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# Limits of classes of finite structures in model theory

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In Prague, 19th July 2019 David Bouška

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Abstract: A limit of a class of structures is an object which captures the asymptotic properties of the class. In this thesis we examine several methods of construction of such objects. We present the necessary theory for each method and then explore a few examples for the sake of a comparison. The methods we chose to examine are Fraïssé's amalgamation, a method based on the compactness theorem and a variant of the forcing method.

Regarding Fraïssé's amalgamation, we prove that Fraïssé's limit of finite linear orders with one unary predicate without any restrictions is isomorphic to rational numbers with a dense and co-dense unary predicate. Regarding the second method, we utilize Ehrenfeucht-Fraïssé games to prove that all models of the limit theory are elementarily equivalent and we observe that it is possible to avoid contact with infinite objects, should we choose to analyze the limit further. In the third main chapter we provide some basic results about the enforceability of predicate density in two types of classes of finite linear orders with a unary predicate.

Keywords: limit, discrete linear orders, Fraïssé's amalgamation, forcing, finite structure, model theory

# Contents

# Introduction

In mathematics, there are several known constructions which come to mind when the word ”limit” is said. The most elementary one is the metric limit, which we get familiar with during an introductory analysis course. We can mention the categorical limit or the ultraproduct from the more advanced ones.

While metric limit is limited to metric spaces, the categorical limit and the ultraproduct both handle structures – categories and indexed families of structures respectively. The aim of this thesis is to explore other available methods of construction of limits of indexed families of structures.

I did my best to keep the preliminary chapter very brief. The main focus was to mention relevant terminology and notation in order to minimize possible confusion later on, as I believe that 10 pages long recap of introduction to first order logic would not target the same group of readers as the rest of the thesis.

The three main chapters of the thesis are built up with a certain format in mind – in each chapter we present necessary theory and our definition of the limit, possibly with some elementary general results and then move on to examples in hopes to get a better idea of how the limit construction behaves in common situations.

# 1. Preliminaries

## 1.1 A summary of basics of first-order logic

A *language*, usually denoted by $\mathcal{L}$, is a triplet of disjoint sets $(F_\mathcal{L}, R_\mathcal{L}, C_\mathcal{L})$ – the set of *function symbols*, the set of *relational symbols* and the set of *constant symbols* – along with an *arity* function $n : R_\mathcal{L} \cup C_\mathcal{L} \to \mathbb{N}^+$.

The set of all $\mathcal{L}$-*terms* is the smallest set which contains every constant from $C_\mathcal{L}$, every variable and which is closed upon composition of terms: if $f$ is a function symbol and $t_1, \ldots t_{n(f)}$ are terms, then $f(t_1, \ldots, t_{n(f)})$ is a term.

The set of *atomic* $\mathcal{L}$-formulas consists of strings of the form $s = t$, where $s, t$ are $\mathcal{L}$-terms and of strings of the form $R(t_1, \ldots t_{n(R)})$, where $R$ is a relational symbol and $t_1, \ldots t_{n(R)}$ are $\mathcal{L}$-terms.

The set of all $\mathcal{L}$-formulas is the smallest set containing all atomic $\mathcal{L}$-formulas which is closed upon logical connectives and quantifiers - it contains $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$, $\phi \to \psi$, $\phi \leftrightarrow \psi$, $\forall x : \phi$ and $\exists x : \phi$ whenever it contains $\phi$ and $\psi$. An $\mathcal{L}$-formula is called an $\mathcal{L}$-*sentence* if it only contains variables which are quantified over by $\forall$ or $\exists$.

An $\mathcal{L}$-*structure* $\mathbb{A}$ is a tuple consisting of a set $A$, a function $f^\mathbb{A} : A^{n(f)} \to A$ for each function symbol $f \in F_\mathcal{L}$, a relation $R^\mathbb{A} \subseteq A^{n(R)}$ for each relational symbol $R \in R_\mathcal{L}$ and an element $c^\mathbb{A} \in A$ for each constant symbol $c \in C_\mathcal{L}$. *Truth value* of a formula in a structure is defined inductively by Tarski's definition of truth.

An $\mathcal{L}$-*theory* is a set of $\mathcal{L}$-sentences. A structure $\mathbb{A}$ is a *model* of theory $T$ if it satisfies every sentence of $T$. $\mathrm{Th}_\mathcal{L}(\mathbb{A})$ denotes the set of all $\mathcal{L}$-sentences which are true in $\mathbb{A}$.

We say that $\phi$ is a *consequence* of a theory $T$ if every model of $T$ satisfies $\phi$. An $\mathcal{L}$-theory $T$ is said to be *complete* if either $\phi$ is a consequence of $T$ or $\neg\phi$ is a consequence of $T$ for every $\mathcal{L}$-sentence $\phi$.

We will rely on one classical result of mathematical logic, which is presented during most introductory courses in the subject.

**Theorem 1** (The compactness theorem)**.** *A theory $T$ has a model if and only if every finite subtheory $S \subseteq T$ has a model.*

We will also use the notion of embeddings. An embedding of an $\mathcal{L}$-structure $\mathbb{A}$ into an $\mathcal{L}$-structure $\mathbb{B}$ is an injective map $F : A \to B$ which preserves functions, relations and constants of $\mathbb{A}$, that is, for every choice of terms $t_i$, every relational symbol $R$, every function symbol $f$ and every constant symbol $c$ we have:

- $R^\mathbb{B}(F(t_1), \ldots, F(t_{n(R)}))$ iff $R^\mathbb{A}(t_1, \ldots, t_{n(R)})$,
- $f^\mathbb{B}(F(t_1), \ldots, F(t_{n(f)})) = F(f^\mathbb{A}(t_1, \ldots, t_{n(f)}))$,
- $c^\mathbb{B} = F(c^\mathbb{A})$.

An isomorphism is a surjective embedding.

## 1.2 Discrete linear orders with endpoints

The theme of the entire thesis is an analysis of different methods of limit construction through examples. We will spend a lot of time with linear orders.

Let $\mathcal{L}$ be the language of orders, consisting of a single binary relational symbol $\leq$. Sometimes we will write $x \geq y$ instead of $y \leq x$ and we shall abbreviate $x \leq y \wedge x \neq y$ by $x < y$.

The theory of discrete linear orders with endpoints, denoted by $DiLO_{ep}$, consists of the following axioms:

(REF) $\forall x : x \leq x$

(WA) $\forall x, y : (x \leq y \wedge x \geq y) \longrightarrow x = y$

(TR) $\forall x, y, z : (x \leq y \wedge y \leq z) \longrightarrow x \leq z$

(LIN) $\forall x, y : (x \leq y \vee x \geq y)$

(MIN) $\exists x \forall y : x \leq y$

(MAX) $\exists x \forall y : x \geq y$

(S) $\forall x : (\exists y : x < y) \longrightarrow \exists y : (x < y \wedge (\forall z : x < z \longrightarrow y \leq z)))$

(P) $\forall x : (\exists y : x > y) \longrightarrow \exists y : (x > y \wedge (\forall z : x > z \longrightarrow y \geq z)))$

The two elements asserted to exist in axioms (MIN) and (MAX) are uniquely determined and we shall refer to them as **MIN** and **MAX** respectively, however, we will not consider these symbols as a part of our language. Axioms (S) and (P) allow us to define two partial functions in any model of the theory. If $a$ is not the minimum element of a model, then by $p(a)$ we denote the predecessor of $a$ and if $b$ is not the maximum element of a model, then by $s(b)$ we denote the successor of $b$. By $p^n$ and $s^n$ we denote the $n$ times repeated application of functions $p$ and $s$, respectively. Let us observe that for every (except the minimum) element $a$ of any model of the theory we have $s(p(a)) = a$ and similarly for every (except the maximum) element $b$ of any model of the theory we have $p(s(b)) = b$.

Throughout the entire thesis we will denote the finite linearly ordered set $1 < 2 < \cdots < i$ by $L_i$.

# 2. Fraïssé's amalgamation

In this chapter we will summarize the theory related to Fraïssé's amalgamation and we will go through two examples of application of the theory.

## 2.1 The theory

The entire section is based upon [2, pp.158-165]. In the following sections we will only apply the theory to relational structures. However, we do not save ourselves a meaningful amount of work by limiting the theory to those and so, in this regard, we present the theory in general.

**Definition 2.** *Let $\mathcal{L}$ be a language and let $\mathbb{D}$ be an $\mathcal{L}$-structure. The* age *of $\mathbb{D}$ is the class of all finitely generated structures embeddable into $\mathbb{D}$.*

*A class of structures is called an age if it is an age of some structure.*

Ages have several note-worthy properties. Two of them are the following:

**(HP)** The hereditary property:

If $\mathbb{K}$ is an age, $\mathbb{A}$ is a structure in $\mathbb{K}$ and $\mathbb{B}$ is a finitely generated structure embeddable into $\mathbb{A}$, then $\mathbb{B}$ is also in $\mathbb{K}$.

**(JEP)** The joint-embedding property:

If $\mathbb{K}$ is an age and $\mathbb{A}, \mathbb{B}$ are structures in $\mathbb{K}$, then there is a structure $\mathbb{C}$ in $\mathbb{K}$ such that both $\mathbb{A}$ and $\mathbb{B}$ are embeddable into $\mathbb{C}$.

According to the following theorem, we can characterize ages of countable structures by these two properties.

**Theorem 3** (Fraïssé, 1954)**.** *Let $\mathcal{L}$ be a language and let $\mathbb{K}$ be a nonempty, up to isomorphisms countable class of finitely generated $\mathcal{L}$-structures. If $\mathbb{K}$ satisfies the properties* (**HP**) *and* (**JEP**)*, then $\mathbb{K}$ is the age of a countable $\mathcal{L}$-structure.*

As the text so far suggests, we are interested not in individual structures, but rather in their isomorphism types. "The class $\mathbb{K}$ is up to isomorphisms countable" is short for "there exist countably many structures which represent isomorphism types of all structures in the class $\mathbb{K}$".

The proof of Theorem **??** is presented in [2]. It provides some guidance to constructing the advertised structure and so we have an opportunity to regard this structure as the "limit" of the class. We can, however, achieve some sort of uniqueness result on top of this existential one before we define the limit.

**Definition 4.** *Let $\mathcal{L}$ be a language and let $\mathbb{D}$ be an $\mathcal{L}$-structure. We say that $\mathbb{D}$ is* homogeneous *if every isomorphism between any two finitely generated substructures of $\mathbb{D}$ can be extended to an automorphism of $\mathbb{D}$.*

Let us note that in [2] the author uses the term *ultrahomogeneous*. The property that Fraïssé noticed during the study of linear orders and which allows for the desired uniqueness result is the following:

**(AP)** The amalgamation property:

> If $\mathbb{K}$ is an age, $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ are structures in $\mathbb{K}$ and $b : \mathbb{A} \to \mathbb{B}$ and $c : \mathbb{A} \to \mathbb{C}$ are embeddings, then there is a structure $\mathbb{D}$ in $\mathbb{K}$ and embeddings $b' : \mathbb{B} \to \mathbb{D}$ and $c' : \mathbb{C} \to \mathbb{D}$ such that $b' \circ b = c' \circ c$.

Not every age satisfies this property, but it is not hard to show that ages of homogeneous structures do satisfy it.

**Proposition 5.** *If $\mathbb{D}$ is a homogeneous structure, then the age of $\mathbb{D}$ satisfies* (**AP**)*.*

*Proof.* Let $\mathbb{K}$ be the age of $\mathbb{D}$, let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ be structures from $\mathbb{K}$ and let $b : \mathbb{A} \to \mathbb{B}$ and $c : \mathbb{A} \to \mathbb{C}$ be our chosen embeddings. These embeddings provide us with an isomorphism $c \circ b^{-1} : \mathrm{Im(b)} \to \mathrm{Im(c)}$. We can extend $c \circ b^{-1}$ to an automorphism of $\mathbb{D}$ because $\mathbb{D}$ is homogeneous. Let us call this automorphism $f$.

Define $b'$ as $f_{|B}$ and $c'$ as the canonical embedding of $\mathbb{C}$ into $\mathbb{D}$. Then $b' : \mathbb{B} \to \mathbb{D}$ and $c' : \mathbb{C} \to \mathbb{D}$ are injective mappings. Verification of $b' \circ b = c' \circ c$ is easy, as $b'_{|\mathrm{Im(b)}} = (f_{|B})_{|\mathrm{Im(b)}} = f_{|\mathrm{Im(b)}} = c \circ b^{-1}$. Then $b' \circ b = c \circ b^{-1} \circ b = c$ and $c' \circ c = c$. □

One of the results that Fraïssé demonstrated was that properties (**HP**), (**JEP**) and (**AP**) together form a characterization of ages of homogeneous structures in the countable case and that such ages determine their countable structures up to an isomorphism.

**Theorem 6** (Fraïssé, 1954)**.** *Let $\mathcal{L}$ be a language and let $\mathbb{K}$ be a nonempty, up to isomorphisms countable class of finitely generated $\mathcal{L}$-structures. If $\mathbb{K}$ satisfies properties* (**HP**)*,* (**JEP**) *and* (**AP**)*, then $\mathbb{K}$ is the age of a countable, homogeneous structure which is uniquely determined up to an isomorphism.*

The proof of this theorem is also presented in [2]. This uniquely determined structure is generally called *the Fraïssé limit* of the class $\mathbb{K}$ and we shall denote it by $\lim_F \mathbb{K}$.

Again, the proof itself provides some guidance to constructing the limit. There is, however, a less straightforward approach to the matter which, in some cases, saves us a bit of work: Given a nonempty, up to isomorphisms countable class of $\mathcal{L}$-structures $\mathbb{K}$ which satisfies (**HP**), (**JEP**) and (**AP**), we can guess what the limit is, call it $\mathbb{D}$, and verify that

- $\mathbb{D}$ is countable
- $\mathbb{D}$ is homogeneous
- $\mathbb{K}$ is the age of $\mathbb{D}$

The uniqueness provided by the theorem guarantees that our structure is the desired limit (up to an isomorphism).

## 2.2 Fraïssé's limit of finite linear orders

We will show that the ordered set of rational numbers is the Fraïssé limit of the class of finite linear orders. For the entire section let $\mathcal{L}$ be a language of one binary relational symbol $\leq$, and let $\mathbb{K}$ denote the class of all finite linear orders, that is, of all structures isomorphic to $L_i$ for some $i \in \mathbb{N}^+$.

Let us note that if a language has no function symbols and finitely many constant symbols, then for any structure in that language the terms "finite" and "finitely generated" are synonymous.

We shall now proceed to the verification of properties of the class $\mathbb{K}$.

**Lemma 7.** *The class $\mathbb{K}$ is nonempty, up to isomorphisms countable and it satisfies* (**HP**) *and* (**JEP**).

*Proof.* The class is obviously nonempty and the map from $\mathbb{K}$ to $\mathbb{N}$ which assigns to each given structure its size clearly uniquely determines the isomorphism type.

The satisfaction of both properties is also easy to verify. For (**HP**) let us choose a finite structure $\mathbb{A}$ embeddable into some $L_i$. This structure has finitely many, say $j$, elements and therefore $\mathbb{A}$ is isomorphic to $L_j$.

For (**JEP**) let us choose two structures $L_i$ and $L_j$ and observe that they can be embedded into $L_{max\{i,j\}}$. □

The verification of (**AP**) might be a bit trickier, depending on whether we choose to use category theory or not. There is a short, high-level proof which consists of diagram chasing and a longer, elementary proof which utilizes induction and that is the proof which I chose to present.

In order to avoid awkward wording we shall allow arithmetic and ordering of elements of different finite linear orders by canonically embedding every one of them into natural numbers: Suppose $\mathbb{A} = \{a_1 < \cdots < a_m\}$, $\mathbb{B} = \{b_1 < \cdots < b_n\}$. Then the expression $a_i < b_j$ is to be read as $F_{\mathbb{A}}(a_i) < F_{\mathbb{B}}(b_j)$ where $F_{\mathbb{A}}$ is an embedding of $\mathbb{A}$ into $\mathbb{N}^+$ defined by $F_{\mathbb{A}} : a_i \mapsto i$ and $F_{\mathbb{B}}$ is an embedding of $\mathbb{B}$ into $\mathbb{N}^+$ defined by $F_{\mathbb{B}} : b_i \mapsto i$. Similarly for arithmetic: the map definition

$$b' : i \mapsto i + c(a) - b(a)$$

in the proof below is short for

$$b' : i \mapsto F_{\mathbb{D}}^{-1}(F_{\mathbb{B}}(i) + F_{\mathbb{C}}(c(a)) - F_{\mathbb{B}}(b(a))),$$

where $F_{\mathbb{B}}, F_{\mathbb{C}}, F_{\mathbb{D}}$ are canonical embeddings of $\mathbb{B}, \mathbb{C}, \mathbb{D}$ into $\mathbb{N}^+$.

**Lemma 8.** *The class $\mathbb{K}$ satisfies* (**AP**).

*Proof.* We shall prove this lemma by induction on the size of the structure $\mathbb{A}$. First let $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ be structures in $\mathbb{K}$, let $b : \mathbb{A} \to \mathbb{B}$ and $c : \mathbb{A} \to \mathbb{C}$ be embeddings and assume that $\mathbb{A} = \{a\}$.

Without loss of generality we can assume that $b(a) \leq c(a)$. In that case we define $\mathbb{D}$ to be a linear order of size $\max\{|\mathbb{C}|, |\mathbb{B}| + c(a) - b(a)\}$ and we also define maps $b' : \mathbb{B} \to \mathbb{D}$ and $c' : \mathbb{C} \to \mathbb{D}$ by

$$b' : i \mapsto i + c(a) - b(a),$$

$$c' : i \mapsto i$$

It is straightforward to verify that the structure $\mathbb{D}$ together with embeddings $b', c'$ forms the desired amalgam - $b' \circ b(a) = b(a) + c(a) - b(a) = c(a)$.

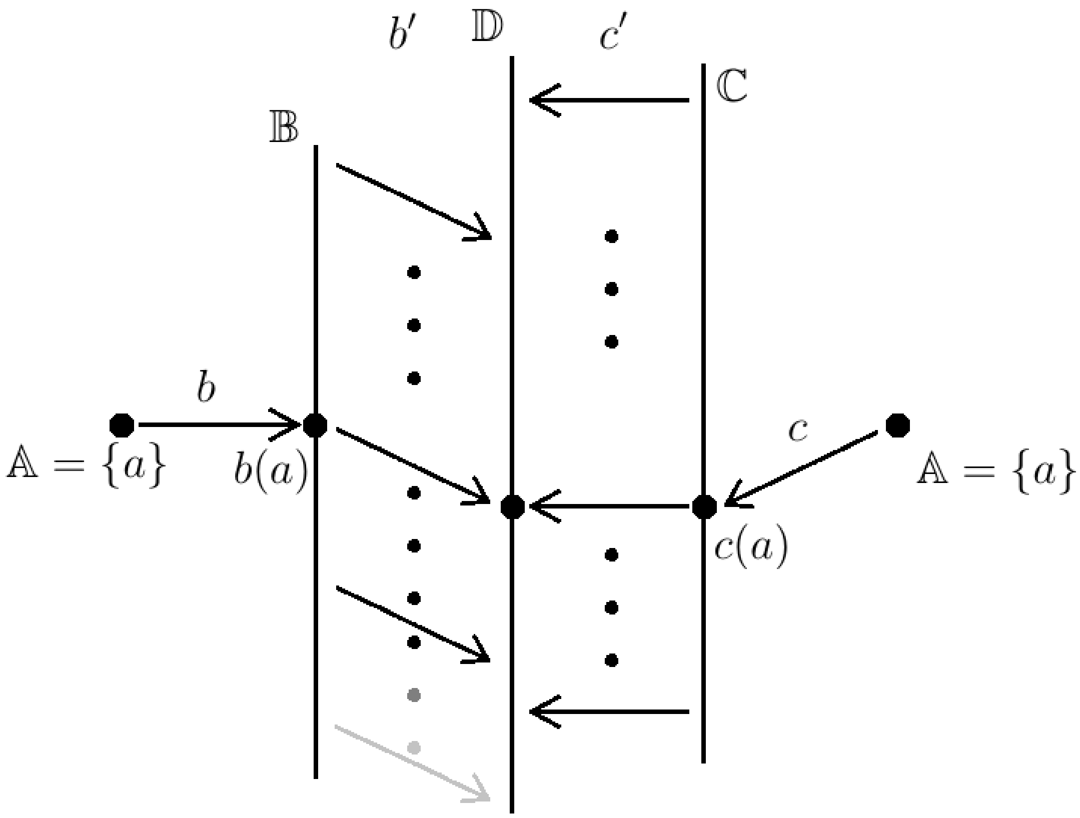
Now let $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ be structures in $\mathbb{K}$, let $b : \mathbb{A} \to \mathbb{B}$ and $c : \mathbb{A} \to \mathbb{C}$ be embeddings and suppose that $\mathbb{A} = \{a_1 \leq \cdots \leq a_n\}$. Then we denote by $\mathbb{A}_1$ the ordered set $\{a_1 \leq \cdots \leq a_{n-1}\}$, by $\mathbb{B}_1$ the ordered set $\{x \in \mathbb{B} | x \leq b(a_{n-1})\}$ and by $\mathbb{C}_1$ the ordered set $\{x \in \mathbb{C} | x \leq c(a_{n-1})\}$.

In accord with the induction hypothesis, let $\mathbb{D}_1$ along with embeddings $c_1', d_1'$ form an amalgam of structures $\mathbb{A}_1, \mathbb{B}_1, \mathbb{C}_1$ with embeddings $b|_{\mathbb{A}_1}$ and $c|_{\mathbb{A}_1}$. Similarly let us denote by $\mathbb{A}_2$ the singleton $\{a_n\}$ and by $\mathbb{B}_2$ and $\mathbb{C}_2$ the ordered sets $\{x \in \mathbb{B} | x > b(a_{n-1})\}$ and $\{x \in \mathbb{C} | x > c(a_{n-1})\}$ respectively. Then, as we did in the first part of the proof, we construct the structure $\mathbb{D}_2$ along with embeddings $b_2'$ and $c_2'$ which forms an amalgam of the structures $\mathbb{A}_2, \mathbb{B}_2, \mathbb{C}_2$ with embeddings $b|_{\mathbb{A}_2}$ and $c|_{\mathbb{A}_2}$.

Let $\mathbb{D}$ be the disjoint union of $\mathbb{D}_1$ and $\mathbb{D}_2$ which inherits orders from those structures and orders elements of $\mathbb{D}_1$ to be less than elements of $\mathbb{D}_2$ and let $b'$ and $c'$ be maps combined from $b_1', b_2'$ and $c_1', c_2'$ respectively. Then it is, again, easy to verify that the resulting structure $\mathbb{D}$ along with embeddings $b'$ and $c'$ forms an amalgam of $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ with embeddings $b, c$.

□

The picture that I usually draw when I want to present the first part of the proof to someone is the following:




Now we know that the Fraïssé limit of the class $\mathbb{K}$ exists. Next we show that the ordered set of rational numbers, as usual denoted by $(\mathbb{Q}, \leq)$, has the aforementioned properties.

The fact that $\mathbb{Q}$ is countable is widely known and finite structures embeddable into $\mathbb{Q}$ correspond to finite linear orders – to the class $\mathbb{K}$.

**Lemma 9.** *The structure* $(\mathbb{Q}, \leq)$ *is homogeneous.*

*Proof.* Let $\mathbb{A}$ and $\mathbb{B}$ be finite substructures of $(\mathbb{Q}, \leq)$ and let $f : \mathbb{A} \to \mathbb{B}$ be an isomorphism between them. Then both $\mathbb{A}$ and $\mathbb{B}$ are isomorphic to some $L_m$. Let us list elements of $\mathbb{A}$ and $\mathbb{B}$ in two increasing sequences $a_1, \dots, a_m$ and $b_1, \dots, b_m$ and for each $i \in \{1, \dots, m-1\}$ let us define a monotonically increasing bijection $f_i : [a_i, a_{i+1}]_{\mathbb{Q}} \to [b_i, b_{i+1}]_{\mathbb{Q}}$ by

$$f_i(x) := b_i + \frac{b_{i+1} - b_i}{a_{i+1} - a_i}(x - a_i)$$

and also monotonically increasing bijections $f_0 : (-\infty, a_0]_{\mathbb{Q}} \to (-\infty, b_0]_{\mathbb{Q}}$ and $f_m : [a_m, \infty)_{\mathbb{Q}} \to [b_m, \infty)_{\mathbb{Q}}$ by

$$f_0(x) := x - a_0 + b_0$$

$$f_m(x) := x - a_m + b_m$$

It is trivial to verify that these functions can be glued together to form an automorphism $F : (\mathbb{Q}, \leq) \to (\mathbb{Q}, \leq)$ which extends $f$. □

Now we may end the section by stating the theorem which we desired to prove.

**Theorem 10.** *Let* $\mathbb{K}$ *be the class of all finite linear orders. Then* $\lim_F \mathbb{K}$ *is isomorphic to* $(\mathbb{Q}, \leq)$*.*

*Proof.* In Lemmas 7 and 8 we have shown that $\mathbb{K}$ satisfies the assumptions of Theorem 6 and so there exists $\lim_F \mathbb{K}$. In Lemma 9 we have shown that $(\mathbb{Q}, \leq)$ is homogeneous and we have commented on countability of $\mathbb{Q}$ and on $\mathbb{K}$ being the age of $(\mathbb{Q}, \leq)$ in the paragraph before that lemma. The uniqueness part of Theorem 6 then guarantees us that $(\mathbb{Q}, \leq)$ is the Fraïssé limit of $\mathbb{K}$, up to an isomorphism. □

## 2.3 Fraïssé's limit of finite linear orders with a unary predicate

Our second example involves introducing a unary predicate to finite linear orders. For the rest of this section let $\mathcal{L}$ be a language consisting of two relational symbols: binary $\leq$ and unary $U$.

By $(L_i, P)$ we shall denote the structure $(\{1, \dots, i\}, \leq, P)$, where $\leq$ is the usual linear order of $\{1, \dots, i\}$ and $P \subseteq \{1, \dots, i\}$ is a unary predicate. Let $\mathbb{K}$ denote the class of all structures isomorphic to some $(L_i, P)$. We will demonstrate that the Fraïssé limit of $\mathbb{K}$ is $(\mathbb{Q}, \leq, P)$, where $P$ is dense and co-dense.

We shall begin by verifying that the class $\mathbb{K}$ satisfies the desired properties.

**Lemma 11.** *The class* $\mathbb{K}$ *is nonempty, up to isomorphisms countable and it satisfies the properties* (**HP**) *and* (**JEP**)*.*

*Proof.* The class is obviously nonempty.

For countability let us observe that each structure $(L_i, P)$ determines a unique sequence over the alphabet $\{0,1\}$ – the $j$-th letter of a sequence is determined by whether the element $j \in \{1,\dots,i\}$ satisfies $P$ or not. A map defined by $w \mapsto 1w$ is injective and each sequence of the form $1w$ encodes a unique natural number and therefore the class $\mathbb{K}$ is countable.

The property (**HP**) is satisfied because if $\mathbb{A}$ is a finite structure embeddable into $(L_i, P)$, say with $j$ elements, then $\mathbb{A}$ is isomorphic to $(L_j, R)$ for some unary predicate $R$ and therefore $\mathbb{A}$ belongs to $\mathbb{K}$.

For (**JEP**) let us choose two structures $(L_i, P)$ and $(L_j, R)$ and observe that they can be embedded into $(L_{i+j}, S)$, where $S(a)$ is defined as $P(a)$ for $a \in \{1,\dots,i\}$ and as $R(a-i)$ for $a \in \{i+1,\dots,i+j\}$. □

**Lemma 12.** *The class* $\mathbb{K}$ *satisfies* (**AP**).

*Proof.* We will prove this claim by induction on the size of $\mathbb{A}$. First let $\mathbb{A}, \mathbb{B}, \mathbb{C}$ be structures in $\mathbb{K}$, let $b : \mathbb{A} \to \mathbb{B}$ and $c : \mathbb{A} \to \mathbb{C}$ be embeddings and assume that $\mathbb{A} = \{a\}$. We define $\mathbb{D}$ to be the linear order of size $|\mathbb{B}| + |\mathbb{C}| - 1$ with a predicate defined by

- $U^{\mathbb{B}}(i)$ for $i \in \{1,\dots,b(a)-1\}$
- $U^{\mathbb{C}}(i-(b(a)-1))$ for $i \in \{b(a),\dots,b(a)-1+c(a)-1\}$
- $U^{\mathbb{B}}(b(a))$ (or, equivalently, as $U^{\mathbb{C}}(c(a))$) for $i = b(a)+c(a)-1$
- $U^{\mathbb{B}}(i-(c(a)-1))$ for $i \in \{b(a)+c(a),\dots,|\mathbb{B}|+c(a)-1\}$
- $U^{\mathbb{C}}(i-|\mathbb{B}|+1)$ for $i \in \{|\mathbb{B}|+c(a)\dots,|\mathbb{B}|+|\mathbb{C}|-1\}$

It is easy to see that the structure $\mathbb{D}$ along with embeddings $b'$ and $c'$ defined by

$$b' : i \mapsto \begin{cases} i & \text{if } i < b(a) \\ i + c(a) - 1 & \text{otherwise} \end{cases},$$

$$c' : i \mapsto \begin{cases} i + b(a) - 1 & \text{if } i < b(a) \\ i + |\mathbb{B}| - 1 & \text{otherwise} \end{cases},$$

forms an amalgam of $\mathbb{A}, \mathbb{B}, \mathbb{C}$ with embeddings $b, c$.

The induction step works in exactly the same fashion as in the proof of Lemma 8. □

We say that the predicate $U^{\mathbb{A}}$ of an $\mathcal{L}$-structure $\mathbb{A}$ is *dense* if $\mathbb{A}$ satisfies the following first-order sentence:

$$\forall x, y : (x < y \longrightarrow \exists z : x < z < y \wedge U(z)).$$

We say that a predicate is *co-dense* whenever its complement is dense.

Let us denote the structure $(\mathbb{Q}, \leq, P)$ by $\mathbb{Q}_P$; $\leq$ is, again, the usual linear order on $\mathbb{Q}$ and $P$ is a unary predicate on $\mathbb{Q}$. In order to prove that $\lim_F \mathbb{K}$ is $\mathbb{Q}_P$ where $P$ is a dense and co-dense unary predicate, we first need to show that some such predicate exists.

**Proposition 13.** *There exists a unary predicate on $\mathbb{Q}$ which is dense and co-dense.*

*Proof.* It suffices to show that there exists such predicate on $[0,1]_{\mathbb{Q}}$.

Define
$$P := \{\frac{p}{2^k} | k \in \mathbb{N}^+, p \in \{1,3,5,\ldots,2^k-1\}\},$$
$$P' := \{\frac{p}{3^k} | k \in \mathbb{N}^+, p \in \{1,2,4,5,7,8,\ldots,3^k-1\}\}.$$
We will show that $P$ and $P'$ are dense and disjoint.

Suppose that $a,b \in [0,1]_{\mathbb{Q}}$ satisfy $a < b$. Then there is $k \in \mathbb{N}$ such that $\frac{1}{2^k} < b-a$. Let $p$ be the smallest odd natural number such that $\frac{p}{2^{k+2}} > a$. It is clear that $\frac{p}{2^{k+2}} \in P$ and if $\frac{p}{2^{k+2}} \geq b$, then $p-2$ would be a smaller number which would still satisfy our two conditions, therefore $a < \frac{p}{2^{k+2}} < b$, and so $P$ is dense.

An analogous argument can be made for the density of $P'$.

Now suppose that $x \in P \cap P'$, that is $x = \frac{p}{2^k}$ for some odd $p$ and $x = \frac{q}{3^l}$ for some $q$ not divisible by 3. If $3^l \cdot p = 2^k \cdot q$, where $p$ is odd, then $k \leq 0$ and so $\frac{p}{2^k} \notin P$, which is a contradiction.

□

We shall now prove that $\mathbb{Q}_P$ satisfies the properties of the Fraïssé limit if $P$ is dense and co-dense. We have already established that rational numbers form a countable structure and it is straightforward to verify that $\mathbb{K}$ is the age of $\mathbb{Q}_P$, as a given finite structure embeddable in $\mathbb{Q}_P$ consisting of elements $a_1 < \cdots < a_n$, of which $a_{r_1}, \ldots a_{r_m}$ satisfy $P$, is isomorphic to $(L_n, R)$, where $R = \{r_1, \ldots, r_m\}$.

**Lemma 14.** *Let $P$ be a dense and co-dense unary predicate on $\mathbb{Q}$ and let $a, b \in \mathbb{Q}$ satisfy $a < b$. Then $(a,b)_{\mathbb{Q}}$ contains infinitely many elements satisfying $P$ and infinitely many elements not satisfying $P$.*

*Proof.* Let us assume there are only finitely many elements in $(a,b)_{\mathbb{Q}}$ satisfying $P$. If there was only one such element $x$, then the predicate $P$ would not be dense because there would be no $P$-element between $a$ and $x$.

If there exist multiple $P$-elements, let us list them all in an increasing order as $x_1, \ldots, x_n$ and observe that there has to be an element $a \in (x_1, x_2)$, which was not listed among $x_1, \ldots x_n$. This is a contradiction.

An analogous argument can be made for the existence of infinitely many elements not in $P$. □

**Theorem 15.** *If $P$ is a dense and co-dense unary predicate on $\mathbb{Q}$, then $\mathbb{Q}_P$ is homogeneous.*

*Proof.* Let $\mathbb{A}$ and $\mathbb{B}$ be finite substructures of $\mathbb{Q}_P$ and let $f : \mathbb{A} \to \mathbb{B}$ be an isomorphism between them. We shall construct an automorphism of $\mathbb{Q}_P$ which extends $f$ using the well known back-and-forth method. Let us list the elements of $\mathbb{A}$ and $\mathbb{B}$ in an increasing order as $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ and extend these lists to enumerations of rational numbers $a_1, \ldots$ and $b_1, \ldots$. We will define two sequences of maps $f_m, f_{m+1}, \ldots$ and $f'_m, f'_{m+1}, \ldots$ in order to construct the desired automorphism. Define the map $f_m$ as the isomorphism $f$ and for each natural $i > m$ where $a_i$ is not in the domain of $f_{i-1}$ define

$$f_i' := f_{i-1} \cup \{(a_i, y)\},$$

where $y$ is an element of $\mathbb{Q}$ satisfying the following properties:

- $y$ is not in the image of $f_{i-1}$
- The addition of the pair $(a_i, y)$ preserves the monotonicity of the map
- $P(a_i) \longleftrightarrow P(y)$

The existence of such element is guaranteed by the previous lemma. If $a_i$ happens to be in the domain of $f_{i-1}$ we set $f_i' := f_{i-1}$.

Similarly, for each natural $i > m$ where $b_i$ is not in the image of $f_i'$, we define

$$f_i := f_i' \cup \{(x, b_i)\},$$

where $x$ is an element of $\mathbb{Q}$ satisfying the following properties:

- $x$ is not in the domain of $f_{i-1}$
- The addition of the pair $(x, b_i)$ preserves the monotonicity of the map
- $P(x) \longleftrightarrow P(b_i)$

The existence of such element is, again, guaranteed by the previous Lemma. If $b_i$ happens to be in the image of $f_i'$, we set $f_i := f_i'$.

Define $F$ to be the union of $f_i$ across all natural $i \geq m$. It remains to show that $F$ is bijective, order-preserving and $P$-preserving. $F$ is $P$-preserving because if there was a pair $(x, y)$ in $F$ such that $P(x), \neg P(y)$ or $\neg P(x), P(y)$, then that pair could be found in some $f_i$, which is not allowed in our construction. Similarly if there were two pairs $(x, y), (t, u)$ in $F$ such that $x < t$ and $y > u$ or $x > t$ and $y < u$, then those pairs could be found in some $f_i$, which is, again, not allowed in the construction. The fact that $F$ is a bijection is easy to see from the way the construction was set up – the domain of $F$ contains each element from the list $a_1, a_2 \ldots$ and the image of $F$ contains each element from the list $b_1, b_2, \ldots$, both of which enumerate all of rational numbers. □

We can now state the main theorem of this section:

**Theorem 16.** *Let $\mathbb{K}$ be the class of all structures isomorphic to $(L_i, P)$ for some positive integer $i$ and some $P \subseteq \{1, \ldots, i\}$. Then $\lim_F \mathbb{K}$ is isomorphic to $\mathbb{Q}_P$, where $P$ is a dense and co-dense unary predicate.*

*Proof.* In Lemmas 11 and 12 we have shown that the class $\mathbb{K}$ satisfies the assumptions of Theorem 6 and so there exists $\lim_F \mathbb{K}$. In Theorem 15 we have shown that $\mathbb{Q}_P$ is homogeneous whenever $P$ is dense and co-dense and we have commented on the countability of $\mathbb{Q}$ and on $\mathbb{K}$ being the age of $\mathbb{Q}_P$ for dense and co-dense $P$ in the paragraph before that lemma. Theorem 6 then guarantees us that $(\mathbb{Q}, \leq, P)$ with dense and co-dense $P$ is isomorphic to the Fraïssé limit of $\mathbb{K}$, if such predicate exists. The existence of such predicate is proven in Proposition 13. □

We have also indirectly proven that any two $\mathbb{Q}_P$ and $\mathbb{Q}_R$, where $P, R$ are dense and co-dense, are isomorphic, however a direct proof of this statement is no more difficult than the proof of Theorem 15.

# 3. The limit by the compactness theorem

In this chapter we will present our own construction which demands less of the class and we will show examples in hope to illustrate differences from other approaches.

## 3.1 The construction

**Definition 17.** *Let $\mathcal{L}$ be a language and let $\mathbb{K}$ denote a sequence $(\mathbb{A}_i | i \in \mathbb{N}^+)$ of finite $\mathcal{L}$-structures, unbounded and non-decreasing in size. We define the* compactness-limit of $\mathbb{K}$*, or* c-limit of $\mathbb{K}$ *for short, denoted by* $\lim_C \mathbb{K}$*, to be the set of all sentences $\phi$ for which there exists a positive integer $n_\phi$ such that every structure of size at least $n_\phi$ in $\mathbb{K}$ satisfies $\phi$.*

Normally one would expect the limit of a class or a sequence of structures to be a specific structure, possibly up to isomorphism. Without demanding additional conditions from $\mathbb{K}$, this does not happen in the simplest of cases – we will later show that the c-limit of finite linear orders has infinitely many countable, non-isomorphic models.

There is, however, a weaker property that is satisfied in some cases, including the one we are interested in. We can demand uniqueness up to elementary equivalence, or in the language of the c-limit itself: we can demand the c-limit to be complete.

One immediate observation is that any model of a c-limit is infinite. This is because the limit contains all sentences of the form

$$\exists x_1, \dots, x_n : \bigwedge_{i \neq j} x_i \neq x_j.$$

We will denote each such sentence by $\psi_n$ and denote the set of all $\psi_n$ by $\Psi$.

To justify the definition we shall first demonstrate that the c-limit is consistent.

**Proposition 18.** *Let $\mathbb{K}$ be a sequence of finite structures. Then* $\lim_C \mathbb{K}$ *has a model.*

*Proof.* We will prove that any finite subtheory of $\lim_C \mathbb{K}$ has a model so that we can apply the compactness theorem.

Let $S$ be a finite subtheory of $\lim_C \mathbb{K}$. For every sentence $\phi$ from $S$ there is a natural number $n_\phi$ such that if a structure from $\mathbb{K}$ is of size at least $n_\phi$, then it satisfies $\phi$. Because $S$ is finite, we can consider the maximum of all $n_\phi$ which we shall denote by $n_S$. Clearly, any structure from $\mathbb{K}$ of size at least $n_S$ satisfies the entire subtheory $S$, therefore $S$ has a model.

The compactness theorem states that if any finite subtheory has a model, then the entire theory has a model. $\square$

This proposition is the reason why we chose to name the limit as we did.

## 3.2 Ehrenfeucht-Fraïssé games and elementary equivalence

Throughout the examples we will rely heavily on Ehrenfeucht-Fraïssé games and so it is justified to briefly recall the topic. A more carefully written introduction can be found in [4, pp.52-56].

Let $\mathcal{L}$ be a language with no function symbols, let $m$ be a natural number and let $\mathbb{A}, \mathbb{B}$ be $\mathcal{L}$-structures. A *$m$-round Ehrenfeucht-Fraïssé game* on structures $\mathbb{A}, \mathbb{B}$ is played between two players – the *Spoiler* and the *Duplicator*.

In each of those $m$ rounds the Spoiler picks an element of either $\mathbb{A}$ or $\mathbb{B}$ and the Duplicator responds by picking an element of the other structure. The resulting sequence of pairs $(a_i, b_i) \in \mathbb{A} \times \mathbb{B}$ is called a *play*. We say that the Duplicator *wins* the play if it forms a partial isomorphism between $\mathbb{A}$ and $\mathbb{B}$. Otherwise we say that the Spoiler wins.

An unfinished play is called a *position*. A *strategy* is a set of rules which tells the player what element to pick in every position in which he can find himself using that ruleset. We say that the strategy is *winning* if the player wins whenever he uses the strategy and we say that the player *has a winning strategy* if there exists some winning strategy.

Now let $Fl_{\mathcal{L}}$ be the set of all formulas in some language $\mathcal{L}$. The *quantifier rank*, denoted by $qr$, is a function on $Fl_{\mathcal{L}}$ defined inductively as follows:

- $qr(\phi) = 0$ if $\phi$ is atomic
- $qr(\phi_1 \wedge \phi_2) = max(qr(\phi_1), qr(\phi_2))$
- $qr(\phi_1 \vee \phi_2) = max(qr(\phi_1), qr(\phi_2))$
- $qr(\neg\phi) = qr(\phi)$
- $qr(\exists x\ \phi) = qr(\phi) + 1$
- $qr(\forall x\ \phi) = qr(\phi) + 1$

**Definition 19.** *Let $m$ be a natural number and let $\mathbb{A}$ and $\mathbb{B}$ be $\mathcal{L}$-structures. We say that $\mathbb{A}$ and $\mathbb{B}$ are* $m$-elementarily equivalent*, and denote it by $\mathbb{A} \equiv_m \mathbb{B}$, if every sentence of quantifier rank at most $m$ holds in $\mathbb{A}$ if and only if it holds in $\mathbb{B}$.*

*We say that $\mathbb{A}$ and $\mathbb{B}$ are* elementarily equivalent *if they are $m$-elementarily equivalent for every positive integer $m$ and we denote it by $\mathbb{A} \equiv \mathbb{B}$.*

The result, attributed to Ehrenfeucht and Fraïssé, which we will use in order to demonstrate the completeness of limits can be found for example in [4, pp.54].

**Lemma 20.** *Let $\mathbb{A}$ and $\mathbb{B}$ be $\mathcal{L}$-structures and let $m$ be a positive integer. Then $\mathbb{A} \equiv_m \mathbb{B}$ if and only if the Duplicator has a winning strategy for the $m$-round Ehrenfeucht-Fraïssé game on $\mathbb{A}$ and $\mathbb{B}$.*

The following theorem is a trivial consequence of this lemma.

**Theorem 21** (Ehrenfeucht, Fraïssé). *Let $\mathbb{A}$ and $\mathbb{B}$ be $\mathcal{L}$-structures. Then $\mathbb{A} \equiv \mathbb{B}$ if and only if the duplicator has a winning strategy for the $m$-round Ehrenfeucht-Fraïssé game on $\mathbb{A}$ and $\mathbb{B}$ for every positive integer $m$.*

Let us note that a theory $T$ is complete if and only if every two models of $T$ are elementarily equivalent. This follows directly from the definition of a consequence of a theory.

## 3.3 The c-limit of finite linear orders

In this section we will present a result about completeness of the c-limit of finite linear orders with endpoints. Our approach is inspired by an approach to a solution of a similar problem presented in [4, pp.56-57].

Let $\mathcal{L}$ be a language consisting of one binary relational symbol $\leq$ and let us denote the sequence of finite linear orders $(L_i | i \in \mathbb{N}^+)$ by $\mathbb{K}$ for the rest of the section. We will demonstrate that $\lim_C \mathbb{K}$ is complete.

Let us note that

- when a consistent theory $T$ has a complete subtheory, then $T$ itself is complete
- consistent theory is complete if and only if any two models of the theory are elementarily equivalent.

Notice that $DiLO_{ep} \cup \Psi$ is a subtheory of $\lim_C \mathbb{K}$. We will show that $\lim_C \mathbb{K}$ is complete by demonstrating that any two infinite models of $DiLO_{ep}$ are elementarily equivalent to each other. Let us begin by describing those models.

Let $\mathbb{A}$ be a model of $DiLO_{ep}$. We define a relation $\cong$ on $\mathbb{A}$ by

$$a \cong b \text{ iff } \exists n \geq 0 : a = s^n(b) \vee a = p^n(b).$$

**Lemma 22.** *The relation $\cong$ on any model $\mathbb{A}$ of $DiLO_{ep}$ is an equivalence.*

*Proof.* All three properties are quite straightforward to prove.

Reflexivity follows from $a = s^0(a)$, the relation is symmetric because if $a = s^n(b)$, then $b = p^n(a)$ and it is also transitive because if $a = s^n(b)$ and $b = s^m(c)$, then $a = s^{m+n}(c)$ and if $a = s^n(b)$ and $b = p^m(c)$, then $a = s^{n-m}(c)$, or $p^{m-n}(c)$, if $n - m$ is negative. Therefore $\cong$ is an equivalence relation. □

In the rest of the section we will deal mostly with ordered sets of the form $\mathbb{Z}^+ \cup (\mathbb{P} \times \mathbb{Z}) \cup \mathbb{Z}^-$, where $\mathbb{Z}^+$ denotes the ordered set of positive integers, $\mathbb{Z}^-$ denotes negative integers and $\mathbb{P}$ is a set ordered linearly by $\leq_{\mathbb{P}}$. Let us denote any such set by $\mathbb{L}_{\mathbb{P}}$.

On $\mathbb{L}_{\mathbb{P}}$ we define a linear order $\leq_{\mathbb{L}_{\mathbb{P}}}$ by the following set of rules:

- The elements of $\mathbb{Z}^+$ inherit the order from $\mathbb{Z}$ and are less than all other elements of $\mathbb{L}_{\mathbb{P}}$.
- The elements of $\mathbb{Z}^-$ inherit the order from $\mathbb{Z}$ and are greater than all other elements of $\mathbb{L}_{\mathbb{P}}$.
- The elements $a = (p, i)$ and $b = (q, j)$ of $\mathbb{P} \times \mathbb{Z}$ are ordered lexicographically:

- If $p <_{\mathbb{P}} q$, then $a <_{\mathbb{L}_{\mathbb{P}}} b$,
- if $p >_{\mathbb{P}} q$, then $a >_{\mathbb{L}_{\mathbb{P}}} b$ and
- if $p = q$, then $i < j$ results in $a <_{\mathbb{L}_{\mathbb{P}}} b$, $i > j$ results in $a >_{\mathbb{L}_{\mathbb{P}}} b$ and $i = j$ results in $a = b$.

Any ordered set $(\mathbb{L}_{\mathbb{P}}, \leq_{\mathbb{L}_{\mathbb{P}}})$ is a model of $DiLO_{ep}$.

**Lemma 23.** *Let $\mathbb{A}$ be an infinite model of $DiLO_{ep}$. Then there exists $\mathbb{P}$ such that $\mathbb{A}$ is isomorphic to $(\mathbb{L}_{\mathbb{P}}, \leq_{\mathbb{L}_{\mathbb{P}}})$.*

*Proof.* Assume that $\mathbb{A} = (A, \leq_A)$ is an infinite, ordered set satisfying $DiLO_{ep}$. The elements **MIN** and **MAX** of the structure $\mathbb{A}$ are not in the $\cong$ relation, as the structure is infinite.

While keeping in mind that two elements $a$ and $b$ are finitely distant from each other if and only if $a \cong b$, it is easy to see that

- $[\mathbf{MIN}]_{\cong}$ with the order inherited from $\mathbb{A}$ is isomorphic to $\mathbb{Z}^+$
- $[\mathbf{MAX}]_{\cong}$ with the order inherited from $\mathbb{A}$ is isomorphic to $\mathbb{Z}^-$
- any other equivalence class is isomorphic to $\mathbb{Z}$.

Let $P$ be a set consisting of every equivalence class but $[\mathbf{MIN}]_{\cong}$ and $[\mathbf{MAX}]_{\cong}$. Then $(\mathbb{A}, \leq_{\mathbb{A}})$ is isomorphic to $[\mathbf{MIN}]_{\cong} \cup P \times \mathbb{Z} \cup [\mathbf{MAX}]_{\cong}$, ordered in a way which fits the description above. ☐

Now let $\mathbb{A}$ be a model of $DiLO_{ep}$. We define a distance function of $a, b$ from $\mathbb{A}$, denoted by $d(a, b)$, as the usual distance of integers if $a \cong b$ and as $\infty$ if $a \ncong b$.

**Theorem 24.** *Let $\mathbb{A}$ and $\mathbb{B}$ be infinite models of $DiLO_{ep}$. Then the Duplicator has a winning strategy for the $m$-round Ehrenfeucht-Fraïssé game on $\mathbb{A}$ and $\mathbb{B}$ for every natural $m$.*

*Proof.* We will demonstrate a strategy which allows the duplicator to ensure that every position listed as $a_1 < \cdots < a_k$, $b_1 < \cdots < b_k$ satisfies the following distance condition for each positive integer $i < k$:

$$(\mathbf{D})\ d(a_i, a_{i+1}) = d(b_i, b_{i+1}) \text{ or both distances are at least } 2^{m-k}.$$

Without loss of generality we can assume that $a_1$ is **MIN** of $\mathbb{A}$, $a_k$ is **MAX** of $\mathbb{A}$, $b_1$ is **MIN** of $\mathbb{B}$ and $b_k$ is **MAX** of $\mathbb{B}$. If those elements are not played, then the Duplicator can imagine that they are played in the first 2 moves and that the length of the game is $m + 2$ instead of $m$.

First let us observe that if two pairs $(a_i, a_{i+1})$ and $(b_i, b_{i+1})$ satisfy the condition $(\mathbf{D})$ in the $k$-th position, then they will satisfy that condition in the $(k+1)$-th position, because if the distances between the elements within the pairs are not the same, then they are at least $2^{m-k}$ and therefore they are at least $2^{m-(k+1)}$. From this it follows that those two pairs will satisfy the distance condition in all future positions, if they do so in the $k$-th position.

Now let the spoiler pick an element $a \in \mathbb{A}$, say between $a_i$ and $a_{i+1}$. We need to select an element $b$ such that the partial isomorphism and the distance conditions are satisfied, that is

- $b_i < b < b_{i+1}$
- $d(a_i, a) = d(b_i, b)$ or both distances are at least $2^{m-k-1}$
- $d(a, a_{i+1}) = d(b, b_{i+1})$ or both distances are at least $2^{m-k-1}$.

We consider the following cases:

- $d(a_i, a_{i+1}) < 2^{m-k}$.

  In that case $d(a_i, a_{i+1}) = d(b_i, b_{i+1})$ and we can set $b := b_i + d(a_i, a)$. Consequently, we have $d(a_i, a) = d(b_i, b)$ and $d(a, a_{i+1}) = d(b, b_{i+1})$ and so the condition is preserved.

- $d(a_i, a_{i+1}) \geq 2^{m-k}$.

  In that case we shall, while keeping in mind that $d(b_i, b_{i+1}) \geq 2^{m-k}$, distinguish further:

  - $d(a_i, a) < 2^{m-k-1}$.

    In this case we can, again, set $b := b_i + d(a_i, a)$ so that $d(a_i, a) = d(b_i, b)$ and we observe that $d(a, a_{i+1}) \geq 2^{m-k-1}$ and $d(b, b_{i+1}) \geq 2^{m-k-1}$.
  - $d(a, a_{i+1}) < 2^{m-k-1}$

    Then we set $b := b_{i+1} - d(a, a_{i+1})$ so that $d(a, a_{i+1}) = d(b, b_{i+1})$ and we observe that both $d(a_i, a)$ and $d(b_i, b)$ are at least $2^{m-k-1}$.
  - $d(a_i, a) \geq 2^{m-k-1}, d(a, a_{i+1}) \geq 2^{m-k-1}$.

    In that case we select an element $b$ between $b_i$ and $b_{i+1}$ that satisfies $d(b_i, b) \geq 2^{m-k-1}$ and $d(b, b_{i+1}) \geq 2^{m-k-1}$. Existence of such element is warranted by the distance condition (**D**) itself.

If the Spoiler decides to pick an element from $\mathbb{B}$, the Duplicator applies the same strategy to $\mathbb{A}$.

After the position is finished and becomes a play, we observe that the play satisfies the partial isomorphism condition and conclude that the Duplicator wins. □

**Theorem 25.** $\lim_C \mathbb{K}$ *is consistent and complete.*

*Proof.* In Proposition 18 we have proven that the c-limit is always consistent. Theorem 24 presents Duplicator's winning strategy for the $m$-round Ehrenfeucht-Fraïssé game on any two infinite models of $DiLO_{ep}$ for arbitrary positive integer $m$. The corollary of Lemma 20 then gives us elementary equivalence of every two infinite models of $DiLO_{ep}$ and so $DiLO_{ep} \cup \Psi$ is complete according to one of our observations. Because $\lim_C \mathbb{K}$ is consistent and contains $DiLO_{ep} \cup \Psi$ as a subtheory, it follows that $\lim_C \mathbb{K}$ is complete as well. □

Let us note that Theorem 24 and its proof work even if we choose to consider finite models of size at least $2^m$ instead of infinite models. The consequence of this is that it is possible to verify the validity of a formula $\phi$ of quantifier rank at most $m$ on a linear order of size $2^m$. According to [3, pp.40], results of this nature were rediscovered many times. The source references texts from Gurevich and Rosenstein as some of the older ones.

Earlier I promised to look into isomorphisms of models of $\lim_C \mathbb{K}$.

**Theorem 26.** $\lim_C \mathbb{K}$ *has infinitely many non-isomorphic countable models.*

*Proof.* We have observed that $\lim_C \mathbb{K}$ and $DiLO_{ep} \cup \Psi$ have the same models ($DiLO_{ep} \cup \Psi$ is complete and $\lim_C \mathbb{K}$ is consistent).

In Lemma 23 we have established that infinite models of $DiLO_{ep}$ are precisely the models isomorphic to $(\mathbb{L}_{\mathbb{P}}, \leq_{\mathbb{L}_{\mathbb{P}}})$ for some $\mathbb{P}$.

Suppose that $\mathbb{P}$ and $\mathbb{P}'$ are finite. It is easy to observe that $(\mathbb{L}_{\mathbb{P}}, \leq_{\mathbb{L}_{\mathbb{P}}})$ and $(\mathbb{L}_{\mathbb{P}'}, \leq_{\mathbb{L}_{\mathbb{P}'}})$ are not isomorphic to each other if $|\mathbb{P}| \neq |\mathbb{P}'|$ and that every such $\mathbb{L}_{\mathbb{P}}$ is countable. □

## 3.4 The c-limit of finite linear orders with a unary predicate

Let us now consider $\mathcal{L}$ to be a language consisting of two relational symbols: binary $\leq$ and unary $U$. For the rest of the section let us denote by $(L_i, P)$ the structure $(\{1, \dots, i\}, \leq, P)$, where $\leq$ is the usual linear order of $\{1, \dots, i\}$ and $P \subseteq \{1, \dots, i\}$ is a unary predicate, and by $\mathbb{K}$ we denote a sequence of $(L_i, P)$ for all positive integers $i$ and all $P \subseteq \{1, \dots, i\}$, as in the last section of chapter 2. It does not matter how we choose to order the individual structures in the sequence, as long as they are nondecreasing in size. One possible way was hinted at in the proof of Lemma 11.

**Observation.** $\lim_C \mathbb{K}$ *is not complete.*

*Proof.* First let us observe that if a sentence $\phi$ is a consequence of $\lim_C \mathbb{K}$, then $\phi \in \lim_C \mathbb{K}$. This follows from the definition of c-limit.

Now consider a nontrivial sentence $\varphi$ about the predicate, for example

$$\forall x : U(x).$$

Clearly there are structures of arbitrarily large size which satisfy this sentence and also structures of arbitrarily large size which do not satisfy it. Therefore neither $\varphi$ nor $\neg\varphi$ can be in $\lim_C \mathbb{K}$ and so they cannot be a consequence of $\lim_C \mathbb{K}$. □

There is a nontrivial way to constrain the predicate in order to achieve completeness.

Let $\alpha_m$ state that any two elements satisfying the predicate must be distanced from each other by at least $m-1$ other elements. Written by a formula:

$$\forall x < y : (U(x) \wedge U(y)) \longrightarrow (\exists x_1 < \dots < x_{m-1} : x < x_1 \wedge x_{m-1} < y \wedge \bigwedge_{i=1}^{m-1} \neg U(x_i))).$$

Also let $\beta_m$ state that there must be at least $m$ elements satisfying the predicate. Written by a formula:

$$\exists x_1, \dots x_m : (\bigwedge_{i \neq j} x_i \neq x_j) \wedge (\bigwedge_{i=1}^{m} U(x_i)).$$

We are interested in a subsequence of $\mathbb{K}$ consisting of structures $\mathbb{A}$ which satisfy the following:

- $U(\mathbf{MIN})$
- $U(\mathbf{MAX})$
- $\alpha_{\lfloor\sqrt{n}\rfloor}$, where $|\mathbb{A}| = n$
- $\beta_{\lfloor\sqrt{n}\rfloor}$, where $|\mathbb{A}| = n$,

where $U(\mathbf{MIN})$ and $U(\mathbf{MAX})$ are abbreviations for the formulas which state that the uniquely determined minimum and maximum elements satisfy $U$. Let us denote such subsequence by $\mathbb{H}$.

The search for a complete subtheory of $\lim_C \mathbb{H}$ has proven to be difficult and so we approach the problem differently compared to the previous section. We will show that for each positive integer $m$ there exists a positive integer $n_m$ such that any formula $\phi$ of quantifier rank $m$ is either satisfied by all structures of size at least $n_m$ from $\mathbb{H}$ or by no structure of size at least $n_m$ from $\mathbb{H}$. Then either $\phi \in \lim_C \mathbb{H}$ or $\neg\phi \in \lim_C \mathbb{H}$ and so we will have proven that $\lim_C \mathbb{H}$ is complete.

We will work with structures satisfying $U(\mathbf{MIN})$ and $U(\mathbf{MAX})$ and for these we define the following notation and terminology:

- For an element $a$ of $(L_i, P)$ let us denote $\max\{b \in (L_i, P) | P(b), b \leq a\}$ by $a^{\downarrow}$ and $\min\{b \in (L_i, P) | P(b), a \leq b\}$ by $a^{\uparrow}$.
- A $P$-successor of an element $a$ of $(L_i, P)$ is the smallest element $b$ satisfying $b > a$ and $P(b)$ and similarly a $P$-predecessor of $a$ is the largest element $c$ satisfying $c < a$ and $P(c)$. $P$-predecessor of $\mathbf{MIN}$ and $P$-successor of $\mathbf{MAX}$ are left undefined, we will not need them.
- Let us also define a $P$-distance of two elements $a \leq b$ of $(L_i, P)$ as the number of times we have to apply the $P$-successor function to $a^{\downarrow}$ in order to obtain $b^{\downarrow}$. We denote this distance by $d^P(a, b)$.

**Theorem 27.** *Let $\mathbb{A}$ and $\mathbb{B}$ be structures in $\mathbb{H}$ of size at least $2^{2m+4}$. Then the Duplicator has a winning strategy for an Ehrenfeucht-Fraïssé game on $\mathbb{A}, \mathbb{B}$ of length $m$.*

*Proof.* We will define a strategy which allows the Duplicator to ensure that every position listed as $a_1 < \cdots < a_k$, $b_1 < \cdots < b_k$, where $U^{\mathbb{A}}(a_i) \leftrightarrow U^{\mathbb{B}}(b_i)$, satisfies the following distance conditions:

(DU) $\forall i < k : d^{U^{\mathbb{A}}}(a_i, a_{i+1}) = d^{U^{\mathbb{B}}}(b_i, b_{i+1})$ or both distances are at least $2^{m-k}$

(D↑ ) $\forall i \leq k : d(a_i, a_i^{\uparrow}) = d(b_i, b_i^{\uparrow})$ or both distances are at least $2^{m-k}$

(D↓ ) $\forall i \leq k : d(a_i^{\downarrow}, a_i) = d(b_i^{\downarrow}, b_i)$ or both distances are at least $2^{m-k}$

Again, without loss of generality we can assume that $a_1$ is $\mathbf{MIN}$ of $\mathbb{A}$, $a_k$ is $\mathbf{MAX}$ of $\mathbb{A}$ and $b_1$ is $\mathbf{MIN}$ of $\mathbb{B}$ and $b_k$ is $\mathbf{MAX}$ of $\mathbb{B}$ as if those moves were not played, the Duplicator can imagine that they are played in the first 2 moves and that the length of the game is $m + 2$ instead of $m$.

It is easy to see that if the distance condition is satisfied by two pairs $(a_i, a_{i+1})$ and $(b_i, b_{i+1})$ in some position, then those pairs satisfy the condition in all future positions, similarly to the proof of Theorem 24. Now let the Spoiler pick an element $a$ from $\mathbb{A}$, say between $a_i$ and $a_{i+1}$. In each of the following cases we will select an element $b^U$ from $\mathbb{B}$ which satisfies $U^{\mathbb{B}}(b^U)$ and some distance condition and then we find the response $b$ to the element $a$ based on the choice of $b^U$ in order to preserve the condition of the play, that is:

(1) $d^{U^{\mathbb{A}}}(a_i, a) = d^{U^{\mathbb{B}}}(b_i, b)$ or both distances are at least $2^{m-k-1}$

(2) $d^{U^{\mathbb{A}}}(a, a_{i+1}) = d^{U^{\mathbb{B}}}(b, b_{i+1})$ or both distances are at least $2^{m-k-1}$

(3) $d(a^{\downarrow}, a) = d(b^{\downarrow}, b)$ or both distances are at least $2^{m-k-1}$

(4) $d(a, a^{\uparrow}) = d(b, b^{\uparrow})$ or both distances are at least $2^{m-k-1}$

We consider the following cases and for each of them we describe how to select the element $b$ and verify that the conditions are preserved.

- $d^{U^{\mathbb{A}}}(a_i, a) < 2^{m-k-1}$,
  - $d(a^{\downarrow}, a) < 2^{m-k-1}$. Then we let $b^U$ be the element which satisfies $U^{\mathbb{B}}(b^U)$ and $d^{U^{\mathbb{B}}}(b_i, b^U) = d^{U^{\mathbb{A}}}(a_i, a)$ and we set $b := b^U + d(a^{\downarrow}, a)$.

    We have selected $b^U$ so that $d^{U^{\mathbb{A}}}(a_i, a) = d^{U^{\mathbb{B}}}(b_i, b^U)$. $d^{U^{\mathbb{B}}}(b_i, b^U) = d^{U^{\mathbb{B}}}(b_i, b)$ holds because $d(a^{\downarrow}, a) < 2^{m-k-1}$, which is smaller than the minimum gap between two $U^{\mathbb{A}}$-elements, therefore condition (1) is preserved.

    Suppose that $d^{U^{\mathbb{A}}}(a_i, a_{i+1}) \geq 2^{m-k}$. Then we know that $d^{U^{\mathbb{A}}}(a, a_{i+1}) \geq 2^{m-k-1}$ and from $d^{U^{\mathbb{B}}}(b_i, b_{i+1}) \geq 2^{m-k}$ we know that $d^{U^{\mathbb{B}}}(b, b_{i+1}) \geq 2^{m-k-1}$. If we assume that $d^{U^{\mathbb{A}}}(a_i, a_{i+1}) < 2^{m-k}$, then we obtain $d^{U^{\mathbb{A}}}(a_i, a_{i+1}) = d^{U^{\mathbb{B}}}(b_i, b_{i+1}) < 2^{m-k}$ and consequently $d^{U^{\mathbb{A}}}(a, a_{i+1}) = d^{U^{\mathbb{B}}}(b, b_{i+1})$, because $d^{U^{\mathbb{B}}}(b_i, b) = d^{U^{\mathbb{A}}}(a_i, a)$. Therefore condition (2) is preserved.

    We have selected $b$ by a formula which makes it easy to see that $d(a^{\downarrow}, a) = d(b^{\downarrow}, b)$ – condition (3) is preserved.

    Suppose that $d(a, a^{\uparrow}) < 2^{m-k}$. Then $a$ is a $U^{\mathbb{A}}$-element because the gaps between $U^{\mathbb{A}}$-elements are more than $2^m$ wide. In that case $b$ is a $U^{\mathbb{B}}$-element and therefore $d(b, b^{\uparrow}) = d(a, a^{\uparrow}) = 0$. Otherwise $d(a, a^{\uparrow}) \geq 2^{m-k-1}$ and $d(b, b^{\uparrow}) \geq 2^{m-k-1}$ because of the minimum gap length. Therefore condition (4) is preserved.
  - $0 < d(a, a^{\uparrow}) < 2^{m-k-1}$. Then let $b^U$ satisfy $d^{U^{\mathbb{B}}}(b_i, b^U) = d^{U^{\mathbb{A}}}(a_i, a) + 1$ and set $b := b^U - d(a, a^{\uparrow})$.

    $d^{U^{\mathbb{A}}}(a_i, a) = d^{U^{\mathbb{B}}}(b_i, b^U) - 1 = d^{U^{\mathbb{B}}}(b_i, b)$ follows from an argument already presented above, and so condition (1) is preserved.

    The argument for $d^{U^{\mathbb{A}}}(a, a_{i+1}) = d^{U^{\mathbb{B}}}(b, b_{i+1})$ or both at least $2^{m-k}$ was also already presented above, and condition (2) is preserved.

    $b$ was selected in a way so it is easy to see that $d(a, a^{\uparrow}) = d(b, b^{\uparrow})$.

    $2^{m-k-1} \leq d(a^{\downarrow}, a) < 2^{m-k}$ is not possible because of the minimum size of gaps between $U^{\mathbb{A}}$-elements. Same goes for $d(b^{\downarrow}, b)$.

– $d(a^{\downarrow}, a) \geq 2^{m-k-1}$ and $d(a, a^{\uparrow}) \geq 2^{m-k-1}$. Then we select $b^U$ so that it satisfies $d^{U^{\mathbb{B}}}(b_i, b^U) = d^{U^{\mathbb{A}}}(a_i, a)$ and observe that our condition warrants existence of an element $b$ which is at least $2^{m-k-1}$ far away from both $b^U$ and the $U^{\mathbb{B}}$-successor of $b^U$.

  As in all previous cases, $b$ was selected so that $d^{U^{\mathbb{A}}}(a_i, a) = d^{U^{\mathbb{B}}}(b_i, b)$, therefore condition (1) is preserved.

  The argument for $d^{U^{\mathbb{A}}}(a, a_{i+1}) = d^{U^{\mathbb{B}}}(b, b_{i+1})$ or both at least $2^{m-k}$ was already presented in the first case and so condition (2) is preserved.

  The conditions (3) and (4) are trivially satisfied.

- $d^{U^{\mathbb{A}}}(a, a_{i+1}) < 2^{m-k-1}$. This case is analogous to the previous one.
- $d^{U^{\mathbb{A}}}(a_i, a) \geq 2^{m-k-1}, d^{U^{\mathbb{A}}}(a, a_{i+1}) \geq 2^{m-k-1}$.

  In all 3 subcases it will be clear that the conditions (1) and (2) are preserved because of our choice of $b^U$ and also it will be clear that the conditions (3) and (4) are preserved as analogous cases were already discussed at the beginning of this case analysis.

  – $d(a^{\downarrow}, a) < 2^{m-k-1}$. Then let $b^U$ be an element which is at least $2^{m-k-1}$ $U^{\mathbb{B}}$-far away from both $b_i$ and $b_{i+1}$ and set $b := b^U + d(a^{\downarrow}, a)$.
  – $0 < d(a, a^{\uparrow}) < 2^{m-k-1}$. Then let $b^U$ be an element which is at least $2^{m-k-1} + 1$ $U^{\mathbb{B}}$-far away from $b_i$ and at least $2^{m-k-1} - 1$ far away from $b_{i+1}$ and set $b := b^U - d(a, a^{\uparrow})$.
  – $d(a^{\downarrow}, a) \geq 2^{m-k-1}$ and $d(a, a^{\uparrow}) \geq 2^{m-k-1}$. Then let $b^U$ be an element at least $2^{m-k-1}$ $U^{\mathbb{B}}$-far away from both $b_i$ and $b_{i+1}$ and choose $b$ to be an element which is at least $2^{m-k-1}$ far away from both $b^U$ and the $U^{\mathbb{B}}$-successor of $b^U$.

After the position becomes a finished play, we can observe that the partial isomorphism condition is satisfied and so the Duplicator wins. □

**Theorem 28.** $\lim_C \mathbb{H}$ *is consistent and complete.*

*Proof.* We have observed that a c-limit is always consistent at the start of the chapter.

For completeness, let us choose a sentence $\phi$ of quantifier rank $m$. In the proof of Theorem 27 we have proven that for $n_m := 2^{2m+4}$ either all or no structures of size at least $n_m$ from $\mathbb{H}$ satisfy $\phi$. Then either $\phi \in \lim_C \mathbb{H}$ or $\neg\phi \in \lim_C \mathbb{H}$ and so $\lim_C \mathbb{H}$ is complete. □

Similarly to our note following Theorem 25, it is possible to verify the validity of a formula $\phi$ of quantifier rank $m$ in the theory $\lim_C \mathbb{H}$ with the help of any structure from $\mathbb{H}$ of size at least $2^{2m+4}$.

# 4. The forcing limit

In this chapter we will present a construction based on games. We will look into a few examples and also mention a relevant alternative approach.

## 4.1 The construction

For the entire chapter let us fix a countable set $A$ consisting of elements $a_i$, where $i \in \mathbb{N}$, and also let us fix a language $\mathcal{L}$ without any function symbols for the entire section. With the help of the set $A$ we extend the language $\mathcal{L}$ by new constants $c^{a_i}$ and denote such extension by $\mathcal{L}(A)$. Consider $\mathbb{K}$ to be a class of finite $\mathcal{L}$-structures $\mathbb{A}_i$ with universes $A_i \subseteq A$.

On this class we will consider the following partial order: If $\mathbb{A}$ and $\mathbb{B}$ are structures in $\mathbb{K}$, then we say that $\mathbb{A} \leq_G \mathbb{B}$ iff there is an embedding of $\mathbb{A}$ as an $\mathcal{L}(\mathbb{A})$-structure into $\mathbb{B}$. We will be working with this partial order and possibly some restrictions of it.

We will consider games of countable length on the class $\mathbb{K}$ played between two players. For convenience, we will stick to calling them the Spoiler and the Duplicator, as one still challenges the other to answer played moves. Each move will have an $\mathcal{L}$-structure $\mathbb{A}_i$ associated with itself. One player starts by choosing a structure $\mathbb{A}_0$ from the class $\mathbb{K}$, and in the $i$-th move, a player chooses a structure of $\mathbb{K}$ which is greater according to the selected order of the class. We define the resulting structure of the game as

$$\bigcup_{i \in \mathbb{N}} \mathbb{A}_i.$$

and denote it by $\mathbb{S}$.

As in our introduction of Ehrenfeucht-Fraïssé games, we will concern ourselves with strategies. A sequence of structures $\mathbb{A}_i$ which results from a game between our two players is called a play. An unfinished play is called a position. A strategy is a set of rules which tells the player what move to make in every position in which he can find himself in using that set of rules.

For a fixed sentence $\phi$, we say that the Duplicator wins a play if and only if the resulting structure $\mathbb{S}$ satisfies $\phi$. We say that a strategy is winning if and only if the player wins whenever he uses it and we say that the player has a winning strategy if such a strategy exists.

**Definition 29.** *Let $\phi$ be a formula in the language $\mathcal{L}$ and let $\mathbb{K}$ be a class of finite $\mathcal{L}$-structures $\mathbb{A}_i$ with universes $A_i \subseteq A$. We say that $\phi$ is enforceable in $\mathbb{K}$ if and only if the Duplicator has a winning strategy for the game on the class $\mathbb{K}$ for the formula $\phi$.*

*We shall denote a game on a class $\mathbb{K}$ in which the Duplicator desires to force $\phi$ utilizing moves $X \subseteq \mathbb{N}$ by $G_X(\phi, \mathbb{K})$.*

If the choice of $X$ is unimportant or clear from the context, then we may write $G(\phi, \mathbb{K})$.

It should be noted that the interesting subset of all games consists of those where the Spoiler moves first and both players have infinitely many moves, or in the language of the set $X$: we are interested in games $G_X(\phi, \mathbb{K})$, where $X \subseteq \mathbb{N}^+$ is infinite and coinfinite. Such games are called regular and they are equivalent (in terms of enforceability of formulas) to games where the Spoiler plays the odd moves and the Duplicator plays the even moves. In the scope of this thesis we will only concern ourselves with these.

There is a more general definition of forcing presented in [1, pp.17-34], where the author allows function symbols in the language. The text contains many details not mentioned here, as well as a more thoroughly written paragraph about regular games.

For the sake of the format of this thesis, we define the forcing limit of a class $\mathbb{K}$, denoted as $\lim_{FO} \mathbb{K}$, to be the set of all formulas enforceable in a regular game.

## 4.2 A few basic properties

For the purposes of this section, we shall treat a strategy for a game as an object $S$ which assigns a structure $S(\mathbb{A}_i, \dots \mathbb{A}_1)$ to any position $\mathbb{A}_i, \dots \mathbb{A}_1$ which can occur if the Duplicator uses $S$ as his strategy.

**Proposition 30.** *Let $\mathcal{L}$ be a language without function symbols, let $\phi_k, k \in \mathbb{N}$ be $\mathcal{L}$-formulas and let $\mathbb{K}$ be a class of $\mathcal{L}$-structures $\mathbb{A}_i$ with universes $A_i \subseteq A$. If every $\phi_k$ is enforceable in $\mathbb{K}$, then all formulas $\phi_k$ are enforceable in $\mathbb{K}$ at once.*

*Proof.* Suppose that $S_{\phi_k}$ are the strategies which allow the Duplicator to win regular games $G(\phi_k, \mathbb{K})$. Then the Duplicator can win every game $G_{X_k}(\phi_k, \mathbb{K})$, where $X_k = \{p_k^l | l \in \mathbb{N}^+\}$, where $p_k$ is the $k$-th prime number. The combined strategy, where the Duplicator applies $S_{\phi_k}$ in his every $p_k^l$-th move, $k \in \mathbb{N}^+, l \in \mathbb{N}^+$, surely allows him to win the regular game $G_X(\phi_k, \mathbb{K})$, where

$$X = \bigcup_{m \in \mathbb{N}} X_m,$$

for every $k$. □

An easily observable consequence of this theorem is that $\phi$ and $\psi$ are not both enforceable if they contradict each other.

In the following section we will also show an example to demonstrate that the enforceability of $\phi \vee \psi$ does not imply the enforceability of either of the two formulas.

**Proposition 31.** *Let $\mathcal{L}$ be a language without function symbols and let $\mathbb{K}$ be any class of finite $\mathcal{L}$-structures $\mathbb{A}_i$ with universes $A_i \subseteq A$. If every structure $\mathbb{A}_i$ of the class $\mathbb{K}$ can be extended by every element $a \in A \setminus A_i$, then there is a strategy which ensures that the universe of the resulting structure $\mathbb{S}$ is $A$.*

*Proof.* Consider an enumeration of $A$, say a sequence $(a_j)_{j=1}^{\infty}$. If the Duplicator extends the structure $\mathbb{A}_i$ by the element $a_j$ in his $j$-th move, then the resulting structure will contain all elements of $A$. □

This strategy can be merged with other strategies as suggested in Proposition 31, despite the fact that we cannot express the desired property by a first-order sentence.

## 4.3 Examples of enforceability in different subclasses of linear orders

Let $\mathcal{L}$ be a language consisting of one binary symbol $\leq$ for the entire section.

**Theorem 32.** *Let $\mathbb{K}$ be a class of all possible finite linear orders $L \subseteq A$. Then the formula*

$$\forall x \exists y : x < y$$

*is enforceable in $\mathbb{K}$.*

*Proof.* We will show that the Duplicator has a winning strategy for a regular game $G(\psi, \mathbb{K})$, where $\psi$ is the formula stated in the theorem.

Let us consider an enumeration of $A$, say $(a_j)_{j=1}^{\infty}$. In each of Duplicator's moves he chooses the first element $a_j$ which has not been used yet and appends it above the maximum element.

The resulting structure does not have a maximum element, for if $x$ is the maximum element, then let us find an index $k$ of that element in the enumeration $(a_j)_{j=1}^{\infty}$ and consider any element with an index larger than $k$ which the Duplicator used. Since the game is regular, there will always be such an element. □

An analogous theorem can be stated for the minimum element, therefore the nonexistence of endpoints is an enforceable property.

Now let us look into whether the density of a linear order is an enforceable property in various settings.

**Theorem 33.** *Let $\mathbb{K}$ be a class of all possible finite linear orders $L \subseteq A$. Then the formula*

$$\forall x, y : (x < y \rightarrow \exists z : x < z < y)$$

*is enforceable in $\mathbb{K}$.*

*Proof.* Let us denote that formula by $\psi$. We will show that the Duplicator has a winning strategy for a regular game $G(\psi, \mathbb{K})$.

Let us fix an enumeration of all elements of the set $A$, say as a sequence $(a_j)_{j=1}^{\infty}$. In each of Duplicator's moves he is faced with a finite linear order $\mathbb{A}_i$ on some of the elements of $A$, say $\mathbb{A}_i = \{x_1 < x_2 < \cdots < x_n\}$. Duplicator's strategy in each move is to select the first $n-1$ elements $a_{j_1}, \ldots, a_{j_{n-1}}$ of the enumeration $(a_j)_{j=1}^{\infty}$ which are not present in $\mathbb{A}_i$ and embed the structure $\mathbb{A}_i$ within $\mathbb{A}_{i+1} := \{x_1 < a_{j_1} < x_2 < a_{j_2} \cdots < a_{j_{n-1}} < x_n\}$.

To prove that the resulting structure $\mathbb{S}$ satisfies the formula $\psi$, we only need to arbitrarily select two elements $x, y$ which satisfy $x < y$ and consider the earliest structure $\mathbb{A}_i$ which contains both of them. There has to be such a structure, because the elements have indices in the enumeration and the Duplicator plays infinitely many moves, as he is playing a regular game. Then $\mathbb{A}_{i+1}$ already contains an element $z$ satisfying $x < z < y$. □

**Theorem 34.** *Let $V_1, \ldots, V_m \subseteq A$ be a disjoint decomposition of $A$ such that every $V_i$ is infinite and let $\mathbb{K}$ be a class of all possible finite linear orders $L \subseteq A$ such that every two elements $x, y \in L$ satisfy $x < y$ whenever $x \in V_i, y \in V_j$ and $i < j$. Then the formula*

$$\forall x, y : (x < y \rightarrow \exists z : x < z < y)$$

*is enforceable in* $\mathbb{K}$.

*Proof.* Let us denote that formula by $\psi$. We will show that the Duplicator has a winning strategy for a regular game $G(\psi, \mathbb{K})$.

Let us fix enumerations of all elements of our sets $V_k$, say as sequences $(v_j^k)_{j=1}^{\infty}$. In each of Duplicator's moves he is faced with a finite linear order $\mathbb{A}_i$ with elements $x_j^k$ from the sets $V_k$, ordered as the theorem states.

In his first move, the Duplicator makes sure that $\mathbb{A}_{i+1}$ coincides with every $V_k$ by adding every missing $v_1^k$ into an appropriate spot according to the order condition stated in the theorem.

The Duplicator also embeds the structure $\mathbb{A}_i$ segment-by-segment in his every move: Suppose that $\{x_1 < \cdots < x_{n_k}\}$ is a segment of $\mathbb{A}_i$ which contains all used elements of $V_k$. Then the corresponding segment of $\mathbb{A}_{i+1}$ is $\{v_{j_1}^k < x_1 < v_{j_2}^k < x_2 < \cdots < x_n < v_{j_{n_k+1}}^k\}$, where $v_{j_1}^k, \ldots, v_{j_{n_k+1}}^k$ are the first $n_k + 1$ elements of the enumeration $(v_j^k)_{j=1}^{\infty}$ which have not been used yet.

To prove that the resulting structure $\mathbb{S}$ satisfies the formula $\psi$, we only need to arbitrarily select two elements $x, y$ which satisfy $x < y$ and consider the earliest structure $\mathbb{A}_i$ which contains both of them. There has to be a structure which contains these two elements, because they have indices in the enumeration and the Duplicator plays infinitely many moves. Then $\mathbb{A}_{i+1}$ already contains an element $z$ satisfying $x < z < y$. □

A slight improvement upon this theorem would be to drop the decomposition requirement and have $V_1, \ldots, V_m$ as just pairwise-disjoint, infinite subsets of $A$. In that case the Duplicator can achieve his goal using the demonstrated strategy without any change.

It should be noted that the strategy demonstrated in this proof also enforces the nonexistence of endpoints, therefore the existence of endpoints is not an enforceable property.

Theorem 34 can also be modified to a version with infinitely many infinite sets $V_i$.

**Theorem 35.** *Let* $V_1, \dots \subseteq A$ *be a disjoint decomposition of* $A$ *such that every* $V_i$ *is infinite and let* $\mathbb{K}$ *be a class of all possible finite linear orders* $L \subseteq A$ *such that for* $x \in V_i, y \in V_j$ *we have* $x < y$ *whenever* $i < j$*. Then the formula*

$$\forall x, y : (x < y \rightarrow \exists z : x < z < y)$$

*is enforceable in* $\mathbb{K}$.

*Proof.* Let us denote that formula by $\psi$. We will show that the Duplicator has a winning strategy for a regular game $G(\psi, \mathbb{K})$.

Let us fix enumerations of all elements of the sets $V_k$, say as sequences $(v_j^k)_{j=1}^{\infty}$. In each of Duplicator's moves he is faced with a finite linear order $\mathbb{A}_i$ with elements $x_j^k$ from sets $V_k$, ordered as the theorem states. Suppose that $k$ is the largest index such that $\mathbb{A}_i$ coincides with $V_k$. The Duplicator first adds missing elements $v_1^l, l = 1, \ldots, k+1$ into their appropriate spots according to the condition stated in the theorem. After this step, he embeds the structure segment-by-segment, as he did in the previous proof: Suppose that $\{x_1 < \cdots < x_{n_k}\}$ is a segment of $\mathbb{A}_i$ which contains all used elements of $V_k$. Then the corresponding segment of $\mathbb{A}_{i+1}$

is $\{v^k_{j_1} < x_1 < v^k_{j_2} < x_2 < \cdots < x_n < v^k_{j_{n+1}}\}$, where $v^k_{j_1}, \ldots, v^k_{j_{n+1}}$ are the first $n+1$ elements of the enumeration $(v^k_j)^\infty_{j=1}$, which have not been used yet.

To prove that the resulting structure $\mathbb{S}$ satisfies the formula $\psi$, we only need to arbitrarily select two elements $x, y$ which satisfy $x < y$ and consider the earliest structure $\mathbb{A}_i$ which contains both of them. There has to be a structure which contains these two elements, because they have indices in the enumeration and the Duplicator plays infinitely many moves. Then $\mathbb{A}_{i+1}$ already contains an element $z$ satisfying $x < z < y$. □

Similarly to the note following Theorem 34, we should note that the assumption of this theorem can be weakened from disjoint decomposition to pairwise-disjoint subsets.

We can mention concrete examples to which Theorems 34 and 35 apply. In this scope, let $A := \mathbb{N}^+$. For the first example, let $\mathbb{K}$ be the class of all linear orders where every even number is smaller than every odd number. For a second one, let $\mathbb{K}$ be the class of all linear orders where $x < y$ iff $x$ has fewer prime divisors than $y$.

To make a non-example, we can consider $\mathbb{K}$ to be the class of all finite linear orders $L \subseteq A$, where the order $\leq_\mathbb{K}$ is a restriction of $\leq_G$ which preserves neighbourships. More formally: $\mathbb{A} \leq_\mathbb{K} \mathbb{B}$ iff $b$ is a successor of $a$ in $\mathbb{B}$ whenever $b$ is a successor of $a$ in $\mathbb{A}$. Let us show that in this setting it is impossible to enforce the density of the linear order:

**Theorem 36.** *Let $\mathbb{K}$ be a class of finite linear orders $L \subseteq A$ and let $\leq_\mathbb{K}$ be the order of $\mathbb{K}$ described in the previous paragraph. Then the density of the linear order is not enforceable.*

*Proof.* We have previously noted that two formulas which are contradictory with each other can not both be enforceable. It is easy to see that the resulting structure of any play is a discrete linear order and therefore it can not be dense. □

I have promised to demonstrate that enforceability of a disjunction does not imply enforceability of either of the disjuncts. Let $A := \mathbb{N}$, consider $\varphi_1$ to be a $\mathcal{L}(1, 2)$-sentence stating that $1 \leq 2$, consider $\varphi_2$ to be a $\mathcal{L}(1, 2)$-sentence stating that $2 \leq 1$ and let $\mathbb{K}$ be the class of all finite linear orders $\mathbb{A}_i \subseteq A$.

We have already demonstrated that it is easy to ensure that all elements of $A$ are used, therefore $\varphi_1 \vee \varphi_2$ is most certainly enforceable, however since the Spoiler is the first one to move in any standard game, it is impossible to enforce either of these two formulas individually.

## 4.4 Enforceability in linear orders with a unary predicate

Let us now extend some of our previous theorems to their predicate versions. For the entire section let $\mathcal{L}$ be a language of two relational symbols: binary $\leq$ and unary $U$.

**Theorem 37.** *Let $\mathbb{K}$ be a class of all possible finite linear orders $L \subseteq A$ with all possible unary predicates. Then the formulas*

$$\forall x, y : (x < y \to \exists z : x < z < y, U(z))$$

*and*

$$\forall x, y : (x < y \to \exists z : x < z < y, \neg U(z))$$

*are enforceable in $\mathbb{K}$.*

*Proof.* The proof is a slight modification of the proof of Theorem 33.

Let us denote the first formula by $\psi$. We will show that the Duplicator has a winning strategy for a regular game $G(\psi, \mathbb{K})$. The winning strategy for the second formula is analogous and we can merge strategies according to Proposition 31.

Let us fix an enumeration of all elements of the set $A$, say as a sequence $(a_j)_{j=1}^{\infty}$. In each of Duplicator's moves he is faced with a finite linear order $\mathbb{A}_i$ on some of the elements of $A$, say $\mathbb{A}_i = \{x_1 < x_2 < \cdots < x_n\}$. The Duplicator's strategy in each move is to select the first $n-1$ elements $a_{j_1}, \ldots, a_{j_{n-1}}$ of the enumeration $(a_j)_{j=1}^{\infty}$ which are not present in $\mathbb{A}_i$ and embed the structure $\mathbb{A}_i$ within $\mathbb{A}_{i+1} := \{x_1 < a_{j_1} < x_2 < a_{j_2} < \cdots < a_{j_{n-1}} < x_n\}$ such that every $a_{j_i}$ satisfies $U$.

To prove that the resulting structure $\mathbb{A}$ satisfies the formula $\psi$, we only need to arbitrarily select two elements $x, y$ which satisfy $x < y$ and consider the earliest structure $\mathbb{A}_i$ which contains both of them. There has to be a structure which contains these two elements, because they have indices in the enumeration and the Duplicator plays infinitely many moves. Then $\mathbb{A}_{i+1}$ already contains an element $z$ satisfying $x < z < y$ and $U(z)$. ☐

**Theorem 38.** *Let $V_1, \ldots, V_m \subseteq A$ be a disjoint decomposition of $A$, let $V^U$ and $V^{\neg U}$ be another disjoint decomposition of $A$ and let $\mathbb{K}$ be a class of all possible finite linear orders $L \subseteq A$ such that for $x \in V_i, y \in V_j$ we have $x < y$ whenever $i < j$ and such that $U(x)$ iff $x \in V^U$ and $\neg U(x)$ iff $x \in V^{\neg U}$. Then the formulas*

$$\forall x, y : (x < y \to \exists z : x < z < y, U(z))$$

*and*

$$\forall x, y : (x < y \to \exists z : x < z < y, \neg U(z))$$

*are enforceable in $\mathbb{K}$ if all $V_i \cap V^U$ and $V_i \cap V^{\neg U}$ are infinite.*

*Proof.* Let us denote the first formula by $\psi$. We will show that the Duplicator has a winning strategy for a regular game $G(\psi, \mathbb{K})$. The winning strategy for the second formula is analogous and we can merge strategies according to Proposition 31.

Let us fix enumerations of all elements of sets $V_k \cap V^U$, say as sequences $(v_j^k)_{j=1}^{\infty}$. In each of Duplicator's moves he is faced with a finite linear order $\mathbb{A}_i$ with elements $x_j^k$ from the sets $V_k$ ordered as the theorem states. In his first move, the Duplicator makes sure that $\mathbb{A}_{i+1}$ coincides with every $V_k$ by adding every missing $v_1^k$ into an appropriate spot according to the condition stated in the theorem.

After this precaution is taken, the strategy is to embed the structure segment-by-segment, as we did previously: Suppose that $\{x_1 < \cdots < x_{n_k}\}$ is a segment of $\mathbb{A}_i$ which contains all used elements of $V_k$. Then the corresponding segment of $\mathbb{A}_{i+1}$ is $\{x_1 < v^k_{j_1} < x_2 < \cdots < v^k_{j_{n-1}} < x_n\}$, where $v^k_{j_1}, \ldots, v^k_{j_{n-1}}$ are the first $n-1$ elements of the enumeration $(v^k_j)^{\infty}_{j=1}$, which have not been used yet.

To prove that the resulting structure $\mathbb{A}$ satisfies the formula $\psi$, we only need to arbitrarily select two elements $x, y$ which satisfy $x < y$ and consider the earliest structure $\mathbb{A}_i$ which contains both of them. There has to be a structure which contains these two elements, because they have indices in the enumeration and the Duplicator plays infinitely many moves. Then $\mathbb{A}_{i+1}$ already contains an element $z$ satisfying $x < z < y$ and $U(z)$. □

Similarly to the format of the previous section, this theorem can be modified into a version with infinitely many infinite sets $V_i$.

**Theorem 39.** *Let $V_1, \dots \subseteq A$ be a disjoint decomposition of $A$, let $V^U$ and $V^{\neg U}$ be another disjoint decomposition of $A$ and let $\mathbb{K}$ be a class of all possible finite linear orders $L \subseteq A$ such that for $x \in V_i, y \in V_j$ we have $x < y$ whenever $i < j$ and such that $U(x)$ iff $x \in V^U$ and $\neg U(x)$ iff $x \in V^{\neg U}$. Then the formulas*

$$\forall x, y : x < y \to \exists z : x < z < y, U(z)$$

*and*

$$\forall x, y : x < y \to \exists z : x < z < y, \neg U(z)$$

*are enforceable in $\mathbb{K}$ if all $A_i \cap A^U$ and $A_i \cap A^{\neg U}$ are infinite.*

*Proof.* Let us denote the first formula by $\psi$. We will show that the Duplicator has a winning strategy for a regular game $G(\psi, \mathbb{K})$. The winning strategy for the second formula is analogous and we can merge strategies according to Proposition 31.

Let us fix enumerations of the elements of the sets $V_k \cap V^U$, say as sequences $(v^k_j)^{\infty}_{j=1}$. In each of Duplicator's moves he is faced with a finite linear order $\mathbb{A}_i$ with elements $x^k_j$ from the sets $V_k$ ordered as the theorem states. Suppose that $k$ is the largest index such that $\mathbb{A}_i$ coincides with $V_k$. The duplicator adds missing elements $v^l_1, l = 1, \ldots, k+1$ into their appropriate spots according to the condition stated in the theorem. The second step of the strategy is to embed the structure segment-by-segment, as we did in previous proofs: Suppose that $\{x_1 < \cdots < x_{n_k}\}$ is a segment of $\mathbb{A}_i$ which contains all used elements of $V_k$. Then the corresponding segment of $\mathbb{A}_{i+1}$ is $\{x_1 < v^k_{j_1} < x_2 < \cdots < v^k_{j_{n-1}} < x_n\}$, where $v^k_{j_1}, \ldots, v^k_{j_n}$ are the first $n$ elements of the enumeration $(v^k_j)^{\infty}_{j=1}$, which have not been used yet.

To prove that the resulting structure $\mathbb{A}$ satisfies the formula $\psi$, we only need to arbitrarily select two elements $x, y$ which satisfy $x < y$ and consider the earliest structure $\mathbb{A}_l$ which contains both of them. There has to be a structure which contains these two elements, because the elements have indices in the enumeration and the Duplicator plays infinitely many moves. Then $\mathbb{A}_{l+1}$ already contains an element $z$ satisfying $x < z < y$ and $U(z)$. □

Both of these theorems can be modified to not require disjoint decompositions but only pairwise-disjoint subsets for both the order and the predicate. The conditions for enforceability can then be weakened to "every $A_i \cap (A \setminus A^{\neg U})$ and every $A_i \cap (A \setminus A^U)$ is infinite" and in both proofs we then enumerate through these sets instead of $A_i \cap A^U$ and $A_i \cap A^{\neg U}$, but this change has no further consequence.

## 4.5 Inductively-defined Forcing in linear orders

We have mentioned some elementary properties of the game-defined forcing in the previous sections, such as the fact that $\phi \wedge \psi$ is enforceable iff $\phi$ is enforceable and $\psi$ is enforceable. This leads to a more abstract definition of forcing, which can be found for example in [5].

Let $\mathcal{L}$ be a language and let $T$ be a theory in $\mathcal{L}$. We call a set of basic $\mathcal{L}(A)$-sentences $P$ a forcing condition iff $T \cup P$ is consistent. The enforceability of an $\mathcal{L}(A)$-formula $\phi$ by the set $P$, from now on denoted as $P \Vdash \phi$, is defined inductively on the formula structure:

- If $\phi$ is atomic, then $P \Vdash \phi$ iff $\phi \in P$
- If $\phi = \phi_1 \wedge \phi_2$, then $P \Vdash \phi$ iff $P \Vdash \phi_1$ and $P \Vdash \phi_2$
- If $\phi = \phi_1 \vee \phi_2$, then $P \Vdash \phi$ iff $P \Vdash \phi_1$ or $P \Vdash \phi_2$
- If $\phi = \neg\psi$, then $P \Vdash \phi$ iff there is no $Q \supseteq P$ such that $Q \Vdash \psi$
- If $\phi = \exists x : \psi(x)$, then $P \Vdash \phi$ iff there is a term $t$ such that $P \Vdash \psi(t)$

We say that a formula $\phi$ is weakly forced by $P$ iff $P \Vdash \neg\neg\phi$ and denote such statement by $P \Vdash^w \phi$.

We will not make an effort to rigorously relate this definition to ours, as we do not desire any additional results based upon it, but I still think that it is worth mentioning as an alternative to our game-related approach. The introductory part of [5] presents a few not entirely trivial results which we do not have access to with our game-based definition.

# Conclusion

This concludes our exploration of methods of limit construction. Regarding the three main chapters, we have summarized the necessary relevant theory in each of them and we have shown how the construction behaves in the case of finite linear orders with endpoints.

In chapter 2, we spent a nontrivial amount of time by verifying the assumptions that the construction requires from the class of structures. That was an investment which turned out to be justified by the uniqueness of the limit up to isomorphism and later by the apparent robustness of the method – the introduction of an additional predicate complicated the situation but did not cause any trouble to the method itself. In this chapter it is impossible to highlight one theorem more noteworthy than the others – the strategy leading up to Theorem 16 was an easy consequence of Theorem 6 and the proof techniques used in the following individual pieces of the puzzle were mostly well known and straightforward.

In chapter 3, we explored a method which posed lesser requirements on the class of structures, but also provided a weaker uniqueness result. The bulk of the effort was concentrated into proofs about the existence of winning strategies in Ehrenfeucht-Fraïssé games. We showed that the method is not as robust as Fraïssé's amalgamation – in the case with a unary predicate there are easily definable sequences of structures which this method does not handle at all. I consider Theorem 27 to be the highlight of this chapter, despite the fact that the proof strategy is a variation of the non-predicate version, which is a variation of a proof strategy demonstrated by [4], which is not new to the world of mathematics.

In chapter 4, we explored a method which required simpler conditions than Fraïssé's amalgamation but, by its nature, was more robust than our compactness-related limit. It, however, generally cannot guarantee uniqueness up to isomorphism or elementary equivalence. The most notable result of the chapter and possibly of the entire thesis is about the enforcability of predicate density – Theorems 38 and 39. The length and difficulty of given proofs does, however, not reflect this opinion.

It turns out that in our examples the forcing limit and the Fraïssé limit give similar results: The Fraïssé limit of the class of finite linear orders is the ordered set of rational numbers, while the forcing limit contains the formula for nonexistence of endpoints and for density of the linear order. It is a well known fact that the only countable model of the theory of dense linear orders without endpoints is the ordered set of rational numbers (up to isomorphism, of course).

After the introduction of a unary predicate, not much changes – the Fraïssé limit of the class of finite linear orders with all possible unary predicates is the ordered set of rational numbers with a dense and co-dense unary predicate, while the forcing limit contains the formulas for density and co-density of the predicate. The forcing method does have one practical advantage over Fraïssé's amalgamation – it allows us to gather information in small pieces. The verification of enforceability of an individual sentence seems to be easier than guessing an entire structure and proceeding to check whether it satisfies the conditions demanded by Fraïssé's amalgamation.

The outcomes of the compactness construction do not seem relatable to the other two approaches. However, out of our three constructions, it is the only one which preserves properties definable by first-order sentences, such as the discreteness of the linear order.

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