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The Stigler-Luckock model for a limit order book

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I would like to express my gratitude to my supervisor Dr. Jan Swart for his guidance and patience with this thesis. My thanks go also to my loving family who supported me throughout my difficult but rewarding times at the university.

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Abstract: One of the types of modern-day markets are so-called order-driven markets whose core component is a database of all incoming buy and sell orders (order book). The main goal of this thesis is to extend the Stigler-Luckock model for order books to give a better insight into the price forming process and behaviour of the market participants themselves. The model introduced in this thesis focuses on a comparison of behaviour and various strategies of market makers who are sophisticated market participants profiting from extensive trading. The market is described using Markov chains, and the strategies are compared using Monte Carlo simulations and game theory. The results showed that market makers' orders should have small spread and large volumes. The final model compares two strategies in which market makers monitor their portfolio. In case of having more cash than asset (or vice versa), they shift prices of their orders to equalise the portfolio. The model recommends checking the market quite often, but acting conservatively, which means not changing prices that frequently and not jumping to conclusions just from a small imbalance in the portfolio.

Keywords: limit order book, market microstructure, market makers, Markov chain, optimal strategy

Contents

Introduction

Financial markets are an essential part of the modern capitalist society. They help for example to allocate resources, define an equilibrium price, boost productivity, or bring capital.

While much of economics focuses on the mechanics of trading from the macroscopic view, this thesis will present the so-called market microstructure approach. It is the area of financial economics that concerns the issues of the basic principles of market structure, investor behaviour, and price-forming process. The market microstructure studies has grown rapidly in the past years due to the fast transformation of the financial market environment driven by technology and globalisation.

The understanding of the formation and consequently finding of the equilibrium price is a difficult task. While creating the price is undoubtedly a complicated process, markets somehow manage to do it automatically, and people exchange goods easily.

A market can be either quota-driven or order-driven. On the one hand, the quota-driven markets only display the bid and ask prices of authorized agents; therefore, the prices are determined in advance. If we decide to buy or sell an asset on a quota-driven market, we are assured that we can find an opposite offer for the announced price. On the other hand, order-driven markets are in a sense more free and open. Orders of all buyers and sellers are shown, and other people may see the price and quantity. However, there is no guarantee that our order will be executed since we are the ones determining the price. There are purely quota-driven or order-driven markets, and also a mix of both principles.

The focus of this thesis are the order-driven markets. Especially the Stigler-Luckock model. The main aim is to slightly improve the model in order to get a more realistic model.

Firstly, the thesis will discuss some economic theory. The law of supply and demand will be explained together with the description of order-driven markets and their principles. The theoretical chapter will be closed by an introduction to the problematic of stochastic processes and game theory. All the theory will be then used in subsequent chapters.

Secondly, the Stigler-Luckock model will be defined in chapter 2. The model itself has its drawbacks, which will be discussed, but it is a great starting point for further exploration. Moreover, an extension with market makers will be described.

Thirdly, the Stigler-Luckock model will be extended by additional features that will be studied to get a somewhat deeper understanding of financial markets in chapter 3. The development and exploration of the new model and its behaviour using Monte Carlo simulations is the core of this thesis. The main contribution of this thesis lies in an investigation of various strategies and finding the best one, which may help us to learn more about the markets.

Hopefully, this thesis will help its reader to understand more how the financial markets function, and how agents trading on them behave.

1. Theoretical background

As proposed in the introduction, the thesis is focusing on market behaviour, its microstructure, and price forming process. Hence, some background theory must be established first. First, the concept of supply and demand will be described followed by some fundamental properties of contemporary markets. Then, a few topics from mathematics related to the financial markets will be touched, namely the theory of stochastic processes and game theory. The theory will be useful in the following parts of the thesis in chapter 2 and chapter 3.

1.1 Supply and demand

People were in need of exchanging goods since the old-aged concept of division of labour. However, there are important questions regarding the process, such as: How to determine the price? How much of one resource is needed to get some amount of another resource? Who decides that?

There is a simple though smart answer—the power of supply and demand. This concept dates back to the 19th century to the French economist Leon Walras who published a comprehensive theory of exchange, production, money, and market in general. Among others, he introduced the theory of supply and demand [Walras, 2014][1]. While supply somehow represents the amount of a product available to be sold (for a specific price), the demand represents the amount of the product desired to be bought (for a specific price). Interaction between these two powers results in the formation of a price. Consequently, supply and demand dictate the allocation of resources and help to distribute assets around the world.

The law of supply and demand is one of the key principles of economics. The main thought is simple: On the one hand, a resource that has high supply and low demand has a lower price. On the other hand, a resource that has low supply and high demand has a higher price.

More rigorously: supply is a function that represents the dependence of the amount of supplied goods on its price, and demand is a function that represents the dependence of the amount of demanded good on its price. Supply is an increasing function of the price, and demand is a decreasing function of the price. To be completely correct, this monotonicity rule applies to the majority of goods, but there are a few exceptions—commodities whose demand is increasing with price or whose supply is decreasing with price—for example, luxurious wares whose purpose is giving social status rather than being used. Namely, a decrement in the price of a painting undoubtedly results in decreased demand, not increased demand. However, those exceptions are quite rare, and we will focus on traditional goods from now on.

The supply and demand are usually considered to be continuous and strictly monotonic functions of the price level. Those assumptions are quite natural and have been used since their recognition by Walras [2014] in the 19th century. The market's behaviour is dependant on the shape and also the mutual position of those 2 curves—they can either intersect or not. In general, they are supposed

[1] The original book dates back to 1874. The source for this thesis was its translated and edited version from 2014.

to be intersecting each other, and the intersection point is called the *equilibrium* (or even the Walrasian equilibrium). The point represents a pair of Walrasian price x_W and Walrasian quantity V_W for which supply and demand are equal (illustrated in Figure 1.1). Let us denote supply by $\lambda_+(x)$ and demand by $\lambda_-(x)$, where x is a price. Then

$$\lambda_+(x_W) = \lambda_-(x_W) = V_W.$$

The price that is determined by the equilibrium is called the *equilibrium price*. The equilibrium price is a goal because it maximizes the possible executable volume of trades. The possible volume of trades as a function of price is always bounded by the minimum of the two supply and demand.

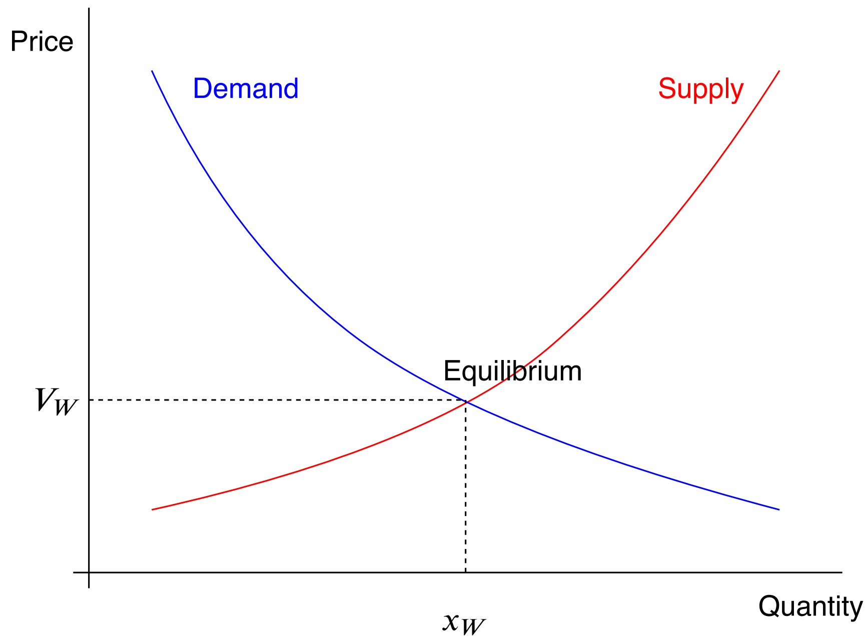


Figure 1.1: Illustration of supply and demand curves with an intersection point.

If the supply and demand curves do not intersect at any point of their domain, there is either excessive supply (demand is entirely lower than supply) or insufficient supply (demand is entirely higher than supply) for all price-quantity pairs (see Figure 1.2).

Note that the continuity assumption is usually not fulfilled in real life. Both prices and quantities are generally discrete variables, which makes it impossible for the functions to be continuous. For example, the so-called "minimum price variation", i.e. the smallest difference between two prices, for quoting and entry of orders in securities traded, on the New York Stock Exchange is [NYSE Rules]:

(a) \$0.01 for securities priced more (or equal) than \$1.00, and

(b) \$0.0001 for securities priced less than \$1.00.

The Walrasian price is determined when supply matches the demand. However, Bouchaud and Donier [2015] remind that this description of a market is only static and one crucial ingredient is absent—the transactions themselves.

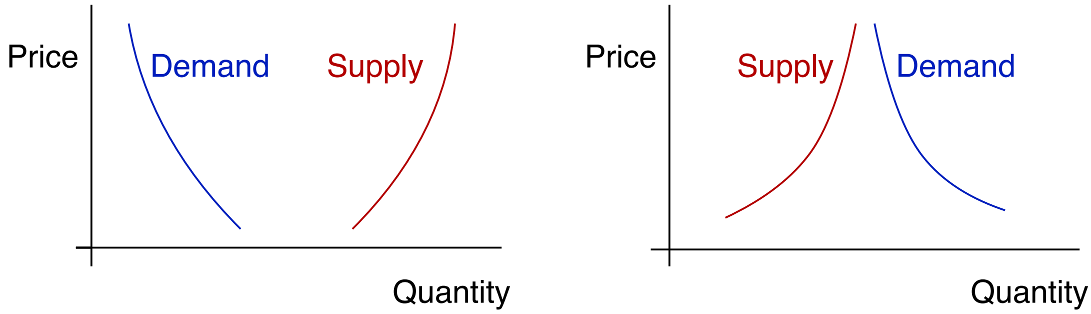


Figure 1.2: Illustration of supply and demand curves without intersection.

Walrasian theory does not explain what happens after the transaction. The Walrasian auction is a concept in which each market participant calculates its demand for all possible prices. That information is sent to an auctioneer who sets the price in order to match the total demand and the total amount of the good. This auction setting can be viewed as a limiting case when the time between transactions is infinitely long. However, in order to understand the market and price-forming process, it is necessary to consider also the dynamics of the market, which is the main goal of this thesis as well.

1.2 Order book

The theory of supply and demand is accepted among economists and finance experts. However, the financial market and price-forming process itself may appear as a black box since the supply and especially the demand are not easily measurable. Nevertheless, price of an asset is formed automatically and "painlessly" despite the fact that all the processes behind it are complicated and complex.

So, how does it work in real life? How is the price determined?

Nowadays, millions of transactions happen every second all around the world. For example, roughly 360 billion shares were traded on the New York Stock Exchange alone in years the 2002 and 2003 [Biais et al., 2005]. The majority of trades happens in an entirely different manner than in the times when the Walrasian supply and demand was defined. Trading increased in volume and speed, and the physical exchange was sidelined. Special institutions—*stock exchanges*—were established to replace traditional markets and to provide a suitable environment for buyers and sellers to meet and realize trades. On top of that, electronic and algorithmic trading is on the rise which is making trading quicker than ever before (see the trend in the example from data from BIS[2] in Figure 1.3).

As mentioned in the beginning, there are initially two types of markets—quota-driven and order-driven. The main focus of this thesis are the order-driven

[2] The Bank for International Settlements is an international financial institution that is owned by 60 member central banks (including the Czech National Bank), which was established in 1930 in order to serve the central banks as an environment for cooperation and to better collectively understand the world economy. Its headquarters are in Basel, Switzerland, and its members represent about 95% of world GDP [BIS brochure].

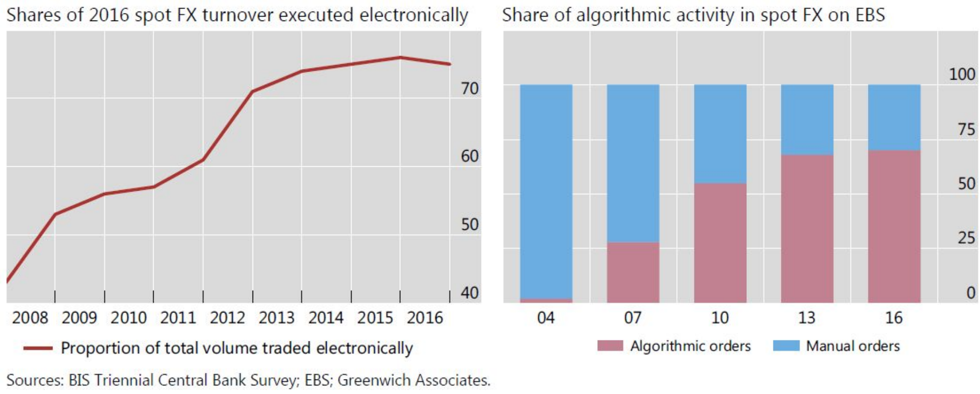


Figure 1.3: Electronification of foreign exchange market in per cents (source: BIS Markets Comittee, 2018).

markets that are characterized by openly displaying all offers of buyer and sellers including prices and amounts.

Each order-driven market has a so-called *order book*. It can be imagined as an electronic database where buyers and sellers record their offers. Those offers are called *orders*, and they express an intention of a trader to carry out some transaction on the market. Abergel et al. [2016] define the order book as a "file in a computer" that is containing all available information about the incoming orders. Each order must have at least these characteristics—a *sign* (buy or sell), a *price*, a *quantity*, and a *time stamp*. Other authors (e.g. Bouchaud and Potters [2002]) simply define that order book is a list of all buy and sell orders with their corresponding price and volume, at a given instant of time.

First, let us distinguish two basic types of orders. This elemental division is used by many authors across the field (e.g. Chakraborti et al. [2011], Maslov [2000], or Luckock [2003]). The first kind is called a *limit order* and is determined by its quantity and price. The trader arrives on the market, and she either likes the current best price and buys (or sells) the asset instantly. Alternatively, the current price is unsatisfactory, and she decides to wait for the price to become more suitable (lower in the case of buying, higher in the case of selling). In other words, the limit order is executed if and only if the price on the market reaches the desired order's price. The second basic type of orders is called *Market order*. Its main characteristic is that the order must be executed immediately. Nevertheless, the definition is not consistent. Maslov [2000] defines the market order as an order of a somewhat impatient trader that needs to sell or buy the asset instantly and who takes without thinking the best price that is currently on the market. Luckock [2003] does not explicitly distinguish between limit and market orders. A new order has a sign, price, and volume, and the classification is determined by the price. If the order can be executed immediately, it is referred to as a market order. This happens if there is a suitable opposite limit order already on the market, i.e. a sell limit order with a lower price or a buy limit order with a higher price. On the contrary, if the order cannot be executed immediately, it is referred to as a limit order. It is placed on the market where it waits for its execution.

Evidently, only limit orders stay on the market, so the order book is sometimes referred to as a *Limit Order Book* (LOB).

Abergel et al. [2016] add a so-called *cancellation order* to the two previously mentioned. This is an order to cancel an existing limit order. Cancelling of orders happens quite often in real life for various reasons, e.g. the limit order is in the order book for some significant time, or the trend has changed, and the price of the limit order is no longer realistic.

However, these terms and procedures will be used in the whole thesis, so it is necessary to give them a proper definition to avoid misunderstanding.

Definition 1. *An order to sell a quantity ω of an asset for a price p that is not settled immediately is called a sell limit order (SLO).*

An order to buy a quantity ω of an asset for a price p that is not settled immediately is called a buy limit order (BLO).

Definition 2. *An order to sell a quantity ω of an asset for a price p that is settled immediately is called a sell market order (SMO).*

An order to buy a quantity ω of an asset for a price p that is settled immediately is called a buy market order (BMO).

Nowadays, each stock market has slightly different procedures and principles, as summarized by Plačková [2011]. There are some strictly order-driven markets, for example: the Computer Assisted Trading System (CATS) in Toronto, the Cotation Assistée en Continu (CAC) in Paris, or the Automatic Order Matching and Execution System (AMS/3) in Hong Kong. By comparison, the New York Stock Exchange (and stock exchanges in the USA in general) uses a different approach. The individual orders are paired by specialists, and any information regarding the content of the order books is not publicly accessible.

Now, let us define some terminology that is common to all markets.

Definition 3. *Let t represent time. Then let*

(i) the bid price $b(t)$ be defined as the maximum of all buy limit orders that are in the LOB at the time t,

(ii) the ask price $a(t)$ be defined as the minimum of all sell limit orders that are in the LOB at the time t,

(iii) the spread $s(t)$ be defined as the difference between the bid and ask at time t, i.e. $s(t) = b(t) - a(t)$,

(iv) the mid price $m(t)$ be defined as the average of the bid and ask. $m(t) = \frac{a(t)+b(t)}{2}$,

(v) the tick (or tick size) be a measure of the minimal possible upward or downward movement of the price of a security

An illustration of an order book with the previously defined terms is in Figure 1.4. Note that some authors use the word "bid" simply for any buy limit order and the word "ask" for any sell limit order.

The previous description of the order book, orders, and their properties was from a financial and economic point of view. However, the so-called market microstructure resembles in many ways the physics of small particles. The field studying the financial markets from the physicists' perspective is called *econophysics*. It has gained quite a lot of attention in the past few decades. Its basis

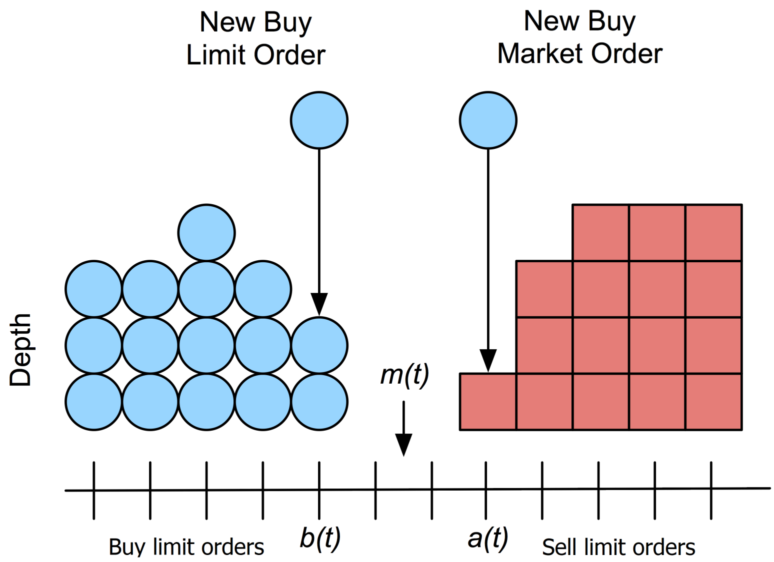


Figure 1.4: Illustration of limit order book with bid and ask (source: Todd et al. [2016]).

are reaction-diffusion processes [Slanina, 2013], and the orders can be considered as particles placed on a line. Naturally, there are two different groups of orders—buy and sell orders. They can be presented as two types of particles with an opposite charge. Challet and Stinchcombe [2001] compare these three analogies between particles and orders:

- *depositing*: placing an order,
- *evaporating*: cancelling an order,
- *annihilation*: transaction (two opposite orders are deposited at the same price and then immediately disappear from the market).

The order book shows the volume of an asset that is requested to by sold or bought exactly for some specific price x. Traders may also be interested in the volume that is on the market for that price or "better" (lower for sell orders and higher for buy orders). That information is derived by simply summing all the volumes of orders with price p and "better". By doing so for all possible prices, we get a so-called *cumulative order book*.

Now, let us take a look at some real-life example of an order book. However, the order book is an ever-changing and unstable system. Its evolution is ruled not only by observable and analytical variables but also by a random noise. In order to reliably capture the shape of the order book, Bouchaud and Donier [2015] chose to gather all orders in the Bitcoin market for five months (from May 2013 to September 2013) every 15 minutes. Then, they obtained a "relative" order book by placing the current mid price in the middle (price 100), and by consequently deriving a "relative" price for all orders, expressed as a ratio to the mid price. Finally, they computed an average order book and average cumulative order book (Figure 1.5). The order book can be linearly approximated around the mid price. They choose the Bitcoin market because traders there are much less strategic than in more mature financial markets and display their orders in the visible order book even quite far from the current price—which leads us to a phenomena broadly seen in most of the financial markets.

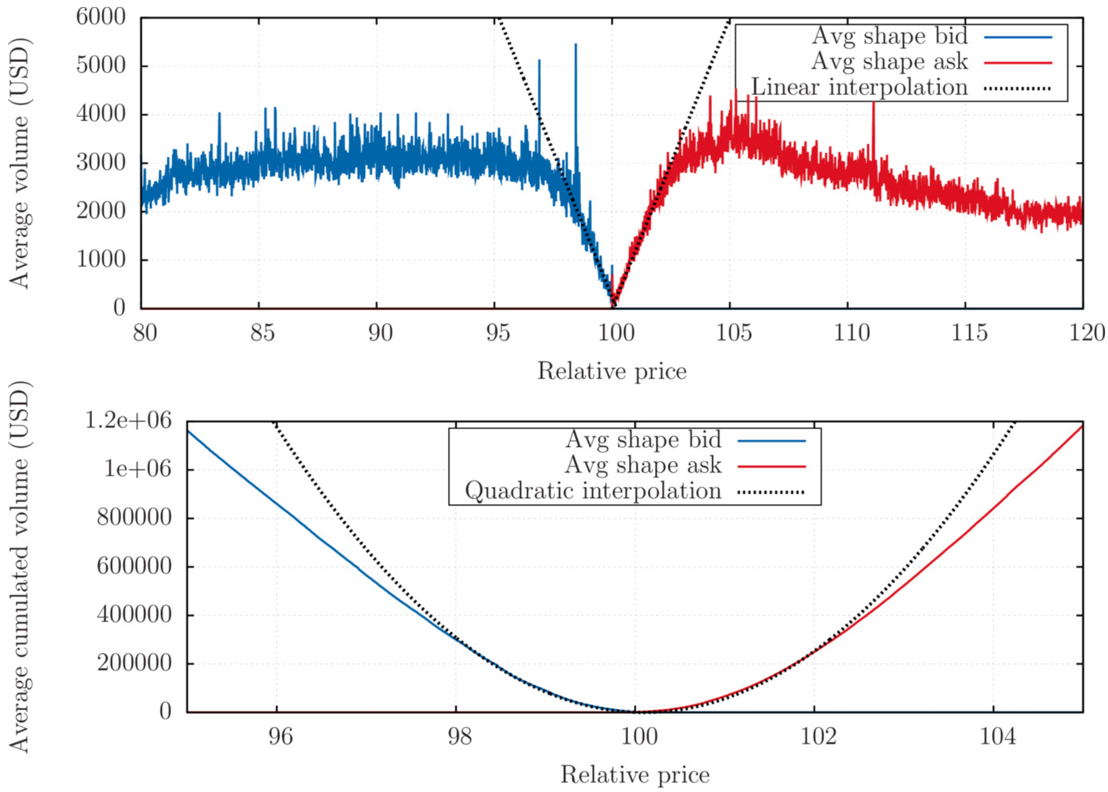


Figure 1.5: Top: The average shape of the limit order book on the Bitcoin market. The data are centred around the current mid-price, and come from 15 minute snapshots of the order book from May 2013 to September 2013. Bottom: Cumulative order book of the same data. (Note that *bid* means buy limit order, and *ask* means sell limit order). Source: Bouchaud and Donier [2015]

Bouchaud and Donier [2015] point out that even though the concept of order books is close to Walras's idealisation of supply and demand, order books still have one fundamental issue: agents do not necessarily reveal their intentions by placing orders. The orders are placed only by agents who are in urgent need to buy or sell. The order book approximates the real supply and demand only in a window very close to the current price. However, the order book around the current price is distorted by influence of market makers and high frequency traders whose orders reveal very little about the true underlying supply or demand. In other words: the classical shape of the order book as a representation of the supply and demand is problematic both far and close to the middle price. The shape of the LOB far from the mid price does not reflect the true supply and demand because traders do not want to show their intentions, and the shape closer to the mid price is affected by trading strategies of other players.

1.2.1 Examples of order books' procedures

Each stock exchange has different rules and procedures. In order to illustrate some basic principles, we will introduce a few stock exchanges with their timetables.

London stock exchange

The leading electronic trading system on the London stock exchange is so-called SETS. It is an order book that combines electronic order-driven trading with integrated market maker liquidity provision[3]. A trading day on the London Stock Exchange is divided into these sections [LSE Trading Hours]:

- **7:50-8:00 Entry phase**: orders enter the market but will not be matched up until 8:00 when the *Opening auction* occurs and a so-called *execution price* is set to maximize the executable volume. The content of the order book remains undisclosed during the Entry phase,
- **8:00-12:00 Continuous trading**: regular trading period when orders may be entered, amended, and cancelled,
- **12:00-12:05 Intraday auction**: the order book's content is undisclosed again, and orders are not matched until the end of the auction (similar to Entry phase),
- **12:05-16:30 Continuous trading**: similar to the one in the morning,
- **16:30-16:35 Closing auction call**: similar to the Opening auction, orders may be entered, amended, and cancelled but will not be matched
- **16:35-16:40 Closing price crossing session**: orders are matched, and consequently the *Closing price* is settled.

New York stock exchange

The most renowned American stock exchange is the one in New York. The trading day there has these sections [NYSE Trading Hours]:

- **7:30-9:30 Pre-opening session**: orders may be entered and cancelled but not matched. At 9:30 all matching orders are executed at a single price that is settled in order to match maximum executable volume of orders,
- **9:30-15:45 Pre-Imbalance**: regular trading period when orders can be entered, amended, and cancelled,
- **15:45-16:00 Post-Imbalance**: continuous trading period in which special orders (*Market-On-Close* and *Limit-On-Close*) may be entered but not cancelled or amended,
- **16:00 Closing Cross Price**: closing auction in which *Closing price* is calculated.

1.3 Stochastic processes

Many parameters regarding the topic of financial markets and order books can be viewed as random processes. Parameters such as the mid price, bid, ask, price of a new order, time between orders, etc. are random variables changing in time. Depending on the model, time can be either discrete or continuous. The theory in this section was obtained from and inspired by the lecture notes by Lachout and Prášková [1998].

[3] liquidity provision will be further explained in section 2.4

Definition 4. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(S, \mathcal{S})$ be a measurable space, and $T \subset \mathbb{R}$. A family of random variables $\{X_t, t \in T\}$ defined on $(\Omega, \mathcal{A}, P)$ with values in S is called stochastic (or random) process.*

If $T = \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ or $T \subset \mathbb{Z}$, $\{X_t, t \in T\}$ is called discrete time stochastic process.

If T is a interval, $\{X_t, t \in T\}$ is continuous time stochastic process.

For any $\omega \in \Omega$, $X_t(\omega)$ is a function on T with values in S which is called the trajectory of the process.

The set S is called the state space of the process $\{X_t, t \in T\}$.

Definition 5. *Let $\{X_t, t \in T\}$ be a stochastic process such that for each $t \in T$ mean value $E\, X_t$ exists. Then the function $\mu_t = E\, X_t$ defined on T is called the mean value of the process $\{X_t, t \in T\}$. If $E\,|X_t|^2 < \infty$ for all $t \in T$, then the function defined on $T \times T$ by $R(s,t) = E(X_s - \mu_s)(X_t - \mu_t)$ is called the autovariance function of the process $\{X_t, t \in T\}$. The value $R(t,t)$ is the variance of the process at time t.*

Example 1 (White Noise). A process $\{W_t, t \in \mathbb{Z}\}$ of uncorrelated random variables with zero mean value and equal finite variance is called White noise. Its name comes from the analogy with physical properties of white light.

△

Example 2 (Brownian motion). Brownian motion is a stochastic process $\{W_t, t \geq 0\}$ such that

- $W_0 = 0$ and $\{W_t, t \geq 0\}$ has continuous trajectories,
- for any times $0 \leq t_1 < t_2 < \dots < t_n$ we have $W_{t_2} - W_{t_1}$,$W_{t_3} - W_{t_2}$, $\dots$,$W_{t_n} - W_{t_{n-1}}$ are independent random variables,
- for any $0 \leq t < s$, $W_s - W_t$ has normal distribution with zero mean and variance $\sigma^2(s-t)$, where $\sigma^2 > 0$ is a constant.

Brownian motion, which is now broadly used in financial theory, was originally derived for the description of the movement of small particles in a liquid.

△

1.3.1 Markov chains

Discrete time Markov chains

Definition 6. *Let $\{X_n, n \in \mathbb{N}_0$[4]$\}$ be a discrete random process with values from $(S, \mathcal{S})$, where S is a countable set[5]. We call $\{X_n, n \in \mathbb{N}_0\}$ a Markov chain if*

$$\begin{aligned} P\Big(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0\Big) &= \\ = P\Big(X_{n+1} = j | X_n = i\Big) &= p_{ij}^{(n,n+1)}, \end{aligned} \tag{1.1}$$

for $i_0, \dots, i_{n-1}, i, j \in S$ that satisfy

$$P\Big(X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0\Big) > 0.$$

[4] $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$

[5] A set S is countable if there exists an injective function f from S to the set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$, i.e. the set S has the same cardinality as some subset of $\mathbb{N}$.

The property in Equation 1.1 is called *Markovian property*. It can be explained as the property guaranteeing that the future of the process is dependant only on the present situation and not on the past.

The probabilities $p_{ij}^{(n,n+1)}$ are called the *transition probabilities* from state i at time n to state j at time $n+1$. It is obviously true that

$$p_{ij} \geq 0,\ \ i,j \in S;\ \sum_{j \in S} p_{ij} = 1,\ \ i \in S. \tag{1.2}$$

The matrix $\mathbb{P}^{(n,n+1)}$ containing all possible combinations of transition probabilities from time n to time $n+1$, i.e.

$$\mathbb{P}^{(n,n+1)} = \left(p_{ij}^{(n,n+1)}\right)_{i,j \in S},$$

is called the *transition matrix*. Since the time step is 1, it is sometimes referred to as the *first order transition matrix*.

Transition probabilities can be naturally generalised, if we want to study evolution of the process over wider steps:

$$p_{ij}^{(n,n+m)} = \mathsf{P}\left(X_{n+m} = j | X_n = i\right),$$

where $m \geq 1$.

If the transition probabilities do not depend on time, we speak about a homogeneous Markov chain.

Definition 7. *A Markov chain $\{X_n, n \in \mathbb{N}_0\}$ is homogeneous if for all $n \in \mathbb{N}$*

$$p_{ij} = p_{ij}^{(n,n+1)}.$$

To finish the specification of Markov chains, we need to define *initial distribution* that determines the probability distribution at time $t = 0$, i.e. a vector of probabilities $\boldsymbol{p} = \{p_i, i \in S\}$, where

$$p_i = \mathsf{P}(X_0 = i),\ \ i \in S.$$

We have

$$p_i \geq 0,\ \ i \in S;\ \sum_{i \in S} p_i = 1. \tag{1.3}$$

Theorem 1 (Characterisation of Markov chains)**.** *Let $\{X_n, n \in \mathbb{N}_0\}$ be a stochastic process with state space S, where S is a countable. Let $\boldsymbol{p} = \{p_i, i \in S\}$ satisfy Equation 1.3, and let $\mathbb{P} = \left(p_{ij}\right)_{i,j \in S}$ be a matrix that satisfies Equation 1.2. Then $(X_n, n \in \mathbb{N}_0)$ is a homogeneous Markov chain with initial distribution $\boldsymbol{p}$ and transition matrix $\mathbb{P}$ if and only if for all finite dimensional distributions of the process we have:*

$$\mathsf{P}\left(X_0 = i_0, X_1 = i_1, \ldots, X_k = i_k\right) = p_{i_0} p_{i_0 i_1} \ldots p_{i_{k-1} i_k},$$

for all $i_0, i_1, \ldots, i_k \in S$ and for all $k \in \mathbb{N}_0$.

Proof. See page 17 in Lachout and Prášková [1998]. $\square$

It can be derived that for homogeneous Markov chains, the transition matrix over n steps $\mathbb{P}^{(n)}$ can be simply computed as

$$\mathbb{P}^{(n)} = \mathbb{P}^n,$$

where $\mathbb{P}^n$ denoted the n-th power of the matrix $\mathbb{P}$. The relation is called the Chapman-Kolmogorov equation.

Another concept is called *stationary distribution*. It gives information about the stability of the random process, and, in certain cases, describes the limiting behaviour of the Markov chain.

Definition 8. *Let $\{X_n, n \in \mathbb{N}_0\}$ be a homogeneous Markov chain with state space S and transition matrix $\mathbb{P}$. Let $\boldsymbol{\pi} = \{\pi_j, j \in S\}$ be some probability distribution on S, i.e. $\pi_j \geq 0$, $j \in S$, $\sum_{j \in S} \pi_j = 1$. Then $\boldsymbol{\pi}$ is called a stationary distribution, if*

$$\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \mathbb{P},$$

where $\boldsymbol{\pi}^T$ denotes the row vector that is the transpose of the column vector $\boldsymbol{\pi}$.

Continuous time Markov chains

Definition 9. *Let $\{X_t, t \geq 0\}$ be a discrete random process with values from $(S, \mathcal{S})$, where S is a countable set. We call $\{X_t, t \geq 0\}$ a continuous Markov chain if*

$$\begin{aligned} \mathsf{P}\Big(X_t = j | X_s = i, X_{t_n} = i_n, \ldots, X_{t_1} = i_1\Big) &= \\ &= \mathsf{P}\Big(X_t = j | X_s = i\Big) = p_{ij}(s,t), \end{aligned} \tag{1.4}$$

for all $i, j, i_1, \ldots, i_n \in S$ and for all $0 \leq t_1 < t_2 < \cdots < t_n < s < t$, such that

$$\mathsf{P}\Big(X_s = i, X_{t_n} = i_n, \ldots, X_{t_1} = i_1\Big) > 0.$$

The probabilities $p_{ij}(s,t)$ are called the transition probabilities from state i at time s to state j at time t, and the probabilities $p_j = \mathsf{P}(X_0 = j)$, $j \in S$, are the initial probabilities.

From now on, we will discuss only homogeneous Markov chains, i.e. processes satisfying

$$p_{ij}(s, s+t) = p_{ij}(t), \;\; s,t \geq 0.$$

Since the time is continuous, we need for all $i, j \in S$ a whole set of probabilities $\{p_{ij}(t), t \geq 0\}$, so that $\sum_{j \in S} p_{ij}(t) = 1$ for all $i \in S$, and consequently a whole system of transition matrices $\{\mathbb{P}(t), t > 0\}$.

The process $\{X_t, t \geq 0\}$ is determined by the vector of initial probabilities $\boldsymbol{p}(0) = \{p_i(0), i \in S\}$ and the system of transition matrices $\{\mathbb{P}(t), t > 0\}$ if for any times $0 < t_1 < t_2 < \cdots < t_k$ and for any states $i_0, i_1, \ldots, i_k \in S$, $k \in \mathbb{N}_0$, it is true that

$$\begin{aligned} \mathsf{P}\left(X_0 = i_0, X_{t_1} = i_1, \ldots, X_{t_k} = i_k\right) &= \\ &= p_{i_0}(0) p_{i_0 i_1}(t_1) p_{i_1 i_2}(t_2 - t_1) \ldots p_{i_{k-1} i_k}(t_{i_k} - t_{i_{k-1}}). \end{aligned}$$

Similarly to discrete time Markov chains, the Chapman-Kolmogorov equation for transition matrices holds:

$$\mathbb{P}(s+t) = \mathbb{P}(s)\mathbb{P}(t),$$

for any $s, t > 0$.

Even though some parts of the theory are just an analogy to the discrete case, some parts must be defined in a different way as the following theorem shows.

Theorem 2. *For each $i \in S$ the following limit exists:*

$$q_i = \lim_{h \to 0_+} \frac{1 - p_{ii}(h)}{h} \leq \infty^{6},$$

for each $i, j \in S$, $i \neq j$, exist the limits:

$$q_{ij} = \lim_{h \to 0_+} \frac{p_{ij}(h)}{h} < \infty,$$

and for all $i \in S$ we have

$$\sum_{i \neq j} q_{ij} \leq q_i.$$

Proof. See page 74 in Lachout and Prášková [1998]. □

The numbers q_{ij} are called the *transition intensities* from state i to state j, and the number q_i is called the *total transition rate* out of state i. The matrix $\mathbb{Q} = \{q_{ij}, i, j \in S\}$, where $q_{ii} = -q_i$ is called the *transition rate matrix* or *generator matrix*. The meaning of the transition intensities is explained in the following theorem.

Theorem 3. *Let $\{X_t, t \geq 0\}$ be a homogeneous Markov chain with countable state space. Then for all $s \geq 0$ and for all $h > 0$:*

$$P(X_t = i, s \leq t \leq s + h | X_s = i) = e^{-q_i h},$$

(where $e^{-q_i h} = 0$ if $q_i = \infty$).

If $q_i = 0$ then $p_{ii}(t) = 1$ for all $t \geq 0$. If $0 < q_i < \infty$ then the time during which the process remains in the state i has an exponential distribution[7] *with parameter q_i.*

Let $0 < q_i < \infty$, and the chain be in a state $i \in S$ at time $t \geq 0$. Then the probability that the chain will firstly go to state j in interval (t, ∞) is q_{ij}/q_i.

Proof. See page 76, 77, and 78 in Lachout and Prášková [1998]. □

[6] The + sign indicates a one-sided limit from right, in particularly a limit as h goes to zero when h ranges through positive values.

[7] A random variable has the exponential distribution with parameter $\lambda > 0$ if it takes values in non-negative numbers, $x \geq 0$, and its probability density function is $f(x; \lambda) = \lambda e^{-\lambda x}$.

Remark. Note that for finite S, it is always true that

$$q_i = \sum_{i \neq j} q_{ij}, \text{ for all } i \in S,$$

because

$$0 = 1 - \sum_{j \in S} p_{ij}(h), \ h \geq 0.$$

By dividing both sides by h and by applying the limit to both side of the equality, we get

$$0 = \lim_{h \to 0_+} \frac{1 - \sum_{j \in S} p_{ij}(h)}{h} = \frac{1 - p_{ii}(h) - \sum_{j \neq i} p_{ij}(h)}{h} = q_i - \sum_{j \neq i} q_{ij},$$

where the last equality holds because limit and sum can be interchanges for finite sums.

△

Definition 10. *Let $\{X_t, t \geq 0\}$ be a random process with a finite state space S defined on $(\Omega, \mathcal{A}, P)$ that has right-continuous trajectories. Let $\mathcal{F}_t$ be a σ-algebra generated by the family of random variables $\{X_s, s \leq t\}$, i.e. $\mathcal{F}_t = \sigma\{X_s, s \leq t\}$. A random variable $\tau : \Omega \to [0, \infty]$ is called a stopping time of the process $\{X_t, t \geq 0\}$, if $[\tau \leq t] \in \mathcal{F}_t$ for every $t \geq 0$.*

Theorem 4. *Let τ be a stopping time of a process $\{X_t, t \geq 0\}$. Let*

$$\mathcal{F}_\infty = \sigma\Big(\bigcup_{t \geq 0} \mathcal{F}_t\Big) = \sigma\{X_t, t \geq 0\}.$$

Then X_τ is $\mathcal{F}_\tau$-measurable random variable, where

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap [\tau \leq t] \in \mathcal{F}_t, t \geq 0\}.$$

Proof. See page 79 in Lachout and Prášková [1998]. □

Now, let us denote the times in which the transitions of a process $\{X_t, t \geq 0\}$ occur by a sequence $J_1, J_2, \ldots$, i.e.

$$\begin{aligned} J_1 &= \inf\{t > 0 : X_t \neq X_0\}, \\ J_2 &= \inf\{t > J_1 : X_t \neq X_{J_1}\}, \\ &\vdots \\ J_{n+1} &= \inf\{t > J_n : X_t \neq X_{J_n}\}, n \geq 0 \end{aligned}$$

Then, J_n, $n \in \mathbb{N}$ are obviously stopping times, because they do not depend on the future development of the original process. Further, let us define a sequence:

$$\begin{aligned} Y_0 &= X_0, \\ Y_n &= X_{J_n}, n \in \mathbb{N}. \end{aligned}$$

If $J_n = \infty$ then we define $Y_\infty = Y_{J_{n-1}}$. According to Theorem 4, Y_n, $n \in \mathbb{N}_0$, are random variables. Then, let us define a matrix $\mathbb{Q}^* = \{q^*_{ij}, i, j \in S\}$ such that:

$$q^*_{ij} = \begin{cases} \frac{q_{ij}}{q_i}, & q_i > 0, \quad i \neq j, \\ 0, & q_i = 0, \quad i \neq j. \end{cases}$$

$$q^*_{ii} = \begin{cases} 0, & q_i > 0, \\ 1, & q_i = 0. \end{cases}$$

It can be proved that the sequence $\{Y_n, n \in \mathbb{N}_0\}$ is a homogeneous discrete-time Markov chain with set space $S = \{0, 1, \dots\}$, and transition probabilities q^*_{ij} [Gichman and Skorochod, 1973]. The Markov chain $\{Y_n, n \in \mathbb{N}_0\}$ is called embedded Markov chain of the process $\{X_t, t \geq 0\}$. We have

$$\mathsf{P}(X_t = i) = \sum_{n=0}^{\infty} \mathsf{P}\left(Y_t = i, J_n \leq t \leq J_{n+1}\right).$$

The embedded Markov chain is used when we are interested in an observation of the changes in states of some continuous-time Markov process where the time perspective is not important.

Theorem 5 (Kolmogorov's differential equations)**.** *Let $q_i < \infty$ for all $i \in S$, and*

$$q_i = \sum_{i \neq j} q_{ij}, \text{ for all } i \in S.$$

Then the transition probabilities $p_{ij}(t)$ are differentiable for all $i, j \in S$ and $t > 0$ and

$$p'_{ij}(t) = -q_i p_{ij}(t) + \sum_{k \neq i} q_{ik} p_{kj}(t) = \sum_{k \in S} q_{ik} p_{kj}(t) \tag{1.5}$$

(Kolmogorov backward equations).

If $\frac{p_{ij}(h)}{h}$ converges to q_{ij} uniformly in i, then for each $i, j \in S$ and $t > 0$

$$p'_{ij}(t) = -p_{ij}(t) q_j + \sum_{k \neq j} p_{ik}(t) q_{kj} = \sum_{k \in S} p_{ik}(t) q_{kj} \tag{1.6}$$

(Kolmogorov forward equation).

In matrix notation:

$$\mathbb{P}'(t) = \mathbb{Q}\mathbb{P}(t),$$
$$\mathbb{P}'(t) = \mathbb{P}(t)\mathbb{Q}.$$

Proof. See page 82 in Lachout and Prášková [1998]. $\square$

Similarly to discrete time Markov chains, we can find the stationary distribution.

Definition 11. *Let $\{X_t, t \geq 0\}$ be a Markov chain with continuous time and countable state space S and transition matrices $\mathbb{P}(t)$, $t \geq 0$. A vector $\boldsymbol{\eta} = \{\eta_i \geq 0, i \in S\}$ such that*

$$\boldsymbol{\eta}^T \mathbb{P}(t) = \boldsymbol{\eta}^T, \ t \geq 0, \tag{1.7}$$

is called a invariant measures of the process $\{X_t, t \geq 0\}$. A probability distribution $\boldsymbol{\pi}$ satisfying Equation 1.7 is called a stationary distribution of the process $\{X_t, t \geq 0\}$.

A probability distribution $\boldsymbol{a} = \{a_i, i \in S\}$ on S is called limiting distribution, if for all $i, j \in S$ we have

$$\lim_{t \to \infty} p_{ij}(t) = a_j.$$

Theorem 6. *If the limiting distribution of a Markov chain exists, it is the unique stationary distribution.*

Proof. See page 90 in Lachout and Prášková [1998]. □

Example 3 (Poisson process). Now, let us demonstrate the theory on a classical example that will be used further in the thesis. The Poisson process is used for the description of the occurrence of random events in a time interval. We are assuming that the probability of one event happening in any interval $(t, t+h]$ is $\lambda h + o(h)$, and the probability of more than one event happening in that interval is $o(h)$[8], where $\lambda > 0$ is a parameter. The numbers of events in disjunct intervals are mutually independent.

Let N_t represent the number of events in the interval $[0, t]$. Then $\{N_t, t \geq 0\}$ is the Poisson process with intensity $\lambda > 0$. Obviously, the state space $S = \mathbb{N}$. It is a classical example of a continuous time Markov chain with independent increments, i.e., for any $0 < t_1 < t_2 < t_3 < t_4$, the numbers of events $N_{t_2} - N_{t_1}$ and $N_{t_4} - N_{t_3}$ in the intervals $(t_1, t_2]$ and $(t_3, t_4]$ are independent random variables. We have for all $t \geq 0$ and $h > 0$:

$$\begin{aligned}
\mathsf{P}(N_{t+h} - N_t = 1) &= \lambda h + o(h) \\
\mathsf{P}(N_{t+h} - N_t = 0) &= 1 - \lambda h + o(h) \\
\mathsf{P}(N_{t+h} - N_t \geq 2) &= o(h)
\end{aligned}$$

Hence, we have for all $t \geq 0$ and $h > 0$:

$$\mathsf{P}(N_{t+h} = j | N_t = i) = p_{ij}(t) = \begin{cases} \lambda h + o(h), & j = i+1, \\ 1 - \lambda h + o(h), & j = i, \\ o(h), & j > i+1, \\ = 0, & j < i. \end{cases}$$

The initial distribution probabilities are simply $p_0(0) = 1$ and $p_j(0) = 0, j > 0$.

[8]Symbol $o(h)$ means that $o(h)/h \to 0$ for $h \to 0_+$

The transition intensities are $q_{i,i+1} = \lambda$, $q_i = -q_{ii} = \lambda$, $q_{ij} = 0$ otherwise. The transition rate matrix is then

$$\mathbb{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

The increments $N_t - N_s$, $0 \leq s < t$, have a Poisson distribution with parameter $\lambda(t-s)$ (see page 100 in Lachout and Prášková [1998]). Therefore, the number of events in any interval $(s,t]$ depends only on the length of the interval.

The embedded Markov chain $\{\widehat{N}_n, n \in \mathbb{N}_0\}$ is a homogeneous discrete-time Markov chain with the initial distribution

$$\boldsymbol{p} = (1, 0, 0, \dots),$$

and the transition matrix

$$\mathbb{Q}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

△

1.4 Game theory

Another part of mathematics that will be further used in the models that are subject of this thesis is so-called *Game Theory*. The theory is somewhere between mathematics and economics, but it is also used to study complicated situations in other fields such as politics, or sociology. It is used to analyse a situation and to find the best strategy or decision.

Game refers to any social situation involving two or more individuals who are called *players* Myerson [1997]. Players are assumed to be *rational* and *intelligent*. Rationality represents the player's consistency in decision-making in order to maximize his/her *utility* (or expected utility), which can be, for example, a monetary pay off or the number of MPs in parliament. A player is intelligent if he/she knows everything about the game, and his/her decision is not random but it is based on facts at any time.

Let us now define some terminology (inspired by Leyton-Brown and Shoham [2008]).

Definition 12. *A (finite, n-person) normal-form game is a triplet (N, X, f), where:*

- *N is a finite set of n players, indexed by i,*

- *$X = X_1 \times X_2 \times \cdots \times X_n$, where X_i is a finite set of actions available to player i. Each vector $x = (x_1, \ldots, x_n)$ is called an action profile or strategy,*
- *$f = (f_1, f_2, \ldots, f_n)$, where $f_i(x_1, \ldots, x_n)$ is a utility function for player i, $f_i : X \to \mathbb{R}$.*

Note that the term *strategy* is sometimes confusing because it may be tempting to use it in the context of just one player, e.g. first player's strategy in roulette is to always bet one dollar on number 3. However, in the context of game theory, strategy will always refer to the choice of all players, i.e. a vector and element of X. The individual choice of any player in the strategy is called action, i.e.the element of X_i for player i.

Both the number of players and the number of strategies is finite by definition. Sometimes, those games are called *finite games*.

Games can be divided into commonly used categories according to:

- **number of players**
  - 2 players
  - n players
- **cooperation of players**
  - Non-cooperative games: all players playing against each other,
  - Cooperative games: there is a possibility of creating of at least one coalition of 2 or more players,
- **pay off**
  - Constant-sum game (or inaccurately Zero-sum game): The overall profit of all players is constant for each strategy. Hence, one player's profit the a result of loss of other(s). Precisely: there exists $k \in \mathbb{R}$ such that $\sum_{i=1}^{n} f_i(x_1, \ldots, x_n) = k$ for all $(x_1, \ldots, x_n) \in X$,
  - Non-constant-sum game.

Non-cooperative games

As mentioned earlier, non-cooperative games are characterized by the absence of any possibility of cooperation among players, so each player considers all other players as enemies in every situation. As a result, players must choose their actions with the highest pay offs regardless of what their opponents do, and this can be tricky because it is not straightforward what the "best" strategy is.

In the year 1951, American mathematician and economist John Nash proposed a solution for finding the "best" strategy [Nash, 1951], which is since then called "Nash equilibrium". The equilibrium is defined as a strategy for which no player can do better by unilaterally changing his/her action (i.e. other's actions remain unchanged).

Definition 13. *Strategy $(x_1, \ldots, x_n) \in X$ is a Nash equilibrium point if for all $i \in \{1, \ldots, n\}$ we have:*

$$f_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \geq f_i(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$$

for all $y_i \in X_i$.

It will be demonstrated later that such a strategy does not always exist. Hence, we define dominance as another criterion used for comparison. One action dominates another if it offers the player a higher pay off no matter what the competitors do.

Definition 14. *Let $i \in \{1, \dots, n\}$. We say that action $x_i \in X_i$ dominates $x_i^{'} \in X_i$ if*

$$f_i(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) > f_i(x_1, \dots, x_{i-1}, x_i^{'}, x_{i+1}, \dots, x_n)$$

for all $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X_1 \times \dots \times X_{i-1} \times X_{i+1} \times \dots \times X_n$.

We say that strategy $x \in X$ dominates strategy $x^{'} \in X$ if

$$f_i(x) \geq f_i(x^{'})$$

for all $i \in \{1, \dots, n\}$, and there exists some $j \in \{1, \dots, n\}$ for which

$$f_j(x) > f_j(x^{'}).$$

The domination of strategies is sometimes called Pareto domination [Leyton-Brown and Shoham, 2008].

Nash also defines the set of *undominated* strategies (or Pareto optimal strategies). However, a Nash equilibrium is not necessarily an undominated strategy and vice versa, which will be be demonstrated later.

Definition 15. *We say that strategy $(x_1, \dots, x_n) \in X$ is undominated if there does not exist any strategy $(y_1, \dots, y_n) \in X$ such that*

$$f_i(y_1, \dots, y_n) \geq f_i(x_1, \dots, x_n)$$

for all $i \in \{1, \dots, n\}$ with at least one strict inequality.

Let us now demonstrate the terms on two examples.

Example 4. The classical example used in game theory is the so-called *Prisoner's dilemma*. Imagine two criminals C1 and C2 arrested by the police for a major crime that they have committed together. They are being questioned in solitary confinement, and they cannot communicate with each other. However, the police officers lack sufficient evidence to prove them guilty, so they need them to confess. In order to get their confession, policemen offer them a deal, 'If you betray you partner, we will set you free, and your partner will be blamed for his and also your crimes'. If none of them confesses, they will both be charged with some minor crime; however, if they both decide to betray, they will both be sentenced. So, both prisoners can choose between staying silent (**S**) and betrayal (**B**). The sentences are:

- if both C1 and C2 betray, they both get 8 years,
- if C1 betrays but C2 remains silent, C1 will be set free, and C2 will serve 10 years (and vice versa),
- if none of them betrays, they will both serve 1 year.

The pay off matrix is then (the numbers are negative because the "best" strategy is the one that maximizes pay off, i.e. the least years in prison):

C1 \ C2	C2 betrays (**B**)	C2 remains silent (**S**)
C1 betrays (**B**)	-8 \ -8	0 \ -10
C1 remains silent (**S**)	-10 \ 0	-1 \ -1

Now, let us analyse all four possible scenarios:

1. (**B**,**B**):
   - C1 switching to **S** $\rightarrow$ worse pay off,
   - C2 switching to **S** $\rightarrow$ worse pay off,

   $\Rightarrow$ Nash equilibrium point, but dominated by (**S**,**S**),
2. (**S**,**S**):
   - C1 switching to **B** $\rightarrow$ improved pay off,
   - C2 switching to **B** $\rightarrow$ improved pay off,

   $\Rightarrow$ undominated strategy, but not a Nash equilibrium point,
3. (**B**,**S**):
   - C1 switching to **S** $\rightarrow$ worse pay off,
   - C2 switching to **B** $\rightarrow$ improved pay off,

   $\Rightarrow$ not a Nash equilibrium point, and not undominated,
4. (**S**,**B**):
   - C1 switching to **B** $\rightarrow$ improved pay off,
   - C2 switching to **S** $\rightarrow$ worse pay off,

   $\Rightarrow$ not a Nash equilibrium point, and not undominated.

The undominated strategy (**S**,**S**) is also the best when comparing total years in prison summed for both prisoners. However, the inability to communicate makes it highly risky, and the safest action is simply to betray the partner.

△

Example 5. An easy example of a game without an equilibrium is Rock-paper-scissors. It is a game of two players P1 and P2 who both can choose between 3 actions—rock (**R**), paper (**P**), or scissors (**S**). Let us suppose a game of one round of rock-paper-scissors after which the winner gets \$1, the loser pays \$1, and nobody wins or loses in case of a tie. The pay off matrix is then

P2 P1	**R**	**P**	**S**
R	0 0	1 -1	-1 1
P	-1 1	0 0	1 -1
S	1 -1	-1 1	0 0

Obviously, there is no equilibrium. For each strategy there exists a different one for which at least one of the players can have higher pay off.

△

The strategies defined in Definition 12 and used up until now are called *pure* strategies. Their extension, so-called *mixed* strategies, are derived by adding a probability distribution to pure strategies.

Definition 16. *Let (N, X, f) be a normal-form game, $X = X_1 \times \cdots \times X_n$, where X_i is a finite set of actions available to player i. Let for any set A be $\Pi(A)$ the set of all probability distributions over A. Then the set of mixed strategies for player i is $S_i = \Pi(X_i)$.*

Even though we showed that Nash equilibrium does not always exist, Nash [1951] proved theorem saying that a Nash equilibrium exists in every finite game with mixed strategies.

Theorem 7. *Any finite game has at least one Nash equilibrium point in the space of mixed strategies.*

Proof. The proof can be found in Nash [1951]. It is achieved by appealing to a fixed-point theorem—a theorem saying that under some conditions a function F has at least one fixed point x, i.e. a point for which $F(x) = x$. □

Example 5 (Continued). Considering the rock-paper-scissors example, there is no equilibrium among pure strategies. However, it is a common knowledge that that the best strategy is to simply randomly choose one of the three options. The best strategy for both players is a vector of probabilities $\big(1/3, 1/3, 1/3\big)$. The expected pay off PO is then:

$$\mathsf{E}\left[PO\right] = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \big(-1\big) = 0.$$

△

2. The Stigler-Luckock model

The Stiegler-Luckock model is the starting point of this thesis, so we will briefly introduce it in this chapter. The main purpose of the model was to examine the behaviour of order books and the process of forming the equilibrium price on markets.

The very first definition of the model was proposed by George J. Stigler in 1964 in The Journal of Business [Stigler, 1964]. This famous American mathematician was awarded the Nobel Memorial Prize in Economic Studies in 1982. His article from 1964 mostly concerns the American stock exchange and its regulations for which he criticized the Securities and Exchange Commission (SEC) because of the imprecise and lax tests regarding the impacts of regulations on financial markets. Moreover, the article introduces a simulation model of the market that helped Stigler to demonstrate his ideas.

A few decades later, Hugh Luckock, a contemporary mathematician focusing on market microstructure and high frequency trading, analysed the same model and expanded it to its full generality [Luckock, 2003]. Interestingly, he was probably not aware of Stigler's model, because he does not cite it in the article. Luckock also analyses the model using computer simulations that were unreachable to Stigler.

Coincidentally, the Stigler-Luckock model was independently reinvented by two other authors—Elena Yudovina in her doctoral thesis [Kelly and Yudovina, 2012] and Jana Plačková in her master thesis [Plačková, 2011]. That fact indicates that this path is reasonable.

However, the model itself shows several drawbacks, which will be demonstrated in the following sections, and needs to be further developed. One of the possible ways of improving the model—a concept of "market-making"—will be introduced, as the plain Stigler-Luckock model is significantly non-liquid.

2.1 Terminology

Even though supply and demand were thoroughly discussed earlier, let us now specify the terminology for the models. Let $\mathcal{P} \subset \mathbb{R}_0^+$[1] be a set representing all possible prices, and let $x \in \mathcal{P}$ be some price. Supply represents the amount of the asset that sellers are prepared to sell for some specific price. Therefore, supply at price x includes all the asset that is available for sale up to price x (i.e. for x and less). Similarly, demand represents the amount of the asset that buyers are ready to buy for some price x, and hence it includes all the asset that is demanded for price x and higher.

We need to give the proper definition of the terms supply and demand since they will be used frequently in the thesis. The definitions were inspired by Luckock [2003].

Definition 17. *Let $x \in \mathcal{P}$ be a price. Then supply $\lambda_+(x)$ is equal to the expected amount of sell orders equal and below the price level x per unit of time, and*

[1] $\mathbb{R}_0^+ = [0, \infty)$

demand $\lambda_-(x)$ is equal to the expected amount of buy orders equal and above the price level x per unit of time.

When analysing the models, we will be also looking at the order book shape. As mentioned earlier, the order book does not perfectly mirror the underlying supply and demand; however, we can expect that if the supply and demand curves have a certain shape and intersect at some point, the mid price will be around the equilibrium price, and the order book will reflect that. Hence, we will analyse the *order book* and also the *cumulative order book*. The cumulative order book was already mentioned in section 1.2, and it is simply just the order book summed from the left-hand side for sell orders and from the right-hand side for buy orders.

2.2 Description of the Stigler-Luckock model

First, let us focus on Stigler's model. As mentioned before, the model was introduced in 1964 in an article focusing on regulations of the security market in the US [Stigler, 1964].

The model is defined in discrete time with discrete prices that are ranging from $\$28\frac{3}{4}$ to $\$31$ with a tick size of $\$\frac{1}{4}$. Therefore, there are in total $L = 10$ possible prices, and the order book is constrained to 10 values.

At each time step $t \in \mathbb{N}$ exactly one new order arrives on the market. Each order is randomly assigned with a sign (buy $\mathbf{B}$ or sell $\mathbf{S}$) that is chosen with equal probability. Let the sign of an order be represented as a random variable A. Then

$$\mathsf{P}(A = \mathbf{B}) = \mathsf{P}(A = \mathbf{S}) = 1/2.$$

The price X of the order is uniformly distributed on the price grid, i.e. each option has the same probability $p = \frac{1}{10}$. Besides that, the quantity of all orders is simply one. The supply and demand functions for price $x \in \{\$28.75, \$29.00, \dots, \$30.75, \$31.00\}$ are then:

$$\lambda_+(x) = \mathsf{E}\left[X \leq x | A = \mathbf{S}\right] = \mathsf{P}(X \leq x) = \sum_{k=28.75}^{x} \mathsf{P}(X = k) = \frac{1}{10}\Big(x - 28.75 + 1\Big),$$

$$\lambda_-(x) = \mathsf{E}\left[X \geq x | A = \mathbf{B}\right] = \mathsf{P}(X \geq x) = \sum_{k=x}^{31.00} \mathsf{P}(X = k) = \frac{1}{10}\Big(31.00 - x + 1\Big).$$

The functions are plotted in Figure 2.1.

The model begins with empty order book, and each order arriving on the market is registered and stays in the LOB. The bid (resp. ask) price is established after the arrival of the first buy (resp. sell) sell order.

Orders are not explicitly distinguished to be limit or market orders. Still, two possible scenarios may occur after the submission of some new order. Firstly, let us analyse a buy order at time t with a price x. Let the current value of the ask price in the LOB be $a(t)$. The possible scenarios are summarized in Table 2.1.

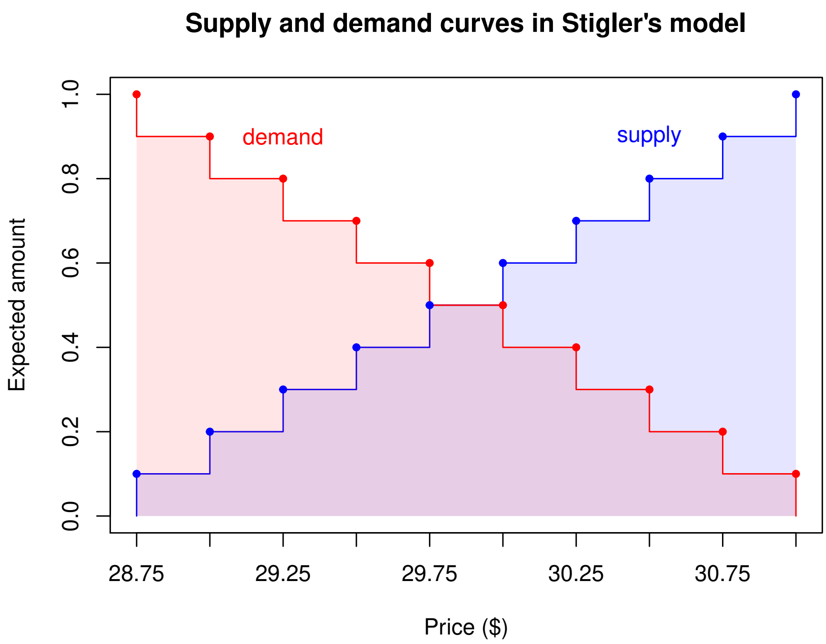


Figure 2.1: Supply and demand curves in Stigler's model.

BUY ORDER		
$x \geq a(t)$	order executed for the price $a(t)$, opposite order removed from LOB	$\rightarrow$ MARKET ORDER
$x < a(t)$	order recorded in LOB, waits there to be executed	$\rightarrow$ LIMIT ORDER

Table 2.1: Possible scenarios in Stigler's model after the submission of a buy order with price x at time t depending on the current ask price $a(t)$.

Similarly, the two possible scenarios for a sell order at time t and a price x are in Table 2.2. (with the bid price $b(t)$).

SELL ORDER		
$x \leq b(t)$	order executed for the price $b(t)$, opposite order removed from LOB	$\rightarrow$ MARKET ORDER
$x > b(t)$	order recorded in LOB waits there to be executed	$\rightarrow$ LIMIT ORDER

Table 2.2: Possible scenarios in Stigler's model after a submission of a sell order with price x at time t depending on the current bid price $b(t)$.

In addition, each order that is not executed after time $N = 25$ after it was placed is automatically cancelled. Hence, there are simultaneously at most 25 orders in the LOB.

Stigler computed manually a set of hundred trades using tables of random numbers [Chakraborti et al., 2011]. The outcome of the simulation demonstrated the evolution of the order book and the movement of the bid and ask prices. It was rather an academic example, but it was a great stepping stone for further research.

A few decades later, Hugh Luckock was able to develop Stigler's model to its full generality [Luckock, 2003]. The article does not cite Stigler, so he was probably not aware of Stigler's model at all. The models is summarized by 5 assumptions:

(A1) all orders are for a single unit, so the buyers only need to specify the maximum price they are prepared to pay, and sellers only specify the minimum price they are prepared to accept,

(A2) there is large number of potential traders; therefore each order is considered to be originated from a different source and submitted independently of others. The arrival of the orders within any specified price range is a Poisson process,

(A3) supply and demand functions are time-independent,

(A4) traders are submitting the orders without considering the current state of the order book; hence, expected order arrival rates are independent of the order book configuration,

(A5) the cancellation rate of unexecuted orders is negligible.

The only major difference from Stigler's model is the assumptions (A5). Luckock argues that cancellation usually happens as a response to changed market conditions, but the modelled market is static, and so it is unnecessary to consider the cancellation rate.

The separation of market and limit orders is done by the same key as in Table 2.1 and Table 2.2.

One of the major generalisations is that Luckock allows a general form of supply and demand functions, and that there are no restrictions upon the range of permissible prices (both discrete and continuous). The assumptions (A2)-(A4) specify that the arrival of new sell orders at prices not exceeding x is a Poisson process with some parameter $\lambda_A(x)$, where $\lambda_A(x) : (0, \infty) \mapsto (0, \infty)$ is a non-decreasing right-continuous function (i.e. supply function). The arrival of new buy orders of at least price x is a Poisson process with parameter $\lambda_B(x)$, where $\lambda_B(x) : (0, \infty) \mapsto (0, \infty)$ is a non-increasing left-continuous function (i.e. demand function). The values $\lambda_A(x)$ and $\lambda_B(x)$ represent the average number of traders submitting orders that are executable at the price x per unit of time. Orders in disjunct price interval are also independent, which gives us 2-dimensional Poisson process, where the dimensions are price and time. Besides that, the model assumes that the excess demand $\lambda_B - \lambda_A$ is positive for some prices and negative for others in order to guarantee the classical supply-demand position resulting in some equilibrium price (illustrated in Figure 1.1).

Models by both authors are characterized by traders who are acting randomly and who are not observing the market before placing their orders. Those models are called "zero-intelligence" models. Even though the non-strategic behaviour in such models seems unrealistic and quite naive, the models are not as theoretical as they might appear. Some practical experiments were performed in which computers with zero-intelligence strategies competed with humans, usually business students, who traded naturally (Gode and Sunder [1993], Gode and Sunder [1997], Tóth et al. [2007]). In spite of both groups having different approaches, the zero-intelligence agents surprisingly demonstrated the same level of efficiency as humans. Since humans performed with the same skill as randomly trading computers, the experiments actually question the validity of one of the fundamental economic assumptions—that people behave like rational optimisers of their own individual profits. In conclusion, the zero-intelligence models, including the Stigler-Luckock model, are more practical and realistic than they may seem at first.

2.3 Simulations

Let us now demonstrate the Stigler-Luckock model's behaviour by some Monte Carlo simulations. Firstly, a proper mathematical description of the simulated model must be given.

The arrival of participants on the market can be described as a Poisson process $\{N_t, t \geq 0\}$ with some overall intensity $R > 0$ that represents the arrival of any market participant, i.e. both buyers and sellers. Let $\mathcal{P}$ be a set of all possible prices, $x_{\max} := \max\{\mathcal{P}\}$, and $x_{\min} := \min\{\mathcal{P}\}$. Let us assume that $\mathcal{P}$ is discrete for simplicity, and let $\lambda_\pm : \mathcal{P} \to [0, \infty)$ be supply and demand functions. More precisely, let μ_+ and μ_- be finite measures on $\mathcal{P}$, then supply and demand at price $x \in \mathcal{P}$ are defined as

$$\lambda_+(x) := \mu_+\Big(\{y \in \mathcal{P}, y \leq x\}\Big),$$
$$\lambda_-(x) := \mu_-\Big(\{y \in \mathcal{P}, y \geq x\}\Big).$$

Then, $\mu_+\big(\{x\}\big)$ is a Poisson intensity at which traders place sell orders at price $x \in \mathcal{P}$, and $\mu_-\big(\{x\}\big)$ is a Poisson intensity of buy orders at a price $x \in \mathcal{P}$. Hence, $\mu_+(\mathcal{P}) = \lambda_+(x_{\max})$ and $\mu_-(\mathcal{P}) = \lambda_-(x_{\min})$ are Poisson intensities of sell and buy orders, respectively. The overall intensity R of the Poisson process $\{N_t, t \geq 0\}$ is then:

$$R = \lambda_+(x_{\max}) + \lambda_-(x_{\min}).$$

Let us denote the probability of arrival of a sell order by $p_{\mathbf{S}}$ and the probability of arrival of a buy order by $p_{\mathbf{B}}$. Then,

$$p_{\mathbf{S}} = \frac{\lambda_+(x_{\max})}{R},$$
$$p_{\mathbf{B}} = \frac{\lambda_-(x_{\min})}{R}.$$

Let us denote the sign of k-th order $\sigma_k \in \{\mathbf{B}, \mathbf{S}\}$, the price of k-th order $X_k \in \mathcal{P}$, and the time of the arrival of the k-th order by $\tau_k \in \mathbb{R}^+$. Then each order is completely determined by the triplet (σ_k, X_k, τ_k), $k \in \mathbb{N}$. Again, the orders are distinguished to be limit or market by their price, and the price of current bid or ask (see Table 2.1 and Table 2.2).

It follows, that the random variables $(\tau_k - \tau_{k-1})_{k\geq 1}$ are exponentially distributed with parameter $1/R$. Moreover, random variables $(\sigma_k, x_k)_{k\geq 1}$ are independent and identically distributed (i.i.d.), and they are also independent of $(\tau_k)_{k\geq 1}$.

We were considering the market in the continuous time up until now. We may however simplify it by studying the discrete embedded Markov chain $\{\widehat{N}_n, n \in \mathbb{N}_0\}$, i.e. the discrete-time Markov chain that is derived by

$$\widehat{N}_0 := 0,$$
$$\widehat{N}_k := N_{\tau_k},\ k \in \mathbb{N}.$$

Now, let us run the Monte Carlo simulations of the Stigler-Luckock model for $\mathcal{P} = \{1, \dots, 100\}$, and measures $\mu_+ = \mu_-$ are uniform distributions on $\mathcal{P}$, i.e.

$$\mu_+\big(\{x\}\big) = \mu_-\big(\{x\}\big) = \frac{1}{100},$$

for all $x \in \mathcal{P}$. Then supply and demand functions are defined as

$$\lambda_+(x) = \mu_+\Big(\{y \in \mathcal{P}, y \leq x\}\Big) = \frac{x}{100},$$
$$\lambda_-(x) = \mu_-\Big(\{y \in \mathcal{P}, y \geq x\}\Big) = \frac{101 - x}{100},$$

for $x \in \{1, \dots, 100\}$.

The overall intensity of the Poisson process is then

$$R = \lambda_+(x_{\max}) + \lambda_-(x_{\min}) = \lambda_+(100) + \lambda_-(1) = 1 + 1 = 2,$$

and the probabilities of arrivals of sell and buy orders are

$$p_{\mathbf{S}} = \frac{\lambda_+(x_{\max})}{R} = \frac{1}{2},$$
$$p_{\mathbf{B}} = \frac{\lambda_-(x_{\min})}{R} = \frac{1}{2}.$$

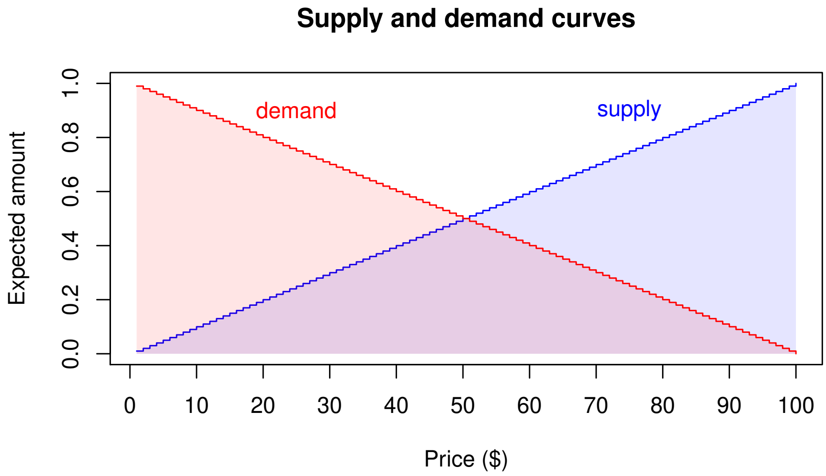
The distribution of price of an order X_k at time τ_k, $k \geq 1$, is then uniform on the price grid, i.e.

$$X_k \sim \text{Unif}\,\{1, \dots, 100\},$$

for $k \geq 1$.

The mutual position of supply and demand curves is in Figure 2.2. Supply and demand "meet" in the middle—between the prices 50 and 51. However, there is no equilibrium price since the supply and demand are not continuous, and their values are never equal. Demand is slightly higher when price equals 50, and supply is slightly higher when price equals 51.

Further, we will only study the embedded process, i.e. the discrete Markov chain for which at each discrete time $n \in \mathbb{N}$ a new trader arrives on the market. Now, let us present the results of simulations. Comparison of resulting order books after $N = 5,000$ and $N = 25,000$ time steps are in Figure 2.3. The buy and sell orders do not meet in the middle as the supply and demand suggest. Moreover, there is a whole interval of prices (from approximately 22 to 78), in which all orders are executed, and orders outside of the interval are remaining in the order book. The window's width is the same for both simulations. The cumulative order books corresponding to the simulations are in Figure 2.4.



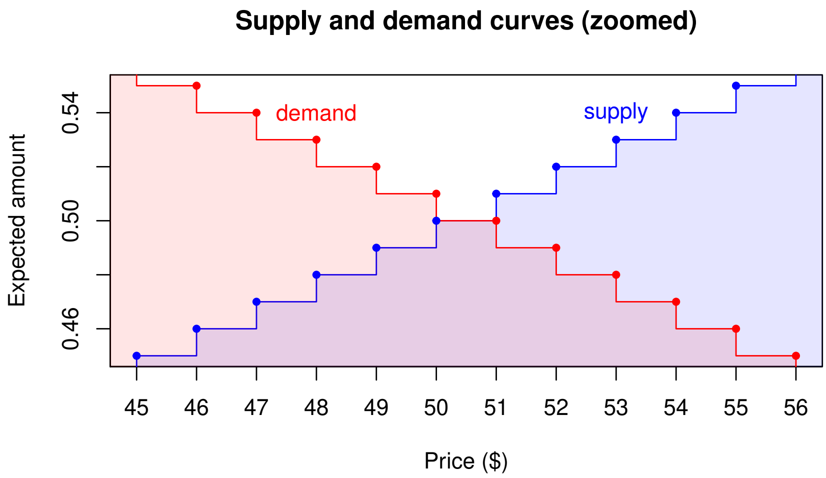


Figure 2.2: Theoretical supply and demand curves in the Stigler-Luckock Model with full x-axis (top) and zoomed in the middle (bottom).

2.3.1 Competitive window

Luckock [2003] refers to the interval in the middle of Figure 2.4 as the *competitive window*. This strange behaviour was then studied by Swart [2018]. He explains that the competitive window appears due to high non-liquidity of the model's market.

The article discusses a continuous-price model in which prices assume values in $I = (0, 1)$. Supply and demand are denoted by λ_+ and λ_-, and they satisfy:

(A1) λ_+ is non-decreasing, λ_- is non-increasing,

(A2) $\lambda_\pm$ are continuous,

(A3) $\lambda_+ - \lambda_-$ is strictly increasing,

(A4) $\lambda_\pm > 0$ on I.

According to Walras [2014], there must exist—under those four assumptions—a so-called *Walrasian price* $x_W \in I$ and *Walrasian volume of trade* $V_W > 0$ such

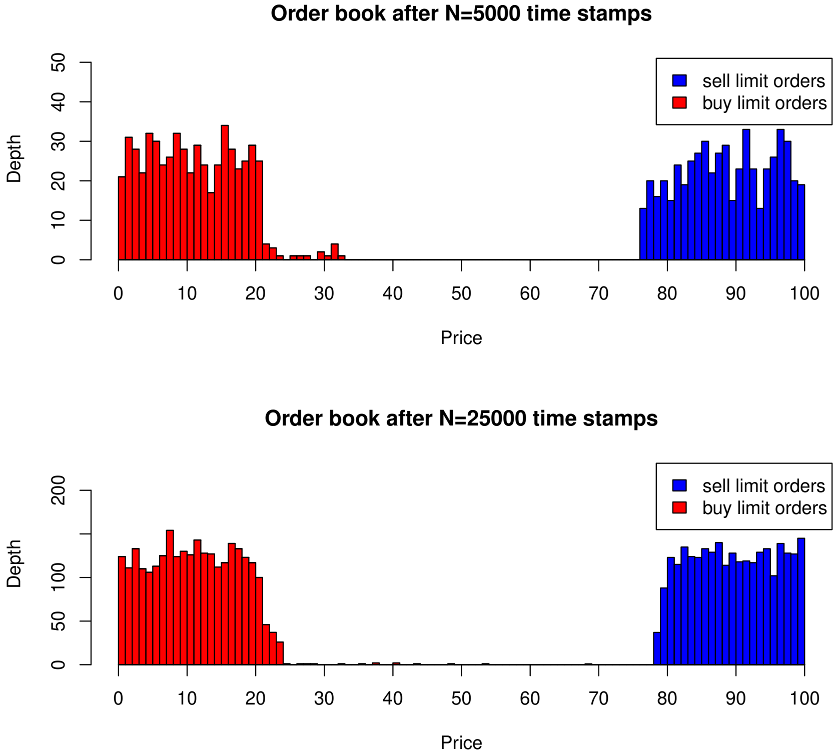


Figure 2.3: Shape of the order book in the Stigler-Luckock model after 5,000 and 25,000 time steps.

that:

$$\lambda_-(x_W) = \lambda_+(x_W) = V_W.$$

Still, this behaviour is not detectable in the Stigler-Luckock model. The bid and ask prices keep oscillating in a competitive window (x_-, x_+), for which $\lambda_-(x_-) = \lambda_+(x_+)$. Therefore, buyers and sellers may wait on the market much longer to execute their order which makes the market highly non-liquid.

Luckock [2003], Swart [2018], and also Kelly and Yudovina [2012] all calculated the precise values of the borders of the competitive window. Subtle difference is how rigorous and how general the assumptions were. The values are

$$x_- \approx 0.218,$$
$$x_+ \approx 0.782.$$

These values are just scaled values as those from the discrete model described earlier (in Figure 2.3). On top of that, the model suggests that the competitive window (x_-, x_+) always appears in the long run. On the one hand, all buy orders below that interval, and all sell orders above the interval are never matched and stay in the order book forever. On the other hand, the rest of the orders are executed eventually. This problem remains open and awaits its proof; however, Kelly and Yudovina [2012] proved it for some models including the one simulated before with discrete price grid.

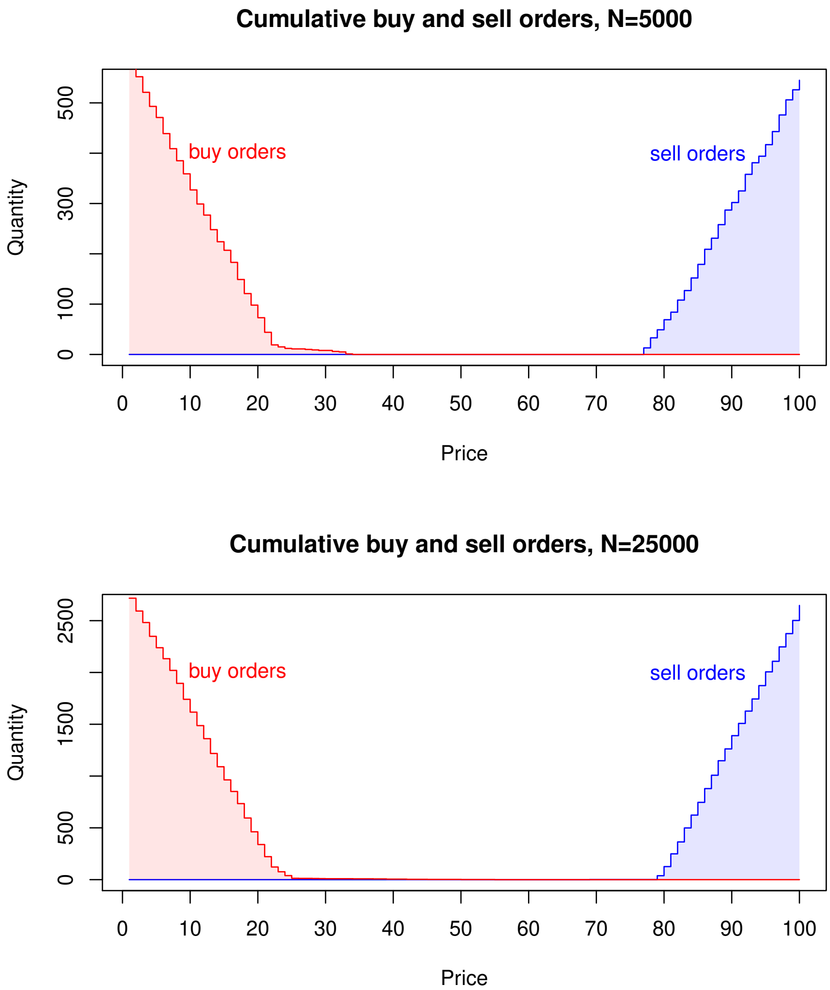


Figure 2.4: Shape of the cumulative order book in the Stigler-Luckock model after 5,000 and 25,000 time steps.

2.4 Introduction of the Market makers

The results cited in the previous section indicate that the model needs some extension to become more realistic. There are undoubtedly many directions in which the model can be improved. The unrealistic nature of the model lies mainly in the simple method of placing orders resulting in non-liquid behaviour.

One direction of improvement could be the introduction of so-called *market makers*. In reality, a significant part of trades is realized because of speculating, i.e. buying or selling the asset not because the traders needs it but because they believes that the price will change and they can make a profit out of it.

Market makers are, as the name suggests, traders whose presence is vital for the market. They "make" the market by bringing the necessary liquidity. They are usually established companies that own enough of the particular asset (and money of course). Consequently, they can consistently buy and sell the asset; and as a result, deliver the liquidity. Their motivation is simple—the fundamental principle is that they buy for less and sell for more. Besides that, it is not extraordinary that stock exchanges themselves pay the market makers a so-called *liquidity provision*. So, not only do the market makers earn money by buying the asset for less and selling it for more, but they also usually get paid by the exchange institution because they are an essential component of the market.

The easiest real-life example of "market-making" are currency exchanges. Their main business focus are currencies, even though they do not want to possess them. They only buy them for less and simultaneously sell them for more. Their source of profit is the difference between those two prices.

The concept of incorporating market makers (or a similar structure) in the models is not new. There are several ways of how authors have approached the problem. For example, Garman [1976] or Glosten and Milgrom [1985] introduced models with only one centralized market-maker who possess a monopoly on all trading. Those models are called a "dealership market". Exchanges are permitted only through that market-maker who determines the price.

Kyle [1985] introduced three types of traders. One single insider who has unique information about the market and provides liquidity, uninformed noise traders who trade randomly, and market makers who have more information than the noise traders and who set prices more efficiently.

Abergel et al. [2016] used different terms for the market makers—agents who submit limit orders are referred to as liquidity providers, while those who submit market orders are referred to as liquidity takers. However, the authors acknowledge that since the various market deregulation waves in the US in 2005 and in the EU in 2007, there are no pure liquidity providers or takers.

Another approach was introduced by Swart and Peržina [2018] who proposed a model with many market makers, not just one. Prior to trading, a Poisson rate $\rho \geq 0$ is defined, which represents the proportion of market makers. For $\rho = 0$, the model is without market makers. The higher the parameter ρ is, the higher is the proportion of market makers.

Then the Poisson process that is representing the arrival of a new market participant is determined by the overall rate R:

$$R = \lambda_+(x_{\max}) + \lambda_-(x_{\min}) + \rho, .$$

where λ_+ and λ_- are supply and demand functions defined on some price grid $\mathcal{P}$ (see section 2.3). Then the probabilities of arrivals of buyer, seller, and market maker are

$$\begin{aligned} p_{\mathbf{B}} &= \frac{\lambda_-(x_{\min})}{R}, \\ p_{\mathbf{S}} &= \frac{\lambda_+(x_{\max})}{R}, \\ p_{\mathbf{MM}} &= \frac{\rho}{R}. \end{aligned}$$

For $\rho = 0$, there are no market makers, and the model is the same as the one in the previous section. For $\rho > 0$, the proportions change and for example when $\rho = 0.5$, the probability of an arrival of a market maker is $p_{\mathbf{MM}} = 20\%$.

Now, let us demonstrate the impact of market makers on simulations. We will continue with the model simulated before (in section 2.3), i.e. $\mathcal{P} = \{1, \ldots, 100\}$, and measures the $\mu_+ = \mu_-$ are the uniform distributions on $\mathcal{P}$. The additional parameter is the rate ρ.

The market makers' strategy is that once they arrive on the market, they place one sell market order for the current bid price and one buy market order for the current ask price. Hence, they help the market to execute more orders and besides that earn money by selling for less and buying for more. If there is no buy limit order or sell limit order (or both) in the order book, the market maker simply skips that.

The crucial point is to find a proper value of the parameter ρ. Shapes of order books with different parameter setting are in Figure 2.5.

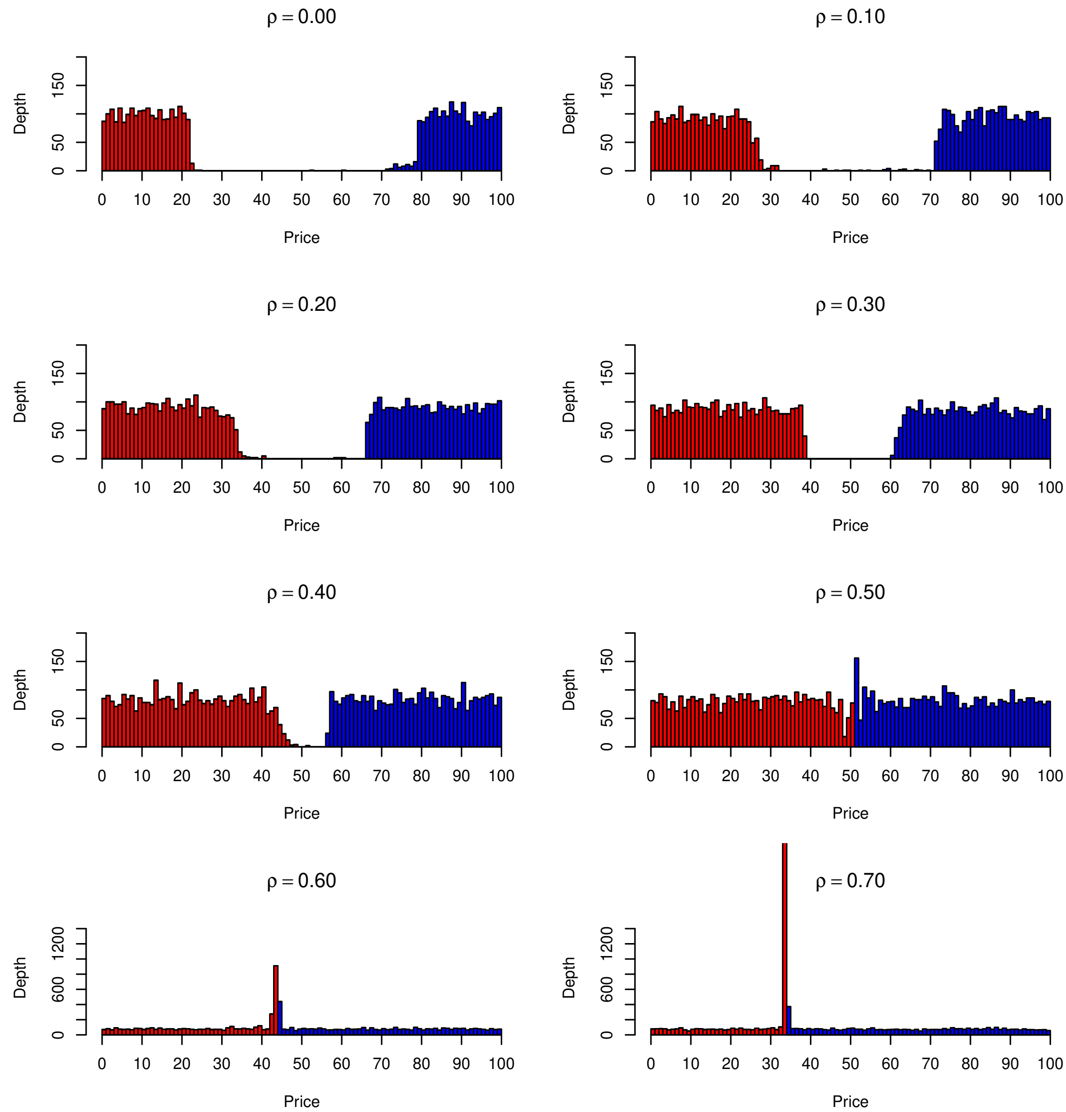


Figure 2.5: Shape of the order book in the Stigler-Luckock model with Market makers after 20,000 time steps for various values of ρ. (Note that the bottom two plots have different scale on the y-axis).

The competitive window closes as the ratio of the market makers ρ increases. It closes eventually in the middle for $\rho = 0.5$. The market is then liquid enough to create one equilibrium price around the middle. For higher values of ρ, the order book acts strangely. The buy and sell limit orders meet on some random price that is usually quite far from the middle, and the order book is stuck on that price forever (more examples are in Figure 2.6).

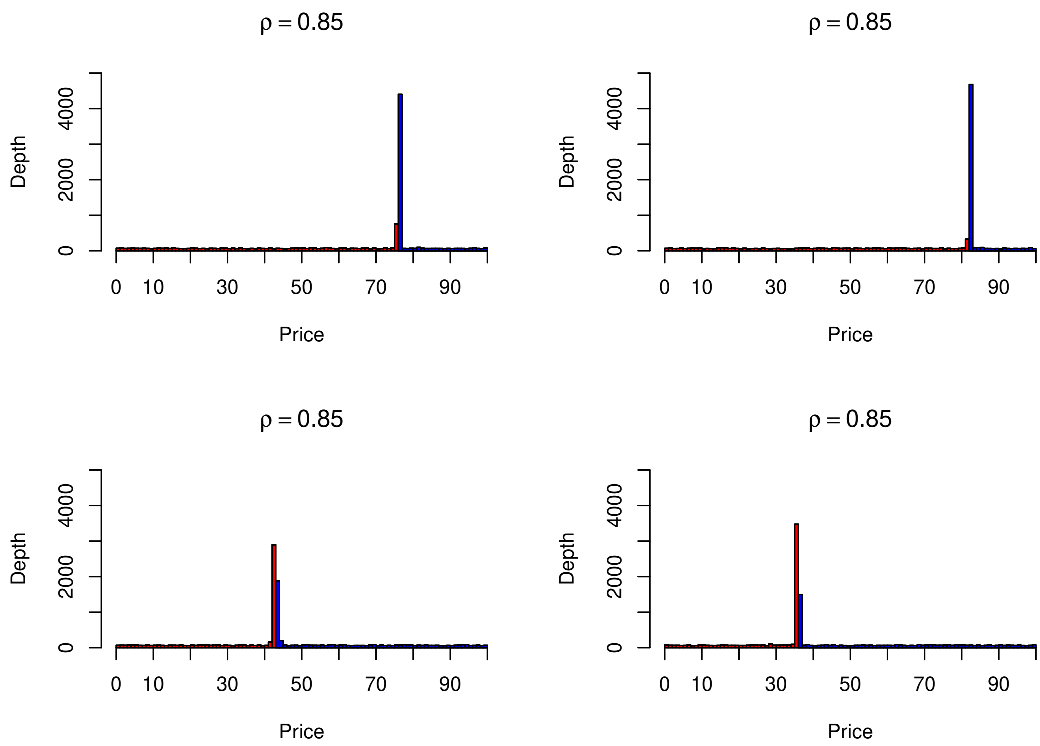


Figure 2.6: Shape of the order book in the Stigler-Luckock model with $\rho = 0.85$ market makers after 20,000 time steps (four equilibrium prices for the same input parameters of the model).

Hence, judging by the simulations, the ideal ratio of market makers on the market is $\rho = 0.5$, i.e. the probability of an arrival of a market maker is 20%. The competitive window closes and both buy and sell orders meet at the Walrasian equilibrium price x_W. Whereas a proper proof of this statement is complicated (can be found in Swart [2018]), insight into the significance of the value can be obtained quite easily.

Please note that the following few paragraphs are only descriptive without proofs, and their purpose is only to illustrate the behaviour of the market when $\rho = 0.5$. For simplicity and clarity, let us now put aside the randomness of the process. It is justifiable by long time and the law of large numbers. Then, the composition of market participants is: 20% of market makers, 40% of sellers, and 40% of buyers. The graphical illustration is in Figure 2.7.

The market makers however place two orders at once. The only exception is in the beginning when the order book is almost empty, and there are not enough limit orders yet. We will not consider this situation, because it is only temporary, and it settles after a few steps. So, the market makers place twice as many orders as the traders, which results in 1/3 of orders being placed by market makers, and 2/3 of orders being placed by traders (1/3 sell orders and 1/3 buy orders). Then, the compositions of orders is as illustrated in Figure 2.8.

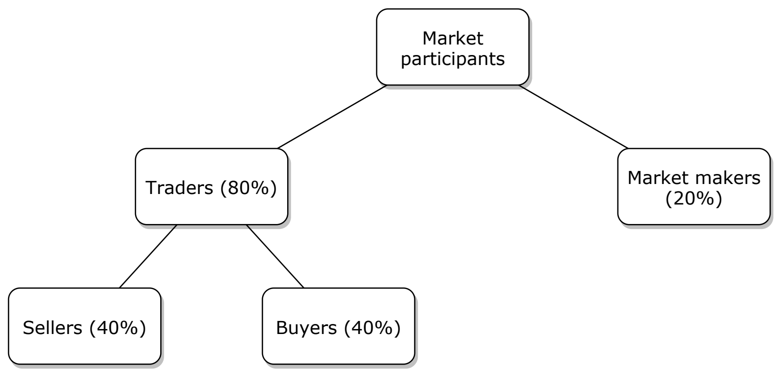


Figure 2.7: Composition of market participants when $\rho = 0.5$.

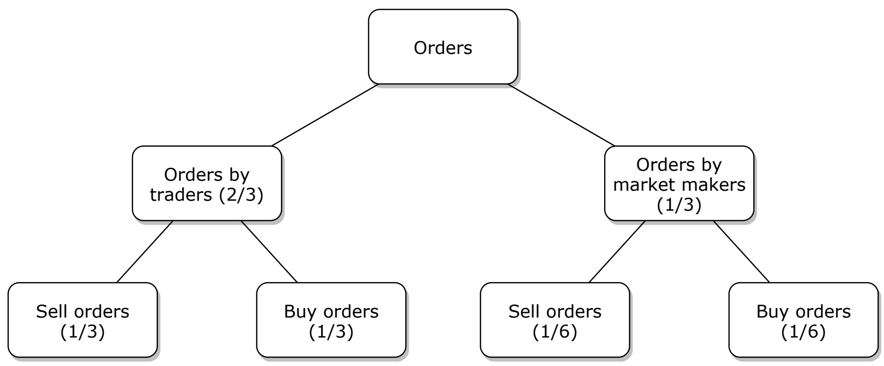


Figure 2.8: Composition of orders when $\rho = 0.5$.

Prices of orders placed by market makers depend on the current situation on the market, but they are always market orders. Prices of orders placed by traders are uniformly distributed, so half of them are placed above the equilibrium price x_W and half of them above. The division of orders is in Figure 2.9. Therefore, 1/6 of all orders are sell orders by traders below equilibrium, and 1/6 of all orders are buy orders by traders above the equilibrium. Besides that, 1/6 of all orders are sell market orders by market makers, and 1/6 of all orders are buy market orders by market makers. The proportion of sell (resp. buy) market orders placed by market makers is equal to the proportion of buy (resp. sell) orders above (resp. below) the Walrasian equilibrium price. Hence, those two groups eventually match, result in trades, and disappear from the order book. The matching groups are highlighted in Figure 2.9 by same colour. Sell limit orders above the equilibrium and buy limit orders below the equilibrium are then the only orders remaining in the order book, which closes the competitive window right in the middle.

Indeed, there are some inaccuracies. For example, the orders submitted by traders may be market orders as well, and they can pair with some limit orders of other traders. This was just to demonstrate the main mechanism behind the significance of the value $\rho = 0.5$. As already mentioned, the proper proof is quite complicated and can be found in Swart [2018].

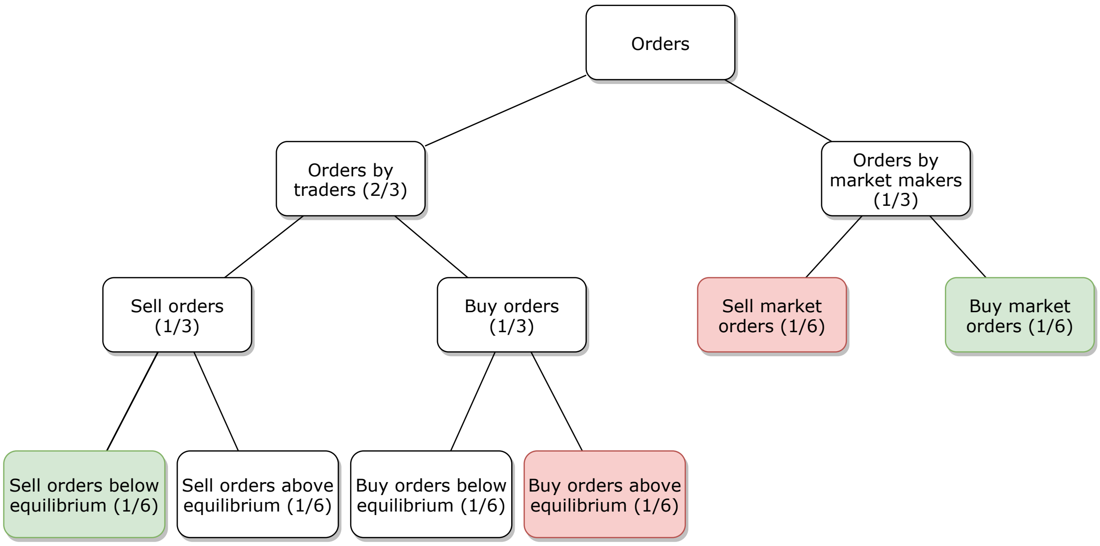


Figure 2.9: Composition of orders when $\rho = 0.5$ with highlighted groups that eventually match and disappear from the order book.

3. Competing Market Makers

The main goal of this thesis is to understand more the structure and mechanics of financial markets using the Stigler-Luckock model as a starting point and inspiration of the analysis. As seen in the previous chapter, the presence of market makers significantly improved the model since they provide highly necessary liquidity. That encourages us to study their behaviour and strategies in more detail and simultaneously to improve the Stigler-Luckock model in order to get a better understanding of the market.

A market maker's primary source of income is buying for less and selling for more. As already mentioned in the previous chapter, some stock exchanges purposedly pay a so-called liquidity provision in order to support the market makers in trading. There is an example of transaction prices on the New York Stock Exchange in Table 3.1. Besides that, market makers also receive a monthly credit when conditions are met.

Category	Adding Liquidity	Removing Liquidity
Securities $\geq$ \$1	\$0.0045 per share	–\$0.0002 per share
Securities $<$ \$1	0.25 % of total dollar value of the transaction	–0.25 % of total dollar value of the transaction

Table 3.1: Transaction fees (negative) and credits (positive) for Electronic Designated Market Makers at New York Stock Exchange [NYSE]

3.1 Definition of the model

First of all, the model and its properties must be defined. Some of the principles are the same as in the Stigler-Luckock model, whereas some of them are new. The goal was simple—find a suitable model with satisfactory good strategies while keeping it as simple as possible.

The market itself, prices, and some other features are derived from the previously described model with market makers. On top of that, there are several approaches and concepts that will be held throughout the following models. Their original inspiration is in the behaviour of real markets, and the aim is to draw the models closer to reality.

The main purpose of this thesis is to compare different strategies of market makers, and analyse the results. In order to do so, there are always at least two different strategies for market makers in each simulation. Each strategy is represented by single market maker, which is a notable difference from the previously cited model. Each market maker has his/her own inventory of cash and asset. The inventories will be compared after each simulation in order to determine which strategy is better, i.e. which strategy increased the inventory more.

The arrival of participants on the market is a Poisson process $\{N_t, t \geq 0\}$ with some overall intensity $R > 0$, which define the arrival rate of all any market participant, i.e. buyers, sellers, and market makers. Let $\mathcal{P}$ be a discrete set of all possible prices, $x_{\max} := \max\{\mathcal{P}\}$, and $x_{\min} := \min\{\mathcal{P}\}$. Let μ_+ and μ_- be finite measures on $\mathcal{P}$, and let $\lambda_\pm : \mathcal{P} \to [0, \infty)$ be supply and demand functions defined as

$$\lambda_+(x) := \mu_+\Big(\{y \in \mathcal{P}, y \leq x\}\Big),$$
$$\lambda_-(x) := \mu_-\Big(\{y \in \mathcal{P}, y \geq x\}\Big).$$

Then, $\mu_+\Big(\{x\}\Big)$ is the Poisson intensity of the arrival of a new sell order by regular traders at price $x \in \mathcal{P}$, and $\mu_-\Big(\{x\}\Big)$ is the Poisson intensity of the arrival of a new buy orders by regular traders at a price $x \in \mathcal{P}$. Therefore, $\mu_+(\mathcal{P}) = \lambda_+(x_{\max})$ and $\mu_-(\mathcal{P}) = \lambda_-(x_{\min})$ are Poisson intensities of sell and buy orders, respectively.

In addition, the Poisson rate of the arrival of a market maker is ρ. Since there are M market makers with different strategies in the model, we will denote the Poisson rate of the arrival of a market maker with i-th strategy as ρ_i,$i \in \{1, \ldots, M\}$. Then,

$$\rho = \sum_{i=1}^{M} \rho_i.$$

The overall intensity R of the Poisson process $\{N_t, t \geq 0\}$ is then:

$$R = \lambda_+(x_{\max}) + \lambda_-(x_{\min}) + \rho = \lambda_+(x_{\max}) + \lambda_-(x_{\min}) + \sum_{i=1}^{M} \rho_i.$$

Let us denote the probability of the arrival of a seller by $p_{\mathbf{S}}$, of a buyer by $p_{\mathbf{B}}$, of a market maker by $p_{\mathbf{MM}}$, and of the i-th market maker by $p_{\mathbf{MM}_i}$, $i \in \{1, \ldots, M\}$. Then,

$$\begin{aligned}
p_{\mathbf{S}} &= \frac{\lambda_+(x_{\max})}{R}, \\
p_{\mathbf{B}} &= \frac{\lambda_-(x_{\min})}{R}, \\
p_{\mathbf{MM}} &= \frac{\rho}{R} = \sum_{i=1}^{M} p_{\mathbf{MM}_i}, \\
p_{\mathbf{MM}_i} &= \frac{\rho_i}{R}.
\end{aligned}$$

Let us denote the time of the arrival of the k-th order by $\tau_k \in \mathbb{R}^+$. The k-th order is specified by its sign $\sigma_k \in \{\mathbf{B}, \mathbf{S}\}$ (buy or sell), its price $X_k \in \mathcal{P}$, and its volume $V_k \in \mathbb{N}_0$[1]. On top of that, it is important to distinguish the orders by the market participant that placed them, i.e. let $\omega_k \in \{\mathbf{T}, \mathbf{MM}\}$, where $\mathbf{T}$ stands for Trader (buyer or seller) and $\mathbf{MM}$ stands for Market Maker. Each order is completely determined by the quintuple $(\omega_k, \sigma_k, X_k, V_k, \tau_k)$, $k \in \mathbb{N}$. The random variables $(\tau_k - \tau_{k-1})_{k \geq 1}$ are exponentially distributed with parameter $1/R$.

[1] The situation when $V_k = 0$ will be explained further in the thesis

Let us denote the embedded Markov chain by $\{\widehat{N}_n, n \in \mathbb{N}_0\}$, which is a discrete-time Markov chain derived as

$$\widehat{N}_0 := 0,$$
$$\widehat{N}_k := N_{\tau_k}, \;\; k \in \mathbb{N},$$

where $\{N_t, t \geq 0\}$ is the Poisson process marking the arrivals of the market participants on the market, and τ_k marks the time of the arrival of k-th order.

Traders

Let us now take a closer look at orders placed by buyers and sellers. Their orders have simple attributes—their price X is uniformly distributed on $\mathcal{P}$, and their volume is always 1.

Markets are usually dominated by market makers who place their offers. Regular traders arrive on the market, and they either like some offer and take it, or they do not find any suitable offer, so they wait in the background, because they do not want to show their intentions to their competitors. Following that principle, we will consider the presence of traders as sort of "hidden". If a regular trader arrives, he/she either likes the current price (i.e. realizes a market order for the current bid or ask), or he/she prefers a different price, and he/she walks away with no record of the order. The possible scenarios are summarized in Table 3.2.

	BUY ORDER	
$x \geq a(t)$	order executed for the price $a(t)$, opposite order removed from LOB	$\rightarrow$ MARKET ORDER
$x < a(t)$	trader walks away, order disappears	$\rightarrow$ NO ORDER
	SELL ORDER	
$x \leq b(t)$	order executed for the price $b(t)$, opposite order removed from LOB	$\rightarrow$ MARKET ORDER
$x > b(t)$	trader walks away, order disappears	$\rightarrow$ NO ORDER

Table 3.2: Possible scenarios after trader's submission of an order with price x at time t. The current ask price is $a(t)$, and the current bid price is $b(t)$.

Market makers

There are M market makers with arrival rates ρ_i, $i \in \{1, \ldots, M\}$. Market makers do not care about the actual price of the asset. Their source of income is the spread s, i.e. difference between buy and sell prices.

Each market maker is characterized with two parameters—*spread* s_i, and *true price* $TP_i(t)$, $t \geq 0$, $i \in \{1, \ldots, M\}$. The true price $TP_i(t)$, $t \geq 0$, represents the market maker's own idea of how much the asset is actually worth, which may change over time. The value at the beginning $TP_i(0)$ is however given to the market maker, which can be compared to the situation after an opening auction in stock exchanges (see subsection 1.2.1) when the opening price is settled.

When market maker i arrives at time t, he/she places randomly a buy or sell order at price

$$X_k = \begin{cases} TP_i(\tau_k) - s_i/2, & \sigma_k = \mathbf{B}, \\ TP_i(\tau_k) + s_i/2, & \sigma_k = \mathbf{S}, \end{cases}$$

Further, we will be focusing on specific strategies of market makers, and subsequently deciding which strategy is better. In order to compare the strategies, we need to keep track of each market maker's wealth. Therefore, market makers are assigned with a so-called *inventory*

$$I_i(t) = \big(A_i(t), C_i(t)\big),$$

where $t \geq 0$, $i \in \{1, \ldots, M\}$. The inventory is a pair of random processes representing the amount of asset $A_i(t)$ and cash $C_i(t)$ that is possessed by the market maker i at time t, and that is available to be invested. This approach was already mentioned by Garman [1976]. Each market maker is assigned with some starting value of the inventory $I_i(0)$ in the beginning.

Besides that, each market maker has also a so-called *market inventory*

$$I_i^{\mathrm{M}}(t) = \big(A_i^{\mathrm{M}}(t), C_i^{\mathrm{M}}(t)\big),$$

where $t \geq 0$, $i \in \{1, \ldots, M\}$. When the market makers places a limit order, the money (or asset) transfers from the inventory to the market inventory. This cash (asset) are still in the possession of the market maker; however, they cannot be invested unless the limit orders are cancelled. So, the market inventory is a pair of random processes representing the amount of asset $A_i^{\mathrm{M}}(t)$ and cash $C_i^{\mathrm{M}}(t)$ that is offered as limit orders on the market by the market maker i at time t. The starting value for all market makers is $I_i^{\mathrm{M}}(0) = (0, 0)$. When a market maker places a limit order, the specific amount of cash or asset is transformed from $I_i(t)$ to $I_i^M(t)$.

Both processes $I_i(t)$ and $I_i^{\mathrm{M}}(t)$ are changing only at Poisson times τ_k when someone arrives on the market.

While traders are only allowed to have unit-sized orders, market makers have various strategies in which the volume V of orders differs. However, if their inventory is insufficient, the order is not executed, which is represented by $V = 0$.

Their orders are distinguished to be limit or market by their price, and the price of current bid or ask (see Table 2.1 and Table 2.2).

Since the true price of each market maker is changing, the market maker needs to controls his/her own invested money and cash in order to avoid collusion. What is meant by that is that when a new order by a market maker i arrives, the market maker needs to check his/her limit orders currently placed on the market, and eventually cancel some of them prior placing the new order.

All the possible situations are in Figure 3.1. In case of a new buy order with some new price, it is reasonable to cancel all other his/her limit buy orders that are currently on the market because they would compete with the new order. The new order can either have a worse price (lower), and nobody would prefer it. Or the new order can have a better price (higher), and nobody would want the older limit orders. Besides that, if the new order has a higher price than the his/her currently placed limit sell orders, the sell orders must be cancelled too. Vice versa for a new sell order.

When a order is cancelled at time t, it moves from the market inventory $I_i(t)$ back to the inventory $I_i^M(t)$.

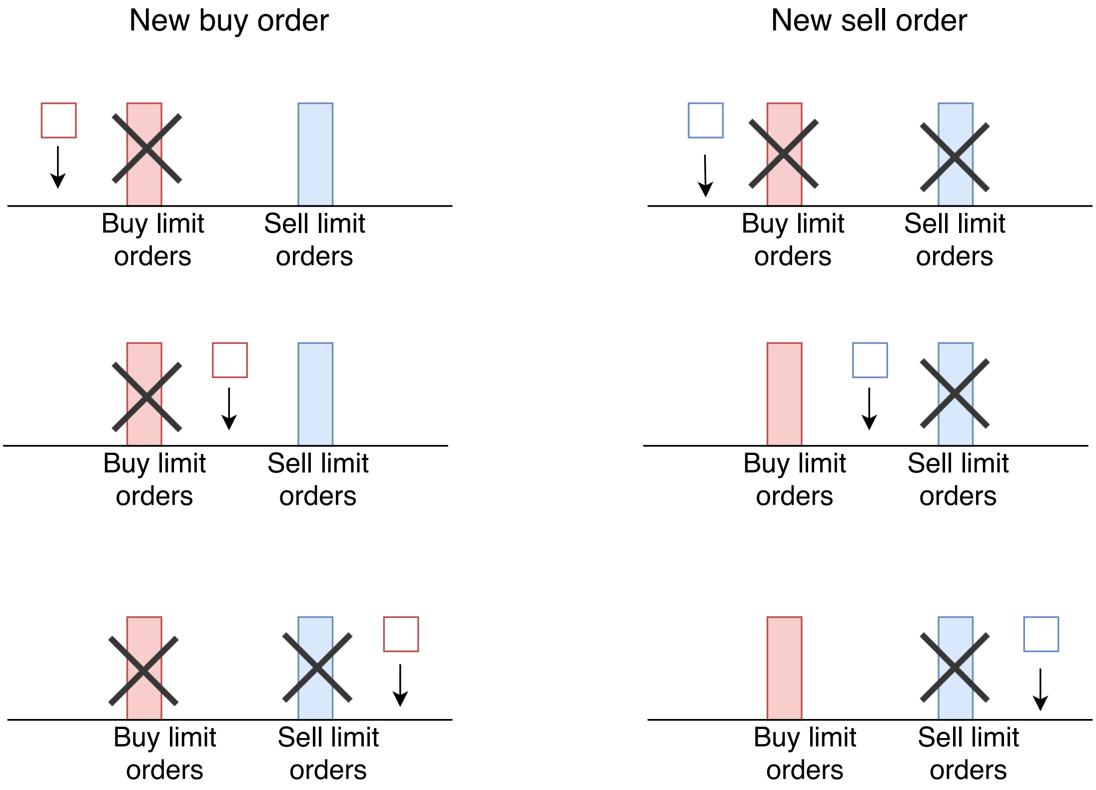


Figure 3.1: Scheme of possible situations before placing a new buy or sell order in which market maker needs to cancel some of his/her own limit orders placed in the past.

Price and time priority

Two concepts will be used in the following models (and were used in models before—for example, Parlour and Seppi [2008] or Roşu [2009]). Markets typically impose two simple rules—price and time priority on the execution of limit orders. Price priority determines that limit orders offering a better price (higher in case of buy orders, lower in case of sell orders) get executed before limit orders with worse prices. This rule is quite straightforward—nobody wants to buy or sell for a worse price if there is a better option. However, more offers may be for the same price. In that case, the time priority comes into action. The queuing rule of “first in, first out” imposes that the oldest limit orders are executed first, and hence the market rewards those who provided the liquidity earlier. With an arrival of a new market order, the price priority is applied and then, if necessary, the time priority.

Comparison of strategies

The main focus of this thesis is to compare various strategies of individual market makers. The specific strategies will be explained further in this chapter. Let us now define the competition by the game theory introduced in section 1.4.

There is a finite set of M players (market makers) indexed by i. Each market maker has a different set of actions. Each player has a utility function f_i, which calculates the value of their portfolio as a money value of their portfolio:

$$f_i(t) = C_i(t) + C_i^{\mathrm{M}}(t) + x_W\Big(A_i(t) + A_i^{\mathrm{M}}(t)\Big),$$

where x_W is the Walrasian equilibrium price which is derived from the supply and demand functions. The price x_W may not be fixed, because it depends on the current supply and demand. Note that utility function are usually considered concave functions of the monetary value.

The game is non-cooperative because market makers cannot create coalitions. It is also an inconstant-sum game because of the arrival of ordinary traders, who also bring/take the asset on/from the market.

The behaviour of the market will be examined by *Monte Carlo* simulations, whose results will be presented and commented on the following pages. All simulations were written and run using R software [R Core Team, 2018].

After each simulation at time T, the utility functions of all market makers $f_i(T)$, $i \in \{1, \dots, M\}$ will be compared in order to find the market maker with highest pay off utility function.

For each combination of strategies, the simulations were run 100 times starting always with an empty order book.

It is however very unlikely to find the optimal strategy, and it does not need to exist at all, as explained in section 1.4. Our main aim is rather to compare the strategies in order to see and explain some behaviour of market makers on real markets.

Discrete time

Further in the thesis, we will also focus on the embedded Markov chains and study the market in discrete time. The embedded Markov chain $(\widehat{N}_n, n \in \mathbb{N}_0)$ describe the market in a way that at each time step $n \in \mathbb{N}$ a new market participant arrives. The embedded Markov chains will always be denoted by the hat sign $\widehat{\ }$. Then we denote the embedded Markov chains to the true price $TP_i(t)$, inventory $I_i(t) = \Big(A_i(t), C_i(t)\Big)$, and market inventory $I_i^{\mathrm{M}}(t) = \Big(A_i^{\mathrm{M}}(t), C_i^{\mathrm{M}}(t)\Big)$ by

$$\begin{aligned}
\widehat{TP}_i(k) &= TP_i(\tau_k), & & \\
\widehat{A}_i(k) &= A_i(\tau_k), & \widehat{A}_i^{\mathrm{M}}(k) &= A_i^{\mathrm{M}}(\tau_k), \\
\widehat{C}_i(k) &= C_i(\tau_k), & \widehat{C}_i^{\mathrm{M}}(k) &= C_i^{\mathrm{M}}(\tau_k), \\
\widehat{I}_i(k) &= \Big(\widehat{A}_i(k), \widehat{C}_i(k)\Big), & \widehat{I}_i^{\mathrm{M}}(k) &= \Big(\widehat{A}_i^{\mathrm{M}}(k), \widehat{C}_i^{\mathrm{M}}(k)\Big),
\end{aligned}$$

where $k \in \mathbb{N}_0$ and τ_k is the time of the arrival of k order ($\tau_0 := 0$) in the original process $(N_t, t \geq 0)$.

3.2 Market with stationary supply and demand

First, let us continue with the same supply and demand functions already introduced in models cited in chapter 2. Let $\mathcal{P} = \{1, \dots, 100\}$, and measures $\mu_+ = \mu_-$ are uniform distributions on $\mathcal{P}$, i.e.

$$\mu_+\Big(\{x\}\Big) = \mu_-\Big(\{x\}\Big) = \frac{1}{100},$$

for all $x \in \mathcal{P}$. Then supply and demand functions are

$$\begin{aligned}\lambda_+(x) &= \mu_+\Big(\{y \in \mathcal{P}, y \leq x\}\Big) = \frac{x}{100},\\ \lambda_-(x) &= \mu_-\Big(\{y \in \mathcal{P}, y \geq x\}\Big) = \frac{101 - x}{100},\end{aligned}$$

for $x \in \{1, \dots, 100\}$.

The overall intensity of the Poisson process is then

$$R = \lambda_+(x_{\max}) + \lambda_-(x_{\min}) + \rho = \lambda_+(100) + \lambda_-(1) + \rho = 1 + 1 = 2 + \rho = 2 + \sum_{i=1}^{M} \rho_i,$$

where ρ is the Poisson rate of market makers, and ρ_i, $i = 1, \dots, M$, are Poisson rates of individual market makers. Let the arrivals of market makers be distributed equally, and let $M = 2$, i.e.

$$\rho_i = \frac{\rho}{M} = \frac{\rho}{2}.$$

The probabilities of arrivals of sellers, buyers and market makers are

$$\begin{aligned}p_{\mathbf{S}} &= \frac{\lambda_+(x_{\max})}{R} = \frac{1}{2+\rho},\\ p_{\mathbf{B}} &= \frac{\lambda_-(x_{\min})}{R} = \frac{1}{2+\rho},\\ p_{\mathbf{MM}} &= \frac{\rho}{R} = \frac{\rho}{2+\rho},\\ p_{\mathbf{MM}_i} &= \frac{\rho_i}{R} = \frac{\rho}{2(2+\rho)}.\end{aligned}$$

The orders by traders are then specified by quintuple

$$(\omega, \sigma, X, V, \tau) = (\mathbf{T}, \sigma, X^{\mathbf{T}}, 1, \tau) =: \Omega^{\mathbf{T}},$$

where $\sigma \in \{\mathbf{B}, \mathbf{S}\}$ (with equal probabilities), $\tau \in \mathbb{R}^+$, and distribution of price $X^{\mathbf{T}}$ is then uniform on the price grid, i.e.

$$X^{\mathbf{T}} \sim \text{Unif}\,\{1, \dots, 100\}.$$

The orders by market makers are then specified by

$$(\omega, \sigma, X, V, \tau) = (\mathbf{MM}, \sigma, X^{\mathbf{MM}}, V^{\mathbf{MM}}, \tau) =: \Omega^{\mathbf{MM}},$$

where $\sigma \in \{\mathbf{B}, \mathbf{S}\}$ (with equal probabilities), $\tau \in \mathbb{R}^+$, and $X^{\mathbf{MM}}$ and $V^{\mathbf{MM}}$ are individually specified.

The Walrasian equilibrium price does not exist because the supply and demand are discrete. However, if we consider them as continuous, we can calculate a pseudo equilibrium price, which we denote by $\bar{x}$

$$\begin{aligned}\frac{\bar{x}}{100} &= \frac{101-\bar{x}}{100},\\ 2\bar{x} &= 101,\\ \bar{x} &= 50.5.\end{aligned}$$

This price will replace the non-existing Walrasian equilibrium price when calculating the utility pay off functions.

3.2.1 Discrete time

Let us now analyse the model in discrete time. This approach is reasonable because the time perspective is not important for our study. Furthermore, the results would be very similar according to the law of large numbers. Let us now focus on the embedded Markov chain $(\widehat{N}_n, n \in \mathbb{N}_0)$.

Let the true price given to the market makers be in the middle

$$\widehat{TP}_1(0) = \widehat{TP}_2(0) = 50.5,$$

and let s_i be a spread of the i-th market maker. Then the price of an order by market maker i at time step k is

$$X_{i,k}^{\mathbf{MM}} = \begin{cases} \widehat{TP}_i(k) - \frac{s_i}{2}, & \sigma_k = \mathbf{B},\\ \widehat{TP}_i(k) + \frac{s_i}{2}, & \sigma_k = \mathbf{S}. \end{cases}$$

The starting inventories of both market makers are the same. They both possess 100 units of the asset and corresponding amount of cash, and they have nothing invested, i.e.

$$\begin{aligned}\widehat{I}_i(0) &= \left(\widehat{C}_i(0), \widehat{A}_i(0)\right) = (100, 50.5\cdot 100) = (100, 5050),\\ \widehat{I}_i^{\mathrm{M}}(0) &= \left(\widehat{C}_i^{\mathrm{M}}(0), \widehat{A}_i^{\mathrm{M}}(0)\right) = (0,0).\end{aligned}$$

Model no. 1

Let us now start with first basic game. Both market makers have the same constant strategy (labelled as `const` in plots) , i.e. $\widehat{TP}_i$, $i = 1,2$, is constant in time. First, we only want to see the difference between wider and narrower spreads. First market maker was given spread $s_1 = 9$, the second $s_2 = 21$. The values were chosen to easily demonstrate the difference between the two strategies, and consequently finding the dominating one (if possible).

Volume of orders of market makers is simply one (labelled as `one` in plots), if the inventory is sufficient. Hence the volume of i-th market maker at time step k is following

$$V_{i,k}^{\mathbf{MM}} = \begin{cases} 1, & \sigma_k = \mathbf{B} \text{ and } \widehat{C}_i(k) \geq X_{i,k}^{\mathbf{MM}}, \\ 1, & \sigma_k = \mathbf{S} \text{ and } \widehat{A}_i(k) \geq 1, \\ 0, & \textit{otherwise}. \end{cases}$$

Results comparing the strategies for various probabilities of an arrival of the market makers $p_{\mathbf{MM}}$ are in Figure 3.2.

Let us now describe the plots. The plot on the left-hand side represents the content of the order book after N time steps, i.e. N arrivals of either traders or market makers. Again, red colour is for buy limit orders and blue colour for sell limit orders. The legend in the top right corner describes the strategy of each market maker

- `spr` stands for the width of the spread,
- `str` stands for strategy, `const` means that they place orders constantly in the same place,
- `am` represents the V variable (`one` means unit quantity every time).

Since the buy/sell prices of both market makers are constant in time in this example, the triangles on the x-axis highlight the prices.

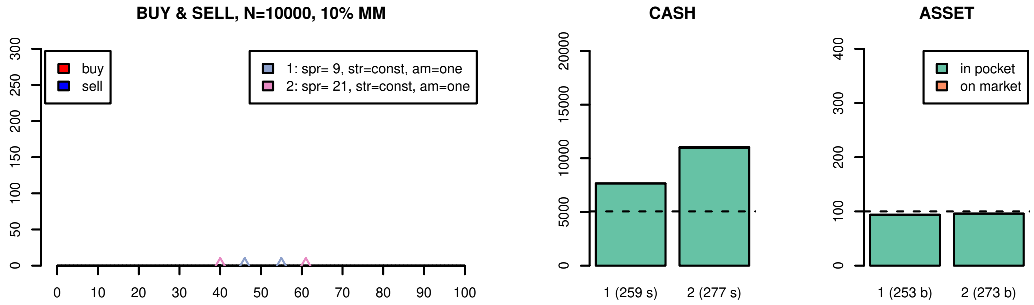
The plots on the right-hand side show the inventories after N time steps. The starting values are marked with dashed lines. The columns can have two colour as explained in the legend. While "on market" represent the proportion of cash/asset offered currently on the market in limit orders ($\widehat{I}_i^{\mathrm{M}}(k)$), "in pocket" represents the proportion of a portfolio that can still be invested, i.e. $\widehat{I}_i(k)$. The bar on the left belongs to the first market maker, and the bar on the right belongs to the second market maker. The numbers in the brackets below the bars represent the total number of realized trades—"b" buys or "s" sells.

In section 2.4, we have seen that the most suitable ratio of market makers was 20%. The previous model was however quite different, because the market makers placed their orders with the smallest spread possible. Therefore, we cannot make the same conclusion here.

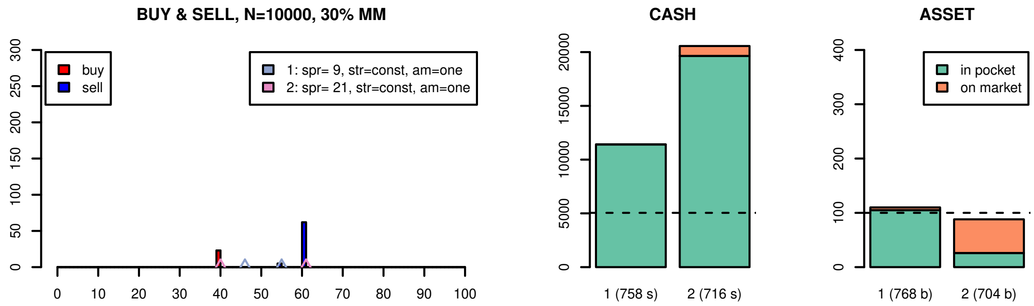
Let us discuss the shape of the order book for a few choices of $p_{\mathbf{MM}}$ (see Figure 3.2). When the probability of an arrival of a market maker ρ is just 10%, the order book is empty. The ratio of market makers is just too small, so they do not place enough limit orders to satisfy all the arriving traders. The second market maker with the wider spread earned more cash than the first one due to the wider spread. The number of trades for both market makers is very similar.

The order book is not empty any more with 30 % of market makers, but there are only limit orders from the second market maker. Despite the fact that the first market maker has slightly more realised trades, the second market maker earned more cash, still due to the wider spread.

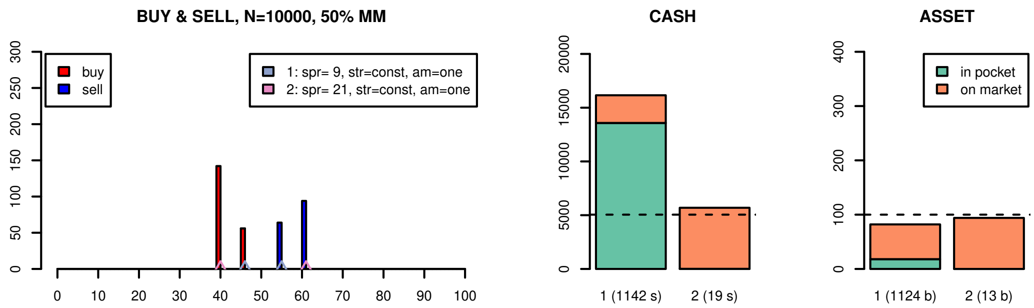
However, the situation turns with 50 % of market makers, when the market is over-saturated. Both market makers have their limit orders in the order book. The market maker with smaller spread always offers better price; therefore, the second market maker has a very small number of trades.



(a) $p_{\mathbf{MM}} = 10\%$



(b) $p_{\mathbf{MM}} = 30\%$



(c) $p_{\mathbf{MM}} = 50\%$

Figure 3.2: State of the order book after $N = 10,000$ time steps with various probabilities of market makers $p_{\mathbf{MM}}$.

All those simulation were run 100 times. In case of 10% and 30% of market makers, the market maker with wider spread had higher utility function

$$f_i(\tau_N) = \widehat{C}_i(N) + \widehat{C}_i^{\mathrm{M}}(N) + \bar{x}\Big(\widehat{A}_i(N) + \widehat{A}_i^{\mathrm{M}}(N)\Big)$$

for all 100 attempts. Therefore, the strategy with wider spread dominated the strategy with narrower spread. However, the market maker with the narrower spread had higher utility function f_i for all 100 simulations for 50 %. So, we have learned that better offering prices, i.e. smaller spread, can get us ahead of other market makers, if the market is saturated. Still, there is always a constraint on the smallest tick size on real markets, so the spread cannot be made smaller infinitely.

Interestingly in all simulations in Figure 3.2, the asset inventory did not change much from its starting value, while on the other hand the cash inventory increased in all cases. This disproportion results from the fact that the probabilities of buying and selling the asset are equal. Hence, the amount of asset fluctuates around 100. On the contrary, the amount of cash is increasing due to the effect of buying for less and selling for more.

Model no. 2

Now, let us focus on the volume variable $V^{\mathbf{MM}}$. Is it possible to beat a competitor with a smaller spread just by increasing the size of orders? We tried a strategy that always keeps specified percentage r of the whole portfolio (not invested and invested together) on the market (in plots represented by the abbreviation `keep_ratio_`r). The volume of the order is chosen such that afterwards, the ratio of invested cash/asset is r. Therefore,

$$V_{i,k}^{\mathbf{MM}} = \begin{cases} \left\lfloor \frac{r \cdot \left(\widehat{C}_i(k) + \widehat{C}_i^{\mathrm{M}}(k)\right) - \widehat{C}_i^{\mathrm{M}}(k)}{X_k^{\mathbf{MM}}} \right\rfloor, & \sigma_k = \mathbf{B}, \\ \left\lfloor r \cdot \left(\widehat{A}_i(k) + \widehat{A}_i^{\mathrm{M}}(k)\right) - \widehat{A}_i^{\mathrm{M}}(k) \right\rfloor, & \sigma_k = \mathbf{S}. \end{cases}$$

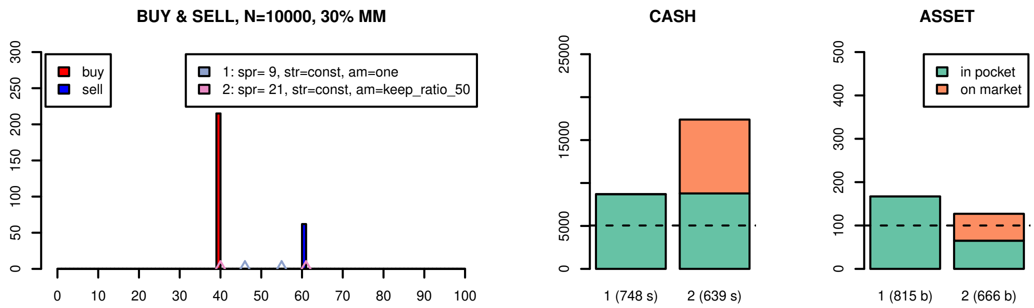
If $V_{i,k}^{\mathbf{MM}} = 0$, no order is placed. Note that the differences can never be negative. When traders take limit orders, the ratio of invested to all cash/asset can only decrease.

In order to compare the strategies, the first market maker has the constant strategy with volume of order just one (`str=const` and `am=one`), and the second market maker has the same strategy (`str=const` and the newly introduced `am=keep_ratio_`r strategy for volume. We simulated the market for $r = 0.5$ and $r = 1$ to demonstrate the strategy. It did not improve the situation in case the of 50 %, because there is simply a large surplus of orders by the market maker with better price. However, it resulted in significantly larger profits in the model with a 30% ratio of market makers, as illustrated in Figure 3.3.

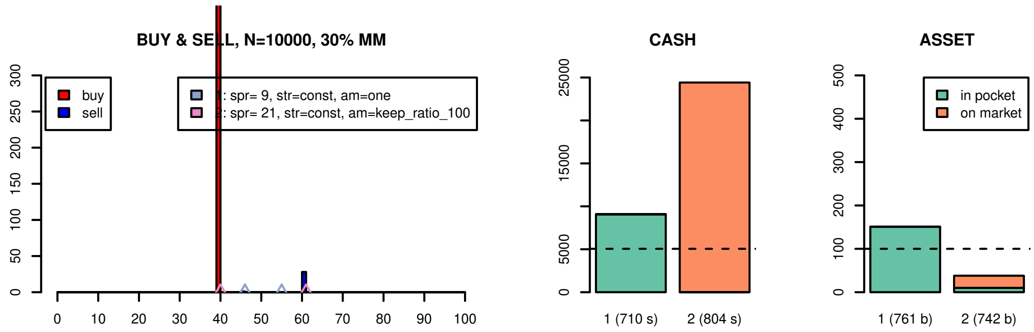
3.3 Market with dynamic supply and demand

The previous section gave us some hints in which direction it may be interesting to develop our study. However, static supply and demand are quite limiting. Prices always change, and a successful market maker's strategy must incorporate techniques that can react to the change reasonably and quickly. Thus, we will change the behaviour of the traders in order to change supply, demand, and consequently equilibrium price.

Let us now suppose two states called *low* and *high*, which are represented by a two state Markov process $(S_t, t \geq 0)$ that jumps between states with rate ζ. In other words, times between changes have exponential distribution with parameter ζ. Demand is decreased in the low state, which results in a lowered price. On the contrary, supply is decreased in the high state to increase the price. The supply and demand are then so-called hidden Markov models. A hidden Markov model is a stochastic process whose parameters are governed by a Markov process which is however not part of the stochastic process we are mainly interested in



(a) $p_{\mathbf{MM}} = 30\%$, $r = 0.5$



(b) $p_{\mathbf{MM}} = 30\%$, $r = 1$

Figure 3.3: State of the order book after $N = 10,000$ time steps. The second market maker has additional parameter r that represents the proportion of asset and cash constantly kept on the market.

Let $\mathcal{P} = \{1, \dots, 100\}$ be the set of possible prices. The changes are achieved by two choices for measures:

$$\mu_+^{\text{low}}(\{x\}) = \frac{1}{100}, \qquad \mu_+^{\text{high}}(\{x\}) = \frac{1}{200},$$
$$\mu_-^{\text{low}}(\{x\}) = \frac{1}{200}, \qquad \mu_+^{\text{high}}(\{x\}) = \frac{1}{100},$$

where $x \in \mathcal{P}$. The supply and demand functions in the low state are

$$\lambda_+^{\text{low}}(x) = \mu_+^{\text{low}}\Big(\{y \in \mathcal{P}, y \leq x\}\Big) = \frac{x}{100},$$
$$\lambda_-^{\text{low}}(x) = \mu_-^{\text{low}}\Big(\{y \in \mathcal{P}, y \geq x\}\Big) = \frac{101 - x}{200},$$

and in the high state are

$$\lambda_+^{\text{high}}(x) = \mu_+^{\text{high}}\Big(\{y \in \mathcal{P}, y \leq x\}\Big) = \frac{x}{200},$$
$$\lambda_-^{\text{high}}(x) = \mu_-^{\text{high}}\Big(\{y \in \mathcal{P}, y \geq x\}\Big) = \frac{101 - x}{100},$$

for $x \in \mathcal{P}$. They are plotted in Figure 3.4.

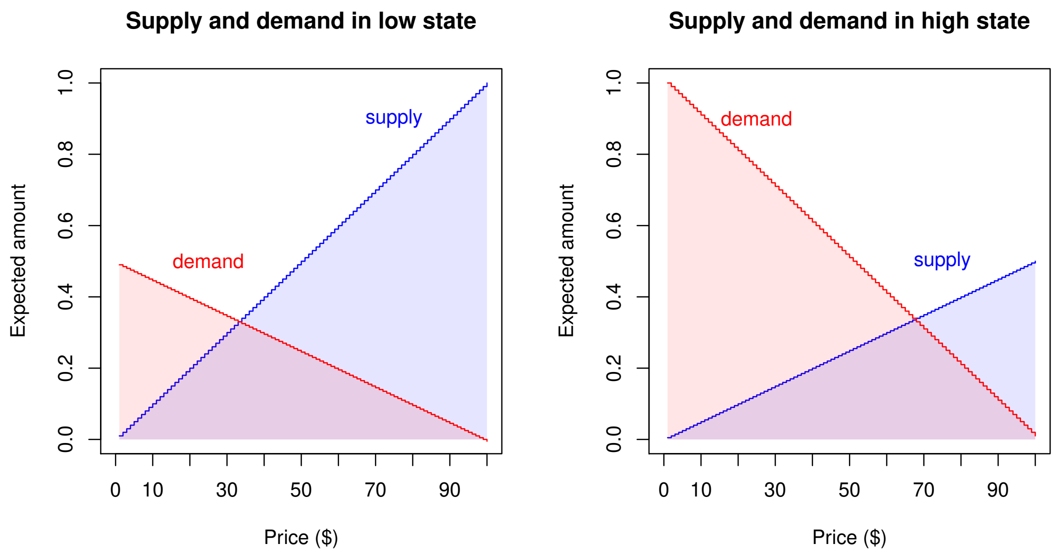


Figure 3.4: Shape of supply and demand in the low and high states

The overall Poisson intensity of the process is

$$\begin{aligned} R &= \lambda_+^{\text{high}}(x_{\max}) + \lambda_-^{\text{high}}(x_{\min}) + \rho \\ &= \lambda_+^{\text{low}}(x_{\max}) + \lambda_-^{\text{low}}(x_{\min}) + \rho \\ &= 1.5 + \rho \\ &= 1.5 + \sum_{i=1}^{M} \rho_i, \end{aligned}$$

where ρ is the Poisson rate of market makers, and ρ_i, $i = 1, \ldots, M$, are Poisson rates of individual market makers. Let $M = 2$, and let the market makers arrive with the same rate, i.e.

$$\rho_i = \frac{\rho}{M} = \frac{\rho}{2}.$$

The probabilities of arrivals of sellers and buyers in low and high states are

$$p_{\mathbf{S}}^{\text{low}} = \frac{\lambda_+^{\text{low}}(x_{\max})}{R} = \frac{1}{1.5+\rho}, \qquad p_{\mathbf{S}}^{\text{high}} = \frac{\lambda_+^{\text{high}}(x_{\max})}{R} = \frac{1}{2(1.5+\rho)}$$

$$p_{\mathbf{B}}^{\text{low}} = \frac{\lambda_-^{\text{low}}(x_{\min})}{R} = \frac{1}{2(1.5+\rho)}, \qquad p_{\mathbf{B}}^{\text{high}} = \frac{\lambda_-^{\text{high}}(x_{\min})}{R} = \frac{1}{1.5+\rho}$$

The probabilities of arrivals of market makers do not depend on the state

$$\begin{aligned} p_{\mathbf{MM}} &= \frac{\rho}{R} = \frac{\rho}{1.5+\rho}, \\ p_{\mathbf{MM}_i} &= \frac{\rho_i}{R} = \frac{\rho}{2(1.5+\rho)}. \end{aligned}$$

The orders by traders are then specified by quintuple

$$(\omega, \sigma, X, V, \tau) = (\mathbf{T}, \sigma^{\mathbf{T}}, X^{\mathbf{T}}, 1, \tau) =: \Omega^{\mathbf{T}},$$

where $\sigma^{\mathbf{T}} \in \{\mathbf{B}, \mathbf{S}\}$, $\tau \in \mathbb{R}^+$, and distribution of price $X^{\mathbf{T}}$ is then uniform on the price grid, i.e.

$$X^{\mathbf{T}} \sim \text{Unif}\,\{1, \ldots, 100\}.$$

The distribution of $\sigma^{\mathbf{T}}$ depends on the state, i.e.

$$\sigma^{\mathbf{T}} = \begin{cases} \mathbf{B}, & \text{with probability } \frac{1}{3} \text{ in low state and } \frac{2}{3} \text{ in high state,} \\ \\ \mathbf{S}, & \text{with probability } \frac{2}{3} \text{ in low state and } \frac{1}{3} \text{ in high state.} \end{cases}$$

The orders by market makers are then specified by

$$(\omega, \sigma, X, V, \tau) = (\mathbf{MM}, \sigma, X^{\mathbf{MM}}, V^{\mathbf{MM}}, \tau) =: \Omega^{\mathbf{MM}},$$

where $\sigma \in \{\mathbf{B}, \mathbf{S}\}$ (with equal probability), $\tau \in \mathbb{R}^+$, and $X^{\mathbf{MM}}$ and $V^{\mathbf{MM}}$ are specified individually.

Again, supply and demand do not intersect because of their discreteness, so there is no Walrasian equilibrium point. We will however need some price to value the pay off functions of market makers. For this purpose, we take the continuous linear interpolation of the supply and demand functions. Then, we can derive values $\bar{x}^{\text{low}}$ and $\bar{x}^{\text{high}}$:

$$\begin{aligned} \frac{\bar{x}^{\text{low}}}{100} &= \frac{101 - \bar{x}^{\text{low}}}{200} & \qquad \frac{\bar{x}^{\text{high}}}{200} &= \frac{101 - \bar{x}^{\text{high}}}{100} \\ 3\bar{x}^{\text{low}} &= 101 & 3\bar{x}^{\text{high}} &= 202 \\ \bar{x}^{\text{low}} &= 33.\bar{6} & \bar{x}^{\text{high}} &= 67.\bar{3} \end{aligned} \tag{3.1}$$

3.3.1 Discrete time

Let us now consider the embedded Markov chain $(\widehat{N}_n, n \in \mathbb{N})$. Let the true price given to the market makers be in the middle

$$\widehat{TP}_1(0) = \widehat{TP}_2(0) = 50.5,$$

and let s_i be a spread of the i-th market maker. Then the price of an order by market maker i at time step k is

$$X^{\mathbf{MM}}_{i,k}, = \begin{cases} \widehat{TP}_i(k) - \frac{s_i}{2}, & \sigma_k = \mathbf{B}, \\ \\ \widehat{TP}_i(k) + \frac{s_i}{2}, & \sigma_k = \mathbf{S}. \end{cases}$$

The starting inventories of both market makers are the same. They both possess 100 units of the asset and corresponding amount of cash, and they have nothing invested, i.e.

$$\begin{aligned} \widehat{I}_i(0) &= \left(\widehat{C}_i(0), \widehat{A}_i(0)\right) = (100, 50.5 \cdot 100) = (100, 5050), \\ \widehat{I}^{\mathrm{M}}_i(0) &= \left(\widehat{C}^{\mathrm{M}}_i(0), \widehat{A}^{\mathrm{M}}_i(0)\right) = (0, 0). \end{aligned}$$

Let us denote the discrete time Markov process that is marking the changes between low and high state by $(S_k^*, k \in \mathbb{N})$, i.e.

$$S_0^* = S_0,$$
$$S_k^* = S_{\tau_k}, k \in \mathbb{N},$$

where τ_k is the time of the arrival of k market participant in the original Poisson process $(N_t, t \geq 0)$. The process $(S_k^*, k \in \mathbb{N})$ has obviously two states {high,low} and some transition matrix

$$\mathbb{P} = \begin{pmatrix} 1-\zeta^* & \zeta^* \\ \zeta^* & 1-\zeta^* \end{pmatrix}.$$

Then the times between changes have a geometric distribution with parameter ζ^*. The expected value is $\frac{1}{\zeta^*}$, and the probability of the time between changes being k is

$$p_k = (\zeta^*)^{k-1}(1-\zeta^*).$$

Simulations were again run for $N = 10,000$ time steps. We want to choose the parameter ζ^* so that on one hand we can observe at least a few changes during $N = 10,000$ steps, and on the other hand we stay in a state for some longer time to actually see its effect. Therefore, the parameter ζ^* was chosen as 1/2000 in our simulations (i.e. 5 changes on average)

Model no. 3

The final model always compares two strategies represented by two market makers. They both have the same starting true price, inventory, market inventory, and spread $s_1 = s_2 = 9$.

Let us now introduce a complicated strategy that will be further analysed. The strategy has a few parameters, and our goal is to find the perfect combination to beat the others. The strategy is labelled `str=trend_final` in plots. The main idea is that the market maker will change his/her value of the true price by analysing his/her inventory and market. For example, if his/her true price is lower than the current equilibrium, he/she should observe more buys than sells and vice versa. Hence, he/she may want to increase his/her true price, in order to equalise the inventory. The equalisation is also an important and realistic aspect since the market maker's profit depends on providing liquidity, and they cannot risk a depletion of one of their resources.

There are four parameters: $(w_i, d_i, m_i, r_i) \in \mathbb{N} \times \mathbb{R}^+ \times \mathbb{N} \times (0,1)$, $i = 1, 2$. The true price of a market maker i is recalculated after every w_i turns (w for wait time) of the market maker. This parameter was incorporated in order to discover whether it is wise to check the true price often or whether it is better to wait for a while and see its effect.

The difference coefficient $d_i \in \mathbb{N}$ represents the market maker's "conservativeness". After each w_i steps, the market makers calculates the disproportion in his inventory. Let us denote the total cash and total asset of a market maker i at time k by

$$\widehat{C}_i^T(k) := \widehat{C}_i(k) + \widehat{C}_i^{\mathrm{M}}(k)$$
$$\widehat{A}_i^T(k) := \widehat{A}_i(k) + \widehat{A}_i^{\mathrm{M}}(k)$$

Let us denote the disproportion in the portfolio of a market maker i at time step k by $h_{k,i}$, which is the difference in the value of cash and asset from previous time step. Then

$$h_{k,i} = \widehat{C}_i^T(k-1) - \widehat{TP}_i(k-1)\widehat{A}_i^T(k-1).$$

The coefficient $d_i \in \mathbb{R}^+$ tells the market maker whether the disproportion $h_{k,i}$ is significant when compared to the total value of his portfolio, i.e. if

$$|h_{k,i}| \geq d_i\Big(\widehat{C}_i^T(k-1) + \widehat{TP}_i(k-1)\widehat{A}_i^T(k-1)\Big), \tag{3.2}$$

then the difference is significant.

If the market maker decides that the difference $h_{k,i}$ is significant, he/she shifts the true price by m_i steps to the left or to the right (depending on the disproportion) if possible. If Equation 3.2 holds then

$$\widehat{TP}_i(k) =$$
$$= \begin{cases} \min\Big\{\widehat{TP}_i(k-1) + m_i, 100 - \frac{s_i}{2}\Big\}, & \widehat{C}_i^T(k-1) > \widehat{TP}_i(k-1)\widehat{A}_i^T(k-1), \\ \max\Big\{\widehat{TP}_i(k-1) - m_i, 1 + \frac{s_i}{2}\Big\}, & \widehat{C}_i^T(k-1) < \widehat{TP}_i(k-1)\widehat{A}_i^T(k-1). \end{cases}$$

The last parameter r_i has the same meaning as in the models with stationary supply and demand. It defines the volume of trade of market maker i at time k as

$$V_{i,k}^{\mathbf{MM}} := \begin{cases} \left\lfloor \frac{r_i \cdot \left(\widehat{C}_i(k) + \widehat{C}_i^{\mathrm{M}}(k)\right) - \widehat{C}_i^{\mathrm{M}}(k)}{X_k^{\mathbf{MM}}} \right\rfloor, & \sigma_k = \mathbf{B}, \\ \left\lfloor r_i \cdot \left(\widehat{A}_i(k) + \widehat{A}_i^{\mathrm{M}}(k)\right) - \widehat{A}_i^{\mathrm{M}}(k) \right\rfloor, & \sigma_k = \mathbf{S}. \end{cases}$$

. That is, r_i is the ratio of cash/asset that the market maker i tries to keep invested in the market.

Results of simulations

Each simulation was run for $N = 10,000$ time steps, i.e. 10.000 arrivals of market participants. After each simulation, the wealth of both market makers was compared using the utility pay off functions

$$f_i(\tau_N) = \widehat{C}_i(N) + \widehat{C}_i^{\mathrm{M}}(N) + \bar{x}\Big(\widehat{A}_i(N) + \widehat{A}_i^{\mathrm{M}}(N)\Big),$$

where $\bar{x} = \bar{x}^{\mathrm{low}}$ in case of market being in the low state at time step N, and $\bar{x} = \bar{x}^{\mathrm{high}}$ in case of market being in the high state at time step N (see Equation 3.1). The logic behind this approach is that we wanted to value the portfolio with the current "equilibrium" price. Then, the asset is valued by the price for which the market maker could hypothetically sell all the asset immediately. There are obviously other different approaches. Further in plots, there will also be a comparison based separately on cash and asset pay off. The market maker with a higher pay off was given a "winning" point. Each pair of market makers competed 100 times in order to determine the winner. The winner was then compared with another market maker competing with a different combination of parameters.

Simulations were performed for different probabilities of the arrival of a market maker ($p_{\mathbf{MM}}$ =10%, 30%, 50%, and 70%). The percentage governs how often the market makers place orders. Therefore, it may be viewed as part of the strategy of the market makers, rather than a property of the market. The values of the parameters were chosen from those sets:

- waiting time $w_i \in \{1, 2, 5, 10, 20, 50, 100\} =: W$,
- difference coef. $d_i \in \{0.005, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5\} =: D$
- shifting parameter $m_i \in \{1, 2, 5, 10, 20\} =: M$,
- ratio of portfolio on the market $r_i \in \{0.2, 0.4, 0.6, 0.8, 1\} =: R$.

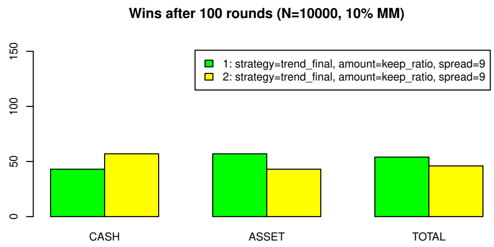
So, we have a finite, 2-person non-cooperative normal-form game with a finite set of actions available to player i $X_i = (w_i, d_i, m_i, r_i) \in W \times D \times M \times R$, and with utility functions f_i, $i = 1, 2$. The is not constant-sum game, because the pay-off function depends also on the behaviour of traders and on the market conditions. We play the game for 100 independent rounds in which the winner gets +1 point. After all 100 rounds, the action with more points is considered as the winning action. The winning action then performed next game with different action. Obviously, not all values of parameters can be compared. We tried many combinations of actions, but it was impossible to try all of them due to high technical and time requirements. Our goal was to find a dominating strategy. However, we most likely did not find a dominating action or a Nash equilibrium for the market makers, because it is simply impossible to try all the possible combinations of parameters. So, the main goal of those simulations was to find the better "direction", i.e. go higher or lower with a certain parameter.

Some parameter were clearly better than other and worked for all probabilities $p_{\mathbf{MM}}$, while some where trickier.

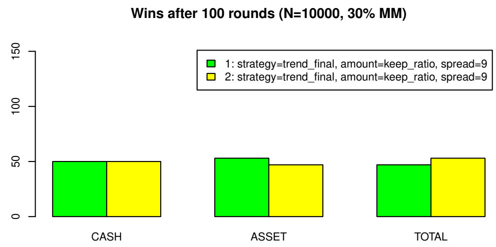
The best option among values of parameter r_i was 1. The market maker who put all his/her portfolio on the market had always higher pay off than his/her competitor with lower proportion. That is quite understandable, because the higher the parameter, the more orders are placed on the market, and the more orders can be executed. The high value of the parameter also guarantees that the market is liquid, and it prevents the situation in Figure 3.2 (a) and (b) from happening even for smaller probability of the arrival of a market maker $p_{\mathbf{MM}}$.

Finding the optimal value of the shifting parameter m_i was also quite straightforward. The lowest possible option (i.e. $m_i = 1$) dominated all other actions that were simulated. The true meaning of it is that shifting the prices of orders by smaller steps is more beneficial than jumping large intervals. This may be explained as a sort of cautious behaviour, that prevents small and random changes in the inventory from causing a large shift. However, the best understanding of the importance of the low value is probably that the parameter m_i goes hand in hand with the waiting parameter w_i.

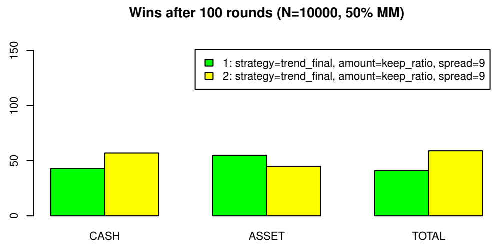
The waiting time w_i between recalculation of the value of the true price performed best pay offs for lower values; however, value 1 did not work. The best options were 2 and 5. An example in which 2 performed better with $p_{\mathbf{MM}} = 10\%$, and 5 performed better with higher values of $p_{\mathbf{MM}}$ is illustrated in Figure 3.5. The parameter w_i determines after how many turns the market maker reevaluates his/her idea of the true price. So, if we lower the percentage of market makers but keep the value of w_i fixed, then measured in real time the market maker controls the value of the true price less often. That may be the reason behind lower w_i performing better with a low percentage of market makers.



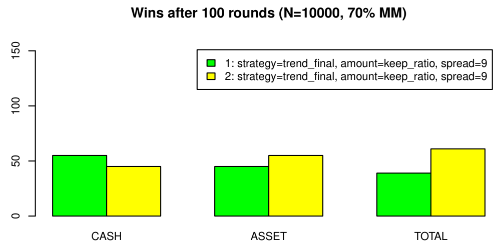
(a) 10 % of market makers



(b) 30 % of market makers



(c) 50 % of market makers



(d) 70 % of market makers

Figure 3.5: Proportion of wins in cash, asset, and both combined for strategies $(w_1, d_1, m_1, r_1) = (2, 0.3, 1, 1)$ and $(w_2, d_2, m_2, r_2) = (5, 0.3, 1, 1)$.

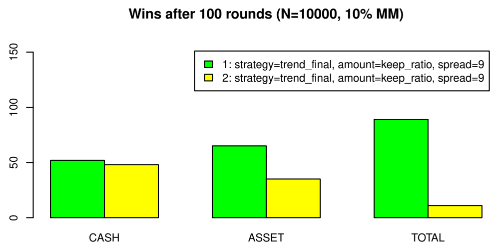
Together with the shifting parameter m_i, we get that making small changes often is more preferable to making large changes less frequently. A comparison is in Figure 3.6.

The last parameter difference coefficient d_i was the most complicated one. At first, the simulations were run for smaller values, because it seemed more reasonable, e.g. $d_i = 0.05$ corresponds to significant difference of 10 assets or 505 cash in the setting of starting values of inventories and true prices. The highest pay offs were however for $d_i = 0.3$ and sometimes even $d_i = 0.4$. The comparison for various values of p_{MM} are in Figure 3.7, where $d_1 = 0.1$ and $d_2 = 0.3$. The reason for higher values being more rewarding may lay in the conservativeness. A more conservative market maker who does not change his opinion just because of a small disproportion gets a higher pay off.

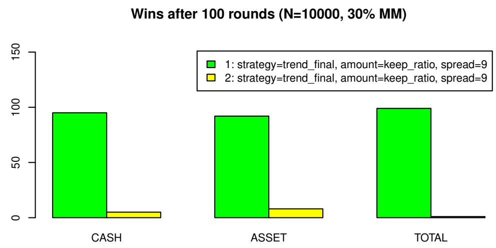
The absolute values of parameters undoubtedly depend on the total steps of simulation N, on the parameter ζ^* that determines how fast do the states change from low to high and vice versa, and also on the spread parameters s_i. Therefore, we are rather focusing on the decision between low and high values of the parameters than on the specific absolute value of the parameter.

Obviously, all the result are based only on simulations. Therefore, we need to be considerate when presenting the results because they are still waiting for its proper proof. We however got a deeper insight into market makers' strategies.

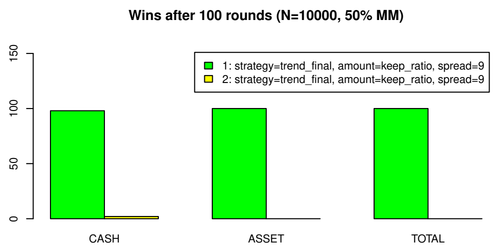
The model can undoubtedly be improved as many options are still open. For example, the traders who do not see a suitable offer simply walk away in our



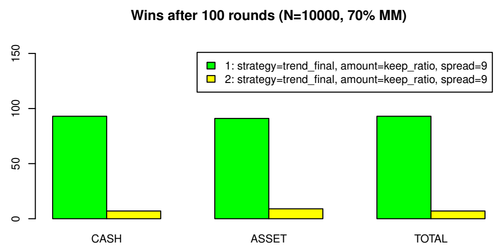
(a) 10 % of market makers



(b) 30 % of market makers

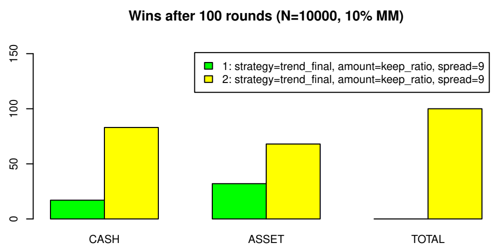


(c) 50 % of market makers

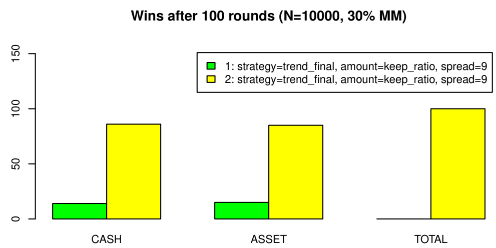


(d) 70 % of market makers

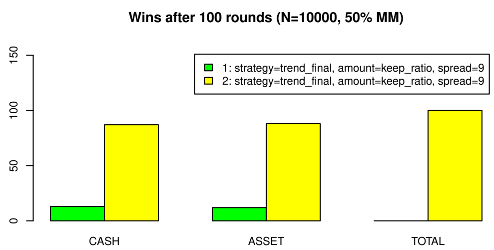
Figure 3.6: Proportion of wins in cash, asset, and both combined for strategies $(w_1, d_1, m_1, r_1) = (2, 0.3, 1, 1)$ and $(w_2, d_2, m_2, r_2) = (10, 0.3, 10, 1)$.



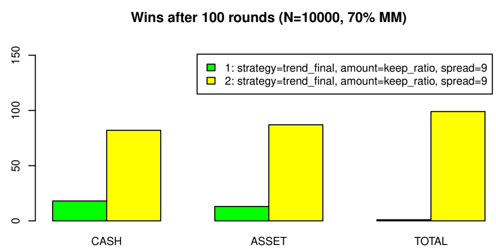
(a) 10 % of market makers



(b) 30 % of market makers



(c) 50 % of market makers



(d) 70 % of market makers

Figure 3.7: Proportion of wins in cash, asset, and both combined for strategies $(w_1, d_1, m_1, r_1) = (5, 0.1, 1, 1)$ and $(w_2, d_2, m_2, r_2) = (5, 0.3, 1, 1)$.

model. In reality, they are just waiting in the background for a suitable offer to appear. For example, Bouchaud and Donier [2015] introduce the idea of modelling two order books. One "*visible*" order book in the traditional sense that can be measures in stock exchanges with the drawback that it is dominated by market makers, and one "*latent*" order book that contains all buy/sell intentions; and as a result, represents the true underlying supply and demand.

Conclusion

The main goal of the thesis was to develop and improve the Stigler-Luckock model, and gain a deeper insight into market maker's strategies.

New approaches were studied in this thesis to improve the original model. Most importantly this concerned the behaviour of so-called market makers. Market makers are an essential component of each properly functioning market. Their purpose is to supply the market with cash and asset; therefore, they are sometimes called "liquidity providers".

The main focus of the thesis were simulations of a model with regular traders, who either take an offer or leave, and with two competitive market makers. The intention was to compare a few strategies and attempt to find the best one in a sense of game theory.

First, a simple market with the equilibrium price in the middle was analysed. The supply and demand functions were constant in time. The outcome was that a market maker with a wider spread wins in market where market makers place a small proportion of all orders, and looses in markets with a higher proportion. Furthermore, an increased volume of orders improved the position of the market maker with wider spread, but only in (unrealistic) markets with little liquidity.

Secondly, a more complicated market was studied—with two states that each have a different equilibrium price. The changes are represented by a hidden Markov model in which supply and demand are not constant in time. The simulations indicated that a reasonable strategy for market makers is to watch their inventory (i.e. cash and asset), and if they observe some disproportion between buys and sells, increase or decrease the prices for which they buy and sell the asset. The results of the simulations suggested that a market maker who controls his/her inventory more frequently, and who changes the prices with just smaller steps, earns more. It was also recommended to place orders with high volumes because it not only provides liquidity but it is also beneficial to the market maker. Finally, a more "conservative" approach was preferred. The meaning of the term is that only larger disproportions in inventories should be reacted to, and the minor ones should not cause panic.

Still, the results were only obtained by simulations. We need to be considerate when presenting the results because they are not properly proved. The proof, if possible at all, is however probably complicated.

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