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Random measurable sets

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I would like to thank my supervisor, Prof. J. Rataj, for repeatedly not letting me get carried away and pointing me in a sensible direction.

Title: Random measurable sets

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Abstract: The aim of this thesis is to compare two major models of random sets, the well established random closed sets (RACS) and the more recent and more general random measurable sets (RAMS). First, we study the topologies underlying the models, showing they are very different. Thereafter, we introduce RAMS and RACS and reformulate prior findings about their relationship. The main result of this thesis is a characterization of those RAMS that do not induce a corresponding RACS. We conclude by some examples of such RAMS, including a construction of a translation invariant RAMS.

Keywords: L^1 space, random set, weak convergence, measurability

Contents

Notation Index

$\mathbb{N}$	The set of all natural numbers (not including 0)
$\mathbb{R}$	The set of all non-negative real numbers
d	A fixed natural number
$(\Omega, \mathcal{A}, \mathbb{P})$	A fixed complete probability space
$\mathcal{B}(\mathbb{R}^d)$	The Borel σ-algebra on $\mathbb{R}^d$
$\mathcal{M}$	The set of all Lebesgue classes of sets in $\mathcal{B}(\mathbb{R}^d)$
$[A]$	The Lebesgue class of a measurable set $A \in \mathcal{B}(\mathbb{R}^d)$
λ^d	The d-dimensional Lebesgue measure
$\mathcal{F}$	The set of all closed subsets of $\mathbb{R}^d$
$\bar{A}$	The closure of a set A
$A \triangle B$	The symmetric difference of sets A and B
1_A	The indicator function of a set A
$B(x, \varepsilon)$	The d-dimensional open ball centered at $x \in \mathbb{R}^d$ with radius $\varepsilon > 0$
$f[A]$	The image of a set A under a function f
$f^{-1}[A]$	The preimage of a set A under a function f
$A_n \nearrow A$	The sequence $(A_n)_{n=1}^{\infty}$ is nondecreasing in $\subseteq$ and $A = \bigcup_{n=1}^{\infty} A_n$

Introduction

While we have many tools for working with random variables and vectors, in many contexts it is be useful to model entire random sets. Originally, the concept of a random set was used in stochastic geometry, but it has found applications in several fields since then.

We mention a few examples from the book Goutsias et al. [2012]. In image processing and analysis, images, as well as clutter, can be modeled as random sets, using algorithms based on the theory of random sets to recover the original image. This has been used to prove that there exists an optimal algorithm for recovering an image from certain types of noise processes.

In artificial intelligence, random sets can be used to model imprecise or vague information. For example, the phrase ”near the tower” can be interpreted as a random disk centered at the tower with probabilities distributed so that disks with smaller radii are more probable. Other applications include theoretical statistics, data fusion and mathematical finance.

Traditionally, random sets are modeled as random closed sets (RACS) and the theory of RACS is well developed. However, in recent years a new framework of random measurable sets (RAMS) has emerged, allowing the manipulation of more irregular functionals. In this thesis, we study the relationship between the two models.

In Chapter 1 we introduce the underlying topologies of both models and provide a few examples. Chapter 2 summarizes results about their relationships and supplies further theory. In Chapter 3 we provide examples of RAMS that do not generate RACS, focusing particularly on stationary sets, an important class of random sets.

1. Preliminaries

The aim of this chapter is to define and compare the topologies underlying RAMS and RACS. First, however, we recall a special case of the Besicovitch derivative theorem (see [Ambrosio et al., 2000, Theorem 2.22]).

Theorem 1. *Let μ be a measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ absolutely continuous with respect to λ^d. Then the limit*

$$\lim_{r \to 0+} \frac{\mu(B(x,r))}{\lambda^d(B(x,r))}$$

exists for λ^d-almost all $x \in \mathbb{R}^d$, is in $L^1(\mathbb{R}^d)$ and is a Radon–Nikodym derivative of μ.

1.1 λ-inner sets

In this section we introduce λ-inner sets. We came up with the notion when trying to find examples of RAMS that do not generate RACS and we also used it for comparing topologies, so we apply it in Chapter 2 (particularly in the proof of Theorem 11), as well as later in this chapter.

Definition 1. *Let $A \in \mathcal{B}(\mathbb{R}^d)$ be a measurable set, then the λ-inner set of A is*

$$\widetilde{A} = \{x \in \mathbb{R}^d \mid \forall \varepsilon > 0 : \lambda^d(A \cap B(x, \varepsilon)) > 0\}.$$

Remark. The λ-inner set of A is actually the support of the restriction of λ^d to A, that is,

$$\widetilde{A} = \mathbb{R}^d \setminus \bigcup \{G \subseteq \mathbb{R}^d \text{ open} \mid \lambda^d(G \cap A) = 0\}.$$

Theorem 2. *Let F be a closed subset of $\mathbb{R}^d$, $F \in \mathcal{F}$. Then, $\widetilde{F}$ is Lebesgue equivalent to F,*

$$\lambda^d(F \triangle \widetilde{F}) = 0.$$

Proof. Let $x \in \widetilde{F}$, then for all $n \in \mathbb{N}$: $\lambda^d(F \cap B(x, \frac{1}{n})) > 0$, and therefore $F \cap B(x, \frac{1}{n})$ is non-empty. By the axiom of choice there exists some sequence $(x_n)_{n=1}^{\infty}$ of members of F such that for all n: $x_n \in B(x, \frac{1}{n})$, so we get $x \in \bar{F} = F$. Hence, $\widetilde{F} \subseteq F$.

To prove the converse, let us consider a measure μ on $\mathbb{R}^d$ defined as the restriction of λ^d to F, that is, for all $B \in \mathcal{B}(\mathbb{R}^d)$

$$\mu(B) = \lambda^d(F \cap B).$$

μ is clearly absolutely continuous with respect to λ^d, $\mu \ll \lambda^d$. By the Besicovitch derivative theorem (Theorem 1), the function $h : \mathbb{R}^d \to \mathbb{R}$ defined as

$$h(x) = \lim_{\varepsilon \to 0+} \frac{\mu(B(x,\varepsilon))}{\lambda^d(B(x,\varepsilon))}$$

exists for λ^d-almost all $x \in \mathbb{R}^d$ and is a Radon–Nikodym derivative of μ with respect to λ^d, $h(x) = \frac{d\mu}{d\lambda^d}$. Therefore,

$$\begin{aligned}
\lambda^d(F \setminus \widetilde{F}) &= \lambda^d(F \cap (\mathbb{R}^d \setminus \widetilde{F})) \\
&= \mu(\mathbb{R}^d \setminus \widetilde{F}) \\
&= \int_{\mathbb{R}^d \setminus \widetilde{F}} \lim_{\varepsilon \to 0+} \frac{\lambda^d(F \cap B(x,\varepsilon))}{\lambda^d(B(x,\varepsilon))} \, d\lambda^d(x) \\
&= \int \lim_{\varepsilon \to 0+} \left(\frac{\lambda^d(F \cap B(x,\varepsilon))}{\lambda^d(B(x,\varepsilon))} 1_{[x \,\mid\, \exists \varepsilon > 0 : \lambda^d(F \cap B(x,\epsilon)) = 0]} \right) d\lambda^d(x) \\
&= \int 0 \; d\lambda^d(x).
\end{aligned}$$

Altogether, we get

$$\lambda^d(F \triangle \widetilde{F}) = 0.$$

□

Example 1. Even though the λ-inner set can be defined for all $A \in \mathcal{B}(\mathbb{R}^d)$, Theorem 2 does not necessarily hold for non-closed sets. For example, let $d = 1$ and let $\mathbb{Q} = \{q_1, q_2, \dots\}$ be some ordering of the rational numbers. Define

$$M = \bigcup_{n=1}^{\infty} B(q_n, \frac{1}{2^n}).$$

Then,

$$\lambda(M) \leq \sum_{n=1}^{\infty} \lambda(B(q_n, \frac{1}{2^n})) = 2 \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

This means that M is not Lebesgue equivalent to $\mathbb{R}$. Furthermore, for all $x \in \mathbb{R}$ and all $\varepsilon > 0$ there exists some q_n in $B(x, \varepsilon)$. Therefore,

$$0 < \lambda(B(q_n, \frac{1}{2^n}) \cap B(x,\varepsilon)) \leq \lambda(M \cap B(x,\varepsilon))$$

Consequently, $\widetilde{M} = \mathbb{R}$ and $\lambda(M \triangle \widetilde{M}) = \lambda(M \triangle \mathbb{R}) > 0$.

Theorem 3. *For all F closed*

$$\widetilde{\widetilde{F}} = \widetilde{F}.$$

Proof. By Theorem 2, for all $x \in \mathbb{R}^d$ and all $\varepsilon > 0$, $\lambda^d(\widetilde{F} \cap B(x,\varepsilon)) = \lambda^d(F \cap B(x,\varepsilon))$. Therefore,

$$\begin{aligned}
\widetilde{\widetilde{F}} &= \{x \in \mathbb{R}^d \mid \forall \varepsilon > 0 : \lambda^d(\widetilde{F} \cap B(x,\varepsilon)) > 0\} \\
&= \{x \in \mathbb{R}^d \mid \forall \varepsilon > 0 : \lambda^d(F \cap B(x,\varepsilon)) > 0\} \\
&= \widetilde{F}.
\end{aligned}$$

□

Theorem 4. *For all $A \in \mathcal{B}(\mathbb{R}^d)$: $\widetilde{A}$ is closed.*

Proof. Let $(x_n)_{n=1}^\infty$ be a sequence in $\widetilde{A}$ such that $x_n \to x$ for some $x \in \mathbb{R}^d$. For all $n \in \mathbb{N}$ and all $\varepsilon > 0$: $\lambda^d(A \cap B(x_n, \varepsilon)) > 0$. Let us fix some $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$: $x_n \in B(x, \frac{\varepsilon}{2})$. Because $B(x_n, \frac{\varepsilon}{2}) \subseteq B(x, \varepsilon)$ and $\lambda^d(A \cap B(x_n, \frac{\varepsilon}{2})) > 0$, we get

$$0 < \lambda^d \left(A \cap B(x_n, \frac{\varepsilon}{2}) \right) \leq \lambda^d(A \cap B(x, \varepsilon)).$$

Therefore, $x \in \widetilde{A}$. □

1.2 The topologies of random sets

In this section we define the topologies of RACS and RAMS, that is, the Fell topology on the space of closed sets and the L^1_{loc} topology on Lebesgue classes of measurable sets, respectively. In the first case we use the definition and results from Matheron [1975], in the second case we follow Galerne and Lachieze-Rey [2015] and Kiderlen and Rataj [2018].

Definition 2. *By $\mathcal{F}$ we denote the set of all closed subsets of $\mathbb{R}^d$. For any set $A \subseteq \mathbb{R}^d$, define the sets*

$$\mathcal{F}_A = \{F \in \mathcal{F} \mid F \cap A \neq \emptyset\}$$

and

$$\mathcal{F}^A = \{F \in \mathcal{F} \mid F \cap A = \emptyset\}.$$

The Fell (or hit-or-miss) topology on $\mathcal{F}$, $\tau_{\mathcal{F}}$, is the topology generated by

$$\{\mathcal{F}^K \mid K \in \mathcal{K}\} \cup \{\mathcal{F}_G \mid G \in \mathcal{G}\},$$

where $\mathcal{K}$ and $\mathcal{G}$ denote the set of all compact and open subsets of $\mathbb{R}^d$, respectively.

Remark. Clearly, the set

$$\mathcal{F}_0 = \{\mathcal{F}^K \cap \bigcap_{i=1}^{n} \mathcal{F}_{G_i} \mid K \in \mathcal{K}, n \in \mathbb{N}, G_1, \ldots, G_n \in \mathcal{G}\}$$

forms a base of $\tau_{\mathcal{F}}$.

Theorem 5 ([Matheron, 1975, Theorem 1-2-2.]). *Let $(F_n)_{n=1}^\infty$ be a sequence in $\mathcal{F}$ and let $F \in \mathcal{F}$. Then, $F_n \to F$ in $\tau_{\mathcal{F}}$, if and only if:*

(i) For all $x \in F$ there exists a sequence $(x_n)_{n=1}^\infty$ in $\mathbb{R}^d$ such that $x_n \to x$ and for all $n \in \mathbb{N}$: $x_n \in F_n$, that is,

$$(\forall x \in F)(\exists (x_n)_{n=1}^\infty \subseteq \mathbb{R}^d)(x_n \to x \ \& \ (\forall n \in \mathbb{N})(x_n \in F_n)),$$

and

(ii) For all increasing sequences $(n_k)_{k=1}^\infty$ in $\mathbb{N}$, all $(x_{n_k})_{k=1}^\infty$ in $\mathbb{R}^d$ and all $x \in \mathbb{R}^d$, if for all $k \in \mathbb{N}$: $x_{n_k} \in F_{n_k}$, and $(x_{n_k})_{k=1}^\infty$ converges, then its limit is in F, that is,

$$(\forall (n_k)_{k=1}^\infty \subseteq \mathbb{N} \text{ incresing})(\forall (x_{n_k})_{k=1}^\infty \subseteq \mathbb{R}^d)(\forall x \in \mathbb{R}^d)$$
$$(x_n \to x \ \& \ (\forall k \in \mathbb{N})(x_{n_k} \in F_{n_k}) \Rightarrow x \in F).$$

Example 2. Let $F_n = \overline{B(0, \frac{1}{n})}$. We will show that the sequence $(F_n)_{n=1}^{\infty}$ converges towards the singleton of zero, $F_n \to \{0\}$, in $\tau_{\mathcal{F}}$.

As for condition (i) of Theorem 5, we can set $(x_n)_{n=1}^{\infty}$ as $(0)_{n=1}^{\infty}$. Then, $0 \to 0$ and for all n: $0 \in F_n$.

On the other hand, let $(x_{n_k})_{k=1}^{\infty}$ be a sequence such that for all k: $x_{n_k} \in F_{n_k}$. For any $\varepsilon > 0$ there exists some k_0 such that for all $k \geq k_0$

$$B(0, \varepsilon) \supseteq \overline{B(0, \frac{1}{n_k})} = F_{n_k} \ni x_{n_k}.$$

Therefore, by Theorem 5 we get $F_n \to \{0\}$ in $\tau_{\mathcal{F}}$.

Example 3. We will show that the mapping $f : F \mapsto \widetilde{F}$ on $\mathcal{F}$ is not continuous in the Fell topology. Again, let $F_n = \overline{B(0, \frac{1}{n})}$ and let $F_0 = \{0\}$. From Example 2 we know that $F_n \to F_0$, so the set $\mathcal{V} = \{F_0, F_1, F_2, \dots\}$ is clearly closed.

Since $\widetilde{F_0} = \emptyset \neq F_0$, by Theorem 3 there is no closed set F such that $\widetilde{F} = F_0$. Therefore, $f^{-1}[\{F_0\}] = \emptyset$. For $n \in \mathbb{N}$, $f^{-1}[\{F_n\}]$ is exactly the Lebesgue class $[F_n]$ of F_n by Theorems 2 and 3. [1] We get

$$f^{-1}[\mathcal{V}] = \bigcup_{n=0}^{\infty} f^{-1}[F_n] = \bigcup_{n=1}^{\infty} [F_n].$$

Therefore, $(F_n)_{n=1}^{\infty}$ is a sequence in $f^{-1}[\mathcal{V}]$ that converges toward $F_0 \notin f^{-1}[\mathcal{V}]$, so $f^{-1}[\mathcal{V}]$ is not closed and f is not continuous.

Definition 3. *By $\mathcal{M}$ we denote the set of Lebesgue classes of all measurable subsets of $\mathbb{R}^d$. We define the L^1_{loc} topology, $\tau_{\mathcal{M}}$, as the topology characterized by the convergence of indicator functions in the space of locally integrable functions $L^1_{loc}(\mathbb{R}^d)$. That is, for all sequences $(A_n)_{n=1}^{\infty}$ in $\mathcal{M}$ and all $A \in \mathcal{M}$*

$$\begin{aligned} A_n \to A \text{ in } \tau_{\mathcal{M}} &\Leftrightarrow 1_{A_n} \to 1_A \text{ in } L^1_{loc}(\mathbb{R}^d) \\ &\Leftrightarrow \int_U (1_{A_n} - 1_A)\, d\lambda \to 0 \text{ for all } U \subseteq \mathbb{R}^d \text{ open and bounded.} \end{aligned}$$

Remark. Clearly, $A_n \to A$ in $\tau_{\mathcal{M}}$ if and only if for all $U \subseteq \mathbb{R}^d$ open and bounded

$$\lambda^d((A_n \triangle A) \cap U) \to 0.$$

Also, we only need to verify the condition for a countable number of sets $U \subseteq \mathbb{R}^d$ (take for example all open balls in $\mathbb{R}^d$ with rational centers and radii).

Theorem 6. *For any $A \in \mathcal{M}$, $U \subseteq \mathbb{R}^d$ open and bounded and $\varepsilon > 0$ denote*

$$B(A, U, \varepsilon) = \{M \in \mathcal{M} \mid \lambda^d((M \triangle A) \cap U) < \varepsilon\}$$

and let

$$\mathcal{M}_0 = \{B(A, U, \varepsilon) \mid A \in \mathcal{M}, U \subseteq \mathbb{R}^d \text{ open and bounded, } \varepsilon > 0\}.$$

Then, $\mathcal{M}_0$ is a base of $\tau_{\mathcal{M}}$.

[1] Here, the notation could be somewhat confusing—while $[F_n]$ denotes the Lebesgue class of F_n, $f^{-1}[\{F_n\}]$ is simply the preimage of the set F_n and is unrelated to Lebesgue classes.

Proof. First let us show that all $B(A, U, \varepsilon)$ are open. Let (M_n) be a sequence in $\mathcal{M} \setminus B(A, U, \varepsilon)$ that converges to some $M \in B(A, U, \varepsilon)$ in $\tau_{\mathcal{M}}$. Then, $\lambda^d((M \triangle A) \cap U) < \varepsilon$ and for all $n \in \mathbb{N}$ larger than some n_0: $\lambda^d((M \triangle M_n) \cap U) < \varepsilon - \lambda^d((M \triangle A) \cap U)$, so

$$\lambda^d((M_n \triangle A) \cap U) \leq \lambda^d((M \triangle A) \cap U) + \lambda^d((M \triangle M_n) \cap U) < \varepsilon.$$

Therefore, $M_n \in B(A, U, \varepsilon)$, which is a contradiction.

Let $\mathcal{G} \subseteq \mathcal{M}$ be open. We will show that, for all $A \in \mathcal{G}$ there exists some $U \subseteq \mathbb{R}^d$ open and bounded and some $\varepsilon > 0$ such that $B(A, U, \varepsilon) \subseteq \mathcal{G}$, implying that $\mathcal{M}_0$ is a base of $\tau_{\mathcal{M}}$. Otherwise it would hold that for all U and all ε: $B(A, U, \varepsilon) \setminus \mathcal{G} \neq \emptyset$. Let $\{U_1, U_2, \dots\}$ be some countable set of open bounded sets sufficient in the definition of L^1_{loc} convergence (see the remark after Definition 3). For all $n \in \mathbb{N}$ choose some element

$$M_n \in B\left(A, \bigcup_{i=1}^{n} U_i, \frac{1}{n}\right) \setminus \mathcal{G}$$

by the axiom of choice. Then, for any U_k and any ε, if $\frac{1}{n_0} < \varepsilon$ and $n_1 = \max(n_0, k)$, then for all $n \geq n_1$

$$\lambda^d((M_n \triangle A) \cap U_k) \leq \lambda^d\left((M_n \triangle A) \cap \bigcup_{i=1}^{n} U_i\right) < \frac{1}{n} < \varepsilon.$$

Therefore, $M_n \to A \in \mathcal{G}$, which, since $\mathcal{G}$ is open, is a contradiction. □

1.3 Comparing the topologies

Here we try to illustrate that the two topologies are very different, showing several negative results about the relationship of their convergences.

Example 4. It does not hold that if for a sequence $(F_n)_{n=1}^{\infty}$ in $\mathcal{F}$ and $F \in \mathcal{F}$

$$F_n \to F \text{ in } \tau_{\mathcal{F}},$$

then

$$[F_n] \to [F] \text{ in } \tau_{\mathcal{M}}$$

($[A]$ denotes the Lebesgue class of A). For $d = 1$, let $\{q_1, q_2, \dots\}$ be some ordering of the rational numbers and let $F_n = \{q_i \mid i \leq n\}$, $F = \mathbb{R}$. Clearly, $[F_n] = [\emptyset]$ for all n, and $[\emptyset]$ does not converge toward $[\mathbb{R}]$ in $\tau_{\mathcal{M}}$. We will show that $F_n \to F$ in $\tau_{\mathcal{F}}$.

For all n, F_n is finite and therefore closed. For any x in $\mathbb{R}$ there exists some sequence (x_n) such that $x_n \in F_n$ and $x_n \to x$. Furthermore, any convergent sequence converges to an element of $\mathbb{R}$. By Theorem 5, $F_n \to F$ in $\tau_{\mathcal{F}}$.

Example 5. It does not hold that if for a sequence $(F_n)_{n=1}^{\infty}$ in $\mathcal{F}$ and $F \in \mathcal{F}$:

$$[F_n] \to [F] \text{ in } \tau_{\mathcal{M}},$$

then

$$F_n \to F \text{ in } \tau_{\mathcal{F}}.$$

Let $F_n = \overline{B(0, \frac{1}{n})}$ and $F = \emptyset$. We know from Example 2 that $F_n \to \{0\} \neq \emptyset$. On the other hand, for any U open and bounded

$$\lambda^d((F_n \triangle \emptyset) \cap U) \leq \lambda^d(\overline{B(0, \frac{1}{n})}) \to 0,$$

so $[F_n] \to [F]$ in $\tau_{\mathcal{M}}$.

Remark. The propositions negated by Examples 4 and 5 can be seen as too strong—the proposition from Example 4 states that the convergence of any sequence in the particular classes is a sufficient condition for the convergence of the classes, while in Example 5 we have that the convergence of the classes ensures the convergence of all sequences in the classes. We can try using λ-inner sets to weaken the propositions to a particular sequence in the classes.

However, Example 5 also shows that it does not hold that if $[F_n] \to [F]$ in $\tau_{\mathcal{M}}$, then $\widetilde{F}_n \to \widetilde{F}$ in $\tau_{\mathcal{F}}$.

Furthermore, Example 4 is a counterexample even for a weaker version of both of the original propositions: If $[F_n] \to [F]$ in $\tau_{\mathcal{M}}$ and $F_n \to F_0$ in $\tau_{\mathcal{F}}$, then $F_0 \in [F]$.

Example 6. It does not hold that if for a sequence $(F_n)_{n=1}^{\infty}$ in $\mathcal{F}$ and $F \in \mathcal{F}$:

$$[F_n] \to [F] \text{ in } \tau_{\mathcal{M}},$$

then there exists some $F_0 \in [F]$ such that

$$\widetilde{F}_n \to F_0 \text{ in } \tau_{\mathcal{F}}.$$

For $d = 2$ let

$$G_k = \{(x, y) \in [0, 1]^2 \mid y \leq x^2\}$$

and let $F_{2k} = G_k$, $F_{2k-1} = \emptyset$. Clearly, $[F_n] \to [\emptyset]$ in $\tau_{\mathcal{M}}$, since $\lambda^2(F_n) \to 0$.

On the other hand, $F_{2k-1} \to \emptyset$ and $F_{2k} \to \{0\} \times [0, 1] \cup [0, 1] \times \{1\}$ in $\tau_{\mathcal{F}}$ by Theorem 5. Neither of those sets is the limit of the whole sequence: As for $\emptyset$, there exists a subsequence $x_{n_k} \in F_{n_k}$ converging to a point outside $\emptyset$ (take for example $x_{2k} = (0, 0)$), contradicting part ii of Theorem 5. For $\{0\} \times [0, 1] \cup [0, 1] \times \{1\}$, there exists no sequence $x_n \in F_n$ such that $x_n \to (0, 0)$. Clearly, F_n does not converge in $\tau_{\mathcal{F}}$.

Example 7. It does not hold that if for a sequence $(F_n)_{n=1}^{\infty}$ in $\mathcal{F}$ and $F \in \mathcal{F}$:

$$\widetilde{F}_n \to F \text{ in } \tau_{\mathcal{F}},$$

then

$$[F_n] \to [F] \text{ in } \tau_{\mathcal{M}}.$$

Similarly as in Example 1, for $d = 1$ take

$$F_n = \bigcup_{i=1}^{n} \overline{B(q_i, \frac{1}{2^{i+2}})},$$

where $\{q_1, q_2, \dots\}$ is some ordering of the rational numbers. Clearly, $\widetilde{F}_n = F_n$ for all n. Let

$$M = \bigcup_{n=1}^{\infty} F_n.$$

Because M is dense in $\mathbb{R}$ and $F_n \nearrow M$, for any $x \in \mathbb{R}$ there exists a sequence $x_n \in F_n$ such that $x_n \to x$. Also, any convergent sequence converges to a point in $\mathbb{R}$, so, by Theorem 5, $\widetilde{F_n} \to \mathbb{R}$ in $\tau_{\mathcal{F}}$.

On the other hand,

$$\begin{aligned}\lambda((F_n \triangle \mathbb{R}) \cap (0,1)) &= \lambda((\mathbb{R} \setminus F_n) \cap (0,1)) \\ &\geq 1 - \sum_{i=1}^{n} \lambda(B(q_i, \frac{1}{2^{i+2}})) \\ &= 1 - \frac{1}{2} \sum_{i=1}^{n} \frac{1}{2^i} \to \frac{1}{2},\end{aligned}$$

so $[F_n]$ does not converge to $\mathbb{R}$ in $\tau_{\mathcal{M}}$.

Remark. In Example 7 it can easily be shown that $[F_n] \to [M]$ in $\tau_{\mathcal{M}}$. Therefore, it is also a counterexample for a weaker proposition: If $\widetilde{F_n} \to F$ in $\tau_{\mathcal{F}}$ and $[F_n] \to [M]$ in $\tau_{\mathcal{M}}$, then $F \in [M]$.

Consequently, the set $[\mathcal{F}]$ of all Lebesgue classes containing a closed set, $[\mathcal{F}] = \{[F] \mid F \in \mathcal{F}\}$, is not closed in $\tau_{\mathcal{M}}$.

2. Random closed sets and random measurable sets

In this chapter we define random closed and measurable sets and try to compare the two models.

Definition 4. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. A random closed set (RACS) is a measurable mapping $Z : (\Omega, \mathcal{A}) \to (\mathcal{F}, \mathcal{B}(\mathcal{F}))$, where $\mathcal{B}(\mathcal{F})$ is the Borel σ-algebra of the topological space $(\mathcal{F}, \tau_{\mathcal{F}})$.*

A random measurable set (RAMS) is a measurable mapping $X : (\Omega, \mathcal{A}) \to (\mathcal{M}, \mathcal{B}(\mathcal{M}))$, where $\mathcal{B}(\mathcal{M})$ is the Borel σ-algebra of the topological space $(\mathcal{M}, \tau_{\mathcal{M}})$.

2.1 Relating RAMS and RACS

In this section we review the results of Galerne and Lachieze-Rey [2015], though the relationship between RAMS and RACS is not the main target of study of the article. We relate RAMS and RACS using the notion of a measurable graph representative.

Definition 5. *A set $Y \subseteq \Omega \times \mathbb{R}^d$ is a measurable graph, if it is a measurable subset of $\Omega \times \mathbb{R}^d$, that is, if $Y \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$.*

A measurable graph Y is closed, if for all $\omega \in \Omega$ the ω-section of Y, $Y_\omega = \{x \in \mathbb{R}^d \mid (\omega, x) \in Y\}$, is a closed subset of $\mathbb{R}^d$.

A measurable graph Y is a measurable graph representative of a RAMS X, if for almost every $\omega \in \Omega$, the ω-section of Y is Lebesgue equivalent to $X(\omega)$, that is, if for a.e. $\omega \in \Omega$,

$$\lambda^d(Y_\omega \triangle X(\omega)) = 0.$$

The following is a theorem by Himmelberg and Parthasarathy [1975] rephrased in terms of random sets by Galerne and Lachieze-Rey [2015]. It states that there is a one to one correspondence between RACS and closed measurable graphs.

Theorem 7 (Himmelberg and Parthasarathy [1975]). *(i) Let Z be a RACS. Then the graph of Z, defined as $\{(\omega, x) \in \Omega \times \mathbb{R}^d \mid x \in Z(\omega)\}$, is a closed measurable graph.*

(ii) For any closed measurable graph Y, the map $Z : \omega \mapsto Y_\omega$ is a RACS.

Remark. Part (ii) of Theorem 7 requires the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to be complete. All other propositions in this section would hold without the completeness assumption.

The proof of the following theorem by Galerne and Lachieze-Rey [2015] uses the Radon–Nikodym derivation theorem for random measures ([Galerne and Lachieze-Rey, 2015, Appendix A]). Here we reformulated the proof, including a proof of a special case of the Radon–Nikodym theorem, as we use its variation in Section 2.2.

Theorem 8 (Galerne and Lachieze-Rey [2015]). *For all RAMS X, there exists a measurable graph representative of X.*

Proof. Let us consider the random Radon measure μ on $\mathbb{R}^d$ defined for all $\omega \in \Omega$ and all $A \in \mathcal{B}(\mathbb{R}^d)$ as

$$\mu(\omega, A) = \lambda^d(X(\omega) \cap A) = \int 1_{X(\omega)} \, d\lambda^d(x).$$

For all $n \in \mathbb{N}$, $\omega \in \Omega$ and $x \in \mathbb{R}^d$, define

$$f_n(\omega, x) = \frac{\mu(\omega, B(x, \frac{1}{n}))}{\lambda^d(B(x, \frac{1}{n}))} = \frac{\lambda^d(X(\omega) \cap B(x, \frac{1}{n}))}{\lambda^d(B(x, \frac{1}{n}))}.$$

For all $n \in \mathbb{N}$ it holds that $f_n(\cdot, \cdot)$ is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable—the proof of this fact is complicated and not relevant to our interests. It is shown that $f_n(\cdot, x)$ is $\mathcal{A}$-measurable for all $x \in \mathbb{R}^d$ and $f_n(\omega, \cdot)$ is continuous for all $\omega \in \Omega$, meaning $f_n(\cdot, \cdot)$ is a Carathéodory function and therefore is, since $\mathbb{R}^d$ is separable, jointly measurable. For more details, see [Galerne and Lachieze-Rey, 2015, Appendix A] and [Aliprantis and Border, 2006, 1, Section 4.10].

Clearly, for all $\omega \in \Omega$ the measure $\mu(\omega, \cdot)$, being a restriction of λ^d, is absolutely continuous with respect to λ^d. By the Besicovitch derivative theorem (Theorem 1), the derivative of $\mu(\omega, \cdot)$, defined as

$$\lim_{\varepsilon \to 0+} \frac{\mu(\omega, B(x, \varepsilon))}{\lambda^d(B(x, \varepsilon))} = \lim_{\varepsilon \to 0+} \frac{\lambda^d(X(\omega) \cap B(x, \varepsilon))}{\lambda^d(B(x, \varepsilon))},$$

exists for λ^d-almost all $x \in \mathbb{R}^d$ and is a Radon–Nikodym derivative of $\mu(\omega, \cdot)$. Consequently, the function f defined for all $\omega \in \Omega$ and $x \in \mathbb{R}^d$ as

$$\begin{aligned} f(\omega, x) &= \limsup_{n \to +\infty} f_n(\omega, x) 1_{[\limsup_{n \to +\infty} f_n(\omega, x) = \liminf_{n \to +\infty} f_n(\omega, x)]} \\ &= \limsup_{n \to +\infty} \frac{\lambda^d(X(\omega) \cap B(x, \frac{1}{n}))}{\lambda^d(B(x, \frac{1}{n}))} 1_{[\limsup_{n \to +\infty} f_n(\omega, x) = \liminf_{n \to +\infty} f_n(\omega, x)]}, \end{aligned}$$

is for all $\omega \in \Omega$ a Radon–Nikodym derivative of $\mu(\omega, \cdot)$. Since the $\limsup$ of a sequence of measurable functions is measurable, f is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. Define

$$Y = \{(\omega, x) \in \Omega \times \mathbb{R}^d \mid f(\omega, x) = 1\}.$$

As $Y = f^{-1}[\{1\}]$ is the preimage of a measurable set under an $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function, it is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, $Y \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$. Therefore, Y is a measurable graph.

For any $\omega \in \Omega$, since $f(\omega, \cdot)$ and $1_{X(\omega)}(\cdot)$ are both Radon–Nikodym derivatives of $\mu(\omega, \cdot)$, they are equal for λ^d-almost all $x \in \mathbb{R}^d$. Hence,

$$\lambda^d(Y_\omega \triangle X(\omega)) \leq \lambda^d(\{x \in \mathbb{R}^d \mid f(\omega, x) \neq 1_{X(\omega)}(x)\}) = 0,$$

that is, Y is a measurable graph representative of X. □

The following theorem was presented by Galerne and Lachieze-Rey [2015] as trivial. Here we supply a proof.

Theorem 9. *For all measurable graphs Y, the map $X : \Omega \to \mathcal{M}$, defined for all $\omega \in \Omega$ as*

$$X(\omega) = [Y_\omega],$$

where $[Y_\omega]$ denotes the Lebesgue class of the ω-section of Y, is a RAMS.

Proof. First, let us show that the mapping $f : \omega \mapsto \lambda^d(Y_\omega)$ is $\mathcal{A}$-measurable. Let $(U_n) \subseteq \mathcal{B}(\mathbb{R}^d)$ be a sequence of measurable sets such that $U_n \nearrow \mathbb{R}^d$ and for all n: $\lambda^d(U_n) < \infty$ and define $\mathcal{V}_n$ as the set of all measurable graphs W such that the mapping $\omega \mapsto \lambda^d(W_\omega \cap U_n)$ is measurable,

$$\mathcal{V}_n = \{W \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d) \mid \omega \mapsto \lambda^d(W_\omega \cap U_n) \text{ is measurable}\}.$$

We will show that $\mathcal{V}_n = \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$. For all $A \in \mathcal{A}$ and $B \in \mathcal{B}(\mathbb{R}^d)$, $A \times B \in \mathcal{V}_n$ and it can be shown that $\mathcal{V}_n$ is a Dynkin system. Therefore, $\mathcal{V}_n$ contains the Dynkin system generated by the set of all measurable rectangles, which is, since that set is closed under finite intersections, equal to the σ-algebra generated by the set, i.e. $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$.

Because $\omega \mapsto \lambda^d(Y_\omega \cap U_n)$ is measurable for all n and $\lambda^d(Y_\omega \cap U_n) \to \lambda^d(Y_\omega) = f(\omega)$, f is measurable.

We will verify the measurability for elements $B(A, U, \varepsilon)$ of the base $\mathcal{M}_0$ from Theorem 6. For any $A \in \mathcal{M}$, $U \subseteq \mathbb{R}^d$ open and bounded and $\varepsilon > 0$

$$X^{-1}[B(A, U, \varepsilon)] = \{\omega \mid \lambda^d((Y_\omega \triangle A) \cap U) < \varepsilon\}.$$

Define $\hat{Y} = \{(\omega, x) \mid x \in (Y_\omega \triangle A) \cap U\}$. $\hat{Y}$ is also a measurable graph and

$$X^{-1}[B(A, U, \varepsilon)] = \{\omega \mid \lambda^d(\hat{Y}) < \varepsilon\} = f^{-1}[[0, \varepsilon)].$$

Since f is measurable, $X^{-1}[B(A, U, \varepsilon)]$ and therefore X are also measurable. □

Corollary 10. *For all RACS Z, the map $X : \Omega \to \mathcal{M}$, defined for all $\omega \in \Omega$ as*

$$X(\omega) = [Z(\omega)],$$

is a RAMS.

Remark. Corollary 10 states that any RACS naturally induces a RAMS. However, many RACS induce the same RAMS—namely all RACS whose Lebesgue classes coincide for all ω.

2.2 Closed RAMS

In this section we study conditions under which a RAMS induces a RACS. Namely, we define closed RAMS and subsequently show that any closed RAMS defines a RACS.

Definition 6. *A RAMS X is closed if for almost every $\omega \in \Omega$ there exists a closed set in the Lebesgue class of $X(\omega)$.*

A RAMS X is surely closed if for all $\omega \in \Omega$ there exists a closed set in the Lebesgue class of $X(\omega)$.

Remark. Examples of non-closed RAMS can be found in Chapter 3.

The following theorem states that all closed RAMS admit a closed measurable graph representative. In the proof, we build on the proof by Galerne and Lachieze-Rey [2015] of Theorem 8.

Theorem 11. *For all closed RAMS X, there exists a closed measurable graph representative of X.*

Proof. As in the proof of Theorem 8 we define for all $n \in \mathbb{N}$, $\omega \in \Omega$ and $x \in \mathbb{R}^d$

$$f_n(\omega, x) = \frac{\lambda^d(X(\omega) \cap B(x, \frac{1}{n}))}{\lambda^d(B(x, \frac{1}{n}))}.$$

Again, for all $n \in \mathbb{N}$, f_n is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. Let us now define

$$g_n(\omega, x) = 1_{[f_n(\omega,x)>0]}.$$

Then, all g_n are also $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable: $g_n^{-1}[\{0\}]$ is equal to $f_n^{-1}[\{0\}]$ and $g_n^{-1}[\{1\}]$ is equal to $f_n^{-1}[(0,\infty)]$, which are both measurable sets due to the measurability of f_n.

We will show that for all $\omega \in \Omega$ and all $x \in \mathbb{R}^d$,

$$\lim_{n\to\infty} g_n(\omega, x) = 1_{\widetilde{X}(\omega)}(x),$$

where $\widetilde{X}(\omega)$ denotes the λ-inner set of $X(\omega)$ (see Chapter 1).

Let $\omega \in \Omega$ be fixed and first let $x \notin \widetilde{X}(\omega)$. Then, there exists $\varepsilon > 0$ such that $\lambda^d(X(\omega) \cap B(x, \varepsilon)) = 0$. For $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$ we get $\lambda^d(X(\omega) \cap B(x, \frac{1}{n_0})) = 0$, so, by the definition of f_n, $f_{n_0}(\omega, x) = 0$. Since f_n are non-increasing in n, for all $n \geq n_0$: $f_n(\omega, x) = 0 = g_n(\omega, x)$.

Now let $x \in \widetilde{X}(\omega)$. This means that for all $\varepsilon > 0$: $\lambda^d(X(\omega) \cap B(x, \varepsilon)) > 0$, so, for all $n \in \mathbb{N}$, $\lambda^d(X(\omega) \cap B(x, \frac{1}{n})) > 0$ and therefore $f_n(\omega, x) > 0$. Hence, for all $n \in \mathbb{N}$: $g_n(\omega, x) = 1$.

Since $1_{\widetilde{X}(\cdot)}(\cdot)$ is the (pointwise) limit of a sequence of $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions, it is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and also

$$\widetilde{X} = \{(\omega, x) \in \Omega \times \mathbb{R}^d \mid x \in \widetilde{X}(\omega)\} = 1_{\widetilde{X}(\cdot)}(\cdot)^{-1}[\{1\}]$$

is an $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable set, that is, a measurable graph.

By Theorem 4, $\widetilde{X}_\omega = \widetilde{X}(\omega)$ is closed for all $\omega \in \Omega$. Therefore, $\widetilde{X}$ is a closed measurable graph.

For any $\omega \in \Omega$ such that there exists a closed set $F \in \mathcal{F}$ Lebesgue equivalent to $X(\omega)$, by Theorem 2 F is Lebesgue equivalent to $\widetilde{F} = \widetilde{X}(\omega)$. Hence, $\lambda^d(\widetilde{X}(\omega) \triangle X(\omega)) = \lambda^d(F \triangle X(\omega)) = 0$. Since the assumption holds for almost all $x \in \mathbb{R}^d$, $\widetilde{X}$ is a representative of X. Altogether, $\widetilde{X}$ is a closed measurable graph representative of X. □

Corollary 12. *For all closed RAMS X there exists a RACS Z such that for almost all $\omega \in \Omega$: $Z(\omega)$ is Lebesgue equivalent to $X(\omega)$.*

Remark. Unlike Theorem 8, the proof of Theorem 11 also gives us a clear way of constructing the RACS in Corrolary 12, that is, $Z : \omega \mapsto \widetilde{X}(\omega)$.

Remark. If a RAMS X is surely closed, $\widetilde{X}(\omega)$ is Lebesgue equivalent to $X(\omega)$ for all $\omega \in \Omega$. Then, the RAMS induced by the graph $\widetilde{X}$ by way of Theorem 9 is clearly the original RAMS X. From this we get a one to one correspondence between surely closed RAMS and RACS with codomain in $\widetilde{\mathcal{F}} = \{\widetilde{F} \mid F \in \mathcal{F}\}$.

More precisely, if we define an equivalence $\sim$ on the set of all RACS as

$$Z_1 \sim Z_2 \Leftrightarrow \forall \omega \in \Omega : \lambda^d(Z_1(\omega) \triangle Z_2(\omega)) = 0,$$

we get a one to one correspondence between the class of surely closed RAMS and equivalence classes of $\sim$, each class $[Z]$ being represented by its member $\widetilde{Z}$.

For a general RAMS X, the RAMS defined as $\omega \mapsto [\widetilde{X(\omega)}]$ is surely closed and coincides with X on all $\omega \in \Omega$ such that $X(\omega)$ contains a closed set.

Corollary 13. *A RAMS X admits a closed measurable graph representative (and induces a RACS) if and only if X is closed.*

Proof. One implication has already been proved as Theorem 11 and the other is trivial: Let Y be a closed representative of X. Then, for almost all $\omega \in \Omega$: $X(\omega)$ contains Y_ω and Y_ω is closed. □

3. RAMS not generating RACS

As a result of Corrolary 12, finding examples of RAMS that do not induce a RACS reduces to finding examples of measurable sets containing no closed sets in their Lebesgue class. We have already used a canonical example of such a set as Example 1. Let us restate it in a general dimension.

Example 8. Let $\{s_1, s_2, \dots\}$ be some ordering of a countable dense set S in $\mathbb{R}^d$. Define

$$M = \bigcup_{n=1}^{\infty} B(s_n, \frac{1}{2^n}).$$

Then,

$$\lambda^d(M) \leq \sum_{n=1}^{\infty} \lambda^d(B(s_n, \frac{1}{2^n})) = \kappa_d \sum_{n=1}^{\infty} \frac{1}{2^{dn}} < \infty,$$

where κ_d is the volume of the d-dimensional unit sphere. Therefore, M is not Lebesgue equivalent to $\mathbb{R}^d$.

For any open set $G \subseteq \mathbb{R}^d$: $\lambda^d(G \cap M) > 0$, since G contains some s_k. Therefore, if M were Lebesgue equivalent to a closed set $F \in \mathcal{F}$, then for all G open: $\lambda^d(G \cap F) = \lambda^d(G \cap M) > 0$. F would have a non-empty intersection with all open sets, meaning F would be dense in $\mathbb{R}^d$. Since F is closed, it would have to be equal to $\mathbb{R}^d$.

3.1 Stationary RAMS

In this section we introduce stationary RAMS and extend Example 8 to this notion, showing that even stationary RAMS can induce no RACS.

Definition 7. *A RAMS X is stationary, if its distribution is translation invariant, that is, if for all $r \in \mathbb{R}^d$ and all $\mathcal{V} \subseteq \mathcal{M}$ measurable in L^1_{loc}:*

$$\mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in \mathcal{V}\}) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in \mathcal{V} + r\}), \tag{3.1}$$

where $\mathcal{V} + r = \{V + r \mid V \in \mathcal{V}\}$ and $V + r = \{v + r \mid v \in V\}$.

Remark. Condition (3.1) is equivalent to $\mathbb{P}_X = \mathbb{P}_{X+r}$ for all $r \in \mathbb{R}^d$, where $\mathbb{P}_X$ is the probability distribution measure of X.

Stationary RAMS are useful due to some of their nice properties. For example, while for a RACS Z we can study the probability of some $x \in \mathbb{R}^d$ belonging to Z, $\mathbb{P}(x \in Z)$, this does not make sense for a general RAMS X, since "$x \in X$" is undefined. However, if X is stationary we can define the volume fraction of X, which can be interpreted as the probability of any fixed x being a member of X (see Kiderlen and Rataj [2018] for more details).

Let us now construct an example of a stationary RAMS that does not admit a closed measurable graph representative.

Example 9. Let $\{s_1, s_2, \dots\}$ be some ordering of a countable dense set S in $(0,1)^d$ such that for all $n \in \mathbb{N}$: $B(s_n, \frac{1}{2^n(\kappa_d+1)}) \subseteq (0,1)^d$, where κ_d is the volume of the d-dimensional unit ball, and define

$$M_0 = \bigcup_{z \in \mathbb{Z}^d} \left(\bigcup_{n=1}^{\infty} B(s_n + z, \frac{1}{2^n(\kappa_d + 1)}) \right),$$

that is, M_0 consists of copies of $\bigcup_{n=1}^{\infty} B(s_n, \frac{1}{2^n(\kappa_d+1)})$ in every hypercube of volume 1 with integer vertices. Again, M_0 is not Lebesgue equivalent to $\mathbb{R}^d$, since

$$\begin{aligned}
\lambda^d(M_0 \cap (0,1)^d) &= \lambda^d(\bigcup_{n=1}^{\infty} B(s_n, \frac{1}{2^n(\kappa_d+1)})) \\
&\leq \sum_{n=1}^{\infty} \lambda^d(B(s_n, \frac{1}{2^n(\kappa_d+1)})) \\
&= \frac{\kappa_d}{\kappa_d+1} \sum_{n=1}^{\infty} \frac{1}{2^{dn}} \\
&< 1 \\
&= \lambda^d(\mathbb{R}^d \cap (0,1)^d).
\end{aligned}$$

By a similar argument as in Example 8, M_0 contains no closed set in its Lebesgue class. We will define X as $M_0 + R$, where R is a random d-vector distributed uniformly on $(0,1)^d$, $R \sim U((0,1)^d)$. In more detail, let us define

$$\mathcal{H} = \{M_0 + r \mid r \in (0,1)^d\},$$

where $M_0 + r = \{m + r \mid m \in M_0\}$. Notice that for $r_1 \neq r_2$, $M_0 + r_1$ is not Lebesgue equivalent to $M_0 + r_2$. Therefore, elements of $\mathcal{H}$ correspond precisely to elements of $(0,1)^d$, that is, $g : (0,1)^d \to \mathcal{H}$, defined for all $r \in (0,1)^d$ as

$$g(r) = M_0 + r,$$

is bijective (even with respect to Lebesgue classes). Next, define a σ-algebra $\mathcal{V}$ on $\mathcal{H}$ as

$$\mathcal{V} = \{g[B] \mid B \in \mathcal{B}((0,1)^d)\} = \{\{M_0 + b \mid b \in B\} \mid B \in \mathcal{B}((0,1)^d)\}$$

and a probability measure μ on $(\mathcal{H}, \mathcal{V})$ such that for all $B \in \mathcal{B}((0,1)^d)$

$$\mu(g[B]) = \lambda^d(B).$$

Clearly, $g : ((0,1)^d, \mathcal{B}((0,1)^d)) \to (\mathcal{H}, \mathcal{V})$ is a strict isomorphism and μ is the pushforward of λ^d via g, $\mu = \lambda^d \circ g^{-1}[\cdot]$.

Finally, define X as identity on $\mathcal{H}$. We will show that X is a RAMS using the base $\mathcal{M}_0$ of $\tau_{\mathcal{M}}$ from Theorem 6. For fixed $A \in \mathcal{M}$ and $U \subseteq \mathbb{R}^d$ open and bounded, the map $f : r \in (0,1)^d \mapsto \lambda^d(((M_0 + r) \triangle A) \cap U)$ is measurable. For any $B(A, U, \varepsilon) \in \mathcal{M}_0$

$$X^{-1}[B(A, U, \varepsilon)] = B(A, U, \varepsilon) \cap \mathcal{H} = \{M_0 + r \mid \lambda^d(((M_0 + r) \triangle A) \cap U) < \varepsilon\}.$$

Then,

$$g^{-1}[B(A, U, \varepsilon) \cap \mathcal{H}] = \{r \mid \lambda^d(((M_0 + r) \triangle A) \cap U) < \varepsilon\} = f^{-1}[[0, \varepsilon)].$$

Since f is measurable, so is $g^{-1}[B(A, U, \varepsilon) \cap \mathcal{H}]$ and therefore also X.

X is stationary: For any $g[B] \in \mathcal{V}$, where $B \in \mathcal{B}((0,1)^d)$, and $r \in \mathbb{R}^d$, define $\widehat{B_r} = B + r$ and

$$B_r = \{b - \lfloor b \rfloor \mid b \in \widehat{B_r}\},$$

where $\lfloor b \rfloor$ is the floor function applied to every coordinate of b. Because M_0 is 1-periodic along the d canonical directions, $g[B] + r = M_0 + (B + r) = g[B_r]$. Furthermore, $\lambda^d(B_r) = \lambda^d(B)$, since λ^d is translation-invariant. Therefore,

$$\mu(g[B] + r) = \mu(g[B_r]) = \lambda^d(B_r) = \lambda^d(B) = \mu(g[B]).$$

Thus, X is a stationary RAMS that does not induce a RACS.

Conclusion

We have given an overview of the relationship between the two most common models of random sets. We introduced the notion of λ-inner sets, proving they can be consistently chosen as a representative of any Lebesgue class containing a closed set. Thereafter, we defined the topologies underlying random sets, illustrated them on some examples and showed counterexamples for propositions about the relationship of their convergences.

Then, we defined RACS, RAMS, and measurable graphs and recapitulated prior results, reformulating one proof and supplying a missing one. The main theorems stated that RACS correspond to measurable graphs, that all RAMS induce a measurable graph and that all measurable graphs induce a RAMS. The main result of this thesis is that every RAMS that is almost surely Lebesgue equivalent to a closed set induces a closed measurable graph. The proof consists of a more careful reinterpretation of the proof that RAMS induce measurable graphs, along with an application of λ-inner sets, which were introduced mainly for this reason. Consequently, RAMS that induce a RACS can be characterized precisely as closed RAMS.

Finally, we presented some examples of RAMS that do not induce RACS, including a construction of such a RAMS that is also translation invariant, where we used a random shift of a set periodic along all canonical directions.

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