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Effects of additional scalar decaplet in the RG evolution of the running gauge couplings in the minimal SO(10) grand unified theory

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Abstract: We begin the thesis with the fundamentals of Lie groups and algebras. Then we introduce basic concepts regarding classical field theory and gauge theory. We formulate the Goldstone theorem and with it we describe the Higgs mechanism. We quantize the classical fields and outline the calculations within the quantum field theory. Using the renormalization procedure we derive the equations for the running coupling of a general gauge theory. We calculate the running gauge couplings in the Standard model and motivate theories beyond it. In particular, we will study the Grand Unified Theories. We discuss a specific Higgs sector with a $45 \oplus 126$ Higgs field. In order to have a more realistic theory we add a scalar Higgs decaplet and study its effects on the running couplings. Finally we discuss the implications of the decaplet for the proton decay.

Keywords: Quantum field theory, gauge field theory, running couplings, grand unification, proton decay

Contents

Introduction

The fundamental structure of matter is one of the oldest scientific questions. Over the centuries and even millennia many theories have been proposed and many experiments have been carried out. The developments of the quantum field theory, the Lie algebras and gauge theories in the last two centuries enabled the formulation the Standard model of particle physics which is the most complete theory of matter we have today.

While the Standard model (SM) is widely successful there still remain reasons to look for a more general theory. One class of such theories form the Grand Unified Theories (GUTs). One of the core aspects of GUTs is the unification of the three gauge interactions present in the SM at high energies. Furthermore, the number of free parameters in the Yukawa sector of GUTs is reduced compared to the SM.

In this thesis we will study the gauge unification. We will begin with a particular theory called the Minimal SO(10) Higgs model. However, the model predicts the wrong masses in the fermion sector. Therefore, we will add a decaplet of scalar fields into the theory which helps to tune those masses. Then we will calculate some of the effects of the new decaplet on the unification and we will discuss the implications it has for one of the major predictions of the theory - the proton decay.

In the first chapter the introduction to Lie groups and Lie algebras is given. It contains basic concepts regarding the representation theory and a few definitions needed later in the thesis.

The second chapter begins with the classical field theory. Afterwards, the quantization of the classical field theory is sketched. This is followed by a section where the principles gauge theory are outlined. Later on, the Goldstone theorem and the Higgs mechanism for an Abelian symmetry is introduced. The chapter ends with the electroweak unification and its connection to the old Fermi theory.

The third chapter is devoted to calculations within the quantum field theory. Namely, various Feynman diagrams will be considered on both the tree level and 1-loop level. The procedure of renormalization will be introduced and used to derive a general formula for the β-function for a gauge theory associated with the running gauge couplings.

In the fourth chapter the β-functions are calculated within the SM. The near convergence of the running couplings starts the motivation to study the Grand Unified Theories.

The fifth chapter is devoted to GUTs. In particular, the Higgs sector is discussed of a particular model called the Minimal SO(10) Higgs model. The addition of a scalar decaplet is motivated and then its effects on the running couplings are calculated. Lastly, the implications of those results for the proton decay and its detection are discussed.

1. Lie groups and Lie algebras

1.1 Lie groups

The mathematical introduction in this chapter follows the structure of [1]. Suppose we have a group G with elements g. Let us denote the group operation simply as $gh = f$ where $g, h, f \in G$. Now let every element $g \in G$ be described by a smooth function of a continuous set of parameters - $g(\alpha)$. Such group is called a Lie group. In this thesis we will work with groups of transformations. We need these transformations to act on various objects. Therefore we define the *representation* of a group as a mapping from G to a set of linear operators acting on a Hilbert space. We denote the representation of $g \in G$ by $R(g)$. The representations must maintain the group operation, i.e. if $gh = f$ then $R(g) R(h) = R(f)$, and if $e \in G$ is the identity element then its representation must be the identity operator.

1.2 Lie algebras

We can now observe that if G is a Lie group and $g(\alpha) \in G$ then its representation is also a smooth function of continuous parameters $R(g(\alpha)) = R(\alpha)$. Let us assume we have parametrized the group elements so that the identity element has all the parameters $\alpha = 0$. This is obviously without any loss of generality as one can always transform the parametrization so that $g(0) = e$. Let $R(G)$ be a representation then we define the *generators* of the representation as

$$X_a = -i \frac{\mathrm{d}R(\alpha)}{\mathrm{d}\alpha}|_{\alpha=0}. \tag{1.1}$$

It is clear that there are as many generators as there are parameters needed to describe the group elements which is what we call the dimension of the gruop. With the generators at hand we can now introduce the *exponential parametrization*, which reads for parameters $\alpha = (\alpha_1, .., \alpha_N)$

$$R(\alpha) = \exp[i\alpha_a X_a], \tag{1.2}$$

where we summed over a (the Einstein summation convention, we shall use it throughout the thesis). Since this is a representation it must conserve the group multiplication law

$$\exp[i\alpha_a X_a] \exp[i\beta_b X_b] = \exp[i\delta_a X_a]. \tag{1.3}$$

Both sides of the equation can be expanded into Taylor series. Then if we compare both sides up to second order we find out that

$$[\alpha_a X_a, \beta_b X_b] = -2i(\delta_c - \alpha_c - \beta_c) X_c \equiv i\alpha_a \beta_b f_{abc} X_c, \tag{1.4}$$

or simply

$$[X_a, X_b] = i f_{abc} X_c. \tag{1.5}$$

The constants f_{abc} defined in (1.4) are called the *structure constants* and it can be shown that one can determine higher orders of Taylor expansion (1.3) using only f_{abc}. From (1.5) it can be easily shown that

$$[X_a, [X_b, X_c]] = [[X_a, X_b], X_c] + [X_b, [X_a, X_c]], \tag{1.6}$$

which is called *Jacobi identity*. As we saw in (1.3) the generators span a vector space which together with (1.5) and (1.6) means the vector space is a *Lie algebra*.

A very important representation is the *adjoint representation*. The matrix components of the operators in this representation are defined as

$$[T_a]_{bc} = -if_{abc} \tag{1.7}$$

and the Hilbert space on which they act is the Lie algebra spanned by the generators X_a (with the dot product induced by trace), from which we constructed the structure constants. We denote the adjoint representation by $R = A$.

An important characteristic of representations is *reducibility*. We say a representation is irreducible if it has no nontrivial invariant subspaces. An *invariant subspace* is a subspace which gets mapped onto itself by $R(g)$ for all $g \in G$ and *nontrivial* means the subspace is neither the whole Hilbert space nor $\{0\}$. Naturally, a representation that is not irreducible is called reducible.

For future references it will be useful to also define the quadratic Casimir $C_2(R)$

$$X_a^R X_a^R = C_2(R)\,\mathbb{1}. \tag{1.8}$$

where X_a^R are the generators of an irreducible representation R (we note the term quadratic Casimir is often defined as the whole operator in (1.8)). The fact that the sum of generators squared is proportional to the identity matrix is due to $X_a^R X_a^R$ commuting with all the generators which is trivial to show thanks to (1.5). The equation (1.8) is then a consequence of Schur's Lemma, which states that every group element commuting with all other group elements in any irreducible representation must have the form $const\mathbb{1}$.

Another useful definition is the *index* of the representation defined as

$$\text{Tr}\left[X_a^R, X_b^R\right] = T(R)\,\delta^{ab}. \tag{1.9}$$

There exists a simple relation between the quadratic casimir and the index which can be derived from (1.9) by setting $a = b$ and summing over a

$$d(R)\,C_2(R) = T(R)\,d(G), \tag{1.10}$$

where $d(R)$ is the dimension of the representation which is the dimension of the Hilbert space it is acting on and $d(G)$ is the dimension of the Lie group which is the number of parameters needed for describing its group elements.

An important example called the Lorentz group is given in Appendix A and the SU(N) groups are introduced in Appendix B.

2. Modern field theory

2.1 Classical field theory

Classical field theory is the fundamental building block of modern particle physics. Once developed it is possible to go on to construct quantum field theory, gauge theories, formulate the Standard Model and so on.

The reason we even talk about the classical and then quantum field theory is because the pure quantum mechanics (QM) is insufficient in some sense. In particular, it does not allow for changes of the number of particles in a closed system, it lacks the deeper connection between spin and Lorentz invariance and so the spin-statistics theorem must be postulated, it does not reflect the Lorentz-symmetry (it is not a relativistic theory). Quantum field theory addresses these issues and in this sense it is a more fundamental theory.

First, let us recall the Lagrangian formalism from classical mechanics where a scalar function L called the Lagrangian is defined for a system of point particles (or more generally for a finite set of parameters describing the system's degrees of freedom). With this function at hand the action S is defined as

$$S = \int L d^0 x. \tag{2.1}$$

Hamilton's principle then states that by extremizing the action one arrives at the desired equations of motion of the system of point particles. The generalization to a field theory is straightforward - substitute *fields* for the parameters representing the degrees of freedom. Conventionally, the system within a field theory is described by a Lagrangian *density* $\mathcal{L}$ from which the Lagrangian is obtained by integrating the density over space

$$L = \int \mathcal{L} d^3 x, \tag{2.2}$$

from which follows the action

$$S = \int \mathcal{L} d^4 x. \tag{2.3}$$

The function $\mathcal{L}$ is often called just the Lagrangian for brevity. From the Hamilton's principle follow the famous Euler-Largange equations

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi \right)} \right). \tag{2.4}$$

The Legendre transform of Lagrangian is the *Hamiltonian density* or just *Hamiltonian*

$$\mathcal{H} \coloneqq \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} \dot{\phi}(x) - \mathcal{L}. \tag{2.5}$$

The quantity $\mathcal{H}$ corresponds to the *energy density* of the system. It is also possible to derive the equations of motion from the Hamiltonian. However, throughout this thesis only relativistic theories will be studied and the Hamiltonian is not a

Lorentz invariant (as is obvious from the fact that energy of a system is dependent on the reference frame). Fortunately the Lagrangian is a Lorentz invariant and therefore Lagrangians will be used almost exclusively.

A very simple example of a field theory is the one of a neutral scalar field ϕ with mass m with the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial^\mu \phi) (\partial_\mu \phi) - \frac{1}{2} m^2 \phi^2. \tag{2.6}$$

Following the procedure outlined above the equations of motion for the field follow from the Euler-Lagrange formula

$$\left(\partial_\mu \partial^\mu + m^2 \right) \phi = 0. \tag{2.7}$$

This is called the Klein-Gordon equation. The general solution of this equation can be written as a Fourier decomposition into plane waves

$$\phi(x,t) = \int \frac{d^3 p}{(2\pi)^3} \left(a_p e^{-ipx} + a_p^\star e^{ipx} \right), \tag{2.8}$$

where a_p and $a_p^\star$ are some coefficients and $p = (\sqrt{\vec{p}^2 + m^2}, \vec{p}) \equiv (\omega_p, \vec{p})$.

Another example is the spinor field. The Lagrangian is

$$\mathcal{L} = \overline{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi = \overline{\psi} \left(i\partial\!\!\!/ - m \right) \psi, \tag{2.9}$$

where $\overline{\psi} := \psi^\dagger \gamma^0$ and $\partial\!\!\!/ := \gamma^\mu \partial_\mu$. The term $\overline{\psi}\gamma^\mu\psi$ can be shown to transform as a 4-vector under the Lorentz transformation which in turn means the Lagrangian really is a Lorentz scalar. The Euler-Lagrange equations lead to

$$\begin{aligned} \left(i\gamma^\mu \partial_\mu - m \right) \psi(x) &= 0 \\ \left(-i\partial_\mu \gamma^\mu - m \right) \overline{\psi} &= 0, \end{aligned} \tag{2.10}$$

where one equation implies the other so on can talk only about the first one called the *Dirac equation*.

2.2 Quantum field theory

In this section we shall briefly talk about a few basic concepts from quantum field theory (QFT). However, this section will not be a comprehensive treatment of the subject but rather an introduction for future references in the thesis. The section follows the textbooks [2] and [3] where can be also found a much more thorough discussion of the theory.

Let us go back to the solution of the Klein-Gordon equation (2.8). If we fixed a point in space of the integrand for an arbitrary momentum, we would be looking at a classical harmonic oscillator solution. Indeed, the solution is essentially describing a harmonic oscillator in each point in space for an unbounded interval of frequencies. Now one of the core ideas behind QFT is to promote the coefficients a_p, resp. $a_p^\star$ to creation, resp. annihilation operators known from quantum harmonic oscillator (basically here we have just quantized the harmonic oscillator as we would in quantum mechanics with ladder operators)

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{i\vec{p}\vec{x}} + a_p^\dagger e^{-i\vec{p}\vec{x}} \right), \tag{2.11}$$

where the factor $\frac{1}{\sqrt{2\omega_p}}$ is added for convenience.

However, as opposed to quantum mechanics, here we interpret the higher excitations as new particles. The operators act on the *Fock space* defined as

$$\mathcal{F} := \oplus_n \mathcal{H}_n, \tag{2.12}$$

where $\mathcal{H}_n$ is the Hilbert space of n-particle states. In other words, the field ϕ (2.11) is associated with the particles it creates and destroys when it acts on a state in Fock space. Formally, this can be written simply as

$$\phi(\vec{x})|0\rangle = |\vec{x}\rangle, \tag{2.13}$$

where created a particle associated with the field ϕ has been created at a point $\vec{x}$ by acting on the *vacuum state* $|0\rangle$.

Following further the analogy of integrating over harmonic oscillators the Hamiltonian of the described system is defined as

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a_p^\dagger a_p + \frac{1}{2} \right), \quad \omega_p = |\vec{p}|. \tag{2.14}$$

What was sketched here is called the **second quantization**.

In the rest of this chapter we will introduce new concepts at the level of classical field theory for more clarity. We do so because the needed concepts can be understood already with classical fields. Chapter 3 is then devoted to showing how to make the calculations on quantum level with help of the Feynman rules.

2.3 Gauge invariance and Yang-Mills theory

We will now discuss the theory for a free complex scalar field (the word *free* meaning there are no interactions)

$$\mathcal{L} = (\partial^\mu \phi)(\partial_\mu \phi^*) - m^2 \phi \phi^*. \tag{2.15}$$

Now consider the transformation

$$\phi \to e^{i\alpha} \phi, \tag{2.16}$$

which clearly leaves the Lagrangian unchanged. The transformation (2.16) is called a *global symmetry* of the Lagrangian. Suppose now that the parameter α is a smooth function of the coordinates x^μ. If acted on the field with the new transformation the Lagrangian would no longer be invariant (or *locally invariant*) thanks to the derivative. However, if the local invariance was required, how to adjust the Lagrangian? Following [4] chap.4, [5], it turns out the local invariance is achieved by introducing a new vector field A_μ with the transformation property

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha(x) . \tag{2.17}$$

Field A_μ is called a gauge field it is added to the Lagrangian as follows

$$\mathcal{L} = (\partial_\mu - ieA_\mu)\,\phi\,(\partial^\mu + ieA^\mu)\,\phi^* - m^2\phi\phi^* . \tag{2.18}$$

The constant e is the gauge *coupling* and it determines the strength of the interaction. The Lagrangian is now locally invariant under the transformation $\phi \to e^{i\alpha(x)}\phi$. It is convenient to define the *covariant derivative*

$$D_\mu \coloneqq \partial_\mu - ieA_\mu . \tag{2.19}$$

The Lagrangian can then be recast as

$$\mathcal{L} = (D_\mu\phi)\,(D^\mu\phi)^* - m^2\phi\phi^* . \tag{2.20}$$

Conventionally the terms in Lagrangians quadratic in the derivatives of the fields are called the *kinetic terms* and terms quadratic in fields are called the *mass terms* (the two terms together form the *free theory*). The rest of the terms are called the *interaction terms*. The reason for this is that the kinetic terms determine the form of free field propagators in the theory while the interaction terms determine how the fields interact with one another. However to show this mathematically is rather non-trivial and beyond the aim of this thesis (we direct the reader at [2] chap.1).

One thing that is still missing from the Lagrangian (2.20) is the kinetic term for the gauge field. For that purpose let us define

$$F_{\mu\nu} \coloneqq [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu , \tag{2.21}$$

where the second equality can be proved by letting it act on a function. The kinetic term of course needs to be gauge invariant since the whole process started with such requirement. The final form of the Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)\,(D^\mu\phi)^* - m^2\phi\phi^* , \tag{2.22}$$

where the first term is the kinetic term for the gauge field.

In the preceeding paragraphs were discussed transformations under the group U(1) (group of all unitary 1×1 matrices). Fortunately it is pretty straightforward to generalize the procedure. Consider the Lagrangian

$$\mathcal{L} = (\partial^\mu\Phi)\,(\partial_\mu\Phi)^\dagger - m^2\Phi\Phi^\dagger , \tag{2.23}$$

where Φ is an N-component object, each component being a complex scalar. The Lagrangian is obviously globally invariant under the transformation

$$\Phi \to e^{i\alpha_a X_a}\Phi , \tag{2.24}$$

where X_a are the generators of vector (defining) representation of the group SU(N) (see Appendix B). Like before we require the Lagrangian to be locally invariant under an SU(N) transformation. This is achieved by introducing N^2-1 gauge fields A^a_μ (one for each independent parameter $\alpha_a(x)$). It is convenient to represent the gauge field as a Lie-algebra-valued field $A_\mu = A^a_\mu X^a$ which then has to have the transforming property

$$A_\mu \to G A_\mu G^{-1} + \frac{i}{g} G \left(\partial_\mu G^{-1}\right), \tag{2.25}$$

where G is the transformation operator. The covariant derivative is defined as

$$D_\mu = \partial_\mu - igA_\mu \tag{2.26}$$

and the kinetic term for gauge fields

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu]\,. \tag{2.27}$$

The components $F_{\mu\nu} = F_{a\mu\nu} X_a$ read

$$F_{a\mu\nu} = \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + g f_{abc} A_{b\mu} A_{b\nu}. \tag{2.28}$$

Finally, the locally invariant Lagrangian under an SU(N) transformation has the form

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + (D_\mu\Phi)(D^\mu\Phi)^\dagger - m^2\Phi\Phi^\dagger. \tag{2.29}$$

Note that there appeared triple-gauge field and quadruple-gauge field interactions in the Lagrangian.

2.4 Goldstone theorem

In this section we shall discuss the *spontaneous symmetry breakdown* phenomenon and how it generates massless, spinless mesons in the spectrum of relativistic QFTs. It is important to note we talk about relativistic systems (as opposed to Galilean-invariant systems where the results would differ). Although our approach will be classical we opt to use the language of quantum field theory.

Let us begin with a simple Lagrangian of a complex scalar field, following Hořejší [4] section 6.2

$$\mathcal{L} = (\partial^\mu\phi)(\partial_\mu\phi^*) - V(\phi)\,, \tag{2.30}$$

where the function V called *potential*, is

$$V(\phi) = -\mu^2\phi\phi^* + \lambda\,(\phi\phi^*)^2\,, \tag{2.31}$$

where λ is the coupling constant and μ is a real parameter with the dimension of mass. Note that the mass term has a "wrong" sign, which can be seen from comparing to (2.5), or (2.6) (if we dismiss the interaction term for a moment). Also, the Lagrangian is obviously invariant with respect to U(1) which will be important later.

Going forward, we wish to study the theory perturbatively. Therefore we want to rewrite the theory so that the fields are expressed in terms of small oscillations around the energy minimum. Furthermore, we want to avoid interaction terms with less than 3 fields participating as well as tachyionic behavior. Tachynic fields are defined as fields with imaginary masses which would imply the existence particles traveling faster than the speed of light. So let us take a look at the energy density of our system, i.e. the Hamiltonian. Using the definition (2.5) the Hamiltonian is

$$\mathcal{H} = \left(\partial^\mu \phi\right)\left(\partial_\mu \phi^*\right) + V\left(\phi\right). \tag{2.32}$$

Apparently the lowest-energy configuration of the system corresponds to a constant field $\phi = \phi_0$ minimizing the potential. We observe that V does not depend on the phase of ϕ, therefore it is useful to define $\rho^2 \coloneqq \phi\phi^*$. The potential then becomes $V\left(\rho\right) = -\mu^2\rho^2 + \lambda\left(\rho^2\right)^2$. The minima of (2.31) satisfy both $V'\left(\rho\right) \stackrel{!}{=} 0$ and $V''\left(\rho\right) \stackrel{!}{>} 0$ and those points are $\rho = \pm\mu\sqrt{2\lambda}$. In terms of our original field the minima are

$$\phi_0 = \frac{\mu}{\sqrt{2\lambda}}\mathrm{e}^{i\alpha} = \frac{v}{\sqrt{2}}\mathrm{e}^{i\alpha}, \tag{2.33}$$

where α is an arbitrary phase and $v = \mu\sqrt{\lambda}$. The quantity $v\sqrt{2}$ is usually referred to as the *vacuum expectation value* or VEV. In order to study the desired low-energy behavior it is convenient to rewrite the fields as

$$\phi\left(x\right) = \rho\left(x\right)\exp\left(\frac{i\pi\left(x\right)}{v}\right), \tag{2.34}$$

where $1/v$ makes sure field π the argument of the exponential is dimensionless. Plugging this into the Lagrangian (2.30) gives

$$\mathcal{L} = \left(\partial^\mu \rho\right)\left(\partial_\mu \rho\right) + \frac{1}{v^2}\rho^2\left(\partial^\mu \pi\right)\left(\partial_\mu \pi\right) - V\left(\rho\right) \tag{2.35}$$

with

$$V\left(\rho\right) = \lambda\left(\rho^2 - \frac{v^2}{2}\right)^2 - \frac{1}{4}\lambda v^4. \tag{2.36}$$

In the future we will drop the last term in the potential as it has no physical implication (it merely shifts the energy density). Now here comes the main point of this procedure - we define a new field σ by

$$\rho = \frac{1}{\sqrt{2}}\left(\sigma + v\right). \tag{2.37}$$

So the field σ essentially describes the perturbations around the vacuum state. We plug this into the Lagrangian and collect the results

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \sigma\right)\left(\partial_\mu \sigma\right) + \frac{1}{2}\left(\partial^\mu \pi\right)\left(\partial_\mu \pi\right) - \lambda v^2\sigma^2 + \text{interactions}. \tag{2.38}$$

The term *interactions* follows the terminology introduced in previous section. The important thing is that we can now clearly see two scalar fields - one massive field σ with the mass (with the right sign) $m_\sigma^2 = 2\lambda v^2$; one massless field π, e.g. $m_\pi = 0$. This is a new result we could not see in the original Lagrangian (2.30). The particle associated with the field π is called a *Goldstone boson*.

It is important to note neither of these descriptions is more "correct" in a any fundamental sense. The transformation (2.34) is simply better suited to study the theory perturbatively around the vacuum.

Let us discuss the symmetry of the example. We started with an invariant Lagrangian under the group U(1) yet the ground state (2.33) clearly does not reflect such invariance. The ground state is continuously degenerate and applying U(1) transformation would get us from one ground state to another. This is the essence of spontaneous symmetry breakdown - the symmetry of the *theory* is no longer manifest in the *physical spectrum*. One might then say the symmetry is rather hidden, but whether *spontaneous* is the right word is semantics not physics. We are now ready to formulate these conclusions more generally.

Goldstone theorem states that if there is a continuous symmetry of the Lagrangian, the non-invariance of its ground state then implies the existence of a massless, spinless particle called the Goldstone boson. This important result will be put to work in the following section.

2.5 Higgs mechanism

In the following paragraphs we will cover the mass-generating mechanism in gauge theories. The reason to employ this mechanism has to do with gauge invariance and renormalizibility. It is easy to see that a term like $\frac{1}{2}m^2AA$ for a gauge field clearly breaks gauge invariance. As for other fields the reasoning is tightly connected to the renormalizibility of the theory. We do not go into further detail on this point here so we only refer the reader to [4] for more details.

Consider a Lagrangian containing a massless complex scalar field with potential (2.36) and an Abelian gauge field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu} + (D_\mu\phi)(D^\mu\phi^*) - \lambda\left(\phi\phi^* - \frac{v^2}{2}\right)^2 . \tag{2.39}$$

The Lagrangian is clearly locally invariant under U(1) transformations

$$\begin{aligned} \phi(x) &\to e^{i\omega(x)}\phi(x) \\ \phi^*(x) &\to e^{-i\omega(x)}\phi^*(x) \\ A_\mu(x) &\to A_\mu(x) + \frac{1}{e}\partial_\mu\omega(x) . \end{aligned} \tag{2.40}$$

If we rewrite the scalar field into the form (2.34) the gauge transformation becomes

$$\begin{aligned} \rho(x) &\to \rho(x) \\ \pi(x) &\to \pi(x) + v\omega(x) \\ A_\mu(x) &\to A_\mu(x) + \frac{1}{e}\partial_\mu\omega(x) \end{aligned} \tag{2.41}$$

Here comes the crucial observation, since the theory is gauge invariant (any physical prediction is independent of the gauge used) it is possible to choose the transformation so that eliminate the field π is eliminated. More specifically we set $\omega(x) := -\pi v$ (this is called the U-gauge) and end up with

$$\begin{aligned} \rho(x) &\to \rho(x) \\ \pi(x) &\to 0 \\ A_\mu(x) &\to A_\mu(x) + \frac{1}{ev}\partial_\mu \pi(x). \end{aligned} \tag{2.42}$$

By doing this we have effectively got rid of the massless Goldstone boson - we gauged it away. Following further the procedure from previous chapter we define a new field σ as the perturbations around the VEV by shifting ρ as in (2.37). For a clearer picture the gauge field in U-gauge is denoted by B_μ with kinetic term $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. The Largangian thus reads

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}(\partial^\mu\sigma)(\partial_\mu\sigma) - \lambda v^2\sigma^2 + \frac{1}{2}g^2v^2B_\mu B^\mu \\ &+ g^2 v\sigma B_\mu B^\mu + g^2\sigma^2 B_\mu B^\mu - \lambda v\sigma^3 - \frac{1}{4}\lambda\sigma^4. \end{aligned} \tag{2.43}$$

Looking at (2.43) a few new interaction arose, namely $\sigma\sigma BB$ and triple and quartic self-interaction of the field σ. The field σ now has a "correct"-sign mass term (which is not surprising, it comes from the same potential as in previous section). However the main point is that the vector field B_μ now has a mass term, which is something we did not see in the original Lagrangian.

So what happened was we begun with a complex scalar field with a potential (we will call it the Higgs potential) which was coupled to a massless gauge field. The complex scalar field was rewritten in terms of radial ρ and angular π variables and after that it was possible to gauge away the angular variable. The radial field ρ was shifted to a minimum of the potential by introducing the field σ describing the perturbations around the minimum. Once we wrote down the new Lagrangian we found out there was no Goldstone boson (we gauged it away) but instead the gauge field acquired mass. This procedure is the famous **Higgs mechanism**.

2.6 Electroweak symmetry breaking

In the previous section we discussed the case of symmetry breaking of an Abelian gauge theory. Here we shall discuss the very important non-Abelian example of breaking chain SU(2)×U(1)$_Y$ →U(1)$_{EM}$. The U(1)$_Y$ symmetry is the so-called hypercharge unlike U(1)$_{EM}$ which generates the well-known electromagentism. This case appears in the SM as the electroweak unification of the weak force with the electromagnetic force. As we shall see the SU(2)×U(1)$_Y$ symmetry group will spontaneously break into U(1) leaving three massive and one massless gauge fields.

Following [2] chap.29.1, consider the Lagrangian of the SU(2)×U(1)$_Y$ gauge theory with a complex scalar field transforming as a dublet under SU(2) and as a singlet under $\mathrm{U}(1)_Y$.

$$\mathcal{L} = \left(\partial_\mu H - igW_\mu^a \tau^a H - \frac{1}{2}ig' B_\mu H\right)^\dagger \left(\partial_\mu H - igW_\mu^a \tau^a H - \frac{1}{2}ig' B_\mu H\right)$$
$$- \frac{1}{4}\left(W_{\mu\nu}^a\right)^2 - \frac{1}{4}B_{\mu\nu}^2 + m^2 H^\dagger H - \lambda \left(H^\dagger H\right)^2, \tag{2.44}$$

where the Higgs potential was included and the SU(2) generators (the Pauli matrices) were denoted by τ^a. The complex scalar called the Higgs field can be rewritten as

$$H = \exp\left(i\frac{\pi^a \tau^a}{v}\right)\begin{pmatrix} 0 \\ \frac{\rho}{\sqrt{2}} \end{pmatrix}. \tag{2.45}$$

Like before we find the VEV $v = m\sqrt{\lambda}$ induced by the Higgs potential. We shift the radial field variable to the minimum of the potential also apply the U-gauge to get rid of the unphysical fields π^a so the complex scalar is simply

$$H = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + \frac{h}{\sqrt{2}} \end{pmatrix}. \tag{2.46}$$

Plugging this into the derivative term in (2.44) gives

$$|D_\mu H|^2 = g^2 \frac{v^2}{8}\left[\left(W_\mu^1\right)^2 + \left(W_\mu^2\right)^2 + \left(\frac{g'}{g}B_\mu - W_\mu^3\right)^2\right]. \tag{2.47}$$

Now those fields themselves do not actually correspond to any particles we measure.

First, one needs look to diagonalize the mass matrix, i.e. the equation (2.47) could be rewritten as a quadratic form (the mass matrix) containing the coefficients acting on a vector of fields involved. The matrix of this form is then diagonalized by transformation

$$Z_\mu := \cos\theta_w W_\mu^3 - \sin\theta_w B_\mu$$
$$A_\mu := \sin\theta_w W_\mu^3 + \cos\theta_w B_\mu, \tag{2.48}$$

where the parameter θ_w is called the *weak mixing angle* and it has to satisfy

$$\tan\theta_w = \frac{g'}{g}. \tag{2.49}$$

Second, the newly massive gauge fields need to correspond to the observed interactions, which introduces further linear combination of the fields present in (2.47). We will not go into much detail regarding this wide topic so we only refer to [4] for an in-depth analysis of the relevant weak V-A charged currents. We simply state the resulting transformation

$$\tau^\pm := \frac{1}{\sqrt{2}}\left(\tau^1 \pm i\tau^2\right)$$
$$W_\mu^\pm := \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right)^2. \tag{2.50}$$

The final form of the gauge masses sector of the Lagrangian is

$$\mathcal{L}_{gauge\ masses} = \frac{1}{2}\left(\frac{1}{2\cos\theta_w}gv\right)^2 Z^\mu Z_\mu + \left(\frac{v}{2}g\right)^2 W_\mu^+ W^{-\mu}, \tag{2.51}$$

where W_μ^+ and W_μ^- are each other's antiparticles and thus have identical masses. From the last expression follow masses of the particles

$$m_W = \frac{1}{2}gv \tag{2.52}$$

$$m_Z = \frac{m_W}{\cos\theta_w} = \frac{1}{2}\left(g^2 + g'^2\right)^{1/2} v. \tag{2.53}$$

The mass of the Higgs field can be easily obtained by plugging (2.45) into (2.44)

$$m_h = \sqrt{2\lambda}v. \tag{2.54}$$

Lastly, we note the vacuum expectation value v is tightly connected to the famous *Fermi constant* G_F by

$$v = \left(G_F\sqrt{2}\right)^{-1/2}. \tag{2.55}$$

G_F is the coupling constant of the old Fermi's interaction. It was proposed by Enrico Fermi to describe beta decay in the 1930's and it is in a good agreement with the experimental data at low energies. However, later it was replaced be the electroweak theory we have just described, which works well also at higher energies. The reason those two constants originating from different theories have a simple relation (2.55) is because v is the only relevant mass scale entering the Higgs mechanism as well as G_F is the only parameter of dimension [mass] appearing in the Fermi theory. From this perspective the relation (2.55) is quite natural.

3. Feynman diagrams and renormalization

3.1 Observables in QFT

So far we have been mostly talking about the classical field theory. In section 2.2 we outlined how the quantum field theory conceptually changes the picture and now we shall use it in practice. To get a prediction out of the theory we need to calculate an observable. The common examples of observables are the differential cross-section $d\sigma$ or the decay rate Γ. The formulas for calculating $d\sigma$ (of two particles with energies E_1, E_2 and velocities $\vec{v}_1$, $\vec{v}_2$) or Γ (for a particle in its rest frame with mass M) read

$$d\sigma = \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|}|\mathcal{M}|^2 d\Pi_{\text{LIPS}}$$
$$d\Gamma = \frac{n}{2M}|\mathcal{M}|^2 d\Pi_{\text{LIPS}}, \tag{3.1}$$

where n is a combinatorial factor accounting for indistinguishable final states and the $\mathcal{M}$-matrix is defined as

$$\mathcal{M}_{i\to f} = i(2\pi)^4\delta(\textstyle\sum p)\langle f|(S - \mathbb{1})|i\rangle, \tag{3.2}$$

where $|f\rangle$ and $|i\rangle$ are the final and initial states respectively, S is the time-evolution operator and the δ-function is there to conserve the overall momentum of the particles involved. The Lorentz invariant phase space element is defined as

$$d\Pi_{\text{LIPS}} := \prod_{\text{final states } j} \frac{d^3p_j}{(2\pi)^3}\frac{1}{2E_{p_j}}(2\pi)^4\delta^4(\Sigma p). \tag{3.3}$$

The matrix elements $\mathcal{M}$ are usually solved perturbatively, i.e. expressed as a power series expansion with respect to the coupling constant. This calculation is done using a pictorial representation called the *Feynman diagrams* associated with the *Feynman rules*. The derivation of these rules is fairly non-trivial and beyond the scope of this thesis but we do need to do these calculations. Therefore, we will show how such a calculation is done on an example that will yield some useful results for later. The actual derivation of the procedure can be found in for example [2] and[3].

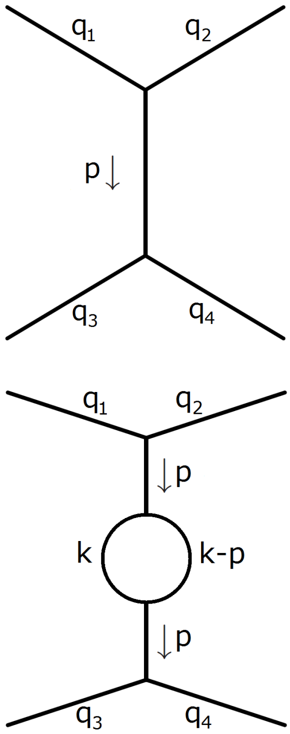
3.2 Scalar theory and renormalization

This introduction follows [2], chap 16.1. Consider this simple Lagrangian of a real scalar massless field

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3. \tag{3.4}$$

The Feynman rules following from this Lagrangian dictate that vertices produce the coupling constant g and internal lines produce the factor of $\frac{i}{p^2+i\epsilon}$. One then must conserve momentum at the vertices and integrate over inner momenta.

One of the simplest diagrams is the tree diagram of t-channel scattering depicted in the first picture in Figure 3.1.



$= (ig)^2 \frac{i}{(q_1+q_2)^2}$

$= (ig)^2 \frac{i}{(q_1+q_2)^2} i\mathcal{M}_{\text{loop}}((q_1+q_2)^2) \frac{i}{(q_1+q_2)^2}$

Figure 3.1: Diargams of t-channel scattering

Now we would like to calculate correction of the leading order. The contribution of the second lowest order in g to the t-scattering amplitude corresponds to the second diagram in Figure 3.1. From now on we shall use $p = q_1 + q_2$ for brevity. Using the Feynman rules given above we can write the contribution of the loop as

$$i\mathcal{M}_{\text{loop}}(p) = \frac{1}{2}(ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k-p)^2 - m^2 + i\varepsilon} \frac{i}{k^2 - m^2 + i\varepsilon} \tag{3.5}$$

where have already conserved the momenta at the vertices. The integral on the right-hand side is clearly divergent as $\int \frac{dk}{k}$ (which diverges as $|k| \to \infty$) but we would like to interpret the resulting matrix element as an observable amplitude of probability density. To do that we will rearrange the integral using a regulator scheme which is going to isolate the formal infinity. Afterwards we will be able to get rid of the formally infinite term. First, we rearrange the integral using the formula with Feynman parameters

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[A + (B-A)x]^2}. \tag{3.6}$$

Now plugging in $A = (k-p)^2 + i\varepsilon$ and $B = k^2 + i\varepsilon$

$$\frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{1}{(k-p)^2 - m^2 + i\varepsilon + [k^2 - (k-p)^2]\,x} = \tag{3.7}$$

$$\frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{1}{[(k-p(1-x))^2 + p^2x(1-x) - m^2 + i\varepsilon]^2} = \tag{3.8}$$

$$\frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{1}{[k^2 - (m^2 - p^2x(1-x)) + i\varepsilon]^2} = \tag{3.9}$$

$$\frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{1}{[k^2 - \Delta + i\varepsilon]^2}, \tag{3.10}$$

where in the first step we completed the square, in the second we shifted $k^\mu \to k^\mu + p^\mu(1-x)$, and in the third we introduced $\Delta \coloneqq m^2 - p^2x(1-x)$ (which is a positive number since the virtual momentum p^μ is spacelike). The integrand in the last expression has poles at $k_0 = (\vec{k}^2 + \Delta)^{1/2} - i\varepsilon$ and $k_0 = -(\vec{k}^2 + \Delta)^{1/2} + i\varepsilon$ which means it is in fact holomorphic in first and third quadrants. Therefore we can apply the Cauchy's integral theorem with the integration curve following the real and imaginary axes and connecting those with an arc in the first and third quadrant. The contributions from the arcs are obviously zero. Therefore the integral along the real axis can be expressed in terms of the integral along the imaginary axis. With this observation we substitute $k_0 \to ik_0$ implying $k^2 \to -k_0^2 - \vec{k}^2 = -k_E^2$, where we defined k_E^2 as the Euclidean norm squared of vector k. This procedure is called the Wick rotation. Therefore the integral becomes

$$\frac{g^2}{2}\int \frac{d^4k_E}{(2\pi)^4}\int_0^1 dx \frac{1}{[k_E^2 + \Delta + i\varepsilon]^2}. \tag{3.11}$$

Now comes the time to introduce the aforementioned regulator. We shall use the Pauli-Villars regularization scheme which introduces a new heavy particle whose contribution to the amplitude will cancel out the divergence of the integral. In order to do that the new particle of large mass $\Lambda \gg m$, called the Pauli-Villars ghost, must have either the wrong statistics (i.e. fermionic or bosonic) or the wrong sign kinetic term (these requirements imply the cancellation of the problematic divergence). This of course makes the particle unphysical, hence the word *ghost*. Once the ghost's mass Λ will be canceled out, it will be taken as an infinte limit in order to restore the original theory. In our case we can introduce a new scalar field $\tilde{\phi}$ with the wrong sign kinetic term. The Lagrangian then becomes

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - \frac{1}{2}\Lambda^2\tilde{\phi}^2 \tag{3.12}$$

The amplitude now has to sum over the real field ϕ and the ghost field $\tilde{\phi}$ (whose propagator has a negative sign due to the wrong sign kinetic term) leaving us with the integral (after the Wick rotation)

$$\int \frac{d^4k_E}{(2\pi)^4}\left[\frac{1}{(k_E^2 + \Delta^2 + i\varepsilon)^2} - \frac{1}{(k_E^2 + \Lambda^2 + i\varepsilon)^2}\right]. \tag{3.13}$$

This integral is convergent as can be seen from Taylor expansion for $k \gg \Lambda, \Delta$ of the integrand $\sim 2\frac{m^2-\Lambda^2}{k_E^6} + O(\frac{1}{k_E^8})$. Thus we arrive at

$$\begin{aligned}
i\mathcal{M}_{\text{loop}}(p) &= \frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\int_0^1 dx \left[\frac{1}{\left(k_E^2+\Delta^2+i\varepsilon\right)^2} - \frac{1}{\left(k_E^2+\Lambda^2+i\varepsilon\right)^2}\right] \\
&= -\frac{ig^2}{32\pi^2}\int_0^1 dx \ln\left(\frac{m^2-p^2x(1-x)}{\Lambda^2}\right) \\
&= \frac{g^2}{32\pi^2}\left[2 - \ln\frac{-p^2}{\Lambda^2}\right] \\
&= -\frac{g^2}{32\pi^2}\ln\frac{Q^2}{\Lambda^2},
\end{aligned} \tag{3.14}$$

where in the second step we plugged in for Δ. In the third step was made an assumption $p \gg m$ which is by no means guaranteed and we will come back to it later. In the last step we observed the factor 2 must be unphysical since redefinig the arbitrary ghost mass $\Lambda^2 \to \Lambda^2 e^{-2}$ gets us rid of it. Also we defined $Q^2 \coloneqq -p^2$. Our matrix element now looks like

$$\begin{aligned}
\mathcal{M}(Q) &= \frac{g^2}{Q^2}\left(1 - \frac{1}{32\pi^2}\frac{g^2}{Q^2}\ln\frac{Q^2}{\Lambda^2} + \mathcal{O}\left(g^4\right)\right) \\
&= \tilde{g}^2\left(1 - \frac{1}{32\pi^2}\tilde{g}^2\ln\frac{Q^2}{\Lambda^2} + \mathcal{O}\left(g^4\right)\right),
\end{aligned} \tag{3.15}$$

where in the last step we defined a dimensionless coupling $\tilde{g}^2 \coloneqq \frac{g^2}{Q^2}$. Now here comes the core idea of renormalization: we define a *renormalized coupling*

$$\tilde{g}_R^2 \coloneqq \mathcal{M}(Q_0) \tag{3.16}$$

at an arbitrary scale Q_0 and calculate matrix elements at different scales with respect to this value. In other words we can measure the amplitude at the scale Q_0 and use that value to predict the amplitude at a different scale. Equation (3.17) is called a *renormalization condition*. To derive the explicit prediction we start by rewriting the definition of renormalized coupling

$$\tilde{g}_R^2 \equiv \mathcal{M}(Q_0) = \tilde{g}^2\left(1 - \frac{1}{32\pi^2}\tilde{g}^2\ln\frac{Q_0^2}{\Lambda^2} + \mathcal{O}\left(\tilde{g}^4\right)\right). \tag{3.17}$$

So we have expressed $\tilde{g}_R^2$ as a power series of $\tilde{g}^2$. We can invert this power series by plugging $\tilde{g}^2 = \tilde{g}_R^2 + a\tilde{g}_R^4$ into (3.17)

$$\tilde{g}^2 = \tilde{g}_R^2 + \frac{1}{32\pi^2}\tilde{g}_R^4\ln\frac{Q_0^2}{\Lambda^2} + \mathcal{O}\left(\tilde{g}_R^6\right). \tag{3.18}$$

Finally we insert (3.18) into the equation for the matrix element (3.15)

$$\mathcal{M}(Q) = \tilde{g}_R^2 - \frac{1}{32\pi^2}\tilde{g}_R^4\ln\frac{Q^2}{Q_0^2} + \mathcal{O}\left(\tilde{g}_R^6\right). \tag{3.19}$$

As we can see the ghost's mass has been successfully canceled out, so by taking the limit $\Lambda \to \infty$ we end up with a physical prediction for our restored original theory.

Let us now go back to the assumption $p \gg m$ in (3.14). Suppose the assumption is not satisfied the integral containing the logarithm is carried further. Like before, the renormalization condition (3.16) would be imposed and finally the matrix element in terms of g_R would have the form

$$\mathcal{M}(Q) = \tilde{g}_R^2 - \frac{1}{32\pi^2}\tilde{g}_R^4 \int_0^1 dx \ln\left(\frac{m^2 + Q^2 x(1-x)}{m^2 + Q_0^2 x(1-x)}\right) + \mathcal{O}\left(\tilde{g}_R^6\right). \tag{3.20}$$

Clearly if the assumption $p \gg m$ was satisfied then (3.20) would turn into (3.19). On other hand if $p \ll m$ the logarithm would vanish altogether. In other words the whole 1-loop correction to the tree-level amplitude disappears and what is left is $\tilde{g}_R^2 \coloneqq \mathcal{M}(Q_0)$.

Looking at the equation (3.19) it is obvious the prediction can work only for Q^2 close to Q_0^2. One way to see this is if Q^2 were sufficiently bigger (or smaller) than Q_0^2 then $|\mathcal{M}|^2$ would become bigger than 1. The way to avoid this is to always have $\tilde{g}_R^2$ at the scale of the desired Q^2. This is called the *running coupling* as the renormalized coupling can be viewed as a function of Q_0^2 defined by (3.16). The crucial observation here is that the matrix element $\mathcal{M}$ in (3.19) is independent of the arbitrary scale Q_0 where we defined $\tilde{g}_R^2$. This is a special case of what is called the *continuum renormalization group* (RG), which states that observables are independent of the scales at which we choose to define the renormalized quantities.

To derive the equation for the running coupling we use the (RG) and take the derivative of (3.19) with respect to the scale Q_0

$$\begin{aligned} 0 = \frac{\partial \mathcal{M}(Q)}{\partial Q_0} &= \frac{\partial}{\partial Q_0}\left[\tilde{g}_R^2\left(1 - \frac{1}{32\pi^2}\tilde{g}_R^2 \ln\frac{Q^2}{Q_0^2}\right)\right] \\ &= 2\tilde{g}_R \frac{\partial \tilde{g}_R}{\partial Q_0} + \frac{\tilde{g}_R^4}{16\pi^2 Q_0} - 2\tilde{g}_R^3 \frac{\partial \tilde{g}_R}{\partial Q_0}\frac{1}{16\pi^2}\ln\frac{Q^2}{Q_0^2}, \end{aligned} \tag{3.21}$$

where we note $\frac{\partial \tilde{g}_R}{\partial Q_0}$ is $\mathcal{O}\left(\tilde{g}_R^3\right)$. From the last equation can be solved for $Q_0 \frac{\partial \tilde{g}_R}{\partial Q_0}$ to order $\mathcal{O}\left(\tilde{g}_R^3\right)$

$$Q_0 \frac{\partial \tilde{g}_R}{\partial Q_0} = -\frac{\tilde{g}_R^3}{32\pi^2}. \tag{3.22}$$

The left-hand side of (3.22) is called the *β-function*.

3.3 Non-Abelian gauge theory renormalization

In the previous section we renormalized the simple scalar theory with Lagrangian (3.6). At the end of the section we derived the *β-function* of the coupling. In this section we will introduce the dimensional regularization (DR) scheme and we shall use it to renormalize a non-Abelian theory. We do that because at the end of the chapter we will derive the *β-function* for the theory and for that we will need the results from this section.

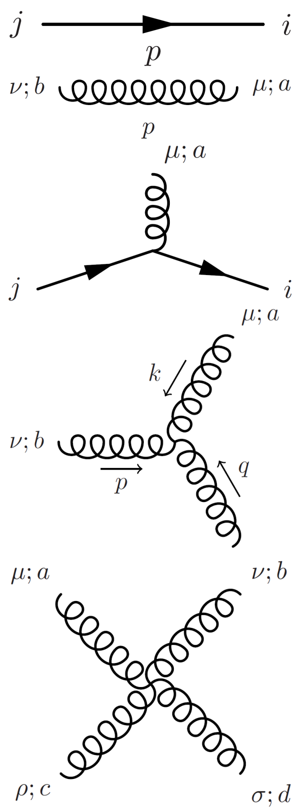
Like before we will calculate 1-loop diagrams and therefore encounter divergent integrals. Using (DR) we will be able to isolate the divergences. It will turn out that in order to derive the β-*function* at the end we only need the divergent terms.

Let us begin by writing down a gauge Lagrangian

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 - \frac{1}{2\xi}\left(\partial_\mu A^a_\mu\right)^2 + \left(\partial_\mu \bar{c}^a\right)\left(\delta^{ac}\partial_\mu + g f^{abc} A^b_\mu\right) c^c \\ &+ \overline{\psi}_i \left(\delta_{ij} i\not{\partial} + g A^a T^a_{ij} - m\delta_{ij}\right)\psi_j \\ &+ \left[\left(\delta_{ki}\partial_\mu - igA^a_\mu T^a_{ki}\right)\phi_i\right]^\star \left[\left(\delta_{kj}\partial_\mu - igA^a_\mu T^a_{kj}\right)\phi_j\right] - M^2\phi^\star_i\phi_i.\end{aligned} \tag{3.23}$$

Recalling section 2.3 there is a kinetic term for the gauge field, the fermionic field, and the complex scalar field. The fermionic and scalar fields are coupled to the gauge fields via the covariant derivative. The second and third term have been added in order to deal with various issues encountered upon quantizing a gauge theory. The scalar field denoted by c is called the *Faddeev–Popov ghosts*. We will not go into why those terms need to appear, an interested reader can find the arguments in [2], chap 25.4.

Some of the Feynman rules (in the so-called Feynman gauge) derived from (3.23) are shown in Figure 3.2.



$$= \frac{i\delta^{ij}}{\not{p} - m + i\epsilon}$$

$$= \frac{-ig^{\mu\nu}\delta^{ab}}{p^2 + i\epsilon}$$

$$= ig\gamma^\mu T^a_{ij}$$

$$\begin{aligned}= g f^{abc}[&g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu \\ &+ g^{\rho\mu}(q-k)^\nu]\end{aligned}$$

$$\begin{aligned}= -ig^2 \times \Big[&f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) \\ &+ f^{ace}f^{bde}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) \\ &+ f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\Big]\end{aligned}$$

Figure 3.2: Gauge theory Feynman rules for two propagators and three vertices.

As in the previous section we will focus on 1-loop level corrections to tree-level diagrams. However, by going into higher order corrections we encounter divergent integrals of the form $\int \frac{d^4k}{k^4}$. This can be solved by introducing *renormalized* fields, couplings, and masses. In this context it is common to refer to Lagrangian (3.23) and the fields, couplings, and masses in it as *bare*. Using a natural notation we define $A^a_B = \sqrt{Z_A} A^a_R, \psi_B = \sqrt{Z_\psi}\psi_R, \phi_B = \sqrt{Z_\phi}\phi_R, m_B = Z_m m_R, M_B = Z_M M_R, c_B = \sqrt{Z_c} c_R$, and $g_B = g_R \frac{Z_g}{Z_\psi \sqrt{Z_A}}$.

The Lagrangian thus reads

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{4}\left(\partial_\mu \sqrt{Z_A} A_{a\nu} - \partial_\nu \sqrt{Z_A} A_{a\mu} + g f_{abc} Z_A A_{b\mu} A_{b\nu}\right)^2 \\
&- \frac{1}{2\xi}\left(\partial_\mu \sqrt{Z_A} A^a_\mu\right)^2 + Z_c\left(\partial_\mu \bar{c}^a\right)\left(\delta^{ac}\partial_\mu + g f^{abc}\sqrt{Z_A} A^b_\mu\right) c^c \\
&+ Z_\psi \overline{\psi}_i \left(\delta_{ij} i\partial\!\!\!/ + g\sqrt{Z_A} A^a T^a_{ij} - \sqrt{Z_m} m \delta_{ij}\right)\psi_j \\
&+ Z_\phi \left[\left(\delta_{ki}\partial_\mu - ig\sqrt{Z_A} A^a_\mu T^a_{ki}\right)\phi_i\right]^\star \left[\left(\delta_{kj}\partial_\mu - ig\sqrt{Z_A} A^a_\mu T^a_{kj}\right)\phi_j\right] \\
&- Z_M M^2 \phi^\star_i \phi_i,
\end{aligned}
\tag{3.24}
$$

where we omitted the subscripts R for brevity.

The renormalization factors are expanded as $Z_i = 1 + \delta_i$ where δ_i is called the *counterterm* and is formally of the order $\mathcal{O}(g_R^2)$. In order to fix the parameters we need the renormalization conditions coming out of the Feynman diagrams. In particular we will be interested in the counterterms $\delta_g, \delta_\psi, \delta_A$ appearing in

$$
g_B = g_R \frac{Z_g}{Z_\psi \sqrt{Z_A}} \tag{3.25}
$$

since from this equation will follow the *β-function.* As we noted at the beginning of the chapter we shall use the dimensional regularization (DR) scheme in our calculations. However there will be quite a few diagrams needed in what follows so rather than calculating them one by one we included the general description of the procedure in Appendix B.

Once the procedure of (DR) is employed and the poles are isolated one can define the counterterms so that only the divergent terms subtract. This is called the *minimal subtraction* scheme and the counterterm looks like $\delta_g \approx \frac{g_R^2}{12\pi^2}\left[-\frac{2}{\varepsilon}\right]$. The matrix elements will then be dependent on the arbitrary scale μ similarly to the dependence of the matrix elements on Λ in the previous section. Following further the analogy, one can get rid of the μ dependence by comparing $\mathcal{M}(Q)$ to $\mathcal{M}(Q_0)$, where Q_0 is a scale at which we defined the renormalized coupling. However, one can imagine setting the counterterm like $\delta_g \approx \frac{g_R^2}{12\pi^2}\left[-\frac{2}{\varepsilon} - \ln\frac{\mu^2}{Q_0^2}\right]$ and setting $\mu \coloneqq Q_0^2$. This effectively gets us rid of the arbitrary scale μ (as well as the unphysical parameter ε) and is equivalent to renormalizing the theory at a scale Q_0 similarly to the previous section.

Based on the previous paragraph we call μ the *renormalization scale* as it plays the role of the scale where we define the renormalized terms like in (3.16).

Let us now go back to the theory (3.24) and fix the counterterms via renormalization conditions. First we will discuss the *vacuum polarization* graphs

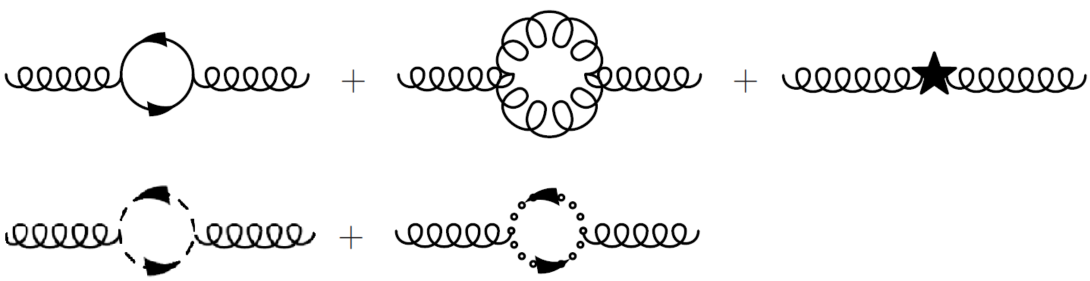


Figure 3.3: Vacuum polarization graphs (from the left): fermion loop, gauge boson loop, counterterm, scalar loop, ghost loop.

$$= i\mathcal{M}^{ab\mu\nu} = i\mathcal{M}_F^{ab\mu\nu} + i\mathcal{M}_3^{ab\mu\nu} + i\mathcal{M}_{cterm}^{ab\mu\nu} + i\mathcal{M}_{sc}^{ab\mu\nu} + i\mathcal{M}_{ghost}^{ab\mu\nu}. \tag{3.26}$$

This is usually called the 2-point function (more accurately, its 1-loop corrections) or a propagator. The a, b are the color indices and μ, ν are the Lorentz indices, all corresponding to the incoming resp. outgoing gluons. We also note that technically we should include the quadruple interaction graphs (schematically) $ggss$ and $gggg$ but these turn out to be zero. Most of the relevant Feynman rules have already been laid out in Figure 3.2 so for these we can go ahead and write down the amplitudes. The occurring integrals will be implicitly solved via (DR). Let us denote the incoming 4-momentum by p and the inner momentum (when there is one) by k.

$$\begin{aligned} i\mathcal{M}_F^{ab\mu\nu} &= i\mathrm{Tr}[T^aT^b](g)^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}[\gamma^\mu(\not{k} - \not{p} + m)\gamma^\nu(\not{k} + m)]}{((p-k)^2 - m^2)(k^2 - m^2)} \\ i\mathcal{M}_3^{ab\mu\nu} &= i\frac{g^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2} \frac{-i}{(k-p)^2} f^{ace} f^{bdf} \delta^{cf} \delta^{ed} \\ &\times \left[g^{\mu\alpha}(p+k)^\rho + g^{\alpha\rho}(p-2k)^\mu + g^{\rho\mu}(k-2p)^\alpha\right] g^{\alpha\beta} g^{\rho\sigma} \\ &\times \left[g^{\nu\beta}(p+k)^\sigma - g^{\beta\sigma}(2k-p)^\nu - g^{\sigma\nu}(2p-k)^\beta\right]. \end{aligned} \tag{3.27}$$

For the scalar graph we need the vertex factor: $ig\left(k^\mu + q^\mu\right) T^a_{ij}$; and the propagator: $\frac{i\delta^{ij}}{p^2 - M^2 + i\varepsilon}$. Similarly for the ghost graph vertex we have a factor: $-gf^{abc}p^\mu$ and the propagator: $\frac{i\delta^{ab}}{p^2 + i\varepsilon}$. The corresponding amplitudes are

$$\begin{aligned} i\mathcal{M}_{cterm}^{ab\mu\nu} &= -i\delta_A\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \\ i\mathcal{M}_{sc}^{ab\mu\nu} &= \mathrm{Tr}[T^aT^b](g)^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2k+p)^\mu(2k+p)^\nu}{((p-k)^2 - M^2)(p^2 - M^2)} \\ i\mathcal{M}_{ghost}^{ab\mu\nu} &= i(-g)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k-p)^2} \frac{i}{k^2} f^{cad} k^\mu f^{dbc}(k-p)^\nu. \end{aligned} \tag{3.28}$$

Now we have to add all the individual amplitudes together. After evaluating all the integrals using the dimensional regularization scheme outlined above the vacuum polarization amplitude reads

$$
\begin{aligned}
\mathcal{M}^{ab\mu\nu} &= \mathcal{M}_F^{ab\mu\nu} + \mathcal{M}_3^{ab\mu\nu} + \mathcal{M}_{cterm}^{ab\mu\nu} + \mathcal{M}_{sc}^{ab\mu\nu} + \mathcal{M}_{ghost}^{ab\mu\nu} \\
&= \delta^{ab}\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\{\frac{g^2}{16\pi^2}[C_2(A)\left(\frac{10}{3\varepsilon}\right) - n_f T(F)\left(\frac{8}{3}\frac{1}{\varepsilon}\right) \\
&\quad - n_{cs}T(F)\left(\frac{2}{3}\frac{1}{\varepsilon}\right)] - \delta_A\}, \qquad (3.29)
\end{aligned}
$$

where we used definitions (1.8) resp. (1.9). From this expression we can write down the renormalization condition

$$
\delta_A = \frac{1}{\varepsilon}\frac{g^2}{16\pi^2}\left[\frac{10}{3}C_2(A) - \frac{8}{3}n_f T(F) - \frac{2}{3}n_{cs}T(F)\right]. \qquad (3.30)
$$

Similarly we are going to fix the other desired counterterms appearing in (3.25). So for δ_ψ we need the fermion self-energy diagram depicted in Figure 3.4. Adding the corresponding counterterms looks the same as before so we can write down the result right away

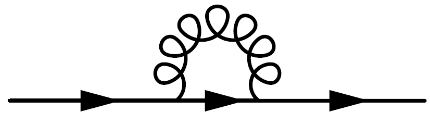


Figure 3.4: Fermion self-energy

$$
\rightarrow i\Sigma_2^{ij}(\not p) = \delta^{ij}\left\{\frac{g^2}{16\pi^2}C_2(F)\left(\frac{2\not p - 8m}{\varepsilon}\right) + \text{finite} + \delta_\psi \not p - (\delta_m + \delta_\psi)\, m\right\},
$$

where the i, j are the color indices of incoming resp. outgoing fermions. Now we can write down the renormalization conditions

$$
\delta_\psi = \frac{1}{\varepsilon}\frac{g^2}{16\pi^2}\left[-2C_2(F)\right] \qquad (3.31)
$$

$$
\delta_m = \frac{1}{\varepsilon}\frac{g^2}{16\pi^2}\left[-6C_2(F)\right]. \qquad (3.32)
$$

Lastly to fix δ_g we need to calculate the 1-loop contributions to the three-point function depicted in Figure 3.5. Together with the counterterms the amplitude amounts to

$$
i\mathcal{M}_{3point} = ig\left(C_2(F) - \frac{1}{2}C_2(A)\right)T_{ij}^a\gamma^\mu\frac{g^2}{16\pi^2}\frac{2}{\varepsilon} + igC_2(A)T_{ij}^a\gamma^\mu\frac{g^2}{16\pi^2}\frac{3}{\varepsilon} + igT_{ij}^a\gamma^\mu\delta_g,
$$

from which follows

$$
\delta_g = \frac{1}{\varepsilon}\left(\frac{g^2}{16\pi^2}\right)\left[-2C_2(F) - 2C_2(A)\right]. \qquad (3.33)
$$

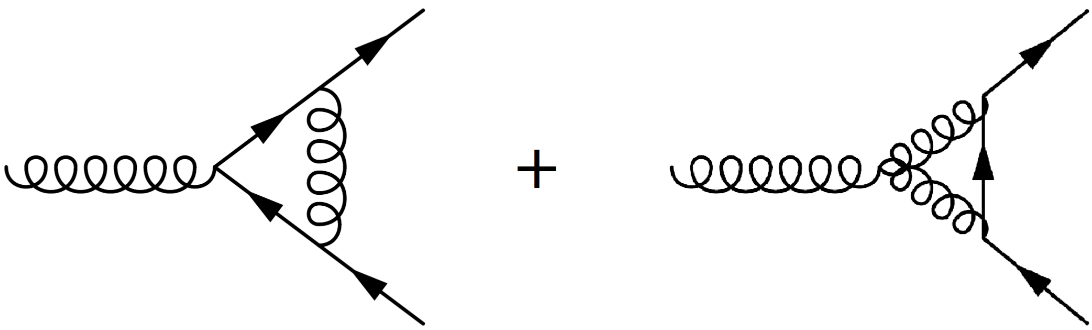


Figure 3.5: 1-loop contributions to the three-point function

3.4 Beta Function

In this section we can use all the hard work from the preceding section. Let us recall the equations (3.25) and (B.4) where we defined the renormalized quantities and introduced the energy scale μ. Since the bare coupling g_B is independent of μ it follows that

$$0 = \mu \frac{d}{d\mu} g_B = \mu \frac{d}{d\mu} \left[g_R \frac{Z_g}{Z_\psi \sqrt{Z_A}} \mu^{\frac{4-d}{2}} \right]. \tag{3.34}$$

Based on this observation we can infer how the coupling g_R evolves with respect to different scales. Recall from (3.22) the definition of the β-function

$$\beta(g_R) = \mu \frac{d}{d\mu} g_R. \tag{3.35}$$

We can plug into (3.35) from (3.34) and from the previous section, where we have derived the forms of the needed counterterms. Solving to the leading order in g_R (note that $\delta_i = \mathcal{O}(g_R^2)$) gives

$$\begin{aligned}
\mu \frac{d}{d\mu} g_R &= -\mu^{1-\frac{\varepsilon}{2}} g_R \frac{d}{d\mu} \left[\frac{Z_g \mu^{\frac{\varepsilon}{2}}}{Z_\psi \sqrt{Z_A}} \right] \\
&= -\mu^{1-\frac{\varepsilon}{2}} g_R \left[\frac{\varepsilon}{2} \mu^{\frac{\varepsilon}{2}-1} + \mu^{\frac{\varepsilon}{2}} \frac{d}{d\mu} [1 + \delta_g - \delta_\psi - \frac{1}{2} \delta_A] \right] \\
&= -g_R \left[\frac{\varepsilon}{2} + \mu \frac{d}{d\mu} \left[\frac{1}{\varepsilon} \left(\frac{\mu^{4-d} g_R^2}{16\pi^2} \right) [-2C_2(F) - 2C_2(A) \right.\right. \\
&\quad \left.\left. + 2C_2(F) - \frac{5}{3} C_2(A) + \frac{4}{3} n_f T(F) + \frac{1}{3} n_{cs} T(F)] \right] \right] \\
&= -g_R \left[\frac{\varepsilon}{2} + \mu^\varepsilon \frac{g_R^2}{16\pi^2} \left[-\frac{11}{3} C_2(A) + \frac{4}{3} n_f T(R_f) + \frac{1}{3} n_{sc} T(R_{sc}) \right] \right].
\end{aligned} \tag{3.36}$$

Now setting $\varepsilon \to 0$ and omitting the subscript R yields the famous result for the running gauge couplings in a non-Abelian theory

$$\beta(g) = \frac{g^3}{(4\pi)^2} \left[-\frac{11}{3} C_2(A) + \frac{4}{3} n_f T(R_f) + \frac{1}{3} n_{sc} T(R_{sc}) \right]. \tag{3.37}$$

4. Standard Model and Grand Unified Theories

4.1 Running of couplings in SM

The Standard Model stands today as the best theory describing the fundamental matter and 3 out of 4 fundamental interactions among them - strong, weak, and electromagnetic with gravity standing outside the SM (one would not notice gravity at all in particle physics experiments). The particles in the SM are the fermions (the quarks and the leptons), the Higgs boson, and the gauge bosons. Mathematically the interactions are expressed as gauge fields associated with Lie groups: strong - SU(3), weak - SU(2), and electromagnetic - U(1). The field content in the SM is summarized in Table 4.1 along with their representations with respect to the gauge groups ordered as (a, b, Y) where a is the represenation under SU(3), b is the representation under SU(2) and Y is the hypercharge associated with U(1). We employing the convention for the hypercharge

$$Q = T_3 + Y. \tag{4.1}$$

Table 4.1: Field content in the SM

Particle type and chirality	Representations
Left-handed Quarks	(3,2,1/6)
Right-handed up-type Quarks	(3,1,2/3)
Right-handed down-type Quarks	(3,1,-1/3)
Left-handed Leptons	(1,2,-1/2)
Right-handed Leptons	(1,1,-1)
Higgs (complex scalar)	(1,2,1/2)

In the previous section we derived the general form of the β-function (3.37). In this section we will calculate the running of the couplings explicitly for the electromagnetic, strong and weak interactions within the SM. The SM is a chiral theory and contains right-handed and left-handed Weyl fermions, which interact differently. When we were deriving the β-function in the previous chapter we worked with Dirac fermions and those contain two Weyl fermions (for both chiralities). Therefore each Weyl fermion contributes half as much as a Dirac fermion, which can be incorporated into (3.37) as

$$\beta(g) = \frac{g^3}{(4\pi)^2}\left[-\frac{11}{3}C_2(A) + \frac{2}{3}n_{fW}T(R_{fW}) + \frac{1}{3}n_{sc}T\left(R_{sc}\right)\right], \tag{4.2}$$

where the subscript W stands for *Weyl*.

The generators of the representations of SU(2) and SU(3) are normalized so that the indices of the representations are $\frac{1}{2}$. For U(1) the index is simply the hypercharge squared. Before we plug into (4.2) we rescale the hypercharge $Y \to \sqrt{\frac{3}{5}}Y$. The reason for this is connected to the desire to embed the SM gauge group into a larger simple group SU(5) (see next section). In order to maintain the normalization of the index of the representation at $\frac{1}{2}$ this rescaling is necessary. One can compensate for the factor by setting $g_1 = \sqrt{\frac{5}{3}}g'$, which leaves the physical predictions unchanged. Now we are ready to read from Table 4.1

$$\begin{array}{llll}
\underline{\text{SU(3)}}: & C_2(A) = 3 & n_f T(R_f) = 6 & n_{sc} T(R_{sc}) = 0 \\
\underline{\text{SU(2)}}: & C_2(A) = 2 & n_f T(R_f) = 6 & n_{sc} T(R_{sc}) = \dfrac{1}{2} \\
\underline{\text{U(1)}}: & C_2(A) = 0 & n_f T(R_f) = 10 & n_{sc} T(R_{sc}) = \dfrac{1}{2}
\end{array}$$

We solve (4.2) by substituting $\alpha \coloneqq \frac{g^2}{4\pi}$ and $t \coloneqq \frac{1}{2\pi} \log \frac{\mu}{M}$ which transforms (4.2) into a first order differential equation for $\alpha^{-1}(t)$. The solution of the equation is

$$\alpha^{-1}(t) = \alpha^{-1}(M) - b(t - t_0) \tag{4.3}$$

where b denotes the coefficient multiplying g^3 in (4.2) and $\alpha^{-1}(M)$ and t_0 are the constants of integration. Finally plugging in the coefficients for the particular groups into (4.3) yields

$$\begin{array}{llll}
\underline{\text{SU(3)}}: & \dfrac{d\alpha_s^{-1}}{dt} = 7 & \longrightarrow & \alpha_s^{-1} = \alpha_s^{-1}(M) + \dfrac{7}{2\pi} \log \dfrac{\mu}{M} \\
\underline{\text{SU(2)}}: & \dfrac{d\alpha^{-1}}{dt} = \dfrac{19}{6} & \longrightarrow & \alpha^{-1} = \alpha^{-1}(M) + \dfrac{19}{12\pi} \log \dfrac{\mu}{M} \\
\underline{\text{U(1)}}: & \dfrac{d\alpha'^{-1}}{dt} = \dfrac{-41}{10} & \longrightarrow & \alpha'^{-1} = \alpha'^{-1}(M) - \dfrac{41}{20\pi} \log \dfrac{\mu}{M}
\end{array} \tag{4.4}$$

where M and $\alpha_i^{-1}(M)$ are the constants of integration. For those we plug in experimentally measured values of the mass of the Z boson M_Z and the coupling constants at the energy corresponding to that mass

$$\begin{array}{ll}
\alpha_s^{-1}(M_Z) = 8.467 \pm 0.667 & \alpha^{-1}(M_Z) = 29.5767 \pm 0.0220 \\
\alpha'^{-1}(M_Z) = 59.0036 \pm 0.0148 & M_Z = (91.1876 \pm 0.0021)\text{GeV}
\end{array} \tag{4.5}$$

The Z-boson mass and the couplings at the corresponing energy were taken from [7], [8]. In Figure 4.1 we show plots of the running couplings in SM given by (4.4).

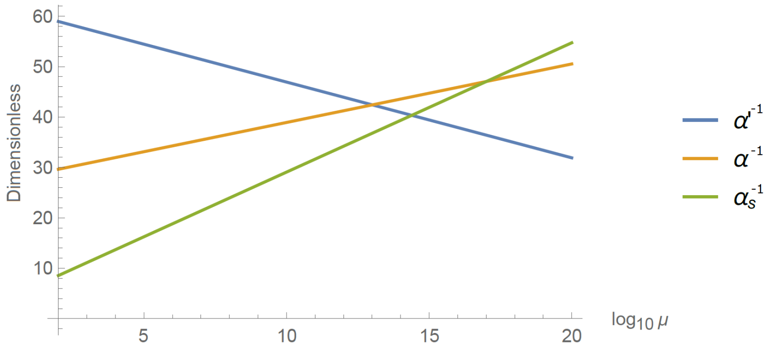


Figure 4.1: Running of the couplings in the SM

4.2 Grand Unified Theories

We can observe in Figure 4.1 at energies around 10^{15} GeV the three SM couplings come close to unifying. This leads to the question whether the couplings actually do unify at these scales thanks to yet undiscovered physics at higher energies. Theoretical models describing such unification are called the Grand Unified Theories (GUTs). The motivation to look for theories beyond the SM goes deeper than this, though. Currently the GSW Standard Model (Glashow-Salam-Weinberg) is widely regarded as incomplete. It does not allow the observed non-zero neutrino masses. Other problems with the SM are the peculiarity of its gauge group SU(3)× SU(2)×U(1) and the number of free parameters.

Recall section 2.6 where we outlined the electroweak symmetry breaking of $SU(2)\times U(1)_Y \rightarrow U(1)_{EM}$. It is possible to construct a theory based on a bigger gauge group with the Higgs mechanism breaking the symmetry down to the SM gauge group. Since the model should unify all the couplings at higher scales it is natural to embed the SM gauge group into a larger simple group associated with one unified gauge coupling.

The smallest simple gauge group conceivable for a GUT is SU(5) [9]. However, there are big problems with SU(5) as is pointed in out in for instance [11] and references therein. The biggest one is that it has been shown the couplings do not in fact unify. That is of course a requirement for GUTs and therefore we will not discuss the model any further.

Another candidate for a GUT gauge group is the special orthogonal group SO(10) (the group of all 10×10 orthogonal matrices M satisfying $\det M = 1$). This theory is much more successful and we will dedicate it the rest of the thesis. The minimalistic versions of the theory containing the $45 \oplus 16$ or $45 \oplus 126$ Higgs field (see next chapter) seemed to have tachyonic instabilities (this was thought in the early '80s [11]). Nevertheless, later it has been shown the instabilities disappear once the 1-loop corrections are calculated. This means the SO(10) GUT is a good candidate for a new theory beyond the Standard model. The SO(10) GUT also incorporates the neutrino non-zero masses quite naturally as well as constraints some of the arbitrariness present in the SM.

SO(10) has rank 5 (as opposed to the SM gauge group with rank 4), where rank is the number of generators in the maximal set of commuting generators in the algebra. It can be shown that this allows for intermediate steps along the symmetry breaking down to the SM group.

The dimension of SO(10) is 45 and therefore there are as many gauge fields mediating interactions. This means the theory predicts new phenomena outside of SM and also allows for its testability. One of the predictions of GUTs is the proton decay. The current limits for proton lifetime are $\tau > 1.6 \times 10^{34}$ [10] and better measurements are to come. Various SO(10) models predict the lifetime to be $\approx 10^{35}$ and therefore it might be possible to observe the decay in years to come, if the theory is correct. Another notable prediction of the theory is the existence of magnetic monopoles.

5. SO(10) Grand Unified Theory

5.1 Higgs sector

When we talk about SO(10) GUTs it is important to realize it represents class of theories rather than just one in particular. One of the most important aspects we distinguish the theories by are their Higgs sectors. As we talked about in the second part of chapter 2 the Higgs sector is responsible for symmetry breaking as well as the mass-generating mechanism. This is how the high-energy behavior of a GUT is connected to the SM and the low-energy experiments. In other words, this is *the* requirement of the GUTs - it must satisfy the low-energy limit described by the SM.

Following the theory in chapter 2, for symmetry breaking to occur we need a Higgs field equipped with a potential. The interaction between the Higgs field and gauge fields occurs through the covariant derivative. To generate the masses of the fermions one needs to include a *Yukawa interaction* to the theory defined as an interaction bilinear in fermion fields and linear in scalar fields

$$\mathcal{L}_{Yukawa} = g_Y \overline{\psi} \Phi \psi + h.c., \tag{5.1}$$

where the second term means hermitian conjugate. When building the Yukawa terms one needs to keep in mind it needs to give a scalar in the end (the Lagrangian is a scalar), it needs to be gauge invariant, and it needs to be renormalizable. The first two are requirements are fairly obvious but the third one is a little more complicated.

We have devoted most of chapter 3 to renormalization and we showed how to calculate a few different loop diagrams. Nevertheless, we have not laid out any actual proof regarding the renormalizibility of a general class of theories and such task is beyond the scope of this thesis. Getting back to the Higgs sector of SO(10) GUT we shall simply state the requirement of renormalizibility of the theory constraints the options of the Yukawa terms containing fermions and Higgs scalars we can include.

For the rest of this thesis we will build on a specific GUT called the Minimal SO(10) Higgs model described in [12]. The fermions of each generation are in a 16-dimensional spinorial representation of SO(10). The Higgs field then sits in the scalar $45 \oplus 126$ representation (where we used the standard notation for direct sum). Of course along with the Higgs field comes the Higgs potential inducing a vacuum expectation value. Clearly the form of the potential determines the symmetry breaking patterns of the theory and there do exist different kinds of breaking chains [11]. Once again though we reiterate the main constraint of the symmetry breaking chains is that its low-energy limit must be the SM gauge group which can be achieved with the mentioned Higgs field.

However, there is a problem with the current setting as pointed out in [13]. The requirement of renormalizibility of the theory mentioned earlier allows only for the Yukawa interaction term

$$\mathcal{L}_{Yukawa} = 16_{\mathrm{F}} \, (\overline{126}_{\mathrm{H}} Y_{126}) \, 16_{\mathrm{F}} + h.c., \tag{5.2}$$

where Y_{126} is a symmetric matrix in the generation space and the fields are denoted by their representations (where the bar in $\overline{126}$ means the conjugate representation). It can be further shown that such a constraint leads to fermion masses contradicting the observations. Namely, it leads to the predictions

$$m_{up} = m_{down} \tag{5.3}$$

$$\frac{m_\tau}{m_b} = \frac{m_\mu}{m_s} = \frac{m_e}{m_d} = 3, \tag{5.4}$$

where $up, down, b, s$ and d denote kinds of quarks and τ, μ and e denote kinds of leptons. Things will get better if we include an additional real scalar decaplet so that the Yukawa term looks like

$$\mathcal{L}_Y = 16_\mathrm{F}\left(10_\mathrm{H} Y_{10} + \overline{126}_\mathrm{H} Y_{126}\right) 16_\mathrm{F} + h.c.. \tag{5.5}$$

The previously degenerate masses of fermions then recieve additional contributions from the new Higgs field. Of course a new field in the theory has other implications. In particular, recalling the β-function (3.37) it changes the running of the gauge couplings. We studied these changes for the gauge couplings in the SM in this thesis and we will discuss them in the next section.

5.2 Effects of additional decuplet on the running couplings

When building the Minimal SO(10) Higgs model [12] one has a certain set of parameters within the theory that need to be tuned. Those parameters occur in for example the aforementioned Higgs potential. There is no reason why there should be just one way to set those parameters so that the low-energy limit of the theory fits the SM. And in fact there exists a big set of *solutions*, i.e. a big subset in the parameter space satisfying the low-energy limit.

Within the research of this thesis we were given this set of solutions and the field contents of the theory (the spectra) associated with all of the particular solutions. One such spectrum is shown in Table 5.1. We note the additional scalars are not yet present in the spectrum in Table 5.1 and we will discuss their addition later on. Using the spectra we calculated the corresponding β-functions associated with the three SM interaction couplings.

Recall the calculation of the 1-loop diagram in the 'ϕ^3 theory' in section 3.2. From equation (3.20) we deduced that for $Q \ll M$ the whole 1-loop contribution to the matrix element is negligible. In a non-Abelian theory the situation is the same and therefore, when we were calculating the right-hand side of (3.37) we accounted for this by adding the field's contribution at the energy corresponding to its mass. From what was said, that is a natural point where to assume the field's relevancy in a given process.

A sample graph of the three running gauge couplings (4.3) calculated in the manner described above is depicted Figure in 5.1 (we note this graph does not correspond to the particular setting in Table 5.1).

As was motivated in the previous section we added an additional decaplet into the spectrum. Now the new scalar is a decaplet under SO(10) but under the SM

Table 5.1: Field content of the SO(10) GUT [12]. In the first column there are the multiplets are denoted by the same convention as in chapter 4. In the second column there the types of the fields, i.e. VB is a vector boson, CS is a complex scalar and so on. In the third column there the contributions to the β-function (the same convention as in chapter is employed). In the last column are the masses of the fields.

multiplet	type	eigenstate	Δb^{321}	mass [GeV]	multiplet	type	eigenstate	Δb^{321}	mass [GeV]
$(\mathbf{8}, \mathbf{2}, +\frac{1}{2})$	CS	1	$(2, \frac{4}{3}, \frac{4}{5})$	$\mathbf{2.3 \times 10^{4}}$	$(\mathbf{3}, \mathbf{2}, -\frac{5}{6})$	**GB**	1	$(\frac{1}{3}, \frac{1}{2}, \frac{5}{6})$	$\mathbf{8.7 \times 10^{15}}$
$(\bar{3}, 1, -\frac{2}{3})$	VB	1	$(-\frac{11}{6}, 0, -\frac{44}{15})$	2.8×10^{13}	$(\bar{\mathbf{3}}, \mathbf{2}, -\frac{1}{6})$	**VB**	1	$(-\frac{11}{3}, -\frac{11}{2}, -\frac{11}{30})$	$\mathbf{8.7 \times 10^{15}}$
$(3, 1, +\frac{2}{3})$	VB	1	$(-\frac{11}{6}, 0, -\frac{44}{15})$	2.8×10^{13}	$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})$	**VB**	1	$(-\frac{11}{3}, -\frac{11}{2}, -\frac{11}{30})$	$\mathbf{8.7 \times 10^{15}}$
$(\bar{3}, 1, -\frac{2}{3})$	GB	1	$(\frac{1}{6}, 0, \frac{4}{15})$	2.8×10^{13}	$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})$	**GB**	1	$(\frac{1}{3}, \frac{1}{2}, \frac{1}{30})$	$\mathbf{8.7 \times 10^{15}}$
$(1, 1, 0)$	VB	1	$(0, 0, 0)$	6.1×10^{13}	$(\bar{3}, 1, +\frac{1}{3})$	CS	1	$(\frac{1}{6}, 0, \frac{1}{15})$	1.1×10^{16}
$(1, 1, 0)$	GB	1	$(0, 0, 0)$	6.1×10^{13}	$(\bar{3}, 1, +\frac{1}{3})$	CS	2	$(\frac{1}{6}, 0, \frac{1}{15})$	1.2×10^{16}
$(3, 2, +\frac{7}{6})$	CS	1	$(\frac{1}{3}, \frac{1}{2}, \frac{49}{30})$	2.6×10^{14}	$(1, 1, +1)$	CS	2	$(0, 0, \frac{1}{5})$	1.6×10^{16}
$(3, 2, +\frac{1}{6})$	CS	3	$(\frac{1}{3}, \frac{1}{2}, \frac{1}{30})$	2.8×10^{14}	$(\bar{3}, 1, +\frac{1}{3})$	CS	3	$(\frac{1}{6}, 0, \frac{1}{15})$	1.6×10^{16}
$(1, 2, +\frac{1}{2})$	RS	1	$(0, \frac{1}{12}, \frac{1}{20})$	3.3×10^{14}	$(\bar{6}, 1, -\frac{1}{3})$	CS	1	$(\frac{5}{6}, 0, \frac{2}{15})$	1.6×10^{16}
$(1, 1, 0)$	RS	2	$(0, 0, 0)$	2.2×10^{15}	$(3, 2, +\frac{7}{6})$	CS	2	$(\frac{1}{3}, \frac{1}{2}, \frac{49}{30})$	1.7×10^{16}
$(\bar{3}, 1, -\frac{2}{3})$	CS	2	$(\frac{1}{6}, 0, \frac{4}{15})$	2.3×10^{15}	$(1, 2, +\frac{1}{2})$	RS	2	$(0, \frac{1}{12}, \frac{1}{20})$	1.7×10^{16}
$(6, 3, +\frac{1}{3})$	CS	1	$(\frac{5}{2}, 4, \frac{2}{5})$	2.3×10^{15}	$(8, 2, +\frac{1}{2})$	CS	2	$(2, \frac{4}{3}, \frac{4}{5})$	1.7×10^{16}
$(3, 3, -\frac{1}{3})$	CS	1	$(\frac{1}{2}, 2, \frac{1}{5})$	2.3×10^{15}	$(3, 2, +\frac{1}{6})$	CS	2	$(\frac{1}{3}, \frac{1}{2}, \frac{1}{30})$	1.7×10^{16}
$(1, 3, -1)$	CS	1	$(0, \frac{2}{3}, \frac{3}{5})$	2.3×10^{15}	$(1, 1, -1)$	VB	1	$(0, 0, -\frac{11}{5})$	1.7×10^{16}
$(\bar{6}, 1, -\frac{4}{3})$	CS	1	$(\frac{5}{6}, 0, \frac{32}{15})$	3.2×10^{15}	$(1, 1, +1)$	VB	1	$(0, 0, -\frac{11}{5})$	1.7×10^{16}
$(1, 1, 0)$	RS	3	$(0, 0, 0)$	3.3×10^{15}	$(1, 1, +1)$	GB	1	$(0, 0, \frac{1}{5})$	1.7×10^{16}
$(8, 1, 0)$	RS	1	$(\frac{1}{2}, 0, 0)$	4.6×10^{15}	$(1, 1, +2)$	CS	1	$(0, 0, \frac{4}{5})$	2.4×10^{16}
$(1, 3, 0)$	RS	1	$(0, \frac{1}{3}, 0)$	6.1×10^{15}	$(\bar{3}, 1, +\frac{4}{3})$	CS	1	$(\frac{1}{6}, 0, \frac{16}{15})$	2.4×10^{16}
$(\bar{\mathbf{3}}, \mathbf{2}, +\frac{5}{6})$	**VB**	1	$(-\frac{11}{3}, -\frac{11}{2}, -\frac{55}{6})$	$\mathbf{8.7 \times 10^{15}}$	$(\bar{6}, 1, +\frac{2}{3})$	CS	1	$(\frac{5}{6}, 0, \frac{8}{15})$	2.4×10^{16}
$(\mathbf{3}, \mathbf{2}, -\frac{5}{6})$	**VB**	1	$(-\frac{11}{3}, -\frac{11}{2}, -\frac{55}{6})$	$\mathbf{8.7 \times 10^{15}}$	$(1, 1, 0)$	RS	4	$(0, 0, 0)$	4.1×10^{16}

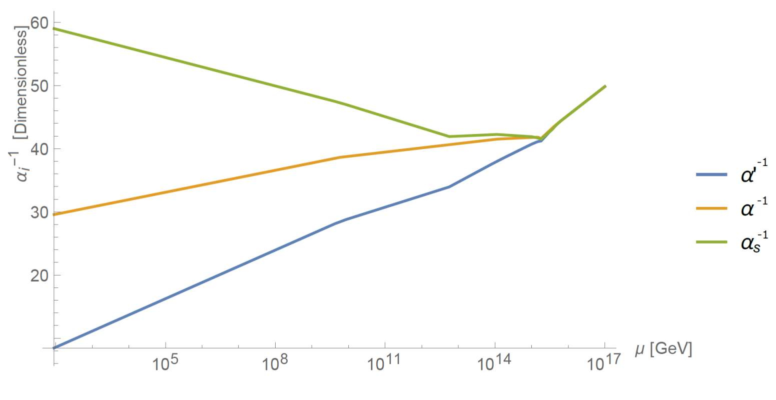


Figure 5.1: Running of the SM couplings in the SO(10) GUT calculated with the original spectra (i.e. without any additional scalars).

gauge group it decomposes into complex scalars (3,1,+$\frac{1}{3}$) and (1,2,-$\frac{1}{2}$) (where we used the representation notation like in chapter 4). From the representations we can easily calculate their contributions to (3.37):

$$\mathbf{(3,1,\frac{1}{3}):} \qquad \underline{\text{SU(3)}}: \rightarrow \frac{1}{6} \qquad \underline{\text{SU(2)}}: \rightarrow 0 \qquad \underline{\text{U(1)}}: \rightarrow \frac{1}{15} \tag{5.6}$$

$$\mathbf{(1,2,-\frac{1}{2}):} \qquad \underline{\text{SU(3)}}: \rightarrow 0 \qquad \underline{\text{SU(2)}}: \rightarrow \frac{1}{6} \qquad \underline{\text{U(1)}}: \rightarrow \frac{1}{10} \tag{5.7}$$

We reiterate that the goal of these fields is to help with the shape of the fermionic masses. However, without redoing the complete Higgs potential minimization it is impossible to deduce the masses of the additional scalars. We add that the masses of the other heavy scalars remain effectively unchanged. This is due to the fact that the extra 10_H must have a small VEV (no more than the electroweak breaking scale where $v \approx 246$GeV) and the contributions to the masses of the heavy scalars are proportional to this VEV.

We tested different mass assignments of the scalars over an interval. Before we state the masses we need to realize that we are working with a large set of solutions in the parameter space - 10198 to be exact. However, different points in the parameter space can and do have different values of the unified coupling α_{GUT}^{-1} (where the three SM constant merge into one) at the points of unification μ_{GUT}. With that the whole spectrum of fields also tends to have higher or lower masses overall based on what solution they are in. Therefore, it only makes sense to set the masses of the additional scalars *relatively* constant with respect to the point of unification of a given solution. In this setting it is much more reasonable to compare the changes of different solutions rather than in a setting where the scalars would have constant *absolute* values of their masses in different solutions.

In our calculations we set the masses of the additional scalars in the range $[10^{-4}\mu_{GUT},\ 10^{-1}\mu_{GUT}]$ where we considered both settings with the triplet being heavier and/or lighter than the doublet. Once the scalars were added to the spectrum the β-function changed and so a solution that used to have all three SM couplings unified at high scales as well as had the SM limit at low scales needed to be fixed. This was done numerically. With new equations for the running couplings (4.3) we set two parameters to be free. One was α_{GUT}^{-1} and the second one was a multiplicative factor μ_{scale} scaling the masses $M \rightarrow \mu_{scale}M$ of all the non-SM fields.

Now that we have specified the free parameters we need to formulate mathematically the problem we set out to solve. As we said earlier we demand to approach the SM in the low-energy limit. Therefore, as a condition we set the values of the SM couplings at the energy corresponding to the Z-boson mass as in (4.5). And finally, we minimized the function

$$\chi^2 = \left(\frac{\tilde{\alpha}_s^{-1} - \alpha_s^{-1}(m_Z)}{\sigma_{\tilde{\alpha}_s^{-1}}}\right)^2 + \left(\frac{\tilde{\alpha}^{-1} - \alpha^{-1}(m_Z)}{\sigma_{\tilde{\alpha}^{-1}}}\right)^2 + \left(\frac{\tilde{\alpha}'^{-1} - \alpha'^{-1}(m_Z)}{\sigma_{\tilde{\alpha}'^{-1}}}\right)^2, \tag{5.8}$$

where $\tilde{\alpha}$ denotes the measured value and the standard deviations σ are the same as in (4.5). A successful minimization of (5.8) is one where χ^2 is small (at least < 1 as that guarantees the SM limit within the measurement errors). All of the results bellow satisfy this condition.

For future reference we first minimized the function (5.8) for the original spectrum (i.e. without any additional scalars) with the only free parameter being α_{GUT}^{-1}. The resulting histogram of the values α_{GUT}^{-1} for each point in the parameter space is depicted in Figure 5.2.

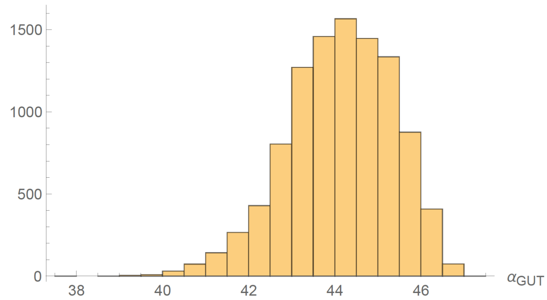


Figure 5.2: Histogram of the values α_{GUT}^{-1} for the given solutions

The histogram in Figure 5.2 is the result of minimizing (5.8) for each given point of the parameter space. It might then come as surprise that at each of those points the minimum of χ^2 has the exact same value but there is a good reason for that. It has to do with the fact that all three couplings have the same values at M_Z given by (4.5) for every point of the parameter space as well as they always converge into one α_{GUT}^{-1}. This in turn means that however α_{GUT}^{-1} changes for a given solution all of the three SM couplings change by exactly the same value. Therefore, χ^2 will have the same minimum only for a different value of α_{GUT}^{-1}.

This can be also seen from (4.3) where one can imagine the free parameter α_{GUT}^{-1} on the right-hand side. The overall contributions from the term $b(t-t_0)$ change for different points in the parameter space by the same value and thus the free parameter α_{GUT}^{-1} completely compensates those changes. Therefore, the minima of χ^2 have the same values for all of the points.

As was described above we added two scalars into the original spectrum and minimized (5.8) via two free parameters α_{GUT}^{-1} and μ_{scale}. The contributions to the β-function of the two additional scalars added together yield $\Delta b = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$. This in turn means that adding the scalars at the same point merely shifts the value of the unified gauge coupling α_{GUT}^{-1} without any other change needed altogether. If the solution fitted the SM in the low-energy limit it will continue to do so. Going back to the equation (4.3) for the running couplings it is easy to see the change of α_{GUT}^{-1} will also remain constant over the whole given set of points in the parameter space for the equal scalar masses $m_{triplet} = m_{dublet} = x\mu_{GUT}$ (for an arbitrary constants x). Because in this case the contribution to the running coupling (in terms of $t = \frac{1}{2\pi}\log\frac{\mu}{M_Z}$) is $\frac{1}{6}(t_{GUT} - t_{triplet}) = \frac{1}{6}(-\frac{1}{2\pi}\log\frac{x}{M_Z})$, i.e. it is a constant which proves the change of α_{GUT}^{-1} due to the addition of the scalars at the same scale will remain constant over the whole set of solutions as well.

More complicated is the case where $m_{triplet} \neq m_{dublet}$. Then there is another term in the right-hand side of (4.3) given by the difference of the masses and the contributions (5.6) or (5.7) depending on which mass is lighter. Once both fields have contributed to the running of the coupling the situation becomes identical to the one described in the previous paragraph. However, yet again it is true the extra term is independent of the particular point in the parameter space because the new contributing term looks like $\Delta b(t_{triplet} - t_{dublet}) = \Delta b(const)$. There are two free parameters $\alpha_{GUT}^{-1}, \mu_{scale}$ that compensate this change in some way so that χ^2 is minimal. The parameter α_{GUT}^{-1} plays the role described in the previous paragraph and also together with μ_{scale} they compensate the constant contributions.

To sum up the two previous paragraphs the quantities μ_{scale} and $\Delta\alpha_{GUT}^{-1} = \hat{\alpha}_{GUT}^{-1} - \alpha_{GUT}^{-1}$ remain constant over the whole set of solutions for a particular mass assignments of the additional scalars (where we denoted value of the unified coupling at μ_{GUT} by α_{GUT}^{-1} for the original spectra and by $\hat{\alpha}_{GUT}^{-1}$ for the spectra with the additional scalars).

Numerical analysis confirmed the propositions laid out in the two previous paragraphs. The results of the calculations are summarized in Figure 5.3. On the left graph in Figure 5.3 there is depicted how μ_{scale} changes with respect to the logarithm of $\Delta M = \frac{m_{triplet}}{m_{dublet}}$. On the right graph in Figure 5.3 there is depicted how $\Delta\alpha_{GUT}^{-1}$ changes with respect to $\log_{10}\Delta M$.

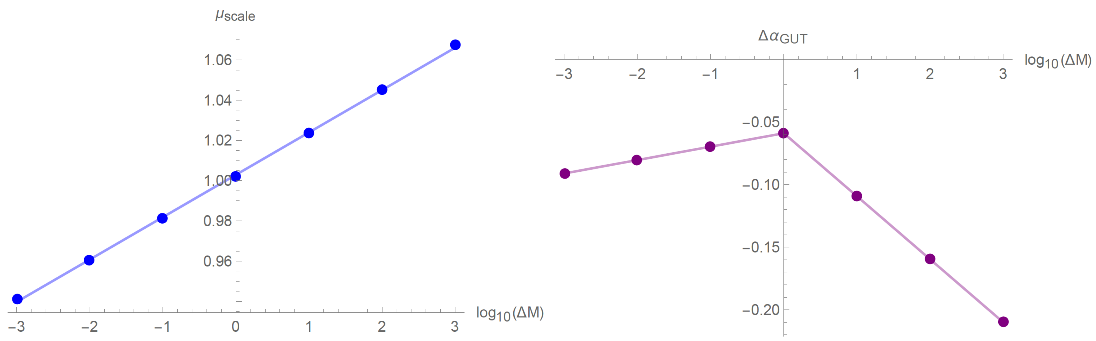


Figure 5.3: On the left there are depicted the values of the multiplicative factor μ_{scale} and on the right there are depicted the values of the difference $\Delta\alpha_{GUT}^{-1}$ for different masses of the additional scalars. The connecting lines are linear fits of the data.

In Figure 5.3 there is an asymmetry between $\log_{10}\Delta M$ being positive and negative. This is due to the fact the additional scalars do not contribute equally to the β-functions and therefore the situation with the triplet being heavier than the doublet differs from the triplet being lighter than the doublet.

As was mentioned in section 4.2 one of the major predictions of GUTs is the proton decay. To have a sense of the implications of the results above we shall discuss the change of the proton lifetime due to the addition of the extra fields. It can be shown that a rough estimate of the proton lifetime is given by [14]

$$\tau \sim \frac{1}{\alpha_{GUT}^2} \frac{M^4}{m_p^5}. \tag{5.9}$$

where m_p is the proton mass and M is the mass of the gauge field mediating the decay.

We have calculated the relative change of the proton lifetime estimate given by (5.9) due to the extra scalar fields. The results are summarized in Figure 5.4.

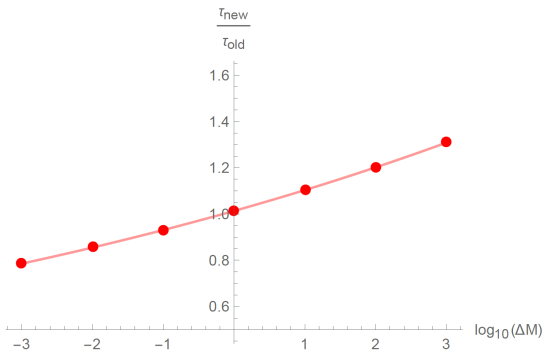


Figure 5.4: The change of the proton lifetime with respect to $\log_{10} \Delta M$. The connecting line is an exponential fit of the data. τ_{old} was calculated with (5.9) using the data from the original spectra and τ_{new} was calculated from the spectra with the additional scalars.

The current lower limit on the proton decay lifetime is $\tau > 1.6 \times 10^{34}$ [10] (in the $p^+ \to e^+ \pi^0$ channel) and the predictions given by SO(10) GUTs are $\tau \approx 10^{35}$ [14] (for some settings). So the current limits and the predictions are fairly close which becomes relevant when it comes to detecting the proton decay.

The currently most precise proton decay observatory is Super-Kamiokande. A new proton decay observatory will be built (or at least it seems very likely as of today) and it will be called Hyper-Kamiokande. When it comes to detecting proton decays the precision is estimated to be ∼10 times better and it should start collecting data in ∼2027 [15].

The maximal value of the relative proton lifetime change in Figure 5.4 is $\frac{\tau_{new}}{\tau_{old}} = 1.31$ (corresponding to $\log_{10} \Delta M = 3$) and the minimal one is $\frac{\tau_{new}}{\tau_{old}} = 0.79$ (corresponding to $\log_{10} \Delta M = -3$) . In light of the figures regarding the future of detection of the proton decay it is now clear the results depicted in Figure 5.4 are fairly relevant. The effect of the additional scalar 10_H can in fact be responsible for the difference between the detection of the proton decay and no detection in the upcoming years, if the theory is in fact close to being correct. From a different perspective we could say the observations at Hyper-Kamiokande will be able to either rule out or confirm the Minimal SO(10) Higgs model even with the additional scalar decaplet.

Conclusion

In this thesis we have introduced basic concepts used in modern particle physics. We have discussed Lie groups and Lie algebras and we have shown how to use them in the context of gauge theories. We have outlined the mass generating mechanism called the Higgs mechanism and how it connects to the Goldstone theorem. Quantization of the classical field theory has been sketched and various Feynman diagrams have been calculated up to 1-loop level. Using those calculations we have derived the β-function for a general gauge theory.

We have discussed the Standard model and in particular the running of the couplings within it. The near unification at the scale 10^{15}GeV of the couplings together with various shortcomings of the SM motivated us to study the Grand Unified Theories. We have discussed the Minimal SO(10) Higgs model and its Higgs sector described in detail in [12]. While the variant $45 \oplus 126$ of the Higgs sector does provide the desired symmetry breaking patterns it contradicts the measured masses of light fermionic fields present in the SM.

This motivated us to add a scalar Higgs decaplet as it contributes to the wrong values of the masses. Therefore, the model with the additional decaplet presents a more realistic theory. However, adding a field to the theory has various implications. One of these implications is explicit in the β-function and so the effects on the running gauge couplings have been studied. The additional decaplet of the scalar fields decomposes into complex scalars (3,1,$\frac{1}{3}$) and (1,2,-$\frac{1}{2}$) under the SM gauge group.

We began with a set of spectra of fields in the Minimal SO(10) GUT Higgs field corresponding to various points of the parameter space of the theory. We added the aforementioned decaplet and calculated the changes of running of the gauge couplings with the new spectra. The constraints of the calculations were both the low-energy and high-energy limits. In the low-energy limit the couplings had to satisfy the measured values of the couplings at the scale M_Z given by [7], [8] and in the high-energy limit the unification was required. To achieve this there were two free parameters that were tuned - the value of the unified coupling at the unification scale α_{GUT} and a multiplicative factor μ_{scale} scaling the masses of fields not present in the SM.

In the last section we discussed those calculations and we described the behavior of the equations governing the running couplings. In particular, we were able to observe the patterns of changes of the free parameters for a particular assignment of the additional scalars' masses. The value of α_{GUT} for every point of the parameter space changes by a constant factor for particular $m_{triplet}, m_{dublet}$. The value of μ_{scale} is dependent purely on $m_{triplet}, m_{dublet}$ and for $\frac{m_{triplet}}{m_{dublet}} = 1$ is $\mu_{scale} = 1$ as can be seen in Figure 5.3.

We calculated the running of the gauge couplings for various mass assignments of the additional fields within the range $[10^{-4}\mu_{GUT}, 10^{-1}\mu_{GUT}]$. With those results we calculated the change of the proton lifetime predicted by the theory with the additional scalars (as opposed to the original theory). The biggest changes of the proton lifetime found in our calculations are $\frac{\tau_{new}}{\tau_{old}} = 1.31$ (corresponding to $\log_{10}\frac{m_{triplet}}{m_{dublet}} = 3$) and $\frac{\tau_{new}}{\tau_{old}} = 0.79$ (corresponding to $\log_{10}\frac{m_{triplet}}{m_{dublet}} = -3$).

Today's limits on the proton lifetime are $\tau > 1.6 \times 10^{34}$ (in the $p^+ \rightarrow$

$e^+\pi^0$ channel) given by Super-Kamiokande [10]. The new observatory Hyper-Kamiokande is estimated to achieve 10 times better precision when it comes to the proton decay detection. The SO(10) GUT predictions on the proton decay are $\tau \approx 10^{35}$ (in some settings). Therefore, the results $\frac{\tau_{new}}{\tau_{old}} = 1.31$ and $\frac{\tau_{new}}{\tau_{old}} = 0.79$ are influential when it comes to the detection of the proton decay. The future observations of the proton decay can then either confirm or disprove this particular setting of the SO(10) GUT.

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List of Abbreviations

SM	Standard Model
QM	quantum mechanics
QFT	Quantum field theory
VEV	vacuum expectation value
RG	renormalization group
DR	dimensional regularization

A. Groups and representations

A.1 Lorentz group

A very important example of a Lie group is the Lorentz group which is the group of Lorentz transformation. We will now introduce the basic representation theory of Lorentz group (following [3]). Throughout this thesis we will use the Minkowski metric in convention $g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$. From special relativity we recall the 4-vector representation of the Lorentz group. Matrices Λ acting on the 4-vectors satisfy the orthogonality relations

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}g_{\mu\nu} = g_{\rho\sigma}. \tag{A.1}$$

The commutation relations of the generators of the Lorentz algebra read

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right). \tag{A.2}$$

A particular set of the 6 generators looks like

$$\left(J^{\mu\nu}\right)_{\alpha\beta} = i\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right) \tag{A.3}$$

Another representation of the Lorentz group is the spinor representation associated with spin-1/2 particles. First, let us define the *Dirac algebra* generated by a set of $n \times n$ γ^{μ} matrices satisfying

$$\{\gamma^{\mu}, \gamma^{\nu}\} \equiv \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} \times \mathbb{1}. \tag{A.4}$$

From the γ^{μ} matrices we can construct the generators of the representation as

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right], \tag{A.5}$$

which satisfy the commutation relations (1.13). Especially interesting are the 4×4 representations of Dirac algebra. One way to satisfy the (1.14) presents the Weyl representation, which reads

$$\gamma^{0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \gamma^{i} = \begin{pmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{pmatrix}. \tag{A.6}$$

We have discussed the operators in the 4-dimensional representation generated by the Dirac algebra. The 4-component object on which these operators act is called a *Dirac spinor*.

A.2 SU(N) groups

The group SU(N) is defined as the group of N×N unitary matrices M with $\det M = 1$. The important examples of this group are the SU(2) and SU(3) from which the gauge group of the SM is built (together with the unitary group U(1), see chapter 4).

The commonly used generators of the group SU(2) are the *Pauli matrices*

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.7}$$

The commonly used generators of the group SU(3) are the *Gell-Mann matrices*

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \tag{A.8}$$

B. Dimensional regularization

The core idea behind DR is the following observation: the integral

$$\int \frac{d^d k}{(2\pi)^d} \frac{g^2}{\left(k^2 - \Delta + i\varepsilon\right)^2} \tag{B.1}$$

diverges for $d \geq 4$ and converges for $d < 4$. Using the identities

$$\int d^d k = \int d\Omega_d \int k^{d-1} dk, \qquad \Omega_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \tag{B.2}$$

one can formally extend the range of d from the natural domain to the real one, since the Γ-function is well defined for real numbers. Its extension around zero will be used in what follows

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon), \tag{B.3}$$

where γ_E is the Euler-Mascheroni constant.

However, in order to have a dimensionless result the coupling constant would have to have mass dimension $[\frac{4-d}{2}]$. Conventionally we set

$$g \to \mu^{\frac{4-d}{2}} g \tag{B.4}$$

rather than having a non-integer dimension coupling. The parameter μ is an arbitrary energy scale.

With these tools at hand, we can outline the calculation of a loop diagram. From loop diagrams come out integrals of the form or similar to (3.5). After using Feynman parameters we arrive at the integral of the form (B.1). We Wick-rotate and substitute $d = 4 - \varepsilon$ for the dimension. Once reduced to a one-dimensional integral via (B.2) we can solve it using identities involving Γ-functions

$$\int dk_E \frac{k_E^a}{\left(k_E^2 + \Delta\right)^b} = \Delta^{\frac{a+1}{2} - b} \frac{\Gamma\left(\frac{a+1}{2}\right) \Gamma\left(b - \frac{a+1}{2}\right)}{2\Gamma(b)}. \tag{B.5}$$

Finally using (B.3) lets us isolate the poles around $\varepsilon = 0$.