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Love-Young Inequality and Its Consequences

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I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Abstract: This thesis is focused on proving the Love-Young inequality and clarifying the manner in which it relates to a fractional Brownian motion. To begin with, several estimates alongside the concept of p-variation of a function are presented. The connection between functions of finite p-variation and regulated functions is then highlighted and used to prove the aforementioned Love-Young inequality. Deficiency of the pathwise approach to stochastic integration is recognised and later discussed amongst the properties of fractional Brownian motions. This constitutes the main application of the featured theory which is the integration with respect to irregular functions.

Keywords: Love-Young inequality, Riemann-Stieltjes integral, p-variation, Wiener process, fractional Brownian motion

Contents

Introduction

In 1936, in his innovative paper, Young [14] proved the existence of the Riemann-Stieltjes integral where the integrator need no longer be of finite variation. He further proved an inequality which, as it has turned out, seems to be of quite the importance and is now referred to as the Love-Young inequality. It should be noted, however, that the original proof of the inequality was obtained by E. R. Love and Young devised its more refined version. It could be said that this paper layed the groundwork on which the pathwise stochastic integration theory stands. A few applications, e.g. theorem on term by term integration and theorem on mean variations of Fourier kernels were demonstrated in Young [14]. But in any event, the paramount application of his paper, which is the integration with respect to stochastic processes, was not addressed. This idea was acknowledged and scrutinized in the later books, such as Lyons et al. [7]. Here the authors provide an introduction to the theory of rough paths which can be made use of in the pathwise approach to stochastic integration (besides other areas of mathematics).

The main goal of this thesis is to give a rigorous and detailed proof of the Love-Young inequality based on the original paper Young [14] and to demonstrate how it can be used to define integrals with respect to stochastic processes and, in particular, with respect to fractional Brownian motions.

We begin with a chapter presenting a heuristic approach to differential equations driven by irregular functions. It also serves as an incentive for integration where the integrator does not have to meet as strict a condition as is the finiteness of total variation.

The second chapter is a treatise on the Love-Young inequality and it entails four subchapters. One ought to pay special attention to Theorem 2.10 in the first subchapter providing the upper bound for the difference of two Riemann sums and to Lemma 2.21 in the second subchapter. This lemma yields the crucial estimate including oscillation of a function. Both of these, amongst the lemmata presented in the subchapter 2.3, will be helpful when proving Theorem 2.30. This particular theorem is what we are striving for. The last section displays the possible shortcomings and restrictions of the Riemann-Stieltjes integration.

The last chapter is focused on the fractional Brownian motion and its properties. We use the Kolmogorov continuity theorem (Theorem 3.5) to prove the finiteness of p-variation of almost all trajectories of fractional Brownian motions. The existence of the Riemann-Stieltjes integral of its trajectories is then explored. Finally, the last section introduces the concept of iterated integrals where the integrator is only of finite p-variation and states an interesting result concerning the existence of solutions to differential equations.

This thesis is of compilatory character. In Motivation, the ideas from the booklet Maslowski et al. [9] have been used. The proof given in Young [14] is presented in the second chapter. We elaborate its steps (see e.g. Lemma 2.14, equalities before Theorem 2.10, Propositions 2.16 and 2.20, Theorem 2.30, etc.), fill all the missing parts (mainly the theory of regulated functions) and generalise some of the conclusions to inner product spaces. The results in this chapter have been taken from Young [14] unless stated otherwise. A few properties of the fractional Brownian motion which are of particular interest are featured in more detail than how they appeared in Nualart [10], Guerra and Nualart [5] and Nualart [11] from where they have been taken. At last, we refer the reader to the monograph Lyons et al. [7] containing a distinct proof of the Love-Young inequality which as well serves as an introduction to the theory of rough paths. The featured results on differential equations and iterated integrals can be found in Lyons et al. [7] and in the seminal article Lyons [8].

1. Motivation

In this chapter, we establish the basic idea behind differential equations driven by irregular functions. Two commonly used approaches to stochastic integration are assessed and one of them is elaborated upon.

Differential equations are of use in many areas since they can model time evolution of a physical system. As is customary, in what follows, we consider the interval $[0, T]$, $T > 0$. Let us consider an example which, alongside others, can be found in Øksendal [13]. It is the ordinary differential equation describing population growth which takes the form

$$\frac{dY(t)}{dt} = \alpha Y(t), \qquad Y(0) = Y_0, \tag{1.1}$$

where $t \in [0, T]$, Y_0 and α are given real numbers. A well-known fact is that the solution to this equation on the interval $[0, T]$ is given by $Y(t) = Y_0 \mathrm{e}^{\alpha t}$. However, rarely can the reality be modelled so simply. Assume that α is not a constant but that it depends on various external effects that cannot always be predicted beforehand. A simple case would be that α is a function of the form $\alpha(t) = \beta(t) +$ "noise", $t \in [0, T]$, where the function β is known but the "noise" term is not. We may know only some of its properties, e.g. its probability distribution. How shall we continue? Let us simplify the situation by considering a discrete time model. Let $\mathcal{D} = \{t_i\}_{i=0}^n$ be a partition of $[0, T]$ and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous (and therefore measurable) function for which we have

$$Y(t_i) - Y(t_{i-1}) = g(Y(t_{i-1}))(t_i - t_{i-1}), \qquad Y(0) = Y_0, \tag{1.2}$$

where $i = 1, 2, \ldots, n$ and Y_0 is given. The solution to equation (1.2) is clearly given by

$$Y(t_j) = Y_0 + \sum_{i=1}^{j} g(Y(t_{i-1}))(t_i - t_{i-1}), \tag{1.3}$$

where $j = 1, 2, \ldots, n$. Now, were we to send the mesh $|\mathcal{D}|$ to 0 (see Definition 2.9), we would have that (1.3) takes the form of the following differential equation in the integral form

$$Y(t) = Y_0 + \int_0^t g(Y(s))\, ds \tag{1.4}$$

where $t \in [0, T]$. The integral in (1.4) can be thought of as a Riemann, or, more generally, a Lebesgue integral.

Let us consider a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which the stochastic processes will be defined. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $\{X_{t_i}, i = 0, 1, \ldots, n\}$ is a random vector. We might introduce the above-mentioned "noise" as $f(Y(t_{i-1}))\, X(t_{i-1}, \omega)(t_i - t_{i-1})$ which is to be added to the right-hand side of equation (1.2). However, this approach carries an unfortunate consequence. If we now try to pass to a continuous time model as above, after imposing sensible requirements on $\{X_t, t \geq 0\}$, we still cannot guarantee its measurability. This technique is explained in more detail in Maslowski et al. [9]. In order to overcome the problem with measurability, consider the noise written as an increment of a process, rather than a process itself. This means that we obtain the equation

$$\begin{aligned} Y(t_i,\omega) - Y(t_{i-1},\omega) &= g(Y(t_{i-1},\omega))(t_i - t_{i-1}) \\ &\quad + f(Y(t_{i-1},\omega))(X(t_i,\omega) - X(t_{i-1},\omega)) \\ Y(0) &= Y_0, \end{aligned} \tag{1.5}$$

where $i = 1, 2, \ldots, n$. The dependence of Y on ω is due to the dependence of $Y(t_j)$ on $X(t_i,\omega)$, $i = 0, 1, \ldots, j$, which is obvious by the following identity:

$$\begin{aligned} Y(t_j,\omega) &= Y_0 + \sum_{i=1}^{j} g(Y(t_{i-1},\omega))(t_i - t_{i-1}) \\ &\quad + \sum_{i=1}^{j} f(Y(t_{i-1},\omega))(X(t_i,\omega) - X(t_{i-1},\omega)), \\ Y(0) &= Y_0, \end{aligned}$$

where $j = 1, 2, \ldots, n$. We thus regard Y as a function of time sooner than a random vector.

We will have some demands for the process X. We will assume that X is a fractional Brownian motion. Chapter 3 serves as an introduction to it. The improvement of the situation is principal since the fractional Brownian motion exists as a measurable process. For its construction, see Nualart [11].

Let us proceed from a discrete time model (1.5) to a continuous one. We would obtain

$$Y(t,\omega) = Y_0 + \int_0^t g(Y(s,\omega))\,ds + \int_0^t f(Y(s,\omega))\,dX(s,\omega), \tag{1.6}$$

for $t \in [0,T]$. The first integral on the right-hand side of equation (1.6) is a Lebesgue integral as previously stated and can be manipulated in the usual manner. On the right-hand side of equation (1.6) we can also see a "derivative" of a fractional Brownian motion. This augurs foreboding problems since almost all trajectories of fractional Brownian motions are nowhere differentiable. But not necessarily does the process X need to have differentiable trajectories. One must give meaning to the second integral in equation (1.6) though.

For a real-valued stochastic process $\{X_t, t \geq 0\}$ two approaches can be considered. One may fix $t \in [0,\infty)$ and obtain a random variable:

$$X_t : \omega \mapsto X_t(\omega), \qquad \omega \in \Omega$$

or, fix $\omega \in \Omega$ and obtain a function

$$X_\omega : t \mapsto X_t(\omega), \qquad t \in [0,\infty). \tag{1.7}$$

The function (1.7) is said to be a sample path, or a trajectory, of the stochastic process X. One may find an intuition behind these outlooks again in Øksendal [13]. We may already observe that it will be possible to approach the integral

$$\int_0^t f(Y(s,\omega))\,dX(s,\omega) \tag{1.8}$$

from two perspectives. On the one hand, the pathwise integration can be considered, that is the existence of the integral $\int_0^t f(Y_s(\omega))\, dX_s(\omega)$ where $\omega \in \Omega$ is given. This scenario involves pointwise convergence of Riemann sums. On the other hand, one can turn to classical stochastic integration (mainly the Itô integral) which entails convergence in probability or convergence in mean square. In this thesis, we shall approach the integral from the pathwise perspective. The stochastic approach will be very briefly mentioned in chapter 3.

The integral (1.8) can be defined pathwise if X, Y and f meet certain regularity conditions. These shall be addressed precisely later, see subchapter 3.3 and subchapter 3.4. The approach from the pathwise perspective allows to give meaning to integrals that cannot be defined in the classical measure theory. Notwithstanding, it carries an underlying downside. Brownian motion B is a fundamental continuous time stochastic process, yet as it is to be seen, its trajectories are not regular enough and the integral $\int_0^T B_t(\omega)\, dB_t(\omega)$ cannot be defined pathwise. The application of this particular integral will be reviewed in chapter 3. Consider the differential equation of the form

$$dY_t = f(Y_t)\, dX_t, \qquad Y(0) = Y_0. \tag{1.9}$$

The functions X, Y and f are said to satisfy equation (1.9) when the integral equation

$$Y_t = Y_0 + \int_0^t f(Y_s)\, dX_s, \qquad t \in [0, T]$$

holds.

We say that X is the control or driving noise, Y_0 is the initial condition and Y is the solution (driven by X). If f is a Lipschitz continuous function and X is of finite variation, equation (1.9) has a unique solution by a well-known theorem (see Lyons et al. [7]). The existence of a solution without its uniqueness is procured as soon as f is continuous. But what happens if X is so irregular that such smoothness cannot be attained? In the following sections, questions such as this will be answered.

2. Love-Young inequality

One usually regards two proofs when considering Love-Young inequality. The original proof was given by Young in 1936. It is quite simple, albeit a tad intricate. The second can be found in Lyons et al. [7]. It makes use of control functions and uses more abstract mathematical objects to obtain the similar result in a more general setting. This chapter is based on the proof of the inequality given by Young [14] and uses its definitions, theorems etc. but presents the proofs and steps in more detail. To begin with, we make some prefatory estimates and introduce the concept of p-variation alongside some of its properties. We then use the theory of regulated functions, relate them to the functions of finite p-variation and obtain a result about the oscillation of a function. Consequently, we prove the Love-Young inequality and give a counterexample to the case when the assumptions for the inequality to hold are not satisfied.

2.1 Preliminary estimates

From now on, we shall consider an inner product space $(V, \langle\cdot,\cdot\rangle)$. We open with a definition of a well-known function that will appear throughout the whole chapter.

Definition 2.1. *For $s \in \mathbb{C}$, $Re(s) > 1$, the Riemann zeta function is defined by the relation*

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

In what follows, the norm is considered to be induced by the dot product $\langle\cdot,\cdot\rangle$ in V.

Lemma 2.2. *Let $n \in \mathbb{N}$ and let $p, q \geq 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} > 1$. For a_i, $b_i \in V$, $i = 1, 2, \ldots, n$, it holds that*

$$\sum_{i=1}^{n} |\langle a_i, b_i\rangle| \leq \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\left(\sum_{i=1}^{n} \|a_i\|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} \|b_i\|^q\right)^{\frac{1}{q}}.$$

Proof. We consider the following sequence of real numbers $|\langle a_i, b_i\rangle|$ arranged in order of decreasing size, that is $n = \operatorname{argmin}_i |\langle a_i, b_i\rangle|$, $i = 1, 2, \ldots, n$. Thus, from the Cauchy-Schwarz inequality and the inequality of arithmetic and geometric means we obtain that

$$\begin{aligned}
|\langle a_n, b_n\rangle| &\leq \left|\prod_{i=1}^{n} \langle a_i, b_i\rangle\right|^{\frac{1}{n}} \\
&\leq \left(\left(\prod_{i=1}^{n} \|a_i\|^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\left(\left(\prod_{i=1}^{n} \|b_i\|^q\right)^{\frac{1}{n}}\right)^{\frac{1}{q}} \\
&\leq \left(\frac{1}{n}\sum_{i=1}^{n} \|a_i\|^p\right)^{\frac{1}{p}}\left(\frac{1}{n}\sum_{i=1}^{n} \|b_i\|^q\right)^{\frac{1}{q}} \\
&= n^{-\left(\frac{1}{p}+\frac{1}{q}\right)}\left(\sum_{i=1}^{n} \|a_i\|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} \|b_i\|^q\right)^{\frac{1}{q}}
\end{aligned}$$

holds. Correspondingly, we get

$$
\begin{aligned}
|\langle a_{n-1}, b_{n-1}\rangle| &\le (n-1)^{-\left(\frac{1}{p}+\frac{1}{q}\right)} \left(\sum_{i=1}^{n-1} \|a_i\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n-1} \|b_i\|^q\right)^{\frac{1}{q}} \\
&\le (n-1)^{-\left(\frac{1}{p}+\frac{1}{q}\right)} \left(\sum_{i=1}^{n} \|a_i\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \|b_i\|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

We continue in this fashion to acquire

$$
\begin{aligned}
\sum_{i=1}^{n} |\langle a_i, b_i\rangle| &\le \left(\sum_{i=1}^{n} i^{-\left(\frac{1}{p}+\frac{1}{q}\right)}\right) \left(\sum_{i=1}^{n} \|a_i\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \|b_i\|^q\right)^{\frac{1}{q}} \\
&\le \zeta\left(\frac{1}{p}+\frac{1}{q}\right) \left(\sum_{i=1}^{n} \|a_i\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \|b_i\|^q\right)^{\frac{1}{q}}
\end{aligned}
$$

which concludes the proof. □

Remark 2.3. *Under the assumptions of Lemma 2.2, there exists $k \in \{1, \dots, n\}$ such that the following inequality holds*

$$
|\langle a_k, b_k\rangle| \le \left(\frac{1}{n}\sum_{i=1}^{n} \|a_i\|^p\right)^{\frac{1}{p}} \left(\frac{1}{n}\sum_{i=1}^{n} \|b_i\|^q\right)^{\frac{1}{q}}.
$$

Proof. Take e.g. $k = \operatorname{argmin}_i |\langle a_i, b_i\rangle|$, $i = 1, 2, \dots, n$. □

Corollary 2.4. *Let a_i, b_i, $i = 1, 2, \dots, n$, be non-negative real numbers. Then, the inequality*

$$
\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}
$$

is satisfied for every $p, q \ge 1$ such that $\frac{1}{p} + \frac{1}{q} \ge 1$.

Proof. The case $\frac{1}{p} + \frac{1}{q} = 1$ reduces to the Hölder inequality. Otherwise, we first observe that

$$
\sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_i a_j b_j = \left(\sum_{i=1}^{n} a_i b_i\right)^2 \tag{2.1}
$$

holds. By Lemma 2.2, we can estimate

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_i a_j b_j &\le \zeta\left(\frac{1}{p}+\frac{1}{q}\right) \left(\sum_{i=1}^{n}\sum_{j=1}^{n} (a_i a_j)^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n}\sum_{j=1}^{n} (b_i b_j)^q\right)^{\frac{1}{q}} \\
&\le \zeta\left(\frac{1}{p}+\frac{1}{q}\right) \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{2}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{2}{q}}.
\end{aligned}
$$

The second inequality follows by (2.1). Hence, it is clear that

$$
\sum_{i=1}^{n} a_i b_i \le \zeta\left(\frac{1}{p}+\frac{1}{q}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}
$$

holds. By induction, one obtains for $m \in \mathbb{N}$:

$$\sum_{i=1}^{n} a_i b_i \leq \zeta \left(\frac{1}{p} + \frac{1}{q} \right)^{\frac{1}{m}} \left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}.$$

The result follows by letting $m \to \infty$.

$\square$

We continue with the definition of a P-partition and use it as a springboard for Lemma 2.8.

Definition 2.5. *Let $n \in \mathbb{N}$, $a_i \in V$, $i = 1, 2, \ldots, n$. We say that for $m \in \mathbb{N}$, $m \leq n$, the set $\{x_1, x_2 \ldots, x_m\}$ is a P-partition of the set $\{a_1, a_2, \ldots, a_n\}$ if there exist $j_0, j_1, \ldots, j_m \in \{0, 1, \ldots, n\}$, $j_0 = 0$, $j_m = n$ such that $x_k = \sum\limits_{i=j_{k-1}+1}^{j_k} a_i$, $k = 1, 2, \ldots, m$.*

Remark 2.6. *One can regard the P-partition as a set of elements derived in such way that its elements are successive sums of the original set.*

Let $a = \{a_1, a_2 \ldots, a_n\}$, $b = \{b_1, b_2 \ldots, b_n\}$ be sets in V. For $n \in \mathbb{N}$ and real numbers $p, q \geq 1$, we denote

$$S_{p,q}(a, b) = \max_{P-partitions} \left\{ \left(\sum_{i=1}^{m_1} \|x_i\|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{m_2} \|y_i\|^q \right)^{\frac{1}{q}} \right\}$$

where $\{x_1, x_2, \ldots, x_{m_1}\}$ and $\{y_1, y_2, \ldots, y_{m_2}\}$ are P-partitions of $\{a_1, a_2 \ldots, a_n\}$ and $\{b_1, b_2 \ldots, b_n\}$, respectively.

Remark 2.7. *For our purposes, it is enough to consider P-partitions with the same number of elements, that is $m_1 = m_2$.*

Lemma 2.8. *Let $p, q \geq 1$ be real numbers satisfying $\frac{1}{p} + \frac{1}{q} > 1$. Assume that $a = \{a_1, a_2 \ldots, a_n\}$, $b = \{b_1, b_2 \ldots, b_n\}$ are sets in V. Then*

$$\left| \sum_{s=1}^{n} \sum_{r=1}^{s} \langle a_r, b_s \rangle \right| \leq \left(1 + \zeta \left(\frac{1}{p} + \frac{1}{q} \right) \right) S_{p,q}(a, b). \tag{2.2}$$

Proof. Consider the following P-partitions of a and b:

$$x_r = \begin{cases} a_r, & 1 \leq r < k, \\ a_r + a_{r+1}, & r = k, \\ a_{r+1}, & k < r < n, \end{cases} \qquad y_r = \begin{cases} b_r, & 1 \leq r < k, \\ b_r + b_{r+1}, & r = k, \\ b_{r+1}, & k < r < n, \end{cases}$$

for $k \in \{1, 2, \ldots, n-1\}$. Let us compute

$$\begin{aligned}
\sum_{s=1}^{n} \left\langle \sum_{i=1}^{s} a_i, b_s \right\rangle &= \sum_{s=1}^{k-1} \left\langle \sum_{i=1}^{s} a_i, b_s \right\rangle + \left\langle \sum_{i=1}^{k} a_i, b_k \right\rangle + \left\langle \sum_{i=1}^{k+1} a_i, b_{k+1} \right\rangle \\
&\quad + \sum_{s=k+1}^{n-1} \left\langle \sum_{i=1}^{s+1} a_i, b_{s+1} \right\rangle \\
&= \sum_{s=1}^{k-1} \left\langle \sum_{i=1}^{s} a_i, b_s \right\rangle + \left\langle \sum_{i=1}^{k+1} a_i, b_k + b_{k+1} \right\rangle - \langle a_{k+1}, b_k \rangle \\
&\quad + \sum_{s=k+1}^{n-1} \left\langle \sum_{i=1}^{s+1} a_i, b_{s+1} \right\rangle \\
&= \sum_{s=1}^{n-1} \left\langle \sum_{i=1}^{s} x_i, y_s \right\rangle - \langle a_{k+1}, b_k \rangle .
\end{aligned}$$

By Remark 2.3, there exists $k \in \{1, 2, \ldots, n-1\}$ such that

$$\begin{aligned}
|\langle a_{k+1}, b_k \rangle| &\leq \left(\frac{1}{n-1} \sum_{i=1}^{n-1} \|a_{i+1}\|^p \right)^{\frac{1}{p}} \left(\frac{1}{n-1} \sum_{i=1}^{n-1} \|b_i\|^q \right)^{\frac{1}{q}} \\
&\leq (n-1)^{-\left(\frac{1}{p}+\frac{1}{q}\right)} S_{p,q}(a, b) .
\end{aligned}$$

Therefore, with this particular k the following estimates hold:

$$\begin{aligned}
\left| \sum_{s=1}^{n} \sum_{i=1}^{s} \langle a_i, b_s \rangle \right| &\leq |\langle a_{k+1}, b_k \rangle| + \left| \sum_{s=1}^{n-1} \left\langle \sum_{i=1}^{s} x_i, y_s \right\rangle \right| \\
&\leq (n-1)^{-\left(\frac{1}{p}+\frac{1}{q}\right)} S_{p,q}(a, b) + \left| \sum_{s=1}^{n-1} \left\langle \sum_{i=1}^{s} x_i, y_s \right\rangle \right| .
\end{aligned}$$

Further, the inequality $S_{p,q}(x, y) \leq S_{p,q}(a, b)$ holds by definition. In the same manner, we can estimate the element

$$\left| \sum_{s=1}^{n-1} \left\langle \sum_{i=1}^{s} x_i, y_s \right\rangle \right| \leq (n-2)^{-\left(\frac{1}{p}+\frac{1}{q}\right)} S_{p,q}(a, b) + \left| \sum_{s=1}^{n-2} \left\langle \sum_{i=1}^{s} u_i, v_s \right\rangle \right| ,$$

where $\{u_1, u_2, \ldots, u_{n-2}\}$ and $\{v_1, v_2, \ldots, v_{n-2}\}$ are P-partitions of the sets $x = \{x_1, x_2, \ldots, x_{n-1}\}$ and $y = \{y_1, y_2, \ldots, y_{n-1}\}$, respectively. By induction, it is proved that the inequality

$$\left| \sum_{s=1}^{n} \sum_{i=1}^{s} \langle a_i, b_s \rangle \right| \leq \left(1 + \sum_{i=1}^{n-1} i^{-\left(\frac{1}{p}+\frac{1}{q}\right)} \right) S_{p,q}(a, b)$$

holds and the result follows. □

Definition 2.9. *Let $[a, b] \subset \mathbb{R}$ be a non-degenerate bounded interval. A partition is a finite set of real numbers $\mathcal{D} = \{t_0, t_1, \ldots, t_n\}$ satisfying $a = t_0 < t_1 < \ldots < t_n = b$. The mesh of the partition $\mathcal{D}$ equals $\max_{i=1,2,\ldots,n} \{t_i - t_{i-1}\}$ and it is denoted by $|\mathcal{D}|$. Partition $\mathcal{F}$ is a refinement of the partition $\mathcal{E}$ if all the points from $\mathcal{E}$ are included amongst the points of $\mathcal{F}$.*

From now on, anytime an interval in $\mathbb{R}$ is being considered, it is assumed to be bounded, closed and non-degenerate unless stated otherwise.

Let $n \in \mathbb{N}$, let $\mathcal{D} = \{t_j\}_{j=0}^{n}$ be a partition of $[a,b]$ and further assume that $X, Y : \mathbb{R} \to V$ are functions defined at least on $[a,b]$. Let us denote

$$F_{\mathcal{D}} = \sum_{s=1}^{n} \langle Y(t_s), X(t_s) - X(t_{s-1}) \rangle,$$

$$S_{p,q}(a,b;X,Y) = \\ = \sup_{\mathcal{D}} \left\{ \left(\sum_{i=1}^{n} \|X(t_i) - X(t_{i-1})\|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \|Y(t_i) - Y(t_{i-1})\|^q \right)^{\frac{1}{q}} \right\},$$

where $p, q \geq 1$ and the supremum is taken over all partitions of $[a,b]$.

The following equalities hold:

$$F_{\mathcal{D}} = \sum_{s=1}^{n} \sum_{r=1}^{s} \langle Y(t_r) - Y(t_{r-1}), X(t_s) - X(t_{s-1}) \rangle + \langle Y(a), X(b) - X(a) \rangle \tag{2.3}$$

$$= \sum_{s=1}^{n} \sum_{r=s+1}^{n} \langle Y(t_{r-1}) - Y(t_r), X(t_s) - X(t_{s-1}) \rangle + \langle Y(b), X(b) - X(a) \rangle. \tag{2.4}$$

Should we choose an arbitrary division point t_j, $j \in \{1, 2, \ldots, n-1\}$, we will immediately obtain:

$$\begin{aligned} F_{\mathcal{D}} &= \sum_{s=1}^{j} \langle Y(t_s), X(t_s) - X(t_{s-1}) \rangle + \sum_{s=j+1}^{n} \langle Y(t_s), X(t_s) - X(t_{s-1}) \rangle \\ &= \sum_{s=1}^{j} \sum_{r=s+1}^{j} \langle Y(t_{r-1}) - Y(t_r), X(t_s) - X(t_{s-1}) \rangle \\ &\quad + \sum_{s=j+1}^{n} \sum_{r=j+1}^{s} \langle Y(t_r) - Y(t_{r-1}), X(t_s) - X(t_{s-1}) \rangle \\ &\quad + \langle Y(t_j), X(b) - X(a) \rangle. \end{aligned} \tag{2.5}$$

The following theorem provides a foundation for the Love-Young inequality.

Theorem 2.10. *Let $n, m \in \mathbb{N}$, $[a,b] \subset \mathbb{R}$, let $\mathcal{D}_1 = \{t_i^1\}_{i=0}^{n}$, $\mathcal{D}_2 = \{t_i^2\}_{i=0}^{m}$ be partitions of the interval $[a,b]$. Assume that ξ_r^1 are such that $\xi_r^1 \in \left[t_{r-1}^1, t_r^1\right]$, $r = 1, 2, \ldots, n$, and ξ_r^2 are such that $\xi_r^2 \in \left[t_{r-1}^2, t_r^2\right]$, $r = 1, 2, \ldots, m$. Let $X, Y : [a,b] \to V$ be functions and let $p, q \geq 1$ be real numbers that satisfy $\frac{1}{p} + \frac{1}{q} > 1$. Then, we have that the inequality*

$$\left| \sum_{r=1}^{n} \left\langle Y\left(\xi_r^1\right), X\left(t_r^1\right) - X\left(t_{r-1}^1\right) \right\rangle - \sum_{s=1}^{m} \left\langle Y\left(\xi_s^2\right), X\left(t_s^2\right) - X\left(t_{s-1}^2\right) \right\rangle \right| \leq \\ \leq 2\left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right) \left(\sum_{i=1}^{n} S_{p,q}\left(t_{i-1}^1, t_i^1; X, Y\right) + \sum_{i=1}^{m} S_{p,q}\left(t_{i-1}^2, t_i^2; X, Y\right) \right) \tag{2.6}$$

holds.

Proof. Consider a refinement of the partitions $\mathcal{D}_1, \mathcal{D}_2$ denoted by $\mathcal{D}$ such that it contains all ξ_r^1, $r = 1, 2, \ldots, n$ and all ξ_r^2, $r = 1, 2, \ldots, m$. We first show that for any point $\xi \in \mathcal{D}$ it holds that

$$|F_{\mathcal{D}} - \langle Y(\xi), X(b) - X(a)\rangle| \leq 2\left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right) S_{p,q}(a, b; X, Y). \tag{2.7}$$

If ξ is equal either a or b, the inequality (2.7) is obtained directly by equalities (2.3), (2.4) and further by applying the inequality (2.2) on $a_r = Y(t_r) - Y(t_{r-1})$, $b_s = X(t_s) - X(t_{s-1})$ in the former situation, and on $a_r = Y(t_{r-1}) - Y(t_r)$, $b_s = X(t_s) - X(t_{s-1})$ in the latter. If ξ is not an end point, by equality (2.5) we get

$$\begin{aligned}
|F_{\mathcal{D}} - \langle Y(\xi), X(b) - X(a)\rangle| &\leq \\
&\leq \left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right)\big(S_{p,q}(a, \xi; X, Y) + S_{p,q}(\xi, b; X, Y)\big) \\
&\leq 2\left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right) S_{p,q}(a, b; X, Y).
\end{aligned}$$

Should the above inequality be applied to intervals $\left[t_{r-1}^1, t_r^1\right]$, we will have

$$\begin{aligned}
&\left|F_{\mathcal{D}} - \sum_{i=1}^{n}\left\langle Y\left(\xi_i^1\right), X\left(t_i^1\right) - X\left(t_{i-1}^1\right)\right\rangle\right| \leq \\
&\qquad\qquad \leq 2\left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right)\sum_{i=1}^{n} S_{p,q}\left(t_{i-1}^1, t_i^1; X, Y\right)
\end{aligned}$$

by triangle inequality. Analogous result can be obtained for the partition $\mathcal{D}_2$. Then the left-hand side of inequality (2.6) is less than or equal to the sum

$$\left|F_{\mathcal{D}} - \sum_{i=1}^{n}\left\langle Y\left(\xi_i^1\right), X\left(t_i^1\right) - X\left(t_{i-1}^1\right)\right\rangle\right| + \left|F_{\mathcal{D}} - \sum_{i=1}^{m}\left\langle Y\left(\xi_i^2\right), X\left(t_i^2\right) - X\left(t_{i-1}^2\right)\right\rangle\right|,$$

which is bounded by

$$2\left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right)\left(\sum_{i=1}^{n} S_{p,q}\left(t_{i-1}^1, t_i^1; X, Y\right) + \sum_{i=1}^{m} S_{p,q}\left(t_{i-1}^2, t_i^2; X, Y\right)\right).$$

This proves the claim. □

2.2 P-variation

Notice that the quantity $S_{p,q}(a, b; X, Y)$ need not be finite. The proved inequalities would hence be relinquished of any meaning. In what follows, we introduce the concept of p-variation of a function to avoid such issues. The p-variation is defined as in Lyons et al. [7].

Definition 2.11. *Let $p \geq 1$ be a real number and let I be an interval. Assume that $X : I \to V$ is a function. We define the p-variation of X on I by the relation*

$$\|X\|_{p,I} = \left(\sup_{\mathcal{D}} \sum_{i=1}^{n} \|X(t_i) - X(t_{i-1})\|^p\right)^{\frac{1}{p}},$$

where the supremum is taken over all partitions of I.

Remark 2.12. *It can be easily verified that* $\|X\|_{p,I} = 0 \Longleftrightarrow X$ *is constant on* I*.*

It should be stressed that the definition is different from

$$[X]_{p,I} = \left(\lim_{|\mathcal{D}| \to 0} \sum_{i=1}^{n} \|X(t_i) - X(t_{i-1})\|^p \right)^{\frac{1}{p}},$$

which is frequently used in stochastic calculus, see e.g. Nualart [11]. A trajectory of a Brownian motion $B : t \to B_t(\omega)$ constitutes a suitable example for the difference in these two definitions. Whereas its 2-variation $\|B\|_{2,[0,T]}$ is almost surely infinite (see Freedman [4]), its quadratic variation $[B]_{2,[0,T]}$, defined as the limit in mean square, is finite and equal to the length of the interval T, see Maslowski et al. [9].

Lemma 2.13. *Whenever* $\|X\|_{q,I} \neq 0$ *for some real* $q \geq 1$*, the function* $p \mapsto \log\left(\|X\|_{p,I}^p\right)$ *is convex.*

Proof. It suffices to show that $h(p) = \log\left(\sum_{i=1}^{n} \|a_i\|^p\right)$ is convex in $p \in [1, \infty)$, where a_i are elements of an inner product space representing the differences $X(t_i) - X(t_{i-1})$ and there exists $j \in \{1, ..., n\}$ satisfying $a_j \neq 0$. By the Hölder inequality we obtain that

$$\sum_{i=1}^{n} \|a_i\|^{\lambda x} \|a_i\|^{(1-\lambda)y} \leq \left(\sum_{i=1}^{n} \|a_i\|^{\frac{\lambda x}{\lambda}} \right)^{\lambda} \left(\sum_{i=1}^{n} \|a_i\|^{\frac{(1-\lambda)y}{1-\lambda}} \right)^{1-\lambda}$$

holds for any real $x, y \geq 1$. Let $\lambda \in (0, 1)$. Then for every $x, y \geq 1$ the following inequality

$$\begin{aligned} h(\lambda x + (1-\lambda) y) &= \log\left(\sum_{i=1}^{n} \|a_i\|^{\lambda x} \|a_i\|^{(1-\lambda)y} \right) \\ &\leq \lambda \log\left(\sum_{i=1}^{n} \|a_i\|^{x} \right) + (1-\lambda) \log\left(\sum_{i=1}^{n} \|a_i\|^{y} \right) \end{aligned}$$

is satisfied. Thus $h(p)$ is indeed convex on $[1, \infty)$. □

The following lemma describes the inequalities concerning p-variation.

Lemma 2.14. *Let* $X : I \to V$ *be a function of finite* p_1*-variation,* $p_1 \geq 1$*.*
(i) For a real number p_2 *satisfying* $p_1 \leq p_2 < \infty$*:*

$$\|X\|_{p_1,I} \geq \|X\|_{p_2,I}.$$

(ii) For real numbers p_2 *and* p_3 *satisfying* $p_1 < p_2 < p_3 < \infty$*:*

$$\left(\|X\|_{p_2,I}^{p_2}\right)^{p_3-p_1} \leq \left(\|X\|_{p_1,I}^{p_1}\right)^{p_3-p_2} \left(\|X\|_{p_3,I}^{p_3}\right)^{p_2-p_1}. \tag{2.8}$$

Proof. Let $a_i \in V$, $i = 1, 2, \ldots, n$, where $n \in \mathbb{N}$. In the first case, it is again enough to show

$$\left(\sum_{i=1}^{n} \|a_i\|^{p_1} \right)^{\frac{1}{p_1}} \geq \left(\sum_{i=1}^{n} \|a_i\|^{p_2} \right)^{\frac{1}{p_2}}. \tag{2.9}$$

We prove this part likewise as in Hardy et al. [6]. We assume that $\sum_{i=1}^{n} \|a_i\|^{p_1} \neq 0$. Otherwise, inequality (2.9) holds trivially. Let us denote

$$Q = \sum_{i=1}^{n} \|a_i\|^{p_1}, \qquad b_i = \frac{\|a_i\|}{Q^{\frac{1}{p_1}}}, \ \ i = 1, 2, \ldots, n.$$

Since $b_i \leq 1$ for each $i = 1, 2, \ldots, n$, it is readily seen that $b_i^{p_1} \geq b_i^{p_2}$. We obtain the following chain of relations:

$$Q = \sum_{i=1}^{n} \|a_i\|^{p_1} = Q \sum_{i=1}^{n} b_i^{p_1} \geq Q \sum_{i=1}^{n} b_i^{p_2} = Q^{1-\frac{p_2}{p_1}} \sum_{i=1}^{n} \|a_i\|^{p_2}$$

This can be rewritten as

$$Q^{\frac{p_2}{p_1}} \geq \sum_{i=1}^{n} \|a_i\|^{p_2},$$

which proves inequality (2.9).

The second part of the claim follows by Lemma 2.13. If X is constant on I, the inequality holds. If not, then from convexity of the function h we obtain

$$\frac{\log\left(\sum_{i=1}^{n} \|a_i\|^{p_2}\right) - \log\left(\sum_{i=1}^{n} \|a_i\|^{p_1}\right)}{p_2 - p_1} \leq \frac{\log\left(\sum_{i=1}^{n} \|a_i\|^{p_3}\right) - \log\left(\sum_{i=1}^{n} \|a_i\|^{p_1}\right)}{p_3 - p_1}.$$

Since the logarithm is non-decreasing, we get

$$\left(\frac{\sum_{i=1}^{n} \|a_i\|^{p_2}}{\sum_{i=1}^{n} \|a_i\|^{p_1}}\right)^{\frac{1}{p_2-p_1}} \leq \left(\frac{\sum_{i=1}^{n} \|a_i\|^{p_3}}{\sum_{i=1}^{n} \|a_i\|^{p_1}}\right)^{\frac{1}{p_3-p_1}}$$

which implies

$$\left(\sum_{i=1}^{n} \|a_i\|^{p_2}\right)^{p_3-p_1} \leq \left(\sum_{i=1}^{n} \|a_i\|^{p_1}\right)^{p_3-p_2} \left(\sum_{i=1}^{n} \|a_i\|^{p_3}\right)^{p_2-p_1}.$$

This is the desired conclusion. □

Definition 2.15. *We say that*

$$\sup_{t,s \in I} \|X(t) - X(s)\|$$

is the oscillation of a function $X : I \to V$ *and we denote it by* $\operatorname{Osc}(X)_I$.

The next proposition connects the oscillation of a function to p-variation.

Proposition 2.16. *Let* $X : [a, b] \to V$ *be a function of finite* q*-variation for some* $q \geq 1$*. Then*

$$\operatorname{Osc}(X)_{[a,b]} = \lim_{p \to \infty} \|X\|_{p,[a,b]}.$$

Proof. We have that for any real $p \geq 1$ the relations

$$\sup_{t,s\in[a,b]} \|X(t) - X(s)\| \leq \left(\sup_{t,s\in[a,b]} \|X(t) - X(s)\|^p \right)^{\frac{1}{p}} \leq \|X\|_{p,[a,b]}$$

hold. By letting $p \to \infty$, one immediately gets the first inequality. In order to prove $\sup_{t,s\in[a,b]} \|X(t) - X(s)\| \geq \lim_{p\to\infty} \|X\|_{p,[a,b]}$, we start with the following estimate. Let $p > q \geq 1$ and let $\mathcal{D} = \{t_i\}_{i=0}^n$ be an arbitrary partition of the interval $[a, b]$. Then it holds that

$$\begin{aligned}
\sum_{i=1}^n \|X(t_i) - X(t_{i-1})\|^p &= \sum_{i=1}^n \|X(t_i) - X(t_{i-1})\|^{p-q} \|X(t_i) - X(t_{i-1})\|^q \\
&\leq \operatorname{Osc}(X)_{[a,b]}^{p-q} \sum_{i=1}^n \|X(t_i) - X(t_{i-1})\|^q \\
&\leq \operatorname{Osc}(X)_{[a,b]}^{p-q} \|X\|_{q,[a,b]}^q
\end{aligned}$$

and we thus obtain

$$\|X\|_{p,[a,b]}^p \leq \|X\|_{q,[a,b]}^q \operatorname{Osc}(X)_{[a,b]}^{p-q} \tag{2.10}$$

which is equivalent to

$$\|X\|_{p,[a,b]} \leq \|X\|_{q,[a,b]}^{\frac{q}{p}} \operatorname{Osc}(X)_{[a,b]}^{1-\frac{q}{p}}.$$

The term $\|X\|_{q,[a,b]}$ is finite by assumption and hence the claim is proved by taking the limit $p \to \infty$.

□

Definition 2.17. *For $p \geq 1$, we denote by $\nu_p(I, V)$ the set of all functions $X : I \to V$ of finite p-variation, that is for which $\|X\|_{p,I} < \infty$.*

Remark 2.18. *Lemma* 2.14 (*i*) *gives $\nu_{p_1}(I, V) \subset \nu_{p_2}(I, V)$ as soon as $1 \leq p_1 \leq p_2 < \infty$.*

Definition 2.19. *We say that the function $X : I \to V$ is Hölder continuous with exponent $\alpha \in (0, 1]$ if there exists a positive constant C with the property that for every pair $s, t \in I$ the inequality $\|X(s) - X(t)\| \leq C|s - t|^\alpha$ holds.*

Proposition 2.20. *Let $p \geq 1$. Assume that $X : [a, b] \to V$ is a Hölder continuous function with exponent $\frac{1}{p}$. Then $X \in \nu_p([a, b], V)$.*

Proof. Choose an arbitrary partition $\mathcal{D} = \{t_i\}_{i=0}^n$ of the interval $[a, b]$. By the definition of Hölder continuity, one may estimate

$$\sum_{i=1}^n \|X(t_i) - X(t_{i-1})\|^p \leq C^p \sum_{i=1}^n |t_i - t_{i-1}| = C^p (b - a). \tag{2.11}$$

The right-hand side of (2.11) is independent of the chosen partition. Having obtained a uniform bound, the inequality $\|X\|_{p,[a,b]} \leq C(b - a)^{\frac{1}{p}}$ follows. This concludes the proof. □

Lemma 2.21. *Let $\mathcal{D} = \{t_i\}_{i=0}^n$ be a partition of the interval $[a,b]$. Assume that $X \in \nu_p([a,b], V)$, $Y \in \nu_q([a,b], V)$ and $\operatorname{Osc}(X)_{[t_{i-1},t_i]} < \gamma$, $i = 1, 2, \ldots, n$. Then if $p_1 > p \geq 1, q_1 > q \geq 1$ satisfy $\frac{1}{p_1} + \frac{1}{q_1} > 1$, we obtain*

$$\sum_{i=1}^n \|X\|_{p_1,[t_{i-1},t_i]} \|Y\|_{q_1,[t_{i-1},t_i]} \leq \gamma^{\frac{p_1-p}{p_1}} \|X\|_{p,[a,b]}^{\frac{p}{p_1}} \|Y\|_{q_1,[a,b]} . \tag{2.12}$$

Proof. First we make the following observation. For any partition $\mathcal{D} = \{t_i\}_{i=0}^n$ of $[a,b]$ the inequality

$$\left(\sum_{i=1}^n \|X\|_{p,[t_{i-1},t_i]}^p\right)^{\frac{1}{p}} \leq \|X\|_{p,[a,b]}$$

is satisfied. By formula (2.10) and Corollary 2.4, the left-hand side of (2.12) is dominated by

$$\sum_{i=1}^n \|X\|_{p,[t_{i-1},t_i]}^{\frac{p}{p_1}} \gamma^{\frac{p_1-p}{p_1}} \|Y\|_{q_1,[t_{i-1},t_i]} \leq$$

$$\leq \gamma^{\frac{p_1-p}{p_1}} \left(\sum_{i=1}^n \|X\|_{p,[t_{i-1},t_i]}^p\right)^{\frac{1}{p_1}} \left(\sum_{i=1}^n \|Y\|_{q_1,[t_{i-1},t_i]}^{q_1}\right)^{\frac{1}{q_1}} . \tag{2.13}$$

By the observation above, the right-hand side of inequality (2.13) can never exceed

$$\gamma^{\frac{p_1-p}{p_1}} \|X\|_{p,[a,b]}^{\frac{p}{p_1}} \|Y\|_{q_1,[a,b]} .$$

□

2.3 Regulated functions

Rigorous framework for the theory of regulated functions can be found in Dudley and Norvaiša [3]. We feature only the basic results from this book which are used in the proof of Theorem 2.30. From now on, we suppose that the space V is additionaly complete, i.e. a Hilbert space. If we say that a limit exists, it means that the limit is finite. Let us begin with a common lemma.

Lemma 2.22. *Let $X : [a,b] \to V$ be a function and let $z \in [a,b)$. The following two statements are equivalent.*
(i) The limit $\lim_{t\to z_+} X(t)$ exists.
(ii) $\forall \epsilon > 0\ \exists \delta > 0\ \forall e, f \in [a,b]$, $0 < e - z < \delta$, $0 < f - z < \delta$ the inequality $\|X(e) - X(f)\| < \epsilon$ holds.

Proof. The forward implication $(i) \Rightarrow (ii)$ is elementary. Assume that (ii) holds and choose an arbitrary $\epsilon > 0$. Consider an arbitrary sequence $\{e_n\}_{n=1}^\infty$ converging to z and satisfying $e_n > z$ for every $n \in \mathbb{N}$. The sequence $\{X(e_n)\}_{n=1}^\infty$ is evidently Cauchy and since V is complete, it is also convergent to some $L \in V$. By assumption, there must exist $\delta > 0$ and $N \in \mathbb{N}$ such that whenever $0 < t - z < \delta$ and $n \geq N$, the inequality $\|X(t) - X(e_n)\| < \epsilon$ holds. Subsequently, we have found δ such that whenever t satisfies $0 < t - z < \delta$, in addition we obtain $\|X(t) - L\| < 2\epsilon$, i.e. L is the wanted limit. The proof is therefore concluded. □

Definition 2.23. *Function $X : [a,b] \to V$ is said to be regulated if for every $z \in [a,b)$ there exists the limit $X(z_+) = \lim_{t\to z_+} X(t)$ and for every $z \in (a,b]$ there exists the limit $X(z_-) = \lim_{t\to z_-} X(t)$.*

Lemma 2.24. *Assume that $X \in \nu_p([a,b], V)$, $p \geq 1$. Then X is regulated.*

Proof. Choose $z \in [a,b)$. We will show that $\lim_{t\to z_+} X(t)$ exists. Suppose the limit does not exist. Then it needs be either infinite or it cannot exist. If there existed a right neighbourhood of z in which X was unbounded, we would have $\|X\|_{p,[a,b]}^p \geq \|X(z) - X(t)\|^p$, $t \in (z,b]$, which contradicts the assumption that $X \in \nu_p([a,b], V)$. On the other hand, if X is bounded in the right neighbourhood of z and $X(z_+)$ does not exist, then by Lemma 2.22, there exist $\epsilon > 0$ and decreasing sequences $\{e_n\}, \{f_n\} \subset (z,b]$ converging to z such that for all $n \in \mathbb{N}$ one has $\|X(e_n) - X(f_n)\| \geq \epsilon$ and where $\max\{e_n, f_n\} < \min\{e_{n-1}, f_{n-1}\}$. If we now denote $T = \{e_1, f_1, e_2, f_2, \ldots\}$, it is an infinite subset of $(z,b]$ of different real numbers. Choose a partition $\mathcal{D} = \{t_i\}_{i=0}^n$ of $[a,b]$ with m division points from T. We immediately get that $\|X\|_{p,[a,b]}^p \geq m\epsilon^p$. Since T is infinite, m can be arbitrarily large, which again contradicts the finiteness of p-variation of X. Thus, X must be a regulated function. □

Remark 2.25. *By Lemma 2.24, we can meaningfully include also the points t_+ and t_- amongst division points of a partition.*

Lemma 2.26. *Let $X : [a,b] \to V$ be a regulated function. Then, for every $\epsilon > 0$ there exist finitely many points $z \in (a,b)$ such that at least one of the three following inequalities holds:*
$\|X(z_+) - X(z)\| \geq \epsilon$ *or* $\|X(z) - X(z_-)\| \geq \epsilon$ *or* $\|X(z_+) - X(z_-)\| \geq \epsilon$.

Proof. We will show only $\|X(z_+) - X(z)\| > \epsilon$, $z \in [a,b)$. The rest can be proved similarly. Let $E_\epsilon = \{z \in [a,b) : \|X(z_+) - X(z)\| \geq \epsilon\}$. Supppose that E_ϵ is infinite and choose an arbitrary sequence $\{z_n\}_{n=1}^\infty \subset E$ of different points (this sequence is bounded). Without loss of generality, we can assume this sequence is decreasing. By the Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{z_{n_k}\}_{k=1}^\infty$ with the limit $Z \in [a,b)$. Nevertheless, for every $k \in \mathbb{N}$ we obtain $\|X(z_{n_k+}) - X(z_{n_k})\| \geq \epsilon$ and $z_{n_k} > Z$. By Lemma 2.22, $X(Z_+)$ does not exist. That contradicts the assumption of the claim. □

Corollary 2.27. *Let $X : [a,b] \to V$ be a regulated function. Then it has at most countable points of discontinuity.*

Proof. Let us denote for $n \in \mathbb{N}$

$$E_n = \Big\{ z \in (a,b) : \max\Big\{ \|X(z_+) - X(z_-)\|, \|X(z_+) - X(z)\|, \|X(z) - X(z_-)\| \Big\} \geq \frac{1}{n} \Big\}.$$

The sets E_n are finite by Lemma 2.26 and hence, their union $\bigcup_{n=1}^\infty E_n$ (which differs from the set of all discontinuity points by a maximal number of two points) is countable. □

The previous proofs were obtained with the help of a few hints. The statement and the proof of the last lemma is the one in the previously mentioned book Dudley and Norvaiša [3].

Lemma 2.28. *Assume that $X : [a,b] \to V$ is a regulated function and let $\epsilon > 0$ be given. Let $[c,d] \subset [a,b]$ be a subinterval such that for all $z \in (c,d)$: $\|X(z_+) - X(z)\| < \epsilon$ and $\|X(z) - X(z_-)\| < \epsilon$. Assume further that $\|X(c_+) - X(c)\| < \epsilon$ and $\|X(d) - X(d_-)\| < \epsilon$. In this case, there exists $\delta > 0$ such that whenever $u, v \in [c,d]$ satisfy $|u - v| < \delta$, the inequality*

$$\|X(u) - X(v)\| < 2\epsilon \tag{2.14}$$

holds.

Proof. First we observe that the inequalities $\|X(z_+) - X(z)\| < \epsilon$ and $\|X(z) - X(z_-)\| < \epsilon$ imply $\|X(z_+) - X(z_-)\| < 2\epsilon$. Were the inequality (2.14) not to hold, there would exist sequences $\{e_n\}_{n=1}^{\infty}$, $\{f_n\}_{n=1}^{\infty} \subset [c,d]$ satisfying $0 < e_n - f_n < \frac{1}{n}$ and $\|X(e_n) - X(f_n)\| \geq 2\epsilon$. By the Bolzano-Weierstrass theorem, a subsequence $\{e_{n_k}\}_{k=1}^{\infty}$ converging to some $y \in [c,d]$ can be found. What is more, $\lim_{k\to\infty} f_{n_k} = y$. Thus, we can find a subsequence $\left\{n_{k_j}\right\}_{j=1}^{\infty} \subset \{n_k\}_{k=1}^{\infty}$, abbreviated to $\{m\}_{m=1}^{\infty}$, such that one of the following possibilities holds: If $e_m < y$, $m \in \mathbb{N}$, then $X(y_-)$ does not exist. If $f_m \geq y$, $m \in \mathbb{N}$, then either $X(y_+)$ does not exist or $\|X(y_+) - X(y)\| \geq 2\epsilon$. If $f_m < y \leq e_m$, $m \in \mathbb{N}$, then either $\|X(y_+) - X(y_-)\| \geq 2\epsilon$ or $\|X(y) - X(y_-)\| \geq 2\epsilon$. All three cases contradict the assumptions of the claim. The inequality (2.14) must therefore hold. □

2.4 Riemann-Stieltjes integral

In this section, conditions for the integrability in the Riemann-Stieltjes sense are presented. For simplicity, we define the integral solely for complex-valued functions. Generalisation to Hilbert spaces and, more generally, Banach spaces, can be made, see Lyons et al. [7].

Definition 2.29. *Assume that $X, Y : [a,b] \to \mathbb{C}$ are functions. Let $\mathcal{D} = \{t_i\}_{i=0}^{n}$ be a partition of $[a,b]$ and for $i = 1, 2, \ldots, n$, let $\xi_i \in [t_{i-1}, t_i]$. The function Y is said to be integrable on the interval $[a,b]$ with respect to the function X in the Riemann-Stieltjes sense if the following limit*

$$\lim_{|\mathcal{D}|\to 0} \sum_{i=1}^{n} Y(\xi_i)\left(X(t_i) - X(t_{i-1})\right)$$

exists. That is, it holds that there exists a complex number $\int_a^b Y dX$ such that for every $\epsilon > 0$ there exists $\delta > 0$ such that for each partition $\mathcal{D} = \{t_i\}_{i=0}^{n}$ of the interval $[a,b]$ satisfying $|\mathcal{D}| < \delta$ and for every $\xi_i \in [t_{i-1}, t_i]$:

$$\left| \int_a^b Y dX - \sum_{i=1}^{n} Y(\xi_i)\left(X(t_i) - X(t_{i-1})\right) \right| < \epsilon.$$

In such event, this limit $\int_a^b Y dX$ is called the Riemann-Stieltjes integral of Y with respect to X on the interval $[a,b]$.

A sufficient condition for the existence of the Riemann-Stieltjes integral of Y with respect to X on the interval $[a,b]$ is that for every $\epsilon > 0$ and any two partitions $\mathcal{D}_1 = \{t_i^1\}_{i=0}^n$, $\mathcal{D}_2 = \{t_i^2\}_{i=0}^m$ of $[a,b]$ with the property $|\mathcal{D}_1|$, $|\mathcal{D}_2| < \delta$, the following inequality

$$\left|\sum_{i=1}^n Y\left(\xi_i^1\right)\left(X\left(t_i^1\right) - X\left(t_{i-1}^1\right)\right) - \sum_{i=1}^m Y\left(\xi_i^2\right)\left(X\left(t_i^2\right) - X\left(t_{i-1}^2\right)\right)\right| < \epsilon$$

holds, where $\xi_i^1 \in \left[t_{i-1}^1, t_i^1\right]$, $i = 1,2,\ldots,n$, $\xi_i^2 \in \left[t_{i-1}^2, t_i^2\right]$, $i = 1,2,\ldots,m$.

2.4.1 Love-Young inequality

We shall now see the results of the previous sections as we bring them all together in order to find sufficient conditions for the Riemann-Stieltjes integrability as well as to prove the Love-Young inequality per se.

Theorem 2.30 (Love-Young)**.** *Let $p,q \geq 1$ satisfy $\frac{1}{p} + \frac{1}{q} > 1$ and let $\xi \in [a,b]$. Assume that the functions $X \in \nu_p([a,b],\mathbb{C})$, $Y \in \nu_q([a,b],\mathbb{C})$ have no common discontinuities. Then Y is integrable with respect to X on the interval $[a,b]$ in the Riemann-Stieltjes sense. Moreover, the inequality*

$$\left|\int_a^b \left(Y(t) - Y(\xi)\right) dX(t)\right| \leq 2\left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right)\|X\|_{p,[a,b]}\|Y\|_{q,[a,b]} \tag{2.15}$$

is satisfied.

Proof. Choose an arbitrary $\epsilon > 0$ and partitions $\mathcal{D}_1 = \{t_i^1\}_{i=0}^n$ and $\mathcal{D}_2 = \{t_i^2\}_{i=0}^m$ such that $|\mathcal{D}_1|$, $|\mathcal{D}_2| < \delta$. δ is to be chosen later. It is obvious that for any subinterval $[c,d] \subset [a,b]$ the inequality $S_{p,q}(c,d;X,Y) \leq \|X\|_{p,[c,d]}\|Y\|_{q,[c,d]}$ holds by definition. Thus, by Theorem 2.10, to prove the existence of the Riemann-Stieltjes integral, it suffices to show that there exist $p_1, q_1 \geq 1$ satisfying $\frac{1}{p_1} + \frac{1}{q_1} > 1$ such that the following bound

$$\sum_{i=1}^n \|X\|_{p_1,\left[t_{i-1}^1,t_i^1\right]}\|Y\|_{q_1,\left[t_{i-1}^1,t_i^1\right]} < \frac{\epsilon}{2\left(1 + \zeta\left(\frac{1}{p_1} + \frac{1}{q_1}\right)\right)} \tag{2.16}$$

holds as well as a corresponding inequality for the partition $\mathcal{D}_2$. By Lemma 2.24 and Lemma 2.26, for every $\gamma_1 > 0$ there exist finitely many discontinuities $\xi_1^1, ..., \xi_n^1 \in [a,b)$ of X satisfying $\|X(\xi_{i,+}) - X(\xi_i)\| \geq \gamma_1$. The other two types of discontinuities, as described in section 2.3, can be treated similarly. By Lemma 2.28, $\delta_1 > 0$ can be found such that for a subinterval $[c,d] \subset [a,b]$: $|d-c| < \delta_1$ it holds that $\|X(d) - X(c)\| < 2\gamma_1$ as soon as $[c,d]$ does not contain any of the discontinuities $\xi_1^1, ..., \xi_n^1$ (and any of the other two types of discontinuities). In a similar manner, $\delta_2 > 0$ can be found for the function Y and any $\gamma_2 > 0$. For there are no common discontinuities of X and Y, we can find $\delta \leq \min\{\delta_1, \delta_2\}$ such that there is no interval in the partition $\mathcal{D}_1$ which would contain a discontinuity point of both X and Y. Subsequently, either $\operatorname{Osc}(X)_{\left[t_{i-1}^1,t_i^1\right]} < 2\gamma_1$ or $\operatorname{Osc}(Y)_{\left[t_{i-1}^1,t_i^1\right]} < 2\gamma_2$ must hold. Let us now choose $p_1 > p, q_1 > q$ still satisfying $\frac{1}{p_1} + \frac{1}{q_1} > 1$ and set

$$\gamma_1 := \frac{1}{2}\left(\frac{\epsilon}{2\|X\|_{p,[a,b]}^{\frac{p}{p_1}}\|Y\|_{q_1,[a,b]}\left(1 + \zeta\left(\frac{1}{p_1} + \frac{1}{q_1}\right)\right)}\right)^{\frac{p_1}{p_1 - p}}.$$

Lemma 2.21 then implies inequality (2.16). Likewise, γ_2 can be obtained on the intervals with the property that $\mathrm{Osc}\,(Y)_{\left[t^1_{i-1},t^1_i\right]} < 2\gamma_2$. Inequality (2.15) is a direct consequence of (2.7). $\square$

Remark 2.31. *By Proposition* 2.20*, instead of the assumptions* $X \in \nu_p\left([a,b],\mathbb{C}\right)$ *and* $Y \in \nu_q\left([a,b],\mathbb{C}\right)$ *it is sufficient that* X *is a Hölder continuous function with exponent* α *and* Y *is Hölder continuous with exponent* β *such that* $\alpha+\beta>1$*. Note, however, this condition is not necessary.*

Remark 2.32. *Young proved a more general version of the integrability of* Y *with respect to* X*. We will address the definitions, though this result will not be reviewed here in more detail and we refer the reader to the original paper Young [14].*

Let $[a,b]$ *be an interval and let* $\mathcal{D}=\{t_i\}_{i=0}^n$ *be its partition. Further let* $\xi_i \in [t_{i-1},t_i]$*,* $i=1,2,\dots,n$*. We say that the integral* $\int_a^b Y dX$ *exists in the so called Moore-Pollard sense if for every* $\epsilon>0$ *there exists a finite set* E *such that it holds that*

$$\left|\int_a^b Y dX - \sum_{i=1}^n Y\left(\xi_i\right)\left(X\left(t_i\right)-X\left(t_{i-1}\right)\right)\right| < \epsilon$$

as soon as the partition $\mathcal{D}$ *includes all points of the set* E*.*

Moreover, if X *and* Y *are regulated functions, the integral* $\int_a^b Y dX$ *is said to exist in the generalised Moore-Pollard sense with the value* $A+B$ *if the integral* $A=\int_a^b Y\left(t_+\right)dX\left(t_-\right)$ *exists in the Moore-Pollard sense and additionally the sum*

$$B=\sum_{t\in F}\left(Y\left(t\right)-Y\left(t_+\right)\right)\left(X\left(t_+\right)-X\left(t_-\right)\right)$$

converges absolutely. Here, F *denotes the set of all shared discontinuity points of* X *and* Y*. Notice that the set* F *is by Corollary 2 countable. Young proved that if* $X\in\nu_p\left([a,b],\mathbb{C}\right)$ *and* $Y\in\nu_q\left([a,b],\mathbb{C}\right)$ *have neither any common discontinuities on the right nor on the left, the integral* $\int_a^b Y dX$ *exists in the Moore-Pollard sense. The integral* $\int_a^b Y dX$ *does exist in the generalised Moore-Pollard sense as soon as it exists in the Moore-Pollard sense or in the Riemann-Stieltjes sense.*

2.4.2 Insufficiency of Riemann-Stieltjes integration

In what follows, we are to provide an example where the conclusion of Theorem 2.10 fails should the condition $\frac{1}{p}+\frac{1}{q}=1$ not be met. Assume that $p=2$, $q=2$, $N\in\mathbb{N}$, $a\in\mathbb{N}$, $a>1$ and let $t\in(0,1)$. Let $X,Y:[0,1]\to\mathbb{C}$ be functions defined in the following way:

$$X\left(t\right)=\sum_{n=1}^N a^{-\frac{n}{2}}\mathrm{e}^{-2\pi i a^n t}, \qquad Y\left(t\right)=\sum_{n=1}^N a^{-\frac{n}{2}}\mathrm{e}^{2\pi i a^n t}.$$

Let $h\in(-1,1)$, $h\neq 0$ be such that we can choose $n_0\in\mathbb{N}$ satisfying $2\pi|h|a^{n_0}\le 1\le 2\pi|h|a^{n_0+1}$. The following relations will be employed:

$$\begin{aligned}\left|\mathrm{e}^{2\pi i a^n h}-1\right| &= \sqrt{\sin^2\left(2\pi a^n h\right)+\left(\cos\left(2\pi a^n h\right)-1\right)^2}\\ &= \sqrt{2-2\cos\left(2\pi a^n h\right)}\\ &= \sqrt{2}\sqrt{1-\cos\left(2\pi a^n h\right)}\\ &= 2\left|\sin\left(\pi a^n h\right)\right|, \qquad n\in\mathbb{N},\end{aligned}$$

$$\sum_{n=1}^{n_0} a^{\frac{n}{2}} = \sqrt{a}\frac{\sqrt{a^{n_0}}-1}{\sqrt{a}-1},$$

$$\sum_{n=0}^{n_0} a^{-\frac{n}{2}} = \frac{\sqrt{a}}{\sqrt{a^{n_0+1}}}\frac{\sqrt{a^{n_0+1}}-1}{\sqrt{a}-1},$$

$$\sum_{n=0}^{\infty} a^{-\frac{n}{2}} = \frac{\sqrt{a}}{\sqrt{a}-1},$$

and

$$\begin{aligned} |Y(t+h)-Y(t)| &\leq \sum_{n=1}^{N} a^{-\frac{n}{2}} \left| \mathrm{e}^{2\pi i a^n t} \mathrm{e}^{2\pi i a^n h} - \mathrm{e}^{2\pi i a^n t} \right| \\ &= \sum_{n=1}^{N} a^{-\frac{n}{2}} \left| \mathrm{e}^{2\pi i a^n h} - 1 \right| \\ &= \sum_{n=1}^{N} a^{-\frac{n}{2}} \left| 2\sin(\pi a^n h) \right|. \end{aligned} \tag{2.17}$$

For every $t \geq 0$ it holds that $\sin(t) \leq t$. Therefore, the right-hand side of inequality (2.17) is at most

$$\begin{aligned} &\sum_{n=1}^{n_0} a^{-\frac{n}{2}} 2\pi a^n |h| + 2 \sum_{n=n_0+1}^{\infty} a^{-\frac{n}{2}} = \\ &\quad = 2\pi|h| \sum_{n=1}^{n_0} a^{\frac{n}{2}} + 2\left(\sum_{n=0}^{\infty} a^{-\frac{n}{2}} - \sum_{n=0}^{n_0} a^{-\frac{n}{2}} \right) \\ &\quad = \frac{\sqrt{|h|}}{\sqrt{a}-1} \left(2\pi\sqrt{|h|}\sqrt{a}\left(\sqrt{a^{n_0}}-1\right) + 2\left(\frac{\sqrt{a}}{\sqrt{|h|}} - \frac{\sqrt{a}}{\sqrt{|h|}}\left(1-\frac{1}{\sqrt{a^{n_0+1}}}\right)\right)\right) \\ &\quad \leq \frac{\sqrt{|h|a}}{\sqrt{a}-1}\left(2\pi\sqrt{|h|}\sqrt{a^{n_0}} + \frac{2}{\sqrt{|h|a^{n_0+1}}}\right) \end{aligned}$$

which is by our choice of n_0 dominated by

$$\frac{\sqrt{|h|a}}{\sqrt{a}-1}\left(\sqrt{2\pi}+2\sqrt{2\pi}\right) = \frac{3\sqrt{2\pi a}}{\sqrt{a}-1}\sqrt{|h|} \leq 32\sqrt{|h|}.$$

Consequently, one has $S_{2,2}(0,1;X,Y) \leq 32^2$. Simultaneously, the equalities

$$\begin{aligned} \int_0^1 Y(t)\,dX(t) &= \int_0^1 Y(t)\,X'(t)\,dt \\ &= \sum_{n=1}^{N}\sum_{m=1}^{N} -2\pi i a^{m-\frac{n+m}{2}} \int_0^1 \mathrm{e}^{2\pi i t(a^n-a^m)} dt \\ &= -2\pi i \sum_{n=1}^{N}\sum_{m=n} 1 \\ &= -2\pi i N \end{aligned}$$

hold. A partition $\mathcal{D} = \{t_i\}_{i=0}^{n}$ of the interval $[0,1]$ can be found such that $|F_{\mathcal{D}} + 2\pi i N| < \epsilon$. It is readily seen that $X(1) = X(0)$ so the left-hand side of (2.7) can indeed exceed any finite bound. Therefore, if $\frac{1}{p} + \frac{1}{q} = 1$, the conclusion of Theorem 2.10 need not hold with any finite bound should the right-hand side of inequality (2.6) be replaced by it.

We finish this section with the integral

$$\int_a^b Y(t)\, dX(t) \tag{2.18}$$

in mind. Suppose the functions X and Y are of finite p-variation for $p < 2$. Then the integral (2.18) exists in the Riemann-Stieltjes sense. However, as we will see, the trajectories of Brownian motion possess finite p-variation solely for $p > 2$. The Riemann-Stieltjes integral will hence not suffice for probabilists in this case of the utmost importance. One might argue that this is a shortcoming of the Riemann-Stieltjes integration theory. Contrariwise, there does not exist a pathwise linear integration theory that could give meaning to the integral (2.18) were X and Y trajectories of a Brownian motion. Lyons et al. [7] explains the problem in more detail.

3. Fractional Brownian motion

Stochastic processes are used as input noises in many applications. These may vary greatly, from biology to financial mathematics. Consequently, the theory of stochastic differential equations had to be developed to find tools for managing these processes. Main focus is on the (in)ability to integrate with respect to stochastic processes. A natural example of a simple stochastic process with reasonable properties is the aforementioned Brownian motion that requires the independence of the increments. Over the last few decades the utility of processes with dependent increments has been shown. Hence, the generalisation of a Brownian motion - the fractional Brownian motion, was unavoidable. As the source of information for this chapter we have used papers Nualart [10], Guerra and Nualart [5] and Nualart [11] on which the following text heavily relies. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. All stochastic processes in this chapter will be defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with the state space $\mathbb{R}$. We will feature some of the elementary properties of fractional Brownian motions.

3.1 Basic properties

Definition 3.1. *We say that a continuous time stochastic process $\{X_t, t \geq 0\}$ is Gaussian if for any finite subset $F \subset [0, \infty)$ the random vector $\{X_t, t \in F\}$ has joint normal distribution.*

Any Gaussian process is given by its expectation function and covariance function. This brings us to the important definition.

Definition 3.2. *A Gaussian process $\{X_t, t \geq 0\}$ is said to be a (standard) fractional Brownian motion if the following criteria are met:*
(i) $X_0 = 0$ $\mathbb{P}$-almost surely.
(ii) $\mathbb{E}X_t = 0$, $t \in [0, \infty)$.
(iii) For $s, t \in [0, \infty)$

$$\mathbb{E}X_s X_t = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \tag{3.1}$$

where $H \in (0, 1)$ denotes the so-called Hurst parameter.

Remark 3.3. *If $H = \frac{1}{2}$, the covariance function is $\mathbb{E}X_s X_t = \min\{s, t\}$. The resulting process is the standard Brownian motion which is also called the Wiener process.*

It is possible to observe one of the significant properties of a fractional Brownian motion - self-similarity as a consequence of formula (3.1). Let $a > 0$ and X be the fractional Brownian motion of Hurst parameter H. Then for any finite set $F \subset [0, \infty)$ the distributions of the random vectors $\left\{a^H X_t, t \in F\right\}$ and $\{X_{at}, t \in F\}$ coincide. Another direct consequence of formula (3.1) is the variance of increments of a fractional Brownian motion, for which it holds that

$$\mathbb{E}\left((X_t - X_s)^2\right) = \mathbb{E}X_t^2 + \mathbb{E}X_s^2 - \left(s^{2H} + t^{2H} - |t-s|^{2H}\right) = |t-s|^{2H}.$$

Therefore, fractional Brownian motions have stationary increments, i.e. the probability distribution of $X_t - X_s$ depends solely on the difference $t - s$. As a result of the standard normal distribution, the even moments satisfy

$$\mathbb{E}\left((X_t - X_s)^{2k}\right) = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1) \left|t-s\right|^{2Hk}, \qquad k \in \mathbb{N}. \tag{3.2}$$

Moreover, consider the increments $X_{t_2} - X_{t_1}$ and $X_{t_4} - X_{t_3}$ of a fractional Brownian motion where $0 \leq t_1 < t_2 \leq t_3 < t_4$. The following equalities hold for its covariance function:

$$\begin{aligned}\mathbb{E}\left((X_{t_4} - X_{t_3})(X_{t_2} - X_{t_1})\right) &= \mathbb{E}\left(X_{t_4}X_{t_2} + X_{t_3}X_{t_1} - X_{t_4}X_{t_1} - X_{t_3}X_{t_2}\right) \\ &= \frac{1}{2}\Big(\left|t_4 - t_1\right|^{2H} + \left|t_3 - t_2\right|^{2H} - \left|t_4 - t_2\right|^{2H} \\ &\qquad - \left|t_3 - t_1\right|^{2H} \Big).\end{aligned} \tag{3.3}$$

Were we to set $t_4 - t_2 = t_3 - t_1 = n$, $n \in \mathbb{N}$ and $t_2 - t_1 = 1$, we would have that the right-hand side of (3.3) equals

$$\rho_H(n) := \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right).$$

This identity can be used to show positive correlation ($\rho_H(n) > 0$ for large n) of the increments for $H > \frac{1}{2}$, negative correlation ($\rho_H(n) < 0$ for large n) of the increments for $H < \frac{1}{2}$ or independence for $H = \frac{1}{2}$ (i.e. Brownian motion). For more details, see e.g. Nualart [10].

3.2 P-variation of the trajectories

One property of the fractional Brownian motion, which is actually crucial for us in order to address its p-variation, has not yet been given. How "continuous" are its trajectories? We shall first see that there exists a "modified" continuous process of a fractional Brownian motion whose trajectories are Hölder continuous.

Definition 3.4. *Let $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ be stochastic processes. The process X is said to be a modification of the process Y if the equality*

$$\mathbb{P}\left(\omega \in \Omega : X_t(\omega) = Y_t(\omega)\right) = 1$$

holds for every $t \geq 0$.

It is not demanding to verify that if a process X is a modification of a process Y, their finite-dimensional distributions are identical. In particular, let X be a modification of a fractional Brownian motion Y. Then X is a fractional Brownian motion as well. Next, we present one of Kolmogorov's results that can be found proved in Bauer [1].

Theorem 3.5 (Kolmogorov continuity theorem)**.** *Let $\{X_t, t \geq 0\}$ be a stochastic process with the state space $\mathbb{R}^n$, $n \in \mathbb{N}$. Assume that there exist real numbers $a, b, c > 0$ such that the inequality*

$$\mathbb{E}\left(\left\|X_t - X_s\right\|^a\right) \leq c\left|t-s\right|^{1+b} \tag{3.4}$$

is satisfied for $s, t \in [0, \infty)$. Then there exists a continuous modification of the process $\{X_t, t \geq 0\}$ still satisfying (3.4). Furthermore, almost all trajectories of this modification $X_\omega : t \mapsto X_t(\omega)$, $t \in [0,T]$, $T > 0$, are Hölder continuous with every exponent $\gamma \in \left(0, \frac{b}{a}\right)$.

Theorem 3.6. *Let $\{X_t, t \in [0,T]\}$ be the fractional Brownian motion of Hurst parameter H. Then there exists its modification almost all trajectories of which have finite p-variation for $p > \frac{1}{H}$.*

Proof. Choose an arbitrary $k \in \mathbb{N}$ and set the constants $a = 2k$, $b = 2Hk - 1$ and $c = 1 \cdot 3 \cdot \ldots \cdot (2k-1)$. The inequality (3.4) is satisfied as a result of formula (3.2). Almost all trajectories of the modification from Theorem 3.5 are Hölder continuous with exponent strictly smaller than

$$\frac{2Hk - 1}{2k} = H - \frac{1}{2k},$$

where $k \in \mathbb{N}$ can be chosen arbitrarily large. Hence, we see that the trajectories are in fact Hölder continuous with every exponent $\gamma \in (0, H)$. Therefore, by Proposition 2.20, the trajectories of the process $\{X_t, t \in [0,T]\}$ are of finite $\frac{1}{\gamma}$-variation as soon as $\gamma \in (0, H)$, which is the desired conclusion. □

Remark 3.7. *The trajectories of a fractional Brownian motion can be deemed to be Hölder continuous with exponent $\gamma \in (0, H)$ $\mathbb{P}$-almost surely. From now on, we consider this modification.*

Remark 3.8. *The Hurst parameter H affects the regularity of the trajectories of fractional Brownian motions in the sense that the larger the H, the smoother the trajectories become.*

We conclude this section with the non-differentiability attribute.

Theorem 3.9. *Almost all trajectories of a fractional Brownian motion are nowhere differentiable.*

Proof. See Biagini et al. [2]. □

3.3 Existence of the integral

Following the ideas in Motivation, two approaches when addressing the integration of a stochastic process are usually considered. The previous section provided a framework to when the pathwise integral

$$\int_0^T Y_t dX_t, \tag{3.5}$$

where $T > 0$ and $t \mapsto X_t$ is a trajectory of a fractional Brownian motion, can be defined as the Riemann-Stieltjes integral. The trajectory $t \mapsto X_t$ is Hölder continuous with every exponent $\gamma \in (0, H)$. Therefore, by Remark 2.31, if $Y : [0,T] \to \mathbb{R}$ is a Hölder continuous function with exponent δ such that the inequality $\gamma + \delta > 1$ holds, the integral (3.5) exists in the Riemann-Stieltjes sense. In particular, it is sufficient for Y to be Hölder continuous with an exponent that is greater than $1 - H$. Similar approach that uses fractional calculus to give meaning to the integral (3.5) can be found in Zähle [15]. However, Theorem 2.30 bears an unfortunate implication for the Brownian motion. In spite of its many applications, we cannot give a meaning to the integral $\int_0^T Z_t dB_t$, where $t \mapsto B_t$ is a trajectory of a Brownian motion, when Z is not regular enough. Brownian motions possess finite p-variation solely for $p > 2$ which clearly prevents us from defining the integral

$$\int_0^T B_t dB_t \tag{3.6}$$

in the Riemann-Stieltjes sense. We shall see the use of (3.6) in the next section. This is a dire disadvantage of the pathwise approach. In such case, one ought to turn to the methods of the Itô calculus. Whilst the Itô integral (3.6) exists, its use is limited to the case $H = \frac{1}{2}$. For a thorough introduction to stochastic calculus and how it relates to the fractional Brownian motion we encourage the reader to take a look at Biagini et al. [2] and Nualart [12]. Moreover, the rough paths theory introduced a way to give meaning to (3.5), where X and Y are trajectories of the fractional Brownian motion of Hurst parameter $H \in \left(0, \frac{1}{2}\right)$, provided extra data is given a priori. See Lyons et al. [7] and Lyons [8].

Regardless, the integral (3.5) is properly defined for any process X satisfying (3.4) and a real-valued function Y that is Hölder continuous with exponent greater than $1 - \frac{b}{a}$. Similarly, the integral (3.5) can be given meaning to when $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ are stochastic processes whose regularities are controlled via the condition (3.4).

3.4 Applications

This section presents some of the results from Lyons [8] and Lyons et al. [7]. Let us first introduce the iterated integrals and their relation to solving differential equations. It shall demonstrate the need for the first-order iterated integral (3.6). The functions in this section, just as the stochastic processes before, are assumed to be real-valued. We start with a proposition describing the p-variation of the Riemann-Stieltjes integral.

Proposition 3.10. *Let $p, q \geq 1$ satisfy $\frac{1}{p} + \frac{1}{q} > 1$. Assume that $X \in \nu_p([a, b], \mathbb{C})$, $Y \in \nu_q([a, b], \mathbb{C})$ are continuous functions. Then the function $z \mapsto \int_0^z Y_t dX_t$ has finite p-variation.*

Proof. See Lyons et al. [7]. □

Definition 3.11. *Let $X : [0, T] \to \mathbb{R}$ be a function of finite p-variation, $p \in [1, 2)$, $n \in \mathbb{N}$. We then define its n-th iterated integral by the relation*

$$\begin{aligned} X_{0,T}^n &= \int_{0<u_n<T} \left(\int_{0<u_{n-1}<u_n} \cdots \left(\int_{0<u_1<u_2} dX_{u_1} \right) \ldots dX_{u_{n-1}} \right) dX_{u_n} \\ &= \int_{0<u_1<u_2<\cdots<u_n<T} dX_{u_1} dX_{u_2} \cdots dX_{u_n}. \end{aligned} \tag{3.7}$$

Remark 3.12. *The integral* (3.7) *exists in the Riemann-Stieltjes sense according to Theorem* 2.30 *and Proposition* 3.10.

Example 3.13. *It can be easily verified that*

$$X_{0,T}^{n} = \frac{(X_T - X_0)^n}{n\,!}$$

holds.

Lemma 3.14. *Let $X : [0,T] \to \mathbb{R}$ be a function of finite p-variation, $p \in [1,2)$. The following estimate holds for each $k \in \mathbb{N}$:*

$$\left|X_{0,T}^{n}\right| \leq \frac{\|X\|_{p,[0,T]}^{n}}{n\,!}. \tag{3.8}$$

Proof.

$$\left|X_{0,T}^{n}\right| \leq \frac{|X_T - X_0|^n}{n\,!} \leq \frac{\mathrm{Osc}\,(X)_{[0,T]}^{n}}{n\,!},$$

which is clearly dominated by the right-hand side of (3.8). □

Without going into much detail, we will present the motivation behind the iterated integrals based on the results from Lyons et al. [7] and Lyons [8] where we refer the reader for a thorough approach. Let $X : [0,T] \to \mathbb{R}$ be a continuous function of bounded 1-variation. We consider the simple case when $f : \mathbb{R} \to \mathbb{R}$ is a function defined by the relation $f\,(u) = au$, a is a real number. Consider the differential equation

$$dY_t = f\,(Y_t)\,dX_t \tag{3.9}$$

that is interpreted in the integral form and Y_0 is a given number (initial condition). The solution to (3.9) can be obtained in the form of the convergent series

$$Y_t = Y_0 + \sum_{n=1}^{\infty} f^n\,(Y_0)\,X_{0,t}^{n},$$

where $f^n\,(Y_0) = \underbrace{f\,(f\,(...\,(Y_0)))}_{\text{n-times}}$, see Lyons et al. [7]. Should we additionally assume that X_t is a smooth function, Y_t is the solution and f is bounded, we obtain the consecutive identities:

$$\begin{aligned}
Y_t &= Y_0 + \int_0^t dY_{u_1} = Y_0 + \int_0^t f\,(Y_{u_1})\,dX_{u_1} \\
&= Y_0 + f\,(Y_0) \int_{0<u_1<t} dX_{u_1} + \int\int_{0<u_1<u_2<t} f\,(f\,(Y_{u_1}))\,dX_{u_1}dX_{u_2} \\
&= Y_0 + \sum_{i=1}^{n} \underbrace{f\,(f\,(\cdots f\,(Y_0)))}_{\text{i-times}} \int\cdots\int_{0<u_1<u_2<\cdots<u_i<t} dX_{u_1}dX_{u_2}\cdots dX_{u_i} \\
&\quad + \int\cdots\int_{0<u_1<u_2<\cdots<u_{n+1}<t} f\,(f\,(\cdots f\,(Y_{u_1})))\,dX_{u_1}dX_{u_2}\cdots dX_{u_{n+1}}.
\end{aligned} \tag{3.10}$$

In a generel case where f is not of the form $f\,(u) = au$, one may come to similar relations using Taylor expansion, again see Lyons et al. [7]. It is readily seen that, by Lemma 3.14, the integral (3.10) is in modulus less than or equal to

$$\sum_{i=n+1}^{\infty} \frac{\left(\|f\|\,\|X\|_{1,[0,T]}\right)^i}{i\,!},$$

which inspires the following remark.

Remark 3.15. *The convergence of the error term* (3.10) *is exceptionally fast. It can be seen that the iterated integrals entail an extensive information about the solution to the equation* (3.9)*. It suffices to find but the first few iterated integrals for the error to be adequately small.*

The aim of the rest of this section is to present a generalisation of the well-known Peano theorem for existence of a solution to a differential equation. Let I be a closed bounded interval. Further, let $X : I \to \mathbb{R}$ be a continuous function, $f : \mathbb{R} \to \mathbb{R}$ be a linear continuous function and $Y_0 \in \mathbb{R}$. Consider now the differential equation

$$Y_t = Y_0 + \int_0^t f(Y_s)\, dX_s. \tag{3.11}$$

Firstly, the integral on the right-hand side of (3.11) must exist. Let hence X be of finite p-variation and let $f \circ Y$ be of finite q-variation such that $\frac{1}{p} + \frac{1}{q} > 1$. However, one would like to impose requirements solely on the functions X and f, not necessarily on the solution Y. This brings us to the following lemma.

Lemma 3.16. *Let $p \geq 1$ be a real number. Let $Y \in \nu_p(I, \mathbb{R})$ be a function. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Hölder continuous function with exponent $\alpha \in (0, 1]$. Then $f \circ Y \in \nu_{\frac{p}{\alpha}}(I, \mathbb{R})$.*

Proof. Let f be Hölder continuous with the constant $C > 0$ and choose an arbitrary partition $\mathcal{D} = \{t_i\}_{i=0}^n$ of I. We then have that

$$\sum_{i=1}^n \left\| f(Y_{t_i}) - f\left(Y_{t_{i-1}}\right) \right\|^{\frac{p}{\alpha}} \leq C^{\frac{p}{\alpha}} \sum_{i=1}^n \left\| Y_{t_i} - Y_{t_{i-1}} \right\|^p \leq C^{\frac{p}{\alpha}} \|Y\|_{p,I}^p .$$

By the uniformity of the bound, when the supremum over all partitions is taken, one gets that $\|f \circ Y\|_{\frac{p}{\alpha},I} \leq C \|Y\|_{p,I}^\alpha$. The claim follows. □

Should we suppose for X to be of finite p-variation, we have that it suffices that $\frac{1}{p} + \frac{\alpha}{p} > 1$ holds for the existence of the Riemann-Stieltjes integral (3.11) (X and f are continuous). Since $\alpha \in (0, 1]$ by definition, we get $p < 2$. Notice that the trajectories of the standard Brownian motion therefore depict a rather troublesome example even if the integral is not precisely in the form of (3.6). The following theorem with its proof can be found in Lyons et al. [7].

Theorem 3.17. *Let $p \in [1, 2)$ and $\alpha \in (p - 1, 1]$. Let $X \in \nu_p(I, \mathbb{R})$ and let $f : \mathbb{R} \to \mathbb{R}$ be a linear Hölder continuous function with exponent α. For every initial condition $Y_0 \in \mathbb{R}$ there exists a solution to the differential equation* (3.11)*.*

Let us turn back to the first example (1.1). For convenience, let $\beta_t \equiv 0$ and let X_t represent the noise so that we get the differential equation

$$Y_t = Y_0 + \int_0^t Y_s dX_s. \tag{3.12}$$

Evidently $f(Y) = Y$ is Hölder continuous with exponent 1. Theorem 3.17 yields that (3.12) indeed admits a solution as soon as X has finite p-variation for some $p \in [1, 2)$.

In Lyons et al. [7], a similar claim concerning the uniqueness of a solution is proved. This however requires getting acquainted with an "alternative" definition of Hölder continuous functions with exponent greater than 1 which do not include only constant functions.

Conclusion

The primary goal of this thesis was to provide the reader with the complete and thorough proof of the Love-Young inequality that appeared in Young [14]. The ideas behind this proof are elementary but intricate at the same time. There exists a shorter proof of the Love-Young inequality, see e.g. Lyons et al. [7]. Had we presented it here, we would have obtained the results in a more general setting, although to the detriment of relying on more advanced techniques.

An example and deterministic models that lead to stochastic integration have been offered, the limitations of the stochastic as well as of the pathwise integration theory have been discussed. As far as consequences of the Love-Young inequality are concerned, the integration theory involving noise of finite p-variation has been our main interest. We have introduced the fractional Brownian motion as a generalisation of the Wiener process and, under certain conditions on regularity, we have given meaning to the Riemann-Stieltjes integrals that involve fractional Brownian motions as integrators. Differential equations and iterated integrals have also been mentioned to show specific implications of the pathwise approach.

The main possible extension of the results given in this thesis is the rough paths theory which is currently in the center of attention and which already has applications across multiple mathematical fields.

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