**FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University**

BACHELOR THESIS

Andrii Zymin

Quasispin models in quantum physics

Institute of Particle and Nuclear Physics

Supervisor of the bachelor thesis:	Mgr. Pavel Stránský, Ph.D.
Study programme:	Physics (B1701)
Study branch:	FOF (1701R026)

Prague 2019

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date .... signature of the author

Dedication.

I would like to thank my supervisor Mgr. Pavel Stránský, Ph.D. for the help and invaluable advice during the whole time of writing of this thesis.

My special thanks go to Paul Wrege Bc. for the help with English grammar, both now and throughout the years.

Last but not least I want to thank my family and friends for their endless support.

Title: Quasispin models in quantum physics

Author: Andrii Zymin

Department: Institute of Particle and Nuclear Physics

Supervisor: Mgr. Pavel Stránský, Ph.D., Institute of Particle and Nuclear Physics

Abstract: The use of symmetries in quantum physics helps in a deeper understanding of physical systems and simplifies numerical calculations. This thesis studies models based on the SU(2) algebra, which, in spite of their apparent simplicity, show rather rich behavior and describe a wide spectrum of physical phenomena. We review various realizations of the SU(2) algebra (namely the spin, boson, and fermion realization) and present the most general quantum hamiltonian with one- and two-body interactions, constructed from the SU(2) generators. We perform the classical limit of the hamiltonian and show a numerical study of several particular examples.

Keywords: dynamical symmetries, Lipkin model, su(2) algebra, quasispin

Název práce: Kvazispinové modely v kvantové fyzice

Autor: Andrii Zymin

Katedra: Ústav částicové a jaderné fyziky

Vedoucí bakalářské práce: Mgr. Pavel Stránský, Ph.D., Ústav částicové a jaderné fyziky

Abstrakt: Využití symetrií v kvantové fyzice pomáhá k hlubšímu porozumění fyzikálních systémů a zjednodušuje numerické výpočty. Tato práce studuje modely založené na algebře SU(2), které i přes zdánlivou jednoduchost vykazují velmi bohaté chování a popisují široké spektrum fyzikálních jevů. Jsou shrnuty různé realizace algebry SU(2) (spinová, bosonová, fermionová). Je předložen nejobecnější tvar kvantového hamiltoniánu zkonstruovaného z generátorů SU(2), který zahrnuje všechny jednočásticové a dvoučásticové interakce, a jeho klasická limita. Několik konkrétních pi̊kladů hamiltoniánu je detailně prezentováno numericky.

Klíčová slova: dynamické symetrie, SU(2) algebra, kvazispin, klasická limita

Contents

Introduction

Presently group theory finds its application in almost all of different fields of theoretical physics. It is one of the most powerful ways of researching symmetries of physical models. By symmetries in this case we mean a set of transformations, that will conserve the system's original state.

Since the beginning of the 20th century symmetry methods have been used to examine many theoretical models. One of the most profitable cases is the so-called Lipkin model first introduced in 1965 [1]. Submitting to the SU(2) algebra, Lipkin model contributes greatly to the research of many-fermion systems.

But Lipkin model is only a specific case of a hamiltonian, which can be expressed in terms if the generators of SU(2). As a matter of fact, SU(2) is a dynamical algebra of much more vast class of hamiltonians. One of the main goals of this thesis is to determine the general form of the hamiltonian, which will allow us to investigate the dynamics of the energy levels of such systems and its dependence on singular parameters.

The SU(2) algebra is very useful in describing simple integrable models, which have a single degree of freedom. In spite of its simplicity, it provides us with the ability to examine several phenomena such as quantum phase transitions, which are transitions between phases at zero temperature, caused by varying of a physical parameter (such as an interaction with an external field)[1, 2], excited-state quantum phase transitions [3, 4], nonhermitian quantum mechanics [2, 5], adiabatic and nonadiabatic dynamics [4, 6].

The SU(2) is an example algebra, used to describe collective modes of complex many-body systems, i.e. modes in which the system behaves as a whole mass, rather than a set of independent interacting particles. Other examples of such algebras are the U(3) and U(4) applied in the research of molecular vibrations and rotations (vibron models); U(6) algebra, which is the interacting boson model (IBM) of nuclear vibrations and rotations.

The first chapter of this thesis is dedicated to the basics of group theory, where we will prepare all the necessary instruments required for solving specific problems concerned with the construction and finding solutions of a SU(2) hamiltonian.

In the second chapter we look at different ways of representing the SU(2) algebra, suitable for our further calculations.

In the third chapter we will discuss the general form of a SU(2) hamiltonian and its matrix representation.

In the fourth chapter we will research the classical limit of such a hamiltonian, which in our case implies performing the Holstein-Primakoff transformation of the hamiltonian and the examination of the system, where the number of particles tends to infinity.

The final chapter is dedicated to examples of numerical solutions of specific cases of the SU(2) hamiltonian and the dependence of its eigenvalues on its parameters.

Chapter 1

The fundamental notions of the group theory and unitary algebras

Before we write a hamiltonian based on a particular group SU(2) let us review some basic concepts of the group theory. This whole chapter is inspired by [7, 8, 9].

The following sections contain the fundamental definitions and statements, which are needed for the further discussion.

1.1 Group theory

Definition 1.1.1. *A group is defined as a set of elements $G = (\hat{G}_1, \hat{G}_2, ..., \hat{G}_n)$, for which the operation of multiplication is defined (implying a consecutive application of the elements), and that satisfy the following conditions:*

1. $\forall \hat{G}_i, \hat{G}_j \in G \Rightarrow \hat{G}_i \hat{G}_j \in G$
2. *The multiplication of the elements satisfies the law of associativity:*

$$\hat{G}_i(\hat{G}_j \hat{G}_k) = (\hat{G}_i \hat{G}_j)\hat{G}_k$$

Generally speaking, elements of a group do not commute:

$$\hat{G}_i \hat{G}_k \neq \hat{G}_k \hat{G}_i$$

3. *There exists an element E such that $\forall \hat{G}_i \in G \Rightarrow E\hat{G}_i = \hat{G}_i$*
4. $\forall \hat{G}_i \in G \;\; \exists \hat{G}_i^{-1}$ *such that* $\hat{G}_i \hat{G}_i^{-1} = \hat{G}_i^{-1} \hat{G}_i = E$

Relying on this definition we define the so-called Lie groups, which are continuous groups.

Definition 1.1.2. *A Lie group is a group with the property that its group operations (multiplication and inversion) are smooth maps:*

$$G(\overrightarrow{\phi})G(\overrightarrow{\psi}) = G(\overrightarrow{\theta}) \tag{1.1}$$

$$G(\overrightarrow{\alpha})^{-1} = G(\overrightarrow{\beta}), \tag{1.2}$$

where $\overrightarrow{\phi} = \{\phi_1, \phi_2, ..., \phi_s\}$ *and* $\overrightarrow{\psi} = \{\psi_1, \psi_2, ..., \psi_s\}$ *are continuous parameters and* $\overrightarrow{\theta} = f(\overrightarrow{\phi}, \overrightarrow{\psi})$, $\overrightarrow{\beta} = h(\overrightarrow{\alpha})$ *are smooth functions.*

An example of a continuous group is addition of two successive rotations by angles ϕ_1 and ϕ_2 in a plane:

$$R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2). \tag{1.3}$$

It can be shown, that in a case of a continuous group, its elements can be obtained in the following way:

$$G_i = e^{i\sum_j \alpha_j \hat{g}_j}, \tag{1.4}$$

where α_j is a constant and $\{\hat{g}_1, \hat{g}_2, ...\}$ is a set of so-called generators, which are linear operators on the given vector space.

Definition 1.1.3. *Linear operators* $(\hat{g}_1, \hat{g}_2, ..., \hat{g}_s)$ *on a s-dimensional vector space define a Lie algebra* $\mathfrak{g}$ *if they satisfy:*

$$[\hat{g}_i, \hat{g}_j] = \sum_k c_{ij}^k \hat{g}_k \tag{1.5}$$

and the Jacobi identity:

$$[\hat{g}_i, [\hat{g}_j, \hat{g}_k]] + [\hat{g}_k, [\hat{g}_i, \hat{g}_j]] + [\hat{g}_j, [\hat{g}_k, \hat{g}_i]] = 0 \tag{1.6}$$

Here the binary operator $[\cdot, \cdot]$ is called commutator (or Lie brackets) and is expressed as follows:

$$[\hat{g}_i, \hat{g}_j] = \hat{g}_i\hat{g}_j - \hat{g}_j\hat{g}_i \tag{1.7}$$

The constants c_{ij}^k are called structure constants. The properties of a Lie algebra and its associated group are determined by the structure constant values.

To illustrate the application of group theory let us give the example of angular momentum operators. It is commonly known that the angular momentum operators close under commutations:

$$[\hat{J}_k, \hat{J}_l] = i\varepsilon_{klm}\hat{J}_m \tag{1.8}$$

The structure constants in (1.8), which are represented by $i\varepsilon_{ijk}$, are enough to derive the spectrum of $\hat{J}_z$ and $\hat{J}^2$ [10]:

$$\begin{aligned} \hat{J}_z |j, m\rangle &= m |j, m\rangle \quad j = 0, 1, 2... \\ \hat{J}^2 |j, m\rangle &= j(j+1) |j, m\rangle \quad m = -j, -j+1, ..., j-1, j, \end{aligned} \tag{1.9}$$

where $|j, m\rangle$ are common eigenstates of $\hat{J}^2$ and $\hat{J}_z$, and j, m are their quantum numbers. The existence of a common eigenspace is a consequence of the fact that this two operators commute: $[\hat{J}^2, \hat{J}_z] = 0$.

Equations (1.8) are identical with the first condition of a Lie algebra. Here we see that the factors $i\varepsilon_{ijk}$ are structure constants. Now, it is obvious that substituting the "inner" commutators accordingly to (1.8) in the Jacobi identity we obtain:

$$[\hat{J}_k, i\hat{J}_k] + [\hat{J}_m, i\hat{J}_m] + [\hat{J}_l, i\hat{J}_l] = 0 \tag{1.10}$$

1.2 Casimir invariants

A vital step in finding the algebraic solution of a many-body hamiltonian is expressing it in terms of the operators, that generate a given algebra. The so-called Casimir invariants form an important set of operators associated with the algebra.

Definition 1.2.1. *Consider an algebra* $\mathfrak{g}$ *which is generated by* $\hat{g}_j$ *operators. Then:*

$$(\forall G)\ \exists C = (\hat{C}_1, \hat{C}_2, ...)$$

such that:

$$(\forall \hat{C}_i \in C)(\forall \hat{g}_j \in \mathfrak{g}) \Rightarrow [\hat{C}_i, \hat{g}_j] = 0$$

Operators $\hat{C}_i$ *are called Casimir invariants.*

Usually, groups are labelled with capital letters, while corresponding algebras with small or fraktur letters. Although this distinction is often omitted in the literature, we will follow this convention in the subsequent text.

Later, taking a closer look at U(2), SU(2) algebras, we will show, that their Casimir invariants are respectively: $\hat{N}$ (the operator of the total number of energy quanta of the system) and $\hat{J}^2$ (the operator of the total angular momentum of the system).

1.3 Dynamical algebras and symmetry algebras

Before we go on with examination of other objects of the group theory, let us have a look at the following example. Consider a hamiltonian $\hat{H}$ that is constructed by a linear combination of operators $\hat{g}_1$, $\hat{g}_2$,... $\hat{g}_s$ (for now we are not interested in a physical motivation of this example), which are generators of some algebra $\mathfrak{g}$:

$$\hat{H} = \alpha\hat{g}_1 + \beta\hat{g}_2 + ... + \omega\hat{g}_s$$

From the definition of a Lie algebra's operators we see, that this hamiltonian does not commute with the generators of this algebra, even though it is constructed purely by them. In this case we say that $\mathfrak{g}$ is the dynamical algebra of the given hamiltonian, since the hamiltonian is expressed in terms of its generators.

The fact that such hamiltonian does not commute with all the generators of $\mathfrak{g}$ tells us, that it does not possess the same symmetry. But let us say, that for a subset of the generators the following is true:

$$[\hat{H}, \hat{g}_l] = 0, \ \ l = 1, ..., r, \ \ r < s$$

If the subset $\{\hat{g}_l\}$ forms an algebra, than $\hat{H}$ is invariant with respect to this algebra. This means, that $\hat{H}$ is invariant in respect to the algebra $\mathfrak{g}'$ generated by $\hat{g}_l$ (we call it a symmetry algebra). In this case we can say, that the algebras $\mathfrak{g}$ and $\mathfrak{g}'$ form a chain

$$\mathfrak{g} \supset \mathfrak{g}',$$

where $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$. Generally, if we have a number of subalgebras $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$,..., they form a chain:

$$\mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \mathfrak{g}_3 \supset ..., \tag{1.11}$$

where the symmetry of the "higher" $\mathfrak{g}_i$ includes all the symmetries of its subalgebras. The smallest one in the chain is the symmetry algebra of the system.

1.4 The basic properties of the U(n) and SU(n) algebras

In the following section we will define some basic properties and objects of the U(n) and SU(n) groups in general and various ways of their realization [7, 11].

Definition 1.4.1. *The unitary group, denoted as U(n), is defined as a set of $n \times n$ unitary matrices, satisfying:*

$$\sum_{j=1}^{n} U_{ij} U_{kj}^{*} = \delta_{ik} \tag{1.12}$$

or in a matrix notation:

$$\hat{U}\hat{U}^{\dagger} = \hat{E}. \tag{1.13}$$

A unitary transformation of a complex vector z_i:

$$z_i' = \sum_{j=1}^{n} U_{if} z_j, z \in \mathbb{C} \tag{1.14}$$

does not change its norm:

$$\sum_i |z_i|^2 = \sum_i |z_i'|^2.$$

Matrix realization

For infinitesimal transformations we can express a unitary operator $\hat{U} = e^{i\epsilon\hat{T}}$ (see equation (1.4)) using a power series:

$$\hat{U} = \hat{E} + i\epsilon\hat{T} + ... \tag{1.15}$$

Taking the property (1.13) in consideration, we find that the operator $\hat{T}$ should be hermitian:

$$\hat{U}\hat{U}^{\dagger} \approx (\hat{E} + i\epsilon\hat{T})(\hat{E} - i\epsilon\hat{T}^{\dagger}) = \hat{E} + i\epsilon(\hat{T} - \hat{T}^{\dagger}) = \hat{E}$$

where we write only the terms with the lowest order in ϵ. This equation shows, that $\hat{T} = \hat{T}^{\dagger}$. Group U(n) has n^2 independent parameters. Generally speaking, a $n \times n$ matrix would depend on $2n^2$ real parameters (n^2 matrix elements, which all have a real and imaginary parts). To determine the number of independent parameters of the U(n) group, lets look at the hermitian matrix $\hat{T}$. The set of equations given by the condition $\hat{T}_{ij}^{\dagger} = \hat{T}_{ji}$ tells us, that the diagonal elements should be purely real. Thus, we can decrease the whole number of diagonal parameters by n.

For the non-diagonal elements we get another $\frac{2(n^2-n)}{2}$ real equations. This means that the number of the parameters can be decreased by another $n^2 - n$.

The generators of the U(n) group can be constructed using matrices G^i_j, that have the following look:

$$G^j_i = \begin{pmatrix} 0 & 0 & ... & 0 & ... & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & ... & 1 & ... & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & ... & 0 & ... & 0 \end{pmatrix} \tag{1.16}$$

In such a matrix the only non-zero element is on the ij position. This matrices satisfy the condition (1.5):

$$[G^i_j, G^m_n] = G^m_j \delta_{in} - G^i_n \delta_{mj} \tag{1.17}$$

Notice that these matrices are not hermitian. There are two ways of creating an hermitian matrix in this case:

$$A^i_j = G^i_j + G^j_i$$
$$B^i_j = i(G^i_j - G^j_i)$$

By constructing different combinations, we construct the generators of the group. There are indeed n^2 of such generators, in accord with the number of independent elements.

Let us now define group SU(n), which is a group of unitary matrices with unit determinant. Starting from the expansion (1.15):

$$\hat{U} = \hat{E} + i\epsilon\hat{T} + ..., \tag{1.18}$$

where we, again, take only the lowest-order terms of ϵ into consideration, we restrict our linear operators to only the ones with a unit determinant, thus obtaining:

$$det(\hat{U}) = det(\hat{E} + i\epsilon\hat{T}) \approx 1 + Tr(i\epsilon\hat{T}) = 1, \tag{1.19}$$

which means that:

$$Tr(\hat{T}) = 0 \tag{1.20}$$

In the equation (1.19) we used a corollary of the Jacobi's formula:

$$det(U) = det(e^{i\epsilon\hat{T}}) = e^{i\epsilon Tr(\hat{T})} \tag{1.21}$$

Coming back to the number of independent parameters we find that the final limitation in case of the SU(n) group is given by (1.20). The total number of parameters is $2n^2 - (n + n^2 - n + 1) = n^2 - 1$.

Differential realization

The action of the operator $\hat{T}$ on an analytic function $f(z_i)$ can be expressed, again, using power series:

$$\hat{T}f(z_i) = f(z_i') = f(z_i + i\epsilon \sum_{j=1}^{n} S_{ij} z_j) = f(z_i) + i\epsilon \sum_{ij=1}^{n} S_{ij} z_j \frac{\partial f}{\partial z_i} + ... \qquad (1.22)$$

Hence, to the first order in ϵ we obtain, that the operator:

$$\hat{\xi}_i^j = z_i \frac{\partial}{\partial z_j} \qquad (1.23)$$

generates the unitary transformation. As stated before, the operator $\hat{T}$ is an arbitrary hermitian operator. That means that there are n^2 linearly independent (1.23) operators.

Now, knowing that $[\frac{\partial}{\partial z_j}, z_i] = \delta_{ij}$, operators (1.23) close under these commutation relations:

$$[\hat{\xi}_j^i, \hat{\xi}_n^m] = \hat{\xi}_j^m \delta_{in} - \hat{\xi}_n^i \delta_{mj} \qquad (1.24)$$

Which, again, corresponds with (1.17), and the δ_{ij} symbols play the roles of the structure constants.

The boson realization

We define the boson creation and annihilation operators $\hat{b}_i$ and $\hat{b}_i^\dagger$, where $i = 1, 2, ...n$. These operators close under commutation relations:

$$[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{ij} \qquad (1.25)$$

All the other commutators are equal to zero. To use the boson operators for generating a U(n) algebra, we construct $\hat{G}_j^i$ operators, which have the same properties as (1.16):

$$\hat{G}_j^i = \hat{b}_i^\dagger \hat{b}_j \qquad (1.26)$$

Now we define the vacuum state $|0\rangle$ that satisfies:

$$\hat{b}_i |0\rangle = 0, \;\; i = 1, 2, ..., n$$

Here we imply the usual normalization $\langle 0|0\rangle = 1$.

The first-order Casimir invariants of a U(n) algebra are defined by:

$$\hat{C}_1[U(n)] \equiv \sum_{i=1}^{n} \hat{\xi}_i^i \qquad (1.27)$$

To prove this we should simply substitute the Casimir operator in a commutator $[\hat{C}_1[U(n)], \hat{\xi}_k^l]$ with (1.27):

$$[\hat{C}_1[U(n)], \hat{\xi}_k^l] = \sum_{i=1}^{n} [\hat{\xi}_i^i, \hat{\xi}_k^l] = \sum_{i=1}^{n} (\hat{\xi}_i^l \delta_{ik} - \hat{\xi}_k^i \delta_{il}) = 0$$

The second-order Casimir invariants of a U(n) algebra are given by:

$$\hat{C}_2[U(n)] = \sum_{ij=1}^{n} \hat{\xi}_i^j \hat{\xi}_j^i \qquad (1.28)$$

The verification of commutation of these operators with the generators of the given U(n) algebra is analogical to the one for the first-order Casimir invariants.

1.5 The SU(2) group and its associated algebra

Finally, let us briefly discuss the properties of the SU(2) algebra, which is crucial for this work. The elements of the SU(2) group are 2×2 complex unitary matrices with an additional condition, that their determinants are equal to 1.

From the relation derived in section 1.4 we see, that in our case of the SU(2) algebra the matrix $\hat{T}$ depends on 3 parameters. Looking back at our conclusions (see section 1.4) about an hermitian operator $\hat{T}$, we can write the operator in the form:

$$\begin{aligned}\hat{T} &= \begin{pmatrix} t_3 & t_1 - it_2 \\ t_1 + it_2 & -t_3 \end{pmatrix} = t_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + t_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= t_1\hat{\sigma}_1 + t_2\hat{\sigma}_2 + t_3\hat{\sigma}_3,\end{aligned} \tag{1.29}$$

where $\hat{\sigma}_1$, $\hat{\sigma}_2$, and $\hat{\sigma}_3$ are the Pauli matrices and the unitary transformation $\hat{U}$ can now be expressed:

$$\hat{U} = e^{i\sum_{n=0}^{\infty} s_n \frac{\sigma_n}{2}} \approx \hat{E} + t_1\frac{\hat{\sigma}_1}{2} + t_2\frac{\hat{\sigma}_2}{2} + t_3\frac{\hat{\sigma}_3}{2} \tag{1.30}$$

The Pauli matrices satisfy the condition of a Lie algebra's generators:

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\varepsilon_{ijk}\hat{\sigma}_k \tag{1.31}$$

Chapter 2

Realizations of the SU(2) algebra

Having constructed the required mathematical apparatus, we can begin direct work with the SU(2) algebra. The following sections are dedicated to different ways of its realization.

2.1 The spin realization

We already know that the Pauli matrices generate the SU(2) algebra. Its realization using these operators is the simplest and the fundamental way of representing the SU(2) algebra. Consider a system of N particles, each of them having two possible projections of spin to a given direction. From the theory of total angular momentum [10] we know that this situation corresponds to a system of N particles with the spin $\frac{1}{2}$.

The angular momentum operator $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$ generate the SO(3) algebra that is isomorfic with the SU(2) algebra, which means that there is an isomorfic mapping between these two groups, and satisfy the relation (1.8).

Now, using the definitions of total spin:

$$\begin{aligned}\hat{J}_x &= \frac{1}{2}\sum_{j=1}^{N}\sigma_j^{(x)}\\ \hat{J}_y &= \frac{1}{2}\sum_{j=1}^{N}\sigma_j^{(y)}\\ \hat{J}_z &= \frac{1}{2}\sum_{j=1}^{N}\sigma_j^{(z)}\end{aligned} \tag{2.1}$$

we can prove that these operators form the SU(2) algebra, by showing that they satisfy the condition (1.8). The Hilbert space $\mathcal{H}$ of such system is a direct product of Hilbert spaces of separate particles:

$$\mathcal{H} = \bigotimes_{j=1}^{N}\mathcal{H}_j \tag{2.2}$$

This means that the operators (2.1) can be written as a sum of single-particle operators:

$$\hat{J}_i = \sum_{j=1}^{N} \hat{I} \bigotimes \hat{I} \bigotimes ... \bigotimes \sigma_j^{(i)} \bigotimes ... \bigotimes \hat{I} \tag{2.3}$$

where $\hat{I}$ is the identity operator and $\sigma_j^{(i)}$ is the one-particle operator that is situated on the i-th position in this direct product. This means that such operator, being applied on a wave function (in the Dirac notation) $|\psi\rangle$, which will also be represented as a direct product of single-particle wave functions: $|\psi\rangle = |\psi_1\rangle \otimes ... \otimes |\psi_N\rangle$, will only operate on the $|\psi_j\rangle$ ket.

This means that the commutation relation (1.31) can be generalized:

$$[\sigma_l^{(i)}, \sigma_m^{(j)}] = 2\varepsilon_{ijk}\delta_{lm}\sigma_m^{(k)} \tag{2.4}$$

Substituting (2.1) in (1.8) we obtain:

$$\begin{aligned}[\hat{J}_x, \hat{J}_y] &= [\frac{1}{2}\sum_{j=1}^{N}\sigma_j^{(x)}, \frac{1}{2}\sum_{k=1}^{N}\sigma_k^{(y)}] = \frac{1}{4}[\sum_{j=1}^{N}\sigma_j^{(x)}, \sum_{k=1}^{N}\sigma_k^{(y)}] \\ &= \frac{1}{4}\sum_{jk}[\sigma_j^{(x)}, \sigma_k^{(y)}] = \frac{1}{4}\sum_{jk}2\delta_{jk}\sigma_k^{(z)} = \frac{1}{2}\sum_{k}^{N}\sigma_k^{(z)} = \hat{J}_z\end{aligned} \tag{2.5}$$

$$\begin{aligned}[\hat{J}_z, \hat{J}_x] &= [\frac{1}{2}\sum_{j=1}^{N}\sigma_j^{(z)}, \frac{1}{2}\sum_{k=1}^{N}\sigma_k^{(x)}] = \frac{1}{4}[\sum_{j=1}^{N}\sigma_j^{(z)}, \sum_{k=1}^{N}\sigma_k^{(x)}] \\ &= \frac{1}{4}\sum_{jk}[\sigma_j^{(z)}, \sigma_k^{(x)}] = \frac{1}{4}\sum_{jk}2\delta_{jk}\sigma_k^{(y)} = \frac{1}{2}\sum_{k}^{N}\sigma_k^{(y)} = \hat{J}_y\end{aligned} \tag{2.6}$$

$$\begin{aligned}[\hat{J}_y, \hat{J}_z] &= [\frac{1}{2}\sum_{j=1}^{N}\sigma_j^{(y)}, \frac{1}{2}\sum_{k=1}^{N}\sigma_k^{(z)}] = \frac{1}{4}[\sum_{j=1}^{N}\sigma_j^{(y)}, \sum_{k=1}^{N}\sigma_k^{(z)}] \\ &= \frac{1}{4}\sum_{jk}[\sigma_j^{(y)}, \sigma_k^{(z)}] = \frac{1}{4}\sum_{jk}2\delta_{jk}\sigma_k^{(x)} = \frac{1}{2}\sum_{k}^{N}\sigma_k^{(x)} = \hat{J}_z\end{aligned} \tag{2.7}$$

2.2 The boson representation

One of the ways to represent the SU(2) algebra is the so-called Schwinger representation.

Consider a system with two types of excitations, which we will denote s and t. The hamiltonian of such a system can be realized for example by a 2 dimensional harmonic oscillator. First we define the creation and annihilation operators for

”s-type” and ”t-type” bosons:

$$\begin{aligned}
\hat{s}^\dagger &= \sqrt{\frac{1}{2}}(\hat{x} - i\hat{p_x}) \\
\hat{t}^\dagger &= \sqrt{\frac{1}{2}}(\hat{y} - i\hat{p_y}) \\
\hat{s} &= \sqrt{\frac{1}{2}}(\hat{x} + i\hat{p_x}) \\
\hat{t} &= \sqrt{\frac{1}{2}}(\hat{y} + i\hat{p_y})
\end{aligned} \tag{2.8}$$

Using the properties of the coordinate and momentum operators, we obtain the following properties of these operators:

$$\begin{aligned}
&[\hat{s}, \hat{s}^\dagger] = 1 \\
&[\hat{t}, \hat{t}^\dagger] = 1 \\
&[\hat{s}^\dagger, \hat{t}^\dagger] = 0 \\
&[\hat{s}, \hat{t}] = 0 \\
&[\hat{s}, \hat{s}] = 0 \\
&[\hat{s}^\dagger, \hat{s}^\dagger] = 0 \\
&[\hat{t}, \hat{t}] = 0 \\
&[\hat{t}^\dagger, \hat{t}^\dagger] = 0
\end{aligned} \tag{2.9}$$

The $\hat{s}^\dagger$ and $\hat{s}$ operators create and annihilate an s-type particle correspondingly. The $\hat{t}^\dagger$ and $\hat{t}$ operatos have an analogous action.

The operators $\hat{s}^\dagger, \hat{s}, \hat{t}^\dagger, \hat{t}$ do not form a Lie algebra by themselves, but we can show that $\hat{s}^\dagger, \hat{s}, \hat{t}^\dagger, \hat{t}$ products do in fact generate a group associated with the SU(2) algebra.

Using the definition of the creation and annihilation operators, we can use them to express the angular momentum operators:

$$\hat{J}_x = \frac{1}{2}(\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t}) \quad \hat{J}_y = \frac{i}{2}(\hat{t}^\dagger\hat{s} - \hat{s}^\dagger\hat{t}) \quad \hat{J}_z = \frac{1}{2}(\hat{s}^\dagger\hat{s} - \hat{t}^\dagger\hat{t}) \tag{2.10}$$

To prove the statement we need to prove that the relation (1.8) is valid for the new definition of the angular momentum operators. Using the commutation relations and basic properties of commutators, particularly:

$$[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}\hat{C}[\hat{B}, \hat{D}] + \hat{A}[\hat{B}, \hat{C}]\hat{D} + \hat{C}[\hat{A}, \hat{D}]\hat{B} + [\hat{A}, \hat{C}]\hat{D}\hat{B} \tag{2.11}$$

We obtain:

$$
\begin{aligned}
[\hat{J}_x, \hat{J}_y] &= [\frac{1}{2}(\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t}), \frac{i}{2}(\hat{t}^\dagger\hat{s} - \hat{s}^\dagger\hat{t})] \\
&= \frac{i}{4}[(\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t}), (\hat{t}^\dagger\hat{s} - \hat{s}^\dagger\hat{t})] \\
&= \frac{i}{4}([\hat{t}^\dagger\hat{s}, (\hat{t}^\dagger\hat{s} - \hat{s}^\dagger\hat{t})] + [\hat{s}^\dagger\hat{t}, (\hat{t}^\dagger\hat{s} - \hat{s}^\dagger\hat{t})]) \\
&= \frac{i}{4}([\hat{t}^\dagger\hat{s}, \hat{t}^\dagger\hat{s}] - [\hat{t}^\dagger\hat{s}, \hat{s}^\dagger\hat{t}] + [\hat{s}^\dagger\hat{t}, \hat{t}^\dagger\hat{s}] - [\hat{s}^\dagger\hat{t}, \hat{s}^\dagger\hat{t}]) \\
&= \frac{i}{4}(\hat{s}^\dagger\hat{s} - \hat{t}^\dagger\hat{t} + \hat{s}^\dagger\hat{s} - \hat{t}^\dagger\hat{t}) = \frac{i}{2}(\hat{s}^\dagger\hat{s} - \hat{t}^\dagger\hat{t}) = i(\frac{1}{2}(\hat{s}^\dagger\hat{s} - \hat{t}^\dagger\hat{t})) = i\hat{J}_z
\end{aligned}
$$

$$
\begin{aligned}
[\hat{J}_z, \hat{J}_x] &= [\frac{1}{2}(\hat{s}^\dagger\hat{s} - \hat{t}^\dagger\hat{t}, \frac{1}{2}(\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t})] \\
&= \frac{1}{4}[(\hat{s}^\dagger\hat{s} - \hat{t}^\dagger\hat{t}, (\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t})] \\
&= \frac{1}{4}([\hat{s}^\dagger\hat{s}, (\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t})] - [\hat{t}^\dagger\hat{t}, (\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t})] \\
&= \frac{1}{4}([\hat{s}^\dagger\hat{s}, \hat{t}^\dagger\hat{s}] + [\hat{s}^\dagger\hat{s}, \hat{s}^\dagger\hat{t}] - [\hat{t}^\dagger\hat{t}, \hat{t}^\dagger\hat{s}] - [\hat{t}^\dagger\hat{t}, \hat{s}^\dagger\hat{t}]) \\
&= \frac{1}{4}(-\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t} - \hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t}) \\
&= \frac{1}{2}(\hat{s}^\dagger\hat{t} - \hat{t}^\dagger\hat{s}) = i(\frac{i}{2}(\hat{t}^\dagger\hat{s} - \hat{s}^\dagger\hat{t})) = i\hat{J}_y
\end{aligned}
$$

$$
\begin{aligned}
[\hat{J}_y, \hat{J}_z] &= [\frac{i}{2}(\hat{t}^\dagger\hat{s} - \hat{s}^\dagger\hat{t}), \frac{1}{2}(\hat{s}^\dagger\hat{s} - \hat{t}^\dagger\hat{t})] \\
&= \frac{i}{4}[(\hat{t}^\dagger\hat{s} - \hat{s}^\dagger\hat{t}), (\hat{s}^\dagger\hat{s} - \hat{t}^\dagger\hat{t})] \\
&= \frac{i}{4}([\hat{t}^\dagger\hat{s}, (\hat{s}^\dagger\hat{s} - \hat{t}^\dagger\hat{t})] - [\hat{s}^\dagger\hat{t}, (\hat{s}^\dagger\hat{s} - \hat{t}^\dagger\hat{t})]) \\
&= \frac{i}{4}([\hat{t}^\dagger\hat{s}, \hat{s}^\dagger\hat{s}] - [\hat{t}^\dagger\hat{s}, \hat{t}^\dagger\hat{t}] - [\hat{s}^\dagger\hat{t}, \hat{s}^\dagger\hat{s}] + [\hat{s}^\dagger\hat{t}, \hat{t}^\dagger\hat{t}]) \\
&= \frac{i}{4}(\hat{t}^\dagger\hat{s} - (-\hat{t}^\dagger\hat{s}) - (-\hat{s}^\dagger\hat{t}) + \hat{s}^\dagger\hat{t}) \\
&= \frac{i}{2}(\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t}) \\
&= i(\frac{1}{2}(\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t})) = i\hat{J}_x
\end{aligned}
$$

We only prove the "positive" order of the operators in the left-side commutators. Changing the order within a commutator would result in the alternation of the indices of the Levi-Civita tensor, thus changing the sign of the result.

Now we define several more operators, that are useful for practical calculations. These are the operator of the number of s-bosons $\hat{n}_s$, the operator of the number of t-bosons $\hat{n}_t$ and the total boson number of the given system $\hat{N}$:

$$
\hat{n}_s = \hat{s}^\dagger\hat{s}, \quad \hat{n}_t = \hat{t}^\dagger\hat{t}, \quad \hat{N} = \hat{n}_s + \hat{n}_t \tag{2.12}
$$

To illustrate the physical meaning of this operators, let us consider a system consisting of n_s s-type boson and n_t t-type bosons: $|n_s, n_t\rangle$. Operators (2.12) are defined in the same way as shown in section 1.4. Which means, that applying them on this state on this state we obtain:

$$\begin{aligned}
\hat{n}_s\,|n_s, n_t\rangle &= n_s\,|n_s, n_t\rangle \\
\hat{n}_t\,|n_s, n_t\rangle &= n_t\,|n_s, n_t\rangle \\
\hat{N}\,|n_s, n_t\rangle &= (n_s + n_t)\,|n_s, n_t\rangle = N\,|n_s, n_t\rangle
\end{aligned} \tag{2.13}$$

where N is the total number of particles in the system.

The operator $\hat{N}$ is in fact a linear Casimir invariant of the SU(2) algebra (see section 1.2):

$$[\hat{N}, \hat{s}^\dagger\hat{t}] = [\hat{N}, \hat{s}^\dagger\hat{s}] = [\hat{N}, \hat{t}^\dagger\hat{s}] = [\hat{N}, \hat{t}^\dagger\hat{t}] = 0$$

A very interesting property of such a system consists in the fact that the second power of the total angular momentum operator $\hat{J}^2$ is a function of the total boson number operator $\hat{N}$.

To prove this we will use the definition of the $\hat{J}^2$ operator:

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \tag{2.14}$$

which is the quadratic Casimir operator of the SU(2) algebra: $[\hat{J}^2, \hat{J}_k] = 0$, $\forall k$.

Now we find the sum on the right side of the equation (2.14) expressed with boson operators and the boson number operators:

$$\begin{aligned}
\hat{J}_x^2 &= \frac{1}{2}(\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t})(\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t}) \\
&= \frac{1}{4}(\hat{t}^\dagger\hat{s}\hat{t}^\dagger\hat{s} + \hat{t}^\dagger\hat{s}\hat{s}^\dagger\hat{t} + \hat{s}^\dagger\hat{t}\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t}\hat{s}^\dagger\hat{t}) \\
&= \frac{1}{4}(\hat{t}^\dagger\hat{s}\hat{t}^\dagger\hat{s} + \hat{t}^\dagger(1 + \hat{n}_s)\hat{t} + \hat{s}^\dagger(1 + \hat{n}_t)\hat{s} + \hat{s}^\dagger\hat{t}\hat{s}^\dagger\hat{t}) \\
&= \frac{1}{4}(\hat{t}^\dagger\hat{s}\hat{t}^\dagger\hat{s} + \hat{t}^\dagger\hat{t} + \hat{t}^\dagger\hat{n}_s\hat{t} + \hat{s}^\dagger\hat{s} + \hat{s}^\dagger\hat{n}_t\hat{s} + \hat{s}^\dagger\hat{t}\hat{s}^\dagger\hat{t}) \\
&= \frac{1}{4}(\hat{t}^\dagger\hat{s}\hat{t}^\dagger\hat{s} + \hat{n}_t + \hat{n}_t\hat{n}_s + \hat{n}_s + \hat{n}_s\hat{n}_t + \hat{s}^\dagger\hat{t}\hat{s}^\dagger\hat{t}) \\
&= \frac{1}{4}(\hat{n}_t + \hat{n}_s + 2\hat{n}_s\hat{n}_t + \hat{t}^\dagger\hat{s}\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t}\hat{s}^\dagger\hat{t}) \\
&= \frac{1}{4}(\hat{N} + 2\hat{n}_s\hat{n}_t + \hat{t}^\dagger\hat{s}\hat{t}^\dagger\hat{s}\hat{s}^\dagger\hat{t}\hat{s}^\dagger\hat{t})
\end{aligned}$$

$$\begin{aligned}
\hat{J}_y^2 &= -\frac{1}{4}(\hat{t}^\dagger\hat{s} - \hat{s}^\dagger\hat{t})(\hat{t}^\dagger\hat{s} - \hat{s}^\dagger\hat{t}) \\
&= -\frac{1}{4}(\hat{t}^\dagger\hat{s}\hat{t}^\dagger\hat{s} - \hat{t}^\dagger\hat{s}\hat{s}^\dagger\hat{t} - \hat{s}^\dagger\hat{t}\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t}\hat{s}^\dagger\hat{t}) \\
&= -\frac{1}{4}(\hat{t}^\dagger\hat{s}\hat{t}^\dagger\hat{s} - \hat{t}^\dagger(1 + \hat{n}_s)\hat{t} - \hat{s}^\dagger(1 + \hat{n}_t)\hat{s} + \hat{s}^\dagger\hat{t}\hat{s}^\dagger\hat{t}) \\
&= -\frac{1}{4}(\hat{t}^\dagger\hat{s}\hat{t}^\dagger\hat{s} - \hat{t}^\dagger\hat{t} - \hat{t}^\dagger\hat{n}_s\hat{t} - \hat{s}^\dagger\hat{s} - \hat{s}^\dagger\hat{n}_t\hat{s} + \hat{s}^\dagger\hat{t}\hat{s}^\dagger\hat{t}) \\
&= -\frac{1}{4}(\hat{t}^\dagger\hat{s}\hat{t}^\dagger\hat{s} - \hat{N} - 2\hat{n}_s\hat{n}_t + \hat{s}^\dagger\hat{t}\hat{s}^\dagger\hat{t})
\end{aligned}$$

$$\hat{J}_z^2 = \frac{1}{4}(\hat{s}^\dagger\hat{s} - \hat{t}^\dagger\hat{t})(\hat{s}^\dagger\hat{s} - \hat{t}^\dagger\hat{t})$$
$$= \frac{1}{4}(\hat{n}_s - \hat{n}_t)(\hat{n}_s - \hat{n}_t) = \frac{1}{4}(\hat{n}_s^2 - \hat{n}_s\hat{n}_t - \hat{n}_t\hat{n}_s + \hat{n}_t^2)$$
$$= \frac{1}{4}(\hat{n}_s^2 - 2\hat{n}_s\hat{n}_t + \hat{n}_t^2)$$

Substituting the obtained expression in (2.14):

$$\hat{J}^2 = \frac{1}{4}(\hat{N} + 2\hat{n}_s\hat{n}_t + \hat{t}^\dagger\hat{s}\hat{t}^\dagger\hat{s} + \hat{s}^\dagger\hat{t}\hat{s}^\dagger\hat{t} + \hat{N} + 2\hat{n}_s\hat{n}_t - \hat{t}^\dagger\hat{s}\hat{t}^\dagger\hat{s} - \hat{s}^\dagger\hat{t}\hat{s}^\dagger\hat{t} + \hat{n}_s - 2\hat{n}_s\hat{n}_t + \hat{n}_t^2)$$
$$= \frac{1}{4}(2\hat{N} + 4\hat{n}_s\hat{n}_t + \hat{n}_s^2 - 2\hat{n}_s\hat{n}_t + \hat{n}_t^2) = \frac{1}{4}(2\hat{N} + 2\hat{n}_s\hat{n}_t + \hat{n}_s^2 + \hat{n}_t^2)$$
$$= \frac{1}{4}(2\hat{N} + (\hat{n}_s + \hat{n}_t)^2) = \frac{1}{4}\hat{N}(\hat{N} + 2)$$

The last equality can be rewritten as: $\frac{\hat{N}}{2}(\frac{\hat{N}}{2} + 1)$. Comparing this to (1.9). We find, that $2j = N$ or $j = \frac{N}{2}$.

2.3 Fermion representation of the SU(2) algebra

The SU(2) algebra can be realized also in terms of fermions. Consider a system with two energy levels, which we denote E_+ and E_- with N positions on each (see fig. 2.1).

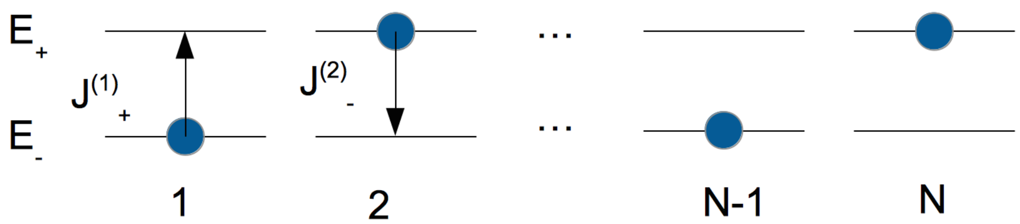

Figure 2.1: A two-energy level system

Analogically with the boson algebra we can define an operator $\hat{a}_{+i}^\dagger$ that will create a particle on the i-th position on the E_+ - energy level, and an operator $\hat{a}_{+i}$ that will correspondingly destroy a particle on that position. Operators $\hat{a}_{-i}^\dagger$ and $\hat{a}_{-i}$ act similarly on the E_- - energy level. These are the so-called fermion operators, which satisfy the following anti-commutation relations:

$$\{\hat{a}_{+i}, \hat{a}_{+j}^\dagger\} = \hat{a}_{+i}\hat{a}_{+j}^\dagger + \hat{a}_{+j}^\dagger\hat{a}_{+i} = \delta_{ij}, \quad \{\hat{a}_{+i}, \hat{a}_{-j}^\dagger\} = 0, \quad \{\hat{a}_{-i}, \hat{a}_{-j}\} = \delta_{ij} \qquad (2.15)$$

Using relations (2.15) we can again prove that linear combinations of operator products $\hat{a}_{+i}^\dagger\hat{a}_{+i}$, $\hat{a}_{-i}^\dagger\hat{a}_{-i}$, $\hat{a}_{+i}^\dagger\hat{a}_{-i}$, $\hat{a}_{-i}^\dagger\hat{a}_{+i}$, $i = (1, ..., N)$:

$$
\begin{aligned}
\hat{J}_z &= \sum_{i=1}^{N} \frac{1}{2}(\hat{a}^\dagger_{+i}\hat{a}_{+i} - \hat{a}^\dagger_{-i}\hat{a}_{-i}) = \sum_{i=1}^{N} \frac{1}{2}\hat{J}^{(i)}_z \\
\hat{J}_+ &= \sum_{i=1}^{N} \hat{a}^\dagger_{+i}\hat{a}_{-i} = \sum_{i=1}^{N} \hat{J}^{(i)}_+ \\
\hat{J}_- &= \hat{J}^+_+ = \sum_{i=1}^{N} \hat{a}^\dagger_{-i}\hat{a}_{+i} = \sum_{i=1}^{N} \hat{J}^{(i)}_-
\end{aligned} \tag{2.16}
$$

generate the SU(2) algebra. Here by $\hat{J}^{(i)}$ we denote single-particle ladder operator that functions on the i-th position.

Since $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ and using the anti-commutation relations (2.15) we find that:

$$
\begin{aligned}
[\hat{J}_x, \hat{J}_y] &= [\frac{1}{2}(\hat{J}_+ + \hat{J}_-), -\frac{i}{2}(\hat{J}_+ - \hat{J}_-)] \\
&= -\frac{i}{4}([\hat{J}_+, \hat{J}_+] - [\hat{J}_+, \hat{J}_-] + [\hat{J}_-, \hat{J}_+] - [\hat{J}_-, \hat{J}_-]) = \frac{i}{4}(2[\hat{J}_+, \hat{J}_-]) = \frac{i}{2}[\hat{J}_+, \hat{J}_-] \\
&= \frac{i}{2}[\sum_{k=1}^{N} \hat{a}^\dagger_{+k}\hat{a}_{-k}, \sum_{m=1}^{N} \hat{a}^\dagger_{-m}\hat{a}_{+m}] = \frac{i}{2}\sum_{k=1}^{N}\sum_{m=1}^{N}[\hat{a}^\dagger_{+k}\hat{a}_{-k}, \hat{a}^\dagger_{-m}\hat{a}_{+m}] \\
&= \frac{i}{2}\sum_{k=1}^{N}\sum_{m=1}^{N}(\hat{a}^\dagger_{+k}\hat{a}_{-k}\hat{a}^\dagger_{-m}\hat{a}_{-m} - \hat{a}^\dagger_{-m}\hat{a}_{+m}\hat{a}^\dagger_{+k}\hat{a}_{-k}) \\
&= \frac{i}{2}\sum_{k=1}^{N}\sum_{m=1}^{N}(\hat{a}^\dagger_{+k}(\delta_{km} - \hat{a}^\dagger_{-m}\hat{a}_{-k})\hat{a}_{+m} - \hat{a}^\dagger_{+m}(\delta_{mk} - \hat{a}^\dagger_{+k}\hat{a}^\dagger_{+m})\hat{a}_{-k}) \\
&= \frac{i}{2}\sum_{k=1}^{N}\sum_{m=1}^{N}(\hat{a}^\dagger_{+k}\delta_{km}\hat{a}_{+m} - \hat{a}^\dagger_{+k}\hat{a}^\dagger_{-m}\hat{a}_{-k}\hat{a}_{+m} - \hat{a}^\dagger_{-m}\delta_{mk}\hat{a}_{-k} + \hat{a}^\dagger_{-m}\hat{a}^\dagger_{+k}\hat{a}_{+m}\hat{a}_{-k}) \\
&= \frac{i}{2}\sum_{k=1}^{N}\sum_{m=1}^{N}(\hat{a}^\dagger_{+k}\delta_{km}\hat{a}_{+m} - \hat{a}^\dagger_{-m}\delta_{mk}\hat{a}_{-k} - \hat{a}^\dagger_{-m}\hat{a}^\dagger_{+k}\hat{a}_{+m}\hat{a}_{-k} + \hat{a}^\dagger_{-m}\hat{a}^\dagger_{+k}\hat{a}_{+m}\hat{a}_{-k}) \\
&= \frac{i}{2}\sum_{k=1}^{N}\sum_{m=1}^{N}(\hat{a}^\dagger_{+k}\delta_{km}\hat{a}_{+m} - \hat{a}^\dagger_{-m}\delta_{mk}\hat{a}_{-k}) = \frac{i}{2}\sum_{k=1}^{N}(\hat{a}^\dagger_{+k}\hat{a}_{+k} - \hat{a}^\dagger_{-k}\hat{a}_{-k}) \\
&= i(\frac{1}{2}\sum_{k=1}^{N}(\hat{a}^\dagger_{+k}\hat{a}_{+k}\hat{a}^\dagger_{-k}\hat{a}_{-k})) = i\hat{J}_z
\end{aligned}
$$

$$
\begin{aligned}
[\hat{J}_z, \hat{J}_x] &= [\hat{J}_z, \frac{1}{2}(\hat{J}_+ + \hat{J}_-)] = [\hat{J}_z, \frac{1}{2}\hat{J}_+] + [\hat{J}_z, \frac{1}{2}\hat{J}_-] \\
&= \frac{1}{4}([\sum_{k=1}^{N}(\hat{a}^\dagger_{+k}\hat{a}_{+k} - \hat{a}^\dagger_{-k}\hat{a}_{-k}), \hat{J}_+] + [\sum_{n=1}^{N}(\hat{a}^\dagger_{+n}\hat{a}^\dagger_{+n} - \hat{a}^\dagger_{-n}\hat{a}_{-n}), \hat{J}_-]) \\
&= \frac{1}{4}([\sum_{k=1}^{N}(\hat{a}^\dagger_{+k}\hat{a}_{+k} - \hat{a}^\dagger_{-k}\hat{a}_{-k}), \sum_{m=1}^{N}\hat{a}^\dagger_{+m}\hat{a}_{-m}] \\
&+ [\sum_{n=1}^{N}(\hat{a}^\dagger_{+n}\hat{a}_{+n} - \hat{a}^\dagger_{-n}\hat{a}_{-n}), \sum_{l=1}^{N}\hat{a}^\dagger_{-l}\hat{a}_{+l}]) = (*)
\end{aligned}
$$

Lets expand the first of these commutators:

$$[\sum_{k=1}^{N}(\hat{a}^\dagger_{+k}\hat{a}_{+k} - \hat{a}^\dagger_{-k}\hat{a}_{-k}), \sum_{m=1}^{N}\hat{a}^\dagger_{+m}\hat{a}_{-m}] =$$

$$\sum_{k=1}^{N}\sum_{m=1}^{N}(\hat{a}^\dagger_{+k}\delta_{km}\hat{a}_{-m} + \hat{a}^\dagger_{+m}\delta_{mk}\hat{a}_{-k}) = \sum_{k=1}^{N}2\hat{a}^\dagger_{+k}\hat{a}_{-k}$$

For the second one we obtain:

$$[\sum_{n=1}^{N}(\hat{a}^\dagger_{+n}\hat{a}_{+n} - \hat{a}^\dagger_{-n}\hat{a}_{-n}), \sum_{l=1}^{N}\hat{a}^\dagger_{-l}\hat{a}_{+l}] =$$

$$\sum_{n=1}^{N}\sum_{l=1}^{N}(-\hat{a}^\dagger_{-l}\delta_{ln}\hat{a}_{+n} - \hat{a}^\dagger_{-n}\delta_{nl}\hat{a}_{+l}) = \sum_{l=1}^{N}(-2\hat{a}^\dagger_{-l}\hat{a}_{+l})$$

Therefore:

$$(*) = \frac{1}{4}(\sum_{k=1}^{N}2\hat{a}^\dagger_{+k}\hat{a}_{-k} - \sum_{l=1}^{N}2\hat{a}^\dagger_{-l}\hat{a}_{+l}) = \frac{1}{2}(\sum_{k=1}^{N}(\hat{a}^\dagger_{+k}\hat{a}_{-k} - \hat{a}^\dagger_{-k}\hat{a}_{+k})$$
$$= i(-\frac{i}{2}\sum_{k=1}^{N}(\hat{a}^\dagger_{+k}\hat{a}_{-k} - \hat{a}^\dagger_{-k}\hat{a}_{+k}) = i(-\frac{i}{2}(\hat{J}_+ - \hat{J}_-)) = i\hat{J}_y$$

$$[\hat{J}_y, \hat{J}_z] = [\frac{i}{2}(\hat{J}_- - \hat{J}_+), \hat{J}_z] = [\frac{i}{2}\hat{J}_-, \hat{J}_z] - [\frac{i}{2}\hat{J}_+, \hat{J}_z]$$
$$= \frac{i}{4}([\sum_{m=1}^{N}\hat{a}^\dagger_{-m}\hat{a}_{+m}\sum_{k=1}^{N}(\hat{a}^\dagger_{+k}\hat{a}_{+k} - \hat{a}^\dagger_{-k}\hat{a}_{-k})]$$
$$- [\sum_{l=1}^{N}\hat{a}^\dagger_{+l}\hat{a}_{-l}, \sum_{n=1}^{N}(\hat{a}^\dagger_{+n}\hat{a}_{+n} - \hat{a}^\dagger_{-n}\hat{a}_{-n})]) = (**)$$

Again, expanding the first commutator:

$$[\sum_{m=1}^{N}\hat{a}^\dagger_{-m}\hat{a}_{+m}, \sum_{k=1}^{N}(\hat{a}^\dagger_{+k}\hat{a}_{+k} - \hat{a}^\dagger_{-k}\hat{a}_{-k})] =$$

$$\sum_{k=1}^{N}\sum_{m=1}^{N}(\hat{a}^\dagger_{-m}\delta_{mk}\hat{a}_{+k} + \hat{a}^\dagger_{-k}\delta_{km}\hat{a}_{+m}) = \sum_{k=1}^{N}2\hat{a}^\dagger_{-k}\hat{a}_{+k}$$

The second commutator:

$$[\sum_{l=1}^{N}\hat{a}^\dagger_{+l}\hat{a}_{-l}, \sum_{n=1}^{N}(\hat{a}^\dagger_{+n}\hat{a}_{+n} - \hat{a}^\dagger_{-n}\hat{a}_{-n})] =$$

$$\sum_{l=1}^{N}\sum_{n=1}^{N}(-\hat{a}^\dagger_{+n}\delta_{nl}\hat{a}_{-l} - \hat{a}^\dagger_{+l}\delta_{ln}\hat{a}_{-n}) = \sum_{l=1}^{N}(-2\hat{a}^\dagger_{+l}\hat{a}_{-l})$$

Thus we obtain:

$$
\begin{aligned}
(**) &= \frac{i}{4}(\sum_{k=1}^{N} 2\hat{a}_{-k}^{\dagger}\hat{a}_{+k} + \sum_{l=1}^{N} 2\hat{a}_{+l}^{\dagger}\hat{a}_{-l}) \\
&= \frac{i}{2}\sum_{k=1}^{N}(\hat{a}_{-k}^{\dagger}\hat{a}_{+k} + \hat{a}_{+k}^{\dagger}\hat{a}_{-k}) = i(\frac{1}{2}\sum_{k=1}^{N}(\hat{a}_{-k}^{\dagger}\hat{a}_{+k} + \hat{a}_{+k}^{\dagger}\hat{a}_{-k})) \\
&= i(\frac{1}{2}(\hat{J}_{+} + \hat{J}_{-})) = i\hat{J}_{x}
\end{aligned}
$$

Chapter 3

The general form of a SU(2) hamiltonian

Using the definitions of the angular momentum operators (2.10), we can write the general hamiltonian with one- and two-body interactions:

$$\hat{H} = A\hat{J}_x + B\hat{J}_y + C\hat{J}_z + D\hat{J}_x^2 + E\hat{J}_y^2 + F(\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x) + G(\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x) + I(\hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y) \tag{3.1}$$

where A, B, C, D, E, F, G, I are constants.

It is worth mentioning, that the term $K\hat{J}_z^2$ (where K is a constant analogous to $A...I$) is not a compulsory part of the hamiltonian. The fact that:

$$\hat{J}^2 = \frac{1}{4}\hat{N}(\hat{N} + 2) \tag{3.2}$$

and the condition, that the number of particles in the given system doesn't change, allows us to exclude this term. We can express it as:

$$K\hat{J}_z^2 = K(\hat{J}^2 - \hat{J}_x^2 - \hat{J}_y^2) = K\left(\frac{1}{4}\hat{N}(\hat{N} + 2) - \hat{J}_x^2 - \hat{J}_y^2\right) \tag{3.3}$$

The last two terms in the equation (3.3) will obviously just alter the constants D and E: $D' = D - K$, $E' = E - K$.

The operator $\hat{J}^2$, being applied on a state vector, will shift the hamiltonian by a scalar number, that is proportional to the number of particles (which we consider to be constant), thus shifting all the energy levels equally.

There is a similar reason for the exclusion of the antisymmetric terms such as $\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x$. These terms are commutators of each pair of the components of the angular momentum, which, according to (3.1) will contribute to the single-body terms.

The choice of operators used to express the hamiltonian can be explained by the second-quantizations theory and, again, (2.10). A one- and two-body hamiltonian for N particles is:

$$\hat{H} = \sum_{i=1}^{N}(\frac{1}{2}\hat{p}_i^2 + \hat{U}_i) + \sum_{i<j}^{N} V_{ij}, \tag{3.4}$$

where the first sum represents the particles' own kinetic and potential energies and the second one represents the two-body interactions. This hamiltonian can be rewritten in terms of one- and two-body matrix elements:

$$\begin{aligned}\sum_{i=1}^{N}(\frac{1}{2}\hat{p}_i^2+\hat{U}_i) &\rightarrow \sum_{\alpha\alpha'}\langle\alpha'|\frac{1}{2}\hat{p}_i^2+\hat{U}_i|\alpha\rangle\,\hat{b}^\dagger_{\alpha'}\hat{b}_\alpha \\ \sum_{i<j}^{N}V_{ij} &\rightarrow \frac{1}{2}\sum_{\alpha_1\alpha_2}\sum_{\alpha'_1\alpha'_2}\langle\alpha'_1\alpha'_2|V_{ij}|\alpha_1\alpha_2\rangle\,\hat{b}^\dagger_{\alpha'_1}\hat{b}^\dagger_{\alpha'_2}\hat{b}_{\alpha_2}\hat{b}_{\alpha_1}\end{aligned} \tag{3.5}$$

where $\hat{b}^\dagger_\alpha$ and $\hat{b}_\alpha$ are the creation and annihilation operators and $|\alpha_i\rangle$ and $|\alpha_i\alpha_j\rangle$ kets form a complete set of one- and two-particle states in the original configuration space.

In our case, the matrix elements are included in the constants $A...I$. The first-order operators in (2.14) are the SU(2) representations of the one-body part of (3.3). The second-order and the "mixed" ones represent the two-body part. The presence of all "mixed" operators guarantee the all possible combinations of the s- and t- creation and annihilations operators.

Now, expressing the $\hat{J}_x$ and $\hat{J}_y$ via $\hat{J}_+$ and $\hat{J}_-$:

$$\begin{aligned}\hat{J}_\pm=\hat{J}_x\pm i\hat{J}_y\rightarrow \hat{J}_x &= \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right)\\ \hat{J}_y &= \frac{i}{2}\left(\hat{J}_y-\hat{J}_x\right),\end{aligned}$$

and applying the obtained hamiltonian on the eigenkets of $\hat{J}_z$:

$$\hat{J}_\pm\,|j,m\rangle=\sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm 1\rangle\,,$$

we can construct the hamiltonian matrix in the basis $|j,m\rangle$, where j is fixed, diagonalize it and find the energy spectrum.

To illustrate, let's consider a system of two particles with a hamiltonian:

$$\hat{H}=A\hat{J}_z^2+B\hat{J}_x \tag{3.6}$$

Here, the $\hat{J}_x$ operator represents an interaction with a magnetic field, and $\hat{J}_z^2$ represents the mutual interaction between particles.

From (3.2) we can determine that the total angular momentum quantum number is $j=\frac{N}{2}=\frac{2}{2}=1$. That means that the matrix representation of (3.6) is a $2j+1\times 2j+1=3\times 3$ matrix in the basis of the $\hat{J}_z$ eigenkets. We find:

$$\hat{H}=\begin{pmatrix} A & \frac{B}{\sqrt{2}} & 0\\ \frac{B}{\sqrt{2}} & 0 & \frac{B}{\sqrt{2}}\\ 0 & \frac{B}{\sqrt{2}} & A\end{pmatrix}$$

Hence, the eigenvalues of this hamiltonian are:

$$\begin{aligned}E_1&=A,\\ E_2&=\frac{1}{2}(A-\sqrt{A^2+4B^2}),\\ E_3&=\frac{1}{2}(A+\sqrt{A^2+4B^2})\end{aligned}$$

Chapter 4

The classical limit of the general hamiltonian

4.1 Coherent states

In the following section we want to investigate the behavior energy levels of the given SU(2) hamiltonian in the classical limit.

The standart way to make a transition from quantum mechanics to classical mechanics is to look at the system in the limit:

$$\lim_{h \to 0} \hat{H} = H_{cl} \tag{4.1}$$

where h is the Planck's constant, $\hat{H}$ and H_{cl} are quantum and classical hamiltonians respectively.

This transition is analogous to the one in optics: we get from wave optics to geometrical optics putting the same condition on the wave length λ: $\lambda \to 0$. Hence $|\overrightarrow{k}| \to 0$, where $\overrightarrow{k}$ is the wave vector [8].

In our case a more useful way of obtaining the classical hamiltonian is to use the *Glauber* coherent states.

Definition 4.1.1. *Glauber states $|z\rangle$ are defined by:*

$$|z\rangle = e^{z\hat{a}^\dagger - \bar{z}\hat{a}} |0\rangle = e^{-\frac{|z|}{2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} z^n |n\rangle \tag{4.2}$$

where the first equality is a unitary transformation of the vacuum state $|0\rangle$.

These states have several notable properties:

$$\langle z_1 | z_2 \rangle = e^{-\frac{1}{2}|z_1|^2 + z_1 \bar{z}_2 - \frac{1}{2}|z_2|^2}$$
$$\frac{1}{\pi} \int |z\rangle \langle z| \, d^2 z = I$$
$$\hat{a} |z\rangle = z |z\rangle$$

The first property tells us that these states are not orthogonal, but from the second one we know that they satisfy the resolution of identity. The last one shows us, that the states $|z\rangle$ are the eigenkets of the annihilation operator $\hat{a}$.

Consider a hamiltonian, which is a function of the SU(2) algebra generators. We can show that the mean values of a hamiltonian in the Glauber states $|z\rangle$ form a continuous spectrum of the classical limit of the hamiltonian.

To prove the this statement let's perform the calculation of $\langle z|\hat{H}|z\rangle$. Without the loss of generality we assume that the states $|n\rangle$ from the definition of the Glauber states are the eigenkets if the given hamiltonian with the corresponding eigenvalues n: $\hat{H}|n\rangle = n|n\rangle$. By calculating we obtain:

$$\begin{aligned}
\langle z|\hat{H}|z\rangle &= \left(e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{(\bar{z})^n}{\sqrt{n!}}\langle n|\right)\hat{H}\left(e^{-\frac{1}{2}|z|^2}\sum_{k=0}^{\infty}\frac{z^k}{\sqrt{k!}}|k\rangle\right) \\
&= \left(e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{(\bar{z})^n}{\sqrt{n!}}\langle n|\right)\left(e^{-\frac{1}{2}|z|^2}\sum_{k=0}^{\infty}\frac{z^k}{\sqrt{k!}}\hat{H}|k\rangle\right) \\
&= \left(e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{(\bar{z})^n}{\sqrt{n!}}\langle n|\right)\left(e^{-\frac{1}{2}|z|^2}\sum_{k=0}^{\infty}\frac{z^k}{\sqrt{k!}}k|k\rangle\right) \\
&= \left(e^{-|z|^2}\sum_{n,k}\frac{(\bar{z})^n z^k}{\sqrt{n!k!}}k\langle n|k\rangle\right) = \left(e^{-|z|^2}\sum_{n,k}\frac{(\bar{z})^n z^k}{\sqrt{n!k!}}k\delta_{nk}\right) \\
&= \left(e^{-|z|^2}\sum_{k=0}^{\infty}\frac{(\bar{z}z)^k}{k!}k\right) = e^{-|z|^2}\sum_{k=0}^{\infty}\frac{|z|^{2k}}{k!}k \\
&= e^{-r^2}\sum_{k=1}^{\infty}\frac{(r^2)^k}{(k-1)!} = e^{-r^2}r^2\sum_{k=1}^{\infty}\frac{(r^2)^{(k-1)}}{(k-1)!} \\
&= e^{-r^2}r^2\sum_{l=0}^{\infty}\frac{(r^2)^l}{l!} = e^{-r^2}r^2e^{r^2} = r^2
\end{aligned}$$

where we denote $|z| = r$ and $l = k-1$. From the resulting relation $\langle z|\hat{H}|z\rangle = r^2 = |z|^2$ we see that we indeed obtain a continuous spectrum.

Another interesting property, which follows the ones listed above, is the mean value of the position and momentum operators in states $|z\rangle$. If we express them in the terms of ladder operators, again denoting the total number of particles as N:

$$\begin{aligned}
\hat{b}^\dagger &= \sqrt{\frac{N}{2}}(\hat{x} - i\hat{p}) \\
\hat{b} &= \sqrt{\frac{N}{2}}(\hat{x} + i\hat{p})
\end{aligned}$$

we obtain:

$$\begin{aligned}
\hat{x} &= \frac{1}{\sqrt{2N}}\left(\hat{b}^\dagger + \hat{b}\right) \\
\hat{p} &= \frac{i}{\sqrt{2N}}\left(\hat{b}^\dagger - \hat{b}\right)
\end{aligned}$$

Now we can determine their mean values in the state $|z\rangle$:

$$
\begin{aligned}
\langle z|\hat{x}|z\rangle &= \langle z|\frac{1}{\sqrt{2N}}\left(\hat{b}^\dagger + \hat{b}\right)|z\rangle = \frac{1}{\sqrt{2N}}\left(\langle z\hat{b}^\dagger|z\rangle + \langle z|\hat{b}z\rangle\right) \\
&= \frac{1}{\sqrt{2N}}\left(\bar{z}\langle z|z\rangle + z\langle z|z\rangle\right) = \sqrt{\frac{2}{N}}Re\{z\}
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
\langle z|\hat{p}|z\rangle &= \langle z|\frac{i}{\sqrt{2N}}\left(\hat{b}^\dagger - \hat{b}\right)|z\rangle = \frac{i}{\sqrt{2N}}\left(\langle z|\hat{a}^\dagger z\rangle - \langle z\hat{a}|z\rangle\right) \\
&= \frac{i}{\sqrt{2N}}\left(z\langle z|z\rangle - \bar{z}\langle z|z\rangle\right) = -\sqrt{\frac{2}{N}}Im\{z\}
\end{aligned} \tag{4.4}
$$

4.2 Holstein-Primakoff mapping

Let us return to the general form of an SU(2) hamiltonian (3.1). To investigate its classical limit we are going to use the Holstein-Primakoff transformation, which is defined by introducing new creation and annihilation operators:

$$
\begin{aligned}
\hat{b}^\dagger &= \sqrt{\frac{N}{2}}(\hat{x} - i\hat{p}) \\
\hat{b} &= \sqrt{\frac{N}{2}}(\hat{x} + i\hat{p}) \\
\hat{b}^\dagger\hat{b} &= \frac{N}{2}(\hat{x}^2 + \hat{p}^2) \\
[\hat{b}, \hat{b}^\dagger] &= 1
\end{aligned} \tag{4.5}
$$

4.2.1 Holstein-Primakoff transformation

To construct our hamiltonian using these new operators, we need to express the $\hat{J}_z$, $\hat{J}_+$, and $\hat{J}_-$ operators:

$$
\begin{aligned}
\hat{J}_+ &= \hat{b}^\dagger\sqrt{N - \hat{b}^\dagger\hat{b}} \\
\hat{J}_- &= \sqrt{N - \hat{b}^\dagger\hat{b}}\hat{b} \\
\hat{J}_z &= \hat{b}^\dagger\hat{b} - \frac{N}{2}
\end{aligned} \tag{4.6}
$$

Using the (4.5) relations, we can find:

$$
\begin{aligned}
\hat{J}_z &= \frac{N}{2}(\hat{x}^2 + \hat{p}^2 + 1) \\
\hat{J}_+ &= \frac{N}{2}(\hat{x} + i\hat{p})\sqrt{2 - (\hat{x}^2 + \hat{p}^2)} \\
\hat{J}_- &= \frac{N}{2}(x - ip)\sqrt{2 - (\hat{x}^2 + \hat{p}^2)}
\end{aligned} \tag{4.7}
$$

The $\hat{J}_x$ and $\hat{J}_y$ operators are constructed as usually:

$$
\begin{aligned}
\hat{J}_x &= \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right) = \frac{N}{2} x\sqrt{2-(\hat{x}^2+\hat{p}^2)} \\
\hat{J}_y &= \frac{i}{2}\left(\hat{J}_- - \hat{J}_+\right) = \frac{N}{2} p\sqrt{2-(\hat{x}^2+\hat{p}^2)}
\end{aligned}
\tag{4.8}
$$

From the relations for $\hat{J}_-$, $\hat{J}_+$, $\hat{J}_x$, and $\hat{J}_y$ we see, that the values of x and p should lie within a circumference with a radius $\sqrt{2}$ in a one-dimensional space (or a hypersphere of the same radius in a n-dimensional space).

Let us prove that after the Holstein-Primakoff transformation the angular momentum operators generate an algebra. To prove this statement we will concentrate on the commutators of $\hat{J}_+$, $\hat{J}_-$, and $\hat{J}_z$. Performing the transformation we obtain and using the formula (2.11) and (4.5):

$$
\begin{aligned}
[\hat{J}_+, \hat{J}_-] &= \left[\hat{b}^\dagger\sqrt{N-\hat{b}^\dagger\hat{b}}, \sqrt{N-\hat{b}^\dagger\hat{b}}\hat{b}\right] = \hat{b}^\dagger\sqrt{N-\hat{b}^\dagger\hat{b}}\left[\sqrt{N-\hat{b}^\dagger\hat{b}}, b\right] \\
&+ \sqrt{N-\hat{b}^\dagger\hat{b}}\left[\hat{b}^\dagger, \hat{b}\right]\sqrt{N-\hat{b}^\dagger\hat{b}} + \left[\hat{b}^\dagger, \sqrt{N-\hat{b}^\dagger\hat{b}}\right]\hat{b}\sqrt{N-\hat{b}^\dagger\hat{b}} \\
&= \hat{b}^\dagger\sqrt{N-\hat{b}^\dagger\hat{b}}\left(\sqrt{N-\hat{b}^\dagger\hat{b}}\hat{b} - \hat{b}\sqrt{N-\hat{b}^\dagger\hat{b}}\right) - N + \hat{b}^\dagger\hat{b} \\
&+ \left(\hat{b}^\dagger\sqrt{N-\hat{b}^\dagger\hat{b}} - \sqrt{N-\hat{b}^\dagger\hat{b}}\hat{b}^\dagger\right)\hat{b}\sqrt{N-\hat{b}^\dagger\hat{b}} \\
&= \hat{b}^\dagger\left(N-\hat{b}^\dagger\hat{b}\right)\hat{b} - N + \hat{b}^\dagger\hat{b} - \sqrt{N-\hat{b}^\dagger\hat{b}}\hat{b}^\dagger\hat{b}\sqrt{N-\hat{b}^\dagger\hat{b}} \\
&= N\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b} - N + \hat{b}^\dagger\hat{b} - \left(N-\hat{b}^\dagger\hat{b}\right)\hat{b}^\dagger\hat{b} \\
&= N\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{b}^\dagger\hat{b} + \hat{b}^\dagger\hat{b} - N + \hat{b}^\dagger\hat{b} - \left(N-\hat{b}^\dagger\hat{b}\right)\hat{b}^\dagger\hat{b} \\
&= 2\left(\hat{b}^\dagger\hat{b} - \frac{N}{2}\right) = 2\hat{J}_z
\end{aligned}
$$

$$
\begin{aligned}
\left[\hat{J}_z, \hat{J}_+\right] &= \left[\hat{b}^\dagger\hat{b} - \frac{N}{2}, \hat{b}^\dagger\sqrt{N-\hat{b}^\dagger\hat{b}}\right] = \left[\hat{b}^\dagger\hat{b}, \hat{b}^\dagger\sqrt{N-\hat{b}^\dagger\hat{b}}\right] \\
&= \hat{b}^\dagger\left[\hat{b}^\dagger\hat{b}, \sqrt{N-\hat{b}^\dagger\hat{b}}\right] + \left[\hat{b}^\dagger\hat{b}, \hat{b}^\dagger\right]\sqrt{N-\hat{b}^\dagger\hat{b}} \\
&= \left[\hat{b}^\dagger\hat{b}, \hat{b}^\dagger\right]\sqrt{N-\hat{b}^\dagger\hat{b}} = \hat{b}^\dagger\sqrt{N-\hat{b}^\dagger\hat{b}} = \hat{J}_+
\end{aligned}
$$

$$
\begin{aligned}
\left[\hat{J}_z, \hat{J}_-\right] &= \left[\hat{b}^\dagger\hat{b} - \frac{N}{2}, \sqrt{N-\hat{b}^\dagger\hat{b}}\hat{b}\right] = \left[\hat{b}^\dagger\hat{b}, \sqrt{N-\hat{b}^\dagger\hat{b}}\right] \\
&= \sqrt{N-\hat{b}^\dagger\hat{b}}\left[\hat{b}^\dagger\hat{b}, \hat{b}\right] + \left[\hat{b}^\dagger\hat{b}, \sqrt{N-\hat{b}^\dagger\hat{b}}\right]\hat{b} \\
&= \sqrt{N-\hat{b}^\dagger\hat{b}}\left[\hat{b}^\dagger\hat{b}, \hat{b}\right] = -\sqrt{N-\hat{b}^\dagger\hat{b}}\hat{b} = -\hat{J}_-
\end{aligned}
\tag{4.9}
$$

In all the calculations we imply that functions of operators can be expressed as a power series of the operator in their argument.

Taking in consideration, that this operators behave as expected, we assume that all the other combinations (like the one used to prove the Schwinger and the fermion realizations) do as well.

Now we can obtain $\hat{H}$ in terms of coordinates x and their conjugated momenta p. Usually, calculating such hamiltonian would imply calculating commutators, that would reduce to commutators of x and p. But using the fact, that for multiple-particle systems:

$$[\hat{x}, \hat{p}] = \frac{i}{N} \tag{4.10}$$

and taking in consideration the classical limit we obtain:

$$\lim_{N\to\infty} [\hat{x}, \hat{p}] = 0 \tag{4.11}$$

This equation allows us to manipulate with all terms of the obtained hamiltonian as if they all commute with each other. This way we get:

$$\begin{aligned} H = {} & \frac{AN}{2} x\sqrt{2-(x^2+p^2)} + \frac{BN}{2}\sqrt{2-(x^2+p^2)} + \frac{CN}{2}(x^2+p^2+1) \\ & + \frac{DN^2}{4} x^2[2-(x^2+p^2)] + \frac{EN^2}{4} p^2[2-(x^2+p^2)] \\ & + \frac{FN^2}{2} xp[2-(x^2+p^2)] + \frac{GN^2}{2} x\sqrt{2-(x^2+p^2)}[2-(x^2+p^2)] \\ & + \frac{IN^2}{2} p\sqrt{2-(x^2+p^2)}[2-(x^2+p^2)] \end{aligned}$$

However, such hamiltonian will diverge in the classical limit with N tending to infinity. Let us return to the hamiltonian (3.1). To resolve the divergence problem, conventionally we set up new coefficients D', E', F', G', I', which are related to the old ones as multiples of $\frac{1}{N}$: $D = \frac{D'}{N}$, $E = \frac{E'}{N}$, etc. This way we obtain a hamiltonian, which converges in the classical limit:

$$\begin{aligned} H_{cl} = \lim_{N\to\infty} \frac{\hat{H}}{N} = {} & \frac{A}{2} x\sqrt{2-(x^2+p^2)} + \frac{B}{2}\sqrt{2-(x^2+p^2)} \\ & + \frac{C}{2}(x^2+p^2+1) + \frac{D'}{4} x^2[2-(x^2+p^2)] + \frac{E'}{4} p^2[2-(x^2+p^2)] \\ & + \frac{F'}{2} xp[2-(x^2+p^2)] + \frac{G'}{2} x\sqrt{2-(x^2+p^2)}[2-(x^2+p^2)] \\ & + \frac{I'}{2} p\sqrt{2-(x^2+p^2)}[2-(x^2+p^2)] \end{aligned}$$

Chapter 5

Numerical calculations

Using software for numerical calculations (see Appendix 1 for the Mathematica code) and the general form of the hamiltonian (3.1) we can research the dynamic of it's energy levels in dependence on the constants $A...I$. Let us have a look at some of interesting special cases.

5.1 The Lipkin model

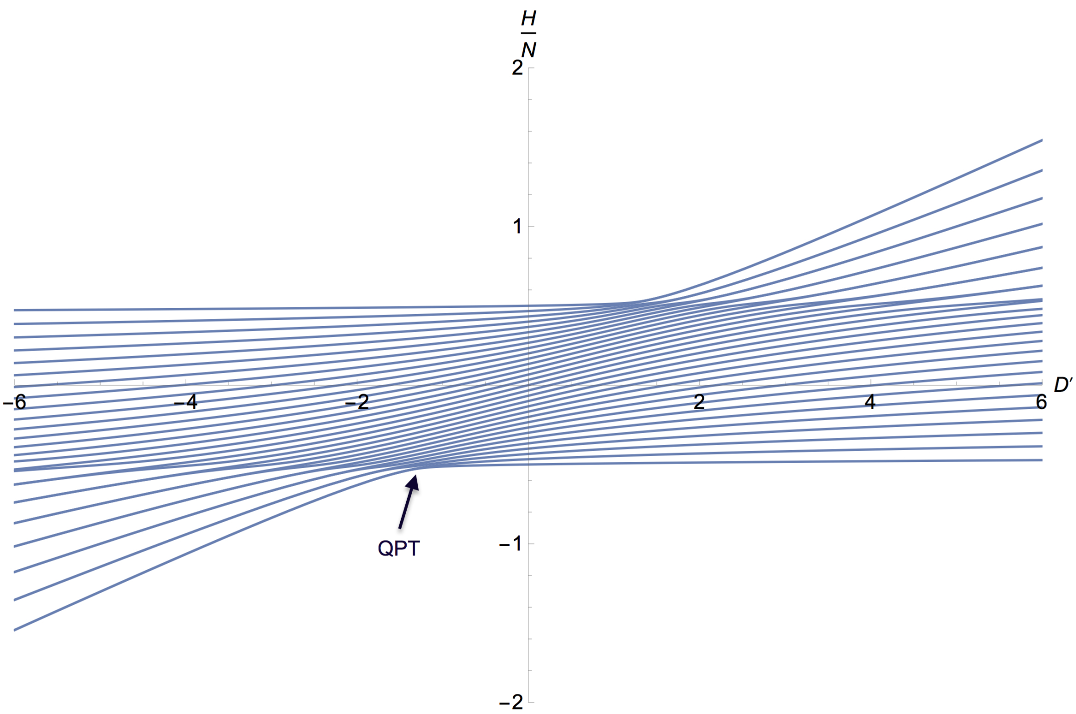

Figure 5.1: Level dynamics of the Lipkin hamiltonian (C = 1, N =30)

The hamiltonian of the so-called Lipkin model [1] is written:

$$\hat{H} = C\hat{J}_z + \frac{D'}{N}\hat{J}_x^2 \tag{5.1}$$

On fig. 5.1 we look at the dynamics of its energy levels of a system consisting of 30 particles with a constant parameter C in dependence on D'. The area that we are interested in is the negative half-plane.The behavior of this case has two notable properties:

The first one is how the energy levels seem to merge under the level $H_i \approx -16$, where we denote the eigenvalues of $\hat{H}$ as H_i. The fact that the lines merge means that at some point the energy levels start forming a degenerated spectrum. In fact, they do not merge completely, but there remains a gap between them, which can be seen on a bigger scale graph.

The second one is more important. From the graph we can see a drastic change in the rate of increase of the functions in the locality of the point $D' \approx -0.05$. Such a change implies a sharp "turn" of the first derivative in this point, thus this is a point where the second derivative is undergoing a non-continuos break. The physical meaning of this break consists in a spasmodic change in the second derivative of the energy level. Such phenomenons are called *second-order quantum phase transitions* (QPT).

Using the Holstein-Primakoff transformation we can also determine the "topography" of the classical limit of energy of the hamiltonian for some specific values of D. The classical hamiltonian in this case has the following form:

$$H_{cl} = \frac{C}{2}(x^2 + p^2 + 1) + \frac{D'}{4}x^2(2 - x^2 - p^2) \tag{5.2}$$

we obtain the following to equations:

Using (5.2) we can determine the general expression for the stationary points of this hamiltonian. Considering the condition of a stationary point of a multi-variable scalar function:

$$\nabla H_{cl} = 0$$

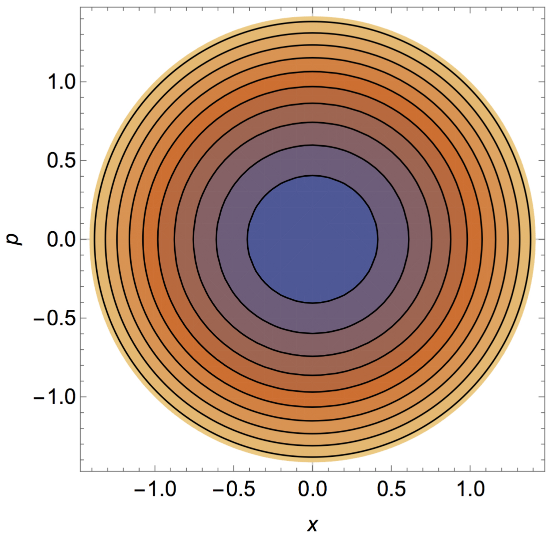

Figure 5.2: Energy "topography" in the point of the QPT (C = 1, D' = -0,05, N =30)

we obtain two equations:

$$\frac{\partial H_{cl}}{\partial x} = Cx + \frac{D'}{2}x(2 - 2x^2 - p^2) = 0$$
$$\frac{\partial H_{cl}}{\partial p} = Cp - \frac{D'}{2}x^2 p = 0,$$

which leads to seven combinations of couples $[x, p]$ and their corresponding energies:

1. $[0, 0]$, $H_1 = \frac{C}{2}$
2. $[\pm\sqrt{\frac{C}{D'} + 1}, 0]$, $H_2 = \frac{C^2 + D'^2}{4D'} + C$
3. $[\pm\sqrt{\frac{2C}{D'}}, \pm\sqrt{2 - \frac{2C}{D'}}]$, $H_3 = \frac{3C}{2}$

Keeping in mind the restrictions on x and p, also because we are interested in cases when $C > 0$, and $D' < 0$, we find, that only the first and second pairs satisfy all the conditions.

For other values of D' we see, that the former point of a global minimum becomes a local maximum and a symmetrical pair of new local minimums appear (fig.5.3).

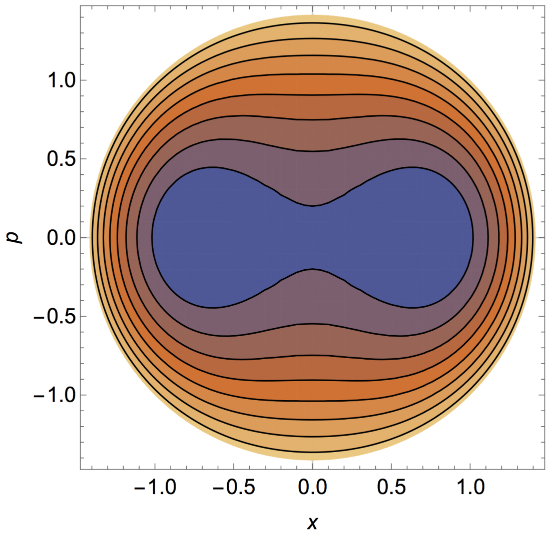

Figure 5.3: Lipkin, C = 1, D' = -2, N=30

Let us finally construct a more general model is described with a hamiltonian:

$$\hat{H} = A\hat{J}_x + B\hat{J}_y + C\hat{J}_z + \frac{D'}{N}\hat{J}_x^2 + \frac{E'}{N}\hat{J}_y^2 \tag{5.3}$$

The dynamic of its energy levels is depicted in fig.5.

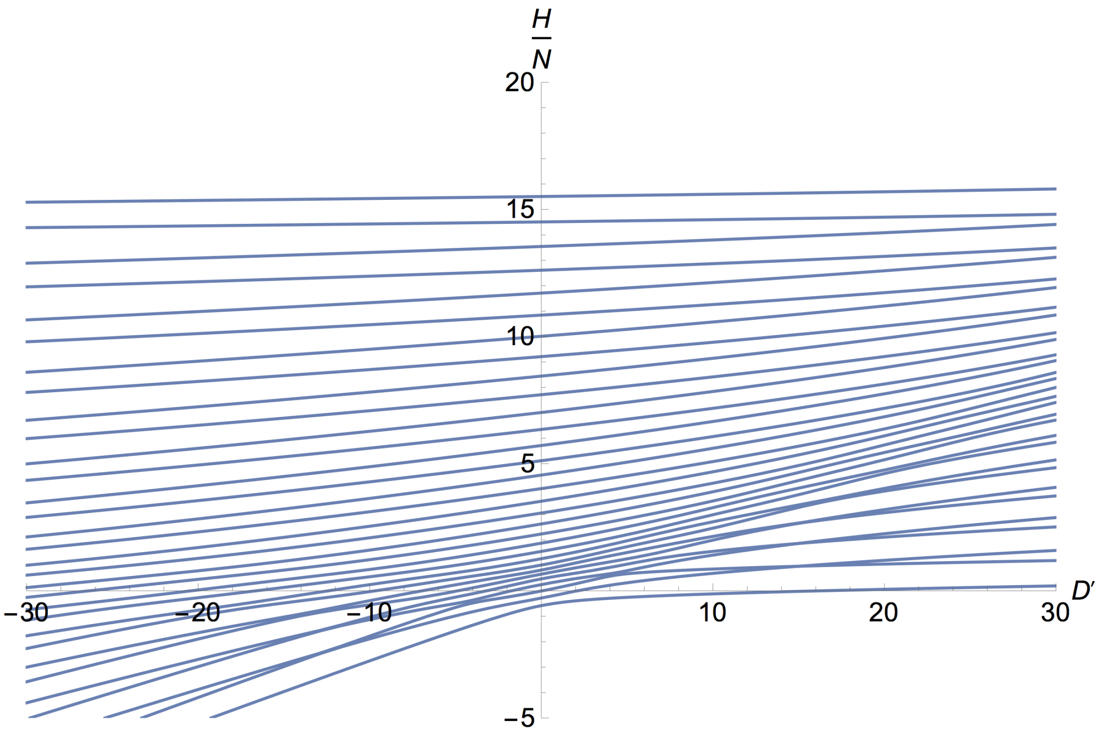

Figure 5.4: A,B,C = 1, E'=2

Now we can determine how the additional terms in (5.3) (compared to (5.2)) affect the "topography" of its energy in the classical limit. Performing the Holstein-Primakoff transformation we obtain a hamiltonian:

$$\begin{aligned} H_{cl} = \frac{A}{2}x\sqrt{2-x^2-p^2} + \frac{B}{2}p\sqrt{2-x^2-p^2} + \frac{C}{2}(x^2+p^2+1) \\ + \frac{D'}{4}x^2(2-x^-p^2) + \frac{E'}{4}p^2(2-x^2-p^2) \end{aligned} \tag{5.4}$$

For $A = B = C = 1$, $E = 2$ and specific values of D we obtain the following graphs.

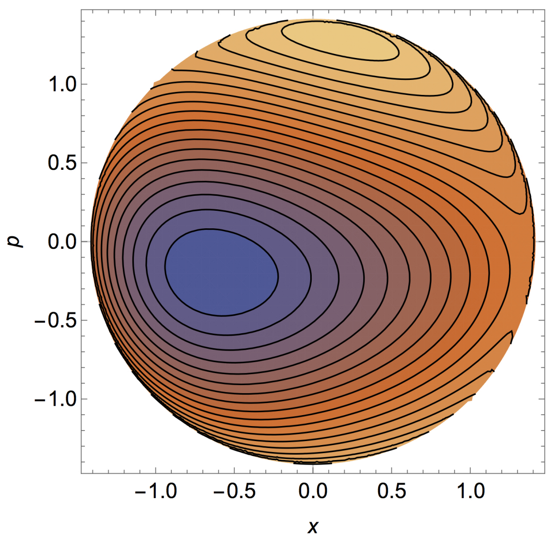

Figure 5.5: A=B=C=1, E' = 2, D' = -0,5, N=30

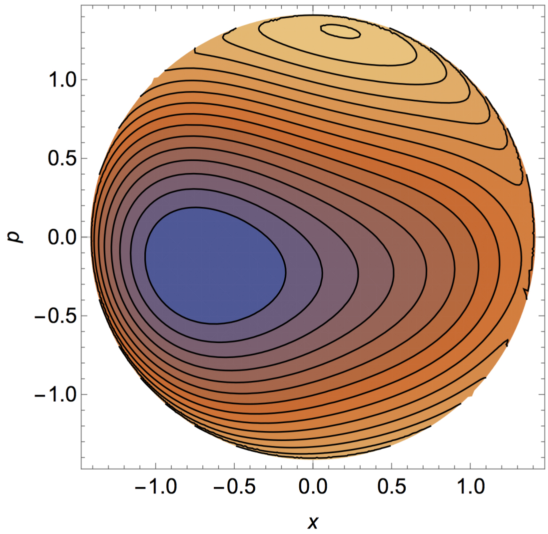

Figure 5.6: A=B=C=1, E'=2, D' = -1,N=30

We see that the additional terms in (5.3) destroy the symmetry with respect to the beginning of the origin.

Conclusion

In conclusion, this thesis focuses on describing systems whose dynamical algebra is SU(2). Our goal was to review its basic notions and use them to research the simplest cases of such systems.

Firstly, we have reiterated the basic notions of the group theory. This was made to construct the formalism, that is required to study the SU(2) hamiltonians and their spectrum.

Secondly, we have reviewed possible ways of physical realization of the SU(2) algebra.

Thirdly, we have determined the general form of a SU(2) hamiltonian with one- and two-body interactions. This was a starting point to the research of spectrums of its specific cases. Because we were also interested in the behavior of the hamiltonians in the classical limit, we found out that some of the constants in its terms should be in some way related to the number of particles of the system.

Finally, using numerical simulations we studied the level dynamics of some simple forms of the SU(2) hamiltonian, including the so-called Lipkin model.

List of Figures

Bibliography

[1] Harry J Lipkin, N Meshkov, and AJ Glick. Validity of many-body approximation methods for a solvable model:(i). exact solutions and perturbation theory. *Nuclear Physics*, 62(2):188–198, 1965.

[2] Pavel Stránský, Martin Dvořák, and Pavel Cejnar. Exceptional points near first-and second-order quantum phase transitions. *Physical Review E*, 97(1):012112, 2018.

[3] MA Caprio, P Cejnar, and F Iachello. Excited state quantum phase transitions in many-body systems. *Annals of Physics*, 323(5):1106–1135, 2008.

[4] Pedro Pérez-Fernández, P Cejnar, Jose M Arias, Jorge Dukelsky, José Enrique García-Ramos, and Armando Relaño. Quantum quench influenced by an excited-state phase transition. *Physical Review A*, 83(3):033802, 2011.

[5] WD Heiss, FG Scholtz, and HB Geyer. The large n behaviour of the lipkin model and exceptional points. *Journal of Physics A: Mathematical and General*, 38(9):1843, 2005.

[6] Wassilij Kopylov, Gernot Schaller, and Tobias Brandes. Nonadiabatic dynamics of the excited states for the lipkin-meshkov-glick model. *Physical Review E*, 96(1):012153, 2017.

[7] Alejandro Frank and Pieter Van Isacker. *Symmetry methods in molecules and nuclei.* S y G, 2005.

[8] Lev Davidovich Landau and Evgenii Mikhailovich Lifshitz. *Quantum mechanics: non-relativistic theory*, volume 3. Elsevier, 2013.

[9] Francesco Iachello. *Lie algebras and applications*, volume 12. Springer, 2006.

[10] Jun John Sakurai and Eugene D Commins. Modern quantum mechanics, revised edition, 1995.

[11] Richard Bellman. *Introduction to matrix analysis*, volume 19. Siam, 1997.

Appendix A

Computer code

Mathematica code used to perform numerical calculations:

```
j = 15; (Total angular momentum of the system, which is
equal to N/2,  where N is the total number of the particles)
A = 0;
B = 0;
Charlie = 1;
(*Delta=;*)
Eho = 0;
F = 0;
G = 0;
H = 0;
India = 0;

Jz = Table[
  m1 KroneckerDelta[m1, m2], {m1, -j, j}, {m2, -j,
   j}]; (Definition of angular momentum
  operators' matrices in the base of eigenkets
  of the Jz operator)

Jplus = Table[
   Sqrt[j (j + 1) - m2 (m2 + 1)] KroneckerDelta[m1, m2 + 1],
   {m1, -j,  j}, {m2, -j, j}];

Jminus = Table[
   Sqrt[j (j + 1) - m2 (m2 - 1)] KroneckerDelta[m1, m2 - 1],
   {m1, -j,  j}, {m2, -j, j}];

Jx = 1/2* (Jplus + Jminus);

Jy = I/2*(Jplus - Jminus);

Hamiltonian[Delta_] :=
 A Jx + B Jy + Charlie Jz + Delta Jx.Jx + Eho Jy.Jy
```

```
 + F Jz.Jz + G (Jx.Jy + Jy.Jx) + H (Jx.Jz + Jz.Jx) +
  India (Jy.Jz + Jz.Jy) (General form of the Hamiltonian)

GeneralEV[Delta_, i_] :=
 Sort[Eigenvalues[N[Hamiltonian[Delta]]]][[
   i]] (i - th eigenvalue of the hamiltonian

     as a function of parameter D)

Plot[GeneralEV[Delta, 1], {Delta, -2, 2}
(Plotting the function)

   Plot[Table[GeneralEV[Delta, i], {i, 1, 2 j + 1}],
   {Delta, -2, 2}, PlotRange -> {{-2, 2}, {-20,
       20}}] (Plotting all energy levels on one graph)
   hamiltonian =
  Jx + Jy + Jz + d Jx.Jx + 2 Jy.Jy
  (A specific form of the hamiltonian)

    Plot[Table[
      Sort[Eigenvalues[N[ham]]][[i]], {i, 1, 2 j + 1}],
      {d, -2, 2}, PlotRange -> {{-2, 2}, {-50,
        50}}] (Plotting all energy levels of the specific
hamiltonian)
```