the matrix A can be (at every point $x \in \Omega$) diagonalised by an orthogonal matrix O such that the diagonal matrix $D = OAO^T$ has positive entries, and that the Hessian matrix $\left(\frac{\partial^2 u}{\partial x_i x_j}\right)_{i,j=1}^n$ is negative definite at the point of a local maximum of u. Some care must be taken when passing from strict to nonstrict inequality and the complete proof can be found in [Evans, 2010, Section 6.4.1]. □

Remark. It is in fact sufficient to only consider *subsolutions* and *supersolutions* to obtain the results in Theorem 1.1. In particular,

(i) if $c \equiv 0$ in Ω, then

$$Lu \leq 0 \text{ in } \Omega \implies \max_{\overline{\Omega}} u = \max_{\partial\Omega} u, \quad Lu \geq 0 \text{ in } \Omega \implies \min_{\overline{\Omega}} u = \min_{\partial\Omega} u,$$

(ii) if $c \geq 0$ in Ω, then

$$Lu \leq 0 \text{ in } \Omega \implies \max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+, \quad Lu \geq 0 \text{ in } \Omega \implies \min_{\overline{\Omega}} u \geq -\max_{\partial\Omega} u^-.$$

Now let us turn our attention to weak solutions. The following theorem has been adopted in a slightly modified form from [Gilbarg and Trudinger, 2001, Section 8.1].

Theorem 1.2 (Weak maximum principle for elliptic equations, weak solutions)**.** *Let L be a differential operator in divergence form*

$$Lu = -\operatorname{div}(Au) + b \cdot \nabla u + cu,$$

where $A \in L^\infty(\Omega, \mathbb{R}^{n\times n})$ is uniformly elliptic, that is for some $\alpha > 0$ and a.e. $x \in \Omega$

$$\forall \xi \in \mathbb{R}^n : A\xi \cdot \xi \geq \alpha \left|\xi\right|^2,$$

$b \in L^\infty(\Omega, \mathbb{R}^n)$, and $c \in L^\infty(\Omega)$, where $c \geq 0$ a.e. in Ω.
Let $u \in W^{1,2}(\Omega)$ be a weak solution of

$$Lu = 0,$$

that is,

$$\forall v \in W_0^{1,2}(\Omega) : \int_\Omega A\nabla u \cdot \nabla v + (b \cdot \nabla u)v + cv = 0. \tag{1.3}$$

Then

(i) If $c = 0$ a.e. in Ω, we have

$$u \leq \operatorname*{ess\,sup}_{\partial\Omega} u =: M, \quad \text{and} \quad u \geq \operatorname*{ess\,inf}_{\partial\Omega} u =: m \quad \text{a.e. in } \Omega.$$

(ii) If $c \geq 0$ a.e. in Ω, we have

$$|u| \leq \operatorname*{ess\,sup}_{\partial\Omega} |u| \quad \text{a.e. in } \Omega.$$

Proof. Assume first that $b = 0$ and $c = 0$ a.e. in Ω. Then $(u - M)^+ \in W^{1,2}(\Omega)$ by Theorem 5.5, and even $(u - M)^+ \in W_0^{1,2}(\Omega)$ because $u \leq M$ on $\partial\Omega$ in the trace sense. Thus we can use it as a test function in (1.3) and obtain

$$0 = \int_\Omega A\nabla u \cdot \nabla(u - M)^+ = \int_\Omega A\nabla(u - M)^+ \cdot \nabla(u - M)^+,$$

since $\nabla u = \nabla(u - M)^+$ on $\operatorname{supp} \nabla(u - M)^+$. Hence, by using the uniform ellipticity of A, we obtain

$$0 \geq \int_\Omega \alpha \left|\nabla(u - M)^+\right|^2 = \alpha \left\|\nabla(u - M)^+\right\|_2^2 \geq \tilde{\alpha} \left\|(u - M)^+\right\|_{1,2}^2,$$

where the last inequality follows from the equivalence of the norms $\|\nabla(\cdot)\|_2$ and $\|\cdot\|_{1,2}$ on $W_0^{1,2}(\Omega)$. Hence $(u - M)^+ = 0$ in $W_0^{1,2}(\Omega)$ and so also $u \leq M$ a.e. in Ω.

Now we allow b to be nonzero and still assume that $c = 0$ a.e. in Ω. We proceed our proof by contradiction and assume $M < \operatorname{ess\,sup}_\Omega u$. We take some ℓ

such that $M \leq \ell < \operatorname{ess\,sup}_\Omega u$. Now as before we have $(u-\ell)^+ \in W_0^{1,2}(\Omega)$ and we use it as a test function in (1.3), and use that $\nabla u = \nabla (u-\ell)^+$ on $\operatorname{supp}(u-\ell)^+$:

$$0 = \int_\Omega A\nabla(u-\ell)^+ \cdot \nabla(u-\ell)^+ + (b\cdot\nabla(u-\ell)^+)(u-\ell)^+.$$

Let us denote $\Omega_\ell := \operatorname{supp}\nabla(u-\ell)^+$. By using the uniform ellipticity of A, boundedness of b and the Hölder inequality,

$$\begin{aligned}\alpha \left\|\nabla(u-\ell)^+\right\|_2^2 &\leq \int_{\Omega_\ell} |b| \left|\nabla(u-\ell)^+\right| \left|(u-\ell)^+\right| \\ &\leq \|b\|_\infty \left\|\nabla(u-\ell)^+\right\|_2 \left\|(u-\ell)^+\right\|_{L^2(\Omega_l)}.\end{aligned}$$

Now necessarily $\|\nabla(u-\ell)^+\|_2 > 0$, because otherwise $\|(u-\ell)^+\| = 0$ (by the equivalence of norms on $W_0^{1,2}(\Omega)$) and that would contradict our choice of ℓ. Hence we can divide and get

$$\alpha \left\|\nabla(u-\ell)^+\right\|_2 \leq \|b\|_\infty \left\|(u-\ell)^+\right\|_{L^2(\Omega_\ell)}. \tag{1.4}$$

Now let us recall the Sobolev embeddings and the embedding of Lebesgue spaces on a bounded domain (see Appendix).

Let $q := \frac{2n}{n-2}$ in the case $n \geq 3$, and let $2 < q < \infty$ in the case $n = 2$. Then we have the continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$. Then, using this embedding and the equivalence of norms on $W_0^{1,2}(\Omega)$, we have

$$\left\|\nabla(u-\ell)^+\right\|_2 \geq C\left\|(u-\ell)^+\right\|_{1,2} \geq \tilde{C}\left\|(u-\ell)^+\right\|_q$$

for some $C, \tilde{C} > 0$.

Now, since $q > 2$ and Ω_ℓ is bounded, we have the continuous embedding $L^q(\Omega_\ell) \hookrightarrow L^2(\Omega_\ell)$ with the constant $|\Omega_\ell|^{1/2-1/q}$, hence

$$\left\|(u-\ell)^+\right\|_{L^2(\Omega_\ell)} \leq |\Omega_\ell|^{\frac{1}{2}-\frac{1}{q}} \left\|(u-\ell)^+\right\|_{L^q(\Omega_\ell)} \leq |\Omega|^{\frac{1}{2}-\frac{1}{q}} \left\|(u-\ell)^+\right\|_q.$$

Putting the last two inequalities into (1.4), we obtain

$$\alpha\tilde{C}\left\|(u-\ell)^+\right\|_q \leq \|b\|_\infty |\Omega_\ell|^{\frac{1}{2}-\frac{1}{q}} \left\|(u-\ell)^+\right\|_q.$$

Again, $\|(u-l)^+\|_q > 0$ because otherwise we would have contradiction with our choice of ℓ, so we can divide and after rearranging the terms

$$\left(\frac{\alpha\tilde{C}}{\|b\|_\infty}\right)^{\frac{2q}{q-2}} \leq |\Omega_\ell|.$$

Note closely that the left-hand side does not depend on ℓ. Thus we can take the limit $\ell \to \operatorname{ess\,sup}_\Omega u$ and obtain

$$\left(\frac{\alpha\tilde{C}}{\|b\|_\infty}\right)^{\frac{2q}{q-2}} \leq |\Omega_{\sup}|, \quad \text{where } \Omega_{\sup} = \bigcap_{\ell<\sup_\Omega u} \Omega_\ell \subset \bigcap_{\ell<\sup_\Omega u} \operatorname{supp}(u-\ell)^+.$$

Therefore we see that necessarily $u = \sup_\Omega u$ on $\Omega_{\sup}$. But since $|\Omega_{\sup}| > 0$, this means that $\nabla u = 0$ on $\Omega_{\sup}$. But this is a contradiction, since by the definition of Ω_ℓ we have that $\nabla(u-\ell)^+ = \nabla u$ is nonzero on $\Omega_\ell \supset \Omega_{\sup}$.

To prove the second inequality, just observe that $-m = \max_{\partial\Omega} -u$ and apply the just proved to $-u$ in place of u. This proves (i).

Finally, we allow $c \geq 0$. Denote $\tilde{M} := \operatorname{ess\,sup}_{\partial\Omega} u^+$ and $\tilde{m} := \operatorname{ess\,inf}_{\partial\Omega} -u^-$. Suppose for contradiction that $\tilde{M} < \sup_\Omega u$ and take $\tilde{M} \leq \tilde{\ell} < \sup_\Omega u$. Then we use $(u-\tilde{\ell})^+ \in W_0^{1,2}(\Omega)$ as a test function in (1.3) and, similarly as before

$$\int_\Omega A\nabla(u-\tilde{\ell})^+ \cdot \nabla(u-\tilde{\ell})^+ + (b\cdot\nabla(u-\tilde{\ell})^+)(u-\tilde{\ell})^+ = -\int_\Omega cu(u-\tilde{\ell})^+$$

Now we examine the right-hand side. We know $\tilde{\ell} \geq 0$, so if $u < 0$, then also $(u-\tilde{\ell})^+ = 0$, hence the function under the integral sign is everywhere nonnegative and thus

$$-\int_\Omega cu(u-\tilde{\ell})^+ \leq 0.$$

Now we can proceed exactly as in the case $c = 0$ and obtain $u \leq \tilde{M}$ a.e. in Ω. Then, we notice that $-\tilde{m} = \operatorname{ess\,sup}_{\partial\Omega}(-u)^+$, so we have $-u \geq -\tilde{m}$. Thus we have $|u| \leq \max\{\tilde{M}, -\tilde{m}\} = \operatorname{ess\,sup}_\Omega \max\{u^+, u^-\} = \operatorname{ess\,sup}_\Omega |u|$. This proves (ii). □

Remark. As can be seen from the proof, we may again consider only weak *supersolutions* and *subsolutions* to obtain the inequalities. We say that

$$Lu \leq (\geq)\, 0 \quad \text{in the weak sense,}$$

provided that

$$\forall v \in W_0^{1,2}(\Omega), v \geq 0 \text{ a.e.} : \int_\Omega A\nabla u \cdot \nabla v + (b\cdot\nabla u)v + cv \leq (\geq)\, 0. \tag{1.5}$$

Then we have (inequalities $Lu \geq (\leq)\, 0$ below are in the weak sense)

(i) if $c = 0$ a.e. in Ω, then

$$Lu \leq 0 \implies u \leq \operatorname*{ess\,sup}_{\partial\Omega} u, \quad Lu \geq 0 \text{ in } \Omega \implies u \geq \operatorname*{ess\,inf}_{\partial\Omega} u \text{ a.e. in } \Omega,$$

(ii) if $c \geq 0$ a.e. in Ω, then

$$Lu \leq 0 \implies u \leq \operatorname*{ess\,sup}_{\partial\Omega} u^+, \quad Lu \geq 0 \text{ in } \Omega \implies u \geq \operatorname*{ess\,inf}_{\partial\Omega} -u^- \text{ a.e. in } \Omega.$$

Now we also mention the maximum principle for a "prototypical" nonlinear elliptic equation, namely, the p-Laplace equation, where $1 < p < \infty$:

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\, \nabla u) = 0. \tag{1.6}$$

A function $u \in W^{1,p}(\Omega)$ is a weak solution of (1.6) if

$$\forall v \in W_0^{1,2}(\Omega) : \int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla v = 0.$$

This equation arises as the Euler-Lagrange equation for the functional

$$I(w) = \int_\Omega |\nabla w|^p\,, \quad w \in W^{1,p}(\Omega).$$

The solutions of the scalar p-Laplace equation satisfy the (both weak and strong) maximum principle (see for instance Tolksdorf [1984], Lindqvist [2006]).

We conclude this section by summarising the types of maximum principles for various types of scalar equations, and the analogous notions for systems of equations. Note that by "boundary" we mean for elliptic equations (on Ω) simply $\partial\Omega$, but for parabolic equations (on $[0,T] \times \Omega$), we mean the parabolic boundary $(\{0\} \times \Omega) \cup ([0,T] \times \partial\Omega)$.

- **Weak maximum principle.** The values of every solution do not exceed its maximum on the boundary.

- **Strong maximum principle.** A solution cannot attain its maximum in Ω, unless it is constant.

- **Maximum modulus principle.** For any solution u, the function $|u|$ satisfies the weak maximum principle.

The two maximum principles have a minimum principle version, just by replacing "maximum" with "minimum". Of course, one has to be more careful about the meaning of "maximum on the boundary" and "attain its maximum" when talking about weak solutions, because by their very definition they are in some Sobolev spaces and don't have pointwise values.

We will not say much about strong maximum principle, the interested reader may refer for instance to Evans [2010] or Gilbarg and Trudinger [2001].

Our main interest lies in the weak maximum principle and its generalisation to systems of equations as the *convex hull property*. We can state the weak maximum and minimum principles at once by words saying "the solution lies in between its boundary values". This sentence can be immediately understood in the setting of systems of equations, when the solutions attain values in $\mathbb{R}^N$ – just interpreting "lies in between" as "is a convex combination". A strong version of the convex hull property is possible, we list it below but do not investigate it any further in the thesis. Another possibility is considering $|u|$ instead of u to obtain a scalar function and to have maximum modulus principle as above.

Hence for systems of equations, we have

- **Convex hull property.** The values of any solution lie in the convex hull of its boundary values.

- **Strong convex hull property.** If a solution attains an extremal point of its boundary values, then it is constant (recall that $x \in M \subset \mathbb{R}^N$ is an *extremal point of M* if it is not an interior point of a line segment contained in M).

- **Maximum modulus principle.** For any solution u, the function $|u|$ satisfies the weak maximum principle.

Again, if we are talking about weak solutions only, some care must be taken when giving precise meaning to these words. The convex hull property will be rigorously stated in the corresponding theorems to follow. We shall not be concerned with the strong convex hull property and the maximum modulus principle will only be mentioned when referring to a book on this topic. Our main area of study will be the convex hull property for systems of equations.

1.2 Some results on the convex hull property

Here we will review some of the research that has been done on the convex hull property. Recall that by $\operatorname{conv} M$ we denote the (closed) convex hull of the set M.

First we take a look on the paper Bildhauer and Fuchs [2002], where the authors prove the convex hull property for minimizers of certain variational problems. Their result is the following.

Theorem 1.3. *Let $f \in \mathcal{C}^2(\mathbb{R}^{n\times N}, [0,\infty))$ satisfy the condition*

$$\forall Z, U \in \mathbb{R}^{n\times N} : \lambda(1+|Z|^2)^{\frac{s-2}{2}} |U|^2 \leq \nabla^2 f(Z)(U,U) \leq \Lambda(1+|Z|^2)^{\frac{q-2}{2}} |U|^2$$

with constants $0 < \lambda < \Lambda$ and exponents $2 \leq s < q$. Consider the functional

$$J(u) = \int_\Omega f(\nabla u), \quad u \in W^{1,s}(\Omega, \mathbb{R}^N).$$

Let $u_0 \in W^{1,s}(\Omega, \mathbb{R}^N)$ satisfy $J[u_0] < \infty$. Then there exists a unique minimizer

$$u_{\min} = \operatorname*{argmin}_{w \in u_0 + W_0^{1,s}(\Omega, \mathbb{R}^N)} J(w)$$

and it satisfies the following convex hull property: If $u_0(\Omega) \subset K$, where $K \subset \mathbb{R}^N$ is a compact convex set, then also $u_{\min}(\Omega) \subset K$.

Proof. See Bildhauer and Fuchs [2002]. □

In another paper Diening et al. [2013], the convex hull property for *finite element* minimizers is established. It assumes $\Omega \subset \mathbb{R}^n$ to have polyhedral boundary and that Ω is triangulated into (finitely many) n-simplexes by $\mathcal{T}$. The (finite dimensional) space $\mathbb{V}(\mathcal{T})^N \subset \mathcal{C}(\overline{\Omega}, \mathbb{R}^N)$ then consists of piecewise affine functions, affine on each of the simplexes in $\mathcal{T}$. Further $\mathbb{V}_0(\mathcal{T})^N \subset \mathbb{V}(\mathcal{T})^N$ are the functions with zero boundary values. The minimizing problem is defined by the functional

$$J(V) = \int_\Omega F(x, |\nabla V(x)|)\, \mathrm{d}x, \quad V \in \mathbb{V}(\mathcal{T})^N,$$

where the crucial assumption on F is its monotonicity in the second argument.

Under this assumption, the minimizer of J among all functions in $\mathbb{V}(\mathcal{T})^N$ with the same boundary values satisfies the convex hull property (some additional assumptions to guarantee the existence are also to be found in the paper):

Theorem 1.4. *Let $G \in \mathbb{V}(\mathcal{T})^N$ and suppose that*

$$U = \operatorname*{argmin}_{V \in G + \mathbb{V}_0(\mathcal{T})^N} J(V).$$

Then U satisfies the convex hull property $U(\Omega) \subset \operatorname{conv} U(\partial\Omega)$.

Idea of the proof. The idea is to take the projection of U to the convex set $\operatorname{conv} U(\partial\Omega)$ and showing that U is in fact equal to this projection. For details, see Diening et al. [2013]. □

The approach taken in the proof is the same as we will apply later in Chapter 3. The essential ingredient in making the proof work is the estimate of the gradient of the projection of U. We will show this inequality in a more general setting (for Sobolev functions) in Theorem 2.8.

1.3 Systems of equations

Let us now turn our attention to systems. In the book Kresin and Maz'ya [2012], the authors have identified the structural conditions for elliptic and parabolic systems that are equivalent with the claim that every classical solution u satisfies the weak maximum modulus principle

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u| . \tag{1.7}$$

First the authors discuss the case of elliptic systems with constant coefficients

$$\sum_{j,k=1}^{n} \mathcal{A}_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} = 0, \tag{1.8}$$

where each $\mathcal{A}_{jk} \in \mathbb{R}^{N\times N}$ is a constant matrix. The condition equivalent to the maximum modulus principle is that it is a "scalar elliptic equation repeated N times". More precisely, provided that Ω has $\mathcal{C}^1$ boundary, the condition is that we have the decomposition

$$\mathcal{A}_{jk} = a_{jk}\mathcal{A}, \quad j,k = 1,\dots,n. \tag{1.9}$$

where $\mathcal{A} \in \mathbb{R}^{N\times N}$ and $(a_{jk})_{j,k=1}^n \in \mathbb{R}^{n\times n}$ are positive definite matrices, see [Kresin and Maz'ya, 2012, Theorem 2.4]. (Compare with our counterexample for the convex hull property in linear elliptic systems, Example 4.1.)

Next, there is a result in the case of variable coefficients with lower order terms,

$$\sum_{j,k=1}^{n} \mathcal{A}_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{j=1}^{n} \mathcal{A}_j \frac{\partial u}{\partial x_j} - \mathcal{A}_0 u = 0, \tag{1.10}$$

where now $\mathcal{A}_{jk}$, $\mathcal{A}_j$ and $\mathcal{A}_0$ are $(N\times N)$-matrix valued functions in Ω with sufficient smoothness, and also Ω is sufficiently smooth. Now the condition equivalent to the maximum modulus principle holding *on every subdomain* $\omega \subset \Omega$ is again that we have the decompoisition (1.9) at every point, where $\mathcal{A}$ and $(a_{jk})_{j,k=1}^n$ are positive-definite matrix valued functions, and moreover a certain "ellipticity-like" inequality holds for the lower order terms. This inequality may in the case of no lowest order term, $\mathcal{A}_0 = 0$, be replaced by the requirement that $\mathcal{A}_j$'s are multiples of $\mathcal{A}$:

$$\mathcal{A}_j = a_j\mathcal{A}, \quad j = 1,\dots,n. \tag{1.11}$$

For details, see [Kresin and Maz'ya, 2012, Theorem 2.15, Corollary 2.4]

The authors' result for a parabolic system with variable coefficients resembles the one above, with one exception: the matrix $\mathcal{A}$ in the decomposition *has to be the identity matrix.* More precisely, the equation studied is the same as in the elliptic case, with the t-derivative term added:

$$\partial_t u - \sum_{j,k=1}^{n} \mathcal{A}_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \mathcal{A}_j \frac{\partial u}{\partial x_j} + \mathcal{A}_0 u = 0. \tag{1.12}$$

Then the validity of the classical parabolic maximum modulus principle is the equivalent to the stronger decomposition requirement

$$\mathcal{A}_{jk} = a_{jk} I, \quad j,k = 1,\dots,n,$$

where $(a_{jk})_{j,k=1}^n$ is a positive-definite matrix valued function and $I \in \mathbb{R}^{N\times N}$ is the identity matrix, and an inequality for the lower order terms analogous to the one above. Again, in the case $\mathcal{A}_0 = 0$, this inequality can be replaced by requiring

$$\mathcal{A}_j = a_j I, \quad j = 1, \dots, n. \tag{1.13}$$

For details, see [Kresin and Maz'ya, 2012, Theorem 8.8, Corollary 8.1].

Now let us focus on our main subject of interest, the convex hull property. The convex hull property says that the solution lies in the smallest convex set which contains boundary values, whereas maximum modulus principle says that the solution lies in the smallest ball centered at the origin which contains the boundary values. Thus the convex hull property is clearly a stronger property.

The above cited characterisations of maximum modulus principle suggest that there is not much freedom to be had in the coefficients of linear systems if we want to obtain the convex hull property. In fact, there is even less freedom – we will see in Example 4.1 that we essentially can't allow mixing of the different components of u and hence that will require us to have coefficients which are formed by blocks of n copies of some positive-definite matrix on the diagonal. We will prove the convex hull property for these kind of elliptic systems in Theorem 3.1, under the condition that the matrices are (possibly variable) scalar multiples of the same positive definite matrix. In parabolic systems, we also cannot allow mixing of the components, not even with constant coefficients, as will be shown in Example 4.2.

The twist is, however, even though we cannot do much for the coefficients of linear systems, what we can do is allow some type of non-linearities. The prototypical example here is the already mentioned p-Laplace equation. But, as we will see in our main result for parabolic systems, Theorem 3.6, we can take more general scalar non-linearities – in the highest-order term as well as in lower-order terms (under some suitable assumptions on their growth).

1.4 Motivation: Convex hull property for the Laplace equation in $\mathbb{R}^N$

As a motivation for studying the convex hull property, let us first illustrate it on the example of the Laplace equation in $\mathbb{R}^N$, that is,

$$\Delta u = 0. \tag{1.14}$$

In this case the convex hull property can be obtained by using the maximum principle for scalar Laplace equation in every direction.

Lemma 1.5. *Let $h \in \mathbb{R}^N$. Then for any $u \in \mathcal{C}^2(\Omega, \mathbb{R}^N)$ we have*

$$h \cdot \Delta u = \Delta(h \cdot u)$$

Proof. Simply caluclate

$$h \cdot \Delta u = \sum_{k=1}^N h_k \sum_{i=1}^n \frac{\partial^2 u_k}{\partial x_i \partial x_i} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i \partial x_i} \sum_{k=1}^N h_k u_k = \Delta(h \cdot u).$$

□

Theorem 1.6 (Convex hull property for Laplace equation in $\mathbb{R}^N$)**.** *Suppose that* $u \in \mathcal{C}^2(\Omega, \mathbb{R}^N) \cap \mathcal{C}(\overline{\Omega}, \mathbb{R}^N)$ *solves the Laplace equation*

$$\Delta u = 0 \quad \text{in } \Omega.$$

Then $u(\Omega) \subset \operatorname{conv} u(\partial\Omega)$.

Proof. Choose a vector $h \in \mathbb{R}^N$, $|h| = 1$. Denote $v = h \cdot u$. Then by the previous lemma, $\Delta v = h \cdot \Delta u = 0$. Thus v satisfies the scalar Laplace equation and thus by the maximum principle (Theorem 1.1),

$$v(\Omega) \subset (-\infty, \max_{\partial\Omega} v].$$

This in fact means that, since $v = h \cdot u$,

$$u(\Omega) \subset \left\{ x \in \mathbb{R}^N : h \cdot x \leq \max_{y \in u(\partial\Omega)} h \cdot y \right\}.$$

(See Figure 1.1 for illustration.)

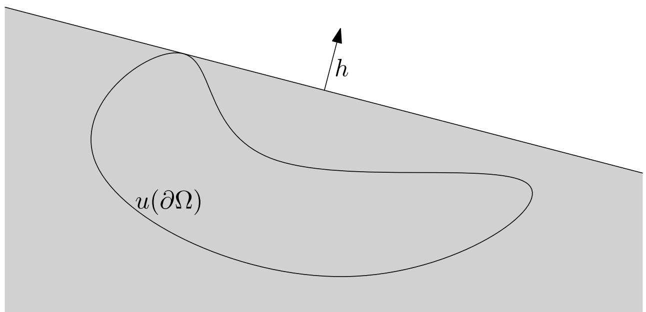

Figure 1.1: Maximum principle in the direction h

Hence

$$u(\Omega) \subset \bigcap_{h \in \mathbb{R}^N, |h|=1} \left\{ x \in \mathbb{R}^N : x \cdot h \leq \max_{y \in u(\partial\Omega)} y \cdot h \right\} =: M.$$

It remains to show that $M = \operatorname{conv} u(\partial\Omega)$.

Clearly $\operatorname{conv} u(\partial\Omega) \subset M$, because M is an intersection of convex sets, each containing $u(\partial\Omega)$.

On the other hand, we know that $\operatorname{conv} u(\partial\Omega)$, is the intersection of all closed half-spaces containing $u(\partial\Omega)$. Every closed half-space is of the form

$$H_{h,\ell} := \{x \in \mathbb{R}^N : x \cdot h \leq \ell\}$$

for some $h \in \mathbb{R}^N$, $|h| = 1$ and $\ell \in \mathbb{R}$. If $\ell < \max_{y \in u(\partial\Omega)} y \cdot h$, then $H_{h,\ell} \not\supset u(\partial\Omega)$. So $M \subset \operatorname{conv} u(\partial\Omega)$ and the proof is finished. ☐

1.5 Introduction to the projection method

To properly motivate the use of projections, we begin this section by redoing the proof of Theorem 1.2 in such a way that the choice of test function can be directly generalised to systems of equations. Also the statement is worded differently to resemble the convex hull property. For simplicity, we omit the lower order terms.

Theorem 1.7 (Maximum principle for linear second-order elliptic PDE)**.** *Consider the problem*

$$-\operatorname{div}(A\nabla u) = 0 \quad \text{in } \Omega,$$

where $A \in L^\infty(\Omega, \mathbb{R}^{N\times N})$ is uniformly elliptic, that is, for some $\alpha > 0$ and a.e. $x \in \Omega$

$$\forall \xi \in \mathbb{R}^N : A\xi \cdot \xi \geq |\xi|^2 . \tag{1.15}$$

Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution of this problem, that is,

$$\forall v \in W_0^{1,2}(\Omega) : \int_\Omega A\nabla u \cdot \nabla v = 0. \tag{1.16}$$

Denote

$$M = \operatorname*{ess\,sup}_{\partial\Omega} u, \quad m = \operatorname*{ess\,inf}_{\partial\Omega} u.$$

Then

$$u \in [m, M] \quad \text{a.e. in } \Omega.$$

Proof. We put $w = (u - M)^+ - (m - u)^-$, then $w \in W^{1,2}(\Omega)$ because positive and negative part are 1-Lipschitz functions and we have Theorem 5.5. Moreover, we see that w has zero trace (since $m \leq u \leq M$ a.e. on $\partial\Omega$), so even $w \in W_0^{1,2}(\Omega)$ and we can use it as a test function in (1.16). So we obtain, since $\nabla u = \nabla(u - M)^+$ on $\operatorname{supp}(u - M)^+$ and $-\nabla u = \nabla(m - u)^+$ on $\operatorname{supp}(m - u)^+$:

$$\begin{aligned} 0 = \int_\Omega A\nabla u \cdot \nabla w = \int_\Omega A\nabla(u - M)^+ \cdot \nabla(u - M)^+ + \int_\Omega A\nabla(m - u)^- \cdot \nabla(m - u)^- \\ \geq \alpha \left\|\nabla(u - M)^+\right\|_2^2 + \alpha \left\|\nabla(m - u)^-\right\|_2^2 \geq \tilde{\alpha} \left\|(u - M)^+\right\|_{1,2}^2 + \tilde{\alpha} \left\|(m - u)^-\right\|_{1,2}^2 . \end{aligned}$$

Hence

$$(u - M)^+ = 0 \quad \text{and} \quad (m - u)^- = 0 \quad \text{a.e. in } \Omega,$$

which simply means that

$$u \in [m, M] \quad \text{a.e. in } \Omega.$$

$\square$

Let us now see how the approach in this proof can be generalised to systems of equations. If we denote $K = [m, M]$, then K is the convex hull of the boundary values of u. Let $\Pi_K : \mathbb{R} \to K$ be the projection to K, that is,

$$\Pi_K(x) = \operatorname*{argmin}_{y\in K} |y - x| = x - (x - M)^+ + (m - x)^-.$$

If $w = (u - M)^+ - (m - u)^-$ as in the proof above, we have $\Pi_K u = u - w$. Hence the test function used in the proof is $w = u - \Pi_K u$. So the idea when u takes values in $\mathbb{R}^N$ can be essentially the same: take K to be the convex hull of the boundary values of u, and if $\Pi_K : \mathbb{R}^N \to K$ is the projection of $\mathbb{R}^N$ to K, use $u - \Pi_K u$ as the test function in the equation (it turns out that $u - \Pi_K u \in W_0^{1,2}$ because Π_K is Lipschitz and $u = \Pi_K u$ on the boundary) and try to prove that $u = \Pi_K u$ in $W^{1,2}(\Omega, \mathbb{R}^N)$.

But there are some caveats. In the proof of Theorem 1.2 we have used that $\nabla u = \nabla(u - M)^+ - \nabla(m - u)^-$, provided that $u \notin [m, M]$. However, in $\mathbb{R}^N$, it is not true that $\nabla u = \nabla(u - \Pi_K u)$ if $u \notin K$. It only holds that (as we will see in Theorem 2.8)

$$\nabla u : \nabla \Pi_K u \geq |\nabla \Pi_K u|^2 . \tag{1.17}$$

We can try to prove the convex hull property only using this inequality instead, but we may run into problems. For instance, if the highest-order term in the equation is $-\mathrm{div}(A\nabla u)$, then the corresponding term after using the test function $u - \Pi_K u$ is $\int_\Omega A\nabla u : \nabla(u - \Pi_K u)$. First, we would want $\nabla \Pi_K u$ in place of ∇u, but if A is linear and elliptic, we may write it as

$$\int_\Omega A\nabla u : \nabla(u-\Pi_K u) = \int_\Omega A\nabla(u-\Pi_K u) : \nabla(u-\Pi_K u)+\int_\Omega A\nabla\Pi_K u : \nabla(u-\Pi_K u)$$

and estimate the first term by ellipticity. But the second term cannot be directly estimated by (1.17) because of the A. However, if the modify the projection Π_K so that it will instead yield the inequality

$$A\nabla\Pi_K u : \nabla(u - \Pi_K u) \geq A\nabla\Pi_K u : \nabla\Pi_K u, \tag{1.18}$$

we could estimate the second term. One plausible idea would be to take a different inner product on $\mathbb{R}^N$ and make the projection with respect to it. But we cannot take "inner product with respect to A" because it even formally doesn't make sense – A is a linear operator $A\colon \mathbb{R}^{n\times N} \to \mathbb{R}^{n\times N}$, not an $N \times N$ matrix. What we can do is to use that the inequality (1.17) holds with $\partial/\partial x_i$ in place of ∇, and to use it for each partial derivative separately to somehow obtain (1.18). What limits us is that we need to use the same projection for all of the partial derivatives (the same one we use for the test function). This somehow forces us to only consider the case that A has diagonal blocks of (possibly variable) scalar multiples of *the same* $N \times N$ matrix. What could further interest us is whether this matrix can be variable. This would lead us to using variable projection, that is projecting with respect to a different projection at every point. Now a different difficulty arises, the inequality (1.18) simply does not hold with variable projection. We will see this in more detail later, just before Example 4.1.

2. Projections to convex sets

In this chapter we develop the theory for projections of Sobolev functions to closed convex sets. First, we prove some properties of projections based on the (geometrically very intuitive) Theorem 2.2. Second, we apply this to Sobolev functions, using the approximation by difference quotients. The crucial result to be used in the proofs in Chapter 3 is the inequality in Theorem 2.8.

2.1 Properties of projections in Hilbert spaces

Even though we will use all the listed properties only in $\mathbb{R}^N$, we will state them in the setting of Hilbert spaces, as it causes no further complications.

Definition 2.1. Let H be a Hilbert space with the inner product $\langle\,,\,\rangle$ and the induced norm $\|\ \|$. Let $K \subset H$ be a nonempty closed convex set. We define the *projection of H to K*, denoted by $\Pi_K\colon H \to K$, by setting for $x \in H$

$$\Pi_K x = \operatorname*{argmin}_{y\in K} \|x-y\| \,.$$

Remark. It is known from the theory of Hilbert spaces that Π_K is well-defined.

Theorem 2.2. *Let H be a Hilbert space and $K \subset H$ be a nonempty closed convex set. Then*

$$\forall x \in H, z \in K : \langle x - \Pi_K x, z - \Pi_K x\rangle \le 0. \tag{2.1}$$

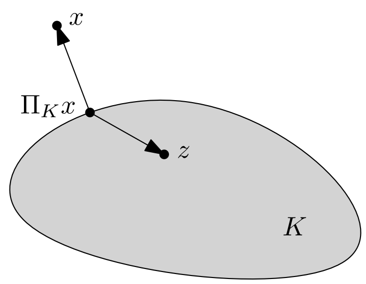

Figure 2.1: Theorem 2.2

Proof. Let $x \in H$ and $z \in K$. We have $\Pi_K x = \operatorname{argmin}_{y\in K} \langle x-y, x-y\rangle$. Thus for any $h \in (0,1)$ we obtain (since $(1-h)\Pi_K x + hz \in K$ from convexity):

$$\begin{aligned}\langle x-\Pi_K x, x-\Pi_K x\rangle &\le \langle x-((1-h)\Pi_K x+hz), x-((1-h)\Pi_K x+hz)\rangle\\ &= \langle (1-h)(x-\Pi_K x)+h(x-z), (1-h)(x-\Pi_K x)+h(x-z)\rangle\\ &= (1-h)^2\langle x-\Pi_K x, x-\Pi_K x\rangle + h^2\langle x-z,x-z\rangle + 2h(1-h)\langle x-\Pi_K x, x-z\rangle\,.\end{aligned}$$

So

$$(2h-h^2)\langle x-\Pi_K x, x-\Pi_K x\rangle \le h^2\langle x-z,x-z\rangle + 2h(1-h)\langle x-\Pi_K x, x-z\rangle\,.$$

Since $h>0$, we can divide by it and get

$$(2-h)\langle x-\Pi_K x, x-\Pi_K x\rangle \le h\langle x-z,x-z\rangle + 2(1-h)\langle x-\Pi_K x, x-z\rangle\,.$$

Passing to the limit as $h \to 0^+$ yields

$$2\langle x - \Pi_K x, x - \Pi_K x\rangle \le 2\langle x - \Pi_K x, x - z\rangle .$$

Finally, subtracting the right hand side and dividing by 2 gives

$$\langle x - \Pi_K x, x - \Pi_K x - (x - z)\rangle = \langle x - \Pi_K x, z - \Pi_K x\rangle \le 0.$$

$\square$

Corollary 2.3. *Let H be a Hilbert space and $K \subset H$ be a nonempty convex closed set. Then for all $x, y \in H$*

$$\langle x - y, \Pi_K x - \Pi_K y\rangle \ge \|\Pi_K x - \Pi_K y\|^2 . \tag{2.2}$$

Proof. From Theorem 2.2, putting $z := \Pi_K y \in K$, we obtain the inequality

$$\langle x - \Pi_K x, \Pi_K x - \Pi_K y\rangle \ge 0.$$

Switching the roles of x and y, we also have

$$\langle y - \Pi_K y, \Pi_K y - \Pi_K x\rangle \ge 0.$$

Adding these two inequalities gives

$$\langle x - y + \Pi_K x - \Pi_K y, \Pi_K x - \Pi_K y\rangle \ge 0.$$

Finally, subtracting $\langle \Pi_K x - \Pi_K y, \Pi_K x - \Pi_K y\rangle = \|\Pi_K x - \Pi_K y\|^2$ yields the result. $\square$

Corollary 2.4. *Let H be a Hilbert space and $K \subset H$ be a nonempty convex closed set. Then $\Pi_K \colon H \to H$ is 1-Lipschitz. That is,*

$$\forall x, y \in H : \|\Pi_K x - \Pi_K y\| \le \|x - y\| .$$

Proof. Let $x, y \in H$. If $\|\Pi_K x - \Pi_K y\| = 0$, the inequality is trivial. Otherwise from Corollary 2.3 and the Cauchy-Schwarz inequality we have

$$\|\Pi_K x - \Pi_K y\|^2 \le \langle x - y, \Pi_K x - \Pi_K y\rangle \le \|x - y\| \|\Pi_K x - \Pi_K y\| .$$

Dividing by $\|\Pi_K x - \Pi_K y\|$ shows the claim. $\square$

2.2 Projections of Sobolev functions

We begin this section by recalling a result from the theory of Sobolev spaces. It will enable us to apply the properties of projections studied above to projections of Sobolev functions having values in $\mathbb{R}^N$.

Definition 2.5 (Difference quotients)**.** For a function $u \colon \Omega \to \mathbb{R}^N$ we define

$$\Delta_i^h u(x) = \frac{u(x + he_i) - u(x)}{h},$$

for all $h > 0$, $i \in \{1, \dots, n\}$, and $x \in \Omega$ such that $x + he_i \in \Omega$.

Theorem 2.6 (Difference quotients and weak partial derivative)**.** *Let $1 \le p < \infty$, and $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. For $\delta > 0$ denote*

$$\Omega_\delta := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

Then for any $\delta > 0$ it holds that

$$\Delta_i^h u \to \frac{\partial u}{\partial x_i} \quad \text{in } L^p(\Omega_\delta), \text{ as } h \to 0.$$

In particular,

$$\Delta_i^h u \to \frac{\partial u}{\partial x_i} \quad \text{a.e. in } \Omega, \text{ as } h \to 0.$$

Proof. Can be found in [Gilbarg and Trudinger, 2001, Section 7.4] □

Following is a special case of Theorem 5.5 on composing Sobolev and Lipschitz functions, applied to the projection Π_K.

Corollary 2.7. *Let $1 \le p \le \infty$ and let $K \subset \mathbb{R}^N$ be nonempty, closed and convex. Then for $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ it holds that $\Pi_K u \in W^{1,p}(\Omega, \mathbb{R}^N)$.*

Proof. Follows directly from Theorem 5.5, using the fact that $\Pi_K \colon \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz by Corollary 2.4. □

Now let us formulate a version of Corollary 2.3 for projections of Sobolev functions.

Theorem 2.8. *Let $1 \le p < \infty$, $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ and let $K \subset \mathbb{R}^N$ be nonempty, closed and convex. Then*

$$\nabla u : \nabla \Pi_K u \ge |\nabla \Pi_K u|^2 \quad \text{a.e. in } \Omega. \tag{2.3}$$

Remark. We will often be using the inequality (2.3) in the form

$$\nabla(u - \Pi_K u) : \nabla \Pi_K u \ge 0.$$

Proof. Let $x \in \Omega$ and choose $i \in \{1, \dots, n\}$. Find $\delta > 0$ such that for every $|h| < \delta$ we have $x + he_i \in \Omega$. Then for such h, Corollary 2.3 gives

$$(u(x + he_i) - u(x)) \cdot (\Pi_K u(x + he_i) - \Pi_K u(x)) \ge |\Pi_K u(x + he_i) - \Pi_K u(x)|^2 .$$

After dividing by h^2, this reads with the difference quotient notation as

$$\Delta_i^h u(x) \cdot \Delta_i^h \Pi_K u(x) \ge \left|\Delta_i^h \Pi_K u(x)\right|^2 .$$

By Corollary 2.7, $\Pi_K u \in W^{1,p}$. Hence by Theorem 2.6, $\Delta_i^h u(x) \to \frac{\partial u}{\partial x_i}(x)$ and $\Delta_i^h \Pi_K u(x) \to \frac{\partial \Pi_K u}{\partial x_i}(x)$ for a.e. $x \in \Omega$, as $h \to 0$. Thus passing to the limit as $h \to 0$ gives for a.e. $x \in \Omega$

$$\frac{\partial u}{\partial x_i}(x) \cdot \frac{\partial \Pi_K u}{\partial x_i}(x) \ge \left|\frac{\partial \Pi_K u}{\partial x_i}(x)\right|^2 .$$

Summing over $i = 1, \dots, n$ gives

$$\sum_{i=1}^n \frac{\partial u}{\partial x_i}(x) \cdot \frac{\partial \Pi_K u}{\partial x_i}(x) \ge \sum_{i=1}^n \left|\frac{\partial \Pi_K u}{\partial x_i}(x)\right|^2$$

for a.e. $x \in \Omega$, which is exactly our claim. □

Definition 2.9. For a matrix $A \in \mathbb{R}^{N\times N}$ we define the bilinear form $\langle\, , \,\rangle_A$ on $\mathbb{R}^N$ by putting $\langle v, w\rangle_A = (Av)\cdot w$. If A is positive definite, then $\langle\, , \,\rangle_A$ is an inner product on $\mathbb{R}^m$ and in this case we denote the induced norm by $|\ |_A$, and for a nonempty closed $K \subset \mathbb{R}^N$ the projection $\Pi^A_K \colon \mathbb{R}^N \to K$ defined by

$$\Pi^A_K(x) = \operatorname*{argmin}_{y\in K} |y-x|_A\,.$$

Theorem 2.10. *Let $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ and let $A \in \mathbb{R}^{N\times N}$ be a positive definite matrix. Let $K \subset \mathbb{R}^N$ be nonempty, and closed convex. Then for a.e. $x \in \Omega$*

$$A\frac{\partial u}{\partial x_i}(x)\cdot\frac{\partial \Pi^A_K u}{\partial x_i}(x) \geq A\frac{\partial \Pi^A_K u}{\partial x_i}(x)\cdot\frac{\partial \Pi^A_K u}{\partial x_i}(x), \quad i=1,\ldots,n.$$

Proof. We can proceed as in the proof of Theorem 2.8, just replacing the Euclidean inner product $\cdot$ with $\langle\, , \,\rangle_A$ and the projection Π_K with Π^A_K.

More precisely, fix $i \in \{1,\ldots,n\}$ and let $x \in \Omega$. Choose $\delta > 0$ such that $x + he_i \in \Omega$ for $|h| < \delta$. Then for all such h, from Corollary 2.3 we obtain

$$\Big\langle u(x+he_i) - u(x), \Pi^A_K u(x+he_i) - \Pi^A_K u(x)\Big\rangle_A \geq \Big|\Pi^A_K u(x+he_i) - \Pi^A_K u(x)\Big|^2_A.$$

Dividing by h^2 and using the difference quotient notation gives

$$\Big\langle \Delta^h_i u(x), \Delta^h_i \Pi^A_K u(x)\Big\rangle_A \geq \Big|\Delta^h_i \Pi^A_K u(x)\Big|^2_A$$

We know by Theorem 2.6 that $\Delta^h_i u(x) \to \frac{\partial u}{\partial x_i}(x)$ and $\Delta^h_i \Pi^A_K u(x) \to \frac{\partial \Pi^A_K u}{\partial x_i}(x)$ for a.e. $x \in \Omega$, as $h \to 0$. Hence for a.e. $x \in \Omega$ we have

$$\left\langle \frac{\partial u}{\partial x_i}(x), \frac{\partial \Pi^A_K u}{\partial x_i}(x)\right\rangle_A \geq \left|\frac{\partial u}{\partial x_i}(x)\right|^2_A.$$

This finishes the proof. ☐

Theorem 2.11. *Let $1 \leq p < \infty$, $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ and let $K \subset \mathbb{R}^N$ be nonempty, closed and convex. Then for every $i \in \{1,\ldots,n\}$*

$$\frac{\partial \Pi_K u}{\partial x_i}\cdot(u - \Pi_K u) = 0 \quad \textit{a.e. in } \Omega. \tag{2.4}$$

Proof. We proceed similarly as before. By applying twice Theorem 2.2 we have (for $x \in \Omega$ and h small enough so that $x \pm he_i \in \Omega$)

$$\begin{aligned}(\Pi_K u(x) - \Pi_K u(x-he_i))\cdot(u(x) - \Pi_K u(x)) \geq 0,\\ (\Pi_K u(x) - \Pi_K u(x+he_i))\cdot(u(x) - \Pi_K u(x)) \geq 0.\end{aligned}$$

Dividing by h and passing to the limit $h \to 0$, Theorem 2.6 gives for a.e. $x \in \Omega$

$$\begin{aligned}-\frac{\partial \Pi_K u}{\partial x_i}(x)\cdot(u(x) - \Pi_K u(x)) \geq 0,\\ \frac{\partial \Pi_K u}{\partial x_i}(x)\cdot(u(x) - \Pi_K u(x)) \geq 0.\end{aligned}$$

Hence the result. ☐

Now since we will be concerned with parabolic equations, we need also to treat the time-derivative term. Extra care must be taken when dealing with the time derivative, this can be overcome by using convolution in time, as will be demonstrated below.

Lemma 2.12. *Let $u \in \mathcal{C}^1((0,T) \times \Omega, \mathbb{R}^N)$ and let $K \subset \mathbb{R}^N$ be nonempty, closed and convex. Then*

$$(u - \Pi_K u) \cdot \partial_t \Pi_K u = 0 \quad a.e. \text{ in } [0,T] \times \Omega.$$

Proof. For $x \in \Omega$, $t \in (0,T)$ and $0 < h < \min\{t, T-t\}$ it follows from Theorem 2.2 that

$$\begin{aligned} \Big(u(t,x) - \Pi_K u(t,x)\Big) \cdot \Big(\Pi_K u(t,x) - \Pi_K u(t-h,x)\Big) &\geq 0, \\ \Big(u(t,x) - \Pi_K u(t,x)\Big) \cdot \Big(\Pi_K u(t,x) - \Pi_K u(t+h,x)\Big) &\geq 0. \end{aligned}$$

Similarly as before, we divide by h and pass to the limit $h \to 0$ to get for a.e. $t \in (0,T)$

$$\begin{aligned} \Big(u(t,x) - \Pi_K u(t,x)\Big) \cdot \partial_t \Pi_K u(t,x) &\geq 0, \\ -\Big(u(t,x) - \Pi_K u(t,x)\Big) \cdot \partial_t \Pi_K u(t,x) &\geq 0. \end{aligned}$$

This proves the claim. □

Theorem 2.13. *If $u \in W^{1,p}([0,T], W^{-1,p}(\Omega, \mathbb{R}^N)) \cap \mathcal{C}([0,T], L^2(\Omega, \mathbb{R}^N))$, then*

$$\int_{t_1}^{t_2} \langle \partial_t u, u - \Pi_K u \rangle \,\mathrm{d}t = \int_\Omega \frac{|u(t_2) - \Pi_K u(t_2)|^2}{2} \,\mathrm{d}x - \int_\Omega \frac{|u(t_1) - \Pi_K u(t_1)|^2}{2} \,\mathrm{d}x,$$

for $0 \leq t_1 < t_2 \leq T$.

Proof. Formally the proof follows by the last lemma. Since we may not write $\partial_t(\Pi_K u)$, we use a mollifier in time. Let first $0 < t_1 < t_2 < T$. Then (for $\delta < \min\{t_1, T - t_2\}$) and $u_\delta(t,x) := \int u(s,x)\psi_\delta(t-s)\,\mathrm{d}s$, where ψ is the standard mollifier. By the lemma above, we find that

$$\int_{t_1}^{t_2} \langle \partial_t u_\delta, u_\delta - \Pi_K u_\delta \rangle \,\mathrm{d}t = \int_\Omega \frac{|u_\delta(t_2) - \Pi_K u_\delta(t_2)|^2}{2} \,\mathrm{d}x - \int_\Omega \frac{|u_\delta(t_1) - \Pi_K u_\delta(t_1)|^2}{2} \,\mathrm{d}x.$$

Passing with $\delta \to 0$ implies the result by the convergences of the convolution operator (see [Evans, 2010, Chapter 5]). Indeed by Corollary 2.4 we find that $\Pi_K u_\delta \to u_\delta$ in the very same spaces as $u_\delta \to u$. □

3. Convex hull property

This chapter covers our main results on the convex hull property.

3.1 Elliptic systems with diagonal blocks

Definition 3.1 (Elliptic convex hull property). Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. Then we say that u *satisfies the convex hull property on* Ω if $u(x) \in \operatorname{conv} u(\partial\Omega)$ for a.e. $x \in \Omega$. Here the set $\operatorname{conv} u(\partial\Omega)$ must be understood in the sense that it is the smallest closed convex set $C \subset \mathbb{R}^N$ such that $u|_{\partial\Omega}(x) \in C$ for a.e. $x \in \partial\Omega$, where $u|_{\partial\Omega}$ is the trace of u and a.e. is meant with respect to the surface measure on $\partial\Omega$.

Theorem 3.2 (Convex hull property for elliptic diagonal systems). *Suppose that* $A = (A_{ij}^{\alpha\beta})_{i,j=1,\dots,n;\alpha,\beta=1,\dots,N} \in L^\infty(\Omega, \mathbb{R}^{(n\times N)\times(n\times N)})$ *has the form*

$$A_{ij}^{\alpha\beta}(x) = \begin{cases} B_{\alpha\beta} a_i(x), & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

where $B = (B_{\alpha\beta})_{\alpha,\beta=1}^N \in \mathbb{R}^{N\times N}$ *is positive definite and* $a_i \in L^\infty(\Omega)$, $i = 1, \dots, n$ *satisfy* $a_i \geq c > 0$ *a.e., where* c *is a constant. Let* $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ *be a weak solution to the equation*

$$-\operatorname{div}(A\nabla u) = 0 \quad \text{in } \Omega, \tag{3.1}$$

that is,

$$\forall v \in W_0^{1,2}(\Omega, \mathbb{R}^N) : \int_\Omega A\nabla u : \nabla v = 0. \tag{3.2}$$

Then u *satisfies the convex hull property on* Ω.

Proof. Put $K = \operatorname{conv} u(\partial\Omega)$, where this set is understood as in Definition 3.1 and take $w = u - \Pi_K^B u$. Now, since Π_K^B is Lipschitz by Corollary 2.4, Theorem 5.5 gives $w \in W^{1,2}(\Omega, \mathbb{R}^N)$. Moreover, because $u = \Pi_K^B u$ a.e. on $\partial\Omega$, we have even $w \in W_0^{1,2}(\Omega, \mathbb{R}^N)$. So we can use w as a test function in (3.2) and obtain

$$0 = \int_\Omega A\nabla u : \nabla w = \int_\Omega A\nabla(u - \Pi_K^B u) : \nabla(u - \Pi_K^B u) + \int_\Omega A\nabla \Pi_K^B u : \nabla(u - \Pi_K^B u). \tag{3.3}$$

From positive definiteness of B we get $\gamma > 0$ such that

$$\forall \xi \in \mathbb{R}^N : B\xi \cdot \xi \geq \gamma \left|\xi\right|^2.$$

Now we consider the first term in (3.3) and compute, using the positive definiteness of B and $a_i \geq c$

$$\begin{aligned} \int_\Omega A\nabla(u - \Pi_K^B u) \cdot \nabla(u - \Pi_K^B u) &= \int_\Omega \sum_{i=1}^n a_i B \frac{\partial(u - \Pi_K^B u)}{\partial x_i} \cdot \frac{\partial(u - \Pi_K^B u)}{\partial x_i} \\ &\geq c\gamma \int \sum_{i=1}^n \left|\frac{\partial(u - \Pi_K^B u)}{\partial x_i}\right|^2 = c\gamma \left\|\nabla(u - \Pi_K^B u)\right\|_2^2 \geq \tilde{\gamma} \left\|u - \Pi_K^B u\right\|_{1,2}^2, \end{aligned}$$

for some $\tilde{\gamma} > 0$, where the last inequality follows from the equivalence of the norms $\|\nabla(\cdot)\|_2$ and $\|\cdot\|_{1,2}$ on $W_0^{1,2}(\Omega, \mathbb{R}^N)$.

Regarding the second term in (3.3), we use Theorem 2.10 and get

$$\int_\Omega A\nabla\Pi_K^B u : \nabla(u - \Pi_K^B u) = \int_\Omega \sum_{i=1}^n a_i B \frac{\partial \Pi_K^B u}{\partial x_i} \cdot \frac{\partial(u - \Pi_K^B u)}{\partial x_i} \geq 0.$$

Hence, looking back at (3.3) we see that $\left\|u - \Pi_K^B u\right\|_{1,2} \leq 0$, so $u = \Pi_K^B u$ a.e. in Ω and thus $u \in K$ a.e. in Ω. This shows that u satisfies the convex hull property. $\square$

3.2 Nonlinear parabolic systems

Now we will consider parabolic equations. Throughout this section, let $T > 0$ and put $Q = (0, T] \times \Omega$.

Definition 3.3 (Parabolic boundary)**.** We denote by

$$\partial_{\mathrm{par}} Q = (\{0\} \times \Omega) \cup ([0, T] \times \partial\Omega)$$

the *parabolic boundary of* Q.

In the following we consider measurable functions $a\colon [0,T] \times \Omega \times \mathbb{R}^N \times \mathbb{R}^{n\times N} \to \mathbb{R}$, such that

$$\lambda\,|\xi|^{p-2} \leq a(t,x,z,\xi) \leq \Lambda\,|\xi|^{p-2} \text{ for all } (t,x,z,\xi) \in [0,T] \times \Omega \times \mathbb{R}^N \times \mathbb{R}^{n\times N}$$

with some $\lambda, \Lambda > 0$ and

$$\begin{aligned}
&b = (b_1, \ldots, b_n), \text{ where } b_i : [0,T] \times \Omega \times \mathbb{R}^N \times \mathbb{R}^{n\times N} \to \mathbb{R},\\
&|b| \leq C a^{\frac{1}{2}} \text{ for some } C > 0,\\
&c = c\colon [0,T] \times \Omega \times \mathbb{R}^N \times \mathbb{R}^{n\times N} \to [c_0, \infty) \text{ with } c_0 \in \mathbb{R}.
\end{aligned}$$

Now we consider weak solutions to

$$\partial_t u - \operatorname{div}(a\nabla u) + b\nabla u + cu = 0 \quad \text{ in } Q. \tag{3.4}$$

Definition 3.4. We call $u \in L^p([0,T), W^{1,p}(\Omega, \mathbb{R}^N)) \cap \mathcal{C}([0,T), L^2(\Omega, \mathbb{R}^N))$ with $\partial_t u \in L^p([0,T), W^{-1,p}(\Omega, \mathbb{R}^N))$ a *weak solution* to the equation (3.4) if

$$\begin{aligned}
&\int_\Omega u(t_2,x)\varphi(t_2,x)\,\mathrm{d}x - \int_\Omega u(t_1,x)\varphi(t_1,x)\,\mathrm{d}x - \int_{t_1}^{t_2}\int_\Omega u(t,x)\partial_t\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_{t_1}^{t_2}\int_\Omega a(t,x,u(t,x),\nabla u(t,x))\nabla u(t,x) : \nabla\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
&= -\int_{t_1}^{t_2}\int_\Omega \nabla u b(t,x,u(t,x),\nabla u(t,x)) \cdot \varphi + c(t,x,u(t,x),\nabla u(t,x))u(t,x) \cdot \varphi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.5}$$

for all $0 \leq t_1 < t_2 \leq T$ and $\varphi \in \mathcal{C}^1([t_1,t_2], W_0^{1,p}(\Omega))$.

We also need to properly define the convex hull of (parabolic) boundary values $u(\partial_{\mathrm{par}} Q)$. The first part in the definition corresponds to $u(\{0\} \times \Omega)$, the second to $u([0,T] \times \partial\Omega)$.

Definition 3.5. Let $u \in L^p([0,T), W^{1,p}(\Omega, \mathbb{R}^N)) \cap \mathcal{C}([0,T), L^2(\Omega, \mathbb{R}^N))$. Then we define $u(\partial_{\mathrm{par}} Q)$ to be the smallest closed convex set $C \subset \mathbb{R}^N$ satisfying the conditions

(1) $u(0,x) \in C$ for a.e. $x \in \Omega$,

(2) $\forall \delta \in (0,T)$: $u_\delta([0, T-\delta] \times \partial\Omega) \subset C$, where $u_\delta \colon [0, T-\delta] \times \partial\Omega \to \mathbb{R}^N$ is the continuous function defined by $u_\delta = \frac{1}{\delta} \int_t^{t+\delta} u(s)|_{\partial\Omega} \, \mathrm{d}s$, where $u(s)|_{\partial\Omega}$ is the trace of $u(s) \in W^{1,p}(\Omega, \mathbb{R}^N)$.

Theorem 3.6 (Convex hull property for certain nonlinear parabolic systems)**.** *Let $u \in L^p([0,T), W^{1,p}(\Omega, \mathbb{R}^N)) \cap \mathcal{C}([0,T), L^2(\Omega, \mathbb{R}^N))$ with $\partial_t u \in L^p([0,T), W^{-1,p}(\Omega, \mathbb{R}^N))$ be a weak solution to (3.4). Then*

(i) If $c = 0$ a.e. and $K = \operatorname{conv} u(\partial_{\mathrm{par}} Q)$, then $u(Q) \subset K$.

(ii) Otherwise if $K = \operatorname{conv}(u(\partial_{\mathrm{par}} Q) \cup \{0\})$, then $u(Q) \subset K$.

Remark. In the case that $a(t,x,u,\nabla u) = |\nabla u|^{p-2}$, $b = 0$, $c = 0$, we obtain the parabolic p-Lalpace equation

$$\partial_t u - \Delta_p u = \partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0.$$

Proof. We use the test function $u - \Pi_K u \in L^p([0,T), W_0^{1,p}(\Omega, \mathbb{R}^N))$ in the weak formulation (3.5) and obtain for fixed $0 \leq t_2 < T$

$$0 = \int_0^{t_2} \Bigg(\underbrace{\langle \partial_t u(t), (u(t) - \Pi_K u(t)) \rangle}_{(t)} + \underbrace{\int_\Omega a \nabla u(t) : \nabla(u(t) - \Pi_K u(t))}_{(a)}$$
$$+ \underbrace{\int_\Omega (b \nabla u(t)) \cdot (u(t) - \Pi_K u(t))}_{(b)} + \underbrace{\int_\Omega c u(t)(u(t) - \Pi_K u(t))}_{(c)} \Bigg) \, \mathrm{d}t.$$

Now let consider each term separately. Using Theorem 2.13 we find

$$(t) = \langle \partial_t u(t), (u(t) - \Pi_K u(t)) \rangle = \frac{1}{2} \partial_t \int_\Omega |u(t) - \Pi_K u(t)|^2.$$

Further, using Theorem 2.8 and the assumption $a \geq 0$ a.e. we have

$$\begin{aligned}(a) &= \int_\Omega a \nabla(u(t) - \Pi_K u(t)) : \nabla(u(t) - \Pi_K u(t)) + \int_\Omega a \nabla \Pi_K u(t) : (\nabla u(t) - \Pi_K u(t)) \\ &\geq \int_\Omega a \left| \nabla(u(t) - \Pi_K u(t)) \right|^2.\end{aligned}$$

Now we use the Cauchy-Schwarz and Young inequalities and the assumption $|b| \leq Ca^{\frac{1}{2}}$ to obtain for every $\varepsilon > 0$:

$$
\begin{aligned}
(b) &= \int_\Omega (b\nabla(u(t) - \Pi_K u(t))) \cdot (u(t) - \Pi_K u(t)) + \int_\Omega (b\nabla \Pi_K u(t)) \cdot (u(t) - \Pi_K u(t)) \\
&\geq - \int_\Omega |b| \, |\nabla(u(t) - \Pi_K u(t))| \, |u(t) - \Pi_K u(t)| \\
&\quad + \int_\Omega \sum_{i=1}^n b_i \underbrace{\frac{\partial \Pi_K u(t)}{\partial x_i} \cdot (u(t) - \Pi_K u(t))}_{=0 \text{ (Theorem 2.11)}} \\
&\geq -C \int_\Omega a^{\frac{1}{2}} \, |\nabla(u(t) - \Pi_K u(t))| \, |u(t) - \Pi_K u(t)| \\
&\geq -C \int_\Omega \varepsilon a \, |\nabla(u(t) - \Pi_K u(t))|^2 - C \int_\Omega \frac{1}{4\varepsilon} \, |u(t) - \Pi_K u(t)|^2 .
\end{aligned}
$$

Finally, for the last term, we distinguish the two cases from the statement of the theorem

(i) Here $c = 0$ a.e., so clearly $(c) = 0$.

(ii) Here $c \geq c_0 \geq -c_0^-$ a.e. and $0 \in K$, so from Corollary 2.3 we have

$$
\begin{aligned}
(c) = \int_\Omega c(u(t) - \Pi_K u(t)) \cdot (u(t) - \Pi_K u(t)) + c(\Pi_K u(t) - 0) \cdot (u(t) - \Pi_K u(t)) \\
\geq -c_0^- \int_\Omega |u(t) - \Pi_K u(t)|^2 .
\end{aligned}
$$

Either way $(c) \geq -c_0^- \int_\Omega |u(t) - \Pi_K u(t)|^2$. So, putting it all together, we obtain

$$
\begin{aligned}
\frac{1}{2}\partial_t \int_\Omega |u(t) - \Pi_K u(t)|^2 + (1 - C\varepsilon) \int_\Omega a \, |\nabla(u(t) - \Pi_K u(t))|^2 \\
- \left(\frac{C}{4\varepsilon} + c_0^-\right) \int_\Omega |u(t) - \Pi_K u(t)|^2 \leq 0.
\end{aligned}
$$

Now we fix $\varepsilon \leq \frac{1}{C}$, so that $1 - C\varepsilon \geq 0$ and we obtain

$$
\partial_t \int_\Omega |u(t) - \Pi_K u(t)|^2 \leq 2 \left(\frac{C}{4\varepsilon} + c_0^-\right) \int_\Omega |u(t) - \Pi_K u(t)|^2 .
$$

If we denote $\eta(t) := \int_\Omega |u(t) - \Pi_K u(t)|^2$, this reads as

$$
\eta'(t) \leq 2 \left(\frac{C}{4\varepsilon} + c_0^-\right) \eta(t).
$$

Since $\eta(0) = 0$, the Gronwall inequality implies $\eta(t) = 0$, $t \in [0, T)$. Thus $\|u(t) - \Pi_K u(t)\|_{L^2(\Omega, \mathbb{R}^N)} = 0$ for a.e. t, in particular $u(t) = \Pi_K u(t)$ a.e. in Ω for a.e. t, which finishes the proof. $\square$

4. Counterexamples for linear systems

Here we show two counterexamples to the convex hull property. They illustrate that we cannot obtain the convex hull property neither for elliptic linear systems nor for parabolic linear systems, if we do not make any other assumption than ellipticity. In the parabolic case, even the assumption of constant coefficients is not enough.

4.1 Elliptic systems

The projection approach will not in general work for systems of elliptic equations

$$-\mathrm{div}(A\nabla u) = 0, \tag{4.1}$$

where $A\colon \mathbb{R}^n \to \mathbb{R}^{(n\times N)\times(n\times N)}$ is uniformly elliptic, that is

$$\forall \xi \in \mathbb{R}^{n\times N} : A(x)\xi : \xi \geq \alpha\, |\xi|^2 .$$

We illustrate it on the following example

$$\begin{aligned} u_1'' &= 0, \\ (au_1' + u_2')' &= 0, \end{aligned}$$

which is (4.1) for $n = 1$, $N = 2$, $A(x) = \begin{pmatrix} 1 & 0 \\ a(x) & 1 \end{pmatrix}$. It is straightforward to check that the matrix A is elliptic, provided $|a(x)| < 2$. Here it would be desirable to take at every point x the projection $\Pi_K^{A(x)}$ defined by setting $\Pi_K^{A(x)} y = \mathrm{argmin}_{z\in K} |z - y|_{A(x)}$, where $|\ |_{A(x)}$ is the norm defined by the elliptic matrix $A(x)$, that is $|w|_{A(x)} = w_1^2 + \lambda(x)w_1w_2 + w_2^2$. Now the issue is that this family of projections does not satisfy the crucial inequality (analoguos to Theorem 2.10)

$$A(x)\nabla u(x) : \nabla\Pi_K^{A(x)} u(x) \geq A(x)\nabla\Pi_K^{A(x)} u(x) : \nabla\Pi_K^{A(x)} u(x), \tag{4.2}$$

or even the weaker version (obtained from the previous one by Cauchy-Schwarz inequality)

$$|\nabla u(x)|_{A(x)} \geq |\nabla\Pi_K^{A(x)} u(x)|_{A(x)}. \tag{4.3}$$

To see this, take $K = \mathbb{R} \times (-\infty, 0]$ and the constant function $u(x) = (0, 1)$. Then clearly $\nabla u = 0$, but by computation, for $z = (z_1, 0) \in \partial K$ we have

$$|u(x) - z|_{A(x)}^2 = z_1^2 + a(x)z_1 + 1 = (z_1 + a(x)/2)^2 - a(x)^2/4 + 1,$$

so we see that the minimum is attained at $z_1 = -a(x)/2$, so $\Pi_K u(x) = (-a/2, 0)$. Therefore $\Pi_K^{A(\cdot)} u(\cdot)$ is not constant if a is not constant, hence its gradient is nonzero somewhere and (4.3) cannot hold.

So far we have only showed that our particular method fails. But the convex hull property for the solutions of (4.1) under no other assumptions in fact does not hold, as the following example shows.

Example 4.1. Consider (4.1) for $n = 1$, $N = 2$, $A(x) = \begin{pmatrix} 1 & 0 \\ a(x) & 1 \end{pmatrix}$, that is,

$$\begin{aligned} u_1'' &= 0, \\ (a(x)u_1' + u_2')' &= 0. \end{aligned}$$

As we noted above, $A(x)$ is elliptic if $|a(x)| < 2$. We put $a(x) = -x$ and take $\varepsilon \in (0, 1)$, so that $A(x)$ is elliptic on the interval $\Omega = (0, 2\varepsilon)$. Then

$$u_1(x) = x, \quad u_2(x) = \tfrac{1}{2}x^2 - \varepsilon x$$

is a smooth solution which on the interval $\Omega = (0, 2\varepsilon)$ satisfies the boundary condition

$$\begin{aligned} u_1(0) = 0, &\quad u_1(2\varepsilon) = 2\varepsilon, \\ u_2(0) = 0, &\quad u_2(2\varepsilon) = 0. \end{aligned}$$

Then $\operatorname{conv} u(\partial\Omega) = \operatorname{conv}\{u(0), u(2\varepsilon)\} = \{(2\varepsilon t, 0) : t \in [0, 1]\}$, in particular $\operatorname{conv} u(\partial\Omega) \subset \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. But $u_2(x) = x(\frac{1}{2}x - \varepsilon) < 0$ for $x \in \Omega$. Therefore the convex hull property $u(\Omega) \subset \operatorname{conv} u(\partial\Omega)$ is not satisfied. Moreover, notice that u does not even satisfy the weaker "component-wise maximum principle" $u(\Omega) \subset \left[\min_{\partial\Omega} u_1, \max_{\partial\Omega} u_1\right] \times \left[\min_{\partial\Omega} u_2, \max_{\partial\Omega} u_2\right]$.

Remark. We may make the matrix in the above example symmetric. We keep all the notation from Example 4.1. Fix $C > 1$ and consider the system of equations

$$\begin{aligned} (Cv_1' + a(x)v_2')' &= 0, \\ (a(x)v_1' + v_2')' &= 0. \end{aligned}$$

This system corresponds to the symmetric matrix $\begin{pmatrix} C & a(x) \\ a(x) & 1 \end{pmatrix}$, which is elliptic for $|a(x)|^2 < C$. Or, after dividing the first equation by C, this system corresponds to the matrix $\tilde{A}(x) = \begin{pmatrix} 1 & \frac{a(x)}{C} \\ a(x) & 1 \end{pmatrix}$. Let v be a solution to this system on the interval $\Omega = (0, 2\varepsilon)$ that satisfies the same boundary condition as u,

$$\begin{aligned} v_1(0) = 0, &\quad v_1(2\varepsilon) = 2\varepsilon, \\ v_2(0) = 0, &\quad v_2(2\varepsilon) = 0. \end{aligned}$$

Now we have

$$-(Au')' = 0, \quad -(\tilde{A}v')' = 0. \tag{4.4}$$

We want to show that if C is large enough, then v does not satisfy the convex hull property, either.

For this, notice from (4.4) that

$$-(\tilde{A}(u' - v'))' = -(\tilde{A}u')' = -((\tilde{A} - A)u')'$$

Now multiply this by $u - v$, integrate by parts and obtain (recall that $u(0) = v(0)$, $u(2\varepsilon) = v(2\varepsilon)$)

$$\int_0^{2\varepsilon} \tilde{A}(u' - v') \cdot (u' - v') = \int_0^{2\varepsilon} (\tilde{A} - A)u' \cdot (u' - v') = \int_0^{2\varepsilon} \frac{a}{C} u_2'(u_1' - v_1') \tag{4.5}$$

On the left-hand side we use the uniform ellipticity of $\tilde{A}$, so for some $\alpha > 0$

$$\alpha \int_0^{2\varepsilon} |u' - v'|^2 \leq \int_0^{2\varepsilon} \tilde{A}(u' - v') \cdot (u' - v').$$

On the right-hand side we employ the Hölder inequality and also the Young inequality to obtain

$$\int_0^{2\varepsilon} \frac{a}{C} u_2'(u_1' - v_1') \leq \frac{1}{C} \|a\|_\infty \|u'\|_2 \|u' - v'\|_2 \leq \frac{1}{C^2} \|a\|_\infty^2 \frac{1}{2\alpha} \|u'\|_2^2 + \frac{\alpha}{2} \|u' - v'\|_2^2 .$$

Putting this together yields

$$\|u' - v'\|_2 \leq \frac{1}{C} \|a\|_\infty \frac{1}{\alpha} \|u'\|_2$$

Since $\|u'\|_2$ does not depend on C and $u = v$ on the boundary, we find (by the fundamental theorem of calculus) that $\|u - v\|_\infty \leq \|u' - v'\|_2$. Hence with $C \to \infty$ we find that $\|u - v\|_\infty \to 0$. Therefore for C large enough we have a point where v is also negative, so v does not satisfy the convex hull property (or even the component-wise maximum principle), either.

4.2 Parabolic systems

In the case of linear parabolic systems

$$\partial_t u - \operatorname{div}(A \nabla u) = 0$$

we present a counterexample with constant coefficients $A \in \mathbb{R}^{(n \times N) \times (n \times N)}$.

Example 4.2. Consider the following initial/boundary value problem.

$$\partial_t u_1 - \frac{\partial^2 u_1}{\partial x^2} = 0 \quad \text{in } Q = (0, \pi) \times (0, \infty) \tag{4.6}$$

$$\partial_t u_2 - a \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_2}{\partial x^2} = 0 \quad \text{in } Q \tag{4.7}$$

$$u_1(x, 0) = \sin x, \quad x \in (0, \pi) \tag{4.8}$$

$$u_1(0, t) = u_1(\pi, t) = 0, \quad t \in (0, \infty) \tag{4.9}$$

$$u_2(x, 0) = \sin x, \quad x \in (0, \pi) \tag{4.10}$$

$$u_2(0, t) = u_2(\pi, t) = 0, \quad t \in (0, \infty) \tag{4.11}$$

where $0 < a < 2$ is constant. Then the linear system is parabolic, similarly as in the previous section. Now u_1 is a solution of the heat equation (4.6) subject to the initial condition (4.8) and homogeneous boundary conditions (4.9). We obtain, for instance by separation of variables, that

$$u_1(x, t) = e^{-t} \sin x.$$

Plugging this into the second equation (4.7), we have

$$\partial_t u_2 - \frac{\partial^2 u_2}{\partial x^2} = f, \quad \text{where } f(x, t) = a \frac{\partial^2 u_1}{\partial x^2}(x, t) = -a e^{-t} \sin x. \tag{4.12}$$

So u_2 is the solution of the nonhomogeneous heat equation (4.12) subject to the initial condition (4.10) and homogeneous boundary condition (4.11). Hence we obtain, for instance by the method of variation of constants in the homogeneous solution, that

$$u_2(x,t) = e^{-t} \sin x (1 - at).$$

Now for $t_0 > 1/a$ and any $x_0 \in (0,\pi)$ we have $u_2(x_0,t_0) < 0$. But $u(\partial_{\mathrm{par}} Q) \subset [0,1] \times [0,1]$, so necessarily $u(x_0,t_0) \notin \operatorname{conv} u(\partial_{\mathrm{par}} Q)$. (We use the notation $\partial_{\mathrm{par}} Q = ((0,\pi) \times \{0\}) \cup (\{0,\pi\} \times [0,\infty)$.) This shows that u does not satisfy the parabolic convex hull property and, as before, not even the weaker componentwise maximum principle $u(Q) \subset \left[\min_{\partial_{\mathrm{par}} Q} u_1, \max_{\partial_{\mathrm{par}} Q} u_1\right] \times \left[\min_{\partial_{\mathrm{par}} Q} u_2, \max_{\partial_{\mathrm{par}} Q} u_2\right]$.

5. Conclusion

As is demonstrated above, the convex hull property poses a rigid requirement on the coupling of the system. It is curious that we could prove the convex hull property for linear elliptic systems under the (up to an orthogonal change of variables) same condition as Kresin and Maz'ya [2012] did for systems with constant coefficients. This indicates that convex hull property for linear elliptic systems is equivalent to the (a priori weaker) maximum modulus principle. Recall also that our parabolic counterexample was with constant coefficients which again reflects the very same conditions for parabolic systems (with constant coefficients) as in Kresin and Maz'ya [2012] for the maximum modulus principle. This motivates the following

Conjecture. *For linear systems (both elliptic and parabolic), the structure conditions on the coefficients for the convex hull property is equivalent to the one for the maximum modulus principle.*

A further question could be whether these structure conditions are equivalent to a-priori L^∞-estimates. That is, the estimate of the L^∞ norm of u by the L^∞ norm of its boundary values (possibly up to a constant).

As for non-linear equations, we have seen in Theorem 3.6 that a non-linearity (as a scalar multiplier) does not break the convex hull property. So it seems reasonable to conclude with the following take-away message:

It is the structure of the coupling that destroys the convex hull property and not the non-linearity.

Appendix

Here is a summary of the standard results referred to from the thesis. They can be found for instance in Evans [2010] or Gilbarg and Trudinger [2001].

Theorem 5.1 (Equivalent norms on $W_0^{1,p}$)**.** *The two norms* $\|\cdot\|_{1,2}$ *and* $\|\nabla(\cdot)\|_2$ *are equivalent on* $W_0^{1,p}(\Omega, \mathbb{R}^N)$.

Lemma 5.2 (Young's inequality (with ε, $p = 2$))**.** *For* $a, b \in \mathbb{R}$ *and* $\varepsilon > 0$ *it holds that*

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

Theorem 5.3 (Hölder's inequality)**.** *For* $u \in L^p(\Omega, \mathbb{R}^N)$, $v \in L^{p'}(\Omega, \mathbb{R}^N)$, *where* $1/p + 1/p' = 1$, *we have*

$$\|uv\|_1 \leq \|u\|_p \|v\|_{p'}.$$

Theorem 5.4 (Gronwall's inequality)**.** *Let* η *be a nonnegative absolutely continuous function on* $[0, T]$ *and* $\varphi, \psi \in L^1([0, T])$ *satisfy* $\varphi, \psi \geq 0$ *a.e. and*

$$\eta'(t) \leq \varphi(t)\eta(t) + \psi(t), \quad a.e.\ t \in [0, T].$$

Then

$$\eta(t) \leq e^{\int_0^t \varphi} \left(\eta(0) + \int_0^t \psi \right), \quad t \in [0, T].$$

Theorem 5.5 (Composition of $W^{1,p}$ and Lipschitz functions)**.** *Let* $1 \leq p \leq \infty$ *and let* $f \colon \mathbb{R}^m \to \mathbb{R}^k$ *be Lipschitz. Then for any* $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ *it holds that* $f \circ u \in W^{1,p}(\Omega, \mathbb{R}^k)$. *Moreover,*

$$\frac{\partial (f \circ u)}{\partial x_i} = f'(u) \frac{\partial u}{\partial x_i}$$

a.e. on the set where the derivative $f'(u(x))$ *exists.*

Theorem 5.6 (Embedding of Lebesgue spaces on bounded domain)**.** *Let* $1 \leq p < q < \infty$ *and* $|\Omega| < \infty$. *Then*

$$\forall u \in L^p(\Omega) : \|u\|_q \leq |\Omega|^{\frac{1}{p} - \frac{1}{q}} \|u\|_p.$$

Theorem 5.7 (Embeddings of Sobolev spaces)**.**

(i) If $1 \leq p < n$, *then there is a constant* C *such that*

$$\forall u \in W^{1,p}(\Omega) : \|u\|_{\frac{np}{n-p}} \leq C \|u\|_{1,p}.$$

(ii) If $1 \leq q < \infty$, *then there is a constant* C *such that*

$$\forall u \in W^{1,n}(\Omega) : \|u\|_q \leq C \|u\|_{1,n}.$$

Bibliography

Michael Bildhauer and Martin Fuchs. Partial regularity for a class of anisotropic variational integrals with convex hull property. *Asymptotic Analysis*, 32, 2002.

Lars Diening, Christian Kreuzer, and Sebastian Schwarzacher. Convex hull property and maximum principle for finite element minimisers of general convex functionals. *Numerische Mathematik*, 124(4):685–700, 2013.

Lawrence C. Evans. *Partial differential equations.* American Mathematical Society, Providence, R.I, 2010. ISBN 978-0-8218-4974-3.

David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer Berlin Heidelberg, 2001. ISBN 3540411607.

Gershon Kresin and Vladimir Maz'ya. *Maximum Principles and Sharp Constants for Solutions of Elliptic and Parabolic Systems.* American Mathematical Society, 2012. ISBN 0821889818.

Peter Lindqvist. Notes on the p-laplace equation. *Report. University of Jyväskylä Department of Mathematics and Statistics*, 102, 2006.

Peter Tolksdorf. Regularity for a more general class of quasilinear elliptic equations. *Journal of Differential Equations*, 51(1):126 – 150, 1984. ISSN 0022-0396.