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Generalized Complex Geometry

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Title: Generalized Complex Geometry

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Abstract: In an attempt to unify the underlying geometry of Hamilton's equations with the language of complex geometry, we motivate the research of generalized complex geometry. We construct the structure of a Courant algebroid on the direct sum of the tangent and cotangent bundle $TM \oplus T^*M$ as we research the Courant bracket. The key notion of an involutive fibre-wise isotropic subbundle, a Dirac structure, is introduced and serves to specify a generalized complex structure. Generalized complex submanifolds are mentioned as well as the process of Dirac redution. Generalized complex geometry and the natural mechanisms in the Courant algebroid setting are then utilised as an interpretational tool in mathematical physics and related areas. We study a reduction of the symplectic structure of a harmonic oscillator, reflect on the nature of the Dirac bracket in string theory and relate a solution of a PDE to a generalized complex submanifold through the Monge-Ampère equations.

Keywords: Courant algebroid Dirac structure generalized complex structure Dirac reduction Dirac bracket Monge-Ampère equations

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Introduction

There is no guarantee that a mathematical physicist understands *how the world works.* Indeed, many object which constitute the world are simply well beyond the reach of their field. Objects such as a toaster, medieval Japanese poetry and people, to name a few. And yet, mathematical physicists seem to seek some sort of a universal understanding as they strongly tend to *generalize*, blurring the edges of Popper's notion of science. *A serious case of platonism*, one can almost hear the judging voice of either an experimental physicist or a metaphysician. In a reaction to such categorizations, the author wishes to remark on a different motivation for investigations in the field of mathematical physics.[1]

The author does not believe there is a mythical *deeper truth* in highly mathematical formulations of physical theories. The statement is that a *richer epistemology* unfolds. It is essential to differentiate between *richness* and *redundancy.* A rich mathematical formalism both simplifies the considered physical statements and unfurls beyond them.

The intriguing object of research is the very interplay of different approaches, a more physically oriented one and an abstracted mathematical one. It is not the truths of the objective world we aim to study, on the contrary, we intend to research the subject itself, the human. The purpose is not to understand the world, but to understand human's understanding of the world and the role of language (mathematics) in this context. This thesis will serve as an illustration of such approach with the standard narrative: a motivation rooted in physics, development of a general mathematical theory, examples of applications in physics. The topic is the modern theory of generalized geometry. We will study Dirac structures, Dirac reduction and in particular, generalized complex stuctures.

Geometry indeed has the long-standing tradition of being considered a somewhat platonic frame of reference. From the days of ancient Greece to coordinate-free formulations of general relativity. *Generalized geometry* has roots in the work of Theodore James Courant, its early pioneer (see Courant [1990]), among whose motivations was to formulate Hamilton's equations with general constraints. The proper perspective turned out to be considering the Whitney sum of the tangent and the cotangent bundle $TM \oplus T^*M$. Further development soon began, the essential figures were Zhang-Ju Liu, Alan Weinstein and Ping Xu (see Liu et al. [1995]). The central notion of the Courant algebroid was introduced. In early 2000s, Nigel Hitchin (see Hitchin [2002]) together with his students (e.g. Gualtieri [2004]) developed the complex branches of generalized geometry. Complex and symplectic geometry was unified. Emerging applications in mathematical physics include topological field thoeries, Cattaneo et al. [2010], or supergravity, Jurčo and Vysoký [2016].

The author hopes this thesis may serve as an introduction to generalized complex geometry for anyone with the knowledge of basic differential geometry.

[1] And even more so, *physical mathematics.*

1. Differential geometry preliminaries and notation remarks

We presuppose knowledge of elementary differential geometry. Regarding notation and conventions, we refer to the textbook of Fecko [2006], chapters 1-13, 17, 19 and Nakahara [2003], chapter 9. However, we wish to remark on a few choices of notation which might otherwise be unclear or ambiguous.

- The image of a morphism f is denoted by $\mathrm{Im}\,(f)$ and the kernel of f is denoted by $\ker\,(f)$.
- A span of a collection of vectors $X_1, \dots, X_n$ is denoted by $\mathrm{Span}(X_1, \dots, X_n)$. The dual of a vector space V is denoted by V^*, the natural annihilator is $\mathrm{Ann}(V)$. Isomorphic vector spaces V, W are denoted $V \simeq W$. A *factorspace* or quotient of two vector spaces[1] V, W is denoted as A/B or $\frac{A}{B}$.
- The notion of an *exact sequence*[2] will be used for a collection of morphisms f, g and vector spaces A, B, C

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

  such that f is a monomorphism and g is an epimorphism and $\mathrm{Im}\,(f) = \ker g$. Note that for such a sequence, $C \simeq B/\mathrm{Im}\,(f)$.
- We say manifolds M, N are equivalent if and only if they are *diffeomorphic*, which is denoted as $M \approx N$. By a *chart*, we will refer either to the pair (U, φ), where $U \subset M$ is homeomorphic to $\mathbb{R}^n$ and $\varphi : U \longrightarrow \mathbb{R}^n$ is the coordinate homeomorphism, or to φ itself. We will refer to U as the domain of the chart. It is imporant that even though we might choose to describe any manifold by local complex coordinate charts, we do not presuppose complex differentiability of the transition maps.
- The *algebra of smooth functions* on a manifold M, $f : M \longrightarrow \mathbb{R}$ is denoted as $C^\infty(M)$.
- A *vector bundle* over a manifold M denoted as $\pi : E \longrightarrow M$ or simply E.
- When mentioning *fibre-wise* properties of a vector bundle, they are satisfied for each single fibre F_x separately. Vector bundles E, E' over the same base manifold which are equivalent as bundles and also fibre-wise isomorphic are denoted as $E \simeq E'$. A *subbundle* L of E is denoted as $L < E$.
- The *space of smooth sections* of a vector bundle E is denoted as $\Gamma(E)$.

[1] Or vector bundles, where the quotient is considered fibre-wise.
[2] We will only consider *short* exact sequences.

- The *contraction* or *pairing* of elements X, ξ of two mutually dual bundles is equivalently denoted as: $\xi\left(X\right) \equiv \langle \xi, X \rangle \equiv i_X \xi$. For A being a tensor section $A \in \Gamma\left(T^k_l M\right)$ of type (k,l), by $i_X A$, $A(X, \bullet, \ldots)$ or $A(X)$ we denote inserting X into the first argument.

- We will often think of *tensor sections as bundle morphisms*. E.g. for a 2-form $\omega \in \Omega^2\left(M\right)$, which is a section $\omega : \Lambda^2\left(M\right) \longrightarrow M$ we consider the morphism $\omega : TM \longrightarrow T^*M$ defined as $\omega(X)$ for $X \in TM$. The composition of such morphism is, for clarity[3], denoted by the $\circ$ symbol, e.g. $\omega \circ \omega^{-1} = 1$. A *dual morphism* ω^* is defined in such a manner that:

$$\langle \omega(X), Y \rangle = \langle X, \omega^*(Y) \rangle$$

  for $\forall X, Y \in \Gamma\left(TM\right)$. In such formalism, a *symmetric*[4] tensor satisfies $g^* = g$, a *skew* tensor is satisfies $\omega^* = -\omega$.

- Throughout the thesis, *Einstein summation convention* is used. We note that any tensors denoted in index notation are to be thought of as their coordinate forms. In case we wish to formulate formulae in a coordinate-free way, we do so index-free as well.

- The *wedge product* $\wedge : \Omega^p \times \Omega^q \longrightarrow \Omega^{p+q}$ on is defined as:

$$\wedge := \frac{(p+q)!}{p!q!} \mathcal{A} \circ \otimes$$

  where $\mathcal{A}$ is the projection to the space of anti-symmetric p-forms. The *Lie derivative* is defined on $\Omega^p\left(M\right)$ by Cartan's formula: $\mathcal{L}_X := i_X \mathrm{d} + \mathrm{d} i_X$.

- A *restriction* of a function X to a subset of its domain $A \subset B$ is denoted as $X|_A$. It is worth noting that for sections of tensor bundle, we will talk about two different kinds of restrictions denoted similarly.

  For a submanifold $Q \hookrightarrow M$ and a tensor section $\sigma : M \longrightarrow T^k_l M$, the restriction $\sigma|_Q$ is the section evaluated on the submanifold $\sigma\left(q\right)$, $q \in Q \hookrightarrow M$. A stronger kind of restriction is, for example, $\alpha|_{TQ}$ which dictates the arguments of a 2-form α as a morphism $TM \longrightarrow T^*M$ can only be taken from TQ.

[3]To distinguish them from coordinate matrix multiplication or contraction in other indeces.
[4]Here we only consider tensors with two arguments of the same type.

2. Hamiltonian and complex geometry

To introduce one into generalized geometry[1], rather than presenting the technical starting-points in the very beginning, we intend to choose a somewhat anachronic narrative. That is to show traces of a possible generalization of known geometries while the proper technical language will emerge.

We will summarize what defines Poisson and symplectic geometry and how they arise from classical mechanics. Futher on, we will present an overview of complex geometry. In an attempt to unify these seemingly unrelated structures, we find ourselves at the beginning of our studies of generalized geometry.

2.1 Hamilton's equations

It is an awe-inspiring epistemological journey that leads one from the very Newton's laws of motion to the 20th century's geometry of space and momentum. We, however, restrain oureselves to the modern formulations and demonstrate the correspondence of the linear partial differential equations of first order defining dynamics widely known[2] as the *canonical Hamilton's equations* and abstract structures on manifolds, namely the *Poisson structure* and the *symplectic structure*. The references for this chapter are the comprehensive textbooks of Fecko [2006] and Nakahara [2003].

Let x^j, p_j denote the canonically conjugate[3] coordinates (of the position and the momentum) on the **phase space**, which is the $2n$-dimensional cotangent bundle T^*Q of an n-dimensional configuration manifold[4] Q. Let $\dot{x}^j$, $\dot{p}_j$ denote the total time derivative of the coordinates. Then the Hamiltonian function $H \in C^\infty(T^*Q)$, which corresponds to the total energy of a system, satisfies the **Hamilton's equations**.

$$\dot{x}^j = \frac{\partial H}{\partial p_j},$$
$$\dot{p}_j = -\frac{\partial H}{\partial x^j},$$

for $\forall j = 1, \dots, n$.

Let us further work only with a time-independent Hamiltonian function and configuration manifold Q

[1] And as is our intention, generalized *complex* geometry in particular.

[2] And widely used across numerous areas of contemporary physics.

[3] For x^j given, p_j are defined in such a way that $p = p_a \mathrm{d}x^a$ is invariant under a change of coordinates $x^j \mapsto \tilde{x}^j(x^j)$.

[4] Where Q is defined by a set of holonomic constraints, i.e. constraints independent of momentum.

One can define the *time evolution field* X_t on the phase space, which is a tangent section $X_t : T^*Q \longrightarrow TT^*Q$ which acts as the total derivative with respect to time on $C^\infty(TT^*Q)$.

$$X_t := \dot{x}^j \frac{\partial}{\partial x^j} + \dot{p}_k \frac{\partial}{\partial p_k} \Longleftrightarrow X_t(f) := \dot{f}, \quad \forall f \in C^\infty(TT^*Q).$$

Then the Hamilton's equations can be read as a specification of X_t.

$$X_t = \frac{\partial H}{\partial p_j}\frac{\partial}{\partial x^j} - \frac{\partial H}{\partial x^k}\frac{\partial}{\partial p_k}.$$

2.1.1 Poisson and symplectic geometry

Definition (Lie bracket)**.** *Let V be a vector space over a field $\mathbb{K}$. Then the operation $[,] : V \times V \longrightarrow V$ is called a **Lie bracket** if it satisfies the following axioms for $\forall A, B, C \in V$, $\forall r, s \in \mathbb{K}$.*

- Linearity[5] $[A, rB + sC] = r[A, B] + s[A, C]$
- Skew-symmetry $[A, B] = -[B, A]$
- Jacobi identity $[[A, B], C] + [[C, A], B] + [[B, C], A] = 0$

Definition (Poisson bracket)**.** *Let M be a manifold. Then the **Poisson bracket** is defined as a Lie bracket $\{,\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ which is a derivation of the $C^\infty(M)$ algebra, i.e.:*

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

for $\forall f, g, h \in C^\infty(M)$.

It is straight-forward to show that the bracket defined in canonical coordinates as:

$$\{f, g\} = \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial x^k} - \frac{\partial f}{\partial x^j}\frac{\partial g}{\partial p_j}$$

is a Poisson bracket on $C^\infty(T^*Q)$.

One can reformulate the Hamilton's equations using the Poisson bracket.

$$\dot{x}^j = \left\{H, x^j\right\},$$
$$\dot{p}_j = \{H, p_j\}.$$

Given the Poisson bracket obeys the chain rule, a time evolution of an arbitrary smooth function $f \in C^\infty(T^*Q)$ is given as:

$$\dot{f} = \{H, f\}.$$

If we consider that any derivation of $C^\infty(T^*Q)$ is a tangent section on T^*Q, we realize that we have obtained a tangent section $\{H, \bullet\}$ which acts as the total derivative in the direction of time evolution. In other words:

$$X_t = \{H, \bullet\}.$$

[5] Skew-symmetry provides the bracket with bilinearity, i.e. linearity in both arguments.

Definition (Poisson structure)**.** *A bivector section* $\Pi \in \Gamma\left(T^2_0 M\right)$ *is said to be a* ***Poisson structure*** *if and only if there exists such Poisson bracket* $\{,\}$ *that:*

$$\{f, g\} = \Pi\left(\mathrm{d}f, \mathrm{d}g\right)$$

for $\forall f, g \in C^{\infty}\left(M\right)$*.*

In canonical coordinates, it can be shown that:

$$\{H, f\} = \frac{\partial}{\partial p_j} \wedge \frac{\partial}{\partial x^j}\left(\mathrm{d}H, \mathrm{d}f\right).$$

In other words, if we define the Poisson structure:

$$\Pi = \frac{\partial}{\partial p_j} \wedge \frac{\partial}{\partial x^j},$$

we can rewrite Hamilton's equations once again:

$$X_t = \Pi\left(\mathrm{d}H\right).$$

Another way to arrive at a coordinate-free formulation is to consider the following coordinates on the phase space T^*Q.

$$\left(z^1, \dots, z^{2n}\right) := \left(x^1, \dots, x^n, p_1, \dots, p_n\right).$$

The Hamilton's equations now read:

$$\begin{aligned} \dot{z}^j &= \frac{\partial H}{\partial z^{n+j}}, \\ \dot{z}^{n+j} &= -\frac{\partial H}{\partial z^j}, \end{aligned}$$

for $\forall j = 1, \dots, n$.

We note there is a mechanism that switches betwen the first n coordinates and the second n coordinates. Let us formulate it by a skew $2n \times 2n$ matrix.

$$\Omega_{ab} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We also note that the right hand side of the Hamilton's equations is the components of the cotangent section $\mathrm{d}H$ and the left side is the components of X_t. Now we can write:

$$\Omega_{ab}\left(X_t\right)^b = \left(\mathrm{d}H\right)_a.$$

One can easily show that in canonical coordinates, the matrix Ω_{ab} corresponds to the components of the non-degenerate closed 2-form:

$$\Omega = \mathrm{d}p_j \wedge \mathrm{d}x^j.$$

The coordinate-free formulation of the Hamilton's equations which we will further consider is the following.

$$\Omega\left(X_t\right) = -\mathrm{d}H.$$

Let us abstract the mathematical structure of Hamilton's equations.

Definition (Symplectic structure)**.** *Let M be an even dimensional manifold. Then a global 2-form $\Omega \in \Omega^2(M)$ is said to be a **symplectic structure** if and only if it is non-degenerate and closed.*

Definition (Hamiltonian field)**.** *Let M be a manifold equipped with a symplectic structure Ω and a fixed function $H \in C^\infty(M)$. Then a tangent section $X_H \in \Gamma(TM)$ is a **Hamiltonian field** with respect to the function H if and only if it satisfies the Hamilton's equations:*

$$\Omega(X_H) = -\mathrm{d}H.$$

We can now pinpoint the physical statement behind Hamilton equations in the following manner:

$$X_t = X_H.$$

The time evolution field X_t is the Hamiltonian field X_H with respect to the Hamiltonian function H, which is the total energy of the system.

Remark. Inverting a non-degenerate 2-form Ω, we obtain a bivector: $-\Omega^{-1} = \Pi$. It can be shown that:

$$\mathrm{d}\Omega = 0 \Longleftrightarrow \Pi(\mathrm{d}\bullet, \mathrm{d}\bullet) \text{ satisfies the Jacobi identity.}$$

In other words, a symplectic structure inverts into a non-degenerate Poisson structure and vice versa.

2.2 Complex geometry

In this section we present an overview of complex geometry, i.e. structures induced by the concept of complex differentiation on manifolds, which are genuinely richer than ordinary differential structures on real manifolds. Throughout the whole section, we follow Nakahara [2003].

Let us begin on the level of linear algebra by introducing an endomorphism which, in a way, "*behaves like the imaginary unit.*" Soon, we will see how this complex-like structure arises naturally on complex manifolds.

The imaginary unit i is characterised by the property:

$$i^2 = -1.$$

Similarly, on a vector space:

Definition (Complex endomorphism)**.** *Let J be an endomorphism of a real vector space V. We say it is a **complex endomorphism** if and only if:*

$$J^2 = -1.$$

We can immediately see its only eigenvalues are $+i$ and $-i$. To work with complex numbers on a real vector space, we must *complexify* it:

Definition (Complexified vector space)**.** *For a real vector space V of dimension m, we call the space V with values in $\mathbb{C}$ a **complexified vector space** $V \otimes \mathbb{C}$ of dimension $2m$.*

Remark. An endomorphism A on V is naturally extended to $V \otimes \mathbb{C}$:

$$A\left(v + iw\right) := A\left(v\right) + iA\left(w\right).$$

We refer to Nakahara [2003] as we state that such an endomorphism exists only on even-dimensional vector spaces and decomposes $V \otimes \mathbb{C}$ into two disjoint eigenspaces of dimension m.

$$V \otimes \mathbb{C} = V^{+} \oplus V^{-}.$$

Now we insert such algebraic structure into each fibre of a manifold.

Definition (Almost complex structure)**.** *Let M be an even-dimensional manifold. Then an **almost complex structure** is defined as a tensor section J of type $(1,1)$ such that in any fibre T_xM of the tangent bundle it is a complex endomorphism.*

If we fibre-wise complexify the tangent bundle into $TM \otimes \mathbb{C}$, we can recognize the two eigenbundles of J:

Proposition 2.2.1 (Complex structure)**.** *Let M be a manifold equipped with a smooth almost complex structure J. Then its complexified tangent bundle $TM \otimes \mathbb{C}$ of dimension $2m$ is decomposed into two disjoint subbundles of dimension m.*

$$\begin{aligned} TM \otimes \mathbb{C} &= TM^{+} \oplus TM^{-}, \\ TM^{\pm} &:= \left\{ X \in TM \otimes \mathbb{C} \mid J\left(X\right) = \pm iX \right\}. \end{aligned}$$

Similarly, we decompose the space of local sections by a diagonal action of J.

Once the decomposition integrates into submanifolds, we call J an integrable complex structure or simply a complex structure.

Definition (Complex structure)**.** *Let M be a manifold equipped with a smooth almost complex structure J. We say J is a **complex structure** if and only if its eigenbundles are involutive.*

$$\left[TM^{+}, TM^{+}\right] \subset TM^{+}, \qquad \left[TM^{-}, TM^{-}\right] \subset TM^{-}.$$

It is simple to see that we can equivalently talk about the involutiveness of TM^{+} alone.

2.2.1 Complex manifolds

Now we set off from a different perspective following Nakahara [2003], sections 8.1, 8.2, 8.3. That is, the perspective of complex analysis on manifolds.

Definition (Holomorphic function)**.** *A smooth function $f : \mathbb{C}^m \longrightarrow \mathbb{C}$ is said to be* ***holomorphic*** *when it satisfies the Cauchy-Riemann equations.*

$$\frac{\partial \Re(f)}{\partial x^j} = \frac{\partial \Im(f)}{\partial y^j}, \qquad \frac{\partial \Im(f)}{\partial x^j} = -\frac{\partial \Re(f)}{\partial y^j}.$$

Definition (Complex manifold)**.** *A manifold M is a a* ***complex manifold*** *if and only if every transition function is a holomorphic function.*

In other words, any pair of complex coordinates $x^j + iy^j$, $u^k + iv^k$, must satisfy the Cauchy–Riemann equations on the intersection of their domains.

$$\frac{\partial u^k}{\partial x^j} = \frac{\partial v^k}{\partial y^j}, \qquad \frac{\partial v^k}{\partial x^j} = -\frac{\partial u^k}{\partial y^j}.$$

We can define a smooth tensor section of type $(1,1)$ by the fibre-wise endomorphism $J : T_xM \longrightarrow T_xM$.

$$J\left(\frac{\partial}{\partial x^j}\right) := \frac{\partial}{\partial y^j}, \qquad J\left(\frac{\partial}{\partial y^j}\right) := -\frac{\partial}{\partial x^j}.$$

Note that $J^2 = -1$. We demonstrate that J is in fact a *globally* defined almost complex structure.

Let us show that J is defined independently of a chart. For simplicity, we formulate only a sketch of the proof: for $m = 1$ with coordinates $x + iy$, $u + iv$ and only for the first defining relation, the general proof is analogous.

$$\begin{aligned} J\left(\frac{\partial}{\partial u}\right) &= J\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial}{\partial y}\right) = \frac{\partial u}{\partial x} J\left(\frac{\partial}{\partial x}\right) + \frac{\partial u}{\partial y} J\left(\frac{\partial}{\partial y}\right) \\ &= \frac{\partial v}{\partial y}\left(\frac{\partial}{\partial y}\right) - \frac{\partial y}{\partial x}\left(-\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial v}. \end{aligned}$$

We made use of the Riemann-Cauchy equations. The above calculations have demonstrated how J can be thought of as a global tensorial incarnation of the Cauchy-Riemann equations. This observation may gently hint towards the following proposition.

Now we state an essential theorem[6] of complex geometry, which relates the *geometric notion* of a complex structure and the *analytic properties* of a complex manifold.

Theorem 2.2.2 (Newlander and Nirenberg [1957])**.** *A manifold is complex if and only if it admits a complex structure.*

[6] Or more precisely, its corollary. See Nakahara [2003], section 8.7 for a brief discussion.

Let us introduce an extremely useful coordinate system on complex manifolds. For local complex coordinates $x^j + iy^j$ we define the **holomorphic** and **antiholomorphic** coordinates, respectively:

$$z := x^j + iy^j, \qquad \overline{z} := x^j - iy^j.$$

This naturally induces the following local frames.

$$\begin{aligned}
\frac{\partial}{\partial z^j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), & \mathrm{d}z^j &= \mathrm{d}x^j + i\mathrm{d}y^j, \\
\frac{\partial}{\partial \overline{z}^j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right), & \mathrm{d}\overline{z}^j &= \mathrm{d}x^j - i\mathrm{d}y^j.
\end{aligned}$$

Now it is clear that a function f is holomorphic if and only if:

$$\frac{\partial f}{\partial \overline{z}^j} = 0 \iff \frac{\partial}{\partial x^j}\left(\Re\left(f\right) + i\Im\left(f\right)\right) + \frac{\partial}{\partial y^j}\left(i\Re\left(f\right) - \Im\left(f\right)\right) = 0.$$

In other words, a holomorphic function depends only of the holomorphic coordinates $f = f\left(z\right)$.

One can easily see that the (anti)holomorphic coordinates diagonalize the complex structure J.

$$J\left(\frac{\partial}{\partial z^j}\right) = i\frac{\partial}{\partial z^j}, \qquad J\left(\frac{\partial}{\partial \overline{z}^j}\right) = -i\frac{\partial}{\partial \overline{z}^j}.$$

In other words:

$$J = i\mathrm{d}z^j \otimes \frac{\partial}{\partial z^j} - i\mathrm{d}\overline{z}^j \otimes \frac{\partial}{\partial \overline{z}^j}.$$

And locally, on a neihbourhood $U \subset M$:

$$\operatorname{Span}\left(\frac{\partial}{\partial z^j}\right) = \Gamma\left(TU^+\right), \qquad \operatorname{Span}\left(\frac{\partial}{\partial \overline{z}^j}\right) = \Gamma\left(TU^-\right).$$

A corollary is that the complex conjugation is a canonical isomorphism of the eigenbundles of J, TM^+ and TM^-.

Now we procede to describe the (anti)holomorphic decomposition of differential forms on a complex manifold. A general p-form will be locally constituted by r holomorphic differentials $\mathrm{d}z^{j_1}, \ldots, \mathrm{d}z^{j_r}$ and s antiholomorphic differentials $\mathrm{d}\overline{z}^{k_1}, \ldots, \mathrm{d}\overline{z}^{k_s}$, such that $r + s = p$.

$$\omega = \frac{1}{r!s!}\omega_{j_1\ldots j_r k_1 \ldots k_s}\mathrm{d}z^{j_1} \wedge \ldots \wedge \mathrm{d}z^{j_r} \wedge \mathrm{d}\overline{z}^{k_1} \wedge \ldots \wedge \mathrm{d}\overline{z}^{k_s}$$

We say ω is a **differential form of degree**[7] (r, s).

$$\omega \in \Omega^{r,s}\left(M\right).$$

[7]Let us clarify that a differential form of *degree* (r, s) is a tensor section of *type* $(0, r+s)$.

For a global coordinate-free definition, we refer to Nakahara [2003].

The exterior derivative d locally acts on $\Omega^{r,s}(M)$ as:

$$\mathrm{d}\omega = \frac{1}{r!s!}\left(\frac{\partial}{\partial z^a}\omega_{j_1\dots j_r k_1\dots k_s}\mathrm{d}z^a + \frac{\partial}{\partial \overline{z}^b}\omega_{j_1\dots j_r k_1\dots k_s}\mathrm{d}\overline{z}^b\right)\wedge\mathrm{d}z^{j_1}\wedge\dots\wedge\mathrm{d}z^{j_r}\wedge\mathrm{d}\overline{z}^{k_1}\wedge\dots\wedge\mathrm{d}\overline{z}^{k_s}.$$

We can decompose the operator $\mathrm{d}:\Omega^{r+s}(M)\longrightarrow\Omega^{r+s+1}(M)$ into two operators changing the holomorphic and the antiholomorphic degree separately. We define the **Dolbeault operators** ∂, $\overline{\partial}$, such that:

$$\mathrm{d} = \partial + \overline{\partial} \qquad \begin{aligned} \partial &: \Omega^{r,s}(M)\longrightarrow\Omega^{r+1,s}(M) \\ \overline{\partial} &: \Omega^{r,s}(M)\longrightarrow\Omega^{r,s+1}(M) \end{aligned}$$

The local action is clearly:

$$\begin{aligned} \partial\omega &= \frac{1}{r!s!}\left(\frac{\partial}{\partial z^a}\omega_{j_1\dots j_r k_1\dots k_s}\mathrm{d}z^a\right)\wedge\mathrm{d}z^{j_1}\wedge\dots\wedge\mathrm{d}z^{j_r}\wedge\mathrm{d}\overline{z}^{k_1}\wedge\dots\wedge\mathrm{d}\overline{z}^{k_s} \\ \overline{\partial}\omega &= \frac{1}{r!s!}\left(\frac{\partial}{\partial \overline{z}^b}\omega_{j_1\dots j_r k_1\dots k_s}\mathrm{d}\overline{z}^b\right)\wedge\mathrm{d}z^{j_1}\wedge\dots\wedge\mathrm{d}z^{j_r}\wedge\mathrm{d}\overline{z}^{k_1}\wedge\dots\wedge\mathrm{d}\overline{z}^{k_s} \end{aligned}$$

Directly from the definition, one shows that the Dolbeault operators are nilpotent just as d is and that they anticommute.

Lemma 2.2.3. *The Dolbault operators satisfy the following relations.*

$$\partial\partial = 0, \qquad \overline{\partial}\overline{\partial} = 0, \qquad \partial\overline{\partial} = -\overline{\partial}\partial$$

Proof. Let $\omega \in \Omega^{r,s}(M)$.

$$0 = \mathrm{d}\mathrm{d}\omega = \left(\partial+\overline{\partial}\right)\left(\partial+\overline{\partial}\right) = \partial\partial\omega + \left(\partial\overline{\partial}+\overline{\partial}\partial\right)\omega + \overline{\partial}\overline{\partial}\omega \tag{2.1}$$

The proposition follows, as the equation above must be satisfied for $\partial\partial\omega \in \Omega^{r+2,s}(M)$, $\overline{\partial}\overline{\partial} \in \Omega^{r,s+2}(M)$ and $\left(\partial\overline{\partial}+\overline{\partial}\partial\right)\omega \in \Omega^{r+1,s+1}(M)$ separately. $\square$

2.2.2 Kähler geometry

In the section, further following Nakahara [2003], sections 8.4, 8.5, we introduce a metric compatible with the complex structure. We show how the two objects can be combined into a symplectic structure.

Definition (Hermitian metric)**.** *Let g be a Riemannian metric on a complex manifold (M,J). We say it is hermitian if and only if:*

$$g(JX,JY) = g(X,Y).$$

It can be shown that every complex manifold admits a hermitian metric.

Definition (Kähler structure)**.** *On a complex manifold (M,J) equipped with a hermitian metric g, we define the **Kähler structure** Ω as:*

$$\Omega(X,Y) := g(JX,Y)$$

for $\forall X,Y \in \Gamma(TM)$.

Properties of the Kähler structure

- It is an antisymmetric 2-form of degree $(1,1)$, $\Omega \in \Omega^{1,1}(M)$.
- $\Omega \wedge \ldots \wedge \Omega$ constitutes an everywhere non-vanishing volume form, i.e. a complex manifold is orientable.
- Ω is a symplectic form if and only if[8] $\mathrm{d}\Omega = 0$. Then we say g is a *Kähler metric* and (M, J, g) is a **Kähler manifold**.
- On a Kähler manifold, we call the triple (J, g, Ω) a **Kähler triple**. This triple constitutes the following commuting diagram.

$$\begin{tikzcd}
TM \arrow[rr, "g"] & & T^*M \\
& TM \arrow[ul, "J"] \arrow[ur, "\Omega"'] &
\end{tikzcd}$$

In an inverse manner, we can obtain a Kähler metric and its inverse from J and Ω.

Lemma 2.2.4. *Let (J, g, Ω) be a Kähler triple on a Kähler manifold M. Then the following diagrams commute.*

$$\begin{tikzcd}
TM & & T^*M \arrow[ll, "g^{-1}"'] \arrow[dl, "\Omega^{-1}"] \\
& TM \arrow[ul, "J"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^*M & & TM \arrow[ll, "g"'] \arrow[dl, "-\Omega"] \\
& T^*M \arrow[ul, "-J^*"] &
\end{tikzcd}$$

Proof. The first diagram is obtained by inverting non-degenerate morphisms. The second diagram is obtained through dualization of the original diagram and inversion of non-degenerate morphisms. Note that $g^* = g$, $\Omega^* = -\Omega$ and $J^{-1} = -J$.
An explicit proof that $J^* \circ \Omega = g$, for $\forall X, Y \in \Gamma(TM)$ is:

$$\begin{aligned}
\langle J^*(\Omega(X)), Y\rangle &= \langle g(J(X)), J(Y)\rangle = g(J(X), J(Y)) = g(X, Y) \\
&= \langle g(X), Y\rangle .
\end{aligned}$$

$\square$

Kähler geometry is the source of another interseting phenomenon. That is, the complex structure of a manifold enables one to construct symplectic 2-forms from scalar potentials.

Definition (Kähler potential)**.** *On a Kähler manifold M equipped with a Kähler structure Ω, a function $\mathcal{K} \in C^\infty(M)$ is a **Kähler potential** when it satisfies:*

$$\Omega = i\partial\overline{\partial}\mathcal{K}.$$

Note that $\partial\overline{\partial}\mathcal{K} \in \Omega^{1,1}(M)$. By lemma 2.2.3, one obtains that $\mathrm{d}\partial\overline{\partial}\mathcal{K} = 0$. It is clear that once $\partial\overline{\partial}\mathcal{K}$ is non-degenerate, one obtains a symplectic structure. We refer to Nakahara [2003] as to why every Kähler structure has a potential $\mathcal{K}$.

[8]Note that both g and J are non-degenerate and Ω is always a 2-form.

Remark. Let us remark on what conditions are to be met for $\mathcal{K}$ to induce a *global* Kähler structure. Let $\mathcal{K}_\alpha$, $\mathcal{K}_\beta$ be functions defined on two overlapping domains of charts defining the coordinates z^j, w^j respectively. Then once

$$\mathcal{K}_\alpha\left(z^j,\overline{z}^j\right)=\mathcal{K}_\beta\left(w^j,\overline{w}^j\right)+\phi_{\alpha\beta}\left(w^j\right)+\psi_{\alpha\beta}\left(\overline{w}^j\right),$$

where $\phi_{\alpha\beta}$ is a holomorphic function of w^j and $\psi_{\alpha\beta}$ is antiholomorphic, a global 2-form is induced as:

$$\partial\overline{\partial}\mathcal{K}_\alpha=\partial\overline{\partial}\mathcal{K}_\beta.$$

2.3 Complex structure of Hamilton's equations

As we have already seen an example of complex and symplectic geometry intertwining in Kähler geometry, we might wonder whether there is a complex structure at work hidden in Hamilton's equations. Generally, we cannot assume the symplectic structure is a Kähler structure. We can, although, try to play with the algebraic structure of the equations themselves and change the perspective.

First observation is: while a complex structure $J^2=-1$ is an endomorphism, Hamilton's equations are constituted by the symplectic 2-form and the Poisson bivector switching between TT^*Q and T^*T^*Q.

$$\begin{aligned}X_H&=\Pi\left(\mathrm{d}H\right),\\ \Omega\left(X_H\right)&=-\mathrm{d}H.\end{aligned}$$

But recall that we already encountered a similar problem before. The original coordinate form of Hamilton's equations switches between the x^j and p_j coordinates. To formulate this mechanism more elegantly, we introduced the (x^j,p_j) coordinates and formulated the "switching mechanism" by an antidiagonal matrix on those new coordinates - the Poisson structure or the symplectic structure.

Analogically, we can consider the pair $(X_H,\mathrm{d}H)$ and the antidiagonal matrix

$$\mathbb{J}_H:=\begin{pmatrix}0&\Pi\\ \Omega&0\end{pmatrix}$$

acting on sections of $TT^*Q\oplus T^*T^*Q$. The action on $(X_H,\mathrm{d}H)$ is determined by Hamilton's equations.

$$\mathbb{J}_H\begin{pmatrix}X_H\\ \mathrm{d}H\end{pmatrix}=\begin{pmatrix}0&\Pi\\ \Omega&0\end{pmatrix}\begin{pmatrix}X_H\\ \mathrm{d}H\end{pmatrix}=\begin{pmatrix}X_H\\ -\mathrm{d}H\end{pmatrix}.$$

We see $\mathbb{J}_H$ is indeed an almost complex structure:

$$\mathbb{J}_H^2=\begin{pmatrix}0&\Pi\\ \Omega&0\end{pmatrix}\begin{pmatrix}0&\Pi\\ \Omega&0\end{pmatrix}=\begin{pmatrix}\Pi\circ\Omega&0\\ 0&\Omega\circ\Pi\end{pmatrix}=\begin{pmatrix}-1&0\\ 0&-1\end{pmatrix}.$$

Note that we can think of the action as a rotation by $\frac{\pi}{2}$ in $TT^*Q\oplus T^*T^*Q$. That is, on the complexified vector bundle $TT^*Q\oplus T^*T^*Q\otimes\mathbb{C}$, this can be translated to the multiplication by the complex unit i. We obtain such action by formally rotating one of the components of $(X_H, \mathrm{d}H)$ into the imaginary direction.

$$(X_H, \mathrm{d}H) \longmapsto (X_H, i\mathrm{d}H) .$$

Now, truly:

$$\mathbb{J}_H \begin{pmatrix} X_H \\ i\mathrm{d}H \end{pmatrix} = \begin{pmatrix} 0 & \Pi \\ \Omega & 0 \end{pmatrix} \begin{pmatrix} X_H \\ i\mathrm{d}H \end{pmatrix} = \begin{pmatrix} i\Pi\,(\mathrm{d}H) \\ \Omega\,(X_H) \end{pmatrix} = \begin{pmatrix} iX_H \\ -\mathrm{d}H \end{pmatrix} = i \begin{pmatrix} X_H \\ i\mathrm{d}H \end{pmatrix} .$$

We can reformulate Hamilton's equations in this complex setting.

Proposition 2.3.1 (Hamilton's equations on $TM \oplus T^*M \otimes \mathbb{C}$)**.** *Let M be the phase space, $H \in C^\infty(M)$ the Hamiltonian function, $\Omega : TM \otimes \mathbb{C} \longrightarrow T^*M \otimes \mathbb{C}$ the canonical symplectic form and $\Pi : T^*M \otimes \mathbb{C} \longrightarrow TM \otimes \mathbb{C}$ the Poisson structure defined by $\Omega \circ \Pi = -1$. Then a tangent section $X_H \in \Gamma(TM \otimes \mathbb{C})$ is the Hamiltonian field if and only if $(X_H, i\mathrm{d}H)$ is a section of the $+i$-eigenbundle of the almost complex structure $\mathbb{J}_H$ on $TM \oplus T^*M \otimes \mathbb{C}$.*

$$\mathbb{J}_H = \begin{pmatrix} 0 & \Pi \\ \Omega & 0 \end{pmatrix} .$$

Remark. Note that for a given vector field X we can generate sections of the $+i$-eigenbundle as $(X, -i\Omega(X))$ or for a given function f as $(-i\Pi(\mathrm{d}f), \mathrm{d}f)$.

Natural questions now arise.

- Is $\mathbb{J}_H$ integrable?
- What do other geometric structures look like on the $TM \oplus T^*M$ bundle?
- What is the internal structure of $TM \oplus T^*M$?

In the following chapter, we shall attempt to show that all of these questions do have surprisingly enriching answers.

3. Generalized geometry

This chapter serves as an introduction to generalized complex geometry. Nothing of vital importance is left out and all[1] propositions are proved carefully. Nevertheless, many interesting aspects of the theory are beyond the scope of this thesis. The entire chapter is but a mere reformulation of the comprehensive and beautifully pedagogical thesis of Marco Gualtieri[2] and selection of the most important topics, the reference is Gualtieri [2004].

We begin with a research of the linear algebra of $V \oplus V^*$ and a natural symmetric product on it. Then we move on to manifolds and $TM \oplus T^*M$ and through the definition of the *Courant bracket* $[\![,]\!]$, we discover the *Courant algebroid* structure. We construct the *generalized complex structure* and in the end, we mention the notion of *generalized submanifolds*.

3.1 Linear algebra of $V \oplus V^*$

We shall consider pairs of vectors and covectors (X, ξ), $X \in V$, $\xi \in V^*$, which are elements of $V \times V^*$ or more suggestively $V \oplus V^*$. We shall denote $(X, \xi) \equiv X \oplus \xi$, as such formalism hints at how the symmetric product will be defined.

3.1.1 Canonical symmetric product

There is indeed a natural canonical way to define a symmetric bilinear product on $V \oplus V^*$.

Definition (Symmetric bilinear product on $V \oplus V^*$). *For a real vector space V of dimension m, we define the **symmetric product**:*

$$\langle , \rangle : V \oplus V^* \longrightarrow \mathbb{R},$$
$$\langle X \oplus \xi, Y \oplus \eta \rangle := \frac{1}{2} \left(i_X \eta + i_Y \xi \right).$$

The product is bilinear, symmetric[3], non-degenerate and its matrix representation is:

$$\left\langle \begin{pmatrix} X \\ \xi \end{pmatrix}, \begin{pmatrix} Y \\ \eta \end{pmatrix} \right\rangle = \frac{1}{2} \begin{pmatrix} X & \xi \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Y \\ \eta \end{pmatrix}.$$

We can identify the dual of $V \oplus V^*$ using the $\langle , \rangle$ product. This might justify that the notation is identical to that of a pairing on a vector space.

[1] Exceptions being one very general modification of the Frobenius' theorem and one very technical lemma with a proof similar to that of an already proved one.

[2] As every definition and proposition in the chapter are credited to Marco Gualtieri , he will not be cited throughout the text.

[3] We could define an antisymmetric product as $\langle X \oplus \xi, Y \oplus \eta \rangle_- := \frac{1}{2} \left(i_X \eta - i_Y \right)$ but it is the symmetric one that turns out to have properties we will prove useful.

The signature of the matrix is (m, m). Therefore, $\langle , \rangle$ defines the **orthogonal group** $O(V \oplus V^*) = O(m, m)$. In other words, we are given symmetries of $V \oplus V^*$ to study[4].

Proposition 3.1.1 (Orthogonal Lie algebra on $V \oplus V^*$)**.** *An element of the Lie algebra $R \in \mathfrak{so}(V \oplus V^*)$ of the orthogonal group $SO(V \oplus V^*)$ is of the form:*

$$R = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}, \qquad \begin{aligned} B^* &= -B, \\ \beta^* &= -\beta, \end{aligned}$$

where $A \in \mathrm{End}(V)$ and the homomorphisms $B : V \longrightarrow V^*$ and $\beta : V^* \longrightarrow V$ are defined by a skew 2-form $B \in \Lambda^2 V^*$ and a skew bivector $\beta \in \Lambda^2 V$.

Proof. Given that a Lie algebra of an orthogonal group on a vector space is its own faithful representation, we can characterize[5] it as:

$$\mathfrak{so}(V \oplus V^*) = \{R \mid \langle R(A), B\rangle = -\langle A, R(B)\rangle, \ A, B \in V \oplus V^*\}$$

Now we directly check the properties of the homomorphisms A, A', B, β in the general form of an endomorphism on $V \oplus V^*$:

$$R = \begin{pmatrix} A & \beta \\ B & -A' \end{pmatrix},$$

We have:

$$\begin{aligned} &\langle A(X) + \beta(\xi) \oplus B(X) - A'(\xi), Y \oplus \eta\rangle \\ =&\langle -A(Y) - \beta(\eta) \oplus -B(Y) + A'(\eta), X \oplus \xi\rangle \end{aligned}$$

Therefore:

$$\begin{aligned} \langle A(X), \eta\rangle &= \langle A'(\eta), X\rangle \\ \langle B(X), Y\rangle &= -\langle B(Y), X\rangle \\ \langle \beta(\xi), \eta\rangle &= -\langle \beta(\eta), \xi\rangle \end{aligned}$$

□

In other words, $\mathfrak{so}(V \oplus V^*) = \mathrm{End}(V) \oplus \Lambda^2 V \oplus \Lambda^2 V^*$.

Now we can recover orthogonal automorphisms by exponentiation. We define two essential symmetries.

Definition (B-transform)**.** *The B-**transform** $e^B \in SO(V \oplus V^*)$ is the exponential[6] of a two-form $B \in \Lambda^2 V^*$.*

$$e^B := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \qquad e^B : X \oplus \xi \longmapsto X \oplus \xi + B(X).$$

[4] In Gualtieri [2004], an orientation is defined easily. And even though we will not need the explicit formulation, we can work with $SO(V \oplus V^*) = SO(m, m)$ as it is a more simple space.

[5] See Fecko [2006], 12.1.10

[6] We will sometimes refer to e^{-B} as a B-transform, for it will turn out to be more a convenient terminology.

Definition $(\beta - transform)$. *The* **β-*transform*** $e^{\beta} \in SO\left(V \oplus V^*\right)$ *is the exponential of a skew bivector* $\beta \in \Lambda^2 V$.

$$e^{\beta} := \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \qquad\qquad e^{\beta} : X \oplus \xi \longmapsto X + \beta\left(\xi\right) \oplus \xi.$$

3.1.2 Maximal isotropic subspaces

As we will be soon able to see, every important structure on $V \oplus V^*$ will be equvialent to the specification of a maximal isotropic subspace with respect to the canonical symmetric pairing $\langle , \rangle$.

Recall we are working in the vector space of dimension $\dim\left(V \oplus V^*\right) = 2m$.

Definition (Maximal isotropic subspace)**.** *A vector subspace* $L < V \oplus V^*$ *is a* ***maximal isotropic subspace*** *if and only if:*

$$\langle A, B \rangle = 0 \qquad \forall A, B \in L \qquad\qquad \dim\left(L\right) = m$$

Example. For $E \leq V$, the space $E \oplus \mathrm{Ann}(E) < V \oplus V^*$ is a maximal isotropic.

Definition. *Let* $E \leq V$ *and* $\omega \in \Lambda^2 E^*$ *we define*

$$L\left(E, \omega\right) := \left\{X \oplus \xi \in E \oplus V^* \mid \xi|_E = \omega\left(E\right)\right\}.$$

Observe that $L\left(E, \omega\right)$ is a maximal isotropic subspace. For $X + \xi, Y + \eta \in L\left(E, \omega\right)$:

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}\left(i_X \eta + i_Y \xi\right) = \frac{1}{2}\left(\omega\left(Y, X\right) + \omega\left(X, Y\right)\right) = 0.$$

The most generic examples are $L\left(0, 0\right) = V^*$ and $L\left(V, 0\right) = V$.

Proposition 3.1.2. *For every maximal isotropic subspace* L*, there exist* $E \leq V$ *and* $\omega \in \Lambda^2 E^*$ *such that* $L = L\left(E, \omega\right)$.

Proof. Let us construct E and ω.
Define $E := \pi_L$, where π_L chooses the V part of an element of $V \oplus V^*$.
Now we define $\omega : E \longrightarrow E^*$ so that it maps any $e \in E$ to $\varepsilon \in E^*$, such that $e \oplus \varepsilon \in L$.

$$\omega : e \longmapsto \pi_{V^*}\left(\pi_V^{-1}\left(e\right) \cap L\right).$$

Or schematically:

$$\begin{tikzcd} e \arrow[r, mapsto, "\pi_V^{-1}"] \arrow[rrr, mapsto, bend right, "\omega"'] & \left\{e \oplus \xi \mid \xi \in V^*\right\} \arrow[r, mapsto, "\pi_L"] & e \oplus \varepsilon \arrow[r, mapsto, "\pi_V^*"] & \varepsilon \end{tikzcd}$$

□

Remark. For $E = V$, the maximal isotropic subspace $L\left(V, \omega\right)$ is the ω-transform of V, $L\left(V, \omega\right) = e^{\omega}\left(V\right)$ which is the graph of ω on V.

Example. For a symplectic manifold (M, Ω) of dimension m, let us consider $n \leq m$ independent local sections $E = \mathrm{Span}(X_1, \ldots, X_n)$. If we take $L\left(E|_x, \Omega\right)$, we obtain a maximal isotropic subspace of $T_x M \oplus T_x^* M$. For $m = n$, we have $L\left(E|_x, \Omega\right) = e^{\Omega}\left(T_x M\right)$.

3.1.3 Generalized complex endomorphisms

We naturally introduce complex endomorphisms on $V \oplus V^*$ so that they respect the $SO(V \oplus V^*)$ structure.

Definition (Generalized complex endomorphism)**.** *We say an endomorphism* $\mathbb{J}: V \oplus V^* \otimes \mathbb{C} \longrightarrow V \oplus V^* \otimes \mathbb{C}$ *is a* ***generalized complex endomorphism*** *if and only if:*

- *It is* complex*:* $\mathbb{J}^2 = -1$.
- *It is* orthogonal*:* $\mathbb{J}\mathbb{J}^* = 1$ *with respect to the canonical symmetric product.*

Acting on the orthogonality formula by $\mathbb{J}$, we obtain an equivalent characterization.

Proposition 3.1.3. $\mathbb{J}$ *is a generalized complex endomorphism if and only if:*

- *It is* complex*:* $\mathbb{J}^2 = -1$.
- *It is* skew*:* $\mathbb{J}^* = -\mathbb{J}$ *with respect to the canonical symmetric product.*

Remark. An endomorphism on $V \oplus V^*$ is skew if and only if it is an element of $\mathfrak{so}(V \oplus V^*)$. See the proof of proposition 3.1.1 and consider that it is the canonical symmetric product $\langle , \rangle$ that identifies $V \oplus V^*$ with its dual. A dual morphism is an $\langle , \rangle$-adjoint morphism.[7]

As a complex endomorphism, $\mathbb{J}$ has two eigenspaces corresponding to the eigenvalues $+i$, $-i$. We will use them to associate a maximal isotropic subspace of $V \oplus V^*$ with $\mathbb{J}$.

Definition (Unreal subspace)**.** *A subspace of the complexified vector space* $L \leq V \oplus V^* \otimes \mathbb{C}$ *is* ***unreal***[8] *if it satisfies* $L \cap \overline{L} = 0$.

Proposition 3.1.4 (Unreal maximal isotropic subspace)**.** *Defining a generalized complex endomorphism is equivalent to specifying an unreal maximal isotropic subspace.*

Proof. Given $\mathbb{J}$ is a complex endomorphism, its $+i$-eigenspace $\equiv L$ is indeed unreal as $\overline{L}$ is the $-i$-eigenspace. We will show it is a maximal isotropic subspace of $V \oplus V^* \times \mathbb{C}$ as well. The dimension of L is m and for $A, B \in L$:

$$\begin{aligned} \langle A, B\rangle &= \langle \mathbb{J}A, \mathbb{J}B\rangle = \langle iA, iB\rangle = -\langle A, B\rangle \\ \Rightarrow \langle A, B\rangle &= 0. \end{aligned}$$

Conversely, given two disjoint m-dimensional subpsaces L, $\overline{L}$, which satisfy the isotropy condition, we can define $\mathbb{J}$ as multiplication by $+i$, $-i$ on L, $\overline{L}$ respectively. □

As we provide important examples of generalized complex endomorphisms, we show how such structures encompass both complex and symplectic structures on V and their modifications.

[7] Equivalently on $V \oplus V^* \otimes \mathbb{C}$, complexification doesn't interfere with the orthogonal structure.
[8] In the terminology of Gualtieri [2004], this corresponds to the *real index zero*.

Example (Symplectic structure as $\mathbb{J}_\Omega$)**.** As we have already demonstrated in section 2.3, the endomorphism:

$$\mathbb{J}_\Omega = \begin{pmatrix} 0 & -\Omega^{-1} \\ \Omega & 0 \end{pmatrix}.$$

defined by a symplectic[9] 2-form $\Omega \in \Lambda^2(V^*)$ is complex: $\mathbb{J}_\Omega^2 = -1$. It is also skew with respect to the canonical pairing: we can check that both the antidiagonal elements satisfy the conditions of proposition 3.1.1. We have also concluded that elements in the form $X \oplus -i\Omega(X)$ constitute the $+i$-eigenbundle, which can be verified easily by action of $\mathbb{J}_\Omega$.

$$L_\Omega = \{X \oplus -i\Omega(X) \mid X \in V \otimes \mathbb{C}\}.$$

L is thus an unreal maximal isotropic subspace of $V \oplus V^* \otimes \mathbb{C}$ defining the symplectic structure on V.

Definition (B-symplectic 2-form)**.** *For a symplectic 2-form Ω on V and $B \in \Lambda^2 V^*$, we define the B-**symplectic 2-form** by the endomorphism:*

$$e^{-B}\mathbb{J}_\Omega e^B = \left(\begin{array}{c|c} -\Omega^{-1}\circ B & -\Omega^{-1} \\ \hline \Omega + B\circ\Omega^{-1}\circ B & B\circ\Omega^{-1} \end{array}\right).$$

It is again a generalized complex endomorphism, as it is complex:

$$e^{-B}\mathbb{J}_\Omega e^B e^{-B}\mathbb{J}_\Omega e^B = e^{-B}(-1)e^B = -1.$$

and skew, as B-transforms are orthogonal. The corresponding unreal maximal isotropic subspace is:

$$e^{-B}(L_\Omega) = \{X \oplus -(B + i\Omega)(X) \mid X \in V \otimes \mathbb{C}\}.$$

Example (Complex endomorphism as $\mathbb{J}_J$)**.** Let us have a complex endomorphism J on V. We define the diagonal endomorphism:

$$\mathbb{J}_J := \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}.$$

$\mathbb{J}_J$ too is a generalized complex endomorphism, as it is complex:

$$\mathbb{J}_J^2 = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}\begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} = \begin{pmatrix} J^2 & 0 \\ 0 & (J^*)^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and skew, it satisfies the conditions of proposition 3.1.1.
We can see the unreal maximal isotropic subspace of $V \oplus V^* \otimes \mathbb{C}$ defining the complex endomorphism on $V \otimes \mathbb{C}$ is the $+i$-eigenbundle:

$$L_J = V^{1,0} \oplus V^{*0,1}$$

A B-transformed generalized complex endomorphism is, similarly:

$$e^{-B}\mathbb{J}_J e^B = \left(\begin{array}{c|c} J & 0 \\ \hline B\circ J + J^*\circ B^* & -J^* \end{array}\right).$$

For the same reasons as in the case of B-symplectic structure, this is clearly again a generalized complex endomorphism. The corresponding $+i$-eigenspace is:

$$e^{-B}(L_J) = \left\{X \oplus \xi - B(X) \mid X \oplus \xi \in V^{1,0} \oplus V^{*0,1}\right\}.$$

[9] On the level of linear algebra, $\mathrm{d}\Omega = 0$ does not make sense. On V, we will use the word *symplectic* to refer to non-degenerate skew 2-forms.

3.2 $TM \oplus T^*M$ as a Courant algebroid

We shall transfer the linear algebra of $V \oplus V^*$ into every fibre over the base manifold M. It is a natural choice to identify V with a tangent space, as we did in section 2.3. A rich structure beyond linear algebra is to be discovered, which provides one with *integrable* generalized complex structures.

3.2.1 Lie algebroids

An essential aspect of geometry, that is, an essential difference between plain vector spaces and the tangent bundle TM, is the ability to view elements of TM as directions on M and the space of sections $\Gamma(TM)$ as derivations of $C^\infty(M)$. Let us generalize this notion.

Definition (Lie algebroid)**.** *A vector bundle ℓ over a smooth manifold M is a* ***Lie algebroid*** *if and only if it is equipped with:*

- *A Lie bracket $[,]_\ell$ involutive on $\Gamma(\ell)$*
- *An* ***anchor*** *$\varrho : \ell \longrightarrow TM$*

The anchor is a smooth bundle map such that on the space of sections, it induces a Lie algebra homomorphism:

$$\varrho([A, B]_\ell) = [\varrho(A), \varrho(B)].$$

for $\forall A, B \in \Gamma(\ell)$.
The anchor mediates an action of sections of ℓ on smooth functions on M.

$$[A, fB]_\ell = f[A, B]_\ell + (\varrho(A) f) B$$

for $\forall A, B \in \Gamma(\ell)$ and $\forall f \in C^\infty(M)$.

These axioms provide a consistent way to perform directional derivatives of $C^\infty(M)$ in directions associated with sections of a quite arbitrary vector bundle ℓ.

Example. The tangent bundle TM itself is a Lie algebroid with the anchor $\varrho = \mathrm{id}$.

Example. An integrable subbundle $TQ \overset{\iota}{\hookrightarrow} TM$ is a Lie algebroid with the anchor $\varrho = \iota$, the natural inclusion.

Remark. A lie algebroid can have both a higher or a lower dimension than the tangent bundle.

One typically constructs involutive structures to produce submanifolds. So will we.

Definition (Generalized foliation)**.** *A* ***generalized foliation*** *disjoints M into leaves of a dimension varying throughtout M. A* ***leaf*** *is a connected submanifold of M.*

Proposition 3.2.1 (Generalized Frobenius' theorem, Sussmann [1973])**.** *Let $\triangle_D$ be a distribution spanned by a collection of smooth vector fields $X_1, \ldots, C_n \in D \subset \Gamma(TM)$. Then if for $\forall X \in D$ there exists a neighbourhood U and a set of functions $c_k^i \in C^\infty(U)$ such that:*

$$\left[X, X^i\right]\Big|_U = \sum_k c_k^i X^k\Big|_U$$

for $i = 1, \ldots, n$, then it is integrable into a generalized foliation.

That is, for $\forall x \in M$, there exists a leaf $Q \hookrightarrow M$ such that $\triangle_D\big|_x = T_x Q$.

Corollary 3.2.2. *A Lie algebroid integrates into a generalized foliation.*

Proof. An anchor produces a distribution $\varrho(\ell) = \triangle_D$. Since ϱ is a smooth bundle map, $\triangle_D$ is spanned by smooth vector fields $\varrho(\Gamma(\ell))$. We can always choose a local basis $A_1, \ldots, A_n \in \Gamma(\ell)$, then $\varrho(A_1), \ldots, \varrho(A_n)$ span $\triangle_D$ and by the Lie algebra homomorphism property of the anchor, we get:

$$[\varrho(A), \varrho(B)] = \varrho([A, B]_\ell) = \varrho\left(\sum_k c_k^{ij} A^k\right) = \sum_k c_k^{ij} \varrho\left(A^k\right).$$

With c_k^{ij} being the structure constants of the Lie algebra $(\ell, [,]_\ell)$. Proposition 3.2.1 provides integrability. ☐

3.2.2 Courant bracket

Our aim is now to introduce a bracket on $TM \oplus T^*M$ to serve two purposes.

- It should be compatible with the canonical orthogonal structure on $TM \oplus T^*M$, which generates fibre-wise generalized complex endomorphisms as isotropic subspaces.
- It should generate Lie algebroids to provide an integrable structure for generalized complex endomorphisms.

In the proofs of the technical lemmata in this section, we will use the following differential geometry identities heavily:

$$\mathcal{L}_X = i_X \mathrm{d} + \mathrm{d} i_X, \qquad \mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y], \qquad i_{[X,Y]} = [\mathcal{L}_X, i_Y].$$

Definition. *On sections of $TM \oplus T^*M \longrightarrow M$ we define the **Courant bracket**:*

$$[\![X \oplus \xi, Y \oplus \eta]\!] := [X, Y] \oplus \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2}\mathrm{d}(i_X \eta - i_Y \xi).$$

Definition. *The **Courant anchor** is the projection $\varrho : TM \oplus T^*M \longrightarrow TM$.*

Properties of the Courant bracket:

- $[\![,]\!]$ is antisymmetric.
- $[\![,]\!]$ is preserved by the Courant anchor $\varrho([\![A, B]\!]) = [\varrho(A), \varrho(B)]$.

- For $X, Y \in \Gamma(TM)$: $[\![X, Y]\!] = [X, Y]$, while for $\xi, \eta \in \Gamma(T^*M)$: $[\![\xi, \eta]\!] = 0$.

Unfortunately, the antisymmetric bracket and the anchor fail to form a Lie algebroid because $[\![,]\!]$ is not a Lie bracket as it does not generally satisfy the Jacobi identity. We will show to which extent it fails and *where* it does not. For this purpose, let us define a different bracket on sections of $TM \oplus T^*M$ in efforts to recover the Jacobi identity.

Definition. *On sections of $TM \oplus T^*M \longrightarrow M$ we define the* ***Dorfman bracket****:*

$$(X \oplus \xi \circ Y \oplus \eta) := [X, Y] \oplus \mathcal{L}_X \eta - i_Y \mathrm{d}\xi.$$

Lemma 3.2.3. *The Dorfmann bracket* $(\circ)$ *satisfies the property:*

$$(A \circ (B \circ C)) = ((A \circ B) \circ C) + (B \circ (A \circ C)).$$

Proof. For $A = X + \xi$, $B = Y + \eta$, $C = Z + \zeta \in \Gamma(TM \oplus T^*M)$.

$$\begin{aligned}
((A \circ B) \circ C) &+ (B \circ (A \circ C)) \\
&= [[X, Y], Z] + [Y, [X, Z]] \oplus \mathcal{L}_{[X,Y]}\zeta - i_Z \mathrm{d}(\mathcal{L}_X \eta - i_Y \mathrm{d}\xi) \\
&\quad + \mathcal{L}_Y(\mathcal{L}_X \zeta - i_Z \mathrm{d}\xi) - i_{[X,Z]}\mathrm{d}\eta \\
&= [X, [Y, Z]] \oplus \mathcal{L}_X \mathcal{L}_Y \zeta - \mathcal{L}_X i_Z \mathrm{d}\eta - \mathcal{L}_Y i_Z \mathrm{d}\xi + i_Z \mathrm{d} i_Y \mathrm{d}\xi \\
&= [X, [Y, Z]] \oplus \mathcal{L}_X(\mathcal{L}_Y \zeta - i_Z d\eta) - i_{[Y,Z]} d\xi \\
&= (A \circ (B \circ C)).
\end{aligned}$$

□

Note that for an antisymmetric bracket, this property is equivalent to the Jacobi identity. Unfortunately, the Dorfman bracket is generally not antisymmetric.

Lemma 3.2.4. *Courant bracket is the antisymmetrization of the Dorfman bracket.*

Proof. Let $A = X \oplus \xi$, $B = Y \oplus \eta \in \Gamma(TM \oplus T^*M)$:

$$\begin{aligned}
\frac{1}{2}((A \circ B) - (B \circ A)) &= \frac{1}{2}([X, Y] - [Y, X]) \oplus \mathcal{L}_X \eta - \mathcal{L}_Y \xi - i_Y \mathrm{d}\xi + i_X \mathrm{d}\eta \\
&= [X, Y] \oplus \frac{1}{2}(\mathcal{L}_X \eta - \mathcal{L}_Y \xi + \mathrm{d} i_Y \eta - \mathcal{L}_Y \xi - \mathrm{d} i_X \eta + \mathcal{L}_X \eta) \\
&= [\![A, B]\!]
\end{aligned}$$

□

However, as it turns our, *under some conditions*, the two brackets merge and their desired properties combine into a Lie bracket.

Lemma 3.2.5. *For $A, B \in \Gamma(TM \oplus T^*M)$:*

$$[\![A, B]\!] = (A \circ B) - \mathrm{d}\langle A, B \rangle.$$

Proof. Let $A = X \oplus \xi,\ B = Y \oplus \eta \in \Gamma(TM \oplus T^*M)$.

$$\begin{aligned}(A \circ B) - \mathrm{d}\langle A, B\rangle &= [X,Y] \oplus \mathcal{L}_X\eta - i_Y\mathrm{d}\xi - \frac{1}{2}(i_X\eta + i_Y\xi) \\ &= [X,Y] \oplus \mathcal{L}_X\eta - i_Y\mathrm{d}\xi - \frac{1}{2}(i_X\eta - i_Y\xi) - \mathrm{d}i_Y\xi \\ &= [X,Y] \oplus \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}(i_X\eta - i_Y\xi) \\ &= [\![A,B]\!]\end{aligned}$$

□

Corollary 3.2.6. *On an isotropic subbundle $L < TM \oplus T^*M$, the Courant bracket is a Lie bracket.*

A last formula defining a Lie algebroid remains to be checked: (*how*) can we perform derivatives of $C^\infty(M)$ in $\varrho(TM \oplus T^*M)$ directions?

Lemma 3.2.7. *For $A, B \in \Gamma(TM \oplus T^*M)$ and $f \in C^\infty(M)$:*

$$[\![A, fB]\!] = f[\![A,B]\!] + (\varrho(A)f)B - \langle A,B\rangle\,\mathrm{d}f.$$

Proof. Let $A = X + \xi,\ B = Y + \eta$.

$$\begin{aligned}[\![X \oplus \xi, f(Y \oplus \eta)]\!] &= [X, fY] \oplus \mathcal{L}_X f\eta - \mathcal{L}_{fY}\xi - \frac{1}{2}\mathrm{d}\left(i_X(f\eta) - i_{fY}\xi\right) \\ &= f[\![X \oplus \xi, Y \oplus \eta]\!] + (Xf)Y \oplus (Xf)\eta \\ &\quad - (i_Y\xi)\,\mathrm{d}f - \frac{1}{2}(i_X\eta - i_Y\xi)\,\mathrm{d}f \\ &= f[\![X \oplus \xi, Y \oplus \eta]\!] + (Xf)(Y \oplus \eta) - \langle X \oplus \xi, Y \oplus \eta\rangle \mathrm{d}f.\end{aligned}$$

□

Corollary 3.2.8. *On an isotropic subbundle L, let $A, B \in \Gamma(L)$, $f \in C^\infty(M)$.*

$$[\![A, fB]\!] = f[\![A,B]\!] + (\varrho(A)f)B.$$

From corollaries 3.2.6, 3.2.8, we conclude that an isotropic subbundle $L < TM \oplus T^*M$ together with the Courant bracket $[\![,]\!]$ and the Courant anchor ϱ is a good candidate for a Lie algebroid. There is, although, another condition of a *global* nature that needs to be satisfied. That is, of course, the *Courant-involutiveness* of L. The following objects will turn out to be perfect indicators of involutive subbundles in section 3.2.4.

Definition. *We define the* ***Jacobiator*** *and the* ***Nijenhuis operator*** *on sections of $TM \oplus T^*M$ as:*

$$\begin{aligned}\mathrm{Jac}(A,B,C) &:= [\![[\![A,B]\!], C]\!] + c.p. \\ \mathrm{Nij}(A,B,C) &:= \frac{1}{3}\left(\langle[\![A,B]\!], C\rangle + c.p.\right)\end{aligned}$$

where $A, B, C \in \Gamma(TM \oplus T^*M)$ and c.p. denotes cyclic permutations.

We observe that $\mathrm{Jac}(A, B, C) = 0$ for $\forall A, B, C \in \Gamma(L)$ if and only if the Jacobi identity is satisfied on the subbundle $L < TM \oplus T^*M$. Now we prove that it is in fact satisfied on $TM \oplus T^*M$ up to an exact term.

Lemma 3.2.9. *For $A, B, C \in \Gamma(TM \oplus T^*M)$ it holds that:*

$$\mathrm{Jac}(A, B, C) = \mathrm{dNij}(A, B, C)$$

Proof. First, we observe that the Dorfman bracket annihilates closed 1-forms: $(\mu \circ C) = 0$ for $\forall C \in \Gamma(TM \oplus T^*M)$, $\mu \in \Omega^1_{closed}(M)$.
Then we prove the following identity using lemma 3.2.5

$$\begin{aligned}
[\![[\![A, B]\!], C]\!] &= ([\![A, B]\!] \circ C) - \mathrm{d}\langle [\![A, B]\!], C \rangle \\
&= (((A \circ B) - \mathrm{d}\langle A, B \rangle) \circ C) - \mathrm{d}\langle [\![A, B]\!], C \rangle \\
&= ((A \circ B) \circ C) - \mathrm{d}\,[\![A, B]\!], C.
\end{aligned}$$

Now we can directly compute $\mathrm{Jac}(A, B, C)$ in terms of $\mathrm{Nij}(A, B, C)$. We will use lemmata 3.2.2 and 3.2.3 and the fact that cyclic permutation *c.p.* annihliates certain terms. In the end, we use the identity we have just proved.

$$\begin{aligned}
\mathrm{Jac}(A, B, C) &= [\![[\![A, B]\!], C]\!] + c.p. \\
&= \frac{1}{4}(((A \circ B) \circ C) - (C \circ (A \circ B)) - ((B \circ A) \circ C) \\
&\quad + (C \circ (B \circ A)) + c.p.) \\
&= \frac{1}{4}((A \circ (B \circ C)) - (B \circ (A \circ C)) - (C \circ (A \circ B)) \\
&\quad - (B \circ (A \circ C)) + (A \circ (B \circ C)) + (C \circ (B \circ A)) + c.p.) \\
&= \frac{1}{4}((A \circ (B \circ C)) - (B \circ (A \circ C)) + c.p.) \\
&= \frac{1}{4}(((A \circ B) \circ C) + c.p.) \\
&= \frac{1}{4}([\![[\![A, B]\!], C]\!] + \mathrm{d}\langle [\![A, B]\!], C \rangle + c.p.) \\
&= \frac{1}{4}(\mathrm{Jac}(A, B, C) + \frac{3}{4}\mathrm{d}(\mathrm{Nij}(A, B, C))).
\end{aligned}$$

□

The following property of the Courant bracket will serve as the last puzzle piece to uniquely specify a general underlying mechanism.

Lemma 3.2.10. *Let $A, B, C \in \Gamma(TM \oplus T^*M)$. Then we have:*

$$\varrho(A)\langle B, C \rangle = \langle [\![A, B]\!] + \mathrm{d}\langle A, B \rangle, C \rangle + \langle B, [\![A, C]\!] + \mathrm{d}\langle A, C \rangle \rangle.$$

Proof. Let $A = X \oplus \xi, B = Y \oplus \eta, C = Z \oplus \zeta$. First, we consider lemma 3.2.5:

$$\langle [\![A, B]\!] + \mathrm{d}\langle A, B \rangle, C \rangle + \langle B, [\![A, C]\!] + \mathrm{d}\langle A, C \rangle \rangle = \langle (A \circ B), C \rangle + \langle B, (A \circ C) \rangle.$$

Then we proceed:

$$\begin{aligned}
&\langle (A \circ B), C \rangle + \langle B, (A \circ C) \rangle \\
&= \frac{1}{2}(i_{[X,Y]}\zeta + i_Z(\mathcal{L}_X \eta - i_Y d\xi) + i_{[X,Z]}\eta + i_Y(\mathcal{L}_X \zeta - i_Z d\xi)) \\
&= \frac{1}{2}(\mathcal{L}_X i_Y \zeta + \mathcal{L}_X i_Z \eta) = \frac{1}{2} i_X d(i_Y \zeta + i_Z \eta) = \varrho(A)\langle B, C \rangle.
\end{aligned}$$

□

3.2.3 Courant algebroid

Let us now define a general structure with the properties of $TM \oplus T^*M$ equipped with the Courant bracket $[\![,]\!]$, the canonical symmetric product $\langle , \rangle$ and the Courant anchor ϱ.

Definition (Courant algebroid)**.** *A **Courant algebroid** is a vector bundle $E \longrightarrow M$ equipped with a skew bracket $[\![,]\!]$ on its sections, a symmetric non-degenerate fibre-wise product $\langle , \rangle$ and a smooth bundle map $\varrho : E \longrightarrow TM$. This induces a differential operator $\mathcal{D} : C^\infty(M) \longrightarrow \Gamma(E)$ via the definition $\langle \mathcal{D}f, A \rangle = \frac{1}{2}\varrho(A) f$. For $\forall A, B, C \in \Gamma(E)$, $\forall f, g \in C^\infty(M)$, these structures must satisfy:*

(*a*) $\varrho([\![A, B]\!]) = [\varrho(A), \varrho(B)]$.

(*b*) $\mathrm{Jac}(A, B, C) = \mathcal{D}(\mathrm{Nij}(A, B, C))$.

(*c*) $[\![A, fB]\!] = f[\![A, B]\!] + (\varrho(A)f)B - \langle A, B\rangle \mathcal{D}f$.

(*d*) $\langle \mathcal{D}f, \mathcal{D}g \rangle = 0$.

(*e*) $\varrho(A)\langle B, C\rangle = \langle [\![A, B]\!] + \mathcal{D}\langle A, B\rangle, C\rangle + \langle B, [\![A, C]\!] + \mathcal{D}\langle A, C\rangle\rangle$.

The collection of data $(TM \oplus T^*M,\ \langle , \rangle,\ [\![,]\!],\ \varrho)$ is a motivating example of a Courant algebroid. The axioms (*b*), (*c*) and (*e*) are satisfied by the lemmata 3.2.9, 3.2.7, 3.2.10 respectively and the induced differential operator is the gradient $\mathcal{D} = \mathrm{d}$.

Remark. A Courant algebroid can be thought of as an attempt to merge the notion of a quadratic Lie algebra $(V,\ [,],\ \langle , \rangle)$, where $\langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle = 0$ and a Lie algebroid $(L,\ [,],\ \varrho)$. By lemma 3.2.10, the original quadratic Lie algebra property is satisfied by the Dorfman bracket on isotropic subbundles.

We might wonder whether the fibre-wise symmetries of the canonical product $\langle , \rangle$ preserve the global structure of a Courant algebroid, the Courant bracket. Let us examine the essential example, the B-transform.

Proposition 3.2.11. *A B-transform is an automorphism of the Courant algebroid $TM \oplus T^*M$ if and only if B is closed.*

Proof. Given the symmetric product snd the anchor are clearly compatible with e^B, we only need to check that the Courant bracket is preserved by B-transforms. Let $X \oplus \xi,\ Y \oplus \eta \in \Gamma(TM \oplus T^*M)$, let B be a smooth 2-form.

$$
\begin{aligned}
\left[\!\left[e^B(X \oplus \xi), e^B(Y \oplus \eta)\right]\!\right] &= [\![X \oplus \xi + i_X B, Y \oplus \eta + i_Y B]\!] \\
&= [\![X \oplus \xi, Y \oplus \eta]\!] + [\![X, i_Y B]\!] + [\![i_X B, Y]\!] \\
&= [\![X \oplus \xi, Y \oplus \eta]\!] + \mathcal{L}_X i_Y B - \frac{1}{2}\mathrm{d} i_X i_Y B \\
&\quad - \mathcal{L}_Y i_X B + \frac{1}{2}\mathrm{d} i_Y i_X B \\
&= [\![X \oplus \xi, Y \oplus \eta]\!] + \mathcal{L}_X i_Y B - i_Y \mathcal{L}_X B + i_Y i_X \mathrm{d}B \\
&= [\![X \oplus \xi, Y \oplus \eta]\!] + i_{[X,Y]} B + i_Y i_X \mathrm{d}B \\
&= e^B([\![X \oplus \xi, Y \oplus \eta]\!]) + i_Y i_X \mathrm{d}B.
\end{aligned}
$$

Thus the Courant bracket is preserved if and only if $i_Y i_X \mathrm{d}B = 0$ for $\forall X, Y \in \Gamma(TM)$. That is, precisely for $\mathrm{d}B = 0$. ☐

3.2.4 Dirac structures

Observation: If we were to find a subbundle $L < TM \oplus T^*M$ closed under the Courant bracket, it would be itself a Courant algebroid. If L happened to be isotropic as well, it would have the structure of a Lie algebroid.[10]

- $\underbrace{[\![A,B]\!]}_{\text{Skew}} = \underbrace{(A \circ B)}_{\text{Jacobi identity}} - \mathrm{d}\underbrace{\langle A, B\rangle}_{=0}$ is a Lie bracket.
- $\varrho\left([\![A,B]\!]\right) = [\varrho(A), \varrho(B)]$.
- $[\![A, fB]\!] = f\,[\![A,B]\!] + (\varrho(A)f)B - \underbrace{\langle A, B\rangle}_{=0}\mathrm{d}f$.

Proposition 3.2.12 (Courant-involutiveness conditions)**.** *Let $L < TM \oplus T^*M$ be an isotropic subbundle. Then the following statements are equivalent:*

- *L is Courant-involutive*
- $\mathrm{Nij}|_L = 0$
- $\mathrm{Jac}|_L = 0$

Proof.

- Let L be involutive. $\implies \langle [\![A,B]\!], C\rangle = 0 \implies \mathrm{Nij}|_L = 0$.
- Let $\mathrm{Nij}|_L = 0$. $\implies$ By lemma 3.2.9: $\mathrm{Jac}|_L = \mathrm{dNij}|_L = 0$
- Let $\mathrm{Jac}|_L = 0$. $\implies L$ is involutive, as we prove by contradiciton:

  Let $\langle [\![A,B]\!], C\rangle \neq 0$. Then for $\forall A, B, C \in \sec L$, $\forall f \in C^\infty(M)$:

  $$0 = \mathrm{Jac}(A,B,fC) = \mathrm{dNij}(A,B,fC) = \frac{1}{3} f\mathrm{d}\langle [\![A,B]\!], C\rangle + \frac{1}{3}\langle [\![A,B]\!], C\rangle\, \mathrm{d}f.$$

  Given that $\mathrm{d}f$ need not be 0 for $\forall f \in C^\infty(M)$, we arrive to a contradiction.

$\square$

Further on, we will study a certain class of isotropic subbundles of Courant algebroids which will induce generalized complex endomorphisms.

Definition (Dirac structures)**.**

- *An **almost Dirac structure** is a maximal isotropic subbundle of a Courant algebroid.*
- *A **Dirac structure** is a Courant-involutive[11] almost Dirac structure.*

Remark. By proposition 3.2.11, B-transform of a Dirac structure is clearly again a Dirac structure for closed B.

Remark. A real Dirac structure is a subbundle $L < E$, a complex Dirac structure is a subbundle $L < E \otimes \mathbb{C}$. Complexification is, of course, compatible with the Courant algebroid structure and isotropy.

[10] See lemmata 3.2.5, 3.2.7 and section 3.2.1.

[11] In other words, it is by proposition 3.2.1 **integrable** into a generalized foliation.

Example (Foliated geometry)**.** Let $\triangle < TM$ be a smooth distribuition of a constant rank. Then

$$L_{\triangle} \equiv \triangle \oplus \mathrm{Ann}(\triangle) < TM \oplus T^{*}M$$

is a maximal isotropic subbundle, i.e. an almost Dirac structure.

Proposition 3.2.13. *$\triangle$ is integrable if and only if $L_{\triangle}$ is a Dirac structure.*

Proof. We need to prove that for $\forall A, B, C \in \sec L_{\triangle}$, $A = X \oplus \xi, B = Y \oplus \eta, C = Z \oplus \zeta$:

$$\mathrm{Nij}|_{L_{\triangle}} = 0 \Longleftrightarrow [\triangle, \triangle] \subset \triangle,$$

as we use proposition 3.2.12 and because an involutive distribution is integrable by the Frobenius' theorem.

$$\begin{aligned}
\langle [\![A, B]\!], C\rangle &= \left\langle [X, Y] \oplus \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}\mathrm{d}\left(i_X\eta - i_Y\xi\right., Z \oplus \zeta \right\rangle \\
&= \frac{1}{2}\left(i_{[X,Y]}\zeta + i_Z\mathcal{L}_X\eta - i_Z\mathcal{L}_Y\xi\right) \\
&= \frac{1}{2}\left(i_{[X,Y]}\zeta + i_Z i_X\mathrm{d}\eta + i_Z\mathrm{d}i_X\eta - i_Z i_Y\mathrm{d}\xi - i_Z\mathrm{d}i_Y\xi\right) \\
&= \frac{1}{2}\left(i_{[X,Y]}\zeta + \eta\left(Z\right)X - \eta\left(X\right)Z - i_{[X,Z]}\eta\right. \\
&\quad \left. - \xi\left(Z\right)Y + \xi\left(Y\right)Z + i_{[Y,Z]}\xi\right) \\
&= \frac{1}{2}\left(i_{[X,Y]}\zeta - i_{[X,Z]}\eta + i_{[Y,Z]}\xi\right).
\end{aligned}$$

Thus if and only if $[\triangle, \triangle] \subset \triangle$ we have $\mathrm{Nij}|_{L_{\triangle}} = 0 \Leftrightarrow L_{\triangle}$ is a Dirac structure. $\square$

Example (Presymplectic geometry)**.** The tangent bundle TM itself is trivially a Dirac structure. A B-transform by a 2-from ω yields a new almost Dirac structure:

$$L_{\omega} = e^{\omega}\left(TM\right).$$

Note that by proposition 3.2.11, L_{ω} is a Dirac structure if and only if ω is a presymplectic structure.

Example (Poisson geometry)**.** Let us consider the symplectic 2-form Ω. If we invert it into a Poisson structure Π, we can modify the Dirac structure $T^{*}M$ by a β-transform.

$$L_{\Pi} = e^{\Pi}\left(T^{*}M\right).$$

We do not a priori know whether a β-transform will preserve the Courant algebroid structure (it does preserve isotropy), but in this case, we can use what we know about the symplectic 2-form $\Omega : TM \longrightarrow T^{*}M$. $e^{\Omega}\left(TM\right)$ is a dirac structure and given that $\Pi : T^{*}M \longrightarrow TM$ is the inverse morphism, it must generate the same pairs $X \oplus \xi$. Thus:

$$L_{\Pi} = e^{\Pi}\left(T^{*}M\right) = e^{\Omega}\left(TM\right) = L_{\Omega}.$$

Remark. The integrability of $e^{\Omega}(TM)$ relies on the closure of $\Omega = \Pi^{-1}$. Hence, so does the integrability of $e^{\Pi}(T^*M)$. But as we know, Π constitutes the Poisson bracket by the formula $\Pi(\mathrm{d}f, \mathrm{d}g) = \{f, g\}$ and the Jacobi identity of $\{,\}$ is equivalent to the closure of Ω. We refer to Gualtieri [2004] for a general proof that $e^{\Pi}(T^*M)$ is a Dirac structure if and only if Π is a Poisson tensor.[12]

Proposition 3.2.14 (Involutiveness of $L(E,\omega)$)**.** *Let L be an almost Dirac structure fibre-wise defined by a subbundle $E < TM$ and a 2-form ω as $L = L(E_x, \omega|_x)$. L is a Dirac structure if and only if E is involutive and ω is closed.*

Proof. Let $\phi : E \longrightarrow TM$ be the inclusion. Let $\sigma \in \Omega^2(M)$ be a smooth extension of ω, so that: $\phi^*\sigma = \omega$. We define d_E on sections of ΛE^* so that $\phi^* \circ \mathrm{d} = \mathrm{d}_E \circ \phi^*$.
Let $A = X \oplus \xi, B = Y \oplus \eta \in \Gamma(L)$ arbitrary, thus $\xi|_E = \omega(X)$, $\eta|_E = \omega(Y)$.
L is involutive if and only if for $[A, B] = K \oplus \kappa$, it holds that $K \in \Gamma(E)$ and $\kappa|_E = \omega(K)$.
Note that $K = [X, Y]$, which mean $K \in \Gamma(E)$ if and only if E is involutive.
Then $\kappa|_E \in [\omega(K)$, it following difference must vanish:

$$
\begin{aligned}
\kappa|_E - i_K\omega &= \phi^*\Big(\mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}\mathrm{d}(i_X\eta - i_Y\xi)\Big) - i_{[X,Y]}\phi^*\sigma \\
&= i_X\mathrm{d}_E\phi^*\eta - i_Y\mathrm{d}_E\phi^*\xi - \frac{1}{2}\mathrm{d}_E(i_Xi_Y\omega - i_Yi_X\omega) - \phi^*[\mathcal{L}_X, i_Y]\sigma \\
&= i_X\mathrm{d}_E\phi^*\eta - i_Y\mathrm{d}_E\phi^*\xi - \mathrm{d}_E(i_Xi_Y\omega) \\
&\quad - \phi^*(i_X\mathrm{d}i_Y\sigma + \mathrm{d}i_Xi_Y\sigma - i_Yi_X\mathrm{d}\sigma - i_Yi_X\sigma) \\
&= i_Yi_X\mathrm{d}_E\omega
\end{aligned}
$$

Thus ω must be closed. □

3.2.5 Generalized complex geometry

The author does indeed hope that the right definition of a generalized complex structure is utterly obvious at this point.

Definition (Generalized complex structure)**.**

- *A* ***generalized almost complex structure*** *is an almost complex structure $\mathbb{J}$ on $TM \oplus T^*M \otimes \mathbb{C}$ orthogonal with respect to the fibre-wise canonical symmetric product $\langle,\rangle$.*
- *A* ***generalized complex structure*** *is a Courant-involutive almost complex structure.*

Remark. By proposition 3.1.3, we know that instead of orthogonality, we can require skewness, which is also equivalent to $\mathbb{J} \in \mathfrak{so}(TM \oplus T^*M \otimes \mathbb{C})$.

Proposition 3.1.4 leads us to its global version: the $+i$-eigenbundle of the generalized complex structure is a Courant-involutive unreal maximal isotropic subbundle $L < TM \oplus T^*M \otimes \mathbb{C}$, in other words:

[12] Even though we will work with general Poisson tensors in the next chapters, any Poisson induced Dirac structure with a physical relevance will be provided its integrability by the canonical symplectic form Ω. Either by inversion, or later, by a "reduction" of the Ω^{-1} tensor.

Proposition 3.2.15.
A generalized complex structure is[13] *an unreal Dirac structure.*

Example (Symplectic geometry)**.** As we know from section 3.1.3, a generalized almost complex structure defined by a non-degenerate 2-form Ω is the fibre-wise endomorphism:

$$\mathbb{J}_\Omega = \begin{pmatrix} 0 & -\Omega^{-1} \\ \Omega & 0 \end{pmatrix}.$$

or equivalently its $+i$-eigenbundle, the almost Dirac structure:

$$L_\Omega = e^{-i\Omega}(TM).$$

The integrability of $e^{-i\Omega}(TM)$ is equivalent to the closure[14] of Ω, as $e^{-i\Omega}$ is a B-transform by $-i\Omega$.

Definition (B-symplectic structure)**.** *A B-symplectic structure is a B-transform of $e^{-i\Omega}(TM)$ by a real 2-form $B \in \Omega^2(M)$.*

$$e^{-B}\mathbb{J}_\Omega e^B = \left(\begin{array}{c|c} -\Omega^{-1} \circ B & -\Omega^{-1} \\ \hline \Omega + B \circ \Omega^{-1} \circ B & B \circ \Omega^{-1} \end{array}\right),$$

$$e^{-B}(L_\Omega) = e^{-(B+i\Omega)}(TM).$$

It is clear that the B-symplectic structure is a generalized complex structure if and only if B is closed.

Example (Complex geometry)**.** The generalized almost complex structure defined by an almost complex structure J is the fibre-wise endomorphism:

$$\mathbb{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}.$$

or equivalently its $+i$-eigenbundle, the almost Dirac structure:

$$L_J = TM^{1,0} \oplus T^*M^{0,1} \equiv TM^{1,0} \oplus \mathrm{Ann}\left(TM^{1,0}\right).$$

Regarding integrability, by proposition 3.2.13 we can see that:

$\left[TM^{0,1}, TM^{0,1}\right] \subset TM^{0,1}$ J is a complex structure.	$\iff$	L_J is a generalized complex structure.

Remark (Generalized Kähler geometry)**.** We only remark on the generalization of Kähler geometry onto a Courant algebroid. Let us have the Kähler triple:

[13] As the word *structure* suggests, generalized complex structure is a notion that is supposed to describe a certain relationship betwen objects (which distinguishes their totality from a mere collection). The author believes it is reasonable to say the structure ***is*** the very decomposition of the Courant algebroid into unreal subbundles as much as it is either one of them or the bundle morphism that realizes the decomposition.

[14] This is equivalent to the fact that Ω defines an integrable distribution $\mathrm{Ann}(\Omega) \subset \Gamma(TM)$.

$$\begin{tikzcd} TM \arrow[rr, "g"] & & T^*M \\ & TM \arrow[ul, "J"] \arrow[ur, "\Omega"'] & \end{tikzcd}$$

We can combine the symplectic Kähler structure Ω and the complex structure J to construct a Riemannian metric g or its inverse g^{-1} (see proposition 2.2.4).

$$\begin{tikzcd} TM & & T^*M \arrow[ll, "g^{-1}"'] \arrow[dl, "\Omega^{-1}"] \\ & TM \arrow[ul, "J"] & \end{tikzcd} \qquad \begin{tikzcd} T^*M & & TM \arrow[ll, "g"'] \arrow[dl, "-\Omega"] \\ & T^*M \arrow[ul, "-J^*"] & \end{tikzcd}$$

We can analogically consider the corresponding generalized complex structures $\mathbb{J}_\Omega$, $\mathbb{J}_J^* = -\mathbb{J}_J$ to construct a Riemannian metric G on $TM \oplus T^*M$.

$$\begin{tikzcd} TM \oplus T^*M & & TM \oplus T^*M \arrow[ll, "G"'] \arrow[dl, "-\mathbb{J}_\Omega"] \\ & TM \oplus T^*M \arrow[ul, "\mathbb{J}_J"] & \end{tikzcd}$$

We can check that:

$$-\mathbb{J}_J\mathbb{J}_\Omega = -\begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}\begin{pmatrix} 0 & -\Omega^{-1} \\ \Omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & J\circ\Omega^{-1} \\ J^*\circ\Omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

In general, one can consider two arbitrary commuting generalized complex structures two produce Riemannian metrics on $TM \oplus T^*M$. For further details, see, of course, Gualtieri [2004].

3.2.6 Twisting

Some almost Dirac structures fail to be Courant-involutive yet might still integrate into a generalized foliation. If we modify the Courant bracket (and thus the whole Courant algebroid structure) we can obtain an involutive structure.

Definition (Twisted Courant bracket)**.** *For a real closed 3-form $H \in \Omega^3_{closed}(M)$, we define the **twisted Courant bracket** on sections of $TM \oplus T^*M$ as:*

$$[\![X\oplus\xi, Y\oplus\eta]\!]_H := [\![X\oplus\xi, Y\oplus\eta]\!] + i_Y i_X H$$

Without a proof [15], we state the following lemma refering to Gualtieri [2004], section 3.7, and Ševera and Weinstein [2001].

Lemma 3.2.16. *Let $A = X\oplus\xi, B = Y\oplus\eta, C = Z\oplus\zeta \in \Gamma(TM\oplus T^*M)$ For the Nijenhuis operator and Jacobiator defined by the twisted Courant bracket $[\![,]\!]_H$, we have:*

$$\mathrm{Jac}_H(A,B,C) = \mathrm{dNij}_H(A,B,C) + i_Z i_Y i_X \mathrm{d}H$$

Proposition 3.2.17 (Twisted Courant algebroid)**.** *The collection of data $(TM\oplus T^*M,\ \langle,\rangle,\ [\![,]\!]_H,\ \varrho)$ defines a Courant algebroid for a closed 3-form H.*

[15] Involving analoguous calculations to those in the proof of lemma 3.2.9.

Proof.

(a) $\varrho\left([\![A,B]\!]_H\right) = [\varrho(A), \varrho(B)]$ is satisfed trivially.

(b) $\mathrm{Jac}_H(A,B,C) = \mathcal{D}(\mathrm{Nij}_H(A,B,C))$ holds for $\mathrm{d}H = 0$.

(c) $[\![A, fB]\!]_H = f\,[\![A,B]\!]_H + (\varrho(A)f)B - \langle A,B\rangle \mathcal{D}f$ is satisfied by the tensoriality of H

(d) $\langle \mathcal{D}f, \mathcal{D}g\rangle = 0$ holds trivially.

(e) $\varrho(A)\langle B,C\rangle = \langle [\![A,B]\!]_H + \mathcal{D}\langle A,B\rangle, C\rangle + \langle B, [\![A,C]\!]_H + \mathcal{D}\langle A,C\rangle\rangle$ is satisfied by the antisymmetry of H.

□

Proposition 3.2.18 (B-transforms of a twisted Courant bracket)**.** *For a 2-form $B \in \Omega^2(M)$ and a closed 3-form $H \in \Omega^3_{closed}(M)$, a B-transformed twisted Courant bracket behaves in the following manner.*

$$\left[\!\!\left[e^B(A), e^B(B)\right]\!\!\right]_H = e^B\left([\![A,B]\!]_{H+\mathrm{d}B}\right)$$

for $\forall A, B \in \Gamma(TM \oplus T^*M)$.

Proof. For $A = X \oplus \xi$, $B = Y \oplus \eta$, see proof of proposition 3.2.11 to obtain:

$$\left[\!\!\left[e^B(A), e^B(B)\right]\!\!\right] = e^B\left([\![A,B]\!]\right) + i_Y i_X \mathrm{d}B,$$

Now we just add the twisting:

$$\left[\!\!\left[e^B(A), e^B(B)\right]\!\!\right] + i_Y i_X H = e^B\left([\![A,B]\!]\right) + i_Y i_X \mathrm{d}B + i_Y i_X H.$$

Considering the action of B-transform on the cotangent elements $i_Y i_X \mathrm{d}B + i_Y i_X H$ is an identity, one obtains the proposition. □

In other words, a B-transform *adds a* $\mathrm{d}B$*-twist.*

For clarity, consider the following example: $\mathrm{d}B = H$ and a Dirac structure L closed under the regular Courant bracket. Now perform the e^{-B} transform, by proposition 3.2.18:

$$\begin{tikzcd}
\Gamma(L) \times \Gamma(L) \arrow[r, "e^{-B}"] \arrow[d, "{[\![,]\!]_{H-\mathrm{d}B}}"] & \Gamma\left(e^{-B}(L)\right) \times \Gamma\left(e^{-B}(L)\right) \arrow[d, "{[\![,]\!]_H}"] \\
\Gamma(L) \arrow[r, "e^{-B}"] & \Gamma\left(e^{-B}(L)\right)
\end{tikzcd}$$

As one substitutes $H - \mathrm{d}B = 0$, we obtain the following proposition:

Proposition 3.2.19. *Let L be an almost Dirac structure. Then it is Courant involutive if and only if $\mathrm{e}^{-B}(L)$ is closed under the* $\mathrm{d}B$*-twisted Courant bracket.*

3.3 Generalized submanifolds

Let us now study how the action generalized complex structures on a Courant algebroid $TM \oplus T^*M$ affects submanifolds of M.

3.3.1 Generalized tangent bundle

A subbundle of TM defines a submanifold Q if it can be thought of as TQ upon restriction. Additional geometric structure of the submanifold can be encoded into the behaviour of TQ under bundle morphisms.

Definition (Complex submanifold)**.** *A submanifold $Q \hookrightarrow M$ is a* ***complex submanifold*** *with respect to a complex structure J on M if and only if:*

$$J : TQ \longrightarrow TQ.$$

This is a natural definition, consider $TQ = TM^{1,0}|_Q$. For a complex structure J, it is an involutive subbundle stable under J and thus integrates into a submanifold.

Definition (Lagrangian submanifold)**.** *A submanifold $Q \hookrightarrow M$ is* ***Lagrangian*** *with respect to a 2-form ω if and only if:*

$$TQ = TQ^\omega \Leftrightarrow TQ^\omega \subseteq TQ, TQ^\omega \supseteq TQ,$$

where

$$TQ^\omega := \left\{X \in \Gamma\left(TM|_Q\right) \mid \omega(X,Y) = 0, \forall Y \in \Gamma\left(TQ\right)\right\}.$$

Proposition 3.3.1. *A submanifold $Q \hookrightarrow M$ is Lagrangian if and only if*

$$\Omega : TQ \longrightarrow \mathrm{Ann}(TQ) \qquad\qquad \Omega^{-1} : \mathrm{Ann}(TQ) \longrightarrow TQ$$

Proof.

- $\Omega : TQ \longrightarrow \mathrm{Ann}(TQ)$ says $\Omega(X)$, $\forall X \in \Gamma(TQ)$ annihilates $\forall Y \in \Gamma(TQ)$. This is equivalent to saying $TQ \subseteq TQ^\Omega$.
- $\Omega^{-1} : \mathrm{Ann}(TQ) \longrightarrow TQ$ provides one with $\Omega^{-1}(\xi) = X, \forall \xi \in \Gamma(\mathrm{Ann}(TQ))$ $X \in \Gamma(TQ)$. By action of Ω, we obtain $\xi = \Omega(X), \forall \xi \in \Gamma(\mathrm{Ann}(TQ))$) $X \in \Gamma(TQ)$. That is $TQ^\Omega \subseteq TQ$.

□

Analogically, we can define a submanifold by a subbundle L of $TM \oplus T^*M$ such that the $\varrho(L)$ can be thought of as TQ. The behaviour of L will, again, encode additional structure.

Definition (Generalized tangent bundle)**.** *For a submanifold $Q \hookrightarrow M$, the generalized tangent bundle is defined as:*[16]

$$\mathbb{T}Q := TQ \oplus \mathrm{Ann}(TQ)$$

[16] For Q_α being a leaf of a foliation, $\bigcup_\alpha \mathbb{T}Q_\alpha$ is a Dirac structure. See proposition 3.2.13. This makes the definition a natural choice, along with the fact that $\mathrm{Ann}(TQ)$ naturally defines TQ itself.

Definition (Generalized complex submanifold)**.** *A submanifold $Q \hookrightarrow M$ is a **generalized complex submanifold**[17] with respect to a generalized complex structure $\mathbb{J}$ on M if and only if:*

$$\mathbb{J} : \mathbb{T}Q \longrightarrow \mathbb{T}Q.$$

Example (Complex submanifold)**.** It is clear that $Q \hookrightarrow M$ is a complex submanifold if and only if the generalized tangent bundle $\mathbb{T}Q$ is stable under the genralized complex structure

$$\mathbb{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix},$$

as J^* naturally decomposes $T^*M|_Q$ into TQ and $\mathrm{Ann}(TQ)$.[18]

Remark. Note that the $+i$-eigenbundle is a generalized tangent bundle.

Example (Lagrangian submanifold)**.** $Q \hookrightarrow M$ is a Lagrangian submanifold if and only if the generalized tangent bundle $\mathbb{T}Q$ is stable under the genralized complex structure

$$\mathbb{J}_\Omega = \begin{pmatrix} 0 & -\Omega^{-1} \\ \Omega & 0 \end{pmatrix}.$$

This fibre-wise endomorphism clearly induces the action:

$$\Omega : TQ \longrightarrow \mathrm{Ann}(TQ) \qquad\qquad \Omega^{-1} : \mathrm{Ann}(TQ) \longrightarrow TQ$$

This is, by proposition 3.3.1, equivalent to Q being Lagrangian.

[17]Gualtieri [2004] uses the term generalized complex submanifold when he refers to a submanifold equipped with a 2-form which "remembers" twisting.

[18]One can think of it as $T^*M|_Q = T^*M^{1,0}|_Q \oplus T^*M^{0,1}|_Q$.

4. Dirac reduction

This chapter is dedicated to the study of reduction of generalized geometry onto submanifolds. We reproduce the reasoning of Courant [1990].

4.1 Induced maximal isotropic subspaces

In the generalized setting, we have described various geometries by Dirac structures, i.e. Courant-involutive maximal isotropic subbundles. First, let us study how a maximal isotropic subspace $L < V \oplus V^*$ passes onto a subspace $W \oplus W^*$ for $W < V$ on the level of linear algebra.

We aim to express the reduced space $L \cap (W \oplus W^*)$ as a quotient space. To obtain such result, consider that we need to get rid of the surplus cotangent part $\mathrm{Ann}(W)$.

$$\mathrm{Ann}(W) = (W \oplus V^*)^{\perp} \subset W \oplus V^*$$

where $\perp$ denotes orthogonality with respect to $\langle , \rangle$. Thus we can form the quotient space:

$$\frac{W \oplus V^*}{(W \oplus V^*)^{\perp}} = \frac{W \oplus V^*}{(\mathrm{Ann}(W))} \simeq W \oplus W^*$$

This relation constitutes an exact sequence:

$$0 \longrightarrow \mathrm{Ann}(W) \xrightarrow{i} W \oplus V^* \xrightarrow{\pi} W \oplus W^* \longrightarrow 0$$

where i is an the natural inclusion and π is defined on $W \oplus V^*$ by: $\pi\,(w \oplus \alpha) = w \oplus \alpha|_W$. We can see the sequence is indeed exact:

$$\ker \pi = \{\alpha \in V^* \mid \alpha|_W = 0\} = \mathrm{Ann}(W) = \mathrm{Im}\,(i)$$

We define L_W as the image of the restricted subspace L under π.

$$L_W := \pi\,(L \cap (W \oplus V^*))$$

We obtain L_W as a quotient space if we take the projection of the above sequence onto L, which constitutes another sequence included in the original sequence such that they commute.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ann}(W) \arrow[r, "i"] & W \oplus V^* \arrow[r, "\pi"] & W \oplus W^* \arrow[r] & 0 \\
0 \arrow[r] & L \cap \mathrm{Ann}(W) \arrow[r, "i"] \arrow[u] & L \cap W \oplus V^* \arrow[r, "\pi"] \arrow[u] & L_W \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

As the new sequence is clearly exact again, we get a natural isomorphism:

$$L_W \simeq \frac{L \cap (W \oplus V^*)}{L \cap \mathrm{Ann}(W)}$$

We procede to prove this guess indeed provides us with a maximal isotropic subspace.

Proposition 4.1.1 (Reduced maximal isotropic subspace, Courant [1990])**.** *Let L be a maximal isotropic subspace. Then L_W is a maximal isotropic subspace.*

Proof. Given the signature of the canonical pairing is split, (m, m), we have to prove that: $L_W = L_W^{\perp}$.

- If we take an arbitrary $x \in L_W$, then there exists $y \in L \cap (W \oplus V^*)$ such that $\pi(y) = x$. Given that $L = L^{\perp}$, we know that $y \in (L \cap (W \oplus V^*))^{\perp}$. Now consider that π preserves $\langle , \rangle$:

$$0 = \langle y_1, y_2 \rangle = \langle \pi(y_1), \pi(y_2) \rangle = \langle x_1, x_2 \rangle$$

Thus: $x \in L_W \Rightarrow x \in L_W^{\perp}$, i.e. $L_W \subset L_W^{\perp}$.

- Consider $a \in L_W^{\perp} \subset W \oplus W^*$ arbitrary. Again, there exists $y \in W \oplus V^*$ such that $\pi(y) = A$.
Given that $\langle , \rangle$ is preserved by π, we obtain $y \in (L \cap (W \oplus V^*))^{\perp}$, indeed:

$$a \perp L_W \Leftrightarrow \pi(y) \perp L_W \Leftrightarrow \pi(y\ ) \perp \pi(L \cap (W \oplus V^*)) \Leftrightarrow y \perp L \cap (W \oplus V^*) .$$

Thus we have:
$$\begin{aligned} y &\in (L \cap (W \oplus V^*))^{\perp} \cap (W \oplus V^*) \\ &= \left(L^{\perp} + (W \oplus V^*)^{\perp}\right) \cap (W \oplus V^*) \\ &= (L + \mathrm{Ann}(W)) \cap (W \oplus V^*) \\ &= L \cap (W \oplus V^*) + \mathrm{Ann}(W) \end{aligned}$$

Therefore we can find $x \in L \cap (W \oplus V^*)$ and $Z \in \mathrm{Ann}(W)$, such that $y = x + z$.
It is easy to see that $\pi(y) = \pi(x + z) = \pi(x)$, therefore:

$$a = \pi(x) \Longrightarrow a \in \pi(L \cap (W \oplus V^*)) .$$

That is: $L_W^{\perp} \subset L_W$.

□

Now we demonstrate that the reduced subspace L_W induces a restricted 2-form.

Proposition 4.1.2 (Courant [1990])**.** *Let $L = L(E, \omega)$ for a subspace $E < V$ and a 2-form $\omega : E \longrightarrow E^*$. Then $L_W = L(E \cap W, \omega|_{E \cap W})$.*

Proof. For $w \oplus \alpha \in L \cap (V \oplus W^*)$ we have $w \in E \cap W$, $\alpha|_{E \cap W} = \omega X$, that is, from ω defined on E, we pass to the reduced 2-form $\omega|_{E \cap W}$.
It is simple to see that the denominator of L_W, $L \cap \mathrm{Ann}(W)$ does not change the induced 2-form: the dual $L \cap \mathrm{Ann}(W)$ elements it factorises out are not an image of any non-zero element of $W \cap E$ under ω. □

Remark. Consider that for L unreal, the reduced structure is clearly unreal as well (unrealness is preserved by intersections and factorisations). Thus this Dirac reduction mechanism generates reduced generalized complex endomorphisms.

4.2 Dirac reduction on manifolds

Let us consider a submanifold $Q \hookrightarrow M$. From a Dirac structure L on M, we can construct the maximal isotropic distribution:

$$L_Q := \frac{L \cap \left(TQ \oplus T^*M|_Q\right)}{L \cap \mathrm{Ann}(TQ)}$$

L_Q is a **reduced Dirac structure** if and only if it is also a subbundle and is Courant-involutive. For L unreal, L_Q is a **reduced generalized complex structure** under the same conditions. The first condition is provided by a fibre-wise constant dimension:

Proposition 4.2.1 (Courant [1990])**.** *Let L be a Dirac structure on M, $Q \hookrightarrow M$.*

$L \cap \left(TQ \oplus T^*M|_Q\right)$ *has constant dimension*	$\Longleftrightarrow$	$L \cap \mathrm{Ann}(TQ)$ *has constant dimension*

Proof. The two distributions are fibre-wise orthogonal complements. ☐

We see that once one of the conditions od proposition 4.2.1 is satisfied, the quotient space L_Q is a bundle.

Proposition 4.2.2 (Integrability of the reduced Dirac structure, Courant [1990])**.** *Let L be an almost Dirac structure on M, L_Q a reduced almost Dirac structure on M. Then L_Q is integrable if L is integrable.*

Proof. We express L_Q as $L\left(TQ \cap L, \iota^*\omega\right)$ for the inclusion $\iota : TQ \cap L \longrightarrow \pi_{TM}L$ and check the conditions of proposition 3.2.14. The tangent part is involutive, because both TQ and $\pi_{TM}L|_Q$ are involutive. The restricted 2-form is closed, because ω is closed and $\mathrm{d}\iota^*\omega = \iota^*\mathrm{d}\omega = 0$. ☐

Proposition 4.2.3 (Characteristic distribution of a reduced Dirac structure, Courant [1990])**.** *Let L be a Dirac structure. Then for the reduced Dirac structure L_Q and the induced 2-form $\tilde{\omega}$ we have:*

$$\mathrm{char}(\tilde{\omega}) = L_Q \cap TQ \simeq \frac{L \cap (TQ \oplus \mathrm{Ann}(TQ))}{L \cap \mathrm{Ann}(TQ)}.$$

Proof. $L_Q \cap TQ$ is the part of TQ which is mapped trivially by the action of $\tilde{\omega}$, i.e. the $\mathrm{char}(\tilde{\omega})$. Furthermore, for $L = L\left(E, \omega\right)$ we have:

$$\begin{aligned} L_Q \cap TQ &\simeq \{X \in L_Q \mid X \in TQ\} \\ &\simeq \frac{\left\{X + \xi \in L \mid X \in TQ,\ \xi|_{TQ} = 0\right\}}{L \cap \mathrm{Ann}(TQ)} \\ &\simeq \frac{\{X + \xi \in L \mid X \in TQ,\ \xi \in \mathrm{Ann}(TQ)\}}{L \cap \mathrm{Ann}(TQ)} \\ &\simeq \frac{L \cap (TQ \oplus \mathrm{Ann}(TQ)}{L \cap \mathrm{Ann}(TQ)}. \end{aligned}$$

☐

4.3 Poisson reduction

Let us investigate the specific case of L being a graph of a Poisson structure Π. The question is: does Dirac reduction yield another graph of a Poisson structure?

Proposition 4.3.1 (Dirac reduction of a Poisson structure, Courant [1990]). *Let $Q \hookrightarrow M$ be a submanifold and $\Pi : T^*M \longrightarrow TM$ a Poisson structure on M with a Dirac structure L being its graph. Then the reduced Dirac structure L_Q defines a Poisson structure on the submanifold Q if the following conditions hold:*

(a) $\ker(\Pi) \cap \mathrm{Ann}(TQ)$ *has a constant dimension.*

(b) $TQ \cap \Pi(\mathrm{Ann}(TQ)) \simeq 0$.

Proof. We will show that the first condition will provide the reduced dirac structure L_Q with a fibre-wise constant dimension. I.e. it is a well-defined bundle and a well-defined Dirac structure.
The second condition ensures us that the induced Dirac structure can be thought of as a graph of a Poisson structure on Q.

(a) By proposition 4.2.1, L_Q is a bundle if $L \cap \mathrm{Ann}(TQ)$ has constant dimension. We also see that for L being a graph of Π:

$$\begin{aligned} L \cap \mathrm{Ann}(TQ) &\simeq \{\xi \in L \mid \xi \in \mathrm{Ann}(TQ)\} \\ &\simeq \{\xi \in \ker \Pi \mid \xi \in \mathrm{Ann}(TQ)\} \\ &\simeq \ker(\Pi) \cap \mathrm{Ann}(TQ) \end{aligned}$$

(b) We observe that $L_Q \cap TQ$, the purely tangent part of L_Q, cannot be constituted as a graph of a bivector, which would necessarily need to map a zero 1-form onto a non-zero tangent vector. Thus we get a condition $L_Q \cap TQ \simeq 0$, which is a consequence of the one given in the proposition:

$$\begin{aligned} 0 \simeq L_Q \cap TQ &\simeq \frac{L \cap (TQ \oplus \mathrm{Ann}(TQ))}{L \cap \mathrm{Ann}(TQ)} \\ \Uparrow \\ 0 \simeq L \cap (TQ \oplus \mathrm{Ann}(TQ)) \\ \simeq TQ \cap \Pi(\mathrm{Ann}(TQ)) \end{aligned}$$

We made use of Proposition 4.2.3. Then we observed that the space of all pairs $X \oplus \xi$ such that $\Pi(\xi) = X$ is isomorphic to the space spanned by X alone.

$\square$

Remark. Consider a non-degenerate Poisson structure which is an inverse of a symplectic structure. As it defines a Dirac structure as a graph of a 2-form, the reduced Poisson structure must be in principle inverse of a symplectic form. The non-degeneracy of the induced symplectic form is precisely the condition $TQ \cap \Pi(\mathrm{Ann}(TQ)) \simeq L_Q \cap TQ \simeq 0$.

5. Generalized geometry in physics

In the final chapter, we provide examples of generalized geometrical structures emerging in mathematical physics (or physical mathematics, one might say).

First, a Dirac reduction of a generalized complex structure will open the way for a research of the underlying geometry of a two-dimensional harmonic oscillator. Then we show that the process of Dirac reduction naturally induces structures used by string theorists to provide a way to work with incompatible objects. Finally, we describe a way to associate generalized complex structures with second order PDEs familiar to a physicist.

5.1 Symplectic reduction of an oscillator

The phase space of a particle in two dimensions is the cotangent bundle $T^*\mathbb{R}^2$ diffeomorphic to $\mathbb{R}^4$. It is equipped with the canonical symplectic form Ω which determines the dynamics through Hamilton's equations. We intend to examine a system with a constant value of energy.

In particular, we are interested in the case of a particle in a parabolic potential, i.e. the *linear harmonic oscillator*. In canonically conjugate coordinates[1], the Hamiltonian reads

$$H = p_1^2 + p_2^2 + x_1^2 + x_2^2$$

up to a rescaling [2]. Setting H constant, without loss of generality $H \overset{!}{=} 1$, we have constrained the phase space onto a unit sphere.

$$S^3 \overset{\iota}{\hookrightarrow} T^*\mathbb{R}^2$$

A natural question of a geometer is: what is $\iota^*(\Omega)$?

It is simple to see that $\iota^*(\Omega)$ must be degenerate, even from the sole fact that a symplectic matrix cannot be defined on an odd dimensional vector space. We will demonstrate what is the obstruction in the language of generalized geometry and Dirac reduction.

First, we recall that we have a generalized complex structure defined by Ω as the $+i$-eigenbundle of $\mathbb{J}_\Omega$ (see section 3.2.5), the graph $L = e^{-i\Omega}(TM)$. Now we will try to reduce this structure onto S^3, having $\mathrm{Ann}(TS^3) = \mathrm{Span}(\mathrm{d}H)$, the induced Dirac structure is L_{S^3} (see proposition **??**).

$$L_{S^3} = \frac{e^{-i\Omega}(TM) \cap (TS^3 \oplus T^*M|_{S^3})}{e^{-i\Omega}(T^*M) \cap \mathrm{Span}(\mathrm{d}H)\,|_{S^3}}$$

[1]The positions of indeces aren't important in this case.

[2]Generally we have a mass factor in the p^2 term and the potential strength factor in the x^2 term. We can factor both out by an opposite rescaling of the x^i and the p_j coordinates, which is a canonical transformation. The scale of the whole function H just changes the scale of the constrained sphere.

To study the induced presymplectic structure $\iota^*(\Omega)$, we compute its characteristic distribution by proposition 4.2.3.

$$\begin{aligned}
\operatorname{char}(\iota^*\Omega) &\cong \frac{e^{-i\Omega}(TM) \cap (TS^3 \oplus \operatorname{Span}(\mathrm{d}H)\,|_{S^3})}{e^{-i\Omega}(TM) \cap \operatorname{Span}(\mathrm{d}H)\,|_{S^3}} \\
&\cong e^{-i\Omega}(TM) \cap \left(TS^3 \oplus \operatorname{Span}(\mathrm{d}H)\,|_{S^3}\right) \\
&\cong \left\{X - i\Omega(X) \in TS^3 \oplus T^*M|_{S^3} \mid -i\Omega(X) \in \operatorname{Span}(\mathrm{d}H)\right\} \\
&\cong \left\{X_H - i\Omega(X_H) \in TS^3 \oplus T^*M|_{S^3}\right\} \\
&\cong \operatorname{Span}(X_H) \cap TS^3.
\end{aligned}$$

Note that $\iota^*(X_H) = X_H$, i.e. it is tangent to the sphere because it is annihilated by $\operatorname{Ann}(TS^3) = \operatorname{Span}(\mathrm{d}H)$,

$$\langle X_H, \mathrm{d}H\rangle = -\Omega(X_H, X_H) = 0.$$

Thus:

$$\operatorname{char}(\iota^*\Omega) \cong \operatorname{Span}(X_H).$$

We have arrived to the information that the characteristic distribution of $\iota^*(\Omega)$ is fibre-wise isomorphic to the span of X_H. We can simply check that it is precisely the span of X_H. First we have:

$$\iota^*(\mathrm{d}H) = \mathrm{d}\iota^*(H) = 0.$$

Hamilton's equations provide us with $\iota^*\Omega(X_H) = \iota^*(-\mathrm{d}H) = 0$.

We could have found that X_H is annihilated by a simple and natural guess, but the result also tells us X_H spans the whole characteristic distribution[3]. The mechanism would also work for multiple Hamiltonian fields $X_{H_1}, \ldots, X_{H_n}$.

5.1.1 Hopf fibration

In what follows, we will demonstrate that X_H is the only field that spans $\operatorname{char}(\iota^*\Omega)$ by constructing a factorspace on which the induced structure is symplectic again. We arrive to such results in four steps:

- Complexification of the phase space.
- Recognition of a Lie group structure of S^3.
- Factorization of S^3 by the action generated by X_H.
- Construction of a Kähler structure on the factorspace.

[3]This is a also one of the results of the theory of symplectic reductions, where the essential mechanism are the *moment maps*.

The underlying geometric structure that will emerge is known as the Hopf fibration.

It is convenient to introduce the following complex coordinates on $T^*\mathbb{R}^2 \approx \mathbb{C}^2$ for the reason that symplectic and complex geometry will intertwine.

$$\begin{aligned} z_1 &:= p_1 + ix_1, \quad \overline{z_1} = p_1 - ix_1, \\ z_2 &:= p_2 + ix_2, \quad \overline{z_2} = p_2 - ix_2. \end{aligned}$$

In the new coordinates, the S^3 constraint reads

$$H = z_1\overline{z_1} + z_2\overline{z_2} \overset{!}{=} 1$$

Following Nakahara [2003], example 4.12, for every point in the phase space, we can define the matrix U:

$$\begin{aligned} (x_1, x_2, p_1, p_2) &\longleftrightarrow \begin{pmatrix} p_1 + ix_1 & -(p_2 + ix_2) \\ p_2 - ix_2 & p_1 - ix_1 \end{pmatrix} \\ (z_1, z_2, \overline{z_1}, \overline{z_2}) &\longleftrightarrow \begin{pmatrix} z_1 & -z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix} \end{aligned}$$

The S^3 constraint now produces matrices with $\det U \overset{!}{=} 1$.

$$\begin{aligned} \det U &= p_1^2 + p_2^2 + x_1^2 + x_2^2 \\ \det U &= z_1\overline{z_1} + z_2\overline{z_2} \end{aligned}$$

We can check that for $\det U \overset{!}{=} 1$, the matrix is also unitary:

$$\begin{pmatrix} z_1 & -z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix} \begin{pmatrix} \overline{z_1} & z_2 \\ -\overline{z_2} & z_1 \end{pmatrix} = \begin{pmatrix} z_1\overline{z_1} + z_2\overline{z_2} & 0 \\ 0 & z_1\overline{z_1} + z_2\overline{z_2} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

That is, we can associate each point on the constant energy sphere in the phase space of a two dimensional oscillator with a special unitary matrix.

$$S^3 \approx SU(2)\,.$$

This hints towards the fact that one can model spin by a two dimensional oscillator. This approach is known as the *Schwinger's model.*

As it will be useful for further calculations, let us now formulate Hamilton's equations explicitely in the complex coordinates.

$$\begin{aligned} \Omega &= \mathrm{d}p_1 \wedge \mathrm{d}x_1 + \mathrm{d}p_2 \wedge \mathrm{d}x_2 \\ &= \frac{1}{2}\mathrm{d}\left(z_1 + \overline{z_1}\right) \wedge \frac{1}{2i}\mathrm{d}\left(z_1 - \overline{z_1}\right) + \frac{1}{2}\mathrm{d}\left(z_2 + \overline{z_2}\right) \wedge \frac{1}{2i}\mathrm{d}\left(z_2 - \overline{z_2}\right) \\ &= -\frac{1}{4i}\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + \frac{1}{4i}\mathrm{d}\overline{z_1} \wedge \mathrm{d}z_1 - \frac{1}{4i}\mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2} + \frac{1}{4i}\mathrm{d}\overline{z_2} \wedge \mathrm{d}z_2 \\ &= \frac{i}{2}\left(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2}\right) \end{aligned}$$

We can see the explicit form of the Hamiltonian field X_H from Hamilton's equations $\Omega\left(X_H\right) = -\mathrm{d}H$.

$$
\begin{aligned}
\mathrm{d}H &= \mathrm{d}\left(z_1\overline{z_1} + z_2\overline{z_2}\right) = z_1\mathrm{d}\overline{z_1} + \overline{z_1}\mathrm{d}z_1 + z_2\mathrm{d}\overline{z_2} + \overline{z_2}\mathrm{d}z_2 \\
\Omega\left(X_H\right) &= \frac{i}{2}\left(X_{z_1}\mathrm{d}\overline{z_1} - X_{\overline{z_1}}\mathrm{d}z_1 + X_{z_2}\mathrm{d}\overline{z_2} - X_{\overline{z_2}}\mathrm{d}z_2\right) \\
&\Downarrow \\
X_H &= 2i\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - \overline{z_1}\frac{\partial}{\partial \overline{z_1}} - \overline{z_2}\frac{\partial}{\partial \overline{z_2}}\right)
\end{aligned}
$$

Now we can get rid of the X_H field on $S^3 \approx SU(2)$. Taking $X_H|_e$ we obtain an element of the Lie algebra $X_H|_e \in \mathfrak{su}\left(2\right)$. By exponentiating this element, we get the Hamiltonian action as a subgroup of $SU\left(2\right)$. We see that this group is $U\left(1\right)$ as its algebra is one-dimensional and we can construct every element as:

$$
\exp\left(itX_H|_e\right) \in U\left(1\right), \qquad t \in \mathbb{R}
$$

That is because X_H is Hermitian $\overline{X_H} = X_H$ and the group elements are unitary.

$$
\overline{\exp\left(itX_H|_e\right)} = \exp\left(-itX_H|_e\right) = \exp\left(itX_H|_e\right)^{-1}.
$$

Therefore, on the coset space $SU(2)/U(1) \xleftarrow{\pi} SU(2)$ we have $\pi^*\left(X_H\right) = 0$ and the characteristic distribution is trivialized.

Now we procede to show that on $SU\left(2\right)/U\left(1\right)$, the symplectic form Ω induces another symplectic form directly - we construct the reduced 2-form.

First, we need to recognize the geometric structure of $SU(2)/U(1)$ that will enable us to introduce a convenient coordinate system.

Proposition 5.1.1 (Fecko [2006], 13.2.8)**.** *$SU(2)/U(1) \approx S^2$.*

Proposition 5.1.2 (Nakahara [2003], example 9.9)**.** *$S^2 \approx \mathbb{CP}^1$.*

On $\mathbb{CP}^1$, we can introduce the coordinates:

$$
z := \frac{z_1}{z_2}, \qquad \overline{z} = \frac{\overline{z_1}}{\overline{z_2}}
$$

on $\mathbb{CP}^1 \setminus \{z_2 = 0\}$. An atlas is completed by a complementary chart

$$
z' := \frac{z_2}{z_1}, \qquad \overline{z}' = \frac{\overline{z_2}}{\overline{z_1}}
$$

on $\mathbb{CP}^1 \setminus \{z_1 = 0\}$.

We follow Nakahara [2003], example 8.8, as we define the function $\mathcal{K}$ on $\mathbb{CP}^1$ and study its properties.

$$
\mathcal{K} := z\overline{z} + 1, \qquad \mathcal{K} = \sum_{j=1}^{2}\left|\frac{z_j}{z_2}\right|^2
$$

We can see that on the z' chart, we have:

$$\mathcal{K}' = z'\overline{z}' + 1, \qquad \mathcal{K}' = \sum_{j=1}^{2} \left| \frac{z_j}{z_1} \right|^2$$

Therefore:

$$\mathcal{K}' = \left| \frac{z_1}{z_2} \right|^2 \mathcal{K}$$

If we then consider $\log \mathcal{K}$, we see it transforms as:

$$\log \mathcal{K}' = \log \mathcal{K} + \log \left(\frac{z_1}{z_2} \right) + \log \left(\frac{\overline{z_1}}{\overline{z_2}} \right).$$

Now, given that $\log\left(\frac{z_1}{z_2}\right)$ is annihilated by $\overline{\partial}$ and $\log\left(\frac{\overline{z_1}}{\overline{z_2}}\right)$ by ∂, $i\partial\overline{\partial}\log\mathcal{K}$ defines a natural global symplectic form on $\mathbb{CP}^1$ (see section 2.2.2).

$$i\partial\overline{\partial}\log\mathcal{K}' = i\partial\overline{\partial}\log\mathcal{K}$$

Definition. *For $\mathcal{K} = z\overline{z}' + 1$, we define the* ***Fubini-study 2-form*** *on $\mathbb{CP}^1$ by:*

$$\Omega_{\mathcal{K}} := \frac{i}{2}\partial\overline{\partial}\log\mathcal{K}.$$

For the inclusion ι and the projection π:

$$T^*\mathbb{R}^2 \xleftarrow{\ \iota\ } S^3 \xrightarrow{\ \pi\ } \mathbb{CP}^1,$$

we formulate the main proposition[4] of this section.

Proposition 5.1.3. *For the canonical symplectic structure $\Omega \in \Omega^2\left(T^*\mathbb{R}^2\right)$ and the Fubini-study 2-form $\Omega_{\mathcal{K}} \in \Omega^2\left(\mathbb{CP}^1\right)$ it holds that:*

$$\pi^*\left(\Omega_{\mathcal{K}}\right) = \iota^*\left(\Omega\right)$$

Proof. For $T^*\mathbb{R}^2 \approx \mathbb{C}^2$, we shall work in the following setting[5]

$$\begin{tikzcd}
\mathbb{C}^2 \setminus \{0\} \arrow[dd, "\tilde{\pi}"'] & \\
& S^3 \arrow[ul, "\iota"'] \arrow[dl, "\pi"] \\
\mathbb{CP}^1 &
\end{tikzcd}$$

In other words, we define $\tilde{\pi}^*$ such that $\iota^* \circ \tilde{\pi}^* = \pi^*$.
The explicit actions are:

$$\begin{aligned}
&\tilde{\pi}^* : f\left(z, \overline{z}\right) \longmapsto f\left(\frac{z_1}{z_2}, \frac{\overline{z_1}}{\overline{z_2}}\right), \\
&\iota^* : \left(z_1\overline{z_1} + z_2\overline{z_2}\right) \longmapsto 1, \\
&\iota^* : \left(z_1 \mathrm{d}\overline{z_1} + \overline{z_1}\mathrm{d}z_1 + z_2\mathrm{d}\overline{z_2} + \overline{z_2}\mathrm{d}z_2\right) \longmapsto 0.
\end{aligned}$$

[4]This is a classical exercise in symplectic geometry, e.g. see the textbook da Silva [2008], homework 20.

[5]We consider $\mathbb{C}^2 \setminus \{0\}$ so that its every point can be mapped onto a complex line.

Now we calculate $\pi^*(\Omega_{\mathcal{K}})$.

$$\Omega_{\mathcal{K}} = \frac{i}{2}\partial\overline{\partial}\log\mathcal{K} = \frac{i}{2}\partial\left(\frac{1}{1+z\overline{z}}\mathrm{d}\overline{z}\right)$$
$$= \frac{i}{2}\left(\frac{1}{1+z\overline{z}} - \frac{z\overline{z}}{(1+z\overline{z})^2}\right)\mathrm{d}z\wedge\mathrm{d}\overline{z} = -\frac{i}{2}\mathcal{K}^{-2}\mathrm{d}z\wedge\mathrm{d}\overline{z},$$
$$\tilde{\pi}^*(\mathrm{d}z\wedge\mathrm{d}\overline{z}) = \mathrm{d}\left(\frac{z_1}{z_2}\right)\wedge\mathrm{d}\left(\frac{\overline{z_1}}{\overline{z_2}}\right) = \left(\frac{1}{z_2}\mathrm{d}z_1 - \frac{z_1}{z_2}\mathrm{d}z_2\right)\wedge\left(\frac{1}{\overline{z_2}}\mathrm{d}\overline{z_1} - \frac{\overline{z_1}}{\overline{z_2}}\mathrm{d}\overline{z_2}\right)$$
$$= \frac{1}{z_2\overline{z_2}}\left(\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1} - \frac{\overline{z_1}}{\overline{z_2}}\mathrm{d}z_1\wedge\mathrm{d}\overline{z_2} - \frac{z_1}{z_2}\mathrm{d}z_2\wedge\mathrm{d}\overline{z_1} + \frac{z_1\overline{z_1}}{z_2\overline{z_2}}\mathrm{d}z_2\wedge\mathrm{d}\overline{z_2}\right).$$

Then we calcualte the action of ι^* on separate terms:

$$\iota^*\left(\frac{1}{z_2\overline{z_2}}\right) = \frac{z_1\overline{z_1}+z_2\overline{z_2}}{z_2\overline{z_2}} = \frac{z_1\overline{z_1}}{z_2\overline{z_2}} + 1 = \tilde{\pi}^*(\mathcal{K})$$
$$\iota^*\left(-\frac{\overline{z_1}}{\overline{z_2}}\mathrm{d}z_1\wedge\mathrm{d}\overline{z_2}\right) = \frac{z_1}{z_2}\mathrm{d}\overline{z_1}\wedge\mathrm{d}\overline{z_2} - \mathrm{d}\overline{z_2}\wedge\mathrm{d}z_2,$$
$$\iota^*\left(-\frac{z_1}{z_2}\mathrm{d}z_2\wedge\mathrm{d}\overline{z_1}\right) = \frac{z_1}{z_2}\left(\frac{z_2}{\overline{z_2}}\mathrm{d}\overline{z_2} + \frac{\overline{z_1}}{\overline{z_2}}\mathrm{d}z_1\right)\wedge\mathrm{d}\overline{z_1}$$
$$= -\frac{z_1}{z_2}\mathrm{d}\overline{z_1}\wedge\mathrm{d}\overline{z_2} + \frac{z_1\overline{z_1}}{z_2\overline{z_2}}\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1}.$$

Thus:

$$\pi^*(\mathrm{d}z\wedge\mathrm{d}\overline{z}) = \iota^*\circ\tilde{\pi}^*(\mathrm{d}z\wedge\mathrm{d}\overline{z})$$
$$= \left(1+\frac{z_1\overline{z_1}}{z_2\overline{z_2}}\right)\left(\left(1+\frac{z_1\overline{z_1}}{z_2\overline{z_2}}\right)\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1} + \left(1+\frac{z_1\overline{z_1}}{z_2\overline{z_2}}\right)\mathrm{d}z_2\wedge\mathrm{d}\overline{z_2}\right)$$
$$= (\pi^*(\mathcal{K}))^2(\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1} + \mathrm{d}z_2\wedge\mathrm{d}\overline{z_2}).$$

Finally, recall that: $\Omega = \frac{i}{2}(\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1} + \mathrm{d}z_2\wedge\mathrm{d}\overline{z_2})$.

$$\pi^*(\Omega_{\mathcal{K}}) = \frac{i}{2}\pi^*\left(\mathcal{K}^{-2}\right)(\pi^*(\mathcal{K}))^2(\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1} + \mathrm{d}z_2\wedge\mathrm{d}\overline{z_2})$$
$$= \frac{i}{2}(\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1} + \mathrm{d}z_2\wedge\mathrm{d}\overline{z_2})$$
$$= \iota^*(\Omega)$$

□

A single thing left to do is to remark on how the structure $S^3 \xrightarrow{\pi} \mathbb{CP}^1$ or equivalently $SU(2) \xrightarrow{\pi} S^2$ constitutes a principle bundle (refering to Fecko [2006]). The $U(1)$ group is the principle fibre inserted into the total space $SU(2)$, the quotient is the base space $SU(2)/U(1) \approx S^2$.

This is known as the **Hopf fibration**, denoted in the form:

$$U(1) \hookrightarrow SU(2) \xrightarrow{\pi} SU(2)/U(1)$$

Or the most usual form:

$$S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$$

It so happens that the Hamiltonian action $\exp(itX_H|_e)$ provides the $U(1) \approx S^1$ fibre of the constant energy sphere S^3 in the phase space of a two-dimensional oscillator, as we have just demonstrated.

5.2 Dirac bracket in string theory

Let us present a situation when a Poisson bracket fails to meet the physical assumptions and a structure called a *Dirac bracket* arises. This may serve as a motivation to show how the mechanisms of Dirac reduction resolve such problems naturally.

On $T^*\mathbb{R}^n$, Poisson brackets are given, induced by the canonical symplectic form, satisfying the canonical relations:

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \qquad \{x^\mu, x^\nu\} = 0, \qquad \{p_\mu, p_\nu\} = 0.$$

We refer to Ardalan et al. [2000] as we present the boundary conditions of an open string ending on a brane.

$$c_\alpha = p_\alpha - F_{\alpha\beta} x^\beta.$$

An antisymmetric 2-form $F_{\alpha\beta}$ is set to be constant and non-degenerate.

A string theorist is troubled at this moment, as the constraints are incompatible with the canonical Poisson structure.

$$\begin{aligned}\{c_\mu, c_\nu\} &= \left\{p_\mu - F_{\mu\alpha}x^\alpha, p_\nu - F_{\nu\beta}x^\beta\right\} = -\{p_\mu, F_{\nu\rho}x^\rho\} - \{F_{\mu\sigma}x^\sigma, p_\nu\} \\ &= -\{p_\mu, F_{\nu\rho}\} - \{p_\mu, x^\rho\} F_{\nu\rho} - \{F_{\mu\sigma}, p_\nu\} x^\sigma - \{x^\sigma, p_\nu\} F_{\mu\sigma} = 2F_{\nu\mu}.\end{aligned}$$

To enforce the constraints is to set $c_\mu = 0, \mu = 1, \dots n$, but we then obtain a contradiction $\{0, 0\} = 2F_{\nu\mu}$ for $F_{\nu\mu}$ non-degenerate. The solution to this problem is a modification of the Poisson bracket.

$$\{f, g\}_D := \{f, g\} - \{f, c_\alpha\} C^{\alpha\beta} \{c_\beta, g\}, \quad C^{\mu\nu} := \{c_\mu, c_\nu\}^{-1}$$

We denote this new bracket with a subscript D, as it is what we will later call a Dirac bracket. We also see that in our case $C^{\mu\nu} = \frac{1}{2}\left(F^{-1}\right)^{\nu\mu}$.

The modified bracket is indeed compatible with the constraints:

$$\{c_\mu, f\}_D = \{c_\mu, f\} - \{c_\mu, c_\alpha\} C^{\alpha\beta} \{c_\beta, f\} = \{c_\mu, f\} - \{c_\mu, f\} = 0.$$

The rather surprising corollary is:

$$\{x^\mu, x^\nu\}_D = \{x^\mu, x^\nu\} - \{x^\mu, c_\rho\} C^{\rho\sigma} \{c_\sigma, x^\nu\} = \frac{1}{2}\delta^\mu_\rho \left(F^{-1}\right)^{\sigma\rho} \delta^\nu_\sigma = \frac{1}{2}\left(F^{-1}\right)^{\nu\mu}.$$

This corresponds to the non-commutativity of the center of mass coordinates of an open string.

Let us now study the discussed mathematical phenomena in a more precise and illuminating setting.

Definition (Poisson-incompatible constraints)**.** *Let $Q \hookrightarrow M$ denote a leaf of a foliation induced by a set of independent constraints $\{c_1, \dots, c_d\}$ and let $\Pi : T^*M \longrightarrow TM$ be a Poisson structure on M, $\{,\}$ its associated Poisson bracket. The constraints are called* ***Poisson-incompatible*** *if and only if the $\{c_\alpha, c_\beta\}$ matrix is non-degenerate.*

Definition (Dirac bracket, cite). *Let $Q \hookrightarrow M$ denote a leaf of a foliation induced by a set of Poisson-incompatible constraints $\{c_1, \ldots, c_d\}$. Then for the incompatible Poisson bracket $\{,\}$ on M, a Poisson bracket $\{,\}_D$ is called a* ***Dirac bracket*** *if and only if:*

$$\{c_\mu, f\}_D\Big|_Q = 0$$

for $\forall f \in C^\infty(M)$. And

$$\{f, g\}_D|_Q = \{f, g\}|_Q$$

for $\forall f, g \in C^\infty(M)$ independent of the constraints.

It is clear that the bracket used in string theory satisfies the conditions of a Dirac bracket. Given its action is either trivial or equvialent to that of the original Poisson bracket, it is indeed itself a Poisson bracket as well.

Remark. In local coordinates, it can be shown that a Dirac bracket is defined uniquely. Indeed, if we locally decompose T^*M into $\{\mathrm{d}c_1, \ldots, \mathrm{d}c_d\}$ and its orthogonal component, we see that the components of its Poisson tensor are defined uniquely. Therefore the $\{,\}_D$ bracket can be refered to as *the* Dirac bracket.

5.2.1 Induced Dirac bracket

Now we intend to show that the Dirac bracket is induced by Dirac reduction. The following reflections will serve two purposes. Firstly, to demonstrate the power of Dirac reduction. Secondly, to illuminate the nature of Dirac bracket and possibly convince the reader that its introduction in string theory is not merely a technical trick not to be trusted but a natural outcome of the interplay of a Poisson structure and incompatible constraints.

We will reformulate the definition of the Dirac bracket in more geometric terms and in the context of Courant's investigations of Dirac reduction treated in section 4.3. First, we will introduce a decomposition of TM used in a similar context by Calvo et al. [2010].

Lemma 5.2.1. *Let $Q \hookrightarrow M$ denote a leaf of a foliation induced by a set of Poisson-incompatible constraints $\{c_1, \ldots, c_d\}$, let $\Pi : T^*M \longrightarrow TM$ be the incompatible Poisson structure on M, $\{,\}$ the associated Poisson bracket. Then*

$$TQ \oplus \Pi(\mathrm{Ann}(TQ)) = TM|_Q$$

Proof. Denote $c \equiv \{c_1, \ldots, c_d\}$ and $\mathrm{d}c \equiv \{\mathrm{d}c_1, \ldots, \mathrm{d}c_d\}$

Consider that $TQ = \mathrm{Ann}(\mathrm{d}c)$ and $\Pi(\mathrm{Ann}(TQ)) = \mathrm{Span}(\{c, \bullet\})$.

First, note $\{c_\alpha, \bullet\}$ is a derivation of $C^\infty(M)$ and can be thus thought of as a section of TM. Given that $\{c_\alpha, c_\beta\}$ is non-degenerate, we obtain $\mathrm{Span}(\{c, \bullet\}) \cap \mathrm{Ann}(\mathrm{d}c) = 0$

Now we only need to check $\mathrm{Ann}(\mathrm{d}c) + \mathrm{Span}(\{c, \bullet\}) = T\ M|_Q$, this is provided by the dimensionality: $\dim(\mathrm{Ann}(\mathrm{d}c)) = m - d$ and $\dim(\mathrm{Span}(\{c, \bullet\})) = d$, while the dimension of the fibre of $TM|_Q$ is precisely m .

□

In the end of our reduction procedure we would like to obtain a bracket that yields zero anytime an incompatible contraint is one of its arguments. Even before the reduction, we observe that mixed inputs - a constraint c_α and a function f independent of constraints - result in trivial results automatically: $\{c_\alpha, f\} = 0$.

We need to realize that the functions independent of constraints are precisely f, such that $\mathrm{d}f \in T^*Q$, while the constraints define the foliation as: $\operatorname{Span}(\mathrm{d}c_1, \ldots, \mathrm{d}c_d) = \operatorname{Ann}(TQ)$. In this setting we can state the following lemma.

Lemma 5.2.2 (Mixed arguments)**.** *For an incompatible Poisson structure* Π *and constraints* $\{c_1, \ldots, c_d\}$ *defining a sumbanifold* $Q \hookrightarrow M$, *it holds that:*

$$\Pi(\mathrm{d}c_\alpha, \mathrm{d}f) = 0$$

for $\forall \mathrm{d}f \in T^*Q, \ \forall \alpha = 1, \ldots, d$

Proof. Consider lemma 5.2.1 and the following linear algebra reasoning; $U = V \oplus W$ and $\operatorname{Ann}(U) = 0$, then $\operatorname{Ann}(V) = W^*$. Now we have

$$TQ \oplus \Pi(\operatorname{Ann}(TQ)) = TM|_Q \Rightarrow T^*Q = \operatorname{Ann}(\Pi(\operatorname{Ann}(TQ)))$$

That is $\mathrm{d}f \in T^*Q \Leftrightarrow \langle \mathrm{d}f, X \rangle = 0$ for $X \in \Pi(\operatorname{Ann}(TQ))$. Once we have $\operatorname{Span}(\mathrm{d}c_1, \ldots, \mathrm{d}c_d) = \operatorname{Ann}(TQ)$ we obtain:

$$0 = \langle \mathrm{d}f, \Pi(\mathrm{d}c_\alpha) \rangle = \Pi(\mathrm{d}c_\alpha, \mathrm{d}f) = \{c_\alpha, f\}$$

□

In other words, we have shown that an incompatible Poisson structure decomposes $T^*M|_Q$ by diagonal action into T^*Q and $\operatorname{Ann}(TQ)$. (A different formulation can be found in Goldberg [1999].)

The definition of the Dirac bracket essentially states that the associated Poisson strucutre Π_D acts *only* on T^*Q and *in the same way* as the original Π. Such structure is indeed induced by Dirac reduction, as we now procede to prove. (A different proof that the Dirac bracket arises upon restriction can be found in Calvo et al. [2010].)

Proposition 5.2.3 (Induced Dirac bracket)**.** *Let $\Pi : T^*M \longrightarrow TM$ be a Poisson structure on $M, \{,\}$ its associated Poisson bracket. Let $Q \hookrightarrow M$ denote a leaf of a foliation induced by a set of Poisson-incompatible constraints.*

Then a Poisson structure $\Pi|_{T^*Q}$ is induced on Q by a Dirac reduction of Π.

Proof. By lemma 5.2.1 we are given the decomposition $TQ \oplus \Pi(\mathrm{Ann}(TQ)) = TM|_Q$.

We see that a Poisson structure is induced as we invoke proposition 4.3.1 and check its conditions:

- A level set Q should satisfy $\ker(\Pi) \cap \mathrm{Ann}(TQ)$ having constant dimension. This is equialent to $\Pi(\mathrm{An}(TQ))$ having constant dimension. Considering proposition 5.2.1, we observe

$$\dim(\Pi(\mathrm{Ann}(TQ))) + \dim(TQ) = \dim\left(TM|_Q\right)$$

  and note that $\dim(TQ)$ and $\dim\left(TM|_Q\right)$ are constant.

- $TQ \cap \Pi(\mathrm{Ann}(TQ)) \simeq 0$ clearly holds as well.

As to why the induced structure is indeed $\Pi|_{T^*Q}$; the induced Dirac structure is

$$L_Q = \frac{L \cap \left(TQ \oplus T^*M|_Q\right)}{L \cap \mathrm{Ann}(TQ)},$$

see chapter 4. Let us first take care of the part that is yet to be factorized. Sections of L are pairs $\Pi(\xi) \oplus \xi$, sections of $L \cap \left(TQ \oplus T^*M|_Q\right)$ are thus

$$\Pi(\xi) \oplus \xi, \qquad \Pi(\xi) \in \Gamma(TQ).$$

Now we make use of lemma 5.2.1 and observe that $\Pi(\mathrm{Ann}(TQ)) \cap TQ = 0$. In other words, $\xi \notin \mathrm{Ann}(TQ)$. Now we can see the factorization does not change anything. Given that $T^*M|_Q = T^*Q \oplus \mathrm{Ann}(TQ)$, we have sections of L_Q:

$$\Pi(\xi) \oplus \xi, \qquad \xi \in \Gamma(T^*Q).$$

The induced Dirac structure is a graph $e^{\Pi}(T^*Q)$, i.e. it defines the Poisson structure $\Pi|_{T^*Q}$. □

Have we, by proposition 5.2.3 really induced a Dirac bracket? We did obtain the desired action on T^*Q but we didn't obtain any action on $\mathrm{Ann}(TQ)$, while we would like it to be trivial. We propose we are free to extend this 'no action' to a trivial action.

The following reasoning is heavily based on the proof that Dirac bracket is induced upon restriction given by Calvo et al. [2010]. In the constrained system, we a priori consider functions $f, g \in C^\infty(Q)$. The action of the restricted Poisson bracket can be written down as $\{\tilde{f}, \tilde{g}\}$, where $\tilde{f}, \tilde{g}$ are extensions of f, g to $C^\infty(M)$. Now we simply consider such extensions that $\mathrm{d}\tilde{f}, \mathrm{d}\tilde{g}$ stay in T^*Q.

Therefore, instead of extending the action of the induced Poisson structure we essentially extend all functions in $C^\infty(Q)$ trivially into $\mathrm{Ann}(TQ)$ (armed with the Occam's razor) and the Poisson bracket we have induced is simply consistent with such a choice as it is only defined on T^*Q. This construction provides one with the Dirac bracket.

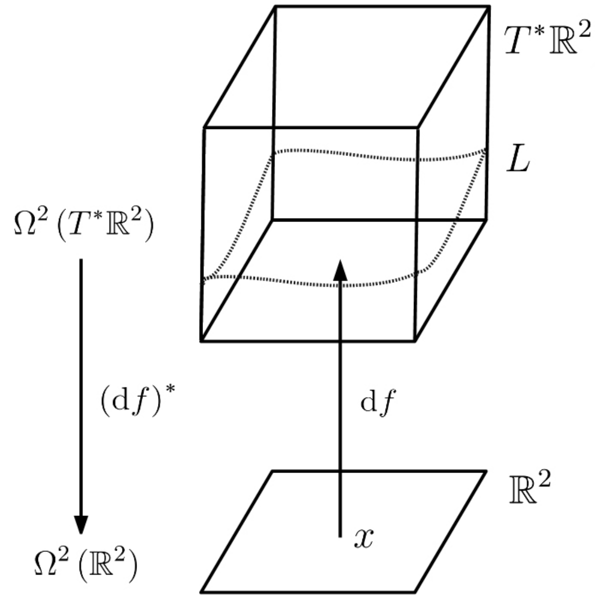
5.3 Monge-Ampère equations

We demonstrate a way to associate a second order partial differential equation (*PDE*) of two real variables with a 2-form on $T^*\mathbb{R}^2$ with rich geometric properties that will later be discussed in the setting of generalized geometry. We refer to the work of Banos [2007] from which all the key concepts and definitions are taken.

The solution of the two dimensional **Monge-Ampère equations** is defined as a function $f \in C^\infty(\mathbb{R}^2)$ such that for a given 2-form $\omega \in \Omega^2(T^*\mathbb{R}^2)$:

$$(\mathrm{d}f)^*\omega = 0$$

To examine what is truly happening, observe that the pull-back $(\mathrm{d}f)^*$ goes in the opposite way to $\mathrm{d}f$, which is a section of $T^*\mathbb{R}^2$.



$T^*\mathbb{R}^2$ is described by the *independent* conjugate coordinates (x^i, p_j). The key to associate the solution f to a solution of a PDE is to realize that as the action of $(\mathrm{d}f)^*$ is to be read as

$$(\mathrm{d}f)^* : \omega\left(x^i, p_j\right) \longmapsto \omega\left(x^i, \frac{\partial f}{\partial x^j}\left(x^i\right)\right).$$

$(\mathrm{d}f)^*$ inserts *dependent* coordinates $\left(x^i, p_j\left(x^i\right) \leftrightarrow \frac{\partial f}{\partial x^j}\left(x^i\right)\right)$. And it is the 2-form ω that specifies the PDE.

An example is called for at this moment. Let us show how the 2-form $\omega = \mathrm{d}p_1 \wedge \mathrm{d}x^2 + \mathrm{d}x^1 \wedge \mathrm{d}p_2$ is associated with the Laplace equation $f_{,11} + f_{,22} = 0$

$$\begin{aligned} 0 &= (\mathrm{d}f)^*\omega = \mathrm{d}f_{,1} \wedge \mathrm{d}x^2 + \mathrm{d}x^1 \wedge \mathrm{d}f_{,2} = f_{,11}\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^1 \wedge f_{,22}\mathrm{d}x^2 \\ &= (f_{,11} + f_{,22})\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \end{aligned}$$

On the cotangent bundle we are provided with the canonical symplectic form $\Omega = \mathrm{d}p_1 \wedge \mathrm{d}x^1 + \mathrm{d}p_2 \wedge \mathrm{d}x^2$; we might ask what PDE it represents.

$$0 = (\mathrm{d}f)^*\Omega = \mathrm{d}f_{,1} \wedge \mathrm{d}x^1 + \mathrm{d}f_{,2} \wedge \mathrm{d}x^2 = (-f_{,12} + f_{,21})\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$$

Any smooth function f satisfies this condition. This observation will be important later in context of the following construction. We can define the 2-form ω by the canonical symplectic form Ω and a tensor $A : T\mathbb{R}^2 \longrightarrow T\mathbb{R}^2$ as:

$$\omega = \Omega(A\bullet, \bullet)$$

Now the Monge-Ampère equations take the following form.

$$(\mathrm{d}f)^*\Omega(A \bullet \bullet) = 0$$

Before we move on to a generalized setting, let us present an example relevant in fluid mechanics.

Example (The Tricomi equation)**.** Let us have the PDE:

$$f_{,11} + x^1 f_{,22} = 0$$

It is clearly associated with $\omega = \mathrm{d}p_1 \wedge \mathrm{d}x^2 + x^1 \mathrm{d}x^1 \wedge \mathrm{d}p_2$.

$$0 = (\mathrm{d}f)^*\omega = \mathrm{d}f_{,1} \wedge \mathrm{d}x^2 + x^1\mathrm{d}x^1 \wedge \mathrm{d}f_{,2} = \left(f_{,11} + x^1 f_{,22}\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2$$

One can easily find such an endomorphism A that $\omega = \Omega(A\bullet, \bullet)$ in (canonical) coordinate formulation.

$$(\omega) = \begin{pmatrix} 0 & 0 & 0 & x^1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -x^1 & 0 & 0 & 0 \end{pmatrix} \qquad (\Omega) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Then we are looking for such A that its component matrix satisfies $(\omega) = (A)^T(\Omega)$. This is satisfied by:

$$\begin{aligned} A = {} & \frac{\partial}{\partial x^1} \otimes \mathrm{d}x^2 - x^1 \frac{\partial}{\partial x^2} \otimes \mathrm{d}x^1 \\ & - x^1 \frac{\partial}{\partial p_1} \otimes \mathrm{d}p_2 + \frac{\partial}{\partial p_2} \otimes \mathrm{d}p_1 \end{aligned} \qquad (A) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -x^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x^1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Indeed:

$$(A)^T(\Omega) = \begin{pmatrix} 0 & -x^1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -x^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & x^1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -x^1 & 0 & 0 & 0 \end{pmatrix}$$

5.3.1 ω-symplectic structure

Further following Banos [2007] we will define the generalized solution of Monge-Ampère equations and present a generalized complex structure $\mathbb{J}_\omega$ such that it will constitute the generalized solution as its generalized complex submanifold. We will demonstrate some of the properties of $\mathbb{J}_\omega$ in a more direct and explicit manner than Banos [2007]. Furthermore, we will show that the generalized complex structure is in fact a B-transform of the anti-diagonal symplectic generalized complex structure and we will use this observation to formulate Monge-Ampère equations in the language of twisted submanifolds.

A geometric way to study the solution f of the Monge-Ampère equations is to examine the *graph* of $\mathrm{d}f$ as a submanifold of $T^*\mathbb{R}^2$. A region of $T^*\mathbb{R}^2$ described by a pair of coordinates $(x^i, p_j \leftrightarrow f_{,j})$ on which ω acts trivially is a graph of $\mathrm{d}f$ for $(\mathrm{d}f)^*\,\omega = 0$. (What ensures that it is truly a manifold will be discussed later.)

A physical interpretation is at hand. One of the many phenomena the Laplace equation describes is incompressible curl-free frictionless fluid. We can then define a potential ϕ such that the fluid velocity field $v_i\,(x^j)$ is its gradient $\phi_{,j}$. The graph of $\mathrm{d}f$ is the manifold defined as $(x^i, v_j \leftrightarrow \phi_{,j})$, *fluid velocity throughout space.*

For simplicity, we will further denote $T^*\mathbb{R}^2 \equiv M$.

Definition (Generalized solution of Monge-Ampère equations, Banos [2007])**.** *A* ***generalized solution*** *of the two dimensional Monge-Ampère equations* $(\mathrm{d}f)^*\omega = 0$ *is a two-dimensional submanifold* $Q \hookrightarrow M$ *lagrangian with respect to* $\omega \in \Omega^2(M)$ *and the canonical symplectic 2-form* $\Omega \in \Omega^2(M)$*. That is*

$$TQ = TQ^\omega \qquad\qquad TQ = TQ^\Omega$$

Note that the graph of $\mathrm{d}f$ for $(\mathrm{d}f)^*\omega = 0$ is a two-dimensional submanifold lagrangian with respect to ω . The lagrangian property says every pair of sections of the tangent bundle of Q is annihilated by ω (i.e. $TQ \subseteq TQ^\omega$) and that it is the complete set of such points in $M \equiv T^*\mathbb{R}^2$ (i.e. $TQ^\omega \subseteq TQ$).

The $TQ = TQ^\Omega$ condition only picks out smooth solutions.

As we introduce the fibre-wise endomorphism A such that $\omega = \Omega(A\bullet, \bullet)$, we see that the generalized solution is a submanifold Q lagrangian with respect to Ω and *stable* under A.

Let us define a generalized almost complex structure by action on $TM \oplus T^*M$.

Definition (Monge-Ampère structure)**.** *We will refer to the fibre-wise endomorphism* $\mathbb{J}_\omega$ *on* $TM \oplus T^*M$ *as the* ***Monge-Ampère structure****.*

$$\mathbb{J}_\omega := \begin{pmatrix} -A & -\Omega^{-1} \\ \tilde{\omega} & A^* \end{pmatrix}$$

where $\tilde{\omega} := \Omega\left((1 + A^2)\bullet, \bullet\right)$ *and* $\omega = \Omega(A\bullet, \bullet)$ *with* Ω *being a symplectic structure on* M*.*

If we think of the tensors as bundle morphisms, the following notation will be useful. $\Omega(A\bullet,\bullet) \equiv \Omega\circ A$, $\Omega\left((1+A^2)\bullet,\bullet\right) \equiv \Omega\circ(1+A^2)$.

Lemma 5.3.1. *For the bundle morphisms defined by the 2-forms Ω, $\tilde{\omega}$ and the tensor A, the following equations hold.*

$$\Omega\circ A = A^*\circ\Omega, \qquad A\circ\Omega^{-1} = \Omega^{-1}\circ A^*, \qquad \tilde{\omega}\circ A = A^*\circ\tilde{\omega}.$$

Proof. The first equation is proved directly:

$$\begin{aligned}\langle\Omega(A(X)),Y\rangle &= \langle A(X),\Omega^*(Y)\rangle = \langle A(X),-\Omega(Y)\rangle = \langle X,-A^*(\Omega(Y))\rangle\,,\\ \langle\Omega(A(X)),Y\rangle &= \langle\omega(X),Y\rangle = \langle X,-\omega(Y)\rangle = \langle X,-\Omega(A(Y))\rangle.\end{aligned}$$

The other two equations are its immediate consequences. □

Proposition 5.3.2. *The Monge-Ampère structure $\mathbb{J}_\omega$ is a generalized almost complex structure.*

Proof. We check the conditions from section 3.2.5.

- It is *skew*: $\mathbb{J}_\omega^* = -\mathbb{J}_\omega$ because $\mathbb{J}_\omega \in \mathfrak{so}\left(TM\oplus T^*M\right)$. Indeed, we check the conditions of proposition 3.1.1 we see that the diagonal elements are bound as required and both $-\Omega^{-1}$ and $\tilde{\omega}$ are skew. ($\tilde{\omega}$ is a composition of a symmetric and a skew morphism.)
- It is *complex*: $\mathbb{J}_\omega^2 = -1$. With the use of lemma 5.3.1 we can compute $\mathbb{J}_\omega^2$ directly:

$$\begin{aligned}\mathbb{J}_\omega^2 &= \begin{pmatrix} -A & -\Omega^{-1} \\ \tilde{\omega} & A^* \end{pmatrix}\begin{pmatrix} -A & -\Omega^{-1} \\ \tilde{\omega} & A^* \end{pmatrix}\\
&= \left(\begin{array}{c|c} A^2-\Omega^{-1}\circ\tilde{\omega} & A\circ\Omega^{-1}-\Omega^{-1}\circ A^* \\ \hline -\tilde{\omega}\circ A + A^*\circ\tilde{\omega} & -\tilde{\omega}\circ\Omega^{-1}+(A^*)^2 \end{array}\right)\\
&= \left(\begin{array}{c|c} A^2-\Omega^{-1}\circ\Omega\circ(A^2+1) & 0 \\ \hline 0 & -\Omega\circ(1+A^2)\circ\Omega^{-1}+(A^*)^2 \end{array}\right)\\
&= \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & -\left(1+(A^*)^2\right)\circ\Omega\circ\Omega^{-1}+(A^*)^2 \end{array}\right) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned}$$

□

Now we can briefly inspect the $+i$-eigenbundle of $\mathbb{J}_\omega$ which is an almost Dirac structure that equivalently defines the generalized almost complex structure. In Banos [2007] it is constructed in a straightforward manner:

$$\{(X\oplus\xi) - i\mathbb{J}_\omega(X\oplus\xi) \mid X\oplus\xi \in \Gamma\left(TM\oplus T^*M\right)\}$$

Given that $\mathbb{J}_\omega^2 = -1$, we can see that multiplication by i is indistinguishable from the action of $\mathbb{J}_\omega$ on this set.

To show under what conditions the generalized complex structure integrates into a submanifold, Banos [2007] refered to general results of Crainic [2004]. We obtain the same result by showing that $\mathbb{J}_\omega$ is in fact a B-transformation of the symplectic generalized complex structure $\mathbb{J}_\Omega$ by the 2-form ω (see section 3.2.5). In other words:

Proposition 5.3.3. *The Monge-Ampère structure $\mathbb{J}_\omega$ is an ω-symplectic structure.*

$$e^{-\omega} J_\Omega e^{\omega} = J_\omega.$$

Proof. We transform the $\mathbb{J}_\omega$ structure in an inverse manner and obtain $\mathbb{J}_\Omega$ with a repeated use of lemma 5.3.1.

$$\begin{aligned}
e^{\omega}\mathbb{J}_\omega e^{-\omega} &= \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \left[\begin{pmatrix} 0 & -\Omega^{-1} \\ \tilde{\omega} & 0 \end{pmatrix} + \begin{pmatrix} -A & 0 \\ 0 & A^* \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \\
&= \left(\begin{array}{c|c} \Omega^{-1}\circ\omega & -\Omega^{-1} \\ \hline \tilde{\omega} + \omega\circ\Omega^{-1}\circ\omega & -\omega\circ\Omega^{-1} \end{array} \right) + \left(\begin{array}{c|c} -A & 0 \\ \hline -\omega\circ A - A^*\circ\omega & A^* \end{array} \right) \\
&= \left(\begin{array}{c|c} \Omega^{-1}\circ\omega - A & -\Omega^{-1} \\ \hline \tilde{\omega} + \omega\circ\Omega^{-1}\circ\omega - \omega\circ A - A^*\circ\omega & -\omega\circ\Omega^{-1} + A^* \end{array} \right) \\
&= \left(\begin{array}{c|c} \Omega^{-1}\circ\Omega\circ A - A & -\Omega^{-1} \\ \hline \Omega\circ A^2 + \Omega + \omega\circ\Omega^{-1}\circ\omega - \omega\circ A - A^*\circ\omega & -\omega\circ\Omega^{-1} + A^* \end{array} \right) \\
&= \left(\begin{array}{c|c} 0 & -\Omega^{-1} \\ \hline +\Omega\circ A^2 + \Omega + \Omega\circ A^2 - \Omega\circ A^2 - \Omega\circ A^2 & -A^*\circ\Omega\circ\Omega^{-1} + A^* \end{array} \right) \\
&= \begin{pmatrix} 0 & -\Omega^{-1} \\ \Omega & 0 \end{pmatrix} = \mathbb{J}_\Omega.
\end{aligned}$$

□

Considering proposition 3.2.11 that says B-transformations by closed 2-forms preserve Courant integrability, we obtain the desired corollary.

Corollary 5.3.4. *The Monge-Ampère structure $\mathbb{J}_\omega$ is integrable if and only if ω is closed.*

Before we prove the link between $\mathbb{J}_\omega$ and the generalized solution of Mong-Ampère equations Q, we remark that ω associated with the Laplace equation $\omega = \mathrm{d}p_1 \wedge \mathrm{d}x^2 + \mathrm{d}x^1 \wedge \mathrm{d}p_2$ and the Tricomi equation $\omega = \mathrm{d}p_1 \wedge \mathrm{d}x^2 + x^1\mathrm{d}x^1 \wedge \mathrm{d}p_2$ are both closed.

Proposition 5.3.5 (Banos [2007])**.** *A generalized complex submanifold of the generalized complex structure* $\mathbb{J}_\omega$ *is a generalized solution* $Q \hookrightarrow M$ *of the Monge-Ampère equations on* M *defined by a closed 2-form* ω*.*

$$\mathbb{J}_\omega : \mathbb{T}Q \longrightarrow \mathbb{T}Q \Longleftrightarrow TQ^{\Omega} = TQ,\ TQ^{\omega} = TQ.$$

Proof. Let us symbolically formulate $\mathbb{J}_\omega$-stability in a matrix form:

$$\begin{aligned} &\begin{pmatrix} -A & -\Omega^{-1} \\ \tilde{\omega} & A^* \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} \\ =&\begin{pmatrix} -A(X) - \Omega^{-1}(\xi) \\ \Omega(X) + \Omega\left(A^2(X)\right) + A^*(\xi) \end{pmatrix} \overset{!}{\in} \begin{pmatrix} TQ \\ \mathrm{Ann}(TQ) \end{pmatrix} \end{aligned}$$

for $\forall X \in \Gamma(TQ)$, $\forall \xi \in \Gamma(\mathrm{Ann}(TQ))$.

In other words, we have the A-stability of TQ :

$$A : TQ \longrightarrow TQ \qquad\qquad A^* : \mathrm{Ann}(TQ) \longrightarrow \mathrm{Ann}(TQ)$$

and the associated A^*-stability of the decompostion of $T^*M|_Q$ into T^*Q and $\mathrm{Ann}(TQ)$. Similarly, the following action of the symplectic form Ω.

$$\Omega : TQ \longrightarrow \mathrm{Ann}(TQ) \qquad\qquad \Omega^{-1} : \mathrm{Ann}(TQ) \longrightarrow TQ$$

which is by proposition 3.3.1 equivalent to $TQ = TQ^{\Omega}$.
We know that $TQ = TQ^{\Omega}$ and A-stability is equivalent to $TQ = TQ^{\Omega}$ and $TQ = TQ^{\omega}$. $\square$

Conclusion

We motivated the studies of generalized geometry in an attempt to unify the framework of complex geometry and the structures defining Hamilton's canonical equations, which led us to a fibre-wise complex endomorphism on $TM \oplus T^*M \otimes \mathbb{C}$. Then we proceeded to introduce generalized complex geometry following the work of Gualtieri [2004]. We presented the canonical symmetric product and the Courant bracket and we showed how they interact on a Courant algebroid, which is the natural structure of $TM \oplus T^*M$. Dirac structures were introduced as a way to describe integrable structures on a Courant algebroid including the generalized complex structure. Afterwards, we mentioned the notion of a generalized complex submanifold and presented Courant's investigations of reductions of Dirac structures onto submanifolds, Courant [1990].

In the final chapter, we presented how the discussed structures and mechanisms can serve as interpretational frames and tools in physics and related areas. We investigated the characteristic bundle of a reduced symplectic structure as a generalized complex structure by the process of Dirac reduction. We rigorously demonstrated how Dirac reduction resolves compatibility issues of Poisson structures and constrained systems, which is a problem known from symplectic geometry and modern string theory. In the end, following Banos [2007], we described how a solution of the Monge-Ampère equations can be thought of as a generalized complex submanifold and proved some partial results in alternative ways.

The author does hope the thesis presents generalized complex geometry as a rich mathematical theory with strong ties to concepts from mathematical physics in a comprehensible way. To decide whether its somewhat platonic and reductionist nature – which invites one to generalize and unify – is appropriate to serve as a principal motivation for a mathematical physicist or is a mere accompanying phenomenon of a rich epistemology is left to the reflective reader.

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