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Efficiency of Representative Portfolios Using Data Envelopment Analysis

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Abstract: In this work, several data envelopment analysis (DEA) models are used to assess efficiency of US representative portfolios. We consider a portfolio to be efficient if no other surpasses it in minimizing risk or maximizing return. This property is precisely defined in the work and it can be well detected by DEA models. DEA models assuming constant return-to-scale (CRS) as well as variable return-to-scale (VRS) are described here. A model with directional measure is also presented. Four of the VRS models are transformed into diversification consistent (DC) models. In the empirical part, CVaRs on multiple levels are used as risk measures and expected return as a return measure typically. Results acquired using different DEA models to assess efficiency of portfolios are compared. DC models are stronger than their classical VRS counterparts. The DC models identified as efficient only the portfolio with the highest expected return. On the contrary, VRS models classified as efficient more portfolios which differ in riskiness. Their results could be interesting if an investor wanted to choose only one portfolio based on its riskiness.

Keywords: Data envelopment analysis, Portfolio efficiency, CVaR

Contents

Introduction

The topic of this thesis arose from an assumption that investors search for the best investment opportunities on financial markets. The aim of this work is, therefore, to describe a tool which can be used to identify them and apply the tool to US representative stock portfolios.

According to Markowitz [1952], the goal should be to recognize portfolios which have high returns and low variance. Artzner et al. [1999] argues that risk measures should be coherent and Pflug [2000] proves coherence of Conditional Value at Risk (CVaR). So, our goal will be to identify portfolios with high return and low CVaR. Our decisions will be based on how the portfolios have performed in the past. We will consider a portfolio efficient if no such portfolio exists that would surpass it in both eliminating risk and maximizing return. A tool which allows us to identify efficient portfolios using multiple risk and return measures is data envelopment analysis (DEA).

DEA was first presented in Charnes et al. [1978] where it was used to assess non-profit organizations. It compares each unit's output-input ratio to the ratio of combinations of all the units. When risk measures are used as inputs, diversification should be considered. Lamb and Tee [2012] modified the method to account for diversification, which allowed its use in rating of investmentment funds. Similar approach can be used for portfolios, too.

In the first chapter, efficiency related to data envelopment analysis is defined, and the original model is introduced, as well as some of its modifications. The second chapter describes CVaR, and possibilities to compute it. The third chapter presents models which consider diversification. These models, as well as some models from the first chapter are used to assess the efficiency of representative portfolios. The results are presented in the fourth chapter.

1. Data Envelopment Analysis

Data envelopment analysis is a method used to assess efficiency of comparable decision making units. A decision making unit (DMU) or a production unit is something (e.g. company, branch of a shop, machine) which uses measurable inputs to produce measurable outputs. We call a set of DMUs comparable if they have the same structure of inputs and outputs, meaning they use the same or equivalent inputs to produce the same or equivalent outputs. Together, the production units comprise a production possibility set. Assumptions of DEA models describe some properties of production possibility set. Following Cook and Seiford [2009] we define it.

Definition 1 (Production Possibility Set)**.** *Production possibility set is a set of ordered pairs of vectors satisfying following. The dimension of the first vector is the same for all of its members and dimension of second vector is also the same for all of its members. The first vector represents the amount of inputs and the second vector the amount of outputs. The amount of inputs represented by the first vector can be used to produce the amount of outputs represented by the second vector from each pair.*

Inputs and outputs have to be measured in such units that greater inputs are less desirable and greater outputs are more desirable.

An important assumption of DEA models is that a production possibility set is comprised of all real units, their linear (or convex) combinations, and anything below them. The term 'below them' can be be formally defined as free disposability. The definition follows Lamb and Tee [2012].

Definition 2 (Free Disposability)**.** *Let* $\boldsymbol{Z}_i = (Z_{1i}, \dots, Z_{Ki})$ *be inputs and* $\boldsymbol{Y}_i = (Y_{1i}, \dots, Y_{Ji})$ *be outputs of* i^{th} *DMU from a given set of comparable DMUs. We say a production possibility set has the trait of free disposability if it includes any unit which uses* $\boldsymbol{Z'} = (Z'_1, \dots, Z'_K)$ *to produce* $\boldsymbol{Y'} = (Y'_1, \dots, Y'_J)$ *whenever such* i *exists that* $Z'_k \geq Z_{ki}$ *and* $Y'_j \leq Y_{ji}$*.*

We will be comparing outputs and inputs. DEA specifies the way in which outputs and inputs are summed for this comparison. Each unit's combination of outputs over a combination of inputs is compared with the same ratio of other units. The higher the ratio is the less inefficient the unit is. If such coefficients can be found that the resulting ratio of i^{th} is not exceeded by any other unit, then i^{th} unit will be considered efficient. Following Cook and Seiford [2009], we define DEA efficiency.

Definition 3 (DEA Efficiency)**.** *Let us have* n *DMUs,* K *inputs and* J *outputs.* Z_{ki} *shall denote* k^{th} *input of* i^{th} *unit, and* Y_{ji} *shall denote* j^{th} *output of* i^{th} *unit. A production unit* H *is called DEA efficient, if such non-negative coefficients* $y_j, \tilde{y}_k$ *exist that*

$$\frac{\sum_{j=1}^{J} y_j Y_{jH}}{\sum_{k=1}^{K} \tilde{y}_k Z_{kH}} \geq \frac{\sum_{j=1}^{J} y_j Y_{ji}}{\sum_{k=1}^{K} \tilde{y}_k Z_{ki}}$$

for $\forall i \in \{1, \dots, n\}$*.*

This definition and the notation used in it will be used in the whole first chapter.

Several DEA models are generally described in the following parts. This overview will be useful to proceed to using DEA for assessing the efficiency of stock portfolios.

1.1 Constant Return-to-Scale Models

The original DEA model, first introduced in Charnes et al. [1978], was assuming constant return-to-scale (CRS). Constant return-to-scale happens when by increasing inputs, outputs are increased proportionally.

Definition 4 (Constant Return-to-Scale)**.** *Let* $\boldsymbol{Z} = (Z_1, \ldots, Z_k)$ *be inputs and* $\boldsymbol{Y} = (Y_1, \ldots, Y_j)$ *be outputs of any given DMU. We are in a situation of constant return-to-scale if for every* $\alpha \geq 0$ *a theoretical unit with* $\alpha \cdot \boldsymbol{Z}$ *inputs and* $\alpha \cdot \boldsymbol{Y}$ *outputs is in the production possibility set.*

To find whether a unit H is DEA efficient, we will be maximizing a ratio of linear combination of outputs over a linear combination of inputs, while keeping several constraints as showed in the following theorem. This ratio has to be maximized for each unit separately. Subscript H is added to coefficients y and $\tilde{y}$ to emphasize that different coefficients may be found for each unit.

Theorem 1. *Consider the following CRS DEA model for* H^{th} *unit:*

$$\Delta_H^{CRS\ I} = \max_{y_{jH}, \tilde{y}_{kH}} \frac{\sum_{j=1}^{J} y_{jH} Y_{jH}}{\sum_{k=1}^{K} \tilde{y}_{kH} Z_{kH}} \tag{1.1}$$

s. t.

$$\frac{\sum_{j=1}^{J} y_{jH} Y_{ji}}{\sum_{k=1}^{K} \tilde{y}_{kH} Z_{ki}} \leq 1, \quad i = 1, \ldots, n$$

$$\tilde{y}_{kH} \geq 0, \quad k = 1, \ldots, K$$

$$y_{jH} \geq 0, \quad j = 1, \ldots, J$$

Suppose:

- *values of input and output measures are non-negative and at least 1 output is positive*
- *production possibility set includes all linear combinations of all real units*
- *production possibility set satisfies free disposability*
- *production possibility set includes nothing besides what is defined by these properties*

Then the production unit H is DEA efficient if and only if

$$\Delta_H^{CRS\ I} = 1$$

Proof. We will show that both implications are valid. If the optimal value is equal to 1, the model found such coefficients $y_{jH}, \tilde{y}_{kH}$ that

$$\frac{\sum_{j=1}^{J} y_{jH} Y_{jH}}{\sum_{k=1}^{K} \tilde{y}_{kH} Z_{kH}} = 1 \geq \frac{\sum_{j=1}^{J} y_{jH} Y_{ji}}{\sum_{k=1}^{K} \tilde{y}_{kH} Z_{ki}}$$

where the inequality follows from the first restriction of the optimization problem. So, according to Definition 3, the production unit H is DEA efficient.

Suppose a unit H is DEA efficient. Such coefficients $y_{jH}, \tilde{y}_{kH}$ can be found that

$$\frac{\sum_{j=1}^{J} y_{jH} Y_{jH}}{\sum_{k=1}^{K} \tilde{y}_{kH} Z_{kH}} \geq \frac{\sum_{j=1}^{J} y_{jH} Y_{ji}}{\sum_{k=1}^{K} \tilde{y}_{kH} Z_{ki}} \qquad \forall i \in \{1, \ldots, n\}$$

By multiplying all y_{jH} by the same appropriate constant C we may receive

$$1 = \frac{\sum_{j=1}^{J} y_{jH} \cdot C \cdot Y_{jH}}{\sum_{k=1}^{K} \tilde{y}_{kH} Z_{kH}} \geq \frac{\sum_{j=1}^{J} y_{jH} \cdot C \cdot Y_{ji}}{\sum_{k=1}^{K} \tilde{y}_{kH} Z_{ki}} \qquad \forall i \in \{1, \ldots, n\}$$

So we know that such coefficients $y_{jH} \cdot C$ exist for which $\frac{\sum_{j=1}^{J} y_{jH} \cdot C \cdot Y_{jH}}{\sum_{k=1}^{K} \tilde{y}_{kH} Z_{kH}} = 1$ while all the constraints from the preceding model are kept. By computing maximum subjected to these constraints, different coefficients may be found but it will not be lower then 1. □

The problem is easier to solve in a following linear formulation, which was also introduced in Charnes et al. [1978]. We set the denominator of the objective function equal to 1, and keep all the other restrictions. The resulting equivalent formulation of the same problem for H^{th} unit using the same notation is:

Theorem 2. *Consider the following linear formulation of CRS DEA model.*

$$\Delta_H^{CRS\ II} = \max_{y_{jH}, \tilde{y}_{kH}} \sum_{j=1}^{J} y_{jH} Y_{jH} \tag{1.2}$$

s. t.

$$\begin{aligned}
&\textstyle\sum_{k=1}^{K} \tilde{y}_{kH} Z_{kH} = 1 \\
&\textstyle\sum_{j=1}^{J} y_{jH} Y_{ji} \leq \sum_{k=1}^{K} \tilde{y}_{kH} Z_{ki}, \quad i = 1, \ldots, n \\
&\tilde{y}_{kH} \geq 0, \quad k = 1, \ldots, K \\
&y_{jH} \geq 0, \quad j = 1, \ldots, J
\end{aligned}$$

Under the same assumptions as in Theorem 1, unit H is DEA efficient if and only if

$$\Delta_H^{CRS\ II} = 1$$

Proof. We will show that Theorem 2 directly follows from Theorem 1.
It can be easily seen that

$$\max_{y_{jH}, \tilde{y}_{kH}} \frac{\sum_{j=1}^{J} y_{jH} Y_{jH}}{\sum_{k=1}^{K} \tilde{y}_{kH} Z_{kH}} = 1.$$

if and only if

$$\max_{y_{jH}, \tilde{y}_{kH}} \sum_{j=1}^{J} y_{jH} Y_{jH} = 1. \quad \text{s. t.} \quad \sum_{k=1}^{K} \tilde{y}_{kH} Z_{kH} = 1.$$

Multiplying the coefficients $y_{jH}, \tilde{y}_{kH}$ by an appropriate constant the denominator can be set equal to 1.

The second constraint of (1.2) is equivalent to

$$\frac{\sum_{j=1}^{J} y_{jH} Y_{ji}}{\sum_{k=1}^{K} \tilde{y}_{kH} Z_{ki}} \leq 1 \qquad i = 1, \ldots, n$$

and the last two constraints, which ensure non-negativity of coefficients, are the same is (1.1). So the model is equivalent to (1.1) and unit H is DEA efficient if and only if

$$\max_{y_{jH}, \tilde{y}_{kH}} \frac{\sum_{j=1}^{J} y_{jH} Y_{jH}}{\sum_{k=1}^{K} \tilde{y}_{kH} Z_{kH}} = 1.$$

□

The optimization problem is usually solved in its dual form, also introduced in Charnes et al. [1978]. We will use the same notation and add a variable θ, which is different for each unit.

Theorem 3. *Consider the following dual CRS DEA model for H^{th} unit:*

$$\Delta_H^{CRS} = \min_{x_i, \theta} \theta \tag{1.3}$$

s. t.

$\sum_{i=1}^{n} x_{iH} Y_{ji} \geq Y_{jH}, \quad j = 1, \ldots, J$

$\sum_{i=1}^{n} x_i Z_{ki} \leq \theta \cdot Z_{kH}, \quad k = 1, \ldots, K$

$x_{iH} \geq 0, \quad i = 1, \ldots, n$

Under the same assumptions as in Theorem 1, Unit H is DEA efficient if and only if

$$\Delta_H^{CRS} = 1$$

Proof. It is a direct consequence of the Strong Duality Theorem in linear programming, see Dupačová and Lachout [2011]. □

Informally described, the restrictions ensure that H^{th} unit's both outputs and inputs are worse than those of any linear combination of other units. By minimizing θ we see how much the inputs of H^{th} unit have to be reduced to achieve efficiency. Therefore $\theta \leq 1$. The closer to one θ of H^{th} unit is, the less inefficient H^{th} unit is. Because the model computes how many times inputs have to be lowered, this type of DEA model is called input-oriented.

An output-oriented model can be also formulated. It was first formulated in Charnes et al. [1985b]. In this case, a variable ψ shows how many times outputs have to be increased to achieve DEA efficiency. The lower the ψ is the less inefficient the unit is. The model minimizes $\frac{1}{\psi}$.

Theorem 4. *Consider the following dual output-oriented CRS DEA model*

$$\Delta_H^{CRS\ O} = \min_{\psi, x_{iH}} \frac{1}{\psi} \tag{1.4}$$

s. t.

$\sum_{i=1}^{n} x_{iH} Y_{ji} \geq \psi \cdot Y_{jH}, \quad j = 1, \ldots, J$

$\sum_{i=1}^{n} x_{iH} Z_{ki} \leq Z_{kH}, \quad k = 1, \ldots, K$

$x_{iH} \geq 0, \quad i = 1, \ldots, n$

Under the same assumptions as in Theorem 1, unit H is DEA efficient if and only if

$$\Delta_H^{CRS\ O} = 1$$

Proof. We will show that it is a modification of Theorem 3. $\theta \neq 0$ because at least one input is positive so we set $\psi = \frac{1}{\theta}$ and $\tilde{x}_{iH} = \psi \cdot x_{iH}$ in (1.3). Then we have

$$\min_{\psi, x_{iH}} \frac{1}{\psi}$$

s. t.

$\sum_{i=1}^{n} \frac{\tilde{x}_{iH}}{\psi} Y_{ji} \geq Y_{jH}, \quad j = 1, \ldots, J$

$\sum_{i=1}^{n} \frac{\tilde{x}_{iH}}{\psi} Z_{ki} \leq \frac{1}{\psi} \cdot Z_{kH}, \quad k = 1, \ldots, K$

$\frac{\tilde{x}_{iH}}{\psi} \geq 0, \quad i = 1, \ldots, n$

By multiplying all inequalities by ψ we receive (1.4). □

1.2 Variable Return-to-Scale Models

Variable return-to-scale (VRS) DEA models are applicable in situations where we do not believe that outputs increase proportionally to inputs. We, however, do assume that all convex combinations of real units are an achievable production possibility. In comparison to the CRS dual model, we add a restriction $\sum_{i=1}^{n} x_i = 1$. This assures that we compare each unit to convex combinations of all units only, instead of comparing it to all non-negative linear combinations of all units. This type of DEA models was first introduced in Banker et al. [1984] The definition of DEA efficiency remains the same as previously.
The resulting model in dual form follows.

Theorem 5. *Consider the following dual VRS DEA model for H^{th}.*

$$\Delta_H^{VRS\ I} = \min_{\theta, x_{iH}} \theta \tag{1.5}$$

s. t.

$\sum_{i=1}^{n} x_{iH} Y_{ji} \geq Y_{jH}, \quad j = 1, \ldots, J$

$\sum_{i=1}^{n} x_{iH} Z_{ki} \leq \theta \cdot Z_{kH}, \quad k = 1, \ldots, K$

$\sum_{i=1}^{n} x_{iH} = 1, x_{iH} \geq 0, \quad i = 1, \ldots, n$

Suppose:

- *values of input and output measures are non-negative and at least 1 output is positive*
- *production possibility set includes all* ***convex*** *combinations of all real unit*
- *production possibility set satisfies free disposability*

- *production possibility set does not include anything which is not a consequence of the two properties described above.*

Then a production unit H is DEA efficient if and only if

$$\Delta_H^{VRS\ I} = 1$$

Proof. It is a variable return-to-scale modification of Theorem 3. For more details see Banker et al. [1984]. □

We can modify the input-oriented model to look for different θ_k with respect to each input separately. The following model is stronger then the preceding one - by which we mean that the objective values achieved by this model are going to be lower then or equal to those in the preceding one. Even some DEA efficient units may not be identified by the following model. The model was presented in Ruggiero and Bretschneider [1998]. In the following situation, θ_k marks how much the k$^{\text{th}}$ input of H$^{\text{th}}$ unit has to be decreased to achieve efficiency.

Theorem 6. *Consider the following VRS DEA model for H^{th} unit.*

$$\Delta_H^{VRS\ I\ 2} = \min_{\theta, x_{iH}} \frac{1}{K} \sum_{k=1}^{K} \theta_k \tag{1.6}$$

s. t.

$$\begin{aligned}
&\textstyle\sum_{i=1}^{n} x_{iH} Y_{ji} \geq Y_{jH}, \quad j = 1, \dots, J \\
&\textstyle\sum_{i=1}^{n} x_{iH} Z_{ki} \leq \theta_k \cdot Z_{kH}, \quad k = 1, \dots, K \\
&\textstyle\sum_{i=1}^{n} x_{iH} = 1, x_{iH} \geq 0, \quad i = 1, \dots, n \\
&0 \leq \theta_k \leq 1, \quad k = 1, \dots, K
\end{aligned}$$

Under the same assumptions as in Theorem 5, if

$$\Delta_H^{VRS\ I\ 2} = 1$$

then unit H is DEA efficient.

Proof. Suppose

$$\min_{\theta, x_{iH}} \frac{1}{K} \sum_{k=1}^{K} \theta_k = 1$$

Since $\theta_k \in [0,1]$ so $\sum_{k=1}^{K} \theta_k =$ K if and only if θ_k=1 for each k. Then the conditions of the Theorem 5 are fulfilled so unit H is DEA efficient. □

An example showing why the opposite implication is not valid is following. Suppose unit H is found to be efficient by (1.5). Suppose unit G has the exactly same outputs as unit H and its inputs differ in only one case in which unit G needs more inputs than unit H. Then unit G is DEA efficient but it is not going to result from (1.6).

An output-oriented VRS model can be also formulated.

Theorem 7. *Consider the following dual output-oriented VRS DEA model*

$$\Delta_H^{VRS\ O} = \min_{\psi, x_{iH}} \frac{1}{\psi} \tag{1.7}$$

s. t.

$\sum_{i=1}^{n} x_{iH} Y_{ji} \geq \psi \cdot Y_{jH}, \quad j = 1, \ldots, J$

$\sum_{i=1}^{n} x_{iH} Z_{ki} \leq Z_{kH}, \quad k = 1, \ldots, K$

$\sum_{i=1}^{n} x_{iH} = 1, x_{iH} \geq 0, \quad i = 1, \ldots, n$

Under the same assumptions as in Theorem 5, unit H is DEA efficient if and only if

$$\Delta_H^{VRS\ O} = 1$$

Proof. It is a variable return-to-scale modification of Theorem 4. □

Similar adjustment as in Theorem 6 can be done to achieve a stronger output-oriented model with respect to each output separately.

These approaches can be combined resulting in an input-output oriented model, which searches for possible improvement in both inputs and outputs at the same time. The model was described in Pastor et al. [1999].

Theorem 8. *Consider the following dual input-output oriented VRS DEA model for H^{th} unit.*

$$\Delta_H^{VRS\ I-O\ 1} = \min_{\theta, \psi, x_{iH}} \frac{1}{2}\left(\theta + \frac{1}{\psi}\right) \tag{1.8}$$

s. t.

$\sum_{i=1}^{n} x_{iH} Y_{ji} \geq \psi \cdot Y_{jH}, \quad j = 1, \ldots, J$

$\sum_{i=1}^{n} x_{iH} Z_{ki} \leq \theta \cdot Z_{kH}, \quad k = 1, \ldots, K$

$\sum_{i=1}^{n} x_{iH} = 1, x_{iH} \geq 0, \quad i = 1, \ldots, n$

Under the same assumptions a in Theorem 5, a unit H is DEA efficient if and only if

$$\Delta_H^{VRS\ I-O\ 1} = 1$$

Proof. It is a combination of approaches presented in Theorems 5 and 7. For more details, see Pastor et al. [1999]. □

Note that this theorem implies that unit H is efficient if and only if no such $\mathbf{x} = (x_1, \ldots, x_n)$ exists that $\sum_{i=1}^{n} x_{iH} Y_{ji} > Y_{jH}, j = 1, \ldots, J$ or $\sum_{i=1}^{n} x_{iH} Z_{ki} < Z_{kH}, k = 1, \ldots, K$

Combining the approach of model (1.6) and (1.8), the model can be also adjusted to search for different θ_k and ψ_j for each unit making it a stronger model then the preceding one.

Theorem 9. *Consider the following input-output oriented VRS DEA model for H^{th} unit.*

$$\Delta_H^{VRS\ I-O\ 2} = \min_{\theta_k, \psi_j, x_{iH}} \frac{1}{K+J}\left(\sum_{k=1}^{K} \theta_k + \sum_{j=1}^{J} \frac{1}{\psi_j}\right) \tag{1.9}$$

s. t.

$\sum_{i=1}^{n} x_{iH} Y_{ji} \geq \psi_j \cdot Y_{jH}, \quad j = 1, \ldots, J$

$\sum_{i=1}^{n} x_{iH} Z_{ki} \leq \theta_k \cdot Z_{kH}, \quad k = 1, \ldots, K$

$\sum_{i=1}^{n} x_{iH} = 1, x_{iH} \geq 0, \quad i = 1, \ldots, n$

$0 \leq \theta_k \leq 1, \quad k = 1, \ldots, K$

$\psi_j \geq 1, \quad j = 1, \ldots, J$

Under the same assumptions as are in Theorem 5, if

$$\Delta_H^{VRS\ I-O\ 2} = 1$$

then a unit H is DEA efficient.

Proof. Suppose the following minimum is found by the model for H$^{\text{th}}$ unit.

$$\min_{\theta_k, \psi_j, x_{iH}} \frac{1}{K+J} \Big(\sum_{k=1}^{K} \theta_k + \sum_{j=1}^{J} \frac{1}{\psi_j} \Big) = 1$$

Due to the constrains imposed on θ_k and ψ_j, this may happen only when $\theta_k = 1$, $k = 1, \ldots, K$ and $\psi_j = 1$, $j = 1, \ldots, J$. Then it satisfies Theorem 8 and unit H is DEA efficient. □

1.3 DEA Models with Directional Measure

DEA models with directional measure not only compute which DMUs are efficient, they also compute the 'direction' in which the inefficient ones can be improved. Instead of searching for parameters which represent how many times must input (or output) be improved to achieve efficiency, it searches for parameters representing how much has to be subtracted from an input (or added to an output) to achieve efficiency. These parameters are called directional measure. This type of a model was first introduced in Charnes et al. [1985b]

These models can be input or output as well as input-output oriented. We will show a VRS input-output oriented DEA model with directional measure. We will present it in the version which does not allow different values of θ and ψ for each input and output, but such model could be also formulated. Our goal will be to maximize ψ and minimize θ, which is achieved by the objective function in the following theorem. If unit H is not efficient, then the minimized value of the objective function represents a ratio of improvement in inputs over improvement in outputs which are necessary to achieve efficiency, see Branda [2015].

An advantage of the following model is that negative inputs and outputs may be considered. It is allowed by how the directional measures e_j and d_k are defined - they are always non-negative, for details see Silva Portela et al. [2004].

Theorem 10. *Consider the following VRS input-output oriented DEA model with directional measure.*

$$\Delta_H^{VRS\ DM} = \min_{\theta, x_{iH}, \psi} \frac{1-\theta}{1+\psi} \tag{1.10}$$

s. t.

$\sum_{i=1}^{n} x_{iH} Y_{ji} \geq Y_{jH} + \psi \cdot e_j(H), \quad j = 1, \ldots, J$

$\sum_{i=1}^{n} x_{iH} Z_{ki} \leq Z_{kH} - \theta \cdot d_k(H), \quad k = 1, \ldots, K$

$\sum_{i=1}^{n} x_{iH} = 1, x_{iH} \geq 0, \quad i = 1, \ldots, n$

$\theta \geq 0, \ \psi \geq 0$

$e_j(H) = \max_{i=1,\ldots,n} Y_{ji} - Y_{jH}, \quad j = 1, \ldots, J$

$d_k(H) = Z_{kH} - \min_{i=1,\ldots,n} Z_{ki}, \quad k = 1, \ldots, K$

Suppose:

- *production possibility set includes all convex combinations of all real unit*
- *production possibility set satisfies free disposability*
- *production possibility set does not include anything which is not a consequence of the two properties described above*

Then unit H is efficient if and only if

$$\Delta_H^{VRS\ DM} = 1$$

Proof. Suppose non-negativity of inputs and outputs. Justification for cases when inputs and outputs can be negative can be found in Silva Portela et al. [2004].

$$\min_{\theta, x_{iH}, \psi} \frac{1-\theta}{1+\psi} = 1 \text{ if and only if } \theta = -\psi$$

Since they are both non-negative, $\theta = -\psi$ if and only if $\theta = \psi = 0$. That happens if and only if no such $\mathbf{x} = (x_1, \ldots, x_n)$ exists that $\sum_{i=1}^{n} x_{iH} Y_{ji} > Y_{jH}, j = 1, \ldots, J$ or $\sum_{i=1}^{n} x_{iH} Z_{ki} < Z_{kH}, k = 1, \ldots, K$ According to note following the Theorem 9 that happens if and only if unit H is DEA efficient. □

2. Conditional Value at Risk

It is necessary to be able to measure risk when assessing portfolio efficiency. We will start with a very general definition of risk measure and proceed to one particular measure of risk - Conditional Value at Risk (CVaR).

Following Rockafellar and Uryasev [2000] we use the definition of risk measure below.

Definition 5 (Risk Measure)**.** *Let Ω be a probability space, $\mathcal{F}$ be a σ-algebra on Ω and P be a probability measure. Risk measure is any mapping ρ: $(\Omega, \mathcal{F}, \mathsf{P}) \rightarrow \mathbb{R}$.*

We will use random loss variables in this section. If X is a random variable representing returns, then $-X$ is a random loss variable.

Lamb and Tee [2012] showed that risk measures used as inputs of production units in DEA models, which we will use to asses portfolio efficiency, have to be coherent. It ensures following. By adding to the loss variable, we add to the value of risk measure; the greater the loss is, the greater the value of the risk measure is; sum of random loss variables does not have greater risk then the sum of the risks of random variables; and risk is proportional to the size of investment. The definition of a coherent risk measure was presented in Artzner et al. [1999].

Definition 6 (Coherent Risk Measure)**.** *Let X, Y be random loss variables on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let ρ be a risk measure on $(\Omega, \mathcal{F}, \mathsf{P})$. We say that ρ is a coherent risk measure if it fulfills following properties for a real constant α and non-negative constant β:*

$\rho(X + \alpha) = \rho(X) + \alpha$ *(translation invariance)*
$\rho(X) \leq \rho(Z), whenever X \leq Z$ *(monotonicity)*
$\rho(X + Y) \leq \rho(X) + \rho(Y)$ *(subadditivity)*
$\rho(\beta X) = \beta\rho(X)$ *(positive homogeneity)*

Note that positive homogeneity implies that $\rho(0) = 0$. Positive homogeneity together with subadditivity imply convexity:

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y) \quad \lambda \in (0, 1)$$

If a risk measure is used as an output, we need superadditivity instead of subadditivity, as described in Lamb and Tee [2012]:

$$\rho(X + Y) \geq \rho(X) + \rho(Y)$$

This may be achieved by using negative values of the risk measures.

A widely used measure of risk is Value at Risk. It is measured on an α probability level. We define it according to Rockafellar and Uryasev [2000].

Definition 7 (VaR$_\alpha$)**.** *Let X be a random variable, $\mathbb{E}\,|X| < \infty$. VaR of X at probability level α is defined as an α quantile as follows:*

$$VaR_\alpha(X) = inf\{x \in \mathbb{R} : P(X \leq x) \geq \alpha\}$$

VaR fulfills some properties of coherent risk measures, however, it misses subadditivity. In these cases VaR may discourage diversification inappropriately.

A coherent risk measure is CVAR. Similarly as VaR, it is measured probability level α. We will use its definition presented in Rockafellar and Uryasev [2002]

Definition 8 (CVaR$_\alpha$). *Let X be a random loss variable. CVaR(X) at a probability level α is defined as follows*

$$CVaR_\alpha(X) = \min_{\zeta \in \mathbb{R}} \left\{ \zeta + \frac{1}{1-\alpha} \mathbb{E}[\max(X - \zeta, 0)] \right\}$$

Under the assumption of continuous distribution, CVaR is the expected loss behind VaR in the given time period. See Rockafellar and Uryasev [2002].

$$CVaR_\alpha(X) = \mathbb{E}[X \mid X > VaR_\alpha(X)]$$

Proof of coherence of CVaR can be found in Pflug [2000]. More risk measures are coherent but CVaR is the most widely used and we will use it in the numerical part. Due to that, it will be useful to know more about its properties.
The following expression of CVaR was presented in Rockafellar and Uryasev [2002]

Theorem 11. *Suppose X is a random variable, which takes T equiprobable scenarios x_t, $t = 1, \ldots, T$. Then CVaR(X) can be rewritten as a solution of a following problem:*

$$CVaR_\alpha(X) = \min_{\zeta \in \mathbb{R}} \left\{ \zeta + \frac{1}{(1-\alpha) \cdot T} \sum_{t=1}^{T} w_t \right\}$$

s. t.

$$w_t \geq x_t - \zeta$$
$$w_t \geq 0$$

Proof. The objective function is minimized for minimal ζ and minimal w_t. We know that $w_t \geq \max(x_t - \zeta, 0)$ So the optimal value will be achieved for $w_t = \max(x_t - \zeta, 0)$. Since

$$\mathbb{E}[\max(X - \zeta, 0)] = \frac{1}{T} \sum_{t=1}^{T} \max(x_t - \zeta, 0)$$

we have proved that this expression is equivalent to the definition of CVaR$_\alpha(X)$ □

The following theorem about CVaR computation is presented and proved in Kopa and Chovanec [2008]. It is useful for the computation of CVaR of a given portfolio.

Theorem 12. *Let X be a random loss variable which takes T equiprobable scenarios x_t, $t = 1, \ldots, T$. Let $x^{[k]}$ be the k^{th} smallest element out of $x^1, \ldots, x^T$ For $\alpha \in [\frac{k}{T}, \frac{k+1}{T})$, $\alpha \neq 1$*

$$CVaR_\alpha(X) = x^{[k+1]} + \frac{1}{(1-\alpha) \cdot T} \sum_{t=k+1}^{T} \left(x^{[t]} - x^{[k+1]} \right)$$

for $k = 0, 1, \ldots, T-1$. $CVaR_1(X) = x^{[T]}$

Proof. In Rockafellar and Uryasev [2002] the following is proved: Consider a discrete random loss variable X which takes values $x^t, t = 1, \dots, T$ with probabilities $p_1, p_2, \dots, p_T$. For a chosen α we define $j_\alpha \in \mathbb{N}$ as follows

$$\alpha \in \left[\sum_{j=1}^{j_\alpha - 1} p_j, \sum_{j=1}^{j_\alpha} p_j \right)$$

Then

$$CVaR_\alpha(X) = \frac{1}{1-\alpha} \left[\left(\sum_{j=1}^{j_\alpha} p_j - \alpha \right) x^{[j_\alpha]} + \sum_{j=j_\alpha+1}^{T} p_j x^{[j]} \right].$$

$p_t = \frac{1}{T}$ for every $t = 1, \dots, T$ and we set $j_\alpha = k + 1$. Then

$$\begin{aligned} CVaR_\alpha(X) &= \frac{1}{1-\alpha} \left[\left(\sum_{j=1}^{j_\alpha} p_j - \alpha \right) x^{[j_\alpha]} + \sum_{j=j_\alpha+1}^{T} p_j x^{[j]} \right] \\ &= \frac{1}{1-\alpha} \left[\left(\sum_{j=1}^{k+1} \frac{1}{T} - \alpha \right) x^{[k+1]} + \sum_{j=k+2}^{T} \frac{1}{T} x^{[j]} \right] \\ &= \frac{1}{1-\alpha} \left[\left(\frac{(k+1)}{T} - \frac{\alpha T}{T} \right) x^{[k+1]} + \frac{1}{T} \sum_{j=k+2}^{T} x^{[j]} \right] \\ &= \frac{(k+1) - \alpha T + T - T}{(1-\alpha)T} x^{[k+1]} + \frac{1}{(1-\alpha)T} \sum_{j=k+2}^{T} x^{[j]} \\ &= \frac{(1-\alpha)T}{(1-\alpha)T} x^{[k+1]} - \frac{T-k-1}{(1-\alpha)T} x^{[k+1]} + \frac{1}{(1-\alpha)T} \sum_{j=k+2}^{T} x^{[j]} \\ &= x^{[k+1]} + \frac{1}{(1-\alpha) \cdot T} \sum_{i=k+1}^{T} \left(x^{[i]} - x^{[k+1]} \right) \end{aligned}$$

and so the theorem is proved. □

The following theorem can be formulated more generally than it is here but for the purpose of this work it is sufficient as follows

Corollary. Let X be a random loss variable which takes T equiprobable scenarios x_t, $t = 1, \dots, T$. Let $x^{[k]}$ be the k^{th} smallest element out of $x^1, \dots, x^T$. Then $CVaR_0(X) = \mathbb{E}\, X$

Proof. Using the preceding theorem for $\alpha = 0$ we have

$$\begin{aligned} CVaR_\alpha(X) &= x^{[k+1]} + \frac{1}{(1-\alpha) \cdot T} \sum_{t=k+1}^{T} \left(x^{[t]} - x^{[k+1]} \right) = x^{[1]} + \frac{1}{T} \sum_{t=1}^{T} \left(x^{[t]} - x^{[1]} \right) \\ &= \frac{1}{T} \sum_{t=1}^{T} x^{[t]} = \mathbb{E}\, X \end{aligned}$$

□

3. Diversification Consistent DEA Models

We will now focus on DEA models which can be used to measure the efficiency of stock portfolios. The portfolios will be considered to be our decision making units. Usually, risk measures are used as inputs, and return measures as outputs, see Lamb and Tee [2012]

Using risk measures as inputs of DEA models is complicated because the risk of a combination of portfolios is (usually) not equal to the combination of risks of portfolios. Therefore, we have to use a different definition of DEA efficiency in this section. We suppose that risk and return measures are such that the lowest values of risk measures and highest values of return measures are the most desirable. Following Branda and Kopa [2014] we define DEA risk efficiency.

Definition 9 (DEA Risk Efficiency)**.** *Let R_i, $i = 1, \dots, n$ be random variables representing rates of return of i^{th} portfolio. Let x_i, $i = 1, \dots, n$ be such non-negative coefficients that $\sum_{i=1}^{n} x_i = 1$. Mark $R = \sum_{i=1}^{n} x_i R_i$ Let ρ_k, $k = 1, \dots, K$ be risk measures and $\mathcal{E}_j$, $j = 1, \dots, J$ be return measures. Then a portfolio H is DEA risk efficient, if such non-negative coefficients $y_j, \tilde{y_k}$ exist that*

$$\frac{\sum_{j=1}^{J} y_j \mathcal{E}_j(R_H)}{\sum_{k=1}^{K} \tilde{y_k} \rho_k(R_H)} \geq \frac{\sum_{j=1}^{J} y_j \mathcal{E}_j(R)}{\sum_{k=1}^{K} \tilde{y_k} \rho_k(R)}$$

The risk and return measures are based on the rates of returns which represent a percentage of change which has happened. Therefore, we cannot produce infinitely higher returns by accepting proportionally more risk. So we will use only VRS DEA models.

As briefly sketched above, for coherent risk measures the following is true.

$$\rho(\sum_{i=1}^{n} x_i R_i) \leq \sum_{i=1}^{n} \rho(x_i R_i)$$

Models which deal with this fact are called diversification-consistent (DC). This means that the production possibility set includes all convex combinations of portfolios and free disposability is also kept. However, the production possibility set may be even greater than classical DEA models assume. DC DEA models solve the problem by computing the risk of a linear combination of units instead of computing the convex combination of risks of units.

A model suggested by Lamb and Tee [2012] uses risk measures as inputs and expected return as the only output. Lamb and Tee [2012] presented the model supposing a situation of nonincreasing returns to scale (NIRS). They replaced $\sum_{i=1}^{n} x_i = 1$ by $\sum_{i=1}^{n} x_i \leq 1$. In Branda and Kopa [2012] the VRS convexity condition was used, which is kept here as well.

Theorem 13. *Consider the following DC input-oriented VRS DEA model for H^{th} portfolio:*

$$\Delta_H^{DC\ I} = \min_{\theta, x_i} \theta \tag{3.1}$$

s. t.

$\sum_{i=1}^{n} x_i \, \mathbb{E}\, R_i \geq \mathbb{E}\, R_H$

$\rho_k(\sum_{i=1}^{n} x_i R_i) \leq \theta \cdot \rho_k(R_H), \quad k = 1, \ldots, K$

$\sum_{i=1}^{n} x_i = 1, x_i \geq 0, \quad i = 1, \ldots, n$

Suppose:

- *expected values and risk measures are non-negative and at least one risk measure is positive*
- *production possibility set includes all convex combinations of all real units*
- *production possibility set satisfies free disposability*

Then H^{th} portfolio is DEA risk efficient if and only if

$$\Delta_H^{DC\ I} = 1$$

Proof. This model is a diversification consistent modification of VRS model (1.5). A complete proof can be found in Lamb and Tee [2012]. □

While this model addresses the issue of diversification, it does not deal with the possibility of certain risk measures being negative. This issue can be addressed by setting negative risk measures equal to 0, but that can be misleading as it does not adjust to how far below 0 the value is.

This issue is addressed in a model suggested in Branda and Kopa [2012]. Instead of setting negative values of risk measure equal to 0, we can use their opposite values as outputs. We use the following notation: $\rho_k^+ = \max\{0, \rho_k\}$ and $\rho_k^- = \min\{0, \rho_k\}$

Theorem 14. *Consider the following DC input-output oriented VRS DEA model. Suppose ρ_k is such measure of risk that $\rho_k(R_H) < 0$ for $k = 1, \ldots, J$ and $\rho_k(R_H) > 0$ for $k = J+1, \ldots K$*

$$\Delta_H^{DC\ I-O\ 1} = \min_{\theta, \psi, x_i} \frac{1}{2}(\theta + \frac{1}{\psi}) \tag{3.2}$$

s. t.

$-\rho_k(\sum_{i=1}^{n} x_i R_i) \geq -\psi \cdot \rho_k(R_H), \quad k = 1, \ldots, J$

$\rho_k(\sum_{i=1}^{n} x_i R_i) \leq \theta \cdot \rho_k(R_H), \quad k = J+1, \ldots, K$

$0 \leq \theta \leq 1, \ \psi \geq 1$

$\sum_{i=1}^{n} x_i = 1, x_i \geq 0, \quad i = 1, \ldots, n$

Suppose:

- *production possibility set includes all convex combinations of all real units*
- *production possibility set satisfies free disposability*

Then portfolio H is DEA risk efficient if and only if

$$\Delta_H^{DC\ I-O\ 1} = 1$$

Proof. It is a diversification consistent modification of (1.8), for more details see Branda and Kopa [2012]. □

As showed in Chapter 1, it can be adjusted to search for a different θ_k and ψ_k for each input and output, which results in a stronger model. This model was also introduced in Branda and Kopa [2012].

Theorem 15. *Consider the following DC input-output oriented VRS DEA model. Suppose ρ_k is such measure of risk that $\rho_k(R_H) < 0$ for $k = 1, \dots, J$ and $\rho_k(R_H) > 0$ for $k = J+1, \dots K$*

$$\Delta_H^{DC\ I-O2} = \min_{\theta_k, \psi_k, x_i} \frac{1}{K}\Big(\sum_{k=J+1}^{K} \theta_k + \sum_{k=1}^{J} \frac{1}{\psi_j}\Big) \tag{3.3}$$

s. t.

$$-\rho_k(\textstyle\sum_{i=1}^n x_i R_i) \geq -\psi_k \cdot \rho_k(R_H), \quad k = 1, \dots, J$$
$$\rho_k(\textstyle\sum_{i=1}^n x_i R_i) \leq \theta_k \cdot \rho_k(R_H), \quad k = J+1, \dots, K$$
$$0 \leq \theta_k \leq 1,\ \psi_j \geq 1, \quad k = J+1, \dots, K,\ j = 1, \dots, J$$
$$\textstyle\sum_{i=1}^n x_i = 1, x_i \geq 0, \quad i = 1, \dots, n$$

Under the same assumptions as in Theorem 13, if

$$\Delta_H^{DC\ I-O2} = 1$$

then portfolio H is DEA risk efficient.

Proof. It is a diversification consistent modification of (1.9), for more details see Branda and Kopa [2012]. □

A different approach to dealing with negative input and output values of DC DEA models was presented in Branda [2015]. It is a diversification consistent modification of DEA model with directional measure. In addition to the adjustment that risk and returns of the combination of units are computed, we have to adjust also the way directional measures are computed. When $\mathcal{E}_j$ is maximized (or ρ_k minimized) it has to be over the whole production possibility set. It is so due to the fact that combination of portfolios may have greater (or lower) returns (or risks) than any single portfolio. This model allows using positive as well as negative values of risk and return measures. It enables us to decide in advance what will be the risk and what the return measures.

Theorem 16. *Consider the following DC input-output oriented VRS DEA model with directional measure. Suppose $\mathcal{E}_j$ are return measures and ρ_k are risk measures.*

$$\Delta_H^{DC\ DM} = \min_{\theta, \psi, x_i} \frac{1-\theta}{1+\psi} \tag{3.4}$$

s. t.

$$\mathcal{E}_j(\textstyle\sum_{i=1}^n x_i R_i) \geq \mathcal{E}_j(R_H) + \psi \cdot e_j(H), \quad j = 1, \dots, J$$
$$\rho_k(\textstyle\sum_{i=1}^n x_i R_i) \leq \rho_k(R_H) - \theta \cdot d_k(H), \quad k = 1, \dots, K$$
$$0 \leq \theta,\ 0 \leq \psi$$

$$\sum_{i=1}^{n} x_i = 1, x_i \geq 0, \quad i = 1, \ldots, n$$
$$e_j(H) = \max_{x'_i | \sum_{i=1}^{n} x'_i = 1} \mathcal{E}_j(x'_i R_i) - \mathcal{E}_j(R_H), \quad j = 1, \ldots, J$$
$$d_k(H) = \rho_k(R_H) - \min_{x''_i | \sum_{i=1}^{n} x''_i = 1} \rho_k(x''_i R_i), \quad k = 1, \ldots, K$$

Under the same assumptions as in Theorem 13, a portfolio H is DEA risk efficient if and only if

$$\Delta_H^{DC\ DM} = 1$$

Proof. It is a DC modification of (1.10). A complete proof can be found in Branda [2015]. □

4. Empirical study

To compare the four diversification consistent models presented in the preceding chapter and models presented in the first chapter, we will use them to assess DEA risk efficiency of 49 US representative industry portfolios. We will use data from Kenneth French Library and consider yearly returns from 1970 to 2018 (49 observations).

In (1.3), (1.4), (1.10), (3.1) and (3.4), expected return is used as a return measure. Following Branda and Kopa [2012] CVaRs on levels 0.5, 0.75, 0.9 and 0.95 are used as risk measures. Models (1.3), (1.4) and (3.1) do not allow negative inputs so only the positive parts of values of CVaRs are used. In (1.5), (1.8), (3.2) and (3.3), which were used not distinguishing input and output measures ahead, CVaRs on the same probability levels are used and CVaR_0 is added in order to account for expected return as well (as shown in corollary following Theorem 12). Note that the formulations of models in Chapter 3 use random returns. In order to apply the properties of CVaR presented in the second chapter, we have to use random losses.

The models were solved using the modeling system GAMS. The DEA scores achieved by the nine models are presented in the table below. Values indicating efficiency are in bold.

	CRS I	VRS I	VRS I-O 1	VRS I-O 2	VRS DM	DC I	DC I-O 1	DC I-O 2	DC DM
Model	(1.3)	(1.5)	(1.8)	(1.9)	(1.10)	(3.1)	(3.2)	(3.3)	(3.4)
Agric	0.64	0.75	0.80	0.60	0.27	0.48	0.64	0.40	0.10
Food	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	0.69	0.83	0.54	0.24
Soda	0.60	0.61	0.79	0.56	0.23	0.32	0.66	0.39	0.11
Beer	0.78	0.81	0.90	0.74	0.48	0.44	0.72	0.46	0.17
Smoke	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	0.88	0.94	0.84	0.62
Toys	0.35	0.51	0.60	0.41	0.10	0.32	0.49	0.29	0.05
Fun	0.61	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**
Books	0.41	0.49	0.66	0.46	0.15	0.29	0.54	0.32	0.06
Hshld	0.61	0.82	0.78	0.65	0.30	0.45	0.59	0.40	0.10
Clths	0.60	0.63	0.81	0.55	0.21	0.34	0.67	0.39	0.10
Hlth	0.38	0.38	0.68	0.41	0.11	0.23	0.58	0.31	0.05
MedEq	0.69	0.72	0.84	0.62	0.23	0.47	0.59	0.42	0.12
Drugs	0.98	**1.00**	**1.00**	**1.00**	**1.00**	0.66	0.80	0.52	0.18
Chems	0.80	0.85	0.90	0.68	0.38	0.49	0.70	0.44	0.13
Rubbr	0.61	0.67	0.80	0.60	0.25	0.41	0.64	0.40	0.10
Txtls	0.40	0.43	0.68	0.43	0.13	0.28	0.58	0.32	0.06
BldMt	0.55	0.65	0.75	0.56	0.20	0.40	0.61	0.38	0.08
Cnstr	0.42	0.51	0.66	0.45	0.13	0.35	0.57	0.33	0.06
Steel	0.29	0.47	0.55	0.39	0.11	0.30	0.45	0.27	0.05
FabPr	0.31	0.46	0.57	0.40	0.11	0.32	0.48	0.28	0.05
Mach	0.58	0.66	0.77	0.54	0.20	0.39	0.61	0.37	0.08
ElcEq	0.61	0.62	0.81	0.56	0.24	0.35	0.67	0.39	0.11
Autos	0.41	0.49	0.66	0.43	0.11	0.29	0.54	0.31	0.05
Aero	0.61	0.73	0.87	0.61	0.45	0.51	0.76	0.46	0.30

Ships	0.51	0.54	0.74	0.51	0.19	0.36	0.63	0.37	0.08
Guns	0.82	0.98	0.99	0.76	0.94	0.68	0.84	0.55	0.47
Gold	0.34	0.48	0.59	0.40	0.09	0.33	0.50	0.29	0.04
Mines	0.45	0.48	0.70	0.46	0.13	0.32	0.60	0.34	0.06
Coal	0.36	0.37	0.67	0.40	0.09	0.21	0.60	0.31	0.05
Oil	0.68	0.75	0.83	0.64	0.33	0.49	0.68	0.42	0.12
Util	0.73	0.86	0.85	0.72	0.45	0.57	0.69	0.44	0.14
Telcm	0.57	0.64	0.77	0.57	0.23	0.38	0.62	0.38	0.09
PerSv	0.25	0.45	0.51	0.37	0.10	0.27	0.41	0.25	0.05
BusSv	0.46	0.55	0.70	0.52	0.21	0.36	0.59	0.35	0.08
Hardw	0.40	0.48	0.66	0.44	0.13	0.31	0.56	0.32	0.06
Softw	0.32	0.33	0.67	0.39	0.11	0.20	0.60	0.31	0.05
Chips	0.45	0.45	0.72	0.46	0.15	0.30	0.63	0.35	0.07
LabEq	0.46	0.49	0.71	0.48	0.17	0.34	0.62	0.35	0.07
Paper	0.71	0.83	0.84	0.68	0.35	0.45	0.65	0.42	0.11
Boxes	0.61	0.71	0.78	0.60	0.27	0.44	0.62	0.39	0.10
Trans	0.71	0.80	0.84	0.64	0.27	0.52	0.68	0.43	0.11
Whlsl	0.56	0.65	0.75	0.56	0.23	0.41	0.62	0.38	0.09
Rtail	0.68	0.68	0.84	0.62	0.33	0.42	0.70	0.42	0.13
Meals	0.57	0.57	0.78	0.55	0.23	0.32	0.65	0.38	0.10
Banks	0.54	0.60	0.75	0.53	0.21	0.36	0.61	0.37	0.08
Insur	0.69	0.71	0.84	0.67	0.37	0.39	0.67	0.41	0.13
RlEst	0.20	0.35	0.46	0.31	0.07	0.23	0.38	0.22	0.03
Fin	0.54	0.58	0.79	0.53	0.23	0.34	0.67	0.39	0.12
Other	0.19	0.40	0.43	0.33	0.10	0.27	0.36	0.22	0.04

Table 4.1: Results of DEA models for representative portfolios

All diversification consistent models classified only one portfolio as efficient. In all cases it was the portfolio Fun which was also the one with the highest expected return. It is foreseeable because non of these portfolios is well diversified. Therefore a combination of them will likely surpass any single one of them in minimizing risk. The maximum expected return is the only measure of all input and output measures which cannot be surpassed.

All VRS DEA models (the ones which are not diversification consistent) classified this portfolio as efficient as well, and they classified another three portfolios as efficient, too.

The CRS DEA model (1.3) classified two portfolios as efficient. They are among those classified as efficient by the VRS models. But, neither of them is classified as efficient by DC models. (1.5) is a VRS modification of (1.3) so it is appropriate that scores of (1.5) are always higher than or equal to those of (1.3). The CRS model assessed as efficient portfolios which have the highest return-to-risk ratio with respect to at least one of the risk measures. Portfolio Food achieved highest values of this ratio for CVaR on levels 0.9 and 0.95, portfolio Smoke for CVaR on level 0.75. For CVaR on level 0.5, portfolio Food achieved the highest return-to-risk, however, the ratio was negative in case of portfolio Smoke.

We should note therefore that this ratio is of interest only for non-negative values of risk measures. The model coped with it by not considering CVaR on level 0.5 when assessing efficiency of the portfolio Smoke. The same was done by model (1.5) which could in some cases lead to misleading results.

DEA scores achieved by (3.2) or (1.8) are always higher than or equal to those achieved by (3.3) or (1.9) respectively. It is appropriate because (3.3) or (1.9) are stronger versions of (3.2) or (1.8). Further, the scores of diversification consistent models are always lower then or equal to their VRS counterparts. It is understandable because the combinations, which each portfolio is compared to, are considered less risky by DC models than by the VRS models.

The following table compares DEA scores of DC models depending on the number of levels which CVaR was used on. The first four columns of the table below present scores achieved using CVaRs on the same levels as previously. The last four present results of models which used CVaRs on levels $\frac{1}{49}, \frac{1}{49}, \ldots, \frac{48}{49}$ In case of models (3.2) and (3.3) also CVaR_0 was used. This choice of levels measures CVaR based on all possible numbers of observed values.

	CVaR on 5 levels				CVaR on 49 levels			
Model	(3.1)	(3.2)	(3.3)	(3.4)	(3.1)	(3.2)	(3.3)	(3.4)
Agric	0.48	0.64	0.40	0.10	0.50	0.66	0.26	0.23
Food	0.69	0.83	0.54	0.24	0.75	0.86	0.47	0.50
Soda	0.32	0.66	0.39	0.11	0.39	0.69	0.30	0.24
Beer	0.44	0.72	0.46	0.17	0.46	0.73	0.42	0.36
Smoke	0.88	0.94	0.84	0.62	0.90	0.95	0.76	0.62
Toys	0.32	0.49	0.29	0.05	0.34	0.50	0.15	0.14
Fun	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**
Books	0.29	0.54	0.32	0.06	0.31	0.56	0.23	0.16
Hshld	0.45	0.59	0.40	0.10	0.48	0.62	0.32	0.22
Clths	0.34	0.67	0.39	0.10	0.38	0.69	0.28	0.24
Hlth	0.23	0.58	0.31	0.05	0.26	0.60	0.17	0.16
MedEq	0.47	0.59	0.42	0.12	0.50	0.72	0.31	0.29
Drugs	0.66	0.80	0.52	0.18	0.70	0.83	0.40	0.45
Chems	0.49	0.70	0.44	0.13	0.50	0.70	0.37	0.29
Rubbr	0.41	0.64	0.40	0.10	0.45	0.68	0.32	0.26
Textls	0.28	0.58	0.32	0.06	0.31	0.60	0.20	0.18
BldMt	0.40	0.61	0.38	0.08	0.44	0.64	0.29	0.22
Cnstr	0.35	0.57	0.33	0.06	0.38	0.59	0.17	0.18
Steel	0.30	0.45	0.27	0.05	0.32	0.46	0.14	0.13
FabPr	0.32	0.48	0.28	0.05	0.35	0.50	0.15	0.14
Mach	0.39	0.61	0.37	0.08	0.40	0.63	0.28	0.22
ElcEq	0.35	0.67	0.39	0.11	0.41	0.70	0.35	0.29
Autos	0.29	0.54	0.31	0.05	0.30	0.56	0.17	0.16
Aero	0.51	0.76	0.46	0.30	0.55	0.77	0.37	0.35
Ships	0.36	0.63	0.37	0.08	0.39	0.65	0.27	0.22
Guns	0.68	0.84	0.55	0.47	0.70	0.85	0.46	0.51
Gold	0.33	0.50	0.29	0.04	0.35	0.51	0.11	0.14
Mines	0.32	0.60	0.34	0.06	0.33	0.61	0.19	0.18
Coal	0.21	0.60	0.31	0.05	0.23	0.60	0.15	0.15

Oil	0.49	0.68	0.42	0.12	0.52	0.70	0.35	0.28
Util	0.57	0.69	0.44	0.14	0.61	0.71	0.39	0.26
Telcm	0.38	0.62	0.38	0.09	0.43	0.67	0.31	0.24
PerSv	0.27	0.41	0.25	0.05	0.29	0.43	0.14	0.12
BusSv	0.36	0.59	0.35	0.08	0.41	0.62	0.27	0.20
Hardw	0.31	0.56	0.32	0.06	0.34	0.58	0.17	0.17
Softw	0.20	0.60	0.31	0.05	0.23	0.60	0.16	0.16
Chips	0.30	0.63	0.35	0.07	0.33	0.65	0.22	0.20
LabEq	0.34	0.62	0.35	0.07	0.37	0.64	0.24	0.21
Paper	0.45	0.65	0.42	0.11	0.47	0.67	0.34	0.25
Boxes	0.44	0.62	0.39	0.10	0.49	0.67	0.33	0.25
Trans	0.52	0.68	0.43	0.11	0.57	0.72	0.32	0.30
Whlsl	0.41	0.62	0.38	0.09	0.45	0.65	0.28	0.23
Rtail	0.42	0.70	0.42	0.13	0.46	0.73	0.33	0.31
Meals	0.32	0.65	0.38	0.10	0.36	0.68	0.31	0.23
Banks	0.36	0.61	0.37	0.08	0.39	0.64	0.27	0.22
Insur	0.39	0.67	0.41	0.13	0.41	0.69	0.39	0.28
RlEstate	0.23	0.38	0.22	0.03	0.25	0.40	0.10	0.10
Fin	0.34	0.67	0.39	0.12	0.36	0.68	0.30	0.25
Other	0.27	0.36	0.22	0.04	0.30	0.38	0.11	0.11

Table 4.2: Comparison of DC DEA models using different levels of CVaRs

All models classified the same portfolio as efficient. In case of (3.1), (3.2) and (3.4), DEA scores achieved using CVaRs on 49 levels were always higher than or equal to those achieved using CVaRs on 4 or 5 levels. In case of (3.3), which used different parameters for each CVaR level, the opposite holds.

More trials were run to observe whether all 5 levels CVaR was measured on are necessary and whether the results will change if we omit some of them. CVaR on level 0 was always kept but CVaRs on the other levels were gradually omitted - leaving three, two and just one of them. We tried all 14 combination of levels of CVaR possible. Models (3.1), (3.2) and (3.4) were used. (3.3) was omitted because it would not classify as efficient anything which was not classified so by (3.2). The portfolio Fun was always classified as efficient by all three of the models, and it was the only one classified so. Models (3.1) and (3.2) were not feasible for portfolio Smoke and CVaR on level 0.5 only because it was negative which resulted in an inability of these models to use any input measure. It seems that 49 observed portfolios are too many for a single one to surpass a well chosen combination of them in minimizing CVaR on any of those 4 levels. If less investment opportunities or better diversified ones were compared, more portfolios may be classified as efficient by the DC models.

Conclusion

In this work, we defined DEA efficiency for the classical models and for the diversification consistent ones. We described the original CRS DEA model and proved that it leads to finding the efficient units. We showed its output-oriented modification and described how it follows from the original model. Several VRS DEA models including one model with directional measure were then described. We continued by discussing risk measures and presenting Conditional Value at Risk, which is used as a risk measure in the following diversification consistent models. In this work, four diversification consistent models were presented. They are modifications of models shown previously, which consider diversification of risk.

The theoretical part is followed by an empirical part where the four DC models as well as their classical counterparts and the CRS input oriented model were used to assess efficiency of US representative portfolios. The DC models are the strongest and they evaluated only one portfolio as efficient. It was the one with the greatest expected value. They failed to identify portfolios with lowest risk because a combination of more portfolios always had lower risk than any one of them separately. The classical VRS models and VRS with directional measure assessed four portfolios as efficient. These models apply better to a situation where investors can only choose one from a set of given portfolios. In such case, it is not appropriate to compare each portfolio to the risk of a combination of all the portfolios because combining them would not be a possibility for the investor. The CRS input oriented model classified two portfolios as efficient. These portfolios have the highest return-to-risk ratio. However, the CRS input oriented model (as well as its direct VRS modification) does not cope well with the possibility of negative input measures. The results received using DC models with CVaR on 5 levels were similar to those received using CVaR on 49 levels. They identified the same portfolios as efficient.

The VRS models provide interesting information about the portfolios by determining which portfolios are in some way the best. The information provided by CRS model is also noteworthy but it can be discovered using significantly simpler methods. It may be interesting to apply the diversification consistent models to a different type of portfolios which are better diversified. In case of the DC models, it may be more interesting to observe the values of $\mathbf{x} = (x_1, \ldots, x_{49})$ which represent the weights of each portfolio in the best convex combination found by the model.

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