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BACHELOR THESIS

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Conservation Laws with respect to Curved Backgrounds associated with Black Holes and Cosmological Models

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I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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To my two best friends, Anela and Miro: Thank you for influencing me and giving me support my entire life. Your good intentions while raising me have shaped my mind in the way that I would never change.

I also want to express my gratitude to my supervisor prof. Bičák. This thesis would not have happened if there were not such interesting topics proposed by the professor. I enjoyed our talks and I attended lectures on relativistic physics with joy.

Title: Conservation laws with respect to curved backgrounds associated with black holes and cosmological models

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Abstract: We review the problem of defining energy, momentum etc. and their conservation in curved spacetimes and a possible solution in the form of a background spacetime. Our focus is set on superpotentials, which, when integrated on a spatial boundary, yield conserved charges, while a conserved vector current is a divergence of a superpotential. Within this thesis, we build a minimal mathematical formalism necessary to prove and interpret Noether's theorem which unites symmetries and conservation laws. We emphasize the significance of Killing vector fields – generators of isometries. After a short historical overview, the KBL superpotential is presented in detail, which makes it possible to define conserved quantities with respect to a curved background spacetime. We then employ its generalization within the Horndeski scalar-tensor theory of gravity. We concentrate on a subclass containing non-minimal derivative coupling of the Einstein tensor and a scalar field. We find superpotentials for spherically symmetric, static spacetimes (e.g. exterior of black holes) and time-dependent cosmological spacetimes, in particular with respect to (Anti-)de Sitter backgrounds.

Keywords: conservation laws, Horndeski gravity, curved backgrounds, superpotentials, conserved currents, spacetime perturbations

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General Introduction

Prelude

What is energy? In curved spacetimes, it might happen that we can't give an answer to this simple question. Among many others, H. Callen notes in his book on thermodynamics [1] that *energy is defined by its conservation law.* This is, indeed, the right definition, for the existence of conserved energy of a physical system reflects its translational symmetry in time; *but what if the system does not posses such a symmetry*? Along with energy, there are analogous issues related to momentum and angular momentum, because these are contained in or constructed from an energy-momentum tensor, residing on the right hand side of Einstein's gravitational field equations, which govern the interplay of matter and spacetime geometry.

At the time of this thesis being written, we were fortunate to witness the unveiling of the first real image of a black hole shadow ever, captured by the Event Horizon Telescope [2]. Prior to this, in 2016, we celebrated the announcement of the first direct detection of gravitational waves, thanks to LIGO and Virgo observatories [3], which told us that two black holes, situated more than a billion of light years away from us, have merged. Both black holes and gravitational waves are interesting predictions of general relativity, which was remarkably formulated by Albert Einstein in 1915 and subsequently extended by ideas of many other notable mathematicians and physicists, including Einstein himself. Up to the present time, as precisely 100 years has passed since the famous solar eclipse expedition[1], general relativity had withstood many experimental tests[2], including the aforementioned recent observations, and currently remains in its place as a main theory of gravity. One important concept, however, has been having many difficulties and controversies — the formulation of conservation laws.

Conserved quantities are of the main importance in describing *physics of the surrounding nature.* Goldberg [6] elucidates:

Conservation laws are important in physics because they allow physical situations to be characterized by a few parameters such as charge, energy, linear and angular momentum, isotopic spin, or the parameters of the current gauge theories.

For instance, a proposition, today famously known as the 'no-hair theorem', was being developed around 1970 independently by Israel, Carter and Hawking [7]. It states, under some assumptions, that a black hole can be completely characterized merely by its mass (energy), electric charge and angular momentum. Hence,

[1] It was a turning point for general relativity, because the Royal Society decided to replace Newton's theory by that of Einstein's, as the eclipse expedition, led by F. Dyson and A. Eddington, had confirmed that the rays of light bend around the Sun in accordance with Einstein's general relativity [4].

[2] For instance, Gravity Probe B, launched in 2004, was a space experiment testing the geodetic and frame-dragging effects, with results confirming the general relativity predictions [5].

a requirement that values of these quantities, once measured and/or calculated, ought to be conserved, is not surprising. Another, very illustrative example is W. Pauli's 'existential postulate' of a very light, neutral, spin-1/2 particle, which he proposed in 1930 in order to explain the apparent non-conservation of energy, momentum and angular momentum in radioactive beta decays [8]. The electron (anti)neutrino is what we call this (anti)particle today. Similarly, in 1905 Einstein was driven by an energy balance in deriving his famous mass-energy equivalence. By going back in time even further, we may recall that Maxwell obtained his equations in their final form by requiring consistency with the equation of continuity. We could continue in this manner almost endlessly, therefore one might conclude that, as physicists, we a priori demand from a physical theory to provide us with conservation laws, because they, in fact, reflect *the true nature of physics*.

A connection between nature and conservation laws is even more profound. Shortly after Einstein's formulation of general relativity, under an influence of F. Klein and D. Hilbert, Emmy Noether enlightened physical and mathematical community with her theorems, which, in modern terminology, connect equations of motion, a physicist's musical instrument, and conserved 'Noether currents'. This connection is a consequence of a more abstract concept of symmetry, which is based on the invariance of the action functional with respect to a certain group of transformations. Because general relativity and other metric theories of gravity are directly related to a geometry of spacetime, the notion 'symmetry' plays a double role here.

However, a generic, dynamical spacetime in general relativity is arbitrarily curved and, thus, lacks symmetries. Maximally symmetric spacetimes are those that have constant curvature. For example, one of the simplest solutions in general relativity — the Schwarzschild black hole — does not have a global timelike symmetry due to the existence of the horizon. In cosmology, the most widely used metric to describe the large-scale geometry of the universe is the Friedmann–Lemaître–Robertson–Walker one; although highly symmetric, it is time-dependent through the scale factor, associated with the currently observed expansion of the universe, due to which there is no time isometry yielding conserved energy. Besides the problem of the global symmetries, there are others closely related to the local ones. More specifically, in a curved spacetime, freely-falling observers feel themselves as if they were in a flat spacetime due to the equivalence principle. It makes definitions of energy, momentum and angular momentum observer-dependent and hence ambiguous, as pointed out by Petrov et al. [9]. Questions arise:

Can we somehow associate the spacetime under consideration with a more symmetrical one?
Is it possible to exploit spacetimes possessing maximal symmetry?

Apart from conservation laws, there are serious issues with interpretation of the present accelerated expansion of the universe. In the standard (ΛCDM) model of Big Bang cosmology, based on theory of general relativity, such expansion is ascribed to Dark Energy, originating from the presence of the cosmological constant in Einstein's field equations. The 'dark' means 'unknown' from the viewpoint of particle physics and it is thus of big interest to find better explanations. One of the approaches is to modify general relativity and recently rediscovered Horndeski theory stands out as the most natural choice; with a scalar field added to the metric tensor as a second field variable, Horndeski theory is the most general scalar-tensor theory of gravity in four dimensions having *second-order field equations*. General relativity and its many well-known modifications (such as Brans-Dicke theory, $f(R)$-gravity, etc.) are special cases of Horndeski theory[3]. The additional scalar field often plays a role of a generator of inflation.

The mentioned detection of gravitational waves gives prospects to investigate possible deviations from Einstein's theory in local strong-gravity regimes and Horndeski theory might be a good candidate to replace general relativity, just as Einstein's general relativity deposed Newton's theory of gravitation, but this is only a matter of speculation, for now.

* * *

Outline of the Thesis

The present work is divided into two main parts which are organized as follows.

The first, larger part is primarily a review of various formulations of conservation laws in general relativity, related difficulties and a possible resolution based on in using as suitable backgrounds highly symmetric spacetimes. The aim is to give a comprehensive survey of the theory both from mathematical and physical perspective.

After giving a simple illustration of the formalism through electrodynamics, we proceed with the first chapter; it is purely mathematical, but other chapters are not independent of it, because it contains important objects and terminology employed later in the text. This involves maps of manifolds and induced maps, flows, Lie derivative, the (gravitational) action functional, etc. This chapter should be regarded as a preliminary one, giving the reader minimal mathematical foundation, necessary for deeper comprehension of other parts of the thesis.

Chapter 2 is a bridge between mathematical and physical picture of conservation laws. We show the importance of isometries and conformal transformations and their generators — (conformal) Killing vector fields. The associated notion of maximal symmetry is demonstrated in detail. We exhibit conserved (physical) quantities which can be constructed in the presence of spacetime symmetries. Noether's theorem is fully presented and applied to diffeomorphism invariance of general relativity. Then we discuss definitions of the energy-momentum tensor and conclude the chapter by explaining the Belinfante symmetrization method.

In the third chapter we give a description of superpotentials and associated pseudotensors (energy-momentum complexes). Their role in defining conserved quantities, non-covariance and historical development is briefly discussed. Important sections are those containing covariant Komar superpotential and the notion of a flat background spacetime.

Chapter 4 deals with a relatively recent formalism of Katz, Bičák and Lynden-Bell (KBL) and further generalizations of their approach, in which conserved quantities are constructed from the Noether's theorem for (possibly large) spacetime perturbations, which are taken with respect to a *curved background spacetime*. Some cosmological applications are mentioned.

[3]There are even theories called 'beyond Horndeski', involving multiple scalar fields, but that is far beyond the scope of the present thesis.

The second, much shorter part of the thesis is an application of the generalized KBL formalism in an interesting subclass of Horndeski theory of gravity. It contains a non-minimal derivative coupling (NDC) of the scalar field to the Einstein tensor.

Within Chapter 5, after a concise introduction of the general Horndeski theory, we concentrate on the NDC model and refer to solutions found in the literature. Then the general formulae for superpotentials for Horndeski theories are presented and the choice of the form of NDC superpotentials is made.

In the final chapter, computation of superpotentials is demonstrated for spherically symmetric, static spacetimes (e.g. black holes) and cosmological (FLRW) spacetimes with respect to the (A)dS background.

* * *

Now we can explain the name of the thesis: 'Conservation Laws with respect to Curved Backgrounds' refers mainly to the fourth chapter of the review part of this thesis, while 'Associated with Black Holes and Cosmological Models' may be assigned to the second part, where computations of superpotentials are carried out.

For an interested reader, 'a purpose of the thesis' is presented at the end of this work as a 'Postlude', because it is a rather long explanation of why is this thesis written in the way it is.

Notation and Conventions

Tensors, although geometric objects independent of a chosen coordinate system, will be denoted as $T^{\mu\ldots\nu}_{\rho\ldots\sigma}$. An exception is Chapter 1. These indices should, however, be understood as abstract indices, telling us what type of a tensor we are dealing with. Only when we specify a particular coordinate system will the Greek indices $\{..., \mu, \nu, \rho, ...\}$ generally run over the four spacetime coordinate labels $0, 1, 2, 3$ (or $1, \ldots, n$ if we work with general dimension n) while the Latin indices $\{..., i, j, k, ...\}$ run over three spatial coordinate labels $1, 2, 3$.

We restrict ourselves only to Levi-Civita connections, which are torsion-free and compatible with the metric. The associated covariant derivative is ∇_μ.
Partial derivatives $\frac{\partial}{\partial x^\mu}$ are mostly denoted as ∂_μ.

Einstein summation convention is present throughout the entire text:

$$\sum_{\mu=0}^{3} A_\mu B^\mu \equiv A_\mu B^\mu$$

A signature of any metric tensor $g_{\mu\nu}$ is assumed to be $(1, 3)$, i.e. $(-, +, +, +)$; except for Chapters 1 and 2, where we deal with general (pseudo-)Riemannian signatures.

Relevant constants:

- Gravitational constant G
- Speed of light in vacuum c
- Einstein's coupling constant $\kappa := 8\pi G/c^4$

List of Abbreviations and Acronyms

Abbreviations and acronyms from the following list are introduced formally in the text only with their first appearance.

(A)dS — (Anti)-de Sitter
EM — Electromagnetic
FLRW — Friedmann–Lemaître–Robertson–Walker
KBL — Katz–Bičák–Lynden-Bell
LL — Landau & Lifshitz
LHS — Left hand side
NDC — Non-minimal derivative coupling
RHS — Right hand side

Part I: A Review

Motivational example: Electrodynamics

We present here, through a very simple example, the main ideas and motivation standing behind the construction of superpotentials. This section is inspired by a review of Deruelle & Uzan [10].

Consider inhomogeneous Maxwell's equations, in a gravitational field,

$$\nabla_\nu F^{\mu\nu} = \mu_0 J^\mu \,, \tag{1}$$

in which the left-hand side (LHS) is a covariant divergence of the electromagnetic (EM) tensor $F_{\mu\nu} \coloneqq \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ defined as the exterior derivative of the four-potential A_μ ($F = \mathrm{d}A$); the right-hand side (RHS) is the EM four-current J^μ multiplied by the permeability of free space μ_0. Eq. (1) is a generalization of its Minkowskian counterpart to curved spacetimes ($\equiv$ presence of gravity), made by replacing the Minkowski metric $\eta_{\mu\nu}$ with $g_{\mu\nu}$ and ∂_μ with ∇_μ ([11], p. 124). Its LHS can be rewritten as

$$\nabla_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} + \underbrace{\Gamma^\mu{}_{(\lambda\nu)} F^{[\lambda\nu]}}_{=0} + \Gamma^\nu{}_{\lambda\nu} F^{\mu\lambda} = \frac{1}{\sqrt{-\mathrm{g}}} \partial_\nu \left(\sqrt{-\mathrm{g}} F^{\mu\nu} \right) \,, \tag{2}$$

where we used the fact that $F^{\mu\nu}$ is antisymmetric, while the Christoffel symbols of the second kind

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu})$$

are symmetric in lower indices[4], and a useful formula (see (A.3a) in Appendix A)

$$\Gamma^\nu{}_{\nu\sigma} = \frac{\partial_\sigma \left(\sqrt{-\mathrm{g}} \right)}{\sqrt{-\mathrm{g}}} \,,$$

where $\mathrm{g} \coloneqq \det(g_{\mu\nu})$ is a determinant of the metric $g_{\mu\nu}$. Multiplying both sides of Eq. (1) by $\sqrt{-\mathrm{g}}$ and using (2) leads to an equation

$$\partial_\nu \left(\sqrt{-\mathrm{g}} F^{\mu\nu} \right) = \mu_0 \sqrt{-\mathrm{g}} J^\mu \,, \tag{3}$$

whose (ordinary) divergence is *identically* zero, due to the antisymmetry of $F^{\mu\nu}$ and a requirement that partial derivatives commute $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$. Indeed, the function $f \coloneqq \sqrt{-\mathrm{g}}$ does not spoil the conservation law; explicitly:

$$\begin{aligned}
\partial_\mu \partial_\nu (f F^{\mu\nu}) &= \partial_\mu \left[F^{\mu\nu} \partial_\nu f + f \partial_\nu F^{\mu\nu} \right] \\
&= \partial_\mu F^{\mu\nu} \partial_\nu f + F^{[\mu\nu]} \partial_{(\mu} \partial_{\nu)} f + \partial_\mu f \partial_\nu F^{\mu\nu} + f \partial_{(\mu} \partial_{\nu)} F^{[\mu\nu]} \\
&= \partial_\mu F^{\mu\nu} \partial_\nu f - \partial_\nu F^{\nu\mu} \partial_\mu f \\
&= 0 \,.
\end{aligned}$$

[4] This way of showing that a term in an equation (involving a contraction of a product of a symmetric and an antisymmetric tensor) is zero will be used throughout the text, sometimes without explicit 'under-braces'. More generally, if we use the (anti)symmetry of a tensor in an equation it will be denoted with its (anti)symmetrization, as in Eq. (2).

Therefore,

$$\partial_\mu \hat{J}^\mu = 0\,. \tag{4}$$

Here we are making use of a new notation, in which a caret (a 'hat') denotes a multiplication by $\sqrt{-\mathrm{g}}$, i.e. $\hat{J}^\mu := \sqrt{-\mathrm{g}}J^\mu$. Equation (4) represents a *differential* (*local*) *conservation law*, which can be interpreted as an equation of continuity. We may integrate it over some region Ω on a spacetime manifold and apply Gauss's theorem to obtain an integral equation. For this purpose, let $\Omega \subset \mathbb{R}^4$, $\Omega = \langle t_1, t_2\rangle \times V$ be a coordinate image[5] of a spacetime region Ω in coordinates $x^\mu = (t, x^i)$ (we set $c = 1$ and x^i are not necessarily Cartesian), where $V \subset \mathbb{R}^3$ is a fixed spatial volume with a boundary ∂V, whose surface element is $\mathrm{d}^2 S$. A boundary of Ω is given as a decomposition $\partial\Omega = V_1 \cup V_2 \cup \sigma$, in which $V_a := \{t_a\}\times V$ are spacelike hypersurfaces (volume V) at different constant times t_a, $a \in \{1, 2\}$, while $\sigma = (t_1, t_2) \times \partial V$ is a timelike hypersurface (Fig. 1). Thereby integrating (4), with n_μ being a normal covector to $\partial\Omega$ with an element $\mathrm{d}^3\Sigma$, yields

$$0 = \int_\Omega \partial_\mu \hat{J}^\mu \;\mathrm{d}^4 x = \oint_{\partial\Omega} \hat{J}^\mu n_\mu \;\mathrm{d}^3\Sigma = \oint_{\partial\Omega} \hat{J}^0 n_0 \;\mathrm{d}^3\Sigma + \oint_{\partial\Omega} \hat{J}^i n_i \;\mathrm{d}^3\Sigma \tag{5a}$$

$$= \int_{V_1\cup V_2} \hat{J}^0 n_0 \;\mathrm{d}^3\Sigma + \int_\sigma \hat{J}^i n_i \;\mathrm{d}^3\Sigma \tag{5b}$$

$$= \int_{V_2} \hat{J}^0 \;\mathrm{d}^3 x - \int_{V_1} \hat{J}^0 \;\mathrm{d}^3 x + \int_{t_1}^{t_2} \left(\oint_{\partial V} \hat{J}^i n_i \;\mathrm{d}^2 S\right) \mathrm{d}t\,. \tag{5c}$$

In (5a) we applied the Gauss's theorem and then separated the product $\hat{J}^\mu n_\mu$ into temporal and spatial part. Next, (5b) and (5c) follow from the fact that, because Ω is a Cartesian product of time and space, the components of the surface normal covector can be taken as

$$n_\mu|_{V_1} = (-1, 0, 0, 0) \quad \& \quad n_\mu|_{V_2} = (1, 0, 0, 0)\,, \tag{6}$$

whereas $n_i \neq 0$ solely on σ (see Fig. 1 below). Finally, in (5c) we used (6) and rewrote the integral over σ using Fubini's theorem. Let us now define a charge contained in the volume V at times t_1, t_2 as

$$Q_V(t_a) := \int_{V_a} \hat{J}^0 \;\mathrm{d}^3 x \qquad a \in \{1, 2\}\;. \tag{7}$$

Then from (5) we obtain

$$Q_V(t_2) - Q_V(t_1) = -\int_{t_1}^{t_2} \left(\oint_{\partial V} \hat{J}^i n_i \;\mathrm{d}^2 S\right) \mathrm{d}t\,, \tag{8}$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_V(t) = -\oint_{\partial V} \hat{J}^i n_i \;\mathrm{d}^2 S\,, \tag{9}$$

[5] Such a decomposition of a spacetime (region) into time *and* space is not generally possible, as one would think at first. The spacetime must be *globally hyperbolic*. It was known since 1970 that such spacetimes are homeomorphic (i.e. topologically equivalent) to $\mathbb{R} \times \Sigma_t$, where Σ_t is a so-called *Cauchy hypersurface* — a spacelike hypersurface of $t = const.$ (precise definitions and theorems can be found in Hawking & Ellis [12]). It holds that globally hyperbolic spacetimes are *foliated* by Cauchy hypersurfaces and contain no singularities. See also an article published in 2003 by Bernal & Sánchez [13], in which they prove the existence of a *diffeomorphism* between globally hyperbolic spacetime and $\mathbb{R} \times \Sigma_t$. In our example, the volume V plays a role of a Cauchy hypersurface.

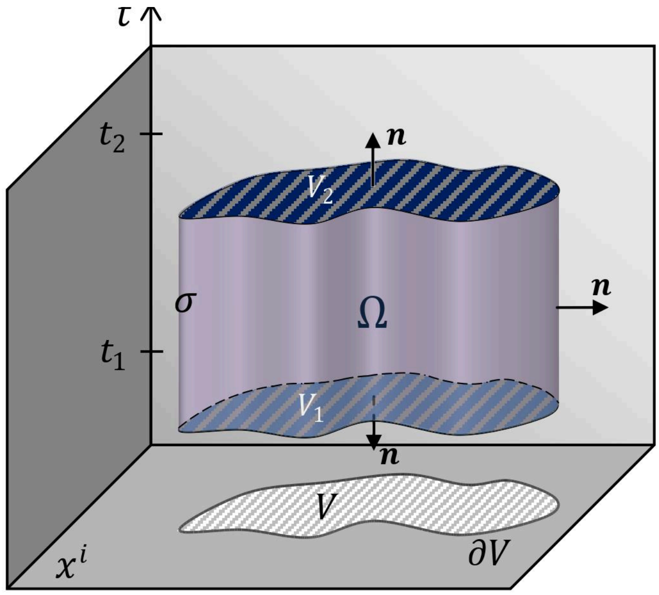

Figure 1: An example of a coordinate image Ω of a spacetime region as a domain of integration with a boundary $\partial\Omega = V_1 \cup V_2 \cup \sigma$ and a normal vector field $\boldsymbol{n}$. One spatial dimension is suppressed.

because, by the fundamental theorem of calculus, the LHS of (8) is equal to

$$\int_{t_1}^{t_2} \dot{Q}_V(t)\; \mathrm{d}t\,,$$

where the dot over Q_V stands for a differentiation with respect to time t. Equation (9) is a well-known *global conservation law*, representing the balance between the total charge contained in the volume V and the outward flux of the current through a surface ∂V. The flux integral on the RHS of (9) may vanish if functions $\hat{J}^i = \hat{J}^i(t, x^k)$ representing sources have a compact support (e.g. an isolated astronomical system), so we can take ∂V to be sufficiently far away from the sources. Then the total charge contained in the volume V is a constant of motion, that is

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_V = 0. \tag{10}$$

It follows from (2) and Maxwell's equations (1) that, by applying Gauss's theorem in (7), the charge can be expressed as a surface integral

$$Q_V = \frac{1}{\mu_0} \oint_{\partial V} \hat{F}^{0i} n_i \; \mathrm{d}^2 S\,. \tag{11}$$

This is very useful, because the surface integral is often easier to calculate than the full one over V. For example, in spherical coordinates $x^i = \{r, \theta, \varphi\}$, there would be no integration over the radial coordinate r. Note that for charge to be expressed in terms of $\hat{F}_{\mu\nu}$, Maxwell's equations are used neither in the interior of Ω nor on V_1 and V_2, but *only* on σ.

To conclude, the antisymmetric EM tensor density $\hat{F}^{\mu\nu}$ generates, through a divergence, identically conserved current $\hat{J}^\mu$. Far away from the sources (i.e. at spatial infinity), a conserved charge Q may be obtained by integrating $\hat{F}^{0i}$ over the closed spacelike surface ∂V.

The superpotential schema

In the given example, $\hat{F}^{\mu\nu}$ can be identified with an object called a *superpotential*. A great deal of this thesis is devoted to gravitational superpotentials $\hat{U}^{\alpha\beta}$ for which the terminology can be generalized from the previous EM example and their role can be represented by the following simple schema:

$$\begin{array}{c} \textit{Local} \\ \textit{conservation} \\ \textit{law} \end{array} \xleftarrow[\partial_\alpha]{} \hat{J}^\alpha \xleftarrow[\partial_\beta]{} \hat{U}^{\alpha\beta} \xrightarrow{\oint_{\partial V}} Q \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \begin{array}{c} \textit{Global} \\ \textit{conservation} \\ \textit{law} \end{array} \tag{12}$$

In words, following Petrov & Katz [14]: superpotential integrated on the boundary of 'space' at a given 'time' gives conserved 'charges'. The divergence of the superpotential is a conserved vector current $\hat{J}^\mu$ which provides local meaning to a global quantity.

In practice, however, it is not that simple. As we will see, there is an ambiguity in construction of superpotentials. Moreover, many superpotentials historically led to non-covariant formulations of conservation laws and most of them work only in asymptotically flat spacetimes. Our aim is to promote covariant expressions applicable to asymptotically curved spacetimes. Nonetheless, we are not 'brushing flatness under the rug'; almost every concept built for curved spacetimes has its flat analogy, so we find it crucial to understand ideas applicable to the (asymptotically) Minkowski spacetimes first, and then generalize these ideas to (asymptotically) curved spacetimes.

The existence of the conserved charge and the conserved current in the motivational example is known to be a consequence of Noether's theorem, because the action from which the Maxwell's equations (1) are obtained

$$\begin{aligned} S_{EM}[g_{\mu\nu}, A_\mu] &= \int_{\mathcal{D}} (J^\mu A_\mu - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}) \sqrt{-\mathrm{g}} \ \mathrm{d}^4 x \\ \Rightarrow \delta S_{EM} &= \int_{\mathcal{D}} (J^\mu - \frac{1}{\mu_0} \nabla_\nu F^{\mu\nu}) \delta A_\mu \sqrt{-\mathrm{g}} \ \mathrm{d}^4 x \,, \end{aligned}$$

is invariant with respect to a gauge transformation

$$A_\mu \mapsto A_\mu + \partial_\mu \alpha \,.$$

where α is a differentiable function. At the same time, the conservation of charge is related to the Lie group U(1), a symmetry (gauge) group of the electromagnetic interaction. In general relativity, the principle of general covariance leads to an infinite dimensional symmetry group consisting of all possible diffeomorphisms of the spacetime manifold, which from the second Noether's theorem lead to the existence of superpotentials. The conserved current in the above schema (12) will be a Noether current.

Theorems of Emmy Noether have a great significance in obtaining and interpreting conservation laws. To deeply understand relation between symmetries and conservation laws hidden in Noether's theorem, we wish to form a minimal mathematical basis in the following chapter.

1

Mathematical Preliminaries

Our main task in the review part of the present work is to give a comprehensive insight to the theory of conservation laws and their relation to symmetries. In order to achieve this, we think it is appropriate to review few important mathematical concepts, necessary for smooth reading of the text. This involves notions of pull-backs of tensors, a Lie derivative, Lie Groups, the variational principle in field theories, etc. Of course, if a person reading this thesis is already familiar with these, then the first chapter may serve as a succinct reminder. Besides, it is a convenient way for us to establish some of the terminology used in the rest of the thesis. One need not to have many prerequisites in order to grasp the main ideas of the first chapter.

With intention to be brief, but mathematically precise, we concentrate in the following sections mainly on subjects utilized later in the text, even though they may be connected to other interesting things or have additional useful properties. Various literature is available for particularities: we personally recommend Fecko [15], Nakahara [16] and Lee [17] from which we drew inspiration for differential geometry part; classical textbooks on gravitation, field theory and cosmology of Wald [18]; Misner, Thorne & Wheeler [7]; Weinberg [11]; Landau & Lifshitz [19] and Hawking & Ellis [12] have been very useful in this context too, as well as for the whole thesis.

1.1 Some aspects of differential geometry

A central object in general relativity is a metric tensor g, which characterizes a geometry of its four-dimensional spacetime — a Lorentzian manifold. Such a manifold is a special case of n-dimensional pseudo-Riemannian manifold, because the signature of any metric tensor in general relativity is $(1, 3) \equiv (-, +, +, +)$[1]. Thus, in definitions and theorems below (and in later chapters too), we will speak about smooth manifolds as such and general (pseudo-)Riemannian ones, with additional structures built upon them. There are some properties of Lorentzian manifolds which do not hold for arbitrary (pseudo-)Riemannian ones, but we will not need those. Also, it is not true that every manifold admits a Lorentz metric ([17], p. 285). We presume that the reader has at least basic knowledge of topology and differential geometry (as did the author have before writing this thesis), so we are going to quickly introduce the nomenclature.

In what follows, by notion 'smooth' we simply mean a C^∞ property, as in [17] and [16]. Of course, in many cases, a C^k property for some appropriate $k > 0$ would be sufficient. A (pseudo-)Riemannian manifold M with a metric g and $\dim M = m$, will be denoted as (M_m, g); if it is unnecessary to indicate a presence of the metric or the dimension of M, the metric g and/or the index m

[1]This, of course, depends on the convention, which might as well be $(+, -, -, -)$.

will be omitted. Such a manifold[2], which we mostly depict on paper as a smooth two-dimensional surface floating in $\mathbb{R}^3$, serves as a playground to additional structures living on it. Smooth functions on M (i.e. $f : M \to \mathbb{R}$ and its coordinate representation is smooth for any chart[3] on M) form a space, which we will symbolize as $\mathcal{F}M$. By assigning one specific tangent vector to every point on M we get a vector field V, a set of linear differential operators (directional derivatives) acting on $\mathcal{F}M$. It is a smooth section[4] of a tangent bundle TM (a disjoint union of all tangent spaces T_pM at points $p \in M$); i.e. $\forall\, f \in \mathcal{F}M \,:\, V[f] \in \mathcal{F}M$ and $\mathfrak{X}M$ stands for a space of all such smooth sections [15]. Smooth sections of a cotangent bundle T^*M (the dual of TM) are covector fields (1-forms). The set of covector fields will be denoted by $\Omega^1 M$. Tensor product enables us to build a space $T^r_s M$ of higher-rank tensor fields (smooth sections of a tensor bundle)[5] of mixed type (r, s), which are multilinear objects mapping r covectors and s vectors to a real number. Coordinate basis vector fields will be denoted as $\frac{\partial}{\partial x^\mu}$, which can be sometimes abbreviated as ∂_μ. For 1-forms, the basis element is represented by $\mathrm{d}x^\mu$. We will many times employ a contraction (canonical pairing) $\langle\cdot,\cdot\rangle : \Omega^1 M \otimes \mathfrak{X}M \to \mathbb{R}$ defined as $\langle \alpha, X\rangle = \alpha_\mu X^\mu$.

1.1.1 Diffeomorphisms, induced maps and flows

The main goal of this subsection is to introduce a pull-back of a tensor field and flows. These are important for proper definitions of the Lie derivative and isometries in later subsections. Let us first recall some basic definitions.

Definition 1. *Consider a map $F : M_m \to N_n$. We a say that F is a* ***smooth map*** *(or differentiable map) if, for any coordinate chart (U, ψ) on M and (V, χ) on N, a composition*

$$\widetilde{F} \coloneqq \chi \circ F \circ \psi^{-1} : \mathbb{R}^m \to \mathbb{R}^n \tag{1.1}$$

is smooth (C^∞). If in addition the inverse map F^{-1} exists and is also smooth, then F is called a ***diffeomorphism****.*

The map $\widetilde{F}$ is called a coordinate presentation of F [16]. The composition (1.1) is more precisely depicted with domains and ranges in the following diagram:

[2]Quick reminder: Smooth manifold (sometimes called differentiable or C^∞-manifold) is a second countable topological Hausdorff space, which is locally Euclidean. In addition, a smooth (or differentiable or C^∞) structure, in the form of a maximal smooth atlas containing a set of smoothly compatible charts covering M, has to be added to the topological properties of M [17]. The existence of a non-degenerate, symmetric, bilinear form g, whose signature is (n, p), makes out of M a pseudo-Riemannian manifold.

[3]Coordinate chart on M_m, or simply a chart, is a pair (U, ψ), in which $U \subset M$ is an open set and $\psi : U \to \psi(U) \subset \mathbb{R}^m$ is a homeomorphism (ψ continuous and $\exists\, \psi^{-1}$ continuous).

[4]A (cross) section of TM is a continuous map $\sigma : M \to TM$ satisfying $\pi \circ \sigma = \mathrm{Id}_M$, where $\pi : TM \to M$ is a natural projection, i.e. $\forall\, (p, X) \in TM \,:\, \pi(p, X) = p$.

[5]Many theories in physics, such as general relativity and gauge theories, are described naturally in terms of a more general framework called a *fibre bundle* [16]. For our purposes, we will be satisfied with the fact that, for instance, a tangent bundle TM is an example of a fibre bundle (M is a base space, T_pM is a fibre at p, projection $\pi : TM \to M$, etc.), so we will not use the terminology of fibre bundles.

$$
\begin{array}{ccc}
U \subset M_m & \xrightarrow{\quad F \quad} & V \subset N_n \\
\Big\downarrow \psi & & \Big\downarrow \chi \\
\psi(U) \subset \mathbb{R}^m & \xrightarrow{\quad \widetilde{F} \quad} & \chi(V) \subset \mathbb{R}^n
\end{array}
\tag{1.2}
$$

A mapping F being a diffeomorphism gains an enchanting geometric interpretation, given, for instance, by Nakahara [16]:

Diffeomorphisms classify spaces into equivalence classes according to whether it is possible to deform one space to another smoothly. Two diffeomorphic spaces are regarded as the same manifold.

Reflexivity, symmetry and transitivity for 'being diffeomorphic' (sometimes denoted as $M \approx N$ or $M \equiv N$) can be easily shown. The central concern of smooth manifold theory is the study of properties of smooth manifolds that are preserved by diffeomorphisms. It is important to note that by setting $M = N$, we obtain a group, denoted $\operatorname{Diff} M$, with a map composition being a group operation [17].

Generally, a (smooth) map $F : M \to N$ induces two *linear* maps: ***push-forward*** F_* and ***pull-back*** F^*. These can assign to vectors and 1-forms, living on one of the manifolds, their counterpart on the other manifold.

DEFINITION 2. *Let $f \in \mathcal{F}N$. We define a pull-back of f, denoted $F^* f$, by a simple composition*

$$
\begin{aligned}
F^* : \mathcal{F}N &\to \mathcal{F}M \\
f &\mapsto F^* f := f \circ F \,.
\end{aligned}
$$

Next, let $V \in T_p M$. A push-forward $F_* V \in T_{F(p)} N$ of V is defined by its action on smooth functions

$$(F_* V)[f] := V[f \circ F] \equiv V[F^* f]\,.$$

Let $\alpha \in T^*_{F(p)} N$. We define a pull-back of α by means of a contraction $\langle \cdot, \cdot \rangle$ with a tangent vector

$$\langle F^* \alpha, V \rangle|_p = \langle \alpha, F_* V \rangle|_{F(p)}\,.$$

The pull-back naturally extends to tensors of type $(0, r)$; $F^* : T^0_{r,F(p)} N \to T^0_{r,p} M$ by the relation

$$(F^* \omega)(V, ..., W) := \omega(F_* V, ..., F_* W)$$

for any $\omega \in T^0_r N$ and $V, ..., W \in T_p M$.

Alternative (equivalent) definition for the push-forward of a vector V, tangent to a curve $\gamma : \mathbb{R} \to M$ at $p \in M$, is to make $F_* V$ tangent to a curve $F \circ \gamma$ at a point $F(p)$. Notice that we are not talking about *fields.* If we knew how to push-forward vector fields, then we could pull-back tensor fields of type $(0, r)$. For shifting vector fields, we need F to be a diffeomorphism ($X \in \mathfrak{X}M$ and $W := F_* X \in \mathfrak{X}N$ are then said to be F-related); for a detailed discussion, see [15], p. 56 or Lemma 4.8 in [17].

The most important thing to note is that the pull-back F^* maps 'backwards' with respect to F, while push-forward goes in the same - 'forward' - direction:

$$\begin{array}{ccc} T_pM & \xrightarrow{\quad F_* \quad} & T_{F(p)}N \\ \vdots & & \vdots \\ M & \xrightarrow{\quad F \quad} & N \\ \vdots & & \vdots \\ T_p^*M & \xleftarrow{\quad F^* \quad} & T_{F(p)}^*N \end{array}$$

We mentioned that F_* and F^* are linear maps. We shall now demonstrate this. From definition 2 and because vectors $V, W \in T_pM$ are (linear differential) functionals on $\mathcal{F}M$, for all $f \in \mathcal{F}N$, $r \in \mathbb{R}$ and $p \in M$

$$\begin{aligned}(F_*(rV+W))[f]|_{F(p)} &= (rV+W)[F^*f]|_p = rV[F^*f]|_p + W[F^*f]|_p \\ &= r(F_*V)[f]|_{F(p)} + (F_*W)[f]|_{F(p)} = (rF_*V + F_*W)[f]|_{F(p)} \,.\end{aligned}$$

Since $\langle \cdot, \cdot \rangle$ is linear, we easily obtain the linearity of F^* too:

$$\begin{aligned}\langle F^*(r\alpha+\beta), V\rangle|_p &= \langle r\alpha+\beta, F_*V\rangle|_{F(p)} = r\langle \alpha, F_*V\rangle|_{F(p)} + \langle \beta, F_*V\rangle|_{F(p)} \\ &= r\langle F^*\alpha, V\rangle|_p + \langle F^*\beta, V\rangle|_p = \langle rF^*\alpha + F^*\beta, V\rangle|_p \,.\end{aligned}$$

Quod erat demonstrandum.

To see what the two induced maps do in coordinates, choose a point $p \in M$ and take two charts, $(U, \psi, \{x^\alpha\})$ on M and $(V, \varphi, \{y^\alpha\})$ on N. Then [16]

$$(F_*V)[f \circ \varphi^{-1}(y)] = V[f \circ F \circ \psi^{-1}(x)] \,, \tag{1.3}$$

where $x = \psi(p)$ and $y = \varphi(F(p))$. If we take $f = y^\alpha$ for $V = V^\mu \partial/\partial x^\mu$ and $F_*V = (F_*V)^\beta \partial/\partial y^\beta$ we obtain the familiar relation

$$V'^\alpha := (F_*V)^\alpha = V^\beta \underbrace{\frac{\partial y^\alpha}{\partial x^\beta}}_{:= J^\alpha_\beta} \,, \tag{1.4}$$

which is just the 'good old' transformation formula for vectors and ${J^\alpha}_\beta$ is the Jacobian matrix. Similarly, for covectors $F^*\alpha = (F^*\alpha)_\mu \mathrm{d}x^\mu$ and $\alpha = \alpha_\nu \mathrm{d}y^\nu$ we have

$$\alpha'_\mu := (F^*\alpha)_\mu = \alpha_\nu \frac{\partial y^\nu}{\partial x^\mu} \,, \tag{1.5}$$

because

$$(F^*\alpha)_\mu V^\mu \equiv \langle F^*\alpha, V\rangle \equiv \langle \alpha, F_*V\rangle \equiv \alpha_\nu (F_*V)^\nu = \alpha_\nu \frac{\partial y^\nu}{\partial x^\mu} V^\mu \,.$$

We can equivalently write

$$F_* \frac{\partial}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \quad \& \quad F^* \mathrm{d}y^\alpha = \frac{\partial y^\alpha}{\partial x^\mu} \mathrm{d}x^\mu \,. \tag{1.6}$$

There is no natural extension of the induced map for a tensor (field) of mixed type [16], unless $F : M \to N$ is a diffeomorphism, because we then have a map

$F^{-1} : N \to M$, which enables us to pull-back and push-forward mixed tensor fields too. The two isomorphisms are constructed as follows. In definition 2, we see that a vector V is push-forwarded by acting on a pull-backed function $F^* f$ or, similarly, a 1-form α is pull-backed by acting on a push-forwarded vector $F_* V$ through a contraction $\langle \omega, \cdot \rangle$. We also know that a mixed tensor of type (r,s) is an object which acts on r covectors and s vectors to give us a real number. The following definition is thus natural.

Definition 3. *Let $F : M \to N$ be a diffeomorphism and $T \in T^r_s N$. We define a pull-back $F^* : T^r_s N \to T^r_s M$ of T as*

$$(F^*T)[\underbrace{X, ..., Y}_{s}, \underbrace{\alpha, ..., \beta}_{r}] = T[F_*X, ..., F_*Y, (F^{-1})^*\alpha, ..., (F^{-1})^*\beta]$$

for any $X, ..., Y \in \mathfrak{X}M$ and $\alpha, ..., \beta \in \Omega^1 M$. We also define a push-forward of a general tensor field as

$$F_* \coloneqq (F^{-1})^* : T^r_s M \to T^r_s N\,.$$

The pull-back of a tensor field can thus be rewritten as [15]

$$(F^*T)[X, ..., Y, \alpha, ..., \beta] \coloneqq T[F_*X, ..., F_*Y, F_*\alpha, ..., F_*\beta]\,. \tag{1.7}$$

The definition of F_* is consistent with the generalization of the push-forward of a tangent vector (Def. 2) to a vector field X, for which we have

$$(F_*X)[f] \equiv \Big((F^{-1})^*X\Big)[f] \equiv X[(F^{-1})_*f] = X[F^*f]\,,$$

where we used the fact that

$$(F^{-1})_* \equiv \Big((F^{-1})^{-1}\Big)^* = F^*\,.$$

For two charts $(U \subset M, \psi; \{x^\mu\})$, $(V \subset N, \varphi; \{x'^\mu\})$, pull-back of a tensor field corresponds to the well-known transformation formula

$$\begin{aligned} T'^{\mu\ldots}_{\nu\ldots}(x') \coloneqq (F^*T)^{\mu\ldots}_{\nu\ldots}(x') &= T^{\alpha\ldots}_{\beta\ldots} \frac{\partial x'^\mu}{\partial x^\alpha} \cdots \frac{\partial x^\beta}{\partial x'^\nu} \cdots \\ &\equiv T^{\alpha\ldots}_{\beta\ldots}(x) J^\mu{}_\alpha(x) \ldots (J^{-1})^\beta{}_\nu(x) \ldots \end{aligned} \tag{1.8}$$

This can be easily verified by inserting relations (1.4) and (1.5) or (1.6) for vectors and 1-forms into arguments of the tensor T and using definition 3.

It is well-known that smooth vector fields can, instead of directional derivatives on $\mathcal{F}M$, be regarded as tangent to their *integral curves*. If M is a smooth manifold and $J \subset \mathbb{R}$ is an open interval, a smooth (parametrized) curve

$$\begin{aligned} \gamma : J &\to M \\ t &\mapsto p \in M\,, \end{aligned}$$

determines a tangent vector $\dot\gamma(t) \in T_{\gamma(t)}M$ at each point of the curve. Here we want to describe a way to work backwards: Given tangent vectors at each point of the manifold (i.e. a vector field), we seek a family of curves that have those as tangent vectors.

Definition 4. *If $X \in \mathfrak{X}M$, an* ***integral curve*** *of X is a smooth curve γ such that*

$$\dot\gamma(t) \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = X_{\gamma(t)} \quad \forall\, t \in J\,.$$

In other words, the tangent vector to γ at each point is equal to the value of X at that point. If $0 \in J$, the point $p = \gamma(0)$ is called the *starting point* of γ. Combining both views on a tangent vector, its action on a $\mathcal{F}M$-function at 'time' t_0 is given as

$$X_{\gamma(t)}[f] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} (f \circ \gamma)(t)\,. \tag{1.9}$$

The following Lemma is taken from [17] (with a slightly different notation) and it shows how an integral curve can be reparametrized to change its starting point, but to correspond to the same vector field.

Lemma 1. (Translation Lemma). *Let $V \in \mathfrak{X}M$, $J \subset \mathbb{R}$ be an open interval, $\gamma : \mathbb{R} \to M$ and $\dot\gamma(t) = V_{\gamma(t)}$. For any $a \in \mathbb{R}$, let J_a denote the interval*

$$J_a = \{t + a : t \in J\}\,.$$

Then the curve $\gamma_a : J_a \to M$ defined by $\gamma_a = \gamma(t-a)$ is an integral curve of V.

Proof. By the chain rule and the fact that γ is an integral curve of V,

$$\begin{aligned}\dot\gamma_a(t_0)[f] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} (f \circ \gamma_a)(t) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} (f \circ \gamma)(t-a) \\ &= \dot\gamma(t_0 - a)[f] = V_{\gamma(t_0-a)}[f] = V_{\gamma_a(t_0)}[f]\,.\end{aligned}$$

□

A smooth vector field 'tears up' a manifold into a system (a *congruence*) of integral curves. Suppose that for every point $p \in M$ there is a *unique* integral curve $\gamma^{(p)} : \mathbb{R} \to M$ starting at p. For each $t \in \mathbb{R}$, we can define a map Φ_t sending each point $p \in M$ to the point obtained by following the integral curve $\gamma^{(p)}$ for 'time' t [17]:

$$\Phi_t(p) \coloneqq \gamma^{(p)}(t)$$

or equivalently

$$\begin{aligned}&\Phi_t : M \to M \\ &\gamma^{(p)}(t_0) \mapsto \gamma^{(p)}(t_0 + t)\,,\end{aligned} \tag{1.10}$$

where $\gamma^{(p)}(0) = p \equiv \Phi_0(p)$[6]. This defines a *family* of maps $\{\Phi_t\} : M \to M$ for any fixed $t \in \mathbb{R}$. Each such a map 'slides' the entire manifold along the integral

[6] Here we are assuming, for simplicity, that every integral curve is defined for all $t \in \mathbb{R}$, which may not always be the case [17]. The map Φ_t is called the *local flow* generated by the associated vector field. The term local indicates that Φ_t need not be defined for arbitrarily large t, but rather in general only in some neighbourhood of zero and this neighbourhood may, in turn, depend on $p \in M$; If a flow exists for $t \in (-\infty, \infty)$, one speaks about a *global flow*, or simply a flow. A local flow is enough for the definition of the central concept of this section, the Lie derivative. Therefore, in what follows we will often omit the word 'local' and speak about a 'flow' (as in [15]) in spite of it being only local.

curves for time parameter t. If we set $q = \gamma^{(p)}(s) \in M$, the translation Lemma implies[7] that $t \mapsto \gamma^{(p)}(s+t)$ is an integral curve $\gamma^{(q)}$ starting at q; since we are assuming uniqueness of integral curves, we must have

$$\gamma^{(q)}(t) = \gamma^{(p)}(s+t) \tag{1.11}$$

The LHS is $\Phi_t(q) = \Phi_t(\Phi_s(p))$, while the RHS is $\Phi_{s+t}(p)$. Thus, in terms of maps Φ_t, equality (1.11) becomes

$$\Phi_t \circ \Phi_s = \Phi_{s+t}\,.$$

This motivates our next definition.

Definition 5. *A continuous map $\Phi : \mathbb{R} \times M \to M$ satisfying the following properties $\forall\, s,t \in \mathbb{R}$ and $\forall p \in M$:*

$$\Phi(t, \Phi(s,p)) = \Phi(s+t, p), \quad \Phi(0,p) = p\,. \tag{1.12}$$

is called a ***flow*** *on M. Given such a flow, we define two collections of maps as follows: $\forall\, t \in \mathbb{R}$, define $\Phi_t : M \to M$ by*

$$\Phi_t \coloneqq \Phi(t,p) \tag{1.13}$$

and $\forall\, p \in M$, define a curve $\gamma^{(p)} : \mathbb{R} \to M$ by

$$\gamma^{(p)}(t) \coloneqq \Phi(t,p)\,. \tag{1.14}$$

The defining properties (1.12) and (1.13) give rise to a *commutative (Abelian) group* structure, since

$$\begin{aligned}
&i)\ \Phi_{r+s} \circ \Phi_t = \Phi_{r+s+t} = \Phi_r \circ \Phi_{s+t}\,, && \text{(Associativity)}\\
&ii)\ \Phi_0 = \mathrm{Id}_M\,, && (\exists\,\text{unit element})\\
&iii)\ \Phi_{-t} \circ \Phi_t = \Phi_0 \Rightarrow \Phi_t^{-1} \equiv \Phi_{-t}\,, && (\exists\,\text{inverse element})\\
&iv)\ \Phi_t \circ \Phi_s = \Phi_{s+t} = \Phi_{t+s} = \Phi_s \circ \Phi_t\,. && \text{(Commutativity)}
\end{aligned} \tag{1.15}$$

It is because of this that the flow is also known as a *one-parameter group of transformations* [16]. Each map $\Phi_t : M \to M$ is by definition a homeomorphism, and if Φ is smooth, then Φ_t is a diffeomorphism. With this, we also designate a flow as a *(local) 1-parameter group of (local) diffeomorphisms* [20] and hence a subgroup of Diff M. The presence of the term ‘local’ is explained in the footnote 6 on page 20. From now on, we will assume that the flow is smooth and thus Φ_t a diffeomorphism.

Because of (1.14), a smooth (global) flow can be directly associated with the integral curves $\{\gamma^{(p)} : p \in M\}$ of a vector field $X \in \mathfrak{X}M$ by

$$X \equiv \frac{\mathrm{d}}{\mathrm{d}t}\gamma^{(\cdot)}(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,\cdot) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(\cdot)\,. \tag{1.16}$$

In this context, X is called the *infinitesimal generator* of the flow.

[7] One can set $\gamma^{(q)} \equiv \gamma_s^{(p)}$ to use the Lemma.

To clarify the name '1-parameter group of transformations' for a flow, take Φ_ε with $\varepsilon \ll 1$ and combine (1.9) and (1.16); one can then write for any $f \in \mathcal{F}M$ and $p \in M$

$$f(\Phi_\varepsilon(p)) = f(\underbrace{\Phi_0(p)}_{=p}) + \varepsilon X[f]|_p + \mathcal{O}\Big(\varepsilon^2\Big)\,. \tag{1.17}$$

By taking functions ψ^a of the coordinate map ψ of some chart (U, ψ), instead of f, we get an *infinitesimal coordinate transformation*

$$x'^a_{(p)} = x^a_{(p)} + \varepsilon X^a\,, \tag{1.18}$$

where we assigned coordinates $\{x^a\}$ to points $\psi(\Phi_\varepsilon(p))$ and $\psi(p)$ in $\mathbb{R}^m$.

There are many theorems dealing with the relation of flows and corresponding vector fields (mainly uniqueness and existence), which we will not show here. They can be found, for instance, in Spivak [20].

1.1.2 Lie derivative, Lie bracket, Lie groups et cetera

This subsection is all about mathematical objects and structures, which are named after Sophus Lie (1842-1899), a Norwegian mathematician, who largely contributed to the theory of groups and symmetries. First we begin with something that is closely related to flows from the previous subsection.

Definition 6. *A* ***Lie group*** *is a smooth manifold G which is endowed with a group structure in the algebraic sense, such that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$, given by*

$$m(g,h) = gh\,, \qquad i(g) = g^{-1}$$

are smooth. The letter e will denote the unit element of G.

If G and H are Lie groups, a *Lie group homomorphism* is a smooth map $F : G \to H$ that is also a group homomorphism (i.e. $F(g_1)F(g_2) = F(g_1 g_2)$ for all $g_1, g_2 \in G$). Alternative characterization of the smoothness is that, if the smooth manifold G has a group structure such that the map

$$\begin{aligned} a : G \times G &\to G \\ (g,h) &\mapsto gh^{-1} \end{aligned} \tag{1.19}$$

is smooth, then G is a Lie Group [17]. To prove this, we notice that the multiplication map and the inversion map can be expressed in terms of the map a: $a(e,h) = h^{-1} = i(h)$ and $a(g, a(e,h)) = gh = m(g,h)$, *Q.E.D.*

The real number field $\mathbb{R}$ is a Lie group under addition because the coordinates of $x - y$ ($\equiv gh^{-1}$) are smooth (linear functions of (x,y)). If V is any real or complex vector space, $\mathrm{GL}(V)$ denotes the set of invertible linear transformations from V to itself. The general linear group $\mathrm{GL}(n,\mathbb{R})$ is a set of invertible $n \times n$ matrices with real entries. In special relativity, we are familiar with the Lorentz group [16]

$$\mathrm{O}(1,3) \coloneqq \{\Lambda \in \mathrm{GL}(4,\mathbb{R}) | \Lambda^T \eta \Lambda = \eta \coloneqq \mathrm{diag}(-1,1,1,1)\}\,.$$

As described in [17], the most important applications of Lie groups involve actions of Lie groups on other manifolds. These typically arise in situations involving some kind of *symmetry*. For example, if M is a vector space or a smooth manifold endowed with a certain geometric structure (such as inner product, a norm, a metric or a distinguished (co)vector field), the set $P \subset \text{Diff}M$ that preserves the structure (called the *symmetry group* of the structure) frequently turns out to be a Lie group acting smoothly on M. In physics, a Lie group often appears as the set of transformations acting on a manifold. For example, SO(3) is the group of rotations in $\mathbb{R}^3$, while the Poincaré group is the set of transformations acting on the Minkowski spacetime [16]. The properties of the group action can shed considerable light on the properties of the structure acted upon.

Definition 7. *If G is a group and M is a set (usually a manifold), a* ***left action*** *of G on M is a map $A : G \times M \to M$ that satisfies*

$$\begin{aligned} A_{g_1}(A_{g_2}(p)) &= A_{g_1 g_2}(p)\,, \qquad && \forall g_1, g_2 \in G \text{ and } p \in M\,, \\ A_e(p) &= p\,, && \forall p \in M\,. \end{aligned} \tag{1.20}$$

The right action can be defined analogously. If M is a smooth manifold and the action is smooth, M is called a *smooth G-space*. We will exclusively work with left actions.

We can now easily observe that the flow (1.12) is a left action of $\mathbb{R}$ on M[8]. If this action is smooth, then M is a smooth P-space, where P is the 1-parameter group of diffeomorphisms (1.15), a Lie subgroup of Diff M acting on M.

Let us now return to vector fields. When a vector field $X \in \mathfrak{X}M$ acts on a $\mathcal{F}M$-function f, we get (from its definition, see p. 15) a new $\mathcal{F}M$-function $X[f]$. Let $Y \in \mathfrak{X}M$ be another vector field. Is $Y[X[f]]$ a new smooth function (or equivalently, is $Y[X[\cdot]]$ a new vector field)? *The answer is no.* As a linear differential operator, a vector field must obey the Leibniz's rule, which YX clearly does not satisfy:

$$\begin{aligned} Y[X[fg]] &= Y[X[f]g + fX[g]] = Y[X[f]]g + \underbrace{X[f]Y[g] + Y[f]X[g]}_{\star} + fY[X[g]] \\ &\neq Y[X[f]]g + fY[X[g]]\,. \end{aligned} \tag{1.21}$$

Nevertheless, there is a way (notice the term $\star$) how to combine two vector fields to obtain a new vector field.

Definition 8. *Let $X, Y \in \mathfrak{X}M$ and $f \in \mathcal{F}M$. Define the* ***Lie bracket*** *$[X, Y]$ by*

$$[X, Y][f] = X[Y[f]] - Y[X[f]] \tag{1.22}$$

It is obvious from (1.21), that if we perform the same calculation for XY and then subtract $Y[X[fg]]$, the undesirable $\star$-terms will cancel out and therefore the Lie bracket $[X, Y]$ is, indeed, a vector field on M. It is straightforward to show that the Lie bracket, as a map $[\cdot,\cdot] : \mathfrak{X}M \times \mathfrak{X}M \to \mathfrak{X}M$, satisfies

i) (bi)linearity

$$[rX + sY, Z] = r[X, Z] + s[Y, Z] \tag{1.23}$$

[8] It is mentioned in [16], on page 190, that the flow looks locally like the additive group $\mathbb{R}$, although it may not be isomorphic to $\mathbb{R}$ globally. It can happen to be isomorphic to the multiplicative group U(1) or SO(2) (special orthogonal 2×2 matrices; fundamentally rotations). This occurs if the flow is periodic with some period T.

ii) antisymmetry

$$[X, Y] = -[Y, X] \tag{1.24}$$

iii) the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{1.25}$$

for all $X, Y, Z \in \mathfrak{X}M$ and $r, s \in \mathbb{R}$ (the linearity in the second argument follows from the antisymmetry property). Geometrically, the Lie bracket shows the non-commutativity of two flows (see [16], p. 193). The most important application of Lie brackets occurs in the context of Lie groups.

Definition 9. *Let g and h be elements of a Lie group G. The* ***left translation*** *$L_g : G \to G$ and the* ***right translation*** *$R_g : G \to G$ of h by g are defined by*

$$L_g(h) = gh$$
$$R_g(h) = hg\,.$$

By definition, both translations are elements of Diff G. For the left translation, which we will employ, we essentially have $L_g = m(g, \cdot)$, where m is the multiplication map (see Def. 6) and $L_{g^{-1}}$ is a smooth inverse for it. For better understanding of the notion 'translation', observe that we can get from any point $h \in G$ to $g \in G$ with the left-translation by gh^{-1} (for which we know it is a smooth assignment from (1.19)). This is even more evident if we take for G a 1-parameter group of diffeomorphisms $\{\Phi_t\}$ (a flow). The left translation is

$$L_{\Phi_t}\Phi_s = \Phi_t \circ \Phi_s = \Phi_{t+s}\,. \tag{1.26}$$

It is a right translation too, because $\{\Phi_t\}$ is an Abelian group. Since $L_e = \mathrm{Id}_G$, a natural place where to start with the translation is the tangent space T_eG at the unit element of G (which is Φ_0 for a flow). The left translation is also a left action of a Lie group on itself.

The map L_g is accompanied with two induced isomorphisms: a push-forward $L_{g*} : T_hG \to T_{gh}G$ and a pull-back L_g^* (see Definition 2 in subsection 1.1.1). On the usual manifold there is no canonical way of discriminating some vector fields from the others, but given a Lie group, there exists a special class of vector fields characterized by an invariance under group action.

Definition 10. *Let G be a Lie group and $X \in \mathfrak{X}G$. A vector field is said to be* ***left-invariant*** *if*

$$L_{g*}X = X \quad \forall g \in G$$

A vector $V \in T_eG$ defines a unique left-invariant vector field X_V throughout G by [16]

$$X_V|_g = L_{g*}V \quad \forall g \in G\,. \tag{1.27}$$

Conversely, a left-invariant vector field X defines a unique vector $V = X|_e \in T_eG$ (See also Theorem 4.20. in [17]). Let us denote the set of left-invariant vector fields on G by $\mathfrak{g}$. Because L_{g*} is linear, the map $T_eG \to \mathfrak{g}$ defined by $V \mapsto X_V$ is an isomorphism and hence the set of all smooth left-invariant vector fields on G forms a *vector (linear) subspace* of $\mathfrak{X}G$ (with dim $\mathfrak{g}$ = dim G) isomorphic to T_eG. It is even much more than that...

LEMMA 2. *Suppose $X, Y \in \mathfrak{X}G$ are left-invariant. Then $[X, Y]$ is left-invariant.*

This Lemma is just a corollary of the following proposition, which can be interpreted as the *naturality* of the Lie bracket [17].

PROPOSITION 3. *Suppose $F : M \to N$ is a diffeomorphism and $X, Y \in \mathfrak{X}M$. Then*

$$F_*[X, Y] = [F_*X, F_*Y] \tag{1.28}$$

Proof. For any $f \in \mathcal{F}N$

$$\begin{aligned} F^*\{\underbrace{(F_*[X,Y])[f]}_{\in \mathcal{F}N}\} &= \underbrace{[X,Y][F^*f]}_{\in \mathcal{F}M} \equiv X[\underbrace{Y[F^*f]}_{\in \mathcal{F}M}] - Y[\underbrace{X[F^*f]}_{\in \mathcal{F}M}] \\ &= X[F^*\{(F_*Y)[f]\}] - Y[F^*\{(F_*X)[f]\}] \\ &= F^*\{(F_*X)[(F_*Y)[f]]\} - F^*\{(F_*Y)[(F_*X)[f]]\} \end{aligned}$$

and because F^* is linear,

$$F^*\{(F_*[X, Y] - [F_*X, F_*Y])[f]\} = 0\,.$$

Hence the argument of F^* must be zero, which completes the proof. □

In other words [17], if $X, Y \in \mathfrak{X}M$ and $F_*X, F_*Y \in \mathfrak{X}N$ are F-related, then $[X, Y]$ and $F_*[X, Y]$ are F-related. To prove the Lemma 2, we can set $F = L_g$ (with $M = G = N$) and then use the left-invariances of X and Y. The mentioned left-invariant linear subspace of $\mathfrak{X}G$ is therefore closed under the Lie bracket.

DEFINITION 11. *The set of left-invariant vector fields $\mathfrak{g}$ with the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called the **Lie algebra** of a Lie group G.*

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is called a Lie algebra homomorphism if it preserves brackets: $\phi[X, Y] = [\phi X, \phi Y]$.

Because of the mentioned isomorphism $\mathfrak{g} \simeq T_eG$, the Lie algebra of G can be identified with the tangent space at the unit element e. Sometimes the definition of the Lie algebra is formulated in a 'generalized' form, in which $\mathfrak{g}$ is a real vector space with an operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is bilinear, antisymmetric and satisfies the Jacobi identity (1.25). Vector fields on a smooth manifold M then form a Lie algebra $\mathfrak{X}M$ under the Lie bracket. If V is any real or complex vector space, $\mathfrak{gl}(V)$ denotes the set of all linear maps from V to itself (endomorphisms). As a specific example may serve the Lie algebra of the Lorentz group (1.1.2):

$$\mathfrak{so}(1, 3) := \{X \in \mathrm{M}(4, \mathbb{R}) |\, e^{tX} \in \mathrm{O}(1, 3) \quad \forall\, t \in \mathbb{R}\}\,,$$

such that

$$\eta X = -(\eta X)^T\,, \tag{1.29}$$

where $\mathrm{M}(n, \mathbb{R})$ is a vector space of $n \times n$ real matrices, which is n^2-dimensional Lie algebra of the Lie group $\mathrm{GL}(n, \mathbb{R})$ under the *commutator* $[A, B] = AB - BA$, usually denoted $\mathfrak{gl}(n, \mathbb{R})$ [21].

It is now very instructive to demonstrate a noteworthy property of infinitesimal generators. By identifying the map L_g with Φ_t, vector fields are automatically left-invariant with respect to their flow, that is

$$\Phi_{t*}X = X\,. \tag{1.30}$$

Indeed, by virtue of (1.16) the equality

$$(\Phi_{t*}X)\,|_{\Phi_t(p)} = \Phi_{t*}\left(X|_p\right)\,,$$

becomes

$$(\Phi_{t*}X)\,|_{\Phi_t(p)} = \Phi_{t*}\,\frac{\mathrm{d}}{\mathrm{d}s}\Phi_s(p)\bigg|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\Phi_s\left(\Phi_t(p)\right)\bigg|_{s=0} \equiv X|_{\Phi_t(p)}\,. \tag{1.31}$$

One could wonder: Why to bother with Lie algebras, if we already have Lie groups which act on smooth manifolds? It turns out that it is in most cases easier to work with infinitesimal generators than with the elements of the group. We will see the verity of this fact in the context of isometries (subsection 2.1).

We are now going to proceed to the most important object of this subsection — the Lie derivative. In order to characterize behaviour of a tensor field $T \in T^r_s M$, we wish to compare its properties at different points of the underlying manifold M. Any kind of derivative is known to be a provider of an infinitesimal measure of change. An issue arises immediately: For two different points $p, q \in M$, regardless how 'close' they are to each other, tensors $T|_p$ and $T|_q$ are objects residing in different linear spaces, so we cannot compare them directly by taking their difference (needed for a derivative). If, however, $q = F(p)$ for $F \in \mathrm{Diff}\, M$, then we can pull-back $T|_q$ to the point p and take the difference $F^*T|_{F(p)} - T|_p$ in the same space $T^r_{s,p}M$. The only thing that still might bother us, is that we don't have a specific way of choosing points at which we want to compare our tensors.

We may naturally ask how does a tensor change along the flow generated by a vector field X.

Definition 12. *Let $X \in \mathfrak{X}M$ be an infinitesimal generator of a flow Φ_t. The* ***Lie derivative*** *of a tensor field along the flow of X is defined as*

$$\mathfrak{L}_X T \coloneqq \lim_{t\to 0}\frac{1}{t}\left[\Phi^*_t T - T\right] \equiv \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\Phi^*_t T\,. \tag{1.32}$$

The idea of *transporting* a tensor field by Φ^*_t a parametric distance t, along the integral curves of the field X *against* the direction of the flow Φ_t, is known as *Lie dragging.* Tensor field, in general, need not satisfy $\Phi^*_t T = T$ [15]. The Lie derivative $\mathfrak{L}_X$ is therefore a *coordinate independent* measure of non-invariance with respect to the flow generated by the vector field X. In Definition 12, we did not write explicitly the presence of a point $p \in M$ at which $\mathfrak{L}_X$ is evaluated at, because we wish to emphasize that we drag the whole tensor *field* along all integral curves of X. The Lie derivative can be written by means of a push-forward too, because (recall Def. 3)

$$\Phi^*_t = (\Phi^{-1}_{-t})^* \equiv \Phi_{-t*}\,. \tag{1.33}$$

This is more appropriate, for example, for vector fields which are push-forwarded.

Right from definition we see that the Lie derivative of a tensor field is a tensor field of the same type. For $\varepsilon \ll 1$ we can approximate the pull-backed (transformed) tensor field with the help of its Lie derivative

$$\Phi^*_\varepsilon T = T + \varepsilon\mathfrak{L}_X T + \mathcal{O}\Big(\varepsilon^2\Big)\,. \tag{1.34}$$

For a vector field Y this becomes

$$\Phi_{-\varepsilon *} Y = Y - \varepsilon \mathfrak{L}_X Y + \mathcal{O}\left(\varepsilon^2\right). \tag{1.35}$$

Lie derivative satisfies $\forall A, B \in T^r_s M$ and $r \in \mathbb{R}$

$$\mathfrak{L}_X(rA + B) = r\mathfrak{L}_X A + \mathfrak{L}_X B\,, \qquad \text{(Linearity)} \tag{1.36}$$
$$\mathfrak{L}_X(A \otimes B) = (\mathfrak{L}_X A) \otimes B + A \otimes (\mathfrak{L}_X B)\,. \qquad \text{(Leibniz's rule \#1)} \tag{1.37}$$

Linearity follows from linearity of Φ^*_t. The Leibniz's rule is non-trivial to show, but if we take, without loss of generality, a tensor product of $Y \in \mathfrak{X}M$ and $\alpha \in \Omega^1 M$, we first pull-it-back ([16], p. 195)

$$\Phi^*_t(Y \otimes \alpha) = (\Phi_{-t*}Y) \otimes (\Phi^*_t \alpha)\,, \tag{1.38}$$

and then manipulate with the definition (by adding a zero in the second equality; tensor product is distributive):

$$\begin{aligned}
\mathfrak{L}_X(Y \otimes \alpha) &\equiv \lim_{t\to 0}[(\Phi_{-t*}Y) \otimes (\Phi^*_t \alpha) - Y \otimes \alpha] \\
&= \lim_{t\to 0}[(\Phi_{-t*}Y) \otimes \{(\Phi^*_t \alpha) - \alpha\} - \{(\Phi_{-t*}Y) - Y\} \otimes \alpha] \\
&= Y \otimes (\mathfrak{L}_X \alpha) + (\mathfrak{L}_X Y) \otimes \alpha\,. \quad Q.E.D.
\end{aligned}$$

This can obviously be extended to more general cases.

For a tensor field $T \in T^r_s M$, acting on its arguments (contracting itself with r 1-forms and s vector fields), it holds

$$\begin{aligned}
\mathfrak{L}_X(T[V, .., \alpha, ...]) = (\mathfrak{L}_X T)[V, ..., \alpha] + T[\mathfrak{L}_X V, ..., \alpha] \\
+ ... + T[V, ..., \mathfrak{L}_X \alpha]\,. \qquad \text{(Leibniz's rule \#2)}
\end{aligned} \tag{1.39}$$

In particular, for $X, Y \in \mathfrak{X}M$ and $\alpha \in \Omega^1 M$ we have

$$\mathfrak{L}_Y \langle \alpha, X\rangle = \langle \mathfrak{L}_Y \alpha, X\rangle + \langle \alpha, \mathfrak{L}_Y X\rangle\,. \tag{1.40}$$

Before we display some other nice properties of the Lie derivative, let us remind the reader that the prescription

$$(rX + Y)[f] = rX[f] + Y[f] \quad r \in \mathbb{R}$$

(the action of a vector field $rX + Y \in \mathfrak{X}M$ on an $\mathcal{F}M$-function) can be regarded as a linear map $\mathfrak{X}M \to \mathbb{R}$, when evaluated at some point $p \in M$. Hence it can be represented by a 1-form $\mathrm{d}f \in \Omega^1 M$ defined as

$$X[f] = \langle \mathrm{d}f, X\rangle \quad \forall X \in \mathfrak{X}M\,, \tag{1.41}$$

where

$$\begin{aligned}
\mathrm{d} : \mathcal{F}M &\to \Omega^1 M \\
f &\mapsto \mathrm{d}f\,,
\end{aligned}$$

is the gradient of the function f. In coordinates, $\mathrm{d}f = \partial_a f \mathrm{d}x^a$. It is also very important that, for a smooth map (or a diffeomorphism) $F : M \to N$, the pull-back commutes with the gradient, i.e. $\forall f \in \mathcal{F}M$

$$F^*\mathrm{d}f = \mathrm{d}(F^* f)\ . \tag{1.42}$$

We can show that this is true by recalling Def. 2 and using (1.41). For every $X \in \mathfrak{X}M$ we have

$$\begin{aligned}\langle F^*(\mathrm{d}f), X\rangle &= \langle \mathrm{d}f, F_*X\rangle = (F_*X)[f] \\ &= X[F^*f] = \langle \mathrm{d}(F^*f)\,, X\rangle \quad \textit{Q.E.D.}\end{aligned}$$

Theorem 1. (Properties of the Lie derivative).
Suppose $X, Y, Z \in \mathfrak{X}M$ *and* $f \in \mathcal{F}M$. *Then*

$$\begin{aligned}(i)\, &\mathfrak{L}_X f = X[f]\,, \\ (ii)\, &\mathfrak{L}_X \mathrm{d} = \mathrm{d}\mathfrak{L}_X\,, \\ (iii)\, &\mathfrak{L}_X Y = [X, Y]\,, \\ (iv)\, &\mathfrak{L}_{X+rY} = \mathfrak{L}_X + r\mathfrak{L}_Y \quad \forall\, r \in \mathbb{R}\,, \\ (v)\, &\mathfrak{L}_{[X,Y]} = [\mathfrak{L}_X, \mathfrak{L}_Y]\,, \\ (vi)\, &\mathfrak{L}_Z[X, Y] = [\mathfrak{L}_Z X, Y] + [X, \mathfrak{L}_Z Y]\,.\end{aligned}$$

Proof. Ad (i): Since $\Phi_t^* f = f \circ \Phi_t$ (Def. 2), it simply follows that

$$\mathfrak{L}_X f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \Phi_t) \equiv X[f]\,.$$

Ad (ii): This is a consequence of the relation (1.42). It yields the desired commutation after taking derivative with respect to t at $t = 0$. Ad (iii): Employing (1.41) and (1.40) gives

$$\begin{aligned}(\mathfrak{L}_X Y)[f] &= \langle \mathrm{d}f, \mathfrak{L}_X Y\rangle = \mathfrak{L}_X\langle \mathrm{d}f, Y\rangle - \langle \mathfrak{L}_X \mathrm{d}f, Y\rangle \\ &\overset{(ii)}{=} \mathfrak{L}_X Y[f] - \langle \mathrm{d}\mathfrak{L}_X f, Y\rangle = \mathfrak{L}_X Y[f] - Y[\mathfrak{L}_X f] \\ &= X[Y[f]] - Y[X[f]] = [X, Y][f]\,.\end{aligned}$$

Ad (iv): We can use property (i);

$$\mathfrak{L}_{X+rY} f \overset{(i)}{=} (X + rY)[f] = X[f] + rY[f] = \mathfrak{L}_X f + r\mathfrak{L}_Y f\,.$$

Ad (v): Property (i) again makes the deal:

$$\begin{aligned}\mathfrak{L}_{[X,Y]} f &= [X, Y][f] = X[Y[f]] - Y[X[f]] \\ &= X[\mathfrak{L}_Y f] - Y[\mathfrak{L}_X f] = \mathfrak{L}_X \mathfrak{L}_Y f - \mathfrak{L}_Y \mathfrak{L}_X f \\ &= [\mathfrak{L}_X, \mathfrak{L}_Y][f]\,.\end{aligned}$$

Ad (vi): Utilizing property (iii), antisymmetry of the Lie bracket and the Jacobi identity yields

$$\begin{aligned}\mathfrak{L}_Z[X, Y] &= [Z, [X, Y]] = -[X, [Y, Z]] - [Y, [Z, X]] \\ &= [[Z, X], Y] + [X, [Z, Y]] = [\mathfrak{L}_Z X, Y] + [X, \mathfrak{L}_Z Y]\,.\end{aligned}$$

□

In the given proof, properties (*iv*) and (*v*) are, for simplicity, shown only for $\mathcal{F}M$-functions. More general proofs can be found in [20] and [17].

It is worth mentioning that the Lie derivative as a map

$$\begin{aligned} \mathfrak{L} : \mathfrak{X}M &\to \mathrm{End}(\mathfrak{X}M) \\ X &\mapsto \mathfrak{L}_X \,, \end{aligned} \tag{1.43}$$

may be regarded as an infinite-dimensional *representation*[9] of the Lie algebra $\mathfrak{X}M$, because of its properties (*iii*)-(*vi*) in Theorem 1. More specifically, it corresponds to the representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \equiv \mathrm{End}(\mathfrak{g})$ defined by $\mathrm{ad}_X(Y) = [X, Y]$. The associated group representation is $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ defined by $\mathrm{Ad}_g \coloneqq C_{g*} : \mathfrak{g} \to \mathfrak{g}$, where $C_g(h) = ghg^{-1}$ is the conjugation map. Both maps ad and Ad are called *adjoint representations* of a Lie algebra and of a Lie group, respectively (see [17] for details). We can regard pull-back Φ_{t*} (which is linear) as a representation of the action of a group $\{\Phi_t\}$ because it assigns to every tensor a set of Jacobi matrices (operators) which transform the tensor 'in the right way' (see (1.8)). Hence, we have a map $\mathfrak{X}M \mapsto \mathrm{Aut}(T^r_s M)$. Since the Lie derivative of a tensor field is a new tensor field of the same type, we can generalize the map (1.43) as $\mathfrak{X}M \to \mathrm{End}(T^r_s M)$. Then we have a relation

$$\text{Action of a Lie } \textit{algebra} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \text{Action of a Lie } \textit{group} \,. \tag{1.44}$$

To conclude this section, we present the formula on how to practically compute the Lie derivative which is defined with no reference to any coordinate system. We can compute a Lie derivative of a tensor field in coordinates both with partial derivatives and covariant ones. The general formula for a tensor $T \in T^r_s M$ reads

$$(\mathfrak{L}_X T)^{\mu\ldots}_{\nu\ldots} = X^\alpha \partial_\alpha T^{\mu\ldots}_{\nu\ldots} - T^{\alpha\ldots}_{\mu\ldots} \partial_\alpha X^\mu - \ldots + T^{\mu\ldots}_{\alpha\ldots} \partial_\nu X^\alpha + \ldots \tag{1.45}$$

$$= X^\alpha \nabla_\alpha T^{\mu\ldots}_{\nu\ldots} - T^{\alpha\ldots}_{\mu\ldots} \nabla_\alpha X^\mu - \ldots + T^{\mu\ldots}_{\alpha\ldots} \nabla_\nu X^\alpha + \ldots \,. \tag{1.46}$$

This rule can be obtained from the property (1.39) by inserting basis vectors and covectors into arguments of tensor T. The second property in Theorem 1 implies that in coordinates the Lie derivative commutes with partial derivatives

$$\mathfrak{L}_X \partial_\mu = \partial_\mu \mathfrak{L}_X \,, \tag{1.47}$$

but not covariant ones.

[9] A representation of a Lie group is a Lie group homomorphism $\rho : G \to \mathrm{GL}(V) \equiv Aut(V)$, where V is a real or complex vector space. If $\dim V$ is ∞ or finite, then the representation is said to be ∞ or finite-dimensional. This yields a smooth left action (see Def. 7) of G on V defined by $\rho(g)v$. For a Lie algebra we have analogous definition with $\mathfrak{g}$ and $\mathfrak{gl}(V)$. Recall that a Lie algebra homomorphism needs to be compatible with the bracket (see paragraph under Def.11), which Lie derivative is (Theorem 1, property (*v*). In fact, one of the chief applications of representation theory is to exploit symmetry. If a system has a symmetry, the set of symmetries will form a group, and understanding the representations of the symmetry group allows one to use that symmetry to various problems [21].

1.2 Variational principle in field theories

It is well known that one of the principal problems of classical mechanics is to derive the appropriate equations of motion for the generalized coordinates of the particles and subsequently to integrate these equations to find the generalized coordinates as explicit functions of time. The equations of motion can be derived from Hamilton's principle in which the stationary action (a definite integral of the Lagrangian over time) yields Euler-Lagrange equations. This idea has been extended and generalized to physical systems not involving just (point-like) particles (i.e. systems with finite number of degrees of freedom), but to fields (systems with infinite number of degrees of freedom; e.g. scalar or gravitational field). Classical field theories, which will be of our interest here, involve electromagnetism and gravitation. Quantum field theory incorporates, in addition, tools of quantum mechanics.

One of the most beautiful formulations in physics is that the true configuration of a physical system is such that the corresponding action S of the system attains an extremal value, id est

$$\delta S = 0 \,. \tag{1.48}$$

The action functional is in field theories defined as an integral of a Lagrangian *density*[10] $\mathcal{L}$ over a four-dimensional spacetime region D

$$S = \int_D \mathcal{L} \; \mathrm{d}^4 x \,. \tag{1.49}$$

Although variational calculus on smooth manifolds can be elegantly formulated without referring to coordinates, we will be satisfied with the possibility to jump from a smooth manifold into local coordinates where we can vary our action.

Let us first assume that the action S is a functional of a field variable ψ which can be any geometric object (e.g. an $\mathcal{F}M$-function ϕ, a 1-form $A \in \Omega^1 M$, a metric tensor $g \in T^0_2 M$, etc.). For any chart $(U, \{x^\mu\})$ on M, a variable ψ is given in terms of its components ψ^A which are functions of coordinates x^μ, where index A is an economic way to represent all tensorial indices. The field components ψ^A are treated as generalized coordinates and are genuine *dynamical variables* ([9], p. 23). When we fix specific indices, we can regard $\psi^A(x)$ simply as multivariable functions from ordinary calculus mapping from $\mathbb{R}^4$ to $\mathbb{R}$ or $\mathbb{R}^4$. For variational calculus, we need functions $\psi^A(x)$ to be of the class C^1 or pointwise C^2. If the variation of the action is done on some domain D, then functions $\psi^A(x)$ (usually forming a Banach or Hilbert space) need to have *prescribed values* on the boundary ∂D. In addition, one needs to build a (vector) space of functions $h^A(x)$ which are also C^1 and vanish on the boundary with their first derivatives, that is

$$h^A|_{\partial D} = 0 \quad \& \quad \partial_\mu h^A|_{\partial D} = 0 \,. \tag{1.50}$$

Formally, δS is defined as a limit:

$$\delta S := \lim_{\lambda \to 0} \frac{S[\psi^A + \lambda h^A] - S[\psi^A]}{\lambda} \,. \tag{1.51}$$

[10] Scalar density $\mathcal{L}$ is basically a product $\sqrt{-\mathrm{g}}\, L$ in which g is a determinant of the metric tensor and L is a scalar function. The minus sign in the square root is needed because of the Lorentzian signature of the metric (see Notation and conventions). The quantity $\sqrt{-\mathrm{g}}\; \mathrm{d}^4 x$ is invariant with respect to coordinate transformations, so that the whole action S is an invariant.

In order for definition to make sense, we see that the linear combination $\psi^A + \lambda h^A$ must be from the same (vector) space of functions as is ψ^A. The first condition in (1.50) is crucial for this. The variation (1.51) is basically a difference between two integrals and by observing that if we define an auxiliary function $F(\lambda) := S[\psi + \lambda\omega]$, we can rewrite (1.51) as

$$\frac{\mathrm{d}F}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \equiv \frac{\mathrm{d}}{\mathrm{d}\lambda} S[\psi + \lambda h]\bigg|_{\lambda=0} . \tag{1.52}$$

While our discussion is a mere generalization of what we know from 'classical' calculus of variations (see Gelfand & Fomin [22]) and the statements seem to be plausible, allow us to present a more rigorous approach adopted from Szekeres [23] which we somewhat modify.

For integration over an orientable four-dimensional spacetime (M, g) we need a non-vanishing 4-form ω (totally antisymmetric (0,4)-tensor). For any coordinate chart $(U, \varphi; \{x^\mu\})$ on a Lorentzian manifold (M, g) we can write [23] for a *metric volume form*

$$\omega_g = \frac{\sqrt{-\mathrm{g}}}{4!}\epsilon_{\mu\nu\rho\sigma}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \otimes \mathrm{d}x^\rho \otimes \mathrm{d}x^\sigma = \sqrt{-\mathrm{g}}\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 , \tag{1.53}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita permutation symbol and $\wedge$ is the exterior product. Every 4-form Ω can [23] be written as $\Omega = f\omega_g$ for some non-zero $f \in \mathcal{F}M$, and for every regular domain $\mathcal{D} \subset U$

$$\int_{\mathcal{D}} \Omega = \int_{\varphi(\mathcal{D})} \widetilde{f}(x)\sqrt{-\mathrm{g}}\ \mathrm{d}^4x , \quad \mathrm{d}^4x := \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 , \tag{1.54}$$

where $\widetilde{f} = f \circ \varphi^{-1}$ is a coordinate representation of f. This is in most cases implicitly understood, so $\widetilde{f}$ and f are not being explicitly distinguished. The dependence on coordinates $\widetilde{f} = \widetilde{f}(x)$ is also automatically assumed, so we will not write it explicitly either. The image of the domain $\varphi(\mathcal{D}) \subset \mathbb{R}^4$ can be sometimes identified with $\mathcal{D}$ for brevity, but we will use it in this section and denote it $x^\mu(\mathcal{D})$.

If it happens to be $f\sqrt{-\mathrm{g}} = \partial_\mu(\sqrt{-\mathrm{g}}\,V^\mu)$ for some $V \in \mathfrak{X}M$, then

$$\Omega = \mathrm{d}v \quad \text{for} \quad v = V^\mu\,\mathrm{d}^3\Sigma_\mu \quad \text{where} \quad \mathrm{d}^3\Sigma_\mu := \frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma , \tag{1.55}$$

where d is the exterior derivative. By Stokes' theorem

$$\int_{\mathcal{D}} \Omega = \int_{\mathcal{D}} \mathrm{d}v = \int_{\partial D} v . \tag{1.56}$$

Let the mentioned set of fields $\{\psi^A(x)\}$ be defined on a neighbourhood U of regular domain $x \in \mathcal{D} \subset (M, g)$. By a *variation* of these fields is meant a one-parameter family of fields $\widetilde{\psi}^A(x, \lambda)$ on $\mathcal{D}$, such that

$$\begin{aligned} &i) \quad \widetilde{\psi}^A(x, 0) = \psi^A(x) \quad \forall x \in \mathcal{D} , \\ &ii) \quad \widetilde{\psi}^A(x, \lambda) = \psi^A(x) \quad \forall \lambda \in I \text{ and } \forall x \in U \setminus \mathcal{D} . \end{aligned}$$

where $I \subset \mathbb{R}$ is a neighbourhood of zero. The second condition implies that the condition holds on the boundary $\partial\mathcal{D}$ and also all derivatives of the variation field components agree there, i.e.

$$\partial_\mu\widetilde{\psi}^A(x, \lambda) = \partial_\mu\psi^A(x) \quad \forall x \in \partial\mathcal{D} . \tag{1.57}$$

We now define the *variational derivatives of the fields* as

$$\delta\psi^A := \frac{\partial}{\partial\lambda}\widetilde{\psi}^A(x,\lambda)\bigg|_{\lambda=0} . \tag{1.58}$$

These vanish on the boundary $\partial\mathcal{D}$ since $\widetilde{\psi}^A$ are independent of λ there. In practice, variations are given as $\widetilde{\psi}^A(x,\lambda) = \psi^A(x) + \lambda h^A(x)$, so that $\delta\psi^A = h^A$. This is why in many physical texts one finds the variation of the action to be simply $S[\psi + \delta\psi] - S[\psi]$, where the parameter λ is left out and $\delta\psi$ is assumed to be small. One then takes a first order Taylor expansion of the Lagrangian in the action to obtain field equations.

A Lagrangian is a (C^2) function $L = L(x; \psi^A, \partial_\mu\psi^A)$ dependent on the fields and their first derivatives[11]. It defines a 4-form $\Lambda = L\omega_g$ and an associated action

$$S[\psi] = \int_{\mathcal{D}} \Lambda = \int_{\mathcal{D}} L\omega_g = \int_{x^\mu(\mathcal{D})} \underbrace{L\sqrt{-\mathrm{g}}}_{:=\mathcal{L}}\, \mathrm{d}^4x \,. \tag{1.59}$$

The laws of nature are in language of physics governed by equations of motion $\equiv$ field equations $\equiv$ Euler-Lagrange equations. To obtain them, we vary the corresponding action and set it to zero:

$$\begin{aligned}
0 = \delta S = \frac{\mathrm{d}}{\mathrm{d}\lambda}\int_{\mathcal{D}} \Lambda\bigg|_{\lambda=0} &= \int_{x^\mu(\mathcal{D})} \delta\mathcal{L}(\psi^A, \partial_\mu\psi^A)\ \mathrm{d}^4x \\
&= \int_{x^\mu(\mathcal{D})} \left\{ \frac{\partial\mathcal{L}}{\partial\psi^A}\delta\psi^A + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^A)} \underbrace{\delta(\partial_\mu\psi^A)}_{=\partial_\mu(\delta\psi^A)} \right\} \mathrm{d}^4x \\
&= \int_{x^\mu(\mathcal{D})} \left\{ \frac{\partial\mathcal{L}}{\partial\psi^A} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^A)}\right)\right\}\delta\psi^A + \int_{x^\mu(\mathcal{D})} \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^A)}\delta\psi^A\right) \mathrm{d}^4x \\
&\overset{(1.56)}{=} \int_{x^\mu(\mathcal{D})} \frac{\delta\mathcal{L}}{\delta\psi^A}\delta\psi^A\ \mathrm{d}^4x + \underbrace{\int_{\partial\mathcal{D}} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^A)}\delta\psi^A\ \mathrm{d}^3\Sigma_\mu}_{=0} ,
\end{aligned}$$

where the integral on the right vanishes because $\delta\psi^A = 0$ on $\partial\mathcal{D}$ and

$$\frac{\delta\mathcal{L}}{\delta\psi^A} := \frac{\partial\mathcal{L}}{\partial\psi^A} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^A)}\right) \tag{1.60}$$

are the Euler-Lagrange equations for the field variable ψ^A. A symbolic quantity $\delta\mathcal{L}/\delta\psi^A$ is sometimes called variational derivative of the Lagrangian density with respect to ψ^A (or simply Lagrangian derivative even though it is not linear; e.g. in [9]). It is often denoted as $\delta S/\delta\psi^A$ instead, but we will use this for different purpose which we are going to describe in a moment.

Hence, from variation of the action we get an equivalence[12]

$$\delta S = 0 \quad \Leftrightarrow \quad \frac{\delta\mathcal{L}}{\delta\psi^A} = 0 \,. \tag{1.61}$$

[11] We will see below that the Hilbert action for the gravitational field depends on the second derivatives of the metric tensor (= field variable). However, it is possible to brush second derivatives under the rug while leaving the form of the field equations untouched by adding an appropriate divergence term (see (1.66)).

[12] For implication ($\delta S = 0$) $\Rightarrow$ ('Eqs. of motion') to be true we need less conditions than for the converse (see [22]).

Generalization of (1.60) to Lagrangians depending on second derivatives of the field variables is straightforward; one only has to perform an additional integration by parts, use $\partial_\mu(\delta\psi^A) = 0$ on the boundary $\partial\mathcal{D}$ and obtain

$$\frac{\delta\mathcal{L}}{\delta\psi^A} := \frac{\partial\mathcal{L}}{\partial\psi^A} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\psi^A_{,\mu}}\right) + \partial_\nu\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\psi^A_{,\mu\nu}}\right) \tag{1.62}$$

where the comma with index μ stands for ∂_μ. It is shown in [9] that the Euler-Lagrange equations can be rewritten in terms of covariant derivatives, without changing their form (i.e. they can be covariantized).

Up to now, we assumed that the action is a functional of only one field variable. It can be assigned, however, to multiple field variables. For example, take a scalar field[13] ϕ and a metric tensor $g_{\mu\nu}$. Variations of S with respect to these variables need to be carried out separately; while varying ϕ we hold $g_{\mu\nu}$ fixed, and vice versa. For $S = S[g_{\mu\nu}, \phi]$ we define 'partial variations' of the action with respect to ϕ and $g_{\mu\nu}$ as

$$\frac{\delta S}{\delta g_{\mu\nu}} := \frac{\mathrm{d}}{\mathrm{d}\lambda} S[g_{\mu\nu} + \lambda h_{\mu\nu}, \phi]\bigg|_{\lambda=0} , \tag{1.63a}$$

$$\frac{\delta S}{\delta\phi} := \frac{\mathrm{d}}{\mathrm{d}\lambda} S[g_{\mu\nu}, \phi + \lambda h]\bigg|_{\lambda=0} . \tag{1.63b}$$

where $h_{\mu\nu}\,(\equiv \delta g_{\mu\nu})$ is symmetric (see [15]). Both $h_{\mu\nu}$ and $h\,(\equiv \delta\phi)$ satisfy conditions (1.50). We would thus write (symbolically)

$$\frac{\delta S}{\delta g_{\mu\nu}} = \int_{\mathcal{D}} \frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} \quad \& \quad \frac{\delta S}{\delta\phi} = \int_{\mathcal{D}} \frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}\delta\phi\,. \tag{1.64}$$

One can also vary the action with respect to the inverse metric tensor $g^{\mu\nu}$ and the definition would be analogous (with $-h^{\mu\nu}$), because

$$(g_{\mu\nu} + \varepsilon h_{\mu\nu})(g^{\nu\rho} - \varepsilon h^{\nu\rho}) = \delta^\rho_\mu + \mathcal{O}\big(\varepsilon^2\big)\,. \tag{1.65}$$

A 'nota bene' fact is that adding a divergence to a Lagrangian does not spoil the form of equations of motion

$$\mathcal{L}' = \mathcal{L} + \partial_\alpha\left(\sqrt{-\mathrm{g}}\,V^\alpha\right) \quad \Rightarrow \quad \delta S' = \delta S\,, \tag{1.66}$$

because

$$S' = S + \oint_{\partial\mathcal{D}} \sqrt{-\mathrm{g}}\,V^\alpha\ \mathrm{d}^3\Sigma_\alpha \quad \& \quad \delta\oint_{\partial\mathcal{D}} \sqrt{-\mathrm{g}}\,V^\alpha\ \mathrm{d}^3\Sigma_\alpha = 0\,, \tag{1.67}$$

where the quantity $V^\alpha = V^\alpha(\psi^A)$ depends only on fields for a Lagrangian dependent on the first derivatives of ψ^A. It might depend on the first derivatives of the fields too, if the Lagrangian depends on $\partial_\nu\partial_\mu\psi^A$ (then one uses both conditions in

[13]Scalar field may represent mass-energy density, pressure, EM potential etc. It always depends on a specific problem. In modified theories of gravity it can be an additional representative of the gravitational field along with the metric tensor. In cosmology, such a scalar field may be used to provoke inflation.

(1.50)). Therefore, even though the form of the actions will differ by a boundary term, they provide indistinguishable field equations

$$\frac{\delta \mathcal{L}'}{\delta \psi^A} = \frac{\delta \mathcal{L}}{\delta \psi^A} \,. \tag{1.68}$$

Transformation of the Lagrangian by a divergence (1.66) is sometimes called a *gauge transformation of the Lagrangian* [9].

We wish to add another, more geometric view on variations.
Variations of the field variables can be generated by (infinitesimal) coordinate transformations. We know from (1.18) that these can be obtained by infinitesimal flows Φ_ε which are generated by the corresponding vector fields X. In the first approximation (see Eq. (1.34)), the Lie derivative $\mathfrak{L}_X$ is thus an appropriate (coordinate independent) measure of 'variation' of the the field variables under an infinitesimal flow (which in coordinates results as a coordinate transformation). However, the variables must not change ($\mathfrak{L}_X \psi^A = 0$) on the boundary $\partial \mathcal{D}$. This can be done by taking a vector field which vanishes on the boundary (see [15] for details). Hence, we can write

$$\delta \psi^A \equiv \varepsilon h^A \equiv \varepsilon \mathfrak{L}_X \psi^A \,, \tag{1.69}$$

which leads to

$$\frac{\delta S}{\delta \psi^A} = \int_{\mathcal{D}} \frac{\delta \mathcal{L}}{\delta \psi^A} \mathfrak{L}_X \psi^A \, \omega_g \,, \tag{1.70}$$

where $\delta \mathcal{L}/\delta \psi^A$ (defined in (1.60)) is obtained from the 'Taylor expansion'

$$\mathcal{L}(\psi^A + \varepsilon \mathfrak{L}_X \psi^A, \psi^A_{,\mu} + \mathfrak{L}_X \psi^A_{,\mu}) = \mathcal{L} + \varepsilon \frac{\partial \mathcal{L}}{\partial \psi^A} \mathfrak{L}_X \psi + \varepsilon \frac{\partial \mathcal{L}}{\partial \psi^A_{,\mu}} \mathfrak{L}_X \psi^A_{,\mu} + \mathcal{O}\left(\varepsilon^2\right) . \tag{1.71}$$

After integrating this over $\mathcal{D}$, using commutation (1.47), integration by parts and Gauss's (Stokes) theorem, one gets

$$S[\psi^A + \mathfrak{L}_X \psi^A] = S[\psi] + \varepsilon \int_{\mathcal{D}} \frac{\delta \mathcal{L}}{\delta \psi^A} \mathfrak{L}_X \psi^A \omega_g + \int_{\partial \mathcal{D}} \frac{\partial \mathcal{L}}{\partial \psi^A_{,\mu}} \mathfrak{L}_X \psi_A \, \mathrm{d}^3 \Sigma_\mu \,, \tag{1.72}$$

and because the second integral vanishes, the result indeed coalesces with (1.70). Note that, if we subtract $\mathcal{L}$ from both sides of (1.71) and dividing by ε, we basically get a chain rule

$$\mathfrak{L}_X \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi^A} \mathfrak{L}_X \psi^A + \frac{\partial \mathcal{L}}{\partial \psi^A_{,\mu}} \mathfrak{L}_X \psi^A_{,\mu} = \frac{\delta \mathcal{L}}{\delta \psi^A} \mathfrak{L}_X \psi^A + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \psi^A_{,\mu}} \mathfrak{L}_X \psi^A \right) . \tag{1.73}$$

where in the second equality we used Liebniz's rule and definition (1.60). Equation (1.73) will be useful later for the Noether's theorem. We employ both approaches in this thesis (using δ or Lie derivative), depending on what seems to be more convenient to us.

1.2.1 The Hilbert action and Einstein's Lagrangian

The action which Hilbert proposed is very simple:

$$S_H = \frac{1}{2\kappa} \int_{\mathcal{D}} R \sqrt{-\mathrm{g}} \, \mathrm{d}^4 x \,, \quad \kappa := \frac{8\pi G}{c^4} \,, \tag{1.74}$$

with R being a scalar curvature[14]. Misner et al. [7] point out that this is the only possible way to construct such an action, whose Lagrangian is *linear* in second derivatives of the metric $g_{\mu\nu}$ which are not coupled to the first derivatives. Nakahara [16] mentions an important fact: a 'first guess' could also be to have only a metric volume form $\sqrt{-\mathrm{g}}\,\mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^3$ as a Lagrangian, but that would not suffice to describe the dynamics of the gravitational field; we need derivatives of the metric. By varying the S_H with respect to the inverse metric tensor $g^{\mu\nu}$ we can obtain equations of motion of the second order for the metric $g_{\mu\nu}$ — the vacuum Einstein field equations. This is shown in numerous textbooks (see, for instance, [7] and [19]), but we will briefly sketch the variation of S_H, because one step will be important later in Chapter 4. We slide with the variation δ into the integral (1.74) of S_H in the sense of the definitions above and by Leibniz's rule we get

$$R\delta\sqrt{-\mathrm{g}} + \sqrt{-\mathrm{g}}\,\delta R\,. \tag{1.75}$$

The variation of the first term is easy (see (A.3) in Appendix A)

$$\delta\sqrt{-\mathrm{g}} = \frac{1}{2}\sqrt{-\mathrm{g}}\,g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-\mathrm{g}}\,g_{\mu\nu}\delta g^{\mu\nu}\,. \tag{1.76}$$

The second term in (1.75) reads

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\,. \tag{1.77}$$

We pause at this moment with variations and collect what we have calculated:

$$2\kappa\delta S_H = \int_{\mathcal{D}} \sqrt{-\mathrm{g}}\,\{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\}\delta g^{\mu\nu}\ \mathrm{d}^4x + \int_{\mathcal{D}} g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-\mathrm{g}}\ \mathrm{d}^4x\,. \tag{1.78}$$

We see that the first integral yields the LHS of Einstein field equations, provided that the second integral is zero. Let us now demonstrate that the expression $g^{\mu\nu}\delta R_{\mu\nu}$ can be rewritten as a divergence. It is a consequence of

PROPOSITION 4. (The Palatini identity). *Let (M, g) be a (pseudo-)Riemannian manifold and $R_{\mu\nu}$ the Ricci tensor. Consider the variation $g \mapsto g + \delta g$. Then*

$$\delta R_{\mu\nu} = \nabla_\alpha \delta\Gamma^\alpha{}_{\mu\nu} - \nabla_\mu \delta\Gamma^\alpha{}_{\alpha\nu}\,.$$

Proof. Nakahara [16], p. 298. □

The identity has the same form for the variation of the inverse metric tensor. Therefore,

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &\overset{Palatini}{=} 2g^{\mu\nu}\nabla_{[\alpha}\delta\Gamma^\alpha{}_{\mu]\nu} = 2\nabla_{[\alpha}\left(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu]\nu}\right) \\ &= 2\nabla_\alpha\left(g^{\nu[\mu}\delta\Gamma^{\alpha]}{}_{\mu\nu}\right) = \frac{1}{\sqrt{-\mathrm{g}}}\partial_\alpha\left(\hat{g}^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - \hat{g}^{\alpha\nu}\delta\Gamma^\mu{}_{\mu\nu}\right)\,, \end{aligned} \tag{1.79}$$

where $\hat{g}^{\mu\nu} = \sqrt{-\mathrm{g}}\,g^{\mu\nu}$. Eq. (1.79) is the result that we promised to exploit later in Chapter 4.

[14]There is also a convention with a minus sign for the proportionality constant $1/2\kappa$. We follow the convention of references Misner et al. [7]

Hence, applying the Gauss's (Stokes') theorem on the second integral in (1.78) and using boundary conditions imposed on the variations of the inverse metric components (the same would apply to the metric tensor of course)

$$\delta g^{\mu\nu} \mid_{\partial\mathcal{D}} = 0 \quad \& \quad \partial_\alpha \delta g^{\mu\nu} \mid_{\partial\mathcal{D}} = 0 \tag{1.80}$$

permits us to celebrate:

$$\delta S_H = 0 \Leftrightarrow \frac{2\kappa}{\sqrt{-\mathrm{g}}} \frac{\delta \mathcal{L}_H}{\delta g^{\mu\nu}} = G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \tag{1.81}$$

To obtain full Einstein field equations, we must add to the Hilbert's scalar curvature a Lagrangian $\mathcal{L}_M$ representing other fields (such as EM field) and matter. Then we define an energy-momentum tensor as

$$T_{\mu\nu} := -\frac{2}{\sqrt{-\mathrm{g}}} \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} . \tag{1.82}$$

It is automatically symmetric due to this definition (or, more precisely, only its symmetric part is effective), but there are other ways to define $T_{\mu\nu}$ which may include its antisymmetric part (see next chapter). The total action $S_{EH} = S_H + S_M$ — the Einstein-Hilbert action — yields gravitational field equations. If, in addition, we wish to pursue cosmological adventures we can add a term $-\Lambda/\kappa$ to the Einstein-Hilbert action.

The Einstein '$\Gamma\Gamma - \Gamma\Gamma$' Lagrangian

As we already said, Hilbert Lagrangian leads to the second-order field equations, in spite of its linear dependence on the second derivatives of the metric. Scalar curvature R is in terms of Christoffel symbols given as

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} \left(\partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\alpha} + \Gamma^\alpha{}_{\lambda\alpha} \Gamma^\lambda{}_{\mu\nu} - \Gamma^\alpha_{\lambda\nu} \Gamma^\lambda{}_{\mu\alpha} \right) . \tag{1.83}$$

Only first two terms in the parentheses contain second derivatives of the metric. One can easily get rid of them by adding appropriate divergence term to the Hilbert Lagrangian $\mathcal{L}_H = \sqrt{-\mathrm{g}}\, R$. We know from (1.66) that such term will not affect the field equations. Einstein noticed that, and so his Lagrangian is made of the '$\Gamma\Gamma - \Gamma\Gamma$' term in the parentheses of (1.83):

$$\mathcal{L}_E = \frac{1}{2\kappa} \sqrt{-\mathrm{g}}\, g^{\mu\nu} \left(\Gamma^\alpha{}_{\lambda\alpha} \Gamma^\lambda{}_{\mu\nu} - \Gamma^\alpha{}_{\lambda\nu} \Gamma^\lambda{}_{\mu\alpha} \right) \tag{1.84}$$

Appropriate divergence, which cancels all second derivatives of $g_{\mu\nu}$ in R, relates the two Lagrangians in the following way

$$\mathcal{L}_E = \mathcal{L}_H - \frac{1}{2\kappa} \partial_\alpha \hat{k}^\alpha , \tag{1.85}$$

while the associated actions differ only by a surface integral due to the Gauss's theorem,

$$S_E = S_H - \frac{1}{2\kappa} \oint_{\partial\mathcal{D}} \hat{k}^\alpha \ \mathrm{d}^3\Sigma_\mu \tag{1.86}$$

where
$$\hat{k}^{\alpha} = \hat{g}^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} - \hat{g}^{\mu\alpha}\Gamma^{\nu}{}_{\mu\nu}\,. \tag{1.87}$$

It is evident that the divergence of k^{μ} resembles the first two terms in (1.83), which after straightforward calculation, indeed cancels second derivatives in $\mathcal{L}_H$.

However, as emphasized in [9], the Einstein Lagrangian (1.84) *is not covariant* because its coordinate transformation is not tensorial[15]. Hence, it can be nullified at any given point of spacetime manifold by stepping into a falling elevator. For this reason, the '$\Gamma\Gamma - \Gamma\Gamma$' Lagrangian makes sense only in the integral expression for the action functional. But why would Einstein destroy covariance, which scalar curvature R has? This has to do with the problem of defining energy of weak gravitational waves, when the field equations are derived from the Hilbert Lagrangian [9].

The surface integral in (1.86) vanishes because of the boundary conditions (1.80) and does not contribute to the variation of the action. Hence, from the point of view of variational principle, both types of the action are equivalent for the purpose of derivation of the Einstein field equations. It is interesting to notice that if we derive the Einstein equations from the action

$$S_E = \frac{1}{2\kappa}\int_{\mathcal{D}} \mathcal{L}_E \; \mathrm{d}^4 x\,, \tag{1.88}$$

the first boundary condition in (1.80) is sufficient.

[15]Here, it is meant that the scalar Lagrangian $L_E = \mathcal{L}_E/\sqrt{-\mathrm{g}}$ does not transform as a tensor of a type $(0,0)$. Lagrangian *densities* $\mathcal{L}$ are not tensors from their definition.

2

Conservation Laws & Symmetries

"Symmetry, as wide or narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection."

— Hermann Weyl, *Symmetry*, 1952

In this chapter we manifest how conservation laws are tightly related to symmetries. To be more 'specific', symmetry is an *invariance* of (the form of) something under some kind of transformation (automorphism) applied to it. The core of this idea is comprised in the Noether's theorem and in the existence of mappings which preserve the spacetime geometry — isometries. The preceding mathematical chapter enables us to describe these in an elegant way.

2.1 Isometries and Killing vector fields

In general relativity, the prominent symmetries are — purely geometric — spacetime symmetries. The geometry of time and space is, as we know, encoded in the metric tensor.

Definition 13. *Let (M_d, g) and (N_d, h) be two (pseudo-)Riemannian manifolds. A diffeomorphism $\Phi : M \to N$ is an* ***isometry*** *if*

$$\Phi^* h = g\,.$$

In terms of vector fields $X, Y \in \mathfrak{X}M$ this means that

$$\underbrace{g(X,Y)}_{\in \mathcal{F}M} = (\Phi^* h)(X,Y) = \Phi^*(\underbrace{h(\Phi_* X, \Phi_* Y)}_{\in \mathcal{F}N}) \equiv h(\Phi_* X, \Phi_* Y) \circ \Phi\,. \tag{2.1}$$

A map Φ is thus an isometry if calculating values of lengths and angles[1] 'here' is equivalent to calculating them 'there' and then pulling-them-back 'here' by Φ^*.

The definition 13 is too general for our purposes, so by setting $(M, g) \equiv (N, h)$, we obtain a condition

$$\Phi^* g = g\,. \tag{2.2}$$

Then (2.1) can be written even more instructively; for any point $p \in M$ we have

$$g(X,Y)|_p = (\Phi^* g)(X,Y)|_p = g(\Phi_* X, \Phi_* Y)|_{\Phi(p)}\,. \tag{2.3}$$

Therefore, isometries preserve the (way of determining) geometry of the manifold. Because isometries are diffeomorphisms, a set of isometries of the manifold (M, g) forms a subgroup of $\operatorname{Diff} M$ and hence a group by itself.

[1] An instructive exercise can be found in [15] on page 81.

One can thus naturally ask whether vector fields can generate (a group of) isometries. Let us suppose that Φ_t is a flow generated by a vector field ξ. It follows then immediately, from the definition of the Lie derivative (Def. 12), that Φ_t is an isometry of (M, g) if and only if

$$\mathfrak{L}_\xi g = 0\,. \tag{2.4}$$

Vector fields along which the metric tensor does not change are called *Killing vector fields*[2] of the metric g. The fourth and the fifth property of the Theorem 1 tell us that any linear combination of Killing fields $\{\xi_{(i)}\}$ is a Killing field:

$$\mathfrak{L}_{\xi_{(i)}+r\xi_{(j)}} g = \mathfrak{L}_{\xi_{(i)}} g + r\mathfrak{L}_{\xi_{(j)}} g = 0 \tag{2.5}$$

and, remarkably, the Lie bracket of any two Killing vector fields is also a Killing one:

$$\mathfrak{L}_{[\xi_{(i)},\xi_{(j)}]} g = \mathfrak{L}_{\xi_{(i)}}\mathfrak{L}_{\xi_{(j)}} g - \mathfrak{L}_{\xi_{(j)}}\mathfrak{L}_{\xi_{(i)}} g = 0\,. \tag{2.6}$$

Therefore, Killing vector fields form a Lie subalgebra of all vector fields on (M, g) and thus a Lie algebra by itself, corresponding to a Lie group of isometries acting on M.

Sometimes, when we are satisfied with preservation of angles only, a weaker requirement than (2.2) is sufficient [15]; if for arbitrary function $\Omega : M \to \mathbb{R}$ holds

$$\Phi^* g = \Omega^2 g\,, \tag{2.7}$$

then Φ is called a *conformal transformation* of (M, g). For $\Omega = const.$ these are called *homotheties* and for $\Omega \equiv 1$ they evidently reduce to isometries. In terms of flows, associated infinitesimal generators ζ, for which

$$\mathfrak{L}_\zeta g = \Omega^2 g\,, \tag{2.8}$$

are called conformal (or homothetic) Killing vectors. As (Lie) groups, isometries are a subgroup of homotheties which are a subgroup of conformal transformations [15]. The associated Lie (sub)algebras are ordered in the same way.

In coordinates, (2.4) reads

$$\xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu \xi^\lambda + g_{\mu\lambda}\partial_\nu \xi^\lambda = 0\,. \tag{2.9}$$

We know that we can rewrite this in terms of the covariant derivative, so by using $\nabla_\lambda g_{\mu\nu} = 0$ (the Levi-Civita connection) we obtain *Killing equations*[3]

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0\,. \tag{2.10}$$

The fact that Killing vectors (the space of solutions of the partial differential equations under consideration — Killing equations) form the Lie algebra may be used, sometimes, for finding new solutions. When only some solutions are known, we simply form all the possible commutators of the solutions which are known

[2]They are named after a German mathematician Wilhelm Killing.

[3]As far as we know, Killing equation, Killing's equation and Killing's equations are different names of the same thing that different authors choose. We follow the designation chosen in the book [15].

so far and, if we are lucky enough, a new solution drops out in this way (linear combinations are trivial, since the equation (2.10) is linear).

In most classical textbooks (for instance [11]) Killing equations (2.10) are obtained by considering an infinitesimal transformation $x \mapsto x'(x)$

$$x'^{\mu}(x) = x^{\mu} + \varepsilon\xi^{\mu}\,, \quad \varepsilon \ll 1\,. \tag{2.11}$$

We already saw this in Eq. (1.18) in the context of an infinitesimal flow Φ_ε. For this reason, transformation (2.11) is often accompanied by words: 'consider an infinitesimal diffeomorphism' (i.e. differing slightly from the identity).

In general, the metric tensor (or any other tensor) changes correspondingly

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = g'_{\mu\nu}(x + \varepsilon\xi)\,. \tag{2.12}$$

We can now ask whether this transformation keeps the form of the metric tensor components unchanged:

$$g_{\mu\nu}(x) \stackrel{?}{=} g_{\mu\nu}(x')\,.$$

After substituting this question into a general transformation law

$$g_{\mu\nu}(x) = g'_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}\,,$$

using (2.11), performing a Taylor expansion of (2.12) around zero, while neglecting the $\mathcal{O}(\varepsilon^2)$ terms and after some additional manipulation, one obtains an equivalence

$$g_{\mu\nu}(x) = g_{\mu\nu}(x') \quad \Leftrightarrow \quad \nabla_{(\mu}\xi_{\nu)} = 0\,, \tag{2.13}$$

which says that the functional form of the components of the metric tensor is the same (up to the first order in ε) in both coordinates x and x' (related by (2.11)) *if and only if* the Killing equations hold. For example, if $g_{12}(x) = \sin(x^3)$ then $g_{12}(x') = \sin(x'^3)$. The advantage of the infinitesimal approach is that one can imagine, for example, a rotation by a finite angle, say α, to be composed of many small angles α/N (with N being large) [15]. By iteration of infinitesimal isometries (smooth deformations of the trivial isometry), one obtains a whole flow of isometries.

An immediate corollary of (2.9) is that if the functions $g_{\mu\nu}(x)$ are independent of some coordinate x^α, then choosing ξ as the basis vector ∂_α, with components $\xi^\mu = \delta^\mu_\alpha$, yields (for fixed α)

$$(\mathfrak{L}_\xi g)_{\mu\nu} = \partial_\alpha g_{\mu\nu} = 0\,. \tag{2.14}$$

Thus, the Killing vector ∂_α in the direction of the (spacetime) coordinate x^α indicates the existence of the symmetry in this direction. If such symmetry happens to occur locally, it clearly may not be so globally. An example may be a torus (surface of a doughnut) which has only one global Killing field, but it enables other, Euclidean ones, locally[4].

Killing equations may be used in the opposite manner: instead of looking for ξ while having g given, we can find the most general form of g by prescribing

[4] A 'flat' torus can be embedded in a four-dimensional Euclidean space by making a tube out of a rectangle, which can then be stretched in a circular way to connect the endpoints (see [15], p. 61 and p. 86).

a set of isometries (and therefore Killing vectors)[5]. For instance, when solving field equations, we often choose a set of symmetries for the examined system in the form of an 'ansatz'. This restricts the form of a solution to a specific class of solutions which allows us to solve the equations more quickly.

Isometry condition (2.2) in components turns into the equality of two *symmetric* matrices which, in n dimensions, puts $n(n+1)/2$ (number of independent components of a symmetric matrix) conditions on n unknown functions $\xi^a(x)$ through the Killing equations. A component expression of the Killing equations (2.9) is thus an overdetermined system of equations. It is far from guaranteed that a non-trivial solution exists. The Lie algebra of Killing vectors is thus always finite-dimensional and one can show that the maximum number of independent Killing vectors in n dimensions is at most $n(n+1)/2$ (see below).

Killing equations are very easy to solve when (M_n, η) is flat; then (2.10) reduces to

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0\,. \tag{2.15}$$

Differentiating with respect to x^ρ and cyclically permuting the indices gives three conditions

$$\partial_\rho \partial_\mu \xi_\nu = -\partial_\rho \partial_\nu \xi_\mu\,, \quad \partial_\nu \partial_\rho \xi_\mu = -\partial_\nu \partial_\mu \xi_\rho\,, \quad \partial_\mu \partial_\nu \xi_\rho = -\partial_\mu \partial_\rho \xi_\nu\,,$$

which, combined with $\partial_\mu \partial_\rho = \partial_\rho \partial_\mu$, yield (see also Eq. (2.28) below)

$$\partial_\rho \partial_\mu \xi_\nu = 0\,. \tag{2.16}$$

Therefore,

$$\xi^\alpha = A^\alpha{}_\beta x^\beta + b^\alpha\,, \tag{2.17}$$

where $A^\alpha{}_\beta$ is a constant matrix with real entries and b^α is a constant vector. Inserting ξ^α back into flat Killing equations (2.15) as $\xi_\nu = \eta_{\nu\alpha}\xi^\alpha$ restricts the matrix A:

$$\partial_\mu \xi_\nu = \underbrace{\eta_{\nu\alpha} A^\alpha{}_\mu}_{=(\eta A)_{\mu\nu}} = -\partial_\nu \xi_\mu = -(\eta A)_{\nu\mu} = -(\eta A)^T_{\mu\nu}\,. \tag{2.18}$$

This tells us that the matrix ηA is antisymmetric and thus has $n(n-1)/2$ independent components. Since multiplying A with η (from the left) only changes the sign of the first row of the matrix A, we need at most $n(n-1)/2$ parameters $A^\alpha{}_\beta$ to describe A. The vector b may be arbitrary and therefore have n parameters b^a which constitute him. The $n(n+1)/2$ basis vectors of the Killing algebra are of the form [15]

$$m_{\{\mu\nu\}} := x_\mu \partial_\nu - x_\nu \partial_\mu \quad \text{where} \quad x_\mu = \eta_{\mu\alpha} x^\alpha \quad \& \quad p_{\{\mu\}} := \partial_\mu \tag{2.19}$$

[5] If the reader still doesn't appreciate the significance of Killing vectors, our recommendation is to listen to the first verse of the Michael Jackson's *Earth song* which, with only minor changes, sounds as an ode to the infinitesimal generators of isometries

What about sunrise	*What about Killing fields*
What about rain	*Is there a time isometry*
What about all the things	*What about all the things*
That you said we were to gain	*That you said were yours and mine*

Braces around indices emphasize that it is a set of generators assigned to fixed indices μ and ν (i.e. to specific coordinates). Vectors $\{m_{\mu\nu}\}$ generate $n-1$ spatial rotations (for $\eta_{\mu\mu}\eta_{\nu\nu} = +1$) and $(n-1)(n-2)/2$ hyperbolic rotations ('Lorentz boosts'; for $\eta_{\mu\mu}\eta_{\nu\nu} = -1$), while $\{p_\mu\}$ generate n translations. General Killing vector from the Lie algebra of isometries can be written as a linear combination with $n(n+1)/2$ coefficients

$$\xi = (A\eta)^{\mu\nu} m_{\{\mu\nu\}} + b^\mu p_{\{\mu\}} \,. \tag{2.20}$$

For the Minkowski spacetime $(M_4, \eta = \operatorname{diag}(-1,1,1,1))$ the matrix A is from the Lie algebra $\mathfrak{so}(1,3)$ (see Eq. (1.29) in Subsection 1.1.2) of the Lorentz group (1.1.2). Poincaré group is a Lie group acting on (M_4, η) equal to the Lorentz group + translations

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + b^\mu \tag{2.21}$$

and it is generated by infinitesimal isometries of (M_4, η) (see (1.18))

$$x \mapsto x' = x + \varepsilon\xi \tag{2.22}$$

characterized by $n(n+1)/2$ parameters.

(Pseudo-)Riemannian manifolds (as spaces) possessing maximum number of Killing vector fields are known as *maximally symmetric* spaces. In special and general relativity, maximal symmetry is represented by Minkowski, de Sitter and Anti-de Sitter spacetimes. These share a common feature — constant curvature.

2.1.1 Maximal symmetry and curvature

It would be inadequate to leave the last assertion without proof. As one would intuitively expect, maximally symmetric geometry is extremely simplified. In other words, the relationship between the curvature tensor and the metric tensor, which we are about to demonstrate, should be simple. Consequently, scalar curvature reduces to a constant.

The first step is to recall the torsion-free formula for the Riemann curvature tensor (Ricci identity)

$$[\nabla_\mu, \nabla_\nu]\xi_\lambda = R^\kappa{}_{\lambda\nu\mu}\xi_\kappa \,. \tag{2.23}$$

Let $\{\lambda\nu\mu\}$ denote summing over cyclic permutation of the indices. Then the first Bianchi identity

$$R^\kappa{}_{\{\lambda\nu\mu\}} \coloneqq R^\kappa{}_{\lambda\nu\mu} + R^\kappa{}_{\mu\lambda\nu} + R^\kappa{}_{\nu\mu\lambda} = 0 \tag{2.24}$$

suggests that we could do the same permutation of indices $\{\lambda, \mu, \nu\}$ in (2.23) and sum all three cyclically permuted results to use the first Bianchi identity. Performing the suggestion yields

$$[\nabla_\mu, \nabla_\nu]\xi_\lambda + [\nabla_\lambda, \nabla_\mu]\xi_\nu + [\nabla_\nu, \nabla_\lambda]\xi_\mu = R^\kappa{}_{\{\mu\nu\lambda\}}\xi_\kappa = 0 \,. \tag{2.25}$$

Next, we write the second and the third commutator explicitly

$$[\nabla_\mu, \nabla_\nu]\xi_\lambda + \nabla_\lambda\nabla_\mu\xi_\nu - \nabla_\mu\nabla_\lambda\xi_\nu + \nabla_\nu\nabla_\lambda\xi_\mu - \nabla_\lambda\nabla_\nu\xi_\mu = 0 \tag{2.26}$$

and after exploiting the antisymmetry of $\nabla_\alpha \xi_\beta$, the property of Killing vectors only, we obtain

$$2\,([\nabla_\mu, \nabla_\nu]\xi_\lambda - \nabla_\lambda \nabla_\nu \xi_\mu) = 0\,. \tag{2.27}$$

Therefore, using (2.23) again,

$$\nabla_\lambda \nabla_\nu \xi_\mu = R^\kappa{}_{\lambda\nu\mu}\xi_\kappa\,. \tag{2.28}$$

This intermediate result is important not only for our derivation of the constancy of curvature, but it also tells us that we can obtain second covariant derivatives of the Killing vectors from the first derivatives. This is noteworthy, because if we know the metric we know the Riemann tensor too, so any higher covariant derivatives of ξ can then be expressed as a linear combination of the first and second derivatives of ξ. The function $\xi^\mu(x)$ can be thus expressed simply as a Taylor series within some finite neighbourhood of any point $p \in (M_n, g)$ in coordinates x^μ. For that, we only need to know ξ^μ and $\nabla_\mu \xi_\nu$ at this specified point p to define the entire Killing vector field in its neighbourhood. Since these form $n + n(n-1)/2 = n(n+1)/2$ linearly independent initial values at p, the maximum number of linearly independent Killing vectors in any neighbourhood of (M_n, g) is $n(n+1)/2$ [23].

Proceeding in the proof of our assertion, we now define an auxiliary 2-form $W_{\nu\mu} := \nabla_\nu \xi_\mu$ and employ the (generalized) Ricci identity on it

$$[\nabla_\rho, \nabla_\lambda] W_{\nu\mu} = R^\kappa{}_{\nu\lambda\rho} W_{\kappa\mu} + R^\kappa{}_{\mu\lambda\rho} W_{\nu\kappa}\,. \tag{2.29}$$

The terms on the LHS are third (covariant) derivatives of the Killing vectors, so we can substitute these by first derivatives of (2.28) to get

$$\nabla_\rho(R^\kappa{}_{\lambda\nu\mu}\xi_\kappa) - \nabla_\lambda(R^\kappa{}_{\rho\nu\mu}\xi_\kappa) = 2\nabla_{[\rho}R^\kappa{}_{\lambda]\nu\mu}\xi_\kappa + R^\kappa{}_{\lambda\nu\mu}W_{\rho\kappa} - R^\kappa{}_{\rho\nu\mu}W_{\lambda\kappa}\,. \tag{2.30}$$

Combining the last two terms with the RHS of Eq. (2.29) and using the antisymmetry of $W_{\alpha\beta}$ leads to

$$\nabla_{[\rho}R^\kappa{}_{\lambda]\nu\mu}\xi_\kappa + (\delta^\sigma_{[\rho}R^\kappa{}_{\lambda]\nu\mu} + \delta^\sigma_{[\mu}R^\kappa{}_{\nu]\lambda\rho})\nabla_\sigma \xi_\kappa = 0\,, \tag{2.31}$$

where we have replaced $W_{\sigma\kappa}$ for $\nabla_\sigma \xi_\kappa$. This identity holds for all $n(n+1)/2$ possible Killing vectors on the maximally symmetric manifold (M_n, g). Let us first take ξ as any of the n coordinate basis vectors, say ∂_α with α arbitrary, but fixed[6]. Its components are then $\xi^\mu = \delta^\mu_\alpha$ and $\xi_\kappa = g_{\alpha\kappa}$. Because its covariant derivative vanishes, it follows then from identity (2.31) that

$$2\nabla_{[\rho}R^\kappa{}_{\lambda]\nu\mu}g_{\alpha\kappa} = \nabla_\rho R_{\alpha\lambda\nu\mu} - \nabla_\lambda R_{\alpha\rho\nu\mu} = \nabla_\rho R_{\alpha\lambda\nu\mu} + \nabla_\lambda R_{\rho\alpha\nu\mu} = 0\,, \tag{2.32}$$

where we used the antisymmetry $R_{\rho\alpha\nu\mu} = -R_{\alpha\rho\nu\mu}$. If we now use the second Bianchi identity

$$\nabla_{\{\rho}R_{\alpha\lambda\}\nu\mu} = 0\,, \tag{2.33}$$

[6] Up to now, our proof/derivation was strongly influenced by Weinberg's textbook [11]. However, a very detailed discussion in the referred book on how to use the identity (2.31) depends on the context and terminology of the whole chapter therein, which is based on homogeneity and isotropy to which we did not wish to allude, yet. In contrast, Szekeres ([23], p. 579), although precise, is very brief, so we chose our own way somewhere in between the two mentioned authors. Nonetheless, Szekeres gives many interesting examples of Lie algebras of Killing fields (and their construction).

then (2.32) implies

$$\nabla_\alpha R_{\lambda\rho\nu\mu} = 0\,. \tag{2.34}$$

Contracting the indices of the Riemann tensor twice with $g^{\lambda\nu}$ and $g^{\rho\mu}$ gives the asserted constancy of the scalar curvature R (with respect to x^α):

$$\nabla_\alpha R = \partial_\alpha R = 0\,. \tag{2.35}$$

because α was chosen arbitrarily, R is constant with respect to any coordinate x^μ, which concludes our proof.

Taking into account (anti)symmetries of the Riemann tensor, it is obvious from its covariant constancy (2.34) that

$$R_{\lambda\rho\nu\mu} \propto g_{\lambda\nu}g_{\rho\mu} - g_{\lambda\mu}g_{\rho\nu}\,. \tag{2.36}$$

We can gain even more specific result from the identity (2.31) if we now take any of the remained

$$\frac{n(n-1)}{2} = \frac{n(n+1)}{2} - n \tag{2.37}$$

Killing vectors forming the basis of the Lie algebra of the isometry group acting on (M_n, g). These have non-trivial components $\xi_\kappa(x)$, but because they satisfy the Killing equations, $\nabla_\sigma \xi_\kappa$ must be antisymmetric and so, independently of the first term, the term in parentheses in Eq. (2.31) must satisfy $n(n-1)/2$ conditions

$$\delta^{[\sigma}_\rho R^{\kappa]}{}_{\lambda\nu\mu} - \delta^{[\sigma}_\lambda R^{\kappa]}{}_{\rho\nu\mu} + \delta^{[\sigma}_\mu R^{\kappa]}{}_{\nu\lambda\rho} - \delta^{[\sigma}_\nu R^{\kappa]}{}_{\mu\lambda\rho} = 0\,. \tag{2.38}$$

Since $\delta^\nu{}_\nu = n$ and $R^\nu{}_{\nu\lambda\rho} = 0$, contracting κ with ν yields

$$2\delta^\sigma_{[\rho}R_{\lambda]\mu} + \underbrace{2R^\sigma{}_{[\rho\lambda]\mu} - R^\sigma{}_{\mu\lambda\rho}}_{=0} + (n-1)R^\sigma{}_{\mu\lambda\rho} = 0\,, \tag{2.39}$$

where the middle term vanishes due to the first Bianchi identity (2.24). Hence, after lowering σ with $g_{\sigma\nu}$,

$$(n-1)R_{\mu\nu\lambda\rho} = g_{\rho\nu}R_{\lambda\mu} - g_{\lambda\nu}R_{\rho\mu}\,. \tag{2.40}$$

We can obtain an expression for the Ricci tensor if we multiply the last equation with $g^{\mu\lambda}$:

$$(n-1)R_{\nu\rho} = Rg_{\rho\nu} - R_{\rho\nu}\,. \tag{2.41}$$

Thus, after rearranging terms and using $R_{\nu\rho} = R_{\rho\nu}$, the Ricci tensor takes the form

$$R_{\rho\nu} = \frac{R}{n}g_{\rho\nu}\,. \tag{2.42}$$

Inserting this in (2.40) gives the desired formula for the Riemann tensor

$$R_{\mu\nu\lambda\rho} = \frac{R}{n(n-1)}(g_{\rho\nu}g_{\lambda\mu} - g_{\lambda\nu}g_{\rho\mu})\,. \tag{2.43}$$

The last two expressions for the Riemann and the Ricci tensor (with $R = \mathit{const.}$) are very simple as we anticipated. They serve as an important tool for characterization of maximally symmetric spacetimes.

There is an interesting corollary of the Eq. (2.28). After rearranging, contracting and raising the indices, we obtain a wave equation for the Killing vector

$$\Box\xi^{\mu} = -R^{\mu}_{\kappa}\xi^{\kappa} \tag{2.44}$$

where $\Box := \nabla^{\nu}\nabla_{\nu}$ is the d'Alembert operator. (2.34).

We showed in Eq. (2.34) that the curvature tensor is constant with respect to covariant differentiation. Actually, curvature tensor is Lie constant along any Killing vector field; Eq. (2.34) is a coordinate expression of this fact for translational Killing vectors.

From Eq. (2.42) we immediately observe that if we are solving vacuum Einstein field equations with a cosmological constant,

$$R_{\mu\nu} = -\Lambda g_{\mu\nu}\,, \tag{2.45}$$

then the spacetime in question has constant curvature. For $\Lambda > 0$ this is de Sitter (dS) spacetime, while for $\Lambda < 0$ Anti-de Sitter (AdS) spacetime. In most applications, a constant K, with dimension of inverse length squared, is introduced by the relation [11]

$$R = -n(n-1)K\,, \tag{2.46}$$

which for (A)dS spacetimes (in four dimensions) means $\Lambda = (4-1)K = 3K$.

2.2 Conserved charges from isometries and conformal transformations

In the process of deriving his equations for the gravitational field, Einstein was searching for a suitable tensor to equate the energy-momentum tensor $T_{\mu\nu}$, for which he had known it needed to satisfy

$$\nabla_{\nu}T^{\mu\nu} = 0\,, \tag{2.47}$$

to represent, in a *covariant* manner, differential *conservation laws* for the sources having mass-energy and momentum. Byers [24] explains that the presence of the metric tensor and its derivatives in this formula shows the energy-momentum transfer between the gravitational fields and the source fields in $T^{\mu\nu}$. True, the presence of the gravitational field is hidden in the *covariant* derivative.

After failing with the first contraction of the Riemann tensor, the Ricci tensor $R_{\mu\nu}$, he concluded that, because of the Levi-Civita connection $\nabla_{\rho}g_{\mu\nu} = 0$ and consequent twice contracted second Bianchi identity

$$\nabla_{\nu}G^{\mu\nu}\,, \tag{2.48}$$

it must be

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}\,, \tag{2.49}$$

where $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, with $R \equiv R^{\alpha}_{\alpha} = g^{\alpha\beta}R_{\alpha\beta}$ being the scalar curvature, G is a gravitational constant corresponding to the Newtonian limit and c is the speed of light in vacuum. By adding a cosmological term we can write Einstein's field equations in all their glory

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,. \tag{2.50}$$

The left hand side (when indices are raised) is covariantly divergence-free for purely geometric reasons, while the same property for the energy-momentum tensor, on the right hand side of (2.50), was imposed upon it by Einstein's physical reasoning.

Although a nice formula, (2.47) does not, however, provide us with conserved quantities by itself. For that, we need an ordinary divergence. This is the case for the Minkowski spacetime; in Cartesian coordinates

$$\partial_\nu T^{\mu\nu} = 0 \quad \Rightarrow \quad \frac{1}{c}\frac{\partial}{\partial t}\int_V T^{\mu 0}\, \mathrm{d}^3 x = -\int_{\partial V} T^{\mu k}\, \mathrm{d}^2 S_k \,. \tag{2.51}$$

By considering an isolated system, we may take the boundary ∂V to be far enough, so that $T^{\mu k} = 0$ on ∂V. It follows then immediately that energy

$$E \equiv P^0 := \int_V T^{00}\, \mathrm{d}^3 x \,, \tag{2.52}$$

is conserved

$$\frac{\partial E}{\partial t} = 0 \,. \tag{2.53}$$

We could alternatively say that energy is well-defined. The same is true for momentum

$$P^i := \int_V T^{0i}\, \mathrm{d}^3 x \,, \tag{2.54}$$

and angular momentum (with respect to the origin $x = 0$)

$$M^{\mu\nu} := \int_V x^{[\mu} T^{\nu]0}\, \mathrm{d}^3 x \,. \tag{2.55}$$

This is because (4-dimensional) Minkowski spacetime possesses maximum number (10) of Killing vectors (see (2.19) above). The soul of this statement is in the following approach.

We mentioned that the covariant divergence of the energy-momentum does not yield conserved quantities. Let us however see, if there is a way to obtain them, by computing the divergence explicitly:

$$\begin{aligned} 0 = \nabla_\beta \hat{T}_\alpha{}^\beta &= \sqrt{-\mathrm{g}}\,\nabla_\beta T_\alpha{}^\beta = \sqrt{-\mathrm{g}}\,\partial_\beta T_\alpha{}^\beta - \Gamma^\mu{}_{\beta\alpha}\hat{T}_\mu{}^\beta + \Gamma^\beta{}_{\beta\mu}\hat{T}_\alpha{}^\mu \\ &= \partial_\beta \hat{T}_\alpha{}^\beta - \partial_\beta(\sqrt{-\mathrm{g}})T_\alpha{}^\beta + \partial_\mu(\sqrt{-\mathrm{g}})T_\alpha{}^\mu - \Gamma^\mu{}_{\beta\alpha}\hat{T}_\mu{}^\beta \\ &= \partial_\beta \hat{T}_\alpha{}^\beta - \frac{1}{2}\hat{T}^{(\nu\beta)}(\partial_\alpha g_{\nu\beta} + \partial_{[\beta} g_{\nu]\alpha}) \\ &= \partial_\beta \hat{T}_\alpha{}^\beta + \frac{1}{2}\hat{T}^{\nu\beta}\partial_\alpha g_{\nu\beta} \,. \end{aligned} \tag{2.56}$$

where in the second equality we used (A.3a) from the Appendix and Leibniz's rule. The obtained result tells us that if the metric tensor is independent of the coordinate x^α, then we have the conservation of a vector density (α is fixed) $\hat{J}^\beta = \hat{T}_\alpha{}^\beta$. We know from (2.14) that this coincides with the existence of a translational Killing vector. Thus, it is a hint on how to extract conservation laws from isometries.

Consider a curved spacetime (M, g) with a conserved, *symmetric* energy-momentum tensor $T = T_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ and suppose that there exists a Killing vector field $\xi = \xi^\mu \partial_\mu$. Construct a 1-form $J = T(\cdot, \xi) = T_{\mu\nu}\xi^\nu \mathrm{d}x^\mu$ (i.e. energy-momentum in the direction of ξ). The corresponding vector field is $J^\mu = g^{\mu\nu}J_\nu$. Because ξ satisfies Killing equations $\nabla_{(\mu}\xi_{\nu)} = 0$, we have

$$\nabla_\mu J^\mu = \nabla_\mu (T^{\mu\nu}\xi_\nu) = \underbrace{(\nabla_\mu T^{\mu\nu})}_{=0}\xi_\nu + T^{(\mu\nu)}\nabla_{[\mu}\xi_{\nu]} = 0\,. \tag{2.57}$$

Since

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}J^\mu\right)\,, \tag{2.58}$$

we obtain a local conservation law

$$\partial_\mu \hat{J}^\mu = \partial_\mu(\hat{T}^{\mu\nu}\xi_\nu) = 0 \tag{2.59}$$

which can be integrated to yield a globally conserved quantity (recall the motivational example at the beginning of the thesis)

$$Q(\xi) = \int_V \hat{T}^{0\nu}\xi_\nu \;\mathrm{d}^3x\,, \tag{2.60}$$

whose physical interpretation derives from ξ.
Example: The angular momentum (2.55) is constructed from the basis vectors of the Minkowski Killing algebra $\xi = m_{\{\mu\nu\}} = x_\mu\partial_\nu - x_\nu\partial_\mu$ generating rotations. Their components are (μ and ν are fixed)

$$\xi^\alpha = \xi[x^\alpha] = x_\mu\delta^\alpha_\nu - x_\nu\delta^\alpha_\mu$$

and ($x_\mu \coloneqq \eta_{\mu\alpha}x^\alpha$)

$$\xi_\beta = \eta_{\alpha\beta}\xi^\alpha = 2\eta_{\alpha[\mu}\eta_{\nu]\beta}x^\alpha\,.$$

Now we construct a vector $T^{\sigma\beta}\xi_\beta$ and obtain $T^\sigma{}_{[\nu}x_{\mu]}$. Its divergence is zero

$$\partial_\sigma T^\sigma{}_{[\nu}x_{\mu]} \overset{(\partial_\sigma T^\sigma_\nu = 0)}{=} T^\sigma_{[\nu}\eta_{\mu]\alpha}\delta^\alpha_\sigma = T_{\mu\nu} - T_{\nu\mu} = 0 \tag{2.61}$$

if $T_{\mu\nu}$ is symmetric. This yields (after integration, with boundary 'at ∞')

$$\frac{\partial}{\partial t}M^{\mu\nu} = 0\,.$$

We also could've just take a divergence of $x^{[\mu}T^{\nu]\sigma}$ to verify the conservation of angular momentum, but the reason we did it with $T^{\sigma\beta}\xi_\beta$ is because we wish to emphasize that the conserved 'charges' are constructed from symmetries.

Interestingly, it turns out that if the energy-momentum tensor happens to be traceless (i.e. $T^\nu{}_\nu = g^{\mu\nu}T_{\mu\nu} = 0$), conformal Killing vector ζ is sufficient. Using symmetry of $T^{\mu\nu}$, $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_{(\mu}\zeta_{\nu)} = \Omega^2 g_{\mu\nu}$ gives

$$\nabla_\mu(T^{\mu\nu}\zeta_\nu) = T^{\mu\nu}\nabla_\mu\zeta_\nu = T^{(\mu\nu)}\nabla_{(\mu}\zeta_{\nu)} = \Omega^2 T^{\mu\nu}g_{\mu\nu} = T^\nu{}_\nu\,. \tag{2.62}$$

Allow us to show a different way (taken from [15]), related to Noether's theorem, which we are about to discuss very soon. For infinitesimal conformal transformation Φ_ε we have (see Eq. (2.8))

$$g \mapsto \Phi_\varepsilon^* g = g + \varepsilon \mathfrak{L}_\zeta g + \mathcal{O}\Big(\varepsilon^2\Big) = g + \underbrace{\varepsilon \Omega^2 g}_{:= \delta g_{\mu\nu}} + \mathcal{O}\Big(\varepsilon^2\Big) . \tag{2.63}$$

The action representing 'matter' (recall (1.82))

$$S_M[g_{\mu\nu}, \psi] = \int_\mathcal{D} \mathcal{L}_M \, \mathrm{d}^4 x \quad \text{such that} \quad \frac{\delta S_M}{\delta g_{\mu\nu}} = - \int_\mathcal{D} \frac{1}{2} T^{\mu\nu} \sqrt{-\mathrm{g}}\, \delta g_{\mu\nu} \, \mathrm{d}^4 x \tag{2.64}$$

changes (is varied) correspondingly

$$S_M[g_{\mu\nu} + \delta g_{\mu\nu}, \psi] = S_M[g_{\mu\nu}, \psi] - \frac{\varepsilon}{2} \int_\mathcal{D} \Omega^2 \underbrace{g_{\mu\nu} T^{\mu\nu}}_{= T^\nu{}_\nu} \sqrt{-\mathrm{g}} \, \mathrm{d}^4 x . \tag{2.65}$$

If we wish that the form of the action S_M remains unchanged under (infinitesimal) conformal transformation, the integral on the RHS of (2.65) must vanish and because $\Omega^2 > 0$ is arbitrary, it must be $T^\nu{}_\nu = 0$. The associated conserved global quantity is then

$$Q(\zeta) = \int_V \hat{T}^{0\nu} \zeta_\nu \, \mathrm{d}^3 x . \tag{2.66}$$

2.3 Intermezzo: Principles

This section serves as a short provider of some terminology. It will be useful for the next section where we present the Noether's theorem and our discussion of impossibility to localize energy-momentum of the gravitational field after introducing pseudotensors in Chapter 3.

2.3.1 General covariance

Principle of general covariance (the 'no prior geometry' demand [7]) which Einstein proposed for his general relativity, states that the general laws of nature must be expressed in the same form for all observers, regardless of their (accelerated) motion and coordinate system. More specifically, the laws of nature are invariant with respect to any coordinate transformation[7].

For a spacetime (M_4, g), two coordinate systems $\{x^\mu\}$ and $\{y^\alpha\}$ are invertibly connected via coordinate presentation $\widetilde{F}$ of a diffeomorphism $F : M_4 \to M_4$ (see diagram (1.2)). This sets 4 functions $y^\alpha(x^\mu)$ characterizing the transformation. Generalization to r dimensions is straightforward. General covariance is thus *diffeomorphism invariance*. General relativity is thus invariant under the action of the Lie group $\mathrm{Diff}(M_4, g)$ which is in this context denoted as $G_{\infty,4}$.

Traversing between different coordinates and restricting oneself to a specific transformation group, suitable for describing physics of the problem under consideration, is our choice, our *gauge*. Coordinate transformations in general relativity are thus *gauge transformations*. In contrast, electromagnetism, for example, has the gauge freedom in the choice of the four-potential (from which we obtain measurable physical quantities), but coordinates remain fixed. Such variations of field variables are called *intrinsic* [9]. This approach is known as an active approach [26]. Invariance of the theory under the change of (the form of) dynamic variables, but not coordinates, is also known as gauge invariance. Representative of such theory is the Yang-Mills theory. Our concern in this thesis is set to extrinsic variations (passive approach [26]) of the field variables, which are generated by coordinate transformations.

Physical quantities are represented by tensors. Diffeomorphisms of (M_4, g) induce transformations of tensors by pull-backs and push-forwards. As we know from Subsection 1.1.1, in coordinates tensors transform with invertible Jacobi matrices. Because the transformations and coordinates are arbitrary, Jacobi matrices are elements of the Lie group $\mathrm{GL}(4, \mathbb{R})$. In special cases these can be elements of the smaller subgroup (e.g. special relativity). The symmetry group of general relativity, the group that preserves the theory, is thus $\mathrm{GL}(4, \mathbb{R})$.

Every flow Φ_t of any vector field $X \in \mathfrak{X}(M, g)$ is a diffeomorphism and there are infinitely many vector fields on M_4. We can thus regard $\mathrm{GL}(4, \mathbb{R})$ (or equivalently $G_{\infty,4}$) being generated by all vector fields of the spacetime manifold.

[7] Of course, for every spacetime configuration there exists a coordinate system in which it is easier to work than in others (e.g. spherical symmetry). Trautman [25] makes a distinction between three (not exactly equivalent) formulations of general covariance: a) All coordinate systems are equally good for stating the laws of physics and they should be treated on the same footing; b) The equations of physics should have tensorial form; c) The equations of physics should have athe same form in all coordinate systems.

2.3.2 The equivalence principle and gravitational energy

It is obvious (from Section 2.2) that the ordinary divergence of the energy-momentum tensor $\partial_\nu T^{\mu\nu}$ is in general different from zero in spacetime regions with gravitational fields. However, the equivalence principle tells us that at any point of the spacetime manifold we can choose normal Riemannian coordinates (such that the connection coefficients $\Gamma^{\alpha}{}_{\beta\gamma}$ vanish and the metric tensor g reduces to the Minkowski metric η), or in different words, we can fall freely. This is consistent with the principle of general covariance, but, as a consequence, it is not meaningful to speak of a localized energy density for gravitational fields [24]. If we try to examine $\partial_\nu T^{\mu\nu}$ at any specific point of (M, g) equivalence principle implies it must be zero, because we are in a flat Minkowski spacetime.

In regions of spacetime near gravitating sources, where the Riemann curvature is non-vanishing, there is failure of a principle of local energy conservation. This may not surprise us if we imagine a dynamical system in which a part of energy is radiated by gravitational waves. In addition, the energy balance locally cannot be discussed independently of the coordinates one uses to calculate it, and consequently different results are obtained in various different coordinate frames. However, though local energy conservation fails, there is a large scale principle of energy-momentum conservation in the general theory. Goldberg [6] states that only the global energy-momentum and angular momentum may be given meaning in general relativity. An example of a situation in which that may be done is a closed system which at large distances, in either spacelike or null directions, approaches Minkowski spacetime and the symmetry group may be restricted to the Poincaré group. Soon after Hilbert's discovery of the variational principle which gives the field equations of general relativity (see Section 1.2.1), the failure of local energy conservation was a problem that concerned people at that time, among them Hilbert, Klein and Einstein. Noether's theorems, in principle, solved this problem [24].

Unhappily, enormous time and effort were devoted in the past to trying to find a magic formula for 'local gravitational energy-momentum'. Misner et al. [7] state that nobody can deny or wants to deny that gravitational forces make a contribution to the mass-energy of a gravitationally interacting system. At issue is not the existence of gravitational energy, but the localizability of gravitational energy. It is not localizable. The equivalence principle forbids.

2.4 Noether's theorem

The following introduction of the theorem is based on the book of Kosmann-Schwarzbach [27][8]. We also drew a lot of inspiration from a very comprehensive Nina Byers' review [24].

Two theorems and their converses, on the connection between symmetries and conservation laws in physics, are collectively referred to as Noether's theorem. In her paper '*Invariante Variationsprobleme*', published in 1918, Emmy Noether considers variational problems permitting an invariance under a continuous Lie group action and proves consequences that are implied for the associated differential equations.

In contemporary terminology, general relativity is a gauge theory whose symmetry group (gauge group) form all continuous coordinate transformations with continuous derivatives, often called the group of general coordinate transformations (see previous section). It is a Lie group that has a continuously infinite number of independent infinitesimal generators. In Noether's terminology such a group is an infinite continuous group. The symmetry group of special relativity, the Poincaré group, is a Lie subgroup of the group of general coordinate transformations (i.e. special relativity is Poincaré invariant). It has a finite number (10) of independent infinitesimal generators. Noether refers to such a group as a finite continuous group[9]. This distinction between a Lie group G_r with a finite (or countably infinite) number of independent infinitesimal generators and an infinite continuous group $G_{\infty,r}$ is what distinguishes Noether's first and second theorem. Theorem I applies when one has a finite continuous group of symmetries, and theorem II when there is an infinite continuous group of symmetries. Field theories with a finite continuous symmetry group have what Hilbert called 'proper energy theorems'. Physically in such theories one has a *localized*, conserved energy density; and one can prove that in any arbitrary volume the net outflow of energy across the boundary is equal to the time rate of decrease of energy within the volume. As will be shown below, this follows from the fact that the energy-momentum tensor of the theory is divergence free. In general relativity, on the other hand, it has no meaning to speak of a definite localization of energy. One may define a quantity which is divergence free analogous to the energy-momentum density tensor of special relativity, but it is gauge dependent: i.e., it is not covariant under general coordinate transformations. Consequently the fact that it is divergence free does not yield a meaningful law of local energy conservation. Thus one has, as Hilbert saw it, in such theories 'improper energy theorems'.

[8] The book comprises translation of the original Noether's paper, generalizations of her approach and contains interesting facts on how the scientific community acknowledged her contribution during the 20th century, so we fully recommend it.

[9] To clarify the (Noether's) terminology, we say that the (transformation) group is a *finite continuous* G_r when its transformations can be expressed in a general form which depends analytically on r essential parameters ε. In the same way, one speaks of an *infinite continuous group* $G_{\infty,r}$ for a group whose most general transformations depend on r essential arbitrary functions $y(x)$ and their derivatives in a way that is analytical or at least continuous and continuously differentiable a finite number of times. An intermediate case is the one in which the groups depend on an infinite number of parameters but not on arbitrary functions [27].

Before we proceed to technicalities of the theorem, allow us to show a very important identity for the Lagrangian density. By taking a Lie derivative of $\mathcal{L} = \sqrt{-\mathrm{g}}\,L$ along ξ we find

$$\mathfrak{L}_\xi \mathcal{L} = \partial_\mu \left(\mathcal{L}\xi^\mu\right) . \tag{2.67}$$

To prove this, we need to use formulae (A.3b) and (A.3c) from the Appendix A and a rule for computing a Lie derivative with covariant derivatives (1.46). Indeed, using the mentioned formulae and $\nabla_\mu \sqrt{-\mathrm{g}} = 0$ yields (here, ξ is in general not a Killing vector)

$$\begin{aligned}
\mathfrak{L}_\xi \mathcal{L} &= \mathfrak{L}_\xi(\sqrt{-\mathrm{g}}\,L) = L\mathfrak{L}_\xi\sqrt{-\mathrm{g}} + \sqrt{-\mathrm{g}}\,\mathfrak{L}_\xi L \\
&\overset{\text{(A.3c)}}{=} \frac{1}{2}L\sqrt{-\mathrm{g}}\,g^{\mu\nu}\mathfrak{L}g_{\mu\nu} + \sqrt{-\mathrm{g}}\,\xi^\mu\nabla_\mu L \\
&\overset{\text{(1.46)}}{=} \frac{1}{2}L\sqrt{-\mathrm{g}}\,g^{\mu\nu}(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu) + \sqrt{-\mathrm{g}}\,\xi^\mu\nabla_\mu L \\
&= \sqrt{-\mathrm{g}}\,L\nabla_\mu\xi^\mu + \sqrt{-\mathrm{g}}\,\xi^\mu\nabla_\mu L = \nabla_\mu\left(\mathcal{L}\xi^\mu\right) \overset{\text{(A.3b)}}{=} \partial_\mu\left(\mathcal{L}\xi^\mu\right) ,
\end{aligned}$$

what was to be shown.

The following technical description of Noether's theorem is done in our own way and it is based on a combination of multiple approaches: A textbook of Petrov et al. [9] contains important identities and detailed derivations; a review of Goldberg [6] has a quick and pragmatic exposition of the theorem and we used some notions from Fecko [15] which are mainly contained in the mathematical Chapter 1.

For any field theory whose field equations are derivable from an action integral defined over spacetime, invariance of its action with respect to a Lie group leads to a differential conservation law for each parameter of the group whenever the resulting field equations are satisfied. Invariance of its action with respect to $G_{\infty,n}$ leads to a set of differential identities for the field equations themselves — one for each independent transformation in the group. When the field equations are satisfied, these identities yield conservation laws.

Consider a field theory with a set of variables ψ^A, where the index A represents both tensorial character (abstract indices) of the variables and their numeration, whose field equations are derivable by the variation of the action

$$S[\psi^A] = \int_{\mathcal{D}} \mathcal{L}(\psi^A, \partial_\mu\psi^A)\ \mathrm{d}^4x, \quad \delta S = 0 \quad \Leftrightarrow \quad \frac{\delta\mathcal{L}}{\delta\psi^A} = 0\,. \tag{2.68}$$

where the Lagrangian depends on fields and their first derivatives[10]. Consider an action of a Lie group on (M, g) in the form of an infinitesimal 1-parameter group of diffeomorphisms Φ_ε (an infinitesimal flow), generated by ξ which is zero on the boundary $\partial\mathcal{D}$, so that $\Phi_\varepsilon(\mathcal{D}) = \mathcal{D}$. The change of field variables is given by their Lie derivative (recall (1.34))

$$\psi'^A := \Phi^*_\varepsilon\psi^A = \psi^A + \varepsilon\mathfrak{L}_\xi\psi^A + \mathcal{O}\left(\varepsilon^2\right) \tag{2.69}$$

[10] We will at least mention how would obtained results look for Lagrangians depending on second derivatives of the fields. We know from Subsection 1.2.1 that the Einstein Lagrangian for the gravitational field, with only first derivatives of the metric, gives identical field equations as the Hilbert Lagrangian which contains second derivatives of the metric.

Now we can ask: What are the consequences if the form of the action S does not change under the action of the flow Φ_ε, i.e.

$$S[\Phi_\varepsilon^*\psi^A] = S[\psi^A]\,. \tag{2.70}$$

Since the action is altered (varied) as

$$S[\Phi_\varepsilon^*\psi^A] = S[\psi^A + \varepsilon\mathfrak{L}_\xi\psi^A] = S[\psi] + \varepsilon\frac{\delta S}{\delta\psi^A} + \int_{\mathcal{D}}\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^A)}\mathfrak{L}_\xi\psi^A\right)\omega_g + \mathcal{O}\big(\varepsilon^2\big)$$

where (review the end of the Section 1.2)

$$\frac{\delta S}{\delta\psi^A} = \int_{\mathcal{D}}\frac{\delta\mathcal{L}}{\delta\psi^A}\mathfrak{L}_\xi\psi^A\omega_g\,, \tag{2.71}$$

we see then, that the action functional remains the same if

$$\frac{\delta\mathcal{L}}{\delta\psi^A}\mathfrak{L}_\xi\psi^A + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^A)}\mathfrak{L}_\xi\psi^A\right) = 0\,. \tag{2.72}$$

This is just the Lie derivative of the Lagrangian $\mathcal{L}$ (see (1.73)). We also showed that (proof is under (2.67))

$$\mathfrak{L}_\xi\mathcal{L} = \partial_\mu(\mathcal{L}\xi^\mu)\,.$$

Therefore, we can write

$$\mathfrak{L}_\xi\mathcal{L} - \partial_\mu(\mathcal{L}\xi^\mu) = \partial_\alpha\hat{B}^\alpha \equiv 0 \tag{2.73}$$

where $\hat{B}^\alpha \neq 0$ is some divergenceless vector density[11]. Hence, merging (2.72) and (2.4) brings us to an identity

$$\frac{\delta\mathcal{L}}{\delta\psi^A}\mathfrak{L}_\xi\psi^A + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^A)}\mathfrak{L}_\xi\psi^A - \mathcal{L}\xi^\mu - \hat{B}^\mu\right] \equiv 0 \tag{2.74}$$

which is usually called *the main Noether identity* [9]. The 'mysteriously' added field B^μ can be very useful. Its particular choice depends on the specific physical problem. We will utilize it later in Section 2.4.3 when symmetrization of the canonical energy-momentum tensor will be needed.

Let us examine what the main Noether identity (2.74) tells us. We see that if the first term vanishes, we obtain a conserved Noether current

$$\partial_\mu\hat{J}^\mu = 0 \tag{2.75}$$

where

$$\hat{J}^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^A)}\mathfrak{L}_\xi\psi^A - \mathcal{L}\xi^\mu\,, \tag{2.76}$$

[11] Such solenoidal field can be chosen in the form of a divergence of some antisymmetric quantity $\hat{W}^{\alpha\beta}$ [9]. Of course, one could put a divergence which does not vanish, but due to Gauss's (Stokes) theorem, the action would differ only by a boundary term. Then the Euler-Lagrange equations would have the same form.

and also a 'corrected' conserved current

$$\partial_\mu(\hat{J}^\mu - \hat{B}^\mu) = 0\,. \tag{2.77}$$

This can happen in three cases[12]: *i*) The field equations $\delta\mathcal{L}/\delta\psi^A$ are satisfied by all variables ψ^A, *ii*) Only some variables satisfy field equations and some are invariant with respect to the action of the group and thus their Lie derivative is zero[13], *iii*) although highly unlikely, all Lie derivatives of ψ^A vanish. That is, either are field equations satisfied by ψ^A or the field variables ψ^A are invariant under the action of ξ from the Lie algebra generating the Lie group which acts on (M,g). In the next section, a possibility *ii*) will play a significant role.

Alternative (similar) derivation is such that one uses the infinitesimal coordinate transformation

$$x^\mu \mapsto x'^\mu = x^\mu + \varepsilon\xi^\mu \tag{2.78}$$

induced by Φ_ε (recall (1.18)). Then one states (and we know it is true from Section 1.2) that the altered action S' (with $\mathcal{L}' \neq \mathcal{L}$) and the original S yield the same field equations

$$\delta S = 0 \quad \Leftrightarrow \quad \delta S' = 0 \tag{2.79}$$

if the associated Lagrangians differ by a divergence

$$\underbrace{\bar{\delta}\mathcal{L}}_{\equiv\varepsilon\mathfrak{L}_\xi\mathcal{L}} := \mathcal{L}'(\psi'^A(x), \partial_\mu\psi'^A(x)) - \mathcal{L}(\psi^A(x), \partial_\mu\psi^A(x)) = \partial_\alpha\hat{B}^\alpha\,. \tag{2.80}$$

where the bar over δ signifies that it is not a variation connected with the variation of the action. The divergence is here taken to be non-zero (see footnote 11). One might, however, run into problems if $\mathcal{L}$ is not a scalar density (remember the derivation of $\mathfrak{L}_\xi\mathcal{L} = \partial_\mu(\mathcal{L}\xi^\mu)$). When the Lagrangian depends only on the field variables and their first derivatives $\mathcal{L}$ may not have simple tensorial character (as tensor density, see Appendix A). We saw this with the Einstein $\Gamma\Gamma - \Gamma\Gamma$ Lagrangian. Nevertheless, it is mostly assumed that such Lagrangian differs from a scalar density by a divergence (as is true for Einstein and Hilbert Lagrangians; Subsection 1.2.1). For such a divergence term $(\partial_\mu\hat{k}^\mu)$, k^μ also depends on the field variables and their first derivatives (Golberg [6]). Then one assumes that the diffeomorphism inducing the coordinate transformation (2.78) is a symmetry of the Lagrangian, i.e. the functional form of the Lagrangians $\mathcal{L}'$ and $\mathcal{L}$ is the same. Thus $\bar{\delta}\mathcal{L} = 0$, which is equivalent to $\mathfrak{L}_\xi\mathcal{L} = 0$. Combining (2.80) and (2.72) one obtains the main Noether identity.

The main Noether identity for Lagrangians depending on second derivatives of ψ^A reads

$$\frac{\delta\mathcal{L}}{\delta\psi^A}\mathfrak{L}_\xi\psi^A + \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\psi^A_{,\mu}}\mathfrak{L}_\xi\psi^A + \frac{\partial\mathcal{L}}{\partial\psi^A_{,\mu\nu}}\mathfrak{L}_\xi\psi^A_{,\nu} - \mathcal{L}\xi^\mu - \hat{B}^\mu\right] \equiv 0 \tag{2.81}$$

where the comma with index denotes partial derivative and

$$\frac{\delta\mathcal{L}}{\delta\psi^A_{,\mu}} := \frac{\partial\mathcal{L}}{\partial\psi^A_{,\mu}} - \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\psi^A_{,\mu\nu}}\right)\,. \tag{2.82}$$

[12]Note that it is being summed over the multi-index A the whole time.

[13]We know from Section 2.1, where we discussed isometries, that the invariance of the metric with respect to the isometry group action is equivalent to vanishing of its Lie derivative along the Killing vector. Generalization to other field variables is trivial.

By specifying the transformation group and using the main Noether identity, one can easily prove theorems of Noether. Because we are mainly interested in superpotentials, a consequence of the $G_{\infty,r}$ invariance (next subsection), we are only going to briefly summarize how is the main identity used in the two theorems (detailed discussion can be found in [9]).

In the case of a finite r-parameter Lie group G_r one has r linearly independent generators from the associated Lie algebra. This gives r linearly independent identities relating equations of motion to a conserved Noether current (this is the first Noether's theorem). As an example, consider 10 Killing vectors of the Minkowski spacetime (2.19) as generators of the Poincaré group. These yield 10 identities and thus 10 conserved currents. These can be integrated (as in the motivational example at the beginning of the thesis) to give us globally conserved quantities. Goldberg [6] speaks of such invariant transformation with constant parameters as an element of a *global* Lie group.

A local gauge group, an infinite Lie group, can be obtained from a finite group G_r by replacing the constant parameters by r continuous fields. Then $G_r \to G_{\infty,r}$. Such a procedure is called the group localization [9]. For instance, as we explained in detail in previous section, the group of all coordinate transformations $x^\mu \mapsto x'^\alpha(x^\mu)$ in spacetime is a Lie group $G_{\infty,4}$. The second Noether theorem can then formulated as follows [9]:

If the action functional S is invariant with respect to transformations of a Lie group $G_{\infty,r}$ parametrized by r differentiable fields and their derivatives up to the order k, then there exist r identical relations between the Lagrangian derivatives and derivatives from them up to the order k.

A detailed proof can be found in [9][14], so we are going to supply only the idea of it. We restrict ourselves to $k = 1$ for simplicity. From the formulae (1.45) for computing the Lie derivative we take the r components of $\xi = \xi^\mu(x)\partial_\mu$ to be r differentiable fields and we introduce the coefficients $\psi^A|^\alpha_\beta$ defined by the Lie derivative (see formula (1.45))

$$\mathfrak{L}_\xi \psi^A = \xi^\mu \partial_\mu \psi^A + \psi^A|^\alpha_\beta \partial_\alpha \xi^\beta \,. \tag{2.83}$$

Substituting this into the (second) main Noether identity (2.81) results in

$$\left[\frac{\delta \mathcal{L}}{\delta \psi^A}\partial_\mu \psi^A - \partial_\alpha \left(\psi^A|^\alpha_\mu \frac{\delta \mathcal{L}}{\delta \psi^A}\right)\right]\xi^\mu + \partial_\mu \left(\hat{J}^\mu - \hat{B}^\mu\right) \equiv 0\,, \tag{2.84}$$

where the current is

$$\hat{J}^\mu = \frac{\delta \mathcal{L}}{\delta \psi^A_{,\mu}}\mathfrak{L}_\xi \psi^A + \frac{\delta \mathcal{L}}{\delta \psi^A_{,\mu\nu}}\mathfrak{L}_\xi \psi^A_{,\nu} - \left(\mathcal{L}\delta^\mu_\alpha + \psi^A|^\mu_\alpha \frac{\delta \mathcal{L}}{\delta \psi^A}\right)\xi^\alpha \,. \tag{2.85}$$

The identity (2.84) can be integrated. After applying Gauss's theorem and assuming that ξ and its derivatives vanish on the boundary one obtains an integral equation from which follows [9] (due to arbitrariness of $\xi^\mu(x)$) a set of r differential equations

$$\frac{\delta \mathcal{L}}{\delta \psi^A}\partial_\mu \psi^A - \partial_\alpha \left(\psi^A|^\alpha_\mu \frac{\delta \mathcal{L}}{\delta \psi^A}\right) \equiv 0\,, \tag{2.86}$$

[14] Notation in the book for this proof is different and the Lie derivative is used later in the book, when the theorem is being exploited, so we simply merged the two notations. Also note that the sign convention for the Lie derivative is different therein.

which prove the second Noether's theorem.
To see an application of the theorem, consider the Hilbert action (1.74). Due to the general covariance, it is invariant under the action of the group $G_{\infty,4}$, so by using (2.86) for $\psi^A = g^{\mu\nu}$ and

$$\frac{\delta \mathcal{L}_H}{\delta g^{\mu\nu}} = \frac{1}{2\kappa} \hat{G}_{\mu\nu}$$

we have

$$\hat{G}_{\mu\nu}\partial_\alpha g^{\mu\nu} - \partial_\beta \left(g^{\mu\nu}|_\alpha^\beta \hat{G}_{\mu\nu} \right) \equiv 0\,. \tag{2.87}$$

Since

$$g^{\mu\nu}|_\alpha^\beta = -2g^{\beta(\nu}\delta_\alpha^{\mu)} \quad \text{and} \quad \partial_\alpha g^{\mu\nu} = -2g^{\beta(\nu}\Gamma^{\mu)}{}_{\alpha\beta} \tag{2.88}$$

it follows from symmetry of $\hat{G}_{\mu\nu}$ that

$$-\hat{G}_\mu{}^\beta \Gamma^\mu{}_{\alpha\beta} + \partial_\beta \hat{G}_\alpha{}^\beta \equiv 0 \tag{2.89}$$

After using $\partial_\beta\sqrt{-\mathrm{g}} = \sqrt{-\mathrm{g}}\Gamma^\mu{}_{\mu\beta}$ ((A.3a) in Appendix A) we get well-known contracted Bianchi identities

$$\nabla_\beta G_\alpha{}^\beta \equiv 0\,. \tag{2.90}$$

When Einstein field equations are being solved, these four equations represent *constraints*, leading only to 6 independent equations. Because of this correspondence with the Hilbert Lagrangian, the identities (2.86) are called *generalized Bianchi identities* .

Remarkably, for both theorems there is a converse and so from a physicist's point of view: for every conservation law there exists a symmetry (of S).

To complete the terminology, Bergmann [28] introduced *weak* and *strong* conservation laws. Weak ones are those which are obtained when the field equations hold, while the strong conservation law can be acquired without satisfying field equations. Very often, especially in quantum field theory, one speaks of *on-shell* and *off-shell* conditions. This means that if the configurations of a physical system satisfy equations of motion, then they are called on-shell, and if not, off-shell.

2.4.1 An assertion of Hilbert and superpotentials

For our purposes, it is the following statement of Noether which she introduces in her paper as 'an assertion of Hilbert', often referred as the third Noether's theorem, that we will need.

If the action S is invariant under an infinite continuous group of transformations $G_{\infty,r}$, then a quantity constructed from the Lagrangian derivatives is expressed through a double divergence of a special quantity, the so-called superpotential.

Let us use the identity (2.84) from the proof of the second theorem. Since the bracket proportional to ξ^μ is identically zero because of (2.86) it must be

$$\partial_\mu(\hat{J}^\mu - \hat{B}^\mu) \equiv 0 \tag{2.91}$$

for all functions ξ^μ in the current (2.85). Therefore, the expression in parentheses must vanish up to a curl

$$\hat{J}^\mu - \hat{B}^\mu = \partial_\nu \hat{U}^{\mu\nu} \tag{2.92}$$

where the *superpotential* $\hat{U}^{\mu\nu} = -\hat{U}^{\nu\mu}$ is completely antisymmetric. Since $\hat{B}^\mu$ itself is defined only up to a curl, superpotential is not uniquely defined. Whatever the definitions, superpotential satisfies the *strong* conservation law

$$\partial_\alpha \partial_\beta \hat{U}^{\alpha\beta} \equiv 0 \,. \tag{2.93}$$

It is an identity and it does not depend on the dynamics of the theory since no field equations were used to obtain it. It carries no physical information [6].

Recall, however, our motivational example at the beginning of the thesis where we expressed charge as a surface integral of $\hat{F}^{0i}$. For that, we used Maxwell's equations only on the timelike part σ of the boundary $\partial\Omega$ of the 4-dimensional domain of integration Ω. The same procedure can be applied herein, but now we are integrating Eq. (2.92) with $\hat{J}^\mu$ given in (2.85). The field equations are contained within the current $\hat{J}^\mu(\xi)$ and if we use them as mentioned, then we obtain relations (for brevity, we take Lagrangians independent of second derivatives of the fields):

$$Q_{V_1} - Q_{V_1} = \int_\sigma \left(\frac{\partial \mathcal{L}}{\partial \psi^A_{,\mu}} \mathfrak{L}_\xi \psi^A - \mathcal{L}\xi^\mu - \hat{B}^\mu \right) \mathrm{d}^3\Sigma_\mu \,, \tag{2.94}$$

$$Q_V(\xi) \coloneqq \oint_{\partial V} \hat{U}^{0i}(\xi) \, \mathrm{d}^2 S_i \,. \tag{2.95}$$

If the flux integral over σ vanishes, as it would [6] with appropriate boundary conditions at spatial infinity, Q_V is a constant of motion. Notably, this statement is dynamical because the field equations are used to obtain (2.94) and the feasible vanishing of the of the flux integral depends on the behaviour of the fields being considered. Because of the strong conservation law (2.93), calculation of the conserved charge can be based solely on our knowledge of asymptotics of the field. *One need not know the field everywhere and the field equations need not be satisfied everywhere.* Also note that we still haven't specified the nature of the vector field ξ. This auxiliary vector is also an element of arbitrariness, but it is mostly taken as a (conformal) Killing vector field.

It was Felix Klein who provided the recipe for construction of superpotentials and it is shown in [9] that the superpotential can be expressed as a linear combination

$$\hat{U}^{\alpha\beta}(\xi) = \left(\frac{2}{3} \partial_\lambda \hat{n}_\sigma{}^{[\alpha\beta]\lambda} - \hat{m}_\sigma{}^{[\alpha\beta]} \right) \xi^\sigma - \frac{4}{3} \hat{n}_\sigma{}^{[\alpha\beta]\lambda} \partial_\lambda \xi^\sigma \,, \tag{2.96}$$

where the coefficients are defined as

$$\hat{m}_\sigma{}^{\alpha\beta} \coloneqq \frac{\delta \mathcal{L}}{\delta \psi^A_{,\alpha}} \psi^A|^\beta_\sigma - \frac{\partial \mathcal{L}}{\partial \psi^A_{,\beta\alpha}} \partial_\sigma \psi^A + \frac{\partial \mathcal{L}}{\partial \psi^A_{,\mu\alpha}} \partial_\mu \left(\psi^A|^\beta_\sigma \right) \tag{2.97a}$$

$$\hat{n}_\sigma{}^{\alpha\beta\lambda} \coloneqq \frac{1}{2} \left[\frac{\partial \mathcal{L}}{\partial \psi^A_{,\alpha\beta}} \psi^A|^\lambda_\sigma + \frac{\partial \mathcal{L}}{\partial \psi^A_{,\alpha\lambda}} \psi^A|^\beta_\sigma \right] = \hat{n}_\sigma{}^{\alpha\lambda\beta} \,. \tag{2.97b}$$

It would be rather tedious to show explicitly how one comes to this form of the superpotential, but let us mention that one simply takes the main Noether identity (2.84) (without $\hat{B}^\mu$) and while using the Leibniz's rule, the terms can be divided in such a way that one part is proportional to ξ and the other part is a divergence. The terms appearing in the divergence are proportional to ξ and its first and second derivatives. In addition, they satisfy a system of (Klein-Noether) identities from which one obtains the formula (2.96). For details, we also recommend Sundermeyer's book [29].

2.4.2 Canonical energy-momentum and spin

Our motivational example at the beginning of the thesis illustrated how significant is the existence of the conserved current. We can apply the same integration procedure as in the electrodynamics example to the Noether currents (provided that the field equations hold) to obtain global conserved quantities, charges.

Charges obtained on the basis of the Noether theorem are called *canonical* conserved quantities [9]. Consider a field theory whose action is a functional of the metric tensor $g_{\mu\nu}$ and other field variables ϕ^A (for example EM four-potential A_μ or a scalar field ϕ satisfying the Klein-Gordon equation) and a Lagrangian depends only on fields and their first derivatives. We can then write the main Noether identity (2.74) in the form

$$\frac{\delta \mathcal{L}}{\delta \phi^A} \mathfrak{L}_\xi \phi^A + \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} \mathfrak{L}_\xi g_{\mu\nu} + \partial_\mu (\hat{J}^\mu - \hat{B}^\mu) \equiv 0 \,, \tag{2.98}$$

with the Noether current being

$$\hat{J}^\mu = \frac{\partial \mathcal{L}}{\partial \phi^A_{,\mu}} \mathfrak{L}_\xi \phi^A + \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta,\mu}} \mathfrak{L}_\xi g_{\alpha\beta} - \mathcal{L} \xi^\mu \,. \tag{2.99}$$

It is emphasized by Petrov et al. [9] that the current is indeed a *covariant* vector density. This is evident for Lie derivatives of the fields, but other terms can also be rewritten with covariant derivatives instead of partial ones.

Now we assume $\hat{B}^\mu = 0$ and that the vector ξ is a Killing vector and so $\mathfrak{L}_\xi g = 0$ and $\nabla_\mu \xi_\nu = \nabla_{[\mu} \xi_{\nu]}$ can simplify the expression (2.99). *If the field equations* for ϕ^A *are satisfied* we have $\partial_\mu \hat{J}^\mu_C = 0$ for the *canonical* Noether current

$$\begin{aligned} \hat{J}^\mu_C(\xi) &= \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi^A)} \mathfrak{L}_\xi \phi^A - \mathcal{L}\xi^\mu = \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi^A)} \left(\xi^\alpha \nabla_\alpha \phi^A + \phi^A|^\alpha_\beta \nabla_\alpha \xi^\beta \right) - \mathcal{L} \xi^\mu \\ &= \hat{\Theta}_\mu{}^\nu \xi^\mu + \hat{\sigma}^{\mu\alpha\beta} \nabla_{[\alpha} \xi_{\beta]} \,, \end{aligned} \tag{2.100}$$

where we have defined the *canonical energy-momentum* tensor density

$$\hat{\Theta}_\mu{}^\nu := \frac{\partial \mathcal{L}}{\partial (\nabla_\nu \phi^A)} \nabla_\mu \phi^A - \mathcal{L} \delta^\nu_\mu \tag{2.101}$$

and a *spin* (or *helicity*) tensor density

$$\hat{\sigma}^{\mu\alpha}{}_\beta := \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi^A)} \phi^A|^\alpha_\beta \,. \tag{2.102}$$

Observe that the canonical energy-momentum is basically a 'Hamiltonian'. From definition of spin it is clear that its antisymmetric part in the last two indices must be non-vanishing. Canonical charges are then

$$Q_C(\xi) = \int_V \hat{J}^0_C(\xi) \; \mathrm{d}^3 x \,. \tag{2.103}$$

Now suppose that the Killing vector ξ is a translational one, say ∂_μ. Its components are constants $\xi^\alpha = \delta^\alpha_\mu$ and so the canonical current (2.100) reads

$$\hat{J}^\nu_C = \hat{\Theta}_\mu{}^\nu \tag{2.104}$$

and its conservation (on-shell) gives a conservation of the canonical energy-momentum

$$\partial_\nu \hat{J}^\nu_C = \nabla_\nu \hat{J}^\nu_C = 0 \quad \Rightarrow \quad \nabla_\nu \hat{\Theta}_\alpha{}^\nu = 0\,. \tag{2.105}$$

However, $\hat{\Theta}_{\alpha\beta}$ *is not symmetric* (in general). If we take a look at Eq. (2.61) it is obvious that for conservation of angular momentum we need a symmetric energy-momentum tensor.

To find out why $\hat{\Theta}_{\alpha\beta}$ is not symmetric, consider a Minkowski spacetime with Killing vectors $m_{\{\mu\nu\}}$ generating rotations and boosts (see (2.19)). Let $\xi^\alpha = m^\alpha_{\mu\nu}$ for μ, ν arbitrary, but fixed. We work in Cartesian coordinates now, so covariant derivatives reduce to partial ones and tensor densities to tensors (since $\det \eta = -1$). From conservation of J^ν_C and $\Theta_\alpha{}^\nu$ and $\partial_\mu \partial_\nu \xi_\rho = 0$ for flat spacetime (see (2.28)) we find that

$$0 = \partial_\nu J^\nu_C = \partial_\nu (\Theta_\alpha{}^\nu \xi^\alpha + \sigma^{\nu\alpha\beta} \partial_{[\alpha} \xi_{\beta]}) = \left(-\Theta^{\alpha\beta} + \partial_\nu \sigma^{\nu[\alpha\beta]}\right) \partial_{[\alpha} \xi_{\beta]}$$

deducing from arbitrariness of $\partial_{[\alpha} \xi_{\beta]}$

$$\Theta^{[\alpha\beta]} = \partial_\nu \sigma^{\nu[\alpha\beta]}\,. \tag{2.106}$$

Thus, if there is a source of spin (that is what the divergence of σ tells us), (2.106) implies that a canonical energy-momentum must have a non-vanishing antisymmetric part. This situation can certainly occur in curved spacetimes too. The term $\Theta_\alpha{}^\nu \xi^\alpha$ is associated with the orbital momentum of the system while $\sigma^{\nu\alpha\beta} \partial_{[\alpha} \xi_{\beta]}$ describes the spin (intrinsic angular momentum) of the system [9].

2.4.3 Belinfante symmetrization procedure

Around 1940 Frederik Belinfante pioneered a method of symmetrization of the canonical quantities in field theories in the Minkowski space in his paper [30]. Later, the method was generalized to curved backgrounds where it furnishes conserved quantities in general relativity and in modified theories.

Belinfante noticed that the conserved canonical Noether current is defined up to a solenoidal vector B^μ (we are now in the Minkowski spacetime) which can be always chosen in the form [9]

$$B^\mu = \partial_\alpha \left(b^{\mu\alpha}{}_\beta \xi^\beta\right) \tag{2.107}$$

where ξ is a Killing vector of the Minkowski spacetime and $b^{\mu\alpha}{}_\beta = b^{[\mu\alpha]}{}_\beta$ is defined as

$$b^{\alpha\beta\gamma} := \sigma^{\gamma[\alpha\beta]} + \sigma^{\alpha[\gamma\beta]} - \sigma^{\beta[\gamma\alpha]}\,. \tag{2.108}$$

and it is known as the *Belinfante correction* tensor. It is evident that due to the antisymmetry of $b^{\mu\alpha}{}_\beta$ and flatness ($\partial_\mu \partial_\nu \xi^\alpha = 0$) we identically have $\partial_\mu B^\mu = 0$. The corrected Belinfante current then reads

$$J^\mu_B = J^\mu_C + B^\mu \tag{2.109}$$

and the *symmetric* Belinfante energy-momentum is

$$_B\Theta^\beta_\alpha = {}_C\Theta_\alpha{}^\beta - \partial_\mu b^{\mu\beta}{}_\alpha \tag{2.110}$$

where ${}_C\Theta_\alpha{}^\beta$ is the canonical energy-momentum tensor defined above. Let's prove the symmetry of ${}_B\Theta^\beta_\alpha$. We wish to get rid of the term $\partial_\mu\sigma^{\mu[\alpha\beta]}$ in Eq. (2.106). First notice that $b^{\mu[\nu\rho]} = \sigma^{\mu[\rho\nu]}$:

$$
\begin{aligned}
2b^{\mu[\nu\rho]} = b^{\mu\nu\rho} - b^{\mu\rho\nu} &= \underbrace{\sigma^{\rho[\mu\nu]}}_{A} + \sigma^{\mu[\rho\nu]} - \underbrace{\sigma^{\nu[\rho\mu]}}_{B} - \underbrace{\sigma^{\nu[\mu\rho]}}_{=-B} - \sigma^{\mu[\nu\rho]} + \underbrace{\sigma^{\rho[\nu\mu]}}_{=-A} \\
&= \sigma^{\mu[\rho\nu]} - \sigma^{\mu[\nu\rho]} = 2\sigma^{\mu[\rho\nu]}\,.
\end{aligned}
$$

Now we take only the antisymmetric part of (2.110), raise the indices (with $\eta^{\alpha\rho}$) and use the last result with (2.106)

$$
{}_B\Theta^{[\rho\beta]} = \partial_\mu\sigma^{\mu[\rho\beta]} - \partial_\mu b^{\mu[\beta\rho]} = \partial_\mu\sigma^{\mu}{}_{[\rho\beta]} - \partial_\mu\sigma^{\mu}{}_{[\beta\rho]} = 0\,. \tag{2.111}
$$

What was to be verified. Because of this, ${}_B\Theta_{\alpha\beta}$ is called *symmetrized energy-momentum* [9]. Obviously, symmetrized energy-momentum is conserved and the Belinfante current is then

$$
J^\mu_B = {}_B\Theta^\mu_\alpha\xi^\alpha \tag{2.112}
$$

The global quantities are constructed in the same manner as for the canonical current

$$
Q_B(\xi) = \int_V \hat{J}^0_B(\xi)\ \mathrm{d}^3x\,. \tag{2.113}
$$

In previous sections we were working with the *metrical* energy-momentum tensor of matter which we defined as

$$
T_{\mu\nu} := -\frac{2}{\sqrt{-\mathrm{g}}}\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} \quad \text{or} \quad T^{\mu\nu} := \frac{2}{\sqrt{-\mathrm{g}}}\frac{\delta\mathcal{L}_M}{\delta g_{\mu\nu}} \tag{2.114}
$$

and it is automatically symmetric. Diffeomorphism invariance of the Einstein-Hilbert action S_{EH} and contracted Bianchi identities say that if the Einstein field equations hold, then the metrical energy-momentum is conserved. However, if we were to define the *total* metrical energy-momentum tensor for gravity and matter

$$
T^{tot}_{\mu\nu} := -\frac{2}{\sqrt{-\mathrm{g}}}\frac{\delta\mathcal{L}_{EH}}{\delta g^{\mu\nu}} = \frac{1}{\kappa}\left(G_{\mu\nu} - \kappa T_{\mu\nu}\right) \tag{2.115}
$$

it identically *disappears* when the Einsten field equations hold. This is one of the sources of theoretical difficulties when one wishes to construct (canonical) conserved quantities associated with gravitational fields. Application of the Noether's theorem thus requires additional manipulation and introduction of objects called *pseudotensors* [9]. Next chapter provides a short overview of these.

3

A Brief History of Conservation Laws in General Relativity

This chapter is rather informative than technical. Its usefulness is in that it furnishes some historical context of various approaches that have been proposed to define conserved quantities. Many articles in physical journals often refer to works of certain physicists and mathematicians that we will briefly mention.

3.1 Superpotentials and pseudotensors

First a phenomenological definition: Pseudotensor[1] is a quantity $\widetilde{\mathcal{T}}^{\mu\nu}$ that satisfies

$$\partial_\nu \left((-\mathrm{g})^{w/2} (T^{\mu\nu} + \widetilde{\mathcal{T}}^{\mu\nu}) \right) = \partial_\nu \partial_\mu \hat{U}^{\mu\nu} = 0, \quad w = 1, 2, \dots \tag{3.1}$$

where $T^{\mu\nu}$ is the energy-momentum tensor residing on the RHS of Einstein field equations and $\hat{U}^{\mu\nu}$ is the superpotential associated with $\widetilde{\mathcal{T}}^{\mu\nu}$. Pseudotensor $\widetilde{\mathcal{T}}^{\mu\nu}$ depends on the metric and its derivatives and is meant to represent the energy-momentum of the gravitational field such that when combined with matter energy-momentum we get a conservation law which can be integrated to obtain global charges. In describing various (historical) constructions of pseudotensors and superpotentials in the following few pages, we follow Petrov et al. [9], Goldberg [6] and Trautman [25].

3.1.1 Canonical approach, Einstein pseudotensor

Take the strong conservation law (2.91) from the previous chapter without $\hat{B}^\mu$ and with the current (2.85) for first order Lagrangians

$$\hat{J}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^A)} \mathfrak{L}_\xi \psi^A - \mathcal{L}\xi^\mu - \frac{\delta \mathcal{L}}{\delta \psi^A} \psi^A|^\mu_\alpha \xi^\alpha \,. \tag{3.2}$$

Now consider the Einstein Lagrangian $\mathcal{L}_E$ (1.84) and take ξ as a translational vector ∂_λ (it need not be a Killing vector). Its components are $\xi^\alpha = \delta^\alpha_\lambda$ and the Jacobi matrix of the transformation $x' = x + \xi$ is an identity matrix. The form of the Lagrangian $\mathcal{L}_E$, which does not depend on coordinates explicitly, will thus stay the same under this (global) transformation[2] and thus, the action S_E remains

[1]Pseudotensors are designated this way because they don't transform as tensors (or tensor densities) under general coordinate transformations. That is what the tilde ('$\sim$') above $\mathcal{T}$ signifies. They mostly have tensorial character under Poincaré (linear, orthogonal) transformations. Sometimes, pseudotensor is called an energy-momentum complex instead, because the notion 'pseudotensor' is also used for a quantity that might change only its sign under coordinate transformations.

[2]Also, non-tensorial parts of the transformation of the Christoffel symbols are zero.

the same, so we can use the Noether's theorem. For $\psi^A = g^{\alpha\beta}$ and $\mathcal{L} = \mathcal{L}_E$ the current (3.2) then reads

$$\begin{aligned}\hat{J}^\mu &= -\frac{\delta\mathcal{L}_E}{\delta g^{\alpha\beta}} g^{\alpha\beta}|^\mu_\lambda + \frac{\partial\mathcal{L}_E}{\partial(\partial_\mu g^{\alpha\beta})}\partial_\lambda g^{\alpha\beta} - \mathcal{L}_E\delta^\mu_\lambda \\ &= -\frac{1}{2\kappa}\hat{G}_{\alpha\beta}g^{\alpha\beta}|^\mu_\lambda + {}_E\widetilde{\mathcal{T}}_\lambda{}^\mu = \frac{1}{\kappa}\hat{G}_\lambda{}^\mu + {}_E\widetilde{\mathcal{T}}_\lambda{}^\mu \end{aligned} \tag{3.3}$$

where we used (2.88) and

$${}_E\widetilde{\mathcal{T}}_\lambda{}^\mu := \frac{\partial\mathcal{L}_E}{\partial(\partial_\mu g^{\alpha\beta})}\partial_\lambda g^{\alpha\beta} - \mathcal{L}_E\delta^\mu_\lambda \tag{3.4}$$

is the *Einstein pseudotensor* which is defined as the canonical energy-momentum tensor (see (2.101)). From the Noether's theorem we have a strong conservation law

$$\partial_\mu\left(\frac{1}{\kappa}\hat{G}_\lambda{}^\mu + {}_E\widetilde{\mathcal{T}}_\lambda{}^\mu\right) \equiv 0\,. \tag{3.5}$$

To give a physical meaning to this identity we must use the Einstein field equations $\hat{G}_\lambda{}^\mu = \kappa\hat{T}_\lambda{}^\mu$. Then we have a weak conservation law

$$\partial_\mu\left(\hat{T}_\lambda{}^\mu + {}_E\widetilde{\mathcal{T}}_\lambda{}^\mu\right) = 0\,. \tag{3.6}$$

In vacuum ($\hat{T}_\lambda{}^\mu = 0$) the pseudotensor ${}_E\widetilde{\mathcal{T}}_\lambda{}^\mu$ would be conserved by itself.

We do not have to write the explicit form of the Einstein pseudotensor to see that it is quadratic in first derivatives of the metric tensor and does not contain second derivatives, since $\mathcal{L}_E$ is quadratic in Christoffel symbols. It can be shown that the Einstein pseudotensor is symmetric. Einstein's motivation for its construction was to describe energy of the linearized gravitational waves. Around 1930, Tolman was first who constructed a superpotential associated with ${}_E\widetilde{\mathcal{T}}^{\mu\nu}$. However, it was not antisymmetric explicitly and, almost ten years later, it was Freud who modified the Tolman superpotential and it reads [9]

$${}_F\hat{U}^{\alpha\beta} = \sqrt{-\mathrm{g}}\left(g^{\rho[\alpha}\Gamma^{\beta]}{}_{\rho\sigma} + \delta^{[\alpha}_\sigma g^{\beta]\nu}\Gamma^\mu{}_{\mu\nu} - \delta^\alpha_\sigma\Gamma^{\beta]}{}_{\mu\nu}g^{\mu\nu}\right)\xi^\sigma\,. \tag{3.7}$$

If we want to use the second order Hilbert Lagrangian $\mathcal{L}_H$ in the same manner as we did it for $\mathcal{L}_E$ above, we must use the form of the current given in Eq. (2.85). Splitting it again in two parts, as in (3.3) yields a ('Hilbert') pseudotensor ${}_H\widetilde{\mathcal{T}}_\lambda{}^\mu$ which satisfies the strong conservation law (3.5) and of course the weak one (3.6) too. The associated superpotential can thus be constructed and acquires a simple form [9]

$${}_M\hat{U}^{\alpha\beta} = \sqrt{-\mathrm{g}}\,g^{\mu[\alpha}\Gamma^{\beta]}{}_{\mu\nu}\xi^\nu\,. \tag{3.8}$$

This superpotential was found around 1957 independently by Mitzkevich and Møller and they used different methods in obtaining it. Since both Einstein and Hilbert pseudotensors are defined as 'canonical energy-momentum tensors' they are not symmetric. This hinders one to construct angular momentum of the system (see Subection 2.4.2 and Eq. (2.61)). Furthermore, they are not covariant. Their application is however possible in asymptotically flat spacetimes which we will discuss later. The advantage of ${}_E\widetilde{\mathcal{T}}^{\mu\nu}$ (and $\mathcal{L}_E$) is that it depends only on first derivatives of the metric. If we recall from Section 1.2 that obtaining Einstein vacuum field equations from $\mathcal{L}_E$ we only need boundary conditions (1.80) for the metric, not for its derivatives. This is actually a Dirichlet boundary condition[3]. Since the action functional S_E generates the same field equations for $\mathcal{L} + div$, where the divergence term depends on the metric and its derivatives of arbitrary order [9], we can obtain by the same procedure other pseudotensors. This leads to an infinite number of possible pseudotensors and superpotentials.

The symmetry problem of the above mentioned pseudotensors was first tackled by Papapetrou in 1948. For the Einstein pseudotensor, he used the Belinfante symmetrization procedure which we discussed in Subsection 2.4.3. However, the symmetrization relied on the use of the Minkowski metric as background metric used for raising and lowering indices. Although symmetric, Papapetrou's pseudotensor contains second derivatives of the metric. The associated Freud superpotential is also changed after the symmetrization, and the resulting superpotential is called Papapetrou's. Their relation is

$$ {}_P\hat{U}^{\alpha\beta} = {}_F\hat{U}^{\alpha\beta} + {}_E b^{\alpha\beta}{}_\sigma \xi^\sigma \, , \tag{3.9} $$

where the quantity ${}_E b^{\alpha\beta}{}_\sigma \xi^\sigma$, needed for symmetrization of ${}_E\widetilde{\mathcal{T}}^{\mu\nu}$, is defined in Eq. (2.107).

Up to now, we have discussed canonical approach emerging from the Noether's theorem. There are, however, other ways to obtain a conservation (3.1). We can construct an arbitrary antisymmetric quantity $\hat{u}^{[\alpha\beta]}{}_\sigma$ from the metric and its derivatives and subsequently define a pseudotensor as

$$ \widetilde{\mathcal{T}}_\sigma{}^\alpha = \partial_\beta \hat{u}_\sigma{}^{[\alpha\beta]} - \frac{1}{\kappa}\hat{G}^\alpha_\sigma \, . \tag{3.10} $$

This method leads to ambiguities of conserved quantities. We can reduce these by making reasonable physical assumptions or restricting pseudotensors to a certain criteria, such as symmetry, absence of higher derivatives, etc.

3.1.2 Symmetric Landau–Lifshitz pseudotensor

The last mentioned method was exploited in 1947 by Landau and Lifshitz (LL) and can be found in their book [19]. Their derivation of the pseudotensor is based on the existence of coordinates in which $g_{\mu\nu}$ is not $\eta_{\mu\nu}$, but all partial derivatives of $g_{\mu\nu}$ vanish. Since

$$ \partial_\mu \sqrt{-\mathrm{g}} = \frac{\sqrt{-\mathrm{g}}}{2} g^{\rho\sigma} \partial_\mu g_{\rho\sigma} \, , $$

from our calculation (2.56) it follows that $\partial_\nu T^{\alpha\beta} = 0$. This is interpreted as an identity and, therefore, $T^{\alpha\beta}$ should be expressible as a divergence of an antisymmetric quantity. After using Einstein field equations and some manipulation with the expressions involved, LL propose two quantities by relations [19]

$$ \begin{aligned} \partial_\rho h^{\mu\nu\rho} &= (-\mathrm{g}) T^{\mu\nu} \, , \\ h^{\mu\nu\rho} &= \partial_\sigma H^{\mu\nu\rho\sigma} = -h^{\mu\rho\nu} \, , \\ H^{\mu\nu\rho\sigma} &= \frac{1}{\kappa}(-\mathrm{g}) g^{\mu[\nu} g^{\rho]\sigma} \, . \end{aligned} $$

[3]For a set of (partial) differential equations which are being solved to find a function f on a domain $\Omega \subset \mathbb{R}^n$, a Dirichlet boundary condition sets the function f to specific values on the boundary of Ω; i.e. $f(x) = h(x)$, $\forall x \in \partial\Omega$, where h is some prescribed function on $\partial\Omega$.

We can notice that the quantity $H^{\mu\nu\rho\sigma}$ has a similar form as a Riemann tensor for maximally symmetric spaces in Eq. (2.43) and it has the same (anti)symmetries. If we transform in some other coordinate system, where $\partial_\rho g_{\mu\nu} \neq 0$, the difference between the divergence of $h^{\mu\nu\rho}$ and $T^{\mu\nu}$ will be, in general, different from zero. That is how the LL pseudotensor is defined:

$$
{}_{LL}\widetilde{\mathcal{T}}^{\mu\nu} := \frac{1}{(-\mathrm{g})}\partial_\rho h^{\mu\nu\rho} - T^{\mu\nu} \quad \Rightarrow \quad \partial_\nu (T^{\mu\nu} + {}_{LL}\widetilde{\mathcal{T}}^{\mu\nu}) \equiv 0\,. \tag{3.11}
$$

The LL pseudotensor turns out to be symmetric and this allows one to define angular momentum of the system. One can show that the +2 weight is the reason for this (i.e. multiplication by $(-\mathrm{g})$, see Appendix A). In addition, it is constructed form metric and its first derivatives only. Because in the absence of the gravitational field ${}_{LL}\widetilde{\mathcal{T}}^{\mu\nu}$ vanishes and in the integral form, conservation law (3.11) reduces to the conservation of the four-momentum, LL identify it ${}_{LL}\widetilde{\mathcal{T}}^{\mu\nu}$ as an energy-momentum pseudotensor of the gravitational field. Although not a tensor, it behaves as a tensor under Poincaré transformations. Thus, it is amenable to asymptotically flat spacetimes. A rather lengthy expression for ${}_{LL}\widetilde{\mathcal{T}}^{\mu\nu}$ can be found in [19].

3.1.3 The Komar superpotential

Shortly after Bergmann [31] had shown that there are infinitely many conservation laws, the first *covariant* expression for the superpotential was found by Arthur Komar in 1959 within his paper 'Covariant Conservation Laws in General Relativity' [32], in which he builds on works of Møller and Bergmann. The Komar superpotential reads[4]

$$
\hat{K}^{\alpha\beta} = \frac{2}{\kappa}\sqrt{-\mathrm{g}}\,\nabla^{[\beta}\xi^{\alpha]}\,, \tag{3.12}
$$

where ξ is an *arbitrary* vector field generating the invariant transformation of the gravitational field action. We can see that if it is a displacement vector generating translation ($\xi^\mu = const.$), then the Komar superpotential is proportional to the Møller superpotential (3.8).

Goldberg [6] stresses that in all cases, one wishes to choose the superpotential so that when ξ is the timelike Killing vector of the Schwarzschild solution, the constant of the motion is mass. Let us see how this works for the Komar superpotential in spherical coordinates $(x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$. We want to evaluate the integral

$$
E := \oint_{\partial V} \hat{K}^{0i}\, \mathrm{d}^2 S_i = \int_0^{2\pi}\int_0^{\pi} K^{01} r^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi\,, \tag{3.13}
$$

where we used that ∂V is given as a surface $r = const.$ and $\sqrt{-\mathrm{g}} = r^2\sin\theta$. Since $\xi^\beta = \delta^\beta_0$ we have

$$
\begin{aligned}
\kappa K^{01} = 2\nabla^{[1}\xi^{0]} = 2g^{\sigma[1}\Gamma^{0]}{}_{\sigma 0} &= g^{\kappa[0}g^{1]\rho}(\partial_0 g_{\rho\sigma} + \partial_\sigma g_{0\rho} - \partial_\rho g_{\sigma 0}) \\
&= g^{\kappa[0}g^{1]\rho}(\partial_\sigma g_{0\rho} - \partial_\rho g_{\sigma 0})\,,
\end{aligned} \tag{3.14}
$$

[4] Our sign convention for the Komar superpotential is such that the mass-energy need not be defined with a minus in front of the integral.

where in the last equality we used the fact that the Schwarzschild metric is static. It is also diagonal (in spherical coordinates) and $g^{00}g^{11} = -1$; then after taking indices $\sigma = 1$ and $\rho = 0$ in (3.14) we obtain

$$K^{01} = -\frac{1}{2\kappa}\partial_1 g_{00} = \frac{1}{2\kappa}\partial_r\left(1 - \frac{2GM}{c^2 r}\right) = \frac{Mc^2}{8\pi r^2}\,. \tag{3.15}$$

where we recovered $\kappa = 8\pi G/c^4$. Doing the same calculation with indices $\sigma = 0$ and $\rho = 1$ in Eq. (3.14) we obtain again $Mc^2/8\pi r^2$. Hence, after collecting both results and substituting them into the integral (3.13), we indeed obtain the mass-energy of the Schwarzschild black hole:

$$E = \frac{2Mc^2}{8\pi}\int_0^{2\pi}\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{Mc^2}{2}\left[-\cos\theta\right]_0^{\pi} = Mc^2\,.$$

There were however some issues associated with the Komar superpotential; for example, it yields twice the correct angular momentum for Kerr black holes. It is known as Komar's anomalous factor. The remedy for this problem was provided by Katz [33] in 1985, where a flat background spacetime is introduced.

3.1.4 Flat background spacetime

What does it mean to have a flat background spacetime? The formalism of this idea was given by Rosen [34] with a motivation to get rid of singularities in general relativity. Rosen proposes a *bimetric* spacetime, where one assumes that there exist two metric tensors at each point of spacetime. The metric $g_{\mu\nu}$ represents a physical one, it describes gravitation and interaction with matter, while $\gamma_{\mu\nu}$, the flat one, is in a relation with $g_{\mu\nu}$, but does not interact with matter. A natural assumption is that far away from matter, both metrics coincide. However, the background metric is not observable. One can regard it as describing geometry that would exist it there were no matter present [34].

Similar idea was introduced much before Rosen, where the whole metric $g_{\mu\nu}$ is decomposed as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.16}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ (a perturbation) is regarded as a *tensor in the Minkowski space* [9] and is assumed to be small with its derivatives too. This is applied, for instance, in linearized gravity for study of weak gravitational waves.

Although the Riemann tensor for $\gamma_{\mu\nu}$ is zero, one can introduce its Christoffel symbols $C^{\alpha}{}_{\rho\sigma}$ in curvilinear coordinates. With this, one has a flat covariant derivative too. The difference of Christoffel symbols of $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ can thus be defined

$$\Delta^{\alpha}{}_{\rho\sigma} := \Gamma^{\alpha}{}_{\rho\sigma} - C^{\alpha}{}_{\rho\sigma}\,, \tag{3.17}$$

and it is a tensor under general coordinate transformation because the non-tensorial parts of the transformation formula for the Christoffel symbols will cancel out. If we denote the covariant derivative associated with $\gamma_{\mu\nu}$ as D_μ, then the formula

$$\Delta^{\alpha}{}_{\rho\sigma} = \frac{1}{2}g^{\alpha\beta}\left(D_\sigma g_{\beta\rho} + D_\rho g_{\sigma\beta} - D_\beta g_{\rho\sigma}\right) \tag{3.18}$$

is a mere modification of the well-known formula for Christoffel symbols characterizing the difference between the covariant and partial derivative. Of course, the formula for the Riemann tensor is then 'generalized' too

$$R^{\alpha}{}_{\beta\rho\sigma} = D_{\rho}\Delta^{\alpha}{}_{\beta\sigma} - D_{\sigma}\Delta^{\alpha}{}_{\beta\rho} + \Delta^{\alpha}{}_{\rho\mu}\Delta^{\mu}{}_{\beta\sigma} - \Delta^{\alpha}{}_{\sigma\mu}\Delta^{\mu}{}_{\beta\rho}\,. \tag{3.19}$$

As a consequence, Einstein's $\Gamma\Gamma - \Gamma\Gamma$ Lagrangian turns into a $\Delta\Delta - \Delta\Delta$ Lagrangian and it is suddenly a true scalar density.

We have mentioned a few times that many pseudotensors may be applied when the spacetime under consideration is asymptotically flat.

Asymptotically flat spacetimes

Asymptotically flat spacetimes are very useful in studying isolated systems; for example, most of the astrophysical objects are isolated systems up to a good approximation. Following Petrov et al. [9] asymptotic flatness can be defined as follows: At the spatial infinity, there exists a homeomorphism assigning coordinates x^{α} such that

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}\Big(r^{-1}\Big)\,, \quad \partial_{\alpha}g_{\mu\nu} = \mathcal{O}\Big(r^{-2}\Big)\,, \quad \text{for} \;\; r \to \infty\,, \tag{3.20}$$

where $r^2 = \eta_{ij}x^i x^j$. There must be no coordinate chart where the fall-off behaviour is stronger than in (3.20). As one would expect, the matter has to behave well too. For energy-momentum tensor we have a condition (in the same coordinates)

$$T_{\mu\nu} = \mathcal{O}\Big(r^{-3-a}\Big) \quad a > 0\,. \tag{3.21}$$

Conserved quantities can be then defined with respect to the spatial infinity, where one can use Killing vectors of the Minkowski spacetime.

Conclusion

In this chapter, we have seen that the definition of energy in general relativity is extremely difficult subject, due to arbitrariness in the choice of superpotentials and transformation generators ξ.

Trautman [25] discusses that it is not a peculiarity of general relativity, but rather an unusual character of the gravitational field. This is hidden in the fact that a general spacetime does not have to possess maximal or high symmetry, as is the case for the Minkowski spacetime.

However, there is no reason to sorrow; although conservation laws are in any physical theory very important and they allow us to describe physical situations by means of constants of motion, it is not a necessary prerequisite to have conservation laws at our disposal. Besides, many global quantities in general relativity can be well-defined for isolated systems of bodies producing gravitational fields which fade away far away from these bodies.

There were many other approaches in defining conserved quantities which we did not illustrate. Some well-known pseudotensors and superpotentials were also constructed by Bergmann, Golberg, Weinberg etc. Hamiltonian formulation of general relativity, introduced by Arnowitt, Deser & Misner, has found many applications in defining conserved quantities in asymptotically flat spacetimes.

Hermann Bondi should be also mentioned in this context. In the next chapter, we are going to show that conserved quantities may be defined with respect to the spatial infinity for asymptotically curved spacetimes too.

4

The KBL Superpotential

In this chapter we reformulated the work of Katz, Bičák and Lynden-Bell [35] to which we refer as KBL hereafter. Notation and sign conventions used here are slightly different from their original work, but coincides with the rest of our thesis. Before we demonstrate the derivation of the KBL superpotential, we briefly discuss some useful notions appearing in the context of the KBL formalism.

4.1 Curved background spacetime

Consider a following situation: Let $(B, \bar{g})$ be a known spacetime which possesses maximal (or high) symmetry and (M, g) be a generic spacetime which is diffeomorphic to $(B, \bar{g})$. The spacetime $(B, \bar{g})$ is fixed and will be called a background spacetime, while (M, g) represents a physical spacetime under consideration.

Thus, there exists a map $\Phi : B \to M$ which deforms the spacetime $(B, \bar{g})$ smoothly into (M, g). If there exists some other diffeomorphism $\Psi : B \to M$, then due to the diffeomorphism invariance of general relativity we should obtain equal results (i.e. the form of the obtained equations) for both maps. This gauge freedom allows us to fix the map Φ and we can denote pull-backed objects from (M, g) on $(B, \bar{g})$ without explicit distinguishing of $\Phi^* T$ and T. For instance, $(\Phi^* g)_{\mu\nu} \equiv g_{\mu\nu}$. All objects originally residing on $(B, \bar{g})$ will be overlined, e.g. background scalar curvature would be $\overline{R}$. We can then work only on $(B, \bar{g}, g)$, where we emphasize that now we have two metrics on B. We assume that Φ is not an isometry and therefore $g := \Phi^* g \neq \bar{g}$. Consequently, we have two Levi-Civita connections on B, which will be also distinguished by a bar; thus, $\bar{\nabla}$ and ∇ are associated with $\bar{g}$ and g, respectively.

Since all objects now live on $(B, \bar{g}, g)$ they can be assigned the same coordinates x^μ. The difference $(g - \bar{g})$, or in coordinates, $g_{\mu\nu}(x) - \bar{g}_{\mu\nu}(x)$, can be regarded as a *perturbation* of the background metric $\bar{g}$. The background metric is given some convenient, a priori known functional form in coordinates x^μ. As Mukhanov et. al [26] explain, this is an active approach to perturbations. The passive approach would generated by coordinate transformation. Let (U, ψ, x^μ) and (V, φ, y^α) be coordinate charts on $(B, \bar{g})$ and (M, g), respectively. The map Φ then has its coordinate representation $\widetilde{\Phi} := \varphi \circ \Phi \circ \psi^{-1}$, or simply $y^\alpha(x^\mu)$ (Figure 4.1). The change $g_{\mu\nu}(x) - \bar{g}_{\mu\nu}(x)$ is then again to be regarded as a perturbation, but emerging from the coordinate transformation. This active approach is analogous to a gauge transformation of the four-potential in electromagnetism. Of course, there might be some other coordinate system x'^μ associated with the background spacetime B and the coordinate transformation $x \mapsto x'(x)$ would affect the form of both the background quantities and the physical ones (pull-backed from M). We have already met with the idea of a background (Subsection 3.1.4) which was flat. We have also mentioned that in many applications the metric

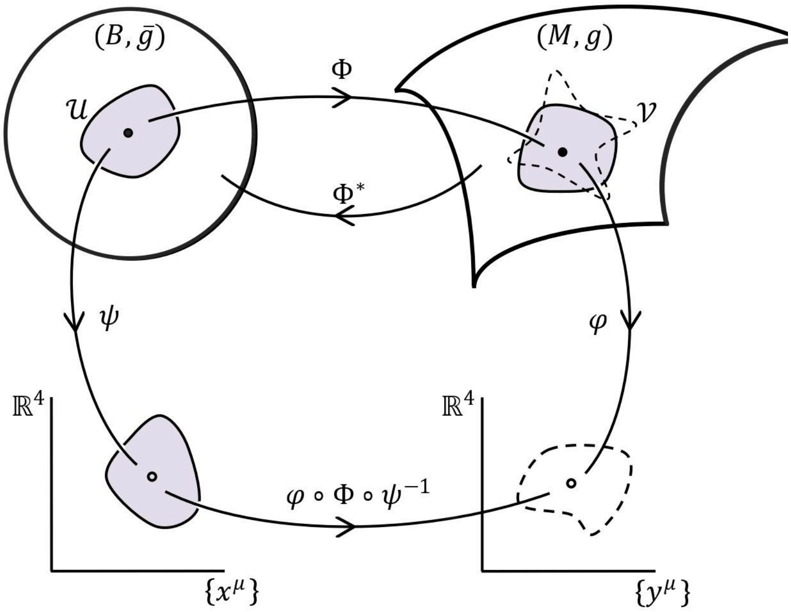

Figure 4.1: Illustration of a mapping between a background spacetime B and a dynamical spacetime M with a choice of coordinates. Highlighted areas represent an open set $\mathcal{U}$ and its images $f(\mathcal{U})$ and $\Phi(\mathcal{U})$, while dashed lines represent an open set $\mathcal{V}$ and its image $\varphi(\mathcal{V})$.

$g_{\mu\nu}$ is decomposed on the Minkowski background as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Here, we generalize this concept to curved backgrounds.

Given the mentioned convention of the same coordinates, such expressions as ${\Gamma^\alpha}_{\rho\sigma}(x) - \overline{\Gamma}^\alpha_{\rho\sigma}(x)$ become true tensors. It is, indeed, a valid fact because these expressions will transform under coordinate transformation as true tensors do. Correct transformation rule is one of the definitions of tensors (see [15], p. 48) which represent physical quantities. For the mentioned difference of the Christoffel symbols, non-tensorial parts of their transformations,

$$\frac{\partial x'^\kappa}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial x'^\alpha \partial x'^\beta} \,,$$

will cancel out.

4.1.1 Perturbations

Application of perturbations is widely used, especially in cosmology. However, one still might ask: Why should we study cosmological or any other perturbations? For a concise answer, we quote a sentence from Mukhanov et al. [26]:

"To model the universe more realistically, we must include the perturbations."

This is not surprising and it should be in agreement with our intuitive understanding of the word perturbation, not necessarily in the context of gravitation. Solutions of Einstein's field equations (second-order non-linear PDEs) are mostly idealized geometries with high degree of symmetry describing relatively 'simple' physical systems. If we want to describe more complex systems, an appropriate way is to introduce perturbations, i.e. something that generates small deviations from the idealized model to the realistic phenomenon.

For example, all cosmological perturbations can be considered as generated by a primordial mechanism which existed at the epoch of the very early universe or as induced by bare perturbations taken in the form of localized astronomical systems formed after the recombination epoch. Petrov et al. [9] state that the geometric structure and time evolution of a background spacetime (or equivalently background metric tensor) are completely known. In addition, the background metric is usually chosen as an exact solution of Einstein equations or as the metric of the Minkowski space. A good example are cosmological spacetimes. A FLRW metric with positive or negative curvature has the same spatial topology as de Sitter (dS) or Anti-de Sitter (AdS) metric, respectively. The (A)dS spacetimes posses maximum number or Killing vectors, while FLRW spacetimes 'only' six out of ten possible. As we will see, thanks to the KBL superpotential one can exploit ten Killing vectors of the (A)dS spacetime to define conserved quantities for the FLRW universe. The number of conformal Killing vectors is even larger; they, too, play an important role.

The notion 'perturbation' is frequently associated with an approximation or a small quantity, but it can be extended formally to the case of finite perturbations, exactly, with no approximation involved. [9].

4.2 Derivation of the KBL superpotential

The purpose of this section is to demonstrate a way to derive the KBL superpotential. It may be useful for anyone interested in the KBL [35] or closely related articles. Because we have managed to build a detailed formalism concerning conservation laws in previous chapters, namely the Noether's theorem, the beginning of the process will be simple. Only index gymnastics will be needed afterwards.

Lagrangian density for perturbations of the gravitational field is defined as a difference between the Lagrangian density of the dynamical field $\mathcal{L}$ and the background Lagrangian density $\overline{\mathcal{L}}$ (for brevity, we will further omit the 'density' and refer to them simply as Lagrangians)

$$\mathcal{L}_P = \mathcal{L} - \overline{\mathcal{L}}, \quad \mathcal{L} = \frac{1}{2\kappa}(\hat{R} - \partial_\mu \hat{k}^\mu), \quad \overline{\mathcal{L}} = \frac{1}{2\kappa}(\overline{\hat{R}} - \partial_\mu \overline{\hat{k}^\mu})\,, \tag{4.1}$$

in which $\hat{R}$ is the scalar curvature density and $\hat{k}^\mu$ is a vector density, whose divergence eliminates second derivatives of the metric in both Lagrangian densities. This is reasonable, because we want to apply Noether's method to $\mathcal{L}_P$ and the needed formulae are much simpler for first order Lagrangians. The idea is the same as for the Einstein Lagrangian (1.85). The particular choice of $\hat{k}^\mu$ gives a particular expression for $\mathcal{L}_P$ (we will return to this later). Our task now is to calculate a Lie derivative of the relative Lagrangian $\mathcal{L}_P$ along the flow of an *arbitrary* vector field ξ (in the spirit of Petrov & Katz [14], with Emmy Noether in the back of our minds), but for more compact calculations we can do it for $\mathcal{L}$ first, then the same procedure applies to $\overline{\mathcal{L}}$ and we can subtract the results afterwards to get an expression for $\mathcal{L}_P$.

From derivation of the main Noether identity (2.74), variation of the Hilbert action (1.74) and a consequence of the Palatini identity (1.79) (with $\delta \to \mathfrak{L}_\xi$), it follows that the main Noether identity for $\mathcal{L}$ reads

$$\frac{1}{2\kappa}\hat{G}_{\mu\nu}\mathfrak{L}_\xi g^{\mu\nu} + \partial_\alpha\left[\frac{1}{2\kappa}\left(\hat{g}^{\mu\nu}\mathfrak{L}_\xi\Gamma^\alpha{}_{\mu\nu} - \hat{g}^{\alpha\nu}\mathfrak{L}_\xi\Gamma^\mu{}_{\mu\nu} - \mathfrak{L}_\xi\hat{k}^\alpha\right) - \mathcal{L}\xi^\alpha\right] \equiv 0\,. \tag{4.2}$$

We can rewrite the first term by using the contracted Bianchi identities and (A.3b) from the Appendix

$$\begin{aligned}\hat{G}_{\mu\nu}\mathfrak{L}_\xi g^{\mu\nu} &= \hat{G}_{\mu\nu}\left(-g^{\alpha\nu}\nabla_\alpha\xi^\mu - g^{\mu\alpha}\nabla_\alpha\xi^\nu\right) = -2\hat{G}^\alpha_\nu\nabla_\alpha\xi^\nu \\ &= -2\nabla_\alpha\left(\hat{G}^\alpha_\nu\xi^\nu\right) = -2\partial_\alpha\left(\hat{G}^\alpha_\nu\xi^\nu\right)\,.\end{aligned}$$

Hence, we obtain a conserved current

$$\hat{J}^\alpha = -\frac{1}{\kappa}\hat{G}^\alpha_\nu\xi^\nu + \frac{1}{2\kappa}\left(\hat{g}^{\mu\nu}\mathfrak{L}_\xi\Gamma^\alpha{}_{\mu\nu} - \hat{g}^{\alpha\nu}\mathfrak{L}_\xi\Gamma^\mu{}_{\mu\nu} - \mathfrak{L}_\xi\hat{k}^\alpha\right) - \mathcal{L}\xi^\alpha\,. \tag{4.3}$$

Applying the same method to the background entities along the same vector field ξ, we can construct a background conserved current $\bar{\hat{J}}^\alpha$. Therefore, the canonical Noether current for $\mathcal{L}_P$ is then

$$\hat{\mathcal{J}} = \hat{J}^\alpha - \bar{\hat{J}}^\alpha\,. \tag{4.4}$$

Since it is conserved identically it must be possible to express it as a divergence of some superpotential

$$\hat{\mathcal{J}} = \partial_\beta\hat{\mathcal{U}}^{\alpha\beta}\,. \tag{4.5}$$

To find the superpotential, we will rewrite the current as a divergence by decomposing individual terms in (4.3). We begin with a Lie derivative of the Christoffel symbols:

$$\begin{aligned}\mathfrak{L}_\xi\Gamma^\alpha{}_{\mu\nu} &= \Gamma_{\beta\mu\nu}\mathfrak{L}_\xi g^{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}\left(\partial_\nu\mathfrak{L}_\xi g_{\beta\mu} + \partial_\mu\mathfrak{L}_\xi g_{\nu\beta} - \partial_\beta\mathfrak{L}_\xi g_{\mu\nu}\right) \\ &= -2\Gamma^\beta{}_{\mu\nu}g^{\alpha\rho}\nabla_{(\rho}\xi_{\beta)} + g^{\alpha\beta}\left(\partial_\nu\nabla_{(\beta}\xi_{\mu)} + \partial_\mu\nabla_{(\nu}\xi_{\beta)} - \partial_\beta\nabla_{(\mu}\xi_{\nu)}\right)\,.\end{aligned} \tag{4.6}$$

If we now rewrite expressions $\partial_\nu\nabla_{(\beta}\xi_{\mu)}$ with covariant derivatives and Christoffel symbols, that is,

$$\partial_\nu\nabla_{(\beta}\xi_{\mu)} = \nabla_\nu\nabla_{(\beta}\xi_{\mu)} + \nabla_{(\rho}\xi_{\mu)}\Gamma^\rho{}_{\beta\nu} + \nabla_{(\beta}\xi_{\rho)}\Gamma^\rho{}_{\mu\nu}\,,$$

most of the terms in (4.6) cancel out and we are left with

$$-2\Gamma^\beta{}_{\mu\nu}g^{\alpha\rho}\nabla_{(\rho}\xi_{\beta)} + g^{\alpha\beta}\left(\nabla_\nu\nabla_{(\beta}\xi_{\mu)} + \nabla_\mu\nabla_{(\nu}\xi_{\beta)} - \nabla_\beta\nabla_{(\mu}\xi_{\nu)} + 2\Gamma^\rho{}_{\mu\nu}\nabla_{(\rho}\xi_{\beta)}\right)\,. \tag{4.7}$$

The first term evidently cancels out with the last term too. Therefore,

$$\mathfrak{L}_\xi\Gamma^\alpha{}_{\mu\nu} = g^{\alpha\beta}\left(\nabla_\nu\nabla_{(\beta}\xi_{\mu)} + \nabla_\mu\nabla_{(\nu}\xi_{\beta)} - \nabla_\beta\nabla_{(\mu}\xi_{\nu)}\right)\,. \tag{4.8}$$

After writing explicitly symmetrized covariant derivatives of ξ, we can observe that two commutators of covariant derivatives emerge. Thus,

$$\mathfrak{L}_\xi\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(R^\rho{}_{\mu\beta\nu}\xi_\rho + R^\rho{}_{\nu\beta\mu}\xi_\rho + \nabla_\nu\nabla_\mu\xi_\beta + \nabla_\mu\nabla_\nu\xi_\beta\right)\,. \tag{4.9}$$

This expression provokes the idea to rewrite the last term as

$$\nabla_\mu \nabla_\nu \xi_\beta = \nabla_\nu \nabla_\mu \xi_\beta + R^\rho{}_{\beta\nu\mu}$$

and use the first Bianchi identity (2.24) to obtain

$$\begin{aligned}\mathfrak{L}_\xi \Gamma^\alpha{}_{\mu\nu} &= \frac{1}{2} g^{\alpha\beta} \left(R^\rho{}_{\nu\beta\mu}\xi_\rho - R^\rho{}_{\nu\mu\beta}\xi_\rho + 2\nabla_\nu \nabla_\mu \xi_\beta \right) \\ &= g^{\alpha\beta} R^\rho{}_{\nu\beta\mu}\xi_\rho + \nabla_\nu \nabla_\mu \xi^\alpha = R^\alpha_{\mu\rho\nu}\xi^\rho + \nabla_\nu \nabla_\mu \xi^\alpha \,. \end{aligned} \tag{4.10}$$

This result is sensible, because if ξ is a Killing vector, then (4.10) will be zero (see Eq. (2.28)) and one easily checks that the RHS is also symmetric in indices μ, ν. Multiplying the last result with $\hat{g}^{\mu\nu}$ gives us one term from the current (4.3):

$$\hat{g}^{\mu\nu} \mathfrak{L}_\xi \Gamma^\alpha{}_{\mu\nu} = \hat{R}^\alpha_\rho \xi^\rho + \nabla_\nu \nabla^\nu \hat{\xi}^\alpha \tag{4.11}$$

Exchanging upper indices μ and α yields

$$\begin{aligned}\hat{g}^{\alpha\nu} \mathfrak{L}_\xi \Gamma^\mu{}_{\mu\nu} &= \hat{g}^{\alpha\nu} g^{(\mu\beta)} \left(R^\rho{}_{\nu[\beta\mu]}\xi^\rho + \nabla_\nu \nabla_\mu \xi_\beta \right) \\ &= \hat{g}^{\alpha\nu} g^{\mu\beta} \nabla_\nu \nabla_\mu \xi_\beta = \hat{g}^{\alpha\nu} g^{\mu\beta} \left(\nabla_\mu \nabla_\nu \xi_\beta + R^\rho{}_{\beta\mu\nu}\xi_\rho \right) \\ &= \nabla_\mu \nabla^\alpha \hat{\xi}^\mu - \hat{R}^\alpha_\rho \xi^\rho \,. \end{aligned} \tag{4.12}$$

Hence, merging (4.11) and (4.12) enables us to write

$$\frac{1}{2\kappa} \left(\hat{g}^{\mu\nu} \mathfrak{L}_\xi \Gamma^\alpha{}_{\mu\nu} - \hat{g}^{\alpha\nu} \mathfrak{L}_\xi \Gamma^\mu{}_{\mu\nu} \right) = \frac{1}{\kappa} \left(\hat{R}^\alpha_\nu \xi^\nu + \nabla_\beta \nabla^{[\beta} \hat{\xi}^{\alpha]} \right) \,. \tag{4.13}$$

A Lie derivative of the vector density $\hat{k}^\alpha$ is:

$$\mathfrak{L}_\xi \hat{k}^\alpha = \hat{k}^\alpha \nabla_\mu \xi^\mu + \xi^\mu \nabla_\mu \hat{k}^\alpha - \hat{k}^\mu \nabla_\mu \xi^\alpha \,. \tag{4.14}$$

The last term in the current (4.3) is explicitly

$$\mathcal{L}\xi^\alpha = \frac{1}{2\kappa} \left(\hat{R} - \partial_\mu \hat{k}^\mu \right) \xi^\alpha = \frac{1}{2\kappa} \left(\hat{R} - \nabla_\mu \hat{k}^\mu \right) \xi^\alpha \,. \tag{4.15}$$

Finally, substituting Eqs. (4.13) - (4.15) into (4.3) results in

$$\begin{aligned}\kappa \hat{J}^\alpha &= -\hat{G}^\alpha_\nu \xi^\nu + \hat{R}^\alpha_\nu \xi^\nu + \nabla_\beta \nabla^{[\beta} \hat{\xi}^{\alpha]} + \hat{R} \delta^\alpha_\nu \xi^\nu + \nabla_\beta \left(\xi^{[\alpha} \hat{k}^{\beta]} \right) \\ &= \nabla_\beta \left[\nabla^{[\beta} \hat{\xi}^{\alpha]} + \xi^{[\alpha} \hat{k}^{\beta]} \right] \end{aligned} \tag{4.16}$$

This means that we have found the associated superpotential, because the covariant divergence of an antisymmetric density is a standard divergence of it. Thus,

$$\hat{U}^{\alpha\beta} := \frac{1}{\kappa} \left[\nabla^{[\beta} \hat{\xi}^{\alpha]} + \xi^{[\alpha} \hat{k}^{\beta]} \right] \quad \& \quad \hat{J}^\alpha = \partial_\beta \hat{U}^{\alpha\beta} \,. \tag{4.17}$$

We immediately notice that the first part of $\hat{\mathcal{U}}^{\alpha\beta}$ is a one half of the Komar superpotential (3.12)[1]. The total perturbation superpotential is then a difference of the above superpotential and a background one:

$$\hat{\mathcal{U}}^{\alpha\beta} = \hat{U}^{\alpha\beta} - \overline{\hat{U}}^{\alpha\beta} = \frac{1}{\kappa} \left(\nabla^{[\beta} \xi^{\alpha]} - \bar{\nabla}^{[\beta} \xi^{\alpha]} \right) + \frac{1}{\kappa} \xi^{[\alpha} \left(\hat{k}^{\beta]} - \bar{\hat{k}}^{\beta]} \right) \,. \tag{4.18}$$

[1]Recall our short mention at the end of the Subsection 3.1.3 that the Komar anomaly was fixed by Katz by introducing a flat background.

This is a generalized KBL superpotential (first obtained by Petrov & Katz [14]). The word 'generalized' is associated with the fact that we were working with arbitrary $\hat{k}^\mu$ and $\hat{\bar{k}}^\mu$. In the original KBL article a natural choice for this vector density was made:

$$\hat{k}^\mu = \hat{g}^{\rho\sigma}\Delta^\mu{}_{\rho\sigma} - \hat{g}^{\mu\rho}\Delta^\sigma{}_{\rho\sigma}\,. \tag{4.19}$$

This is a mere generalization of the vector density (1.87) whose divergence is added to the Hilbert Lagrangian to obtain the Einstein Lagrangian. It does not surprise us that the perturbation Lagrangian then looks as follows

$$\mathcal{L}_P = \frac{1}{2\kappa}\hat{g}^{\mu\nu}\left(\Delta^\rho_{\mu\nu}\Delta^\sigma_{\rho\sigma} - \Delta^\rho_{\mu\sigma}\Delta^\sigma_{\rho\nu}\right) - \frac{1}{2\kappa}\left(\hat{g}^{\mu\nu} - \hat{\bar{g}}^{\mu\nu}\right)\overline{R}_{\mu\nu}\,. \tag{4.20}$$

The background vector density $\hat{\bar{k}}^\mu$ evidently disappears with the choice (4.19) and therefore the KBL superpotential reads

$$\hat{\mathcal{U}}^{\alpha\beta}_{\mathrm{KBL}} = \frac{1}{\kappa}\left(\nabla^{[\beta}\hat{\xi}^{\alpha]} - \bar{\nabla}^{[\beta}\hat{\xi}^{\alpha]}\right) + \frac{1}{\kappa}\xi^{[\alpha}\hat{k}^{\beta]} \tag{4.21}$$

Julia & Silva [36] have shown that Dirichlet boundary conditions (i.e. $g = \bar{g}$ on the boundary) select uniquely the KBL superpotential. This, too, may not surprise us, because we have mentioned in Subsection 3.1.1 that the Dirichlet boundary condition is associated with the Einstein Lagrangian.

Petrov & Katz [14] have later applied Belinfante symmetrization procedure to the canonical KBL superpotential and this led to the to the possibility of making the corrected superpotential independent of $\hat{k}^\mu$ and to the fact that the Belinfante corrected KBL superpotential and the Belinfante corrected Komar superpotential are identical.

We still haven't used Einstein field equations, because we were working with the strong conservation law. We could replace $\hat{G}^\alpha_\nu$ with $\kappa\hat{T}^\alpha_\nu$ in the current (4.3) and obtain a weak conservation law, but with a physical meaning. Please note that our sign of the Lagrangians, conserved current and vector density $\hat{k}^\alpha$ (see also (1.87)) is different than in the original KBL article. We could of course multiply it by minus one and recover their results.

Using the form (4.19) of the vector density $\hat{k}^\mu$ and background covariant derivatives, the KBL superpotential (4.21) can be written as ($\Delta^\alpha{}_{\rho\sigma}$ now has the same role as Christoffel symbols in ordinary life)

$$\kappa\hat{\mathcal{U}}^{\alpha\beta}_{\mathrm{KBL}} = \hat{g}^{\rho[\beta}\bar{\nabla}_\rho\xi^{\alpha]} + \hat{g}^{\rho[\beta}\Delta^{\alpha]}{}_{\rho\sigma}\xi^\sigma + \xi^{[\alpha}\Delta^{\beta]}{}_{\rho\sigma}\hat{g}^{\rho\sigma} - \xi^{[\alpha}\hat{g}^{\beta]\rho}\Delta^\sigma{}_{\rho\sigma}\,. \tag{4.22}$$

Here, the presence of difference tensors $\Delta^\alpha{}_{\rho\sigma}$ makes it even more obvious that we are relating conserved current (and superpotential) with respect to a curved background.

Since the KBL approach is the canonical one (in the sense of Noether's theorem), it is possible to write the conserved current as a combination of terms which are identified as generalized canonical energy-momentum, spin density and some quantity which disappears when the vector ξ is a Killing vector of the background. These expressions are rather lengthy and it would not be very instructive to show them here explicitly. For details and specific formulae, see KBL [35].

Also observe that we haven't assumed in any of the steps in the derivation of the KBL superpotential that perturbations are small. In practice however, as

Petrov & Katz [14] explain, the background will be defined near the boundary only and one shall be interested in boundary surface integrals of the superpotential where $g \to \bar{g}$ (mostly at spatial infinity). However, even local currents can be used to characterize perturbations if we integrate them over some spatial volume within some finite domain with given gauge conditions.

4.3 Features and applications

The first application of the KBL formalism was in the realm of cosmological perturbations, where the FLRW metric is mapped onto de Sitter backgrounds. Ten Killing vectors of the background can be used instead of only six which FLRW spacetimes posses. The remaining four Killing vectors correspond to energy and linear momentum. The corresponding integrals then serve as constraints on perturbations if initial data are given.

Later, Petrov & Katz [14] exploited the fact that the FLRW spacetimes are conformally related with Minkowski spacetime. This provided them with 15 conformal Killing vectors of FLRW spacetime which they took as a background. This yields 15 global charges.

It was shown that the KBL superpotential in special cases yields correct masses which were found before KBL (e.g. when a flat background is taken). It agrees in many cases with charges obtained by means of pseudotensors which we discussed in previous chapter. To be more specific, it gives the right mass for Reissner-Nordstr'om and Kerr black holes with respect to a flat background and a true mass for Schwarzschild-Ads black holes (with the AdS background) [9]. Obviously, the KBL superpotential is covariant.

The approach of Katz, Bičák and Lynden-Bell can be generalized to theories of gravity other than general relativity. One of such theories is that of Horndeski and in the next chapter we will construct superpotentials for two types of spacetimes within an interesting subclass of this theory.

Part II: Superpotentials in Horndeski Gravity

5

Horndeski scalar-tensor theory

5.1 Introduction

For an initiation of this chapter, we convey few words from a recent review [37] on Horndeski theory of gravity given by T. Kobayashi. Some additional comments are taken from Babichev et. al [38], Minamitsuji [39] and Maselli et. al [40].

In recent years, arguably the most major reason for exploring modified theories of gravity arises from the discovery of the accelerated expansion of the present universe. This may be caused by the cosmological constant extremely adjusted in order to fit the observations, but currently it would be better to have other possibilities at hand and a long distance modification of general relativity is one of such possible alternatives. In the early universe, it is quite likely that some scalar field called inflaton provoked inflation and there are a number of models in which the inflaton field is coupled non-minimally to gravity (see references in [37]). Such inflation models are studied within the context of modified gravity.

If we want to test gravity, predictions of theories other than general relativity are also important. Current observations mostly probe the weak-field/slow-motion regime of the theory some of the most interesting strong-field predictions of general relativity are still elusive and difficult to verify. Observational and theoretical issues with Einstein's theory — including the dark matter and dark energy problems, the origin of curvature singularities and the quest for a quantum theory of gravity are other motives for examining other theories than general relativity. The first detection of gravitational waves [3] sets constraints on these theories, because of the propagation speed of these waves. In view of this, modified gravity is worth studying even if general relativity should turn out to be the correct (low-energy effective) description of gravity in the end. Pursuing consistent modifications helps us to learn more deeply about general relativity and gravity. For instance, by studying gravity in higher or lower dimensions one can clarify how special gravity is in four dimensions.

There is a theorem (Lovelock's) which says that the only possible second-order equations of motion derived from a Lagrangian density constructed solely from the metric are the Einstein field equations. The simplest way to loosen the assumptions of the theorem is to introduce a new degree of freedom in the form of a scalar field, but still keeping the order of the field equations. Incorporating higher derivatives may lead to a pathological theory and the correspondence with the Newtonian theory of gravity could be lost.

In 1974, in his paper 'Second-order scalar-tensor field equations in a four-dimensional space' [41], Gregory W. Horndeski proposed a general form of Lagrangian depending on the derivatives of the metric tensor and a scalar field of arbitrary order, yet still leading to second-order Euler-Lagrange equations for these two fields. Much of his work is based on preceding results and statements given by Lovelock.

The theory of Horndeski has gained attention after the equivalence was shown between the so-called generalized Galileon theory and that of Horndeski's. The Galileon is a theory on the Minkowski background with a symmetry under transformation of the scalar field $\phi \mapsto \phi + b_\mu x^\mu + c$; in analogy with Galilei transformation in mechanics. One then introduces covariant Galileon along with gravity by replacing $\partial_\mu \to \nabla_\mu$ and $\eta_{\mu\nu} \to g_{\mu\nu}$ with intention to obtain second-order field equations. This involves addition of appropriate counter terms to remove higher derivatives. However, the shift symmetry for ϕ is then broken within the covariant Galileon and one needs to provide further manipulations to obtain the generalized Galileon theory $\equiv$ Horndeski theory. Between the years 2009 and 2012, the generalized Galileon was first given by Deffayet et. al, equivalence with the Horndeski theory was shown by Kobayashi et. al and a rediscovery of the Horndeski original paper was introduced by Charmousis et. al. References to these authors and their works can be found in Kobayashi [37].

5.1.1 Horndeski Lagrangians

The action of the Horndeski theory, the most general scalar-tensor theory giving rise to second-order field equations, is given as

$$S = \int_{\mathcal{D}} L\sqrt{-\mathrm{g}}\ \mathrm{d}^4x\,, \quad L = \sum_{i=2}^{5} L_i\,, \tag{5.1}$$

in which the constituent Lagrangians[1] L_i are expressed in terms of arbitrary functions $G_i(\phi, X)$ and derivatives of the scalar field:

$$L_2 = G_2(\phi, X)\,, \tag{5.2a}$$
$$L_3 = -\,G_3(\phi, X)\Box\phi\,, \tag{5.2b}$$
$$L_4 = G_4(\phi, X)R + \partial_X G_4\left[(\Box\phi)^2 - \mathrm{Tr}\,\Pi^2\right]\,, \tag{5.2c}$$
$$L_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}\partial_X G_5\left[(\Box\phi)^3 - 3\Box\phi\mathrm{Tr}\,\Pi^2 + 2\mathrm{Tr}\,\Pi^3\right]\,, \tag{5.2d}$$

where

$$X := -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi\,, \tag{5.3}$$

is the so-called kinetic term, $\Box := \nabla^\mu\nabla_\mu$ is the d'Alembert operator and a matrix $\Pi_\nu{}^\mu := \nabla_\nu\nabla^\mu\phi$ enables us to write long expressions in a compact way, such as

$$\nabla_\mu\nabla^\nu\phi\nabla_\nu\nabla^\mu\phi \equiv (\mathrm{Tr}\,\Pi^2)\,,$$

and 'Tr' denotes a trace; R is the scalar curvature and $G_{\mu\nu}$ is the Einstein tensor. Finally, $\partial_X G_i$, $(i = 4, 5)$ are derivatives with respect to the kinetic term X, i.e.

$$\partial_X G_i = \frac{\partial}{\partial X}G_i(\phi, X)\,. \tag{5.4}$$

The same notation is applied to $\partial_\phi G_i$ which will be used below. Note that indices are raised by $g^{\mu\nu}$, so all of the derivatives above in (5.2) contain (an inverse

[1]The reason why enumeration starts from $i = 2$ can be found in [37].

of) the metric tensor. For later purposes we define notation $\nabla^{\mu\nu} := \nabla^\mu \nabla^\nu$ and $\nabla_{\mu\nu} := \nabla_\mu \nabla_\nu$.

To illustrate how wide Horndeski theory is, we can mention that it incorporates general relativity, Brans-Dicke theory, $f(R)$-gravity, quadratic gravity, etc. We will focus on the following theory contained in the Horndeski gravity.

5.2 Non-minimal derivative coupling

Consider a subclass of Horndeski theory with the action of the form

$$S_{\text{NDC}} = \int_{\mathcal{D}} \Big\{ k(R - 2\Lambda) + \alpha X + \beta G_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi \Big\} \sqrt{-\mathrm{g}}\, \mathrm{d}^4 x \,, \tag{5.5}$$

where $k := c^4/16\pi G$, while α and β are coupling constants. The first term in (5.5) forms the usual 'cosmological Hilbert action'. Apart from the kinetic term X, one foremostly notices a coupling of the Einstein tensor with the *first* derivatives of the scalar field

$$G_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi \,. \tag{5.6}$$

This coupling has been employed extensively in the last two decades (especially in cosmology) and it is mostly called *non-minimal derivative coupling* (NDC) in the literature (e.g. [39] and references therein) or also non-minimal kinetic coupling, but we will use the former. In the realm of Horndeski theory, with the action (5.1), NDC can be obtained by setting in (5.2) functions

$$\text{NDC}_1 \begin{cases} G_2 = \alpha X - 2k\Lambda \,, \\ G_3 = 0 \,, \\ G_4 = \beta X + k \,, \\ G_5 = 0 \,. \end{cases} \tag{5.7}$$

or

$$\text{NDC}_2 \begin{cases} G_2 = \alpha X - 2k\Lambda \,, \\ G_3 = 0 \,, \\ G_4 = k \,, \\ G_5 = \ -\phi \,. \end{cases} \tag{5.8}$$

These two choices are obtained by noticing that [42]

$$\begin{aligned} XR + (\Box\phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 &= G_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi + \text{divergence} \\ &= -\phi G_{\mu\nu} \nabla^\mu \nabla^\nu \phi + \text{divergence} \ \,, \end{aligned} \tag{5.9}$$

For example, we may observe that the Einstein tensor is explicitly present in (5.2) only within the Lagrangian L_5 in the form $G_5 G_{\mu\nu} \nabla^\mu \nabla^\nu \phi$. By setting $G_5 = -\phi$ and then using the Leibniz's rule with $\nabla^\mu G_{\mu\nu} = 0$, we obtain

$$-\phi G_{\mu\nu} \nabla^\mu \nabla^\nu \phi = G_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi - \nabla_\mu (\phi G^{\mu\nu} \nabla_\nu \phi) \,.$$

Because the RHS is the desired NDC minus a covariant divergence, which can be expressed as a standard divergence

$$\nabla_\mu (\phi G^{\mu\nu} \nabla_\nu \phi) = \frac{1}{\sqrt{-\mathrm{g}}} \partial_\mu (\sqrt{-\mathrm{g}}\, \phi G^{\mu\nu} \nabla_\nu \phi) \,,$$

it follows then that the NDC action (5.5) will differ from the original Horndeski action (5.1) only by a boundary term

$$\int_{\partial\mathcal{D}} \phi G^{\mu\nu} \nabla_\nu \phi \sqrt{-\mathrm{g}}\, \mathrm{d}^3\Sigma_\mu \,,$$

which does not have an effect on the form of the field equations. Therefore, to sum up, the NDC action (5.5) corresponds to the choice of functions G_i in the Horndeski Lagrangian either from (5.7) or from (5.8). One can use this fact to verify the computations.

Applications of the NDC are primarily in cosmology, but it has been recently considered for black holes too.

5.3 NDC field equations

By varying the action (5.5) with respect to $g^{\mu\nu}$ we obtain equations of motion[2]

$$0 = \frac{\delta S_{\mathrm{NDC}}}{\delta g^{\mu\nu}} \quad \Leftrightarrow \quad G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{\alpha}{2k} T^{(1)}_{\mu\nu} + \frac{\beta}{k} T^{(2)}_{\mu\nu} \,, \tag{5.10}$$

where the 'energy-momentum' tensors are

$$\begin{aligned} T^{(1)}_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2} g_{\mu\nu} \nabla_\lambda\phi\nabla^\lambda\phi \,, \\ T^{(2)}_{\mu\nu} &= \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi R - 2\nabla_\lambda\phi\nabla_{(\mu}\phi R^\lambda_{\nu)} - \nabla^\lambda\phi\nabla^\rho\phi R_{\mu\lambda\nu\rho} \\ &\quad - \left(\nabla_\mu\nabla^\lambda\phi\right)\left(\nabla_\nu\nabla_\lambda\phi\right) + \left(\nabla_\mu\nabla_\nu\phi\right)\Box\phi + \frac{1}{2}G_{\mu\nu}(\nabla\phi)^2 \\ &\quad - g_{\mu\nu}\left[-\frac{1}{2}\left(\nabla^\lambda\nabla^\rho\phi\right)\left(\nabla_\lambda\nabla_\rho\phi\right) + \frac{1}{2}(\Box\phi)^2 - \nabla_\lambda\phi\nabla_\rho\phi R^{\lambda\rho}\right] \,, \end{aligned} \tag{5.11}$$

and since the action S_{NDC} is invariant with respect to the transformation $\phi \mapsto \phi + const.$, we can write field equations for ϕ in the form of a conserved current

$$0 = \frac{\delta S_{\mathrm{NDC}}}{\delta\phi} \quad \Leftrightarrow \quad \nabla_\mu\left[\left(\alpha g^{\mu\nu} - \beta G^{\mu\nu}\right)\nabla_\nu\phi\right] = 0 \,. \tag{5.12}$$

Solutions to the above field equations were found, for instance, by Anabalon et al. [43] and Minamitsuji [39] in the context of black holes. For cosmological solutions there are many references in Kobayashi [37]

[2] We took these from [43] and our β corresponds to $\eta/2$ therein.

5.4 Formulae for superpotentials

We have mentioned a formula (2.96) in Subsection 2.4.1 where the superpotential is expressed compactly in terms of a vector ξ and its first derivatives with the use of coefficients (2.97). This formula has been generalized by Petrov & Lompay [44] where instead of partial derivatives of the vector field ξ one has a covariant derivative $\bar{\nabla}_\mu$ corresponding to the background metric. The general superpotential formula for an arbitrary vector field ξ has the same form

$$\hat{U}^{\alpha\beta} = \left(\frac{2}{3}\bar{\nabla}_\lambda \hat{n}_\sigma^{[\alpha\beta]\lambda} - \hat{m}_\sigma^{[\alpha\beta]}\right)\xi^\sigma - \frac{4}{3}\hat{n}_\sigma^{[\alpha\beta]\lambda}\bar{\nabla}_\lambda \xi^\sigma \,. \tag{5.13}$$

Conserved current is generated as a divergence of the superpotential, $\hat{J}^\alpha = \partial_\beta \hat{U}^{\alpha\beta}$. In Horndeski gravity, we have 2 variables $\psi^A = \{g_{\mu\nu}, \phi\}$ and four individual Lagrangians L_i, $i = 2, 3, 4, 5$; see (5.2) above. General formulae for coeffiecients of conserved currents and superpotentials in the Horndeski theory were derived in a recent paper by Schmidt & Bičák [45]. We are now going to briefly summarize what was done therein, but we focus on superpotentials rather than currents. The first step is to separate the whole superpotential $\hat{U}^{\alpha\beta}$ into metric (g) and scalar field (ϕ) parts, which are both additionally split into four parts corresponding to Lagrangians L_i. The superpotential (5.13) is then

$$\hat{U}^{\alpha\beta} = \sum_{i=2}^{5}\left(\hat{U}^{\alpha\beta}_{(\phi)(i)} + \hat{U}^{\alpha\beta}_{(g)(i)}\right) \,. \tag{5.14}$$

It turns out (see discussion under Eq. (94) in [45]) that $(\forall\, i)$ $\hat{U}^{\alpha\beta}_{(\phi)(i)} = 0$ and $\hat{U}^{\alpha\beta}_{(g)(2)} = 0$. Hence,

$$\hat{U}^{\alpha\beta} = \sum_{i=3}^{5}\hat{U}^{\alpha\beta}_{(g)(i)} \,, \tag{5.15}$$

where[3]

$$\hat{U}^{\alpha\beta}_{(g)(3)} = 2\hat{G}_3 \delta_\sigma^{[\alpha}\nabla^{\beta]}\phi\,\xi^\sigma \,, \tag{5.16a}$$

$$\begin{aligned}\hat{U}^{\alpha\beta}_{(g)(4)} = {} & 2\hat{G}_4 \Delta^{[\alpha}_{\sigma\lambda} g^{\beta]\lambda}\xi^\sigma + 2\hat{G}_4 \delta_\sigma^{[\alpha} g^{\beta]\lambda}\bar{\nabla}_\lambda \xi^\sigma \\ & + 4\Big[\partial_X \hat{G}_4 \left(\delta_\sigma^{[\alpha}\nabla^{\beta]\rho}\phi\nabla_\rho\phi - \delta_\sigma^{[\alpha}\nabla^{\beta]}\phi\Box\phi - \nabla^{[\alpha}\phi\nabla^{\beta]}{}_\sigma\phi\right) \\ & - \partial_\phi \hat{G}_4 \delta_\sigma^{[\alpha}\nabla^{\beta]}\phi\Big]\,\xi^\sigma \,, \end{aligned} \tag{5.16b}$$

$$\begin{aligned}\hat{U}^{\alpha\beta}_{(g)(5)} = {} & \Big[\hat{G}_5\Big(2\delta_\sigma^{[\alpha}\nabla^{\beta]\lambda}{}_\lambda\phi + 2\nabla^{[\alpha\beta]}{}_\sigma\phi - 2\delta_\sigma^{[\alpha}\nabla_\lambda{}^{\beta]\lambda}\phi - \Delta^\rho_{\sigma\kappa} g^{\kappa[\alpha}\nabla^{\beta]}{}_\rho\phi \\ & + \Delta^{[\alpha}_{\sigma\rho}\nabla^{\beta]\rho}\phi - 2G_\sigma^{[\alpha}\nabla^{\beta]}\phi - \Box\phi\Delta^{[\alpha}_{\sigma\lambda}\hat{g}^{\beta]\lambda}\Big) \\ & + 2\partial_\phi \hat{G}_5\left(\delta_\sigma^{[\alpha}\nabla^{\beta]}\phi\Box\phi + \nabla^{[\alpha}\phi\nabla^{\beta]}{}_\sigma\phi - \delta_\sigma^{[\alpha}\nabla^{\beta]\rho}\phi\nabla_\rho\phi\right) \\ & + \partial_X \hat{G}_5\Big(-2\delta_\sigma^{\alpha}\nabla^{\beta]\rho}\phi\nabla_\rho\phi\Box\phi + 2\nabla_\rho\phi\nabla_\sigma{}^{[\alpha}\phi\nabla^{\beta]\rho}\phi \\ & + 2\delta_\sigma^{[\alpha}\nabla^{\beta]\rho}\phi\nabla_{\rho\kappa}\phi\nabla^\kappa\phi + \delta_\sigma^{[\alpha}\nabla^{\beta]}\phi(\Box\phi)^2 - \delta_\sigma^{[\alpha}\nabla^{\beta]}\phi\,\mathrm{Tr}\,\Pi^2 \\ & + 2\nabla^{[\alpha}\phi\nabla^{\beta]}{}_\sigma\phi\Box\phi - 2\nabla^{[\alpha}\phi\nabla^{\beta]\rho}\phi\nabla_{\rho\sigma}\phi\Big)\Big]\xi^\sigma \\ & + \hat{G}_5\left[\delta_\sigma^{[\alpha}\nabla^{\beta]\lambda}\phi - g^{\lambda[\alpha}\nabla^{\beta]}{}_\sigma\phi - \Box\phi\delta_\sigma^{[\alpha} g^{\beta]\lambda}\right]\bar{\nabla}_\lambda\xi^\sigma \,. \end{aligned} \tag{5.16c}$$

[3] We aggregated these from [45] and can be found in Eqs. (95), (99)-(102) and (A7) in the Appendix A therein.

As can be seen, expression for $\hat{U}^{\alpha\beta}_{(g)(5)}$ is lengthy and involves much more terms than (5.16b). This is the main reason why we choose functions G_i from (5.7) to form our NDC Lagrangian in the action (5.5), because the construction of superpotentials in Chapter 6 will be easier. Thus, the total NDC_1 superpotential has a much simpler form:

$$
\begin{aligned}
{}_1\hat{U}^{\alpha\beta} = {}_1\hat{U}^{\alpha\beta}_{(g)(4)} &= 2(k+\beta X)\left[\Delta^{[\alpha}_{\sigma\lambda}\hat{g}^{\beta]\lambda}\xi^{\sigma} + \delta^{[\alpha}_{\sigma}\hat{g}^{\beta]\lambda}\bar{\nabla}_{\lambda}\xi^{\sigma}\right] \\
&\quad + 4\beta\sqrt{-\mathrm{g}}\left(\delta^{[\alpha}_{\sigma}\nabla^{\beta]\rho}\phi\nabla_{\rho}\phi - \delta^{[\alpha}_{\sigma}\nabla^{\beta]}\phi\Box\phi - \nabla^{[\alpha}\phi\nabla^{\beta]}{}_{\sigma}\phi\right)\xi^{\sigma} ,
\end{aligned}
\tag{5.17}
$$

where we used (5.7), (5.15) and

$$
{}_1\hat{U}^{\alpha\beta}_{(g)(3)} = 0 = {}_1\hat{U}^{\alpha\beta}_{(g)(5)} , \qquad \partial_X G_4 = \beta , \qquad \partial_\phi G_4 = 0 .
$$

Note that ${}_2\hat{U}^{\alpha\beta}_{(g)(3)}$ would also be zero. The NDC_2 superpotential would not be a bad choice either because the term with $\partial_X\hat{G}_5$ would vanish and the form of the superpotential would simplify greatly. We can also notice that the presence of the kinetic term X in the NDC action (5.5) has no effect on the form of NDC superpotentials.

6

Construction of NDC Superpotentials

Allow us to write the formula (5.17) for the NDC superpotential once again and let us denote it $\hat{\mathcal{U}}^{\alpha\beta}$ for convenience:

$$\begin{aligned}\hat{\mathcal{U}}^{\alpha\beta} &= 2(k+\beta X)\sqrt{-\mathrm{g}}\left[\Delta^{[\alpha}_{\sigma\lambda}g^{\beta]\lambda}\xi^{\sigma} + \delta^{[\alpha}_{\sigma}g^{\beta]\lambda}\bar{\nabla}_{\lambda}\xi^{\sigma}\right] \\ &\quad + 4\beta\sqrt{-\mathrm{g}}\left(\delta^{[\alpha}_{\sigma}\nabla^{\beta]\rho}\phi\nabla_{\rho}\phi - \delta^{[\alpha}_{\sigma}\nabla^{\beta]}\phi\Box\phi - \nabla^{[\alpha}\phi\nabla^{\beta]}{}_{\sigma}\phi\right)\xi^{\sigma}\,. \end{aligned} \tag{6.1}$$

The superpotential $\hat{\mathcal{U}}^{\alpha\beta}_{B}$ of the background is then obtained by replacing $g_{\mu\nu}$ and ϕ with $\bar{g}_{\mu\nu}$ and $\bar{\phi}$, respectively. Covariant derivatives ∇_{μ} are replaced by background ones $\bar{\nabla}_{\mu}$. The difference of connections $\Delta^{\alpha}_{\rho\sigma}$ thus vanishes and we can write

$$\begin{aligned}\hat{\mathcal{U}}^{\alpha\beta}_{B} &= 2(k+\beta\bar{X})\sqrt{-\bar{\mathrm{g}}}\,\delta^{[\alpha}_{\sigma}\bar{g}^{\beta]\lambda}\bar{\nabla}_{\lambda}\xi^{\sigma} \\ &\quad + 4\beta\sqrt{-\bar{\mathrm{g}}}\left(\delta^{[\alpha}_{\sigma}\bar{\nabla}^{\beta]\rho}\bar{\phi}\bar{\nabla}_{\rho}\bar{\phi} - \delta^{[\alpha}_{\sigma}\bar{\nabla}^{\beta]}\bar{\phi}\bar{\Box}\bar{\phi} - \bar{\nabla}^{[\alpha}\bar{\phi}\bar{\nabla}^{\beta]}{}_{\sigma}\bar{\phi}\right)\xi^{\sigma}\,. \end{aligned} \tag{6.2}$$

The relative superpotential for perturbations is then simply

$$\hat{\mathcal{U}}^{\alpha\beta}_{P} = \hat{\mathcal{U}}^{\alpha\beta} - \hat{\mathcal{U}}^{\alpha\beta}_{B}\,. \tag{6.3}$$

In the following two sections, we are going to perform a step-by-step construction of superpotentials in the NDC subclass of Horndeski theory for *i*) spherically symmetric, static spacetimes (e.g. black holes) with a static, isotropic scalar field and *ii*) isotropic, homogeneous and time-dependent spacetimes (cosmological models) with a homogeneous, time-dependent scalar field. We find general expressions for the superpotential $\hat{\mathcal{U}}^{\alpha\beta}$ in terms of ϕ, its derivatives and functions that are contained in the metric tensor. These expressions can then be used for finding the form of $\hat{\mathcal{U}}^{\alpha\beta}_{B}$. One can substitute particular functions into the relative superpotential after choosing a specific a background and a spacetime regarded as being perturbed with respect to the background one.

6.1 Spherically symmetric, static case

Spacetime coordinates are $(x^0, x^1, x^2, x^3) \equiv (ct, r, \theta, \varphi)$, where $\{r, \theta, \varphi\}$ are spherical coordinates which are in a well-known relation with Cartesian coordinates X^i:

$$X^1 = r\sin(\theta)\cos(\varphi)\,, \quad X^2 = r\sin(\theta)\sin(\varphi)\,, \quad X^3 = r\cos(\theta)\,. \tag{6.4}$$

The metric of the spherically symmetric, static spacetime is such that

$$g_{\mu\nu} = \mathrm{diag}(-\Upsilon(r), \Psi(r), r^2, r^2\sin^2(\theta))\,, \quad \sqrt{-\mathrm{g}} = r^2\sin(\theta)\sqrt{\Upsilon(r)\Psi(r)}\,, \tag{6.5}$$

with $\Upsilon(r)$ and $\Psi(r)$ both > 0 (if black holes are considered, then we work only in their exterior). The matrix of the inverse metric $g^{\mu\nu}$ is also diagonal and it has reciprocal values to $g_{\mu\nu}$. Non-zero independent components of the Christoffel symbols $\Gamma^{\mu}{}_{(\alpha\beta)}$ are

$$\begin{aligned}
\Gamma^0{}_{01} &= \frac{\Upsilon'(r)}{2\Upsilon(r)} \,, && \\
\Gamma^1{}_{00} &= \frac{\Upsilon'(r)}{2\Psi(r)} \,, & \Gamma^1{}_{11} &= \frac{\Psi'(r)}{2\Psi(r)} \,, \\
\Gamma^1{}_{22} &= -\frac{r}{\Psi(r)} \,, & \Gamma^1{}_{33} &= -\frac{r\sin^2(\theta)}{\Psi(r)} \,, \\
\Gamma^2{}_{21} &= 1/r \,, & \Gamma^2{}_{33} &= -\sin(\theta)\cos(\theta) \,, \\
\Gamma^3{}_{31} &= 1/r \,, & \Gamma^3{}_{32} &= \cot(\theta) \,.
\end{aligned} \tag{6.6}$$

The difference tensor is then

$$\Delta^{\alpha}{}_{\rho\sigma} = \Gamma^{\alpha}{}_{\rho\sigma} - \bar{\Gamma}^{\alpha}{}_{\rho\sigma} \,, \tag{6.7}$$

where $\bar{\Gamma}^{\alpha}{}_{\rho\sigma} \coloneqq \Gamma^{\alpha}{}_{\rho\sigma}(\bar{\Upsilon}, \bar{\Psi})$. It evidently has only 5 independent components which do not vanish, corresponding to first five Christoffel symbols above.

Derivatives of the scalar field

From formula (6.1) for the NDC superpotential it is obvious that the derivatives of the scalar field play an important role. For isotropic scalar field $\phi(r)$ the gradients are given as

$$\nabla_{\mu}\phi = \partial_{\mu}\phi = (0, \phi'(r), 0, 0) \,, \qquad \nabla^{\mu}\phi = g^{\mu\nu}\partial_{\nu}\phi = (0, \phi'(r)/\Psi(r), 0, 0) \,, \tag{6.8}$$

where ϕ' denotes differentiation of ϕ with respect to r. From now on, we will for brevity sometimes omit writing explicit arguments of functions even when derivatives are taken, but it should be understood that, for example, Υ' means $\Upsilon'(r)$. The kinetic term then reads

$$X = -\frac{(\phi')^2}{2\Psi} \,. \tag{6.9}$$

Higher derivatives of the scalar field are

$$\nabla_{\mu}\nabla_{\nu}\phi = \partial_{\mu}\partial_{\nu}\phi - \Gamma^{\sigma}{}_{\mu\nu}\partial_{\sigma}\phi \,, \tag{6.10a}$$

$$\nabla^{\mu}{}_{\nu}\phi \coloneqq g^{\mu\rho}\nabla_{\rho}\nabla_{\nu}\phi \,, \tag{6.10b}$$

$$\nabla^{\mu\nu}\phi \coloneqq g^{\mu\rho}g^{\nu\sigma}\nabla_{\rho}\nabla_{\sigma}\phi \,. \tag{6.10c}$$

With the use of the Christoffel symbols (6.6) it follows that[1]

$$\nabla_{\mu}\nabla_{\nu}\phi = \operatorname{diag}\left(-\frac{\Upsilon'\phi'}{2\Psi}, \phi'' - \frac{\Psi'\phi'}{2\Psi}, \frac{r\phi'}{\Psi}, \frac{r\sin^2(\theta)\phi'}{\Psi}\right) \,. \tag{6.11}$$

[1] Interestingly, if the scalar field was a function of angular coordinates θ and φ, the derivatives $\partial_{\mu}\partial_{\nu}$ (and consequently $\nabla_{\mu}\nabla_{\nu}$) *would not commute*, because we work in spherical coordinates. From index notation we could obtain

$$\nabla_{\mu}\nabla_{\nu}\phi = \partial_{\mu}\partial_{\nu}\phi - \Gamma^{\sigma}{}_{\mu\nu}\partial_{\sigma}\phi = \partial_{\nu}\partial_{\mu}\phi - \Gamma^{\sigma}{}_{\nu\mu}\partial_{\sigma}\phi = \nabla_{\nu}\nabla_{\mu}\phi \,,$$

because we are accustomed to the commutation $\partial_{\mu}\partial_{\nu} = \partial_{\nu}\partial_{\mu}$. For example, the basis vectors on the sphere ∂_{θ} and ∂_{φ} (or their combination) generate rotations for which we know (e.g. from quantum mechanics) that they are non-commutative. As infinitesimal generators, they give rise to a non-Abelian group.

The matrix $\nabla^{\mu}{}_{\nu}\phi$ is then obtained simply by a matrix multiplication of two diagonal matrices $g^{\mu\alpha}$ and (6.11). The diagonal elements of $\nabla_\mu\nabla_\nu\phi$ will be thus divided by diagonal elements of $g_{\mu\nu}$. The same thing applies to $\nabla^{\mu\nu}\phi$ and $\nabla^{\mu}{}_{\nu}\phi$. The last derivative we need, the d'Alembertian, gives

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi \tag{6.12}$$

$$= \frac{\Psi\left[2\Upsilon\left(r\phi'' + 2\phi'\right) + r\phi'\Upsilon'\right] - r\phi'\Upsilon\Psi'}{2r\Upsilon\Psi^2}\,. \tag{6.13}$$

It is just a trace of $\nabla^\mu\nabla_\nu\phi$. Note that it is a covariant $\Box$, it cannot be written as $g^{\mu\nu}\partial_\mu\partial_\nu$, in contrast to the kinetic term X, which can be written with both partial and covariant derivatives.

Construction of the superpotential

Now we can already build separate terms from (6.1) residing in two larger parentheses. The term proportional to ξ^σ in the square bracket,

$$\Delta^{[\alpha}{}_{\sigma\lambda}g^{\beta]\lambda}\,,$$

resulted in 3 independent non-vanishing components. One of them is

$$\Delta^{[0}{}_{0\lambda}g^{1]\lambda} = \frac{2\bar{\Psi}\bar{\Upsilon}\Upsilon' - (\Upsilon\bar{\Psi} + \bar{\Upsilon}\Psi)\bar{\Upsilon}'}{4\Upsilon\bar{\Psi}\bar{\Upsilon}\Psi}\,, \tag{6.14}$$

and the other two happen to be equal

$$\Delta^{[1}{}_{2\lambda}g^{2]\lambda} = \Delta^{[1}{}_{3\lambda}g^{3]\lambda} = \frac{\Psi - \bar{\Psi}}{2r\bar{\Psi}\Psi}\,. \tag{6.15}$$

If we want to check the validity of these expressions, we see that they indeed might be correct, because if we set $g_{\mu\nu} = \bar{g}_{\mu\nu}$, then $\Upsilon = \bar{\Upsilon}$ and $\Psi = \bar{\Psi}$; therefore both (6.14) and (6.15) will be zero. Thus, our calculations are sensible so far.

For the next term, proportional to $\bar{\nabla}_\lambda\xi^\sigma$, we have simple expressions for its non-trivial components:

$$\begin{aligned} \delta^{[a}_{a}g^{0]0} &= -\frac{1}{2\Upsilon}\,, & \delta^{[b}_{b}g^{1]1} &= \frac{1}{2\Psi}\,, \\ \delta^{[c}_{c}g^{2]2} &= \frac{1}{2r^2}\,, & \delta^{[d}_{d}g^{3]3} &= \frac{1}{2r^2\sin^2(\theta)}\,. \end{aligned} \tag{6.16}$$

where all Latin indices a, b, c, d are from sets $\{0,1,2,3\} \setminus \beta$, with β being in the corresponding $g^{\beta\beta}$ component. In total, the non-zero superpotential coefficients of $\bar{\nabla}_\lambda\xi^\sigma$ have the form

$$2(k + \beta X)\sqrt{-\mathrm{g}}\,\delta^{[\alpha}_{\alpha}g^{\beta]\beta}\,, \quad \alpha \neq \beta\,, \tag{6.17}$$

with X given in Eq. (6.9) and $\sqrt{-\mathrm{g}}$ in (6.5).

For the second large parenthesis (proportional to ξ^σ) in the superpotential (6.1) we need to combine derivatives of the scalar field which we discussed in detail above. We can write (6.1) as

$$\hat{\mathcal{U}}^{\alpha\beta} = \hat{N}^{\alpha\beta}_{\sigma}\xi^\sigma + \delta^{[\alpha}_{\sigma}g^{\beta]\lambda}\bar{\nabla}_\lambda\xi^\sigma\,, \tag{6.18}$$

where the coefficients $\hat{N}^{\alpha\beta}_{\sigma} = \hat{N}^{[\alpha\beta]}_{\sigma}$ represent all the terms contracted with ξ^{σ} in (6.1). Those that are not zero have the following form

$$\hat{N}^{01}_{0} = \frac{r\sin(\theta)\sqrt{\Upsilon\Psi}}{4\Psi^3}\left\{\frac{r\Psi}{\Upsilon\bar{\Psi}\bar{\Upsilon}}\left[\beta\,(\phi')^2 - 2k\Psi\right]\times \right. \tag{6.19a}$$

$$\left. \times\left[\bar{\Upsilon}\Psi\bar{\Upsilon}' + \bar{\Psi}\left(\Upsilon\bar{\Upsilon}' - 2\bar{\Upsilon}\Upsilon'\right)\right] + 8\beta\phi'\left[2\Psi\,(r\phi'' + \phi') - r\phi'\Psi'\right]\right\},$$

$$\hat{N}^{12}_{2} = \hat{N}^{13}_{3} = \frac{r\sin(\theta)}{2\bar{\Psi}\Psi^2\sqrt{\Upsilon\Psi}}\left\{2k\Upsilon\Psi^2(\Psi - \bar{\Psi}) - 8\beta r\phi'\Upsilon\bar{\Psi}\Psi\phi'' \right. \tag{6.19b}$$

$$\left. - \beta\,(\phi')^2\left[\Psi\left(2r\bar{\Psi}\Upsilon' + \Upsilon(3\bar{\Psi} + \Psi)\right) - 4r\Upsilon\bar{\Psi}\Psi'\right]\right\}.$$

The background superpotential is obtained from the above expressions with replacements $\Upsilon \to \bar{\Upsilon}$, $\Psi \to \bar{\Psi}$ and $\phi \to \bar{\phi}$. It is without the term containing the difference tensor $\Delta^{\alpha}{}_{\rho\sigma}$ (contributing only to the $\hat{\mathcal{U}}^{01}$ component). We assume that there is no need to exhibit similar expressions for the background superpotential.

The obtained components $\hat{\mathcal{U}}^{\alpha\beta}$ can be used to construct a perturbation superpotential, where the perturbations can be large and in any region of spacetime. The coefficient $\hat{N}^{01}_{0}$ would correspond to the timelike Killing vector ∂_0 with components δ^{μ}_{0} and one can integrate the perturbation superpotential (after choosing a specific spacetime and its background) to calculate the mass of the black hole.

6.2 Cosmological case

If we keep the spherical coordinates, we can use the following form of the FLRW metric [11]

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + R^2(t)\left[\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\right], \tag{6.20}$$

where $R(t)$ is an unknown function of time and K is a constant, which by a suitable choice of units for r can be chosen to have value ± 1 or 0. We now take the scalar field as a function of time only $\phi = \phi(t)$. Appropriate curved backgrounds are (Anti)-de Sitter spacetimes, whose metric can be written as [46]

$$\mathrm{d}s^2 = -\left(1 - \frac{\Lambda r^2}{3}\right)c^2\,\mathrm{d}t^2 + \left(1 - \frac{\Lambda r^2}{3}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\,. \tag{6.21}$$

The background scalar field is $\bar{\phi}(t)$. However, because solutions of the NDC field equations will contain coupling parameters from the action (5.5), asymptotic form of the metric need not be precisely (6.21) (see, for instance, [43] and [39]). Coupling parameters may have a role in redefining the effective cosmological constant. Thus, we will write the (A)dS backgrounds as

$$\mathrm{d}s^2 = -f(r)c^2\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\,. \tag{6.22}$$

where the function is $f(r)$ is different for each $K = \pm 1$.

The procedure is then the same as for the case of black holes in the previous section. We first find the Christoffel symbols and then the difference tensor $\Delta^{\alpha}_{\rho\sigma}$ and derivatives of the scalar field can be found afterwards. The computation resulted in 6 independent non-vanishing coefficients $N^{[\alpha\beta]}_{\sigma}$ representing all terms proportional to ξ^{σ} in the formula (6.1). These are

$$N^{10}_{0} = \frac{f'(r)\left(2c^2k + \beta\left(\dot{\phi}(t)\right)^2\right)\left(f^2(r)R^2(t) - Kr^2 + 1\right)}{4c^2 f(r)R^2(t)}, \tag{6.23a}$$

$$N^{i0}_{i} = \frac{-2c^2k\dot{R}(t) + 4\beta\dot{\phi}(t)R(t)\ddot{\phi}(t) + 3\beta\left(\dot{\phi}(t)\right)^2\dot{R}(t)}{c^4R(t)}, \tag{6.23b}$$

$$N^{12}_{2} = \hat{N}^{13}_{3} = \frac{\left(f(r) + Kr^2 - 1\right)\left(2c^2k + \beta\left(\dot{\phi}(t)\right)^2\right)}{2c^2rR^2(t)}. \tag{6.23c}$$

These coefficients need to be multiplied by

$$\sqrt{-\mathrm{g}} = \frac{cr^2R^3(t)\sin^2(\theta)}{\sqrt{1 - Kr^2}}, \tag{6.24}$$

to make out of a superpotential tensor $\mathcal{U}^{\alpha\beta}$ a density $\hat{\mathcal{U}}^{\alpha\beta}$. The coefficients of $\bar{\nabla}_{\lambda}\xi^{\sigma}$ are again very simple as in the spherically symmetric, static case.

Conclusion

Within this thesis we have managed to present both mathematical and physical view on conservation laws in general relativity. Our mathematical foundation, in the form of the preliminary chapter, enabled us to formulate many concepts in an elegant way. We tried to add a more geometric view on the underlying theory.

The connection between symmetries and conservation laws is strongly promoted throughout the whole text. The fact that maximal symmetry coincides with constant curvature has been demonstrated in detail. We presented how the existence of Killing vector fields provides a way to construct conserved quantities. That this is a consequence of the Noether's theorem is shown afterwards in detail. Superpotentials, to which we devoted a lot of space in this thesis are also a consequence of the Noether's theorem and it also illustrates the importance of the first chapter, as a basis for the second chapter where the relation of symmetries and conservation laws is studied. We referred frequently to our motivational example from the beginning of the thesis when we discussed global charges.

After reviewing some concepts of the theory of conservation laws in general relativity, in the form of pseudotensors and superpotentials applicable to asymptotically flat spacetimes, we focused on the KBL formalism. We illustrated the notion of a curved background spacetime and derived the associated KBL superpotential almost independently of the known literature. Concerning the literature, we believe that our Bibliography list can be useful for people interested in conservation laws in general relativity.

We implemented generalized KBL method in a Horndeski gravity subclass, which involves the non-minimal derivative coupling (NDC) of the scalar field to the Einstein tensor. The superpotentials are constructed for *i*) a spherically symmetric and stationary form of both the metric and the scalar field, *ii*) a time-dependent form of both the metric and the scalar field; both cases are constructed in spherical coordinates. The former can be applied, for instance, to black hole spacetimes with a scalar field for calculating the mass of these black holes and the latter may have use in cosmological perturbations. This is a motivation for our future work. Formulae for both types of superpotentials (or their non-zero coefficients) are expressed in terms of functions that are present in the metric tensor, derivatives of these functions and derivatives of the scalar field. When one makes a specific choice of both physical and background spacetimes in the NDC subclass of Horndeski gravity, the particular metric functions and scalar fields can then be substituted into our general NDC superpotential formulae for the mentioned types of spacetimes and scalar fields. The construction of these superpotentials should be regarded as an illustration of our understanding of the theory that we presented in the review part.

Postlude: A Purpose of the Thesis

I have decided to write a review on conservation laws in general relativity (the first part of this work) for several reasons, which are described in the following paragraphs.

Plenty of shorter reviews, in the form of articles or chapters in books, focusing mainly on particular aspects of conservation laws in general relativity, are available. However, there is not much specialized literature or monographs with a general, unifying picture. An exception, published very recently (2017), is a book written by Petrov, Kopeikin, Lompay and Tekin [9], which I truly admire and recommend. In the course of writing this thesis, I had 'enough' time to read most of the book carefully and I exploited many ideas from it (and references therein) to write my thesis. The book of Petrov et al. spans truly an extensive amount of subjects concerning conservation laws in GR and other metric theories of gravity, including perturbations, and it thus might be a bit difficult for a reader/student to absorb all the information that is presented in the book; especially if the reader needs to pick out only certain aspects of the theory. It could be even more time-consuming than just reading the whole book. Furthermore, I think that few things in [9] may have been done differently, in a better way or in more detail. I discovered Sundermeyer's book [29] a bit too late to make a reasonable opinion on it, but it seems to be very instructive.

During my bachelor studies, I realized that physicists do not always utilize a rigorous mathematical notation. A 'physicist's convention' is sometimes comprehensible only when one is already familiar enough with a given subject. Short and compact notation is with no doubt extremely useful, but utilizing it should be accompanied with the true notation in the back of one's mind. That is why the first chapter containing mathematical preliminaries is the second longest chapter of the present thesis. I believe that the rest of the thesis is then more comprehensive.

My wish was to write the review part from a student's perspective: What does a student, familiar with basics of general relativity, need to know in order to understand articles or books concerned with conservation laws? The first, review part of the thesis should prepare anyone to be able to read scholarly articles in scientific journals related to conservation laws in general relativity (or even other theories of gravity).

I personally obey the rule 'the truth is always in the book', but sometimes, when I find a bachelor thesis that gives me some information very quickly and in a comprehensive manner, I wish that there were more of such theses. Many people told me that the bachelor thesis is not that important, and that I should save my energy for higher academic degree theses. But why? Why at least not to try to write something in more detail, or in other way than it is given in textbooks. I am convinced that the best way to learn something is to try to explain or present that something to another person, comprehensively.

Professor Bičák, my thesis supervisor, has been lecturing for many years a subject called ‘Relativistic physics II’, which is a master’s degree course held at the Institute of Theoretical Physics at Charles University. This one semester series of lectures, that I joyfully attended this year, is very mathematical and covers a variational principle in field theories, the Hilbert action, main concepts of conservation laws in general relativity, the initial value (Cauchy) problem, Hamiltonian formalism etc. Many of my ‘personal’ comments in this thesis are strongly influenced by prof. Bičák’s lectures.

If this thesis serves anyhow to anyone in the future, then the purpose of my thesis will be fulfilled.

Mak Pavičević

Appendices

A

Tensor densities

A tensor density D of type (r,s) and (positive) weight $w \in \mathbb{Q}$ is a quantity which transforms under a coordinate transformation $x \mapsto x'(x)$ as

$$D'^{\alpha\ldots\beta}_{\mu\ldots\nu}(x') = J^{-w}(x) D^{\kappa\ldots\lambda}_{\rho\ldots\sigma}(x) \underbrace{\frac{\partial x'^{\alpha}}{\partial x^{\kappa}} \cdots \frac{\partial x'^{\beta}}{\partial x^{\lambda}}}_{r} \underbrace{\frac{\partial x^{\rho}}{\partial x'^{\mu}} \cdots \frac{\partial x^{\sigma}}{\partial x'^{\nu}}}_{s}, \tag{A.1}$$

where

$$J(x) \coloneqq \det\left(\frac{\partial x'^{\alpha}(x)}{\partial x^{\kappa}}\right)$$

is the Jacobian of the transformation $x'(x)$. For brevity, we will sometimes omit writing the coordinate arguments (x) and (x') explicitly.

An important entity is the determinant of the metric $\mathrm{g} \coloneqq \det(g_{\mu\nu})$. By taking a determinant of the transformation rule for the metric

$$g'_{\mu\nu}(x') = g_{\rho\sigma}(x) \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}}$$

and using the formula $\det(AB) = \det A \det B$, we obtain

$$\mathrm{g}' = J^{-2}\mathrm{g}\,.$$

Because we are working with Lorentzian metrics, their determinant is negative and so $-\mathrm{g}$ is positive. We can thus take a square root of it and write

$$\sqrt{-\mathrm{g}'} = J^{-1}\sqrt{-\mathrm{g}}\,. \tag{A.2}$$

In this text, every tensor density is of weight $+1$ and has the form

$$\hat{T} \coloneqq \sqrt{-\mathrm{g}}T$$

where T is a tensor. Some examples: Scalar density is $\sqrt{-\mathrm{g}}f$ for any function f (a scalar; $(0,0)$-tensor); vector density is a quantity $\hat{V}^{\mu} = \sqrt{-\mathrm{g}}\,V^{\mu}$.

There are many useful formulae involving $\sqrt{-\mathrm{g}}$. Let $\hat{V}^{\mu}$ be a vector density and $\Gamma^{\alpha}{}_{\beta\gamma}$ a Christoffel symbol of the metric $g_{\mu\nu}$ with determinant g. Let ∇_{μ} and ∂_{μ} denote covariant and partial derivatives, respectively. Let the symbol D represent any kind of differentiation (∂_{μ}, variation δ, Lie derivative, etc.). Then

$$\Gamma^{\alpha}{}_{\alpha\mu} = \partial_{\mu} \ln\left(\sqrt{-\mathrm{g}}\right), \tag{A.3a}$$

$$\nabla_{\mu}\hat{V}^{\mu} = \partial_{\mu}\hat{V}^{\mu} \tag{A.3b}$$

$$D\sqrt{-\mathrm{g}} = \frac{1}{2}\sqrt{-\mathrm{g}}\,g^{\mu\nu}Dg_{\mu\nu}\,, \tag{A.3c}$$

$$g^{\mu\nu}Dg_{\mu\nu} = -g_{\mu\nu}Dg^{\mu\nu}\,. \tag{A.3d}$$

Proofs of these formulae can be found in Nakahara [16] and Weinberg [11].

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