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BACHELOR THESIS

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Weyl and Conformal Symmetries

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Title: Weyl and Conformal Symmetries

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Abstract: The problem whether scale invariance implies full conformal invariance, for special relativistic classical field theories, was resolved in the literature by exploiting the Weyl and diffeomorphic symmetries held simultaneously by the suitably improved corresponding theories. In a recent study it was conjectured that, for Liouville field theory, the case is different, as such theory could not have both symmetries (Weyl and diffeomorphic) at once. In this thesis we investigate that conjecture. In the first part the conformal symmetry, and the associated Witt and Virasoro algebras, are revised. Next, we review the relationship between scale and conformal invariance, using the approach of Weyl and Ricci gauging. The last part is devoted to the Liouville field theory. We prove there the incompatibility of Weyl and diffeomorphic invariance, by showing that the particular choice of the Weyl potential, necessary to have traceless energy-momentum tensor, gives rise to an action that is not a scalar under diffeomorphisms. A preliminary study of this "classical gravitational anomaly" is also performed.

Keywords: Classical Weyl symmetry Liouville theory

Contents

Introduction

The question of when scale invariance of special relativistic classical field theories implies full conformal invariance was resolved in [7] for spacetimes of any dimension n, including the special case of $n = 2$, and fields of any spin. Nonetheless in [9] the case of $n = 2$ Liouville field theory was reconsidered. There the classical centre charge c (Virasoro) appears. In [9] it is conjectured about the impossibility to have, at once, diffeomophic and Weyl symmetry. In this thesis, we investigate this issue.

The conformal symmetry arises in physics in many diverse areas. One example may be conformal gravity, as an alternative theory of gravitation [11], [10]. Another place where conformal and Weyl symmetries can be found are the Dirac materials, such as graphene [6]. Additional interesting examples and discussion of quantum case can be found in [12].

The flow of the thesis is as follows. In the first chapter we introduce the notation, discuss conformal transformations and associated algebras, for the case of $n > 2$ and for the case of $n = 2$. Then we go through the conditions imposed on the energy-momentum tensor from symmetries of a theory and improvements ensuring these features. For the case of Liouville theory, we illustrate the extension of Witt algebra to Virasoro algebra.

The second chapter is devoted to process of Weyl and Ricci gauging, i.e. promoting rigid scale invariance to local Weyl invariance, while introducing Weyl gauge potential. The equivalence of trasformation properties of particular combinations of introduced Weyl potential, and appropriate combinations of Ricci tensor and Ricci scalar, allows to interchange in some cases the Weyl gauging with geometrical objects, and have locally Weyl invariant theory without Weyl gauge field. The cases for which this is possible, are discussed at the end of the chapter.

The third chapter deals with the classical Liouville theory and the search for proper improvements giving the traceless energy-momentum tensor is presented. It starts with construction of Liouville theory as a dimensional limit of appropriate higher dimension theory which gives rise to additional terms which are being investigated. This implies a condition on the Weyl gauge field. Following this restriction the energy-momentum tensor and transformation properties of the improvement are studied.

1. Geometric transformations and Conformal symmetry

In this chapter we discuss spatiotemporal continuous symmetries of classical fields and actions, and how they shape the components of the energy-momentum tensor. Furthermore, we show some interesting mathematical structures coming from those transformations.

First we define *geometric transformations* for various fields, discuss the Killing equation and its solutions. Then we look at Noether theorem and construct the energy-momentum tensor, which has several interesting properties due to symmetries of a chosen theory.

In the last part we give more details of the energy-momentum tensor, and discuss its improvements that are surface terms necessary for ensuring its properties. We finally show the algebras obeyed by the geometric transformations in two dimensions.

1.1 Notation

Let us start by introducing our notation. The flat space metric $\eta_{\mu\nu}$ for general dimension n

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, \dots), \tag{1.1}$$

with $\mu, \nu = 0, 1, \dots, n-1$. In curved space we name $g_{\mu\nu}$ the metric tensor.

The Christoffel symbols are

$$\Gamma^{\sigma}{}_{\mu\nu} \equiv \frac{1}{2} g^{\sigma\rho} \left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu} \right), \tag{1.2}$$

where Einstein summation convention is understood.

The Riemann tensor is defined as

$$R^{\rho}{}_{\sigma\mu\nu} \equiv \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}, \tag{1.3}$$

while the Ricci tensor is

$$R_{\sigma\nu} \equiv R^{\rho}{}_{\sigma\rho\nu}, \tag{1.4}$$

and the Ricci scalar is

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{1.5}$$

Finally, the diffeomorphic covariant derivative ∇_ρ of general tensor $T^{\alpha\dots\beta}_{\mu\dots\nu}$ is

$$\begin{aligned} \nabla_\rho T^{\alpha\dots\beta}_{\mu\dots\nu} =& \partial_\rho T^{\alpha\dots\beta}_{\mu\dots\nu} + T^{\lambda\dots\beta}_{\mu\dots\nu} \Gamma^{\alpha}{}_{\lambda\rho} + \dots + T^{\alpha\dots\lambda}_{\mu\dots\nu} \Gamma^{\beta}{}_{\lambda\rho} \\ & - T^{\alpha\dots\beta}_{\lambda\dots\nu} \Gamma^{\lambda}{}_{\mu\rho} - \dots - T^{\alpha\dots\beta}_{\mu\dots\lambda} \Gamma^{\lambda}{}_{\nu\rho}. \end{aligned} \tag{1.6}$$

1.2 Conformal transformations

Suppose we have the following coordinate transformations

$$x'^{\mu} = x^{\mu} + \delta x^{\mu} \equiv x^{\mu} - f^{\mu}(x), \tag{1.7}$$

where $\delta x^\mu \equiv -f^\mu$ are infinitesimal, and we wish to know the corresponding field transformations. We define a variation δ of a general field $\Phi_i(x)$ (scalar filed, vector field etc.), excluding spinors, which are discussed later

$$\delta\Phi_i(x) \equiv \Phi'_i(x') - \Phi_i(x). \tag{1.8}$$

For our analysis it is useful to introduce the so-called *geometric transformations* [8]

$$\delta_f\Phi_i(x) \equiv \Phi'_i(x) - \Phi_i(x). \tag{1.9}$$

This transformation clearly commutes with derivative, because the fields are evaluated at the same point. We can easily see

$$\delta_f\Phi_i(x) = \delta\Phi_i(x) - \partial_\mu\Phi_i\delta x^\mu. \tag{1.10}$$

Since for scalar fields φ we have $\varphi'(x') = \varphi(x)$, we get

$$\delta_f\varphi = f^\mu\partial_\mu\varphi. \tag{1.11}$$

For contravariant vector and covaraint vector fields, V^μ and V_μ, we have

$$\begin{aligned} V'^\mu(x') &= \frac{\partial x'^\mu}{\partial x^\nu}V^\nu(x) \\ V'_\mu(x') &= \frac{\partial x^\nu}{\partial x'^\mu}V_\nu(x), \end{aligned} \tag{1.12}$$

hence

$$\begin{aligned} \delta_f V^\mu &= f^\nu\partial_\nu V^\mu - \partial_\nu f^\mu V^\nu \\ \delta_f V_\mu &= f^\nu\partial_\nu V_\mu + \partial_\mu f^\nu V_\nu. \end{aligned} \tag{1.13}$$

From this it is easy to find the geometric transformation formula for a general tensor T. By definition, a tensor transforms like

$$T'^{\alpha\ldots\beta}_{\mu\ldots\nu}(x') = \frac{\partial x'^\alpha}{\partial x^\gamma}\cdots\frac{\partial x'^\beta}{\partial x^\delta}\frac{\partial x^\lambda}{\partial x'^\mu}\cdots\frac{\partial x^\kappa}{\partial x'^\nu}T^{\gamma\ldots\delta}_{\lambda\ldots\kappa}(x). \tag{1.14}$$

The geometric transformation is then

$$\begin{aligned} \delta_f T^{\alpha\ldots\beta}_{\mu\ldots\nu} = f^\lambda\partial_\lambda T^{\alpha\ldots\beta}_{\mu\ldots\nu} &+ \partial_\mu f^\lambda T^{\alpha\ldots\beta}_{\lambda\ldots\nu} + \cdots + \partial_\nu f^\lambda T^{\alpha\ldots\beta}_{\mu\ldots\lambda} - \\ &- \partial_\lambda f^\alpha T^{\lambda\ldots\beta}_{\mu\ldots\nu} - \cdots - \partial_\lambda f^\beta T^{\alpha\ldots\lambda}_{\mu\ldots\nu}. \end{aligned} \tag{1.15}$$

One can recognize that this transformation is nothing but the *Lie derivative* for bosonic fields in coordinate formulation [3] i.e.

$$\delta_f T^{\alpha\ldots\beta}_{\mu\ldots\nu} = \mathbf{L}_f T^{\alpha\ldots\beta}_{\mu\ldots\nu}. \tag{1.16}$$

From our definition of the geometric transformations (1.9) one can see that zero value of $\delta_f\Phi_i$ reveals a symmetry of the field Φ_i under the infinitesimal transformations f^μ. Especially interesting are the symmetries of the metric tensor (called isometries). Let us study

$$\delta_f g_{\mu\nu} = 0 \tag{1.17}$$

as an equation for f^μ.

Using the formula (1.15) we can rewrite the previous condition as

$$\begin{aligned} 0 = \delta_f g_{\mu\nu} &= f^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu f^\alpha g_{\alpha\nu} + \partial_\nu f^\alpha g_{\mu\alpha} \\ &= \partial_\mu f_\nu - \frac{1}{2} f^\alpha \left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right) + (\mu \leftrightarrow \nu) \\ &= \nabla_\mu f_\nu + \nabla_\nu f_\mu, \end{aligned} \tag{1.18}$$

where we used $f^\mu = g^{\mu\nu} f_\nu$ and the expression (1.2) for $\Gamma^\lambda{}_{\mu\nu}$. The last expression is called *Killing equation* and its solutions are Killing vectors.

We derived the condition on infinitesimal coordinate transformations under which the metric is invariant. We could also think of transformations which leave the metric invariant up to an infinitesimal factor $\sigma(x)$ i.e.

$$g'_{\mu\nu}(x) = (1 + \sigma(x)) g_{\mu\nu}(x). \tag{1.19}$$

Although such transformations change the metric, they preserve angles. In particular, under these transformations the causality is preserved. We can again study which conditions f^μ has to satisfy to have

$$\delta_f g_{\mu\nu} = \sigma g_{\mu\nu}. \tag{1.20}$$

This time we get the *conformal Killing equation*

$$\nabla_\mu f_\nu + \nabla_\nu f_\mu = \sigma g_{\mu\nu}. \tag{1.21}$$

Taking the trace of this, we may express σ in terms of f^μ and by substituting this back into (1.21) we obtain

$$\nabla_\mu f_\nu + \nabla_\nu f_\mu = \frac{2}{n} \nabla_\rho f^\rho g_{\mu\nu}. \tag{1.22}$$

Since the conformal Killing equation is a generalization of the Killing equation, the solutions of (1.18) form a subset of solutions of (1.22). In flat space limit the conformal Killing equation becomes

$$\partial_\mu f_\nu + \partial_\nu f_\mu = \frac{2}{n} \partial_\rho f^\rho \eta_{\mu\nu}. \tag{1.23}$$

As our following solution to this problem is rather terse, one can find an extended discussion on this in [5].

Taking the divergence ∂^ν of this equation we get

$$(2 - n)\partial_\mu(\partial \cdot f) = n \Box f_\mu, \tag{1.24}$$

where we denoted $\Box \equiv \partial_\alpha \partial^\alpha$ and $\partial \cdot f \equiv \partial_\alpha f^\alpha$.

Taking another derivative ∂_ν of this equation we get

$$(2 - n)\partial_\nu \partial_\mu(\partial \cdot f) = n \Box \partial_\nu f_\mu. \tag{1.25}$$

Contracting the indices we have

$$(n - 1)\Box(\partial \cdot f) = 0. \tag{1.26}$$

These equations show that something special happens for $n = 1$ and $n = 2$, but let us focus for the moment on $n > 2$.

Combining the equations (1.23), (1.25) and (1.26) we see

$$\partial_\mu \partial_\nu (\partial \cdot f) = 0, \tag{1.27}$$

therefore $\partial \cdot f$ is at most linear in x, but since we have the divergence of f^μ we cannot say much about f^μ itself now. We have to derive another condition on f^μ.

Taking a derivative of (1.22) ∂_ρ we have

$$\partial_\rho \partial_\nu f_\mu + \partial_\rho \partial_\mu f_\nu = \frac{2}{n} \eta_{\mu\nu} \partial_\rho (\partial \cdot f). \tag{1.28}$$

From this we can write

$$n \partial_\mu \partial_\nu f_\rho = \eta_{\rho\mu} \partial_\nu (\partial \cdot f) + \eta_{\nu\rho} \partial_\mu (\partial \cdot f) - \eta_{\mu\nu} \partial_\rho (\partial \cdot f). \tag{1.29}$$

The discussion of (1.27) implies that $\partial_\nu (\partial \cdot f)$ has to be constant, therefore

$$\partial_\mu \partial_\nu f_\rho = constant. \tag{1.30}$$

We see that f^μ is at most quadratic in x and can be written as

$$f^\mu(x) = a^\mu + b^\mu{}_\nu x^\nu + c^\mu{}_\nu x^\nu + d^\mu{}_{\alpha\beta} x^\alpha x^\beta, \tag{1.31}$$

where a^μ, $b^\mu{}_\nu$, $c^\mu{}_\nu$, $d^\mu{}_{\alpha\beta}$ are infinitesimal constants, furthermore $b_{\mu\nu} = -b_{\nu\mu}$, $c_{\mu\nu} = c_{\nu\mu}$ and $d^\mu{}_{\alpha\beta} = d^\mu{}_{\beta\alpha}$ by construction.

By investigating each term separately we try to give every transformation a geometrical meaning. The first two terms represent the notorious Poinacaré transformations. The term a^μ corresponds to translations and the $b^\mu{}_\nu x^\nu$ term are infinitesimal Lorentz transformations. Those two terms are also the solutions of the Killing equation (1.18).

To find a meaning for the third term $c^\mu{}_\nu x^\nu$ we substitute f^μ back to (1.22), this way we see that $c_{\mu\nu}$ is proportional to metric

$$c_{\mu\nu} + c_{\nu\mu} = 2c_{\mu\nu} = \frac{2}{n} c^\rho_\rho \eta_{\mu\nu},$$

therefore this term can be rewritten as cx^μ where c is another small factor. Now it is clear that this transformation corresponds to a scaling of coordinates.

The last term $d^\mu{}_{\alpha\beta} x^\alpha x^\beta$ can be simplified by using (1.29)

$$d_{\rho\mu\nu} = \frac{2}{n}(\eta_{\rho\mu} d^\lambda{}_{\lambda\nu}) + \frac{2}{n}(\eta_{\nu\rho} d^\lambda{}_{\lambda\mu}) - \frac{2}{n}(\eta_{\mu\nu} d^\lambda{}_{\lambda\rho}). \tag{1.32}$$

This means that $d_{\rho\mu\nu}$ can be reduced to four parameters d_μ and the transformation can be written as $2(d \cdot x)x^\mu - d^\mu x^2$. This is called the *special conformal transformation* and it is an inversion, followed by translation, followed by an inversion [2].

In table 1.1 the generators associated with these infinitesimal transformations are provided.

Table 1.1: Infinitesimal conformal transformations in $n > 2$ dimensions

Transf.	Ind. par.	f^μ	Generators
Translations	n	a^μ	$P_\mu = \partial_\mu$
Lorentz transf.	$n(n-1)/2$	$b^\mu{}_\nu x^\nu$	$M_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu$
Dilatation	1	cx^μ	$D = x^\mu \partial_\mu$
Special conformal transf.	n	$2(d \cdot x)x^\mu - d^\mu x^2$	$K_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu$

From the table 1.1 of the infinitesimal transformations, we can compute the commutation relations of $P_\mu, M_{\mu\nu}, D, K_\mu$:

$$[P_\mu, P_\nu] = 0 \tag{1.33}$$
$$[M_{\mu\nu}, P_\rho] = \eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu \tag{1.34}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = M_{\mu\rho}\eta_{\nu\sigma} + M_{\nu\sigma}\eta_{\mu\rho} - M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\rho}\eta_{\mu\sigma} \tag{1.35}$$
$$[P_\mu, D] = P_\mu \tag{1.36}$$
$$[M_{\mu\nu}, D] = 0 \tag{1.37}$$
$$[D, D] = 0 \tag{1.38}$$
$$[P_\mu, K_\nu] = 2\eta_{\mu\nu} D + 2M_{\mu\nu} \tag{1.39}$$
$$[M_{\mu\nu}, K_\rho] = \eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu \tag{1.40}$$
$$[D, K_\mu] = K_\mu \tag{1.41}$$
$$[K_\mu, K_\nu] = 0. \tag{1.42}$$

We see that the commutators are closed on the set of conformal generators, and therefore form the conformal algebra. Moreover the full conformal algebra consists of two nontrivial subalgebras. The first one is Poincaré algebra consisting of translations generators P_μ and Lorentz transformations generators $M_{\mu\nu}$. The second one is Poincaré algebra extended by dilatation generator D. This implies that a theory invariant under Poincaré transformations and dilatations is not necessarily invariant under the special confromal transformations [5], yet many authors interchange the scale and conformal invariance. In many cases the former really implies the latter, for classical case this question was resolved in [7] and it is the subject of the next chapter, however a general proof of this implication for a general quantum case is still missing [12].

The number of independent generators of the full conformal algebra is $(n+2)(n+1)/2$: n for translations, $n(n-1)/2$ for Lorentz transformations, 1 for dilatation, and n for special conformal transformations.

After discussing the conformal transformations in $n > 2$, we focus on the very special case of $n = 2$. We first write the conformal algebra in a compact fashion as a commutator of two transformations, e.g. for a scalar field φ (the result is actually independent from the type of field)

$$[\delta_f, \delta_g]\,\varphi = \delta_f\,(\delta_g \varphi) - \delta_g\,(\delta_f \varphi) \equiv \delta_{[f,g]}\varphi, \tag{1.43}$$

where

$$[f, g]^\mu \equiv f^\nu \partial_\nu g^\mu - g^\nu \partial_\nu f^\mu, \tag{1.44}$$

is the *Lie bracket* of f^μ and g^μ.

We then notice that, in this case the conformal Killing equation is

$$\partial_\mu f_\nu + \partial_\nu f_\mu = (\partial \cdot f)\eta_{\mu\nu}, \tag{1.45}$$

and let us work in Euclidean space i.e. the metric is $\delta_{\mu\nu}$. This does not invalidate the following discussion for Minkowski space, since it is connected to Euclidean one through Wick rotation (that is taking purely imaginary time). Furthermore, it reveals the specialty of the $n = 2$ case in a straightforward way.

With this choice the equation (1.45) splits into

$$\begin{aligned} \partial_0 f_1 + \partial_1 f_0 = 0 \\ \partial_0 f_0 - \partial_1 f_1 = 0. \end{aligned} \tag{1.46}$$

These are the Cauchy-Riemann conditions for the function $f = f_0 + if_1$, defined on the complex plane, to be holomoprhic. Moreover, by taking derivatives of those equations, we see that each f_i has to be harmonic function, i.e. it must satisfy Laplace equation

$$\Delta f_i = 0, \tag{1.47}$$

where $\Delta \equiv \partial_0^2 + \partial_1^2$.

Introducing complex variables[1]

$$\begin{aligned} z =& x^0 + ix^1 & \bar{z} =& x^0 - ix^1 \\ \partial_z =& \frac{1}{2}(\partial_0 - i\partial_1) & \partial_{\bar{z}} =& \frac{1}{2}(\partial_0 + i\partial_1) \\ f =& f_0 + if_1 & \bar{f} =& f_0 - if_1, \end{aligned} \tag{1.48}$$

the Cauchy-Riemann conditions (1.46) become

$$\begin{aligned} \partial_{\bar{z}} f(z,\bar{z}) =& 0 \\ \partial_z \bar{f}(z,\bar{z}) =& 0, \end{aligned} \tag{1.49}$$

therefore $f = f(z)$ and $\bar{f} = \bar{f}(\bar{z})$. In two dimensions any holomorphic map $z \to z - f(z)$ is equivalent to conformal transformation.

Furthermore, any holomorphic function can be expanded in Laurent series, which has infinite number of parameters. This implies the infinite-dimensional algebra of the two dimensional conformal transformations. Let us now derive the associated algebra.

In these variables the geometric transformation is

$$\delta_f \varphi(z,\bar{z}) = \left(f(z)\partial_z + \bar{f}(\bar{z})\partial_{\bar{z}}\right)\varphi(z,\bar{z}). \tag{1.50}$$

Since f $(\bar{f})$ is holomorphic (antiholomorphic) it can be expanded in the Laurent series

$$f(z) = \sum_{n\in\mathbb{Z}} f_n z^{n+1} \qquad \bar{f}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{f}_n \bar{z}^{n+1}, \tag{1.51}$$

where f_n $(\bar{f}_n)$ are infinitesimal constant parameters.

[1]If one insists in having Minkowski space, one has to use light-cone variables.

Introducing generators l_n ($\bar{l}_n$) of the transformation

$$l_n \equiv -z^{n+1}\partial_z \qquad \bar{l}_n \equiv -\bar{z}^{n+1}\partial_{\bar{z}}, \tag{1.52}$$

(1.50) becomes

$$\delta_f \varphi(z,\bar{z}) = -\sum_{n\in\mathbb{Z}} (f_n l_n + \bar{f}_n \bar{l}_n)\varphi(z,\bar{z}). \tag{1.53}$$

The Lie Bracket (1.44) expanded in the Laurent series gives

$$[f,g] = \sum_{n,m} f_n g_m z^{m+n+1}(m-n) \qquad \overline{[f,g]} = \sum_{n,m} \bar{f}_n \bar{g}_m \bar{z}^{m+n+1}(m-n) \tag{1.54}$$

and by recalling the commutator of two geometric transformations $[\delta_f, \delta_g]$ (1.43), we can find the commutation relations of the two dimensional conformal algebra by comparing the terms with the same infinitesimal parameters. The resulting algebra is

$$\begin{aligned} [l_n, l_m] &= (n-m) l_{n+m} \\ [\bar{l}_n, \bar{l}_m] &= (n-m) \bar{l}_{n+m} \\ [l_n, \bar{l}_m] &= 0. \end{aligned} \tag{1.55}$$

This is called the *Witt algebra*.

At this point let us discuss spinors. Although the main interest of this work is in scalar theories, some of the discussed mathematical tools may be used in general case. Let us briefly mention methods for dealing with the spinors [1] [5]. Since they are defined by their transformation properties in a flat space and we would like to work with them on general manifold, where the properties of general coordinate transformations are more restrictive, we have to introduce vielbeins[2], which connect the general coordinate description of a manifold with a local Lorentz frame[3] for every point on the manifold. This way we can define the covariant derivative of spinors, respecting their local Lorentz transformation properties.

At any non-singular point X of a manifold, we can construct a local coordinate basis on the tangent space y^a_X. There the metric is flat i.e. η_{ab}. We use Latin indices for locally flat coordinates (local frame), while for the general coordinates we use Greek indices.

Mathematically speaking we introduce vielbeins e^a_μ through

$$g_{\mu\nu}(X) \equiv e^a_\mu(X) e^b_\nu(X) \eta_{ab}, \tag{1.56}$$

where the summation over Latin indices is understood, and vielbeins can be expressed as

$$e^a_\mu(X) = \left.\frac{\partial y^a_X}{\partial x^\mu}\right|_{x=X}. \tag{1.57}$$

[2]The word "viel" is usually replaced by German word characterizing the dimension number, e.g. for two dimensions we have zweibeins, for four dimensions we speak about vierbeins etc. In some literature also the name tetrads appears for vierbeins

[3]Hence the mathematical name for vielbeins - frame fields

Such object responds to both, the Lorentz transformation of y^a_X and the general coordinate transformation of x^μ. For the former

$$e^a_\mu(X) \to e'^a_\mu(X) = \Lambda^a{}_b(X) e^b_\mu(X) \tag{1.58}$$

and for the latter

$$e^a_\mu(X) \to e'^a_\mu(X) = \frac{\partial x^\nu}{\partial x'^\mu} e^a_\nu(X). \tag{1.59}$$

Contracting objects with vielbeins then results in changing their transformation characteristics, e.g. $e^a_\mu V^\mu$ transforms as four scalars under a general coordinate transformation, but it transforms as a vector under a local Lorentz transformation.

The last step is to introduce a covariant derivative ∇ that works well also for spinors, i.e. in a locally flat frame. We define such derivative as

$$(\nabla_\mu \Phi)^i = \partial_\mu \Phi^i + (S_\mu)^i{}_j \Phi^j, \tag{1.60}$$

where

$$(S_\mu)^i{}_j = \frac{1}{2} s_\mu{}^{ab} (\Sigma_{ab})^i{}_j, \tag{1.61}$$

where matrices Σ_{ab} are the generators of the Lorentz transformations and $s_\mu{}^{ab}$ is called a spin connection and it is related to Christoffel connection through the request that the full covariant derivative of the vielbein is zero

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\lambda{}_{\mu\nu} e^a_\lambda + s_\mu{}^a{}_b e^b_\nu \equiv 0, \tag{1.62}$$

i.e.

$$s_\mu{}^a{}_b = e^a_\lambda (\delta^\lambda_\nu \partial_\mu + \Gamma^\lambda{}_{\mu\nu}) E^\nu_b \tag{1.63}$$

E^ν_b being the inverse vielbein, i.e.

$$\begin{aligned} e^a_\mu E^\nu_a &= \delta^\nu_\mu \\ e^a_\mu E^\mu_b &= \delta^a_b. \end{aligned} \tag{1.64}$$

With this in mind we can now handle fields of any spin.

1.3 Energy-momentum tensor

Here we discuss the energy-momentum tensor. We start by deriving canonical energy-momentum tensor following Noether's argumentation. Then we mention possible improvements and another way used to obtain energy-momentum tensor.

Suppose we have a theory described by a flat space action A with corresponding Lagrangian density $\mathcal{L}$. We restrict ourselves to theories, where Lagrangian density is a functional only of fields Φ_i and their first derivatives $\partial_\mu \Phi_i$, i.e. no explicit dependence on coordinate variable x, and $\Phi_i = \{\varphi, \psi_\alpha, V_\mu, \ldots\}$. We then write

$$A = A[\Phi_i, \partial_\mu \Phi_i, \Omega] = \int_\Omega \mathrm{d}^n x \mathcal{L}[\Phi_i(x), \partial_\mu \Phi_i(x)]. \tag{1.65}$$

Considering some general infinitesimal transformation, which changes coordinates as (1.7)

$$x^\mu \to x'^\mu = x^\mu - f^\mu(x),$$

fields change as (1.8)

$$\Phi_i(x) \to \Phi_i'(x') = \Phi_i(x) + \delta_f \Phi_i(x) - f^\mu \partial_\mu \Phi_i(x),$$

field derivatives change as

$$\partial_\mu \Phi_i(x) \to \partial_\mu' \Phi_i'(x') = \partial_\mu \Phi_i(x) + \delta_f \partial_\mu \Phi_i(x) - f^\nu \partial_\mu \partial_\nu \Phi_i(x)$$

and the corresponding space-time volume $\Omega \to \Omega'$. We denote such transformed action A'

$$A' = \int_{\Omega'} \mathrm{d}^n x' \mathcal{L}[\Phi_i'(x'), \partial_\mu' \Phi_i'(x')]. \tag{1.66}$$

A transformation that leaves the action unchanged, i.e. $A = A'$, we call *symmetry* of the theory and it has some important consequences we would like to examine.

To compare the integrals we have to integrate over the same space-time interval, therefore we have to transform from the primed coordinates back to the original ones. Since we supposed infinitesimal transformation we have

$$\mathrm{d}^n x' = \mathrm{d}^n x \left| \det \left(\frac{\partial x'}{\partial x} \right) \right| = \mathrm{d}^n x (1 - \partial_\alpha f^\alpha). \tag{1.67}$$

Substituting into A' and expanding $\mathcal{L}$ we have

$$\begin{aligned}
A' &= \int_\Omega \mathrm{d}^n x (1 - \partial_\alpha f^\alpha) \left[\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \Phi_i} (\delta_f \Phi_i - f^\mu \partial_\mu \Phi_i) \right. \\
&\qquad \left. + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi_i)} (\delta_f \partial_\nu \Phi_i - f^\mu \partial_\nu \partial_\mu \Phi_i) \right] \\
&= \int_\Omega \mathrm{d}^n x \left[\mathcal{L} - \partial_\mu f^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \Phi_i} (\delta_f \Phi_i - f^\mu \partial_\mu \Phi_i) \right. \\
&\qquad \left. + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi_i)} (\delta_f \partial_\nu \Phi_i - f^\mu \partial_\nu \partial_\mu \Phi_i) \right] \\
&= \int_\Omega \mathrm{d}^n x \left[\mathcal{L} + \delta_f \Phi_i \left(\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \right) \right) - \partial_\mu f^\mu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \Phi_i} f^\mu \partial_\mu \Phi_i \right. \\
&\qquad \left. + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \delta_f \Phi_i \right) - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi_i)} f^\mu \partial_\nu \partial_\mu \Phi_i \right] \\
&= \int_\Omega \mathrm{d}^n x \left[\mathcal{L} + \delta_f \Phi_i \left(\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \right) \right) \right. \\
&\qquad \left. + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \delta_f \Phi_i \right) - \partial_\mu f^\mu \mathcal{L} - f^\mu \partial_\mu \mathcal{L} \right] \\
&= \int_\Omega \mathrm{d}^n x \left[\mathcal{L} + \delta_f \Phi_i \left(\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \right) \right) \right. \\
&\qquad \left. + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \delta_f \Phi_i - f^\mu \mathcal{L} \right) \right].
\end{aligned} \tag{1.68}$$

Demanding now $A' - A = 0$ for every space-time region Ω we see

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \delta_f \Phi_i - f^\mu \mathcal{L} \right) = \left(\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \right) - \frac{\partial \mathcal{L}}{\partial \Phi_i} \right) \delta_f \Phi_i. \tag{1.69}$$

For any infinitesimal transformation which leaves the action invariant there is a relation between Euler-Lagrange equations, the right hand side of (1.69), and divergence of some special object. Hence, on-shell, (i.e., when fields satisfy Euler-Lagrange equations) we have

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \delta_f \Phi_i - f^\mu \mathcal{L} \right) = 0. \tag{1.70}$$

That is a conservation law for the current

$$J^\mu_f \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \delta_f \Phi_i - f^\mu \mathcal{L}, \tag{1.71}$$

related to the Noether current, the energy-momentum tensor, ${\Theta^\mu}_\alpha$ [4] as

$$J^\mu_f \equiv {\Theta^\mu}_\alpha f^\alpha. \tag{1.72}$$

Not only the symmetry transformations leave us with (on-shell) conserved currents, they give us conditions on the energy-momentum tensor.

Considering translation invariance $f^\mu(x) = a^\mu$ of an action, substituting into (1.72) and taking divergence

$$\partial_\mu J^\mu_f = a^\alpha \partial_\mu {\Theta^\mu}_\alpha, \tag{1.73}$$

we see that since on-shell J^μ_f is conserved the energy-momentum tensor Θ^μ_α is also conserved. Furthermore, since for the translation transformation holds $\delta_f \Phi_i = a^\mu \partial_\mu \Phi_i$ for all kinds of fields, we can find the general formula for the energy-momentum tensor by combining (1.71) and (1.72). Such formula has even its own name and it is called the *canonical energy-momentum tensor*

$$\Theta^{can}_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \partial_\nu \Phi_i - \eta_{\mu\nu} \mathcal{L}. \tag{1.74}$$

Considering not only the translation invariance of the action but the full Poincaré invariance, we get another condition for the on-shell energy-momentum tensor

$$\partial_\mu J^\mu_f = \partial_\mu ({\Theta^\mu}_\alpha {b^\alpha}_\beta x^\beta) = \partial_\mu {\Theta^\mu}_\alpha {b^\alpha}_\beta x^\beta + {\Theta^\mu}_\alpha {b^\alpha}_\beta \delta^\beta_\mu = 0 + \Theta^{\mu\nu} b_{\mu\nu} = 0, \tag{1.75}$$

where we used in the third equality that the energy-momentum tensor is conserved (from the translation invariance) and the last equality comes from the conservation of J^μ_f under the symmetry. Since $b_{\mu\nu}$ is antisymmetric and arbitrary we see that the Lorentz (along with translation) invariance implies symmetric energy-momentum tensor $\Theta^{\mu\nu} = \Theta^{\nu\mu}$.

In the previous section we derived a broader set of transformations. Let us see what other conditions the scale a the special conformal transformations put on the table. In a similar fashion we have for scale transformation

$$\partial_\mu J^\mu_f = \partial_\mu ({\Theta^\mu}_\alpha c x^\alpha) = c \Theta^\mu_\mu = 0, \tag{1.76}$$

[4] Usual notation for energy-momentum tensor is ${T^\mu}_\alpha$ which we reserve to a variational definition of the tensor

where we used the same procedure as with the Lorentz transformation. Hence, scale invariance requires on-shell tracelessness of the energy-momentum tensor. Unfortunately, studying special conformal transformation in the same way we do not obtain any new information about energy-momentum tensor.

Knowing all the features of $\Theta_{\mu\nu}$ we may argue that the canonical energy-momentum tensor is not manifestly symmetric nor traceless in a case of broader set of symmetries. In such cases we expect the energy-momentum tensor to be improvable in a way that makes it symmetric and traceless.

One way to improve the energy-momentum tensor was introduced by Belinfante and Rosenfeld [4]. They add a superpotential-like term to the canonical energy-momentum tensor

$$\Theta_{\mu\nu} = \Theta_{\mu\nu}^{can} + \partial_\lambda B^{\lambda}{}_{\mu\nu}, \tag{1.77}$$

where $B^{\lambda\mu\nu} = -B^{\mu\lambda\nu}$ is antisymmetric in the first two indices, and since the new introduced term is a divergence, it does not change the conservation of the current J_f^μ.

The energy-momentum tensor can also be obtained by writing the action in a diffeomorphic invariant manner, $A \to \mathcal{A}$ (see later), hence at least minimally coupled to the metric $g_{\mu\nu}$. In this case we can compute the energy-momentum tensor [3]

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{A}}{\delta g^{\mu\nu}}, \tag{1.78}$$

where factor 2 is only a normalization factor and can be arbitrarily chosen. In the flat space limit both tensors coincide, i.e.

$$T_{\mu\nu}\Big|_{g_{\mu\nu}=\eta_{\mu\nu}} \equiv \Theta_{\mu\nu}. \tag{1.79}$$

Let us note that the energy-momentum tensor given by the variation of an action is in general not traceless, nevertheless, the symmetry is guaranteed because of the definition. In case we deal with scale invariant theory, we usually have to find improvements of the energy-momentum tensor $T_{\mu\nu}$. This is discussed in the next chapter.

1.4 Improvements of the energy-momentum tensor

In this section we are going to illustrate improvements to the canonical energy-momentum tensor for a special case of our interest throughout this thesis. That is a scalar field theory in $n = 2$, named Liouville theory.

We start with the free massless Klein-Gordon action A_{KG} in two dimensions

$$A_{KG} = \int \mathrm{d}^2 x \, \frac{1}{2} \eta^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi. \tag{1.80}$$

For the following study of symmetries, it is useful to know the derivatives of the previously discussed transformations f^μ. They are computed in table 1.2.

Table 1.2: Useful derivatives of infinitesimal conformal transformations

f^μ	a^μ	$b^\mu{}_\nu x^\nu$	cx^μ	$2(d\cdot x)x^\mu - x^2 d^\mu$
$\partial_\alpha f^\mu$	0	$b^\mu{}_\alpha$	$c\delta^\mu_\alpha$	$2d_\alpha x^\mu + 2(d\cdot x)\delta^\mu_\alpha - 2x_\alpha d^\mu$
$\partial_\mu f^\mu$	0	0	nc	$2n(d\cdot x)$
$\partial_\alpha\partial_\mu f^\mu$	0	0	0	$2nd_\alpha$

Taking the infinitesimal variation of (1.80) we have

$$\begin{aligned} \delta_f A_{KG} &= \int \mathrm{d}^2x\, \partial_\mu\varphi\partial^\mu\delta_f\varphi \\ &= \int \mathrm{d}^2x \left(\partial_\mu f^\nu\partial_\nu\varphi\partial^\mu\varphi - \frac{1}{2}\partial_\nu f^\nu\partial_\mu\varphi\partial^\mu\varphi\right), \end{aligned} \tag{1.81}$$

where the second line comes from the partial integration. Substituting in (1.81) the f^μ and recalling the derivatives in table 1.2 we get $\delta_f A_{KG} = 0$ for the full conformal group.

Since it is conformally symmetric theory we must be able to find a traceless energy-momentum tensor. We start by computing the canonical energy-momentum tensor (1.74):

$$\Theta^{can}_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}\partial_\lambda\varphi\partial^\lambda\varphi. \tag{1.82}$$

It is clearly symmetric and taking the trace we see that it is also traceless, therefore we do not need any improvements. The canonical energy-momentum tensor in this case already has all the demanded properties.

Now we move to the Liouville action A_L

$$A_L = \int \mathrm{d}^2x \left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{m^2}{\beta^2}e^{\beta\varphi}\right), \tag{1.83}$$

where m and β are constants.

Studying the symmetries of this action under the field transformations (1.11) we have

$$\delta_f A_L = \delta_f A_{KG} - \frac{m^2}{\beta}\int \mathrm{d}^2x\, e^{\beta\varphi} f^\mu\partial_\mu\varphi. \tag{1.84}$$

Since we already know $\delta_f A_{KG} = 0$, we are left only with the second term to study.

By partial integration we have

$$\delta_f A_L = \frac{m^2}{\beta^2}\int \mathrm{d}^2x\, e^{\beta\varphi}\partial_\mu f^\mu \tag{1.85}$$

and by looking into table 1.2 it is clear that the action A_L is only Poincaré invariant (i.e. $\partial_\mu f^\mu = 0$) under the field transformation $\varphi \to f^\alpha\partial_\alpha\varphi$.

Nevertheless, introducing a new field transformation accompanied by a field independent shift

$$\tilde{\delta}_f\varphi \equiv \delta_f\varphi + \partial_\mu f^\mu = f^\mu\partial_\mu\varphi + \partial_\mu f^\mu, \tag{1.86}$$

we can verify that under such transformation, the Liouville action is fully conformal invariant

$$\begin{aligned} \tilde{\delta}_f A_L &= \int \mathrm{d}^2x \left(\partial^\mu\varphi\partial_\mu\delta_f\varphi + \partial^\mu\varphi\partial_\mu\partial_\nu f^\nu - \frac{m^2}{\beta^2}e^{\beta\varphi}\left(f^\mu\partial_\mu(\beta\varphi) + \partial_\mu f^\mu\right)\right) \\ &= \delta_f A_{KG} - \int \mathrm{d}^2x\, \varphi\partial_\mu\partial^\mu\partial_\nu f^\nu - \int \mathrm{d}^2x\, \frac{m^2}{\beta^2}\partial_\mu\left(e^{\beta\varphi}f^\mu\right). \end{aligned} \tag{1.87}$$

The first term is zero as we showed before, the second term is zero because transformations f^ν are at most quadratic in x and the last term vanishes because it is a surface term. With transformation (1.86) the Liouville theory is invariant under the full conformal group. So again we should be able to find a traceless energy-momentum tensor.

Computing the canonical energy-momentum tensor (1.74) we have

$$\Theta^{can}_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \eta_{\mu\nu}\frac{1}{2}\eta^{\sigma\rho}\partial_\sigma\varphi\partial_\rho\varphi + \eta_{\mu\nu}\frac{m^2}{\beta^2}e^{\beta\varphi}. \tag{1.88}$$

The trace of this canonical energy-momentum tensor is

$$\Theta^\mu_\mu = 2\frac{m^2}{\beta^2}e^{\beta\varphi}, \tag{1.89}$$

but, as we showed, the action is conformally invariant therefore it should be traceless on-shell. Recalling the equation of motion for Liouville theory

$$\Box\varphi + \frac{m^2}{\beta}e^{\beta\varphi} = 0, \tag{1.90}$$

it is clear the trace is non-zero in general.

Therefore we want to find an improvement term to make it traceless. Using the definition of the energy-momentum tensor (1.72) and (1.71) we get

$$\Theta^\mu_\nu f^\nu = f^\nu\left(\partial^\mu\varphi\partial_\nu\varphi - \delta^\mu_\nu\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi + \delta^\mu_\nu\frac{m^2}{\beta^2}e^{\beta\varphi}\right) + \partial^\mu\varphi\partial_\nu f^\nu, \tag{1.91}$$

where the first is just $f^\nu\Theta^{can}{}^\mu_\nu$ and the second term is the new improvement term

$$\tilde{\Theta}^\mu_\nu f^\nu \equiv \partial^\mu\varphi\partial_\nu f^\nu. \tag{1.92}$$

Since the dilatation is responsible for the tracelessness we use $f^\mu = cx^\mu$ in following steps.

$$\tilde{\Theta}^\mu_\nu x^\nu = 2\partial^\mu\varphi. \tag{1.93}$$

Requiring the on-shell conservation of the energy-momentum tensor i.e. $\partial_\mu\Theta^{\mu\nu} = 0$ and because $\partial_\mu\Theta^{can\mu\nu} = 0$, we demand the improvement term has vanishing divergence $\partial_\mu\tilde{\Theta}^{\mu\nu} = 0$.

By taking derivative of the equation (1.93)

$$\partial_\mu(\tilde{\Theta}^\mu_\nu x^\nu) = \tilde{\Theta}^\mu_\mu = 2\Box\varphi, \tag{1.94}$$

we see that $\tilde{\Theta}_{\mu\nu}$ can be written in terms of $\eta_{\mu\nu}\Box\varphi$ and $\partial_\mu\partial_\nu\varphi$, i.e.

$$\tilde{\Theta}_{\mu\nu} = a\eta_{\mu\nu}\Box\varphi + b\partial_\mu\varphi\partial_\nu\varphi. \tag{1.95}$$

The tracelessness of $\Theta_{\mu\nu}$ then gives us the condition on a and b. The improvement term is therefore

$$\tilde{\Theta}_{\mu\nu} = \frac{2}{\beta}\left(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\varphi. \tag{1.96}$$

The improved energy-momentum tensor respecting the conformal symmetry is

$$\begin{aligned}\Theta_{\mu\nu} =&\Theta^{can}_{\mu\nu} + \tilde{\Theta}_{\mu\nu}\\ =&\partial_\mu\varphi\partial_\nu\varphi - \eta_{\mu\nu}\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi + \eta_{\mu\nu}\frac{m^2}{\beta^2}e^{\beta\varphi} + \frac{2}{\beta}\left(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\varphi.\end{aligned} \tag{1.97}$$

It is interesting to look at the properties of the "improved" transformation $\tilde{\delta}_f$. Let us study the commutator of the two such transformations

$$[\tilde{\delta}_f, \tilde{\delta}_g]\varphi = \delta_{[f,g]}\varphi + \partial_\mu[f^\mu(\partial_\nu g^\nu + 1) - g^\mu(\partial_\nu f^\nu + 1)], \tag{1.98}$$

where the $[f, g]$ denotes the Lie bracket (1.44).

We see that commutation relations with this new transformations are different from those of (1.43). A new term, $\partial_\mu[\ldots]^\mu$, arose. We call it central term: a field independent, pure divergence term (hence contributing only for nontrivial boundaries), that commutes with all generators. This new algebra is called *Virasoro algebra*, that is the central extension of the Witt algebra

$$\begin{aligned}[L_n, L_m] &= (n-m)L_{n+m} + cn(n^2-1)\delta_{n,-m}\\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + cn(n^2-1)\delta_{n,-m}\\ [L_n, \bar{L}_m] &= 0\\ [L_n, c] &= [\bar{L}_n, c] = 0.\end{aligned} \tag{1.99}$$

The extra generator c, is called central charge. The Virasoro algebra appears in many areas of physics, from statistical mechanics to black-hole physics [5].

2. Weyl gauging and Ricci gauging

In the previous chapter we introduced the improvement terms for the energy-momentum tensors. Now we describe how to systematically produce such improvement terms. In this chapter we shall review the results of [7]. We assume rigid scale invariance in flat space. Then we demand diffeomorphic invariance and replace rigid scale invariance with rigid Weyl invariance.

Promoting rigid Weyl invariance to local Weyl invariance requires the introduction of gauge fields we call W_μ (this was the first example of gauge fields in history [14]). When W_μ enters the (Weyl) gauged action in specific combinations, such combinations can be replaced by expressions in terms of R and $R_{\mu\nu}$, because the response under Weyl variation is the same. This replacement is called in [7] "Ricci gauging". The necessary and sufficient condition for an action to be Ricci gaugable is that, in flat space, scale invariance implies full conformal invariance.

2.1 Weyl transformations

Assume we have an action A which depends only on fields Φ_i and their first derivatives $\partial_\mu \Phi_i$, where $\Phi_i = \{\varphi, \psi_\alpha, V_\mu, ...\}$, i.e. fields of arbitrary spin, and which is rigid scale invariant, i.e. invariant under transformation

$$\begin{aligned} x^\mu &\to e^\omega x^\mu \\ \Phi_i &\to e^{d_\Phi \omega} \Phi_i, \end{aligned} \tag{2.1}$$

where ω is a constant and d_Φ is the scale dimension of Φ_i, usually determined by kinetic term in the action. We now promote our action A to diffeomorphic invariant form $\mathcal{A}$ in the usual way [1]

$$\mathcal{A} = \int \mathrm{d}^n x \, \sqrt{-g} \mathcal{L} \left(\Phi_i, \nabla_\mu \Phi_i \right). \tag{2.2}$$

Since we deal with fields of arbitrary spin, we have to introduce covariant derivatives that work well also for spinors. Such procedure was discussed in previous chapter, where vielbeins and spin connection were introduced.

We know that our action $\mathcal{A}$ is rigid scale invariant (i.e. $x^\mu \to e^\omega x^\mu$). We will rewrite the rigid scale transformation (2.1) in a more useful form

$$\begin{aligned} e^a_\mu &\to e^\omega e^a_\mu \\ x^\mu &\to x^\mu \\ \Phi_i &\to e^{d_\Phi \omega} \Phi_i. \end{aligned} \tag{2.3}$$

These transformations are called rigid Weyl transformations. Notice that the scaling is translated to e^a_μ and x^μ does not transform, like for the geometric transformations δ_f discussed earlier. To have full equivalence of (2.1) with (2.3) it is understood here that diffeomorphism and Weyl invariance can coexist in any theory. Among the interesting questions opened by [9] is precisely the use of the

Liouville action that can be either diffeomorphism invariant or Weyl invariant, but not both. In what follows in this chapter we shall not consider this issue, but we shall look at it in the chapter 3.

Now we will try to construct actions invariant when these transformations are local, e.g. $\omega \to \omega(x)$. This is the first example of the gauge principle in history, see [14]. For more information about the history of gauge theory see [13].

The action depends on derivatives of field, hence we have to look at derivatives of the transformed fields. For scalar field φ we have

$$\partial_\mu \left(e^{d_\varphi \omega}\varphi\right) = e^{d_\varphi \omega}\left(\partial_\mu + d_\varphi \partial_\mu \omega\right)\varphi \neq e^{d_\varphi \omega}\partial_\mu \varphi, \tag{2.4}$$

thus $\partial_\mu \varphi$ is not covariant when $\varphi \to e^{d_\varphi \omega}\varphi$

We need to introduce Weyl-covariant derivative D_μ

$$D_\mu \equiv \partial_\mu + d_\phi W_\mu, \tag{2.5}$$

with

$$W_\mu \to W_\mu - \partial_\mu \omega \tag{2.6}$$

under the Weyl transformations. With this

$$D_\mu \varphi \to e^{d_\varphi \omega} D_\mu \varphi, \tag{2.7}$$

when $\varphi \to e^{d_\varphi \omega}\varphi$.

For fields of arbitrary spin Φ^i we see that the Weyl variation of the diffeomorphic-covariant derivative is [7]

$$\Delta(\nabla_\mu \Phi)^i = d_\Phi \partial_\mu \omega \Phi^i + 2\partial_\tau \omega (\Sigma^\tau{}_\mu)^i{}_j \Phi^j. \tag{2.8}$$

So the Weyl covariant derivative is

$$D_\mu \equiv \nabla_\mu + \Lambda^\nu{}_\mu W_\nu, \tag{2.9}$$

where W_ν transforms as in (2.6) and

$$\Lambda^\nu{}_\mu \equiv d_\Phi g^\nu_\mu + 2\Sigma^\nu{}_\mu \tag{2.10}$$

2.2 Ricci gauging for n>2

Since we want to couple the fields to curvature tensors, we need to find a relation between the latter and expressions involving Weyl potential. Having in mind Ricci curvature tensor we need to construct second rank tensor from W_μ and $g_{\mu\nu}$. The possible tensors are $W_\mu W_\nu$, $\nabla_\mu W_\nu$ and $g_{\mu\nu}$ itself. Let us investigate the response of those tensors to finite Weyl transformations

$$\Delta\left(W_\mu W_\nu\right) = \left(W_\mu - \omega_\mu\right)\left(W_\nu - \omega_\nu\right) - W_\mu W_\nu = \omega_\mu \omega_\nu - \left(W_\mu \omega_\nu + W_\nu \omega_\mu\right), \tag{2.11}$$

where $\omega_\mu \equiv \partial_\mu \omega$. For $\nabla_\mu W_\nu$ we need to know the Weyl variation of Christoffel symbols (computed in appendix A). We then obtain

$$\Delta\left(\nabla_\mu W_\nu\right) = -\nabla_\mu \omega_\nu - g_{\mu\nu}\omega^\sigma \omega_\sigma + 2\omega_\mu \omega_\nu + g_{\mu\nu} W^\sigma \omega_\sigma - \left(W_\mu \omega_\nu + W_\nu \omega_\mu\right). \tag{2.12}$$

Now we also compute the Weyl variation of $g_{\mu\nu}W_\sigma W^\sigma$

$$\Delta\left(g_{\mu\nu}g^{\rho\sigma}W_\rho W_\sigma\right) = g_{\mu\nu}\left(\omega_\sigma\omega^\sigma - 2W_\sigma\omega^\sigma\right). \tag{2.13}$$

The Weyl variation is linear, therefore we can combine previous tensors to get an expression that under Weyl variation is independent of W^μ. Up to the sign, and a multiplicative constant we have only one solution

$$\Omega_{\mu\nu}[W] = \nabla_\mu W_\nu - W_\mu W_\nu + \frac{1}{2}g_{\mu\nu}W_\sigma W^\sigma. \tag{2.14}$$

Indeed, the Weyl variation of $\Omega_{\mu\nu}[W]$ is

$$\Delta\Omega_{\mu\nu}[W] = -\nabla_\mu\omega_\nu + \omega_\mu\omega_\nu - \frac{1}{2}g_{\mu\nu}\omega_\sigma\omega^\sigma = -\Omega_{\mu\nu}[\omega]. \tag{2.15}$$

Let us now obtain the Weyl variation of Ricci tensor

$$\Delta R_{\mu\nu} = -g_{\mu\nu}\nabla^2\omega - (n-2)\left(\nabla_\mu\omega_\nu - \omega_\mu\omega_\nu + \omega_\sigma\omega^\sigma g_{\mu\nu}\right) \tag{2.16}$$

and of $g_{\mu\nu}R$ where R is Ricci scalar

$$\begin{aligned}\Delta\left(g_{\mu\nu}R\right) &= \Delta\left(g_{\mu\nu}g^{\sigma\rho}R_{\sigma\rho}\right) = g_{\mu\nu}g^{\sigma\rho}\Delta R_{\sigma\rho}\\ &= -g_{\mu\nu}\Big(2(n-1)\nabla^2\omega + (n-2)(n-1)\omega_\rho\omega^\rho\Big).\end{aligned} \tag{2.17}$$

Both previous computations are presented in Appendix A.

Now it is easy to see that the tensor

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{2(n-1)}g_{\mu\nu}R \tag{2.18}$$

has Weyl variation proportional to the Weyl variation of (2.14) i.e.

$$\Delta S_{\mu\nu} = -(n-2)\Omega_{\mu\nu}[\omega]. \tag{2.19}$$

From this we see that Weyl variation of a Weyl-gauged action where W_μ enters only as in the expression (2.14) gives the same result as Weyl variation of an action when $\Omega_{\mu\nu} \to S_{\mu\nu}$. This substitution is called the Ricci gauging in [7].

The form (2.14) is very specific and gives us the wanted coupling. Thus, the question when is this possible naturally arises. To answer, we start by finding connection between Weyl transformations and conformal transformations. Suppose $x^\mu \to y^\mu(x)$ to be conformal transformations

$$\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}g_{\mu\nu}(x) = \hat{g}_{\alpha\beta}(y), \tag{2.20}$$

where

$$\hat{g}_{\alpha\beta}(x) = e^{2\hat{\omega}}g_{\alpha\beta}(x), \tag{2.21}$$

and we look for those $\hat{\omega}$ such that $e^{2\hat{\omega}}$ defines special conformal transformations. Under (2.20) and (2.21), $S_{\mu\nu}$ in (2.18) transforms as follows

$$\hat{S}_{\mu\nu} - S_{\mu\nu} = -(n-2)\Omega_{\mu\nu}[\hat{\omega}]. \tag{2.22}$$

In flat space limit the left hand side vanishes and our condition is

$$\partial_\mu \hat{\omega}_\nu - \hat{\omega}_\mu \hat{\omega}_\nu + \frac{1}{2}\eta_{\mu\nu}\hat{\omega}_\sigma \hat{\omega}^\sigma = 0, \tag{2.23}$$

whose solution is

$$\hat{\omega}(x) = \ln(1 - 2c_\mu x^\mu + c^2 x^2), \tag{2.24}$$

with c^μ a constant vector. This identifies special conformal transformation, as can bee seen in 1.1.

Suppose that we obtain, according to the previously explained procedure, a Ricci gauged action $\mathcal{A}$

$$\mathcal{A}(\Phi_i, S_{\mu\nu}) = \mathcal{A}(\Phi'_i, S_{\mu\nu} + (n-2)\Omega_{\mu\nu}[\omega]). \tag{2.25}$$

For ω such that $\Omega_{\mu\nu}[\omega] = 0$, no other gauging is needed. But this is what we have solved while solving (2.23) in flat space limit. Therefore we have the result

A necessary condition for a Weyl gauged action to admit Ricci gauging is that it be conformally invariant in the flat space limit

We know the necessary condition, so what is the sufficient one? We start with conformal invariant action A_0 which contains only first derivatives of the conformally variant fields. For *infinitesimal* conformal transformation we have

$$\delta A_0 = \int \mathrm{d}^n x\, \hat{\omega}_\mu j^\mu = c_\mu \int \mathrm{d}^n x\, j^\mu = 0, \tag{2.26}$$

where j^μ is the so-called *virial current* [7], that, e.g., for a scalar ϕ is

$$j^\mu \equiv \frac{\delta A_0}{\delta(\partial^\nu \phi)} \Lambda^{\mu\nu} \phi \tag{2.27}$$

and $\hat{\omega} = c_\mu x^\mu$ since it is infinitesimal transformation. Because the action is conformally invariant, $\delta A_0 = 0$ implies

$$j^\mu = \partial_\nu J^{\mu\nu}, \tag{2.28}$$

with $J^{\mu\nu}$ which does not depend on higher derivatives, otherwise lower derivatives would give us information on higher derivatives.

Thus, conformal invariance is possible only for actions which are at most quadratic in the derivatives of conformally variant fields.

Now, we vary the action A_0 under *finite* conformal transformation.

$$\Delta A_0 = \int \mathrm{d}^n x \, \left(\hat{\omega}_\mu j^\mu + \hat{\omega}_\mu \hat{\omega}_\nu T^{\mu\nu}\right), \tag{2.29}$$

where $T^{\mu\nu}$ does not have field derivatives. Using again (2.28), partial integration leads to

$$\Delta A_0 = \int \mathrm{d}^n x \, \left(-J^{\mu\nu}\partial_\mu \hat{\omega}_\nu + \hat{\omega}_\mu \hat{\omega}_\nu T^{\mu\nu}\right) \tag{2.30}$$

and by using (2.23) (that identifies special conformal transformations)

$$\Delta A_0 = \int \mathrm{d}^n x\, \hat{\omega}_\mu \hat{\omega}_\nu \left(T^{\mu\nu} - J^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu} J\right). \tag{2.31}$$

$\Delta A_0 = 0$ implies that

$$T^{\mu\nu} = J^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} J. \tag{2.32}$$

In our hunting for the Ricci gauging, we now promote the above flat space expressions to curved-space ones. Furthermore (2.28) becomes $j^\mu = \nabla_\nu J^{\mu\nu}$ and (2.32) becomes $T^{\mu\nu} = J^{\mu\nu} - \frac{1}{2}g^{\mu\nu}J$. By taking the trace of the last expression we can compute $J^{\mu\nu}$

$$J^{\mu\nu} = T^{\mu\nu} - \frac{1}{n-2}g^{\mu\nu}T. \tag{2.33}$$

Gauging the diffeomorphic-invariant action A_0 we have

$$\mathcal{A} = \mathcal{A}_0 + \int \mathrm{d}^n x \sqrt{-g}\left(W_\mu j^\mu + W_\mu W_\nu T^{\mu\nu}\right). \tag{2.34}$$

Putting everything together

$$\mathcal{A} = \mathcal{A}_0 + \int \mathrm{d}^n x \sqrt{-g}\left(-J^{\mu\nu}\nabla_\nu W_\mu + W_\mu W_\nu J^{\mu\nu} - \frac{1}{2}W_\rho W^\rho g_{\mu\nu} J^{\mu\nu}\right), \tag{2.35}$$

and we recognize $\Omega_{\mu\nu}[W]$ (2.14)

$$\mathcal{A} = \mathcal{A}_0 - \int \mathrm{d}^n x \sqrt{-g} J^{\mu\nu}\Omega_{\mu\nu}[W]. \tag{2.36}$$

We then conclude that *necessary and sufficient condition for a Weyl-invariant action $\mathcal{A}$ to allow for Ricci gauging is that the flat space limit of the ungauged action, A_0, is not only scale invariant, but fully conformally invariant. Furthermore the Ricci gauging is achieved by* (2.36)*.*

2.3 Ricci gauging for n=2

For the case of 2 dimensions we have to do few things differently. First we notice that $S_{\mu\nu}$ in (2.18) is identically zero, because in two dimensions we have

$$R_{\mu\nu} = \frac{R}{2}g_{\mu\nu}. \tag{2.37}$$

This is so because the Ricci tensor is symmetric and in $n = 2$ the Riemann tensor has only one independent component[1]. Therefore, the coupling will be to the Ricci scalar rather then Ricci tensor. Let us now take the trace of (2.15)

$$g^{\mu\nu}\Delta\Omega_{\mu\nu}[W] = -\nabla^2\omega = -\Omega^\mu_\mu[\omega] \tag{2.38}$$

and compare to the Weyl variation of the Ricci scalar:

$$R[e^{2\omega}g_{\mu\nu}] - e^{-2\omega}R[g_{\mu\nu}] = -2e^{-2\omega}\nabla^2\omega = -2e^{-2\omega}\Omega^\mu_\mu[\omega] \tag{2.39}$$

When we found the relation between Weyl and conformal transformations, we got (2.22), so that the condition for conformal invariance is (2.23). Looking at (2.39) we see that the condition is now

$$\nabla^2\hat{\omega} = 0, \tag{2.40}$$

[1]The number of independent components for the Riemann tensor is $n^2(n^2-1)/12$

but in flat space this is just the wave equation, and it has the infinitely many harmonic functions as solutions. For any harmonic function there exists a holomorphic function with its real part equal to the given harmonic function. Henceforth this establishes a relation between (2.40) and the Witt algebra of (1.55).

The discussion on necessary and sufficient condition for Ricci gauging is pretty much the same as for $n > 2$ case. Varying the action as in (2.26) gives us stronger condition since

$$\delta A_0 = \int \mathrm{d}^2x \, \hat{\omega}_\mu j^\mu = -\int \mathrm{d}^2x \, J^{\mu\nu}\partial_\nu\hat{\omega}_\mu = 0. \tag{2.41}$$

where we assumed (2.28).

Now we will do a little sidestep. In appendix B we introduce light cone coordinates

$$x^\pm \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^1).$$

In this notation the solution of (2.40) are either[2] $\hat{\omega}(x^+)$ or $\hat{\omega}(x^-)$. Also we can rewrite d^2x as dx^+dx^-. Now we see that from (2.41) we have

$$\int \mathrm{d}x^+\mathrm{d}x^- J^{++}\nabla_+^2\hat{\omega}(x_+) = 0. \tag{2.42}$$

The only dependence on x^- must be in J^{++} therefore

$$\int \mathrm{d}x^- J^{++} = 0. \tag{2.43}$$

Since J^{++} is a general function this formula holds only for $J^{++} = 0$. With similar arguments one shows that $J^{--} = 0$. In local frame the values of g^{++} and g^{--} are zero (see appendix B). Then

$$J^{\mu\nu} = g^{\mu\nu}J, \tag{2.44}$$

where J is scalar. So now we can see the Virial current

$$j^\mu = \partial^\mu J. \tag{2.45}$$

So again, *conformal invariance is possible only for action which are at most quadratic in the derivatives of conformally variant fields.*

Computing finite Weyl variation of the action A_0 in two dimensions is same as in higher dimensions i.e. (2.29)

$$\Delta A_0 = \int \mathrm{d}^2x \, \left(\hat{\omega}_\mu j^\mu + \hat{\omega}_\mu\hat{\omega}_\nu T^{\mu\nu}\right).$$

The first term in the integrand vanishes due to (2.40) and partial integration. In the second term we assume that $T^{\mu\nu} = g^{\mu\nu}K$, where the constant K is determined by virial current. For the case of two dimensions we have $K = 0$. So again the integrand vanishes in the finite variation of A_0.

Now we still have to do the final step. Rewrite action in covariant form i.e. Weyl gauge. This time it is not different. The gauged action is of form (2.34). As we discussed right now the $T^{\mu\nu}$ does not contribute. So final form is

$$\mathcal{A} = \mathcal{A}_0 - \int \mathrm{d}^2x \, \sqrt{-g}J\nabla^\mu W_\mu = \mathcal{A}_0 - \int \mathrm{d}^2x \, \sqrt{-g}J\Omega^\mu_\mu[W] \tag{2.46}$$

[2] As announced, this is the "real version" of the complex property $f(z)$ or $\bar{f}(\bar{z})$ of (1.49)

and once again, *necessary and sufficient condition for a Weyl-invariant action* $\mathcal{A}$ *to allow for Ricci gauging is that the flat space limit of the ungauged action* A_0 *is not only scale invariant, but fully conformally invariant. Furthermore the Ricci gauging is achieved by* (2.46).

3. Liouville anomaly

In this chapter we investigate the results of the paper [9], where it is proposed that, in the case of Liouville theory, one can only have either diffeomorphic invariance *or* Weyl invariance, but not both.

3.1 Liouville action

As explained in detail earlier Liouville action (1.83)

$$A_L = \int \mathrm{d}^2x \left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{m^2}{\beta^2}e^{\beta\varphi} \right)$$

is invariant under the "improved" conformal transformations $\tilde{\delta}_f\varphi \equiv f^\nu\partial_\nu\varphi + \partial_\nu f^\nu$, with f^μ given in table 1.1, and we derived the associated traceless energy-momentum tensor (1.97). Such tensor can also be obtained by varying the non-minimally coupled diffeomorphic invariant Liouville action $\mathcal{A}_L$

$$\mathcal{A}_L = \int \mathrm{d}^2x \sqrt{-g} \left(\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{m^2}{\beta^2}e^{\beta\varphi} + \frac{1}{\beta}R\varphi \right) \tag{3.1}$$

with respect to the metric $g_{\mu\nu}$

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{A}_L}{\delta g^{\mu\nu}},$$

obtaining

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\frac{1}{2}g^{\sigma\rho}\partial_\sigma\varphi\partial_\rho\varphi + g_{\mu\nu}\frac{m^2}{\beta^2}e^{\beta\varphi} + \frac{2}{\beta}\left(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu\right)\varphi, \tag{3.2}$$

which becomes (1.97) in the flat-space limit.

In fact, the trace of this energy-momentum tensor is

$$T^\mu_\mu = 2\nabla^2\varphi + \frac{2m^2}{\beta}e^{\beta\varphi}, \tag{3.3}$$

which is not zero, since the non-minimal coupling changed equations of motion to

$$\nabla^2\varphi + \frac{m^2}{\beta}e^{\beta\varphi} - \frac{1}{\beta}R = 0. \tag{3.4}$$

The trace is proportional to the Ricci scalar.

The Weyl transformations for the action (3.1) have the form

$$\begin{aligned} \varphi &\to \varphi - \frac{2}{\beta}\omega \\ g_{\mu\nu} &\to e^{2\omega}g_{\mu\nu}. \end{aligned} \tag{3.5}$$

We see that the action (3.1) is Weyl invariant only up to the field independent term

$$\mathcal{A}_L \to \mathcal{A}_L - \frac{2}{\beta^2}\int \mathrm{d}^2x \sqrt{-g}\left(R\omega + g^{\mu\nu}\omega_\mu\omega_\nu\right). \tag{3.6}$$

Although the action is not Weyl invariant the equations of motion are.

3.2 Dimensional limit

Now we will do a little sidestep. Let us introduce a new action A_n

$$A_n = \int \mathrm{d}^n x \left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \lambda\phi^{\frac{2n}{n-2}}\right), \tag{3.7}$$

where $n > 2$ and ϕ is a scalar field.

This action is invariant under

$$\hat{\delta}_f\phi = f^\alpha\partial_\alpha\phi + \frac{n-2}{2n}\partial_\alpha f^\alpha\phi, \tag{3.8}$$

where f^α is from full conformal group given in table 1.1.

Computing the canonical energy-momentum tensor (1.74)

$$\Theta^{can}_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_i)}\partial_\nu\Phi_i - \eta_{\mu\nu}\mathcal{L},$$

we have

$$\Theta^{can}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\left(\frac{1}{2}\eta^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - \lambda\phi^{\frac{2n}{n-2}}\right). \tag{3.9}$$

This is symmetric, but not traceless. Nevertheless, since (3.7) is invariant under full conformal transformations, we should be able to improve it to the traceless form. We can do so similarly to the Liouville case by recalling (1.71) and (1.72). Then we get

$$\Theta_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\left(\frac{1}{2}\eta^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - \lambda\phi^{\frac{2n}{n-2}}\right) + \frac{n-2}{4(n-1)}\left(\eta_{\mu\nu}\Box - \partial_\nu\partial_\mu\right)\phi^2. \tag{3.10}$$

The other way to get this result is through writing the action A_n in a diffeomorphic and Weyl invariant manner and coupling the field non-minimally to the curvature. This we can do by Ricci gauging. The action is invariant under Weyl transformation (2.3) with scale dimension $d_\phi = (2-n)/2$. By Weyl gauging we obtain

$$\begin{aligned}\mathcal{A}_n &= \int \mathrm{d}^n x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\left(\partial_\mu + d_\phi W_\mu\right)\phi\left(\partial_\nu + d_\phi W_\nu\right)\phi - \lambda\phi^{\frac{2n}{n-2}}\right)\\ &= \int \mathrm{d}^n x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \lambda\phi^{\frac{2n}{n-2}}\right.\\ &\qquad\left. + \frac{1}{2}d_\phi\left(-\nabla_\mu W^\mu + d_\phi W_\mu W^\mu\right)\phi^2\right).\end{aligned} \tag{3.11}$$

The last term is clearly the trace of $\Omega_{\mu\nu}[W]$ in (2.14). Since we know $S_{\mu\nu}$ in (2.18), and its Weyl variation (2.19), we can replace it by a multiple of R

$$\mathcal{A}_n = \int \mathrm{d}^n x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \lambda\phi^{\frac{2n}{n-2}} + \frac{n-2}{8(n-1)}R\phi^2\right). \tag{3.12}$$

Varying this action as in (1.78) we have the energy-momentum tensor

$$\begin{aligned}T_{\mu\nu} =& \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - \lambda\phi^{\frac{2n}{n-2}}\right)\\ &+ \frac{n-2}{4(n-1)}\Big(g_{\mu\nu}\nabla^2 - \nabla_\nu\nabla_\mu + G_{\mu\nu}\Big)\phi^2,\end{aligned} \tag{3.13}$$

where $G_{\mu\nu}$ is the Einstein tensor. In flat-space limit $T_{\mu\nu}$ becomes $\Theta_{\mu\nu}$ in (3.10). Moreover $T_{\mu\nu}$ is traceless on-shell. Since the equation of motion is

$$\nabla^2\phi + \lambda\frac{2n}{n-1}\phi^{\frac{n+2}{n-2}} - \frac{n-2}{4(n-1)}R\phi = 0. \tag{3.14}$$

The trace of the energy-momentum tensor

$$T^\mu_\mu = \frac{2-n}{2}\partial^\mu\phi\partial_\mu\phi - n\lambda\phi^{\frac{2n}{n-2}} + \frac{n-2}{4}\nabla^2\phi^2 + \frac{(n-2)(2-n)}{8(n-1)}R\phi^2 \tag{3.15}$$

vanishes.

The action $\mathcal{A}_n$, for $n > 2$, is Weyl and diffeomorphic invariant. In the rest of the thesis we will be investigating where does this property disappear for Liouville, $n = 2$, or if Liouville energy-momentum tensor can be improved to have it.

We begin with construction of Liouville action $\mathcal{A}_L$ from the action $\mathcal{A}_n$. Since ϕ and λ are arbitrary we rewrite the field ϕ

$$\phi = \frac{2n}{\beta(n-2)}e^{\frac{n-2}{2n}\beta\varphi} \tag{3.16}$$

and then we take a dimensional limit $n \to 2$.

Putting this into the action $\mathcal{A}_n$ the term with λ becomes

$$\lambda\phi^{\frac{2n}{n-2}} = \lambda\left(\frac{2n}{\beta(n-2)}\right)^{\frac{2n}{n-2}} e^{\beta\varphi} \xrightarrow{n\to 2} \frac{m^2}{\beta^2}e^{\beta\varphi}, \tag{3.17}$$

where $\xrightarrow{n\to 2}$ denotes the dimensional limit. With [9], in the last step we renormalized λ such that the dimensional dependent factor multiplied by λ became $\frac{m^2}{\beta^2}$. The kinetic term of the action $\mathcal{A}_n$ becomes

$$\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{1}{2}g^{\mu\nu}e^{\frac{n-2}{n}\beta\varphi}\partial_\mu\varphi\partial_\nu\varphi \xrightarrow{n\to 2} \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi. \tag{3.18}$$

The last term can be expanded in the limit $n \to 2$ as

$$\begin{aligned}\frac{n-2}{8(n-1)}R\phi^2 &= \frac{n^2}{2\beta^2(n-1)(n-2)}Re^{\frac{n-2}{n}\beta\varphi}\\ &= \frac{n^2}{2\beta^2(n-1)(n-2)}R + \frac{n}{2\beta(n-1)}R\varphi + \mathcal{O}(n-2)\end{aligned} \tag{3.19}$$

and the $\mathcal{O}(n-2)$ term goes to zero when we take the dimensional limit $n \to 2$. Therefore we have

$$\mathcal{A}_n \xrightarrow{n\to 2} \mathcal{A}_L + \lim_{n\to 2}\frac{2}{\beta^2(n-2)}\int \mathrm{d}^n x\,\sqrt{-g}R. \tag{3.20}$$

In $n = 2$ the integrand is zero[1], therefore we have limit of the type “0/0”. If we throw it away, we have the wanted Liouville action but we lose the Weyl invariance. We have to be more careful here.

[1] It is well known that $\int_\Sigma \mathrm{d}^2x\sqrt{-g}R$ is proportional to the Euler characteristic of Σ. Nonetheless, what matters here is that $\sqrt{-g}R \sim \partial_\mu R^\mu$, hence, as long bulk properties are concerned, we can take this as zero, as done in [9].

We now know that the loss of Weyl invariance is hidden in the extra term in (3.20). Therefore we try to find a way for proper evaluating the limit through Weyl gauging. In (3.11) we Weyl gauged the action $\mathcal{A}_n$ and in the next step we found the proper Ricci gauging, which was

$$\frac{1}{2(n-1)}R = \nabla_\mu W^\mu + \frac{n-2}{2} W_\mu W^\mu. \tag{3.21}$$

Now we can rewrite the second term in (3.20)

$$\begin{aligned}\frac{2}{\beta^2(n-2)} \int \mathrm{d}^n x \sqrt{-g} R =& \frac{4(n-1)}{\beta^2(n-2)} \int \mathrm{d}^n x\, \partial_\mu \left(\sqrt{-g} W^\mu\right) \\ &+ \frac{2(n-1)}{\beta^2} \int \mathrm{d}^n x \sqrt{-g} W_\mu W^\mu.\end{aligned} \tag{3.22}$$

Since the first term is a pure divergence in any dimension it does not contribute in the dimensional limit. Hence

$$\lim_{n\to 2} \frac{2}{\beta^2(n-2)} \int \mathrm{d}^n x \sqrt{-g} R = \frac{2}{\beta^2} \int \mathrm{d}^2 x \sqrt{-g} W_\mu W^\mu, \tag{3.23}$$

where according to (3.21) we have a condition for W_μ

$$R = 2\nabla_\mu W^\mu. \tag{3.24}$$

To make Liouville action Weyl invariant we have to add (3.23), because the Weyl transformation of such term is

$$\frac{2}{\beta^2} \int \mathrm{d}^2 x \sqrt{-g} W_\mu W^\mu \to \frac{2}{\beta^2} \int \mathrm{d}^2 x \sqrt{-g} W_\mu W^\mu + \frac{2}{\beta^2} \int \mathrm{d}^2 x \sqrt{-g} \left(R\omega + g^{\mu\nu}\omega_\mu \omega_\nu\right), \tag{3.25}$$

that exactly cancels the field independent term (3.6) which arises in Liouville action $\mathcal{A}_L$ under Weyl transformation.

In paper [4], the energy-momentum tensor improvements in two dimensions were discussed with emphasis on the trace anomaly and its relation to central charges of Virasoro algebra. Among other things the problem

$$\partial_\mu R^\mu = \sqrt{-g} R \tag{3.26}$$

was discussed as a partial solution to requirements on an improvement. The solution for R^μ was provided and since

$$\nabla_\mu W^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} W^\mu),$$

we see from (3.24) and (3.26)

$$W^\mu = \frac{1}{2\sqrt{-g}} R^\mu. \tag{3.27}$$

The solution W^μ to (3.24) in not unique, rather it has ambiguity and

$$W'^\mu = W^\mu + \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}} \partial_\nu r \tag{3.28}$$

solve the condition as well.

3.3 Useful relations

Before presenting the solution for W^μ, and deriving the energy-momentum tensor, some new parametrizations and tricks have to be added to our mathematical toolkit.

First we introduce new parametrization of the metric $g_{\mu\nu}$

$$\gamma_{\mu\nu} \equiv \frac{g_{\mu\nu}}{\sqrt{-g}} \qquad \gamma^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} \qquad \sqrt{-g} \equiv e^\sigma. \tag{3.29}$$

Such parametrization $\gamma_{\mu\nu}$ has simple relations between its components and components of its inverse

$$\gamma_{00} = -\gamma^{11} \qquad \gamma_{11} = -\gamma^{00} \qquad \gamma_{10} = \gamma_{01} = \gamma^{10} = \gamma^{01}. \tag{3.30}$$

Furthermore $\det \gamma_{\mu\nu} = -1$. Let us stress that $\gamma_{\mu\nu}$ and $\gamma^{\mu\nu}$ are Weyl invariant quantities.

We also introduce

$$\cosh \omega \equiv \gamma_{+-} \qquad e^\gamma \equiv \sqrt{\frac{\gamma_{++}}{\gamma_{--}}}, \tag{3.31}$$

where $+-$ denote light-cone components. More on light-cone coordinates is in appendix.

The other important object is Levi-Civita symbol $\epsilon^{\mu\nu}$ that can be expressed as

$$\epsilon^{\mu\nu} = \gamma^{1\mu}\gamma^{0\nu} - \gamma^{0\mu}\gamma^{1\nu}, \tag{3.32}$$

hence $\epsilon^{01} = -\epsilon^{10} = 1$.

This can be easily seen from product of two Levi-Civita symbols expressed in $g^{\mu\nu}$ or $\gamma^{\mu\nu}$

$$\epsilon^{\mu\nu}\epsilon^{\alpha\beta} = \gamma^{\mu\beta}\gamma^{\nu\alpha} - \gamma^{\mu\alpha}\gamma^{\nu\beta}, \tag{3.33}$$

$$\epsilon^{\mu\nu}\epsilon^{\alpha\beta} = g(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}). \tag{3.34}$$

For Levi-Civita symbols also hold a useful identity

$$V^{\lambda\mu\ldots\nu}\epsilon^{\sigma\rho} + V^{\rho\mu\ldots\nu}\epsilon^{\lambda\sigma} + V^{\sigma\mu\ldots\nu}\epsilon^{\rho\lambda} = 0 \tag{3.35}$$

where $V^{\lambda\mu\ldots\nu}$ is an arbitrary tensor with at least one index.

Since we want to compute energy-momentum tensor in a variational way

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{A}}{\delta g^{\mu\nu}},$$

the variations of some objects are provided.

The variation of the metric determinant is usually needed

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu}, \quad \delta\sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\mu\nu}\delta g_{\mu\nu}, \quad \delta\frac{1}{\sqrt{-g}} = -\frac{1}{2}\frac{1}{\sqrt{-g}} g^{\mu\nu}\delta g_{\mu\nu}. \tag{3.36}$$

The variation of the inverse metric $g^{\mu\nu}$

$$\delta g^{\mu\nu} = -g^{\rho\mu}g^{\nu\sigma}\delta g_{\rho\sigma}. \tag{3.37}$$

Since $\gamma_{\mu\nu}$ has a negative unit determinant, the variation is

$$0 = \delta(-1) = \delta \det \gamma_{\mu\nu} = 2\gamma_{01}\delta\gamma_{01} - \delta(\gamma_{00}\gamma_{11}). \tag{3.38}$$

The variation of parameters (3.31) are

$$\delta \cosh \omega = \frac{1}{2}(\delta\gamma_{00} - \delta\gamma_{11}) = \frac{1}{2\sqrt{-g}}\delta g_{\mu\nu}\left(\delta_0^\mu\delta_0^\nu - \delta_1^\mu\delta_1^\nu - \frac{1}{2}(g_{00} - g_{11})g^{\mu\nu}\right) \tag{3.39}$$

$$\begin{aligned}\delta\gamma =& \frac{\delta\gamma_{00}(\gamma_{00}\gamma_{11} + \gamma_{11}^2 - 2\gamma_{01}^2) + \delta\gamma_{11}(\gamma_{00}^2 + \gamma_{00}\gamma_{11} - 2\gamma_{01}^2)}{\left[(\gamma_{00} + \gamma_{11})^2 - 4\gamma_{01}^2\right]\gamma_{01}} \\ =& \frac{2\delta g_{\mu\nu}}{(g_{00} + g_{11})^2 - 4g_{01}^2}\left[\frac{1}{2}(\delta_0^\mu\delta_1^\nu + \delta_1^\mu\delta_0^\nu)(g_{00} + g_{11}) - g_{01}(\delta_0^\mu\delta_0^\nu + \delta_1^\nu\delta_1^\mu)\right].\end{aligned} \tag{3.40}$$

Defining

$$\delta\gamma = \Gamma^{\mu\nu}\delta g_{\mu\nu}, \tag{3.41}$$

we can write

$$\begin{aligned}\Gamma^{\mu\nu} =& \frac{1}{4g_{++}g_{--}}\begin{pmatrix} g_{--} - g_{++} & g_{--} + g_{++} \\ g_{--} + g_{++} & g_{--} - g_{++}\end{pmatrix} \\ =& \frac{1}{2\sqrt{g_{--}g_{++}}}\begin{pmatrix} -\sinh\gamma & \cosh\gamma \\ \cosh\gamma & -\sinh\gamma\end{pmatrix}.\end{aligned} \tag{3.42}$$

In paper [4] beside the solution for (3.26) a variation identity is provided

$$\begin{aligned}\delta\left[\epsilon^{\mu\nu}(\cosh\sigma - 1)\partial_\nu\gamma\right] - & \partial_\nu\left[\epsilon^{\mu\nu}(\cosh\sigma - 1)\delta\gamma\right] = \\ & = -\frac{1}{2}\gamma^{\mu\nu}\left(\partial_\alpha\gamma_{\nu\beta} + \partial_\beta\gamma_{\nu\alpha} - \partial_\nu\gamma_{\alpha\beta}\right)\delta\gamma^{\alpha\beta}.\end{aligned} \tag{3.43}$$

Since it is non-trivial identity, we would like to prove it. The left-hand side can be written as

$$\begin{aligned}LHS =& \delta\left[\epsilon^{\mu\nu}(\cosh\sigma - 1)\partial_\nu\gamma\right] - \partial_\nu\left[\epsilon^{\mu\nu}(\cosh\sigma - 1)\delta\gamma\right] \\ =& \epsilon^{\mu\nu}(\delta\cosh\sigma\partial_\nu\gamma - \partial_\nu\cosh\sigma\delta\gamma) \\ =& \frac{\epsilon^{\mu\nu}}{2\left[(\gamma_{00} + \gamma_{11})^2 - 4\gamma_{01}^2\right]\gamma_{01}}\Big[(\delta\gamma_{00} - \delta\gamma_{11})(\partial_\nu\gamma_{00}(\gamma_{00}\gamma_{11} + \gamma_{11}^2 - 2\gamma_{01}^2) \\ & + \partial_\nu\gamma_{11}(\gamma_{00}^2 + \gamma_{00}\gamma_{11} - 2\gamma_{01}^2)) \\ & - (\delta\gamma_{00}(\gamma_{00}\gamma_{11} + \gamma_{11}^2 - 2\gamma_{01}^2) + \delta\gamma_{11}(\gamma_{00}^2 + \gamma_{00}\gamma_{11} - 2\gamma_{01}^2))(\partial_\nu\gamma_{00} - \partial_\nu\gamma_{11})\Big] \\ =& \frac{\epsilon^{\mu\nu}}{2\gamma_{01}}(\delta\gamma_{00}\partial_\nu\gamma_{11} - \delta\gamma_{11}\partial_\nu\gamma_{00}).\end{aligned}$$

The right-hand side may be written as

$$\begin{aligned}RHS =& -\frac{1}{2}\gamma^{\mu\nu}\left(\partial_\alpha\gamma_{\nu\beta} + \partial_\beta\gamma_{\nu\alpha} - \partial_\nu\gamma_{\alpha\beta}\right)\delta\gamma^{\alpha\beta} \\ =& \frac{1}{2\gamma_{01}}\Big[(\delta\gamma_{11}\partial_0\gamma_{00} - \delta\gamma_{00}\partial_0\gamma_{11})(\gamma_{11}\gamma^{\mu1} + \gamma_{01}\gamma^{\mu0}) \\ & + (\delta\gamma_{00}\partial_1\gamma_{11} - \delta\gamma_{11}\partial_1\gamma_{00})(\gamma_{00}\gamma^{\mu0} + \gamma_{01}\gamma^{\mu1})\Big].\end{aligned}$$

Using now that $\epsilon^{01} = 1$ we see LHS=RHS and the identity is proved.

3.4 What is our solution for W^μ?

In this section we want to verify the solution provided in paper [4] and then construct the solution to our problem (3.24). The equation to solve is:

$$\partial_\mu R^\mu = \sqrt{-g}R. \tag{3.44}$$

The solution provided in paper [4] is

$$R^\mu = -\gamma^{\mu\nu}\partial_\nu\sigma - \partial_\nu\gamma^{\mu\nu} + \epsilon^{\mu\nu}(\cosh\omega - 1)\partial_\nu\gamma, \tag{3.45}$$

where $\omega, \gamma, \sigma, \gamma^{\mu\nu}$ were introduced earlier.

This can be presented in another fashion, as in [9]

$$R^\mu = \frac{\epsilon^{\mu\nu}\epsilon^{\alpha\beta}}{\sqrt{-g}}\partial_\alpha g_{\beta\nu} + \epsilon^{\mu\nu}(\cosh\omega - 1)\partial_\nu\gamma. \tag{3.46}$$

It is easy to show that the expressions are equivalent. We see that last term in both expressions is the same, therefore we are left to prove

$$\gamma^{\mu\nu}\partial_\nu\sigma + \partial_\nu\gamma^{\mu\nu} = -\frac{\epsilon^{\mu\nu}\epsilon^{\alpha\beta}}{\sqrt{-g}}\partial_\alpha g_{\beta\nu}. \tag{3.47}$$

Recalling (3.34), one has

$$\begin{aligned}
LHS =&\gamma^{\mu\nu}\partial_\nu\sigma + \partial_\nu\gamma^{\mu\nu}\\
=&2g^{\mu\nu}\partial_\nu\sqrt{-g} + \sqrt{-g}\partial_\nu g^{\mu\nu}\\
=&\sqrt{-g}\partial_\nu g_{\alpha\beta}(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})\\
=&-\frac{1}{\sqrt{-g}}\partial_\alpha g_{\beta\nu}\epsilon^{\mu\nu}\epsilon^{\alpha\beta} = RHS.
\end{aligned}$$

Now we verify that R^μ solves the equation (3.44). First we rewrite the right hand side by recalling the definition of R

$$\begin{aligned}
\sqrt{-g}R =&\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}R_{\mu\nu\alpha\beta}\\
=&\frac{\sqrt{-g}}{2}g^{\mu\alpha}g^{\nu\beta}(\partial_\nu\partial_\alpha g_{\mu\beta} + \partial_\mu\partial_\beta g_{\alpha\nu} - \partial_\nu\beta g_{\mu\alpha} - \partial_\mu\partial_\alpha g_{\nu\beta})\\
&+ \sqrt{-g}g^{\mu\alpha}g^{\nu\beta}g_{\sigma\rho}(\Gamma^\rho_{\mu\beta}\Gamma^\sigma_{\nu\alpha} - \Gamma^\rho_{\mu\alpha}\Gamma^\sigma_{\nu\beta})\\
=&\sqrt{-g}\partial_\mu\partial_\alpha g_{\beta\nu}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})\\
&+ \frac{\sqrt{-g}}{4}g^{\mu\alpha}g^{\nu\beta}g^{\sigma\rho}(4\partial_\mu g_{\alpha\sigma}\partial_\rho g_{\beta\nu} - 4\partial_\mu g_{\alpha\sigma}\partial_\nu g_{\beta\rho}\\
&+ 3\partial_\mu g_{\sigma\beta}\partial_\alpha g_{\nu\rho} - 2\partial_\sigma g_{\mu\beta}\partial_\nu g_{\alpha\rho} - \partial_\sigma g_{\mu\alpha}\partial_\rho g_{\beta\nu}).
\end{aligned} \tag{3.48}$$

Our next step is to put the provided form of R^μ, (3.46), inside the left hand side of (3.44), i.e. compute the divergence of R^μ

$$\partial_\mu R^\mu = \frac{\epsilon^{\mu\nu}\epsilon^{\alpha\beta}}{\sqrt{-g}}\partial_\mu\partial_\alpha g_{\beta\nu} - \frac{\epsilon^{\mu\nu}\epsilon^{\alpha\beta}}{2\sqrt{-g}}\partial_\alpha g_{\beta\nu}g^{\sigma\rho}\partial_\mu g_{\sigma\rho} + \partial_\mu[\epsilon^{\mu\nu}(\cosh\omega - 1)\partial_\nu\gamma]. \tag{3.49}$$

The first term in (3.49) is the only one which contains second derivatives, because the last term contains $\epsilon^{\mu\nu}$ and therefore the second derivatives in the last term vanish.

By comparing the terms with the second derivatives on both sides of equation (3.44) and using the (3.34) we see they are identical.

We now rewrite the last term of (3.49) by using variational identity (3.43)

$$\begin{aligned}\partial_\mu[\epsilon^{\mu\nu}(\cosh\omega-1)\partial_\nu\gamma] =& -\frac{1}{4}\gamma^{\mu\nu}(\partial_\alpha\gamma_{\nu\beta}+\partial_\beta\gamma_{\nu\alpha}-\partial_\nu\gamma_{\alpha\beta})\partial_\mu\gamma^{\alpha\beta}\\ =&\frac{\sqrt{-g}}{4}g^{\mu\alpha}g^{\nu\beta}g^{\sigma\rho}\left(2\partial_\alpha g_{\sigma\beta}\partial_\rho g_{\mu\nu}-2\partial_\alpha g_{\nu\mu}\partial_\beta g_{\sigma\rho}\right.\\ &\left.+\partial_\nu g_{\alpha\mu}\partial_\beta g_{\sigma\rho}-\partial_\sigma g_{\alpha\beta}\partial_\rho g_{\mu\nu}\right).\end{aligned} \tag{3.50}$$

The middle term in (3.49) is equal to

$$-\frac{\epsilon^{\mu\nu}\epsilon^{\alpha\beta}}{2\sqrt{-g}}\partial_\alpha g_{\beta\nu}g^{\rho\sigma}\partial_\mu g_{\rho\sigma}=\frac{\sqrt{-g}}{2}g^{\mu\alpha}g^{\nu\beta}g^{\sigma\rho}(\partial_\alpha g_{\beta\nu}\partial_\mu g_{\rho\sigma}-\partial_\beta g_{\alpha\nu}\partial_\mu g_{\rho\sigma}). \tag{3.51}$$

Putting all together we have

$$\begin{aligned}RHS-LHS=&\frac{\sqrt{-g}}{4}g^{\mu\alpha}g^{\nu\beta}g^{\sigma\rho}(4\partial_\mu g_{\alpha\sigma}\partial_\rho g_{\beta\nu}-4\partial_\mu g_{\alpha\sigma}\partial_\nu g_{\beta\rho}\\ &+3\partial_\mu g_{\sigma\beta}\partial_\alpha g_{\nu\rho}-2\partial_\sigma g_{\mu\beta}\partial_\nu g_{\alpha\rho}-\partial_\sigma g_{\mu\alpha}\partial_\rho g_{\beta\nu}\\ &-2\partial_\alpha g_{\beta\nu}\partial_\mu g_{\rho\sigma}+2\partial_\beta g_{\alpha\nu}\partial_\mu g_{\rho\sigma}-2\partial_\alpha g_{\sigma\beta}\partial_\rho g_{\mu\nu}\\ &+2\partial_\alpha g_{\nu\mu}\partial_\beta g_{\sigma\rho}-\partial_\nu g_{\alpha\mu}\partial_\beta g_{\sigma\rho}+\partial_\sigma g_{\alpha\beta}\partial_\rho g_{\mu\nu})\\ =&\sqrt{-g}(-2\partial_\mu g^{\mu\rho}\partial_\rho g_{\beta\nu}g^{\beta\nu}+\partial_\mu g^{\mu\rho}\partial_\nu g_{\beta\rho}g^{\beta\nu}-\partial_\mu g^{\rho\nu}\partial_\alpha g_{\nu\rho}g^{\alpha\mu}\\ &+\partial_\sigma g^{\alpha\nu}\partial_\nu g_{\alpha\rho}g^{\sigma\rho}-\partial_\sigma g^{\mu\alpha}\partial_\rho g_{\nu\beta}g^{\rho\sigma}g^{\mu\alpha}g^{\beta\nu})\\ =&\sqrt{-g}(\partial_\nu g_{\alpha\beta}\partial_\mu[g^{\mu\alpha}g^{\nu\beta}-g^{\mu\nu}g^{\alpha\beta})\\ &+\partial_\mu g^{\mu\nu}\partial_\nu g_{\alpha\beta}g^{\alpha\beta}-\partial_\sigma g^{\mu\nu}\partial_\rho g_{\alpha\beta}g^{\rho\sigma}g^{\mu\nu}g^{\beta\alpha}]=0.\end{aligned} \tag{3.52}$$

In the last step we used (3.34). The object R^μ is indeed the solution and we can use it to construct W^μ.

By combining (3.27) and (3.46) we have

$$W^\mu=\frac{\epsilon^{\mu\nu}}{2\sqrt{-g}}\left(\frac{\epsilon^{\alpha\beta}}{\sqrt{-g}}\partial_\alpha g_{\beta\nu}+(\cosh\omega-1)\partial_\nu\gamma\right). \tag{3.53}$$

One of the first conditions imposed on W^μ in our discussion was its Weyl transformation

$$W_\mu\to W_\mu-\partial_\mu\omega.$$

Let us verify if it holds.

From (3.45) we see that Weyl transformation of R^μ depends only on σ since the rest is Weyl invariant

$$R^\mu\to R^\mu-2\sqrt{-g}g^{\mu\nu}\partial_\nu\omega. \tag{3.54}$$

From this we see that Weyl transformation of W_μ is of desired form

$$W_\mu=\frac{\gamma_{\mu\nu}}{2}R^\nu\to\frac{\gamma_{\mu\nu}}{2}(R^\nu-2\gamma^{\nu\rho}\partial_\rho\omega)=W_\mu-\partial_\mu\omega. \tag{3.55}$$

3.5 Energy-momentum Tensor improvement

We now want to compute the energy-momentum tensor improvement from our new term in the action

$$\Delta I = \frac{2}{\beta^2}\int \mathrm{d}^2x\,\sqrt{-g}g^{\mu\nu}W_\mu W_\nu = \frac{1}{2\beta^2}\int \mathrm{d}^2x\,\gamma_{\mu\nu}R^\mu R^\nu. \tag{3.56}$$

The natural way to compute the energy-momentum tensor of this action is by varying with respect to $\gamma^{\mu\nu}$ since all objects inside the action, and variation identity (3.43), are expressed in this notation

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\Delta I}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{-g}}\frac{\delta\Delta I}{\delta\gamma^{\alpha\beta}}\frac{\delta\gamma^{\alpha\beta}}{\delta g^{\mu\nu}}, \tag{3.57}$$

$$\frac{\delta\gamma^{\alpha\beta}}{\delta g^{\mu\nu}} = \sqrt{-g}\frac{\delta g^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}\delta g^{\rho\sigma}g_{\rho\sigma}}{\delta g^{\mu\nu}}, \tag{3.58}$$

$$T_{\mu\nu} = 2\frac{\delta\Delta I}{\delta\gamma^{\mu\nu}} - \gamma_{\mu\nu}\gamma^{\alpha\beta}\frac{\delta\Delta I}{\delta\gamma^{\alpha\beta}}, \tag{3.59}$$

$$\delta\Delta I = \frac{1}{2\beta^2}\int \mathrm{d}^2x\,\delta\gamma_{\mu\nu}R^\mu R^\nu + \frac{1}{\beta^2}\int \mathrm{d}^2x\,\gamma_{\mu\nu}R^\nu\delta R^\mu, \tag{3.60}$$

where

$$\delta R^\mu = -\delta\gamma^{\mu\nu}\partial_\nu\sigma - \gamma^{\mu\nu}\partial_\nu\delta\sigma - \partial_\nu\delta\gamma^{\mu\nu} + \partial_\nu[\epsilon^{\mu\nu}(\cosh\omega - 1)\delta\gamma] - \bar{\Gamma}^\mu_{\alpha\beta}\delta\gamma^{\alpha\beta}. \tag{3.61}$$

The last two terms emerged by using (3.43) and we used the notation

$$\bar{\Gamma}^\mu_{\alpha\beta} = \frac{1}{2}\gamma^{\mu\nu}\left(\partial_\alpha\gamma_{\nu\beta} + \partial_\beta\gamma_{\nu\alpha} - \partial_\nu\gamma_{\alpha\beta}\right). \tag{3.62}$$

The solution R^μ for equation (3.44) is not unique. In fact it has a freedom $R^\mu \to R^\mu + \epsilon^{\mu\nu}\partial_\nu r$. That is why we add δr in δR

$$\delta R^\mu \to \delta R^\mu + \epsilon^{\mu\nu}\partial_\nu\delta r. \tag{3.63}$$

It is understood that R^μ in the following part contains this superpotential-like term.

Since W^μ has the wanted Weyl transformation form, we demand that this new term does not change it. Therefore we impose a condition to r such that the quantity

$$\frac{g_{\mu\nu}}{2\sqrt{-g}}\epsilon^{\nu\lambda}\partial_\lambda r$$

is Weyl invariant.

Before computation of $T_{\mu\nu}$ let us divide $\delta\Delta I$ into four terms.

$$\delta\Delta I = \delta\Delta I^1 + \delta\Delta I^2 + \delta\Delta I^3 + \delta\Delta I^4, \tag{3.64}$$

where

$$\begin{aligned}
\delta\Delta I^1 =& \frac{1}{2\beta^2}\int \mathrm{d}^2x\,\delta\gamma_{\mu\nu}R^\nu R^\mu\\
\delta\Delta I^2 =& \frac{1}{\beta^2}\int \mathrm{d}^2x\,\gamma_{\mu\nu}R^\nu(-\delta\gamma^{\mu\lambda}\partial_\lambda\sigma-\partial_\lambda\delta\gamma^{\mu\lambda}-\bar{\Gamma}^\mu_{\alpha\beta}\delta\gamma^{\alpha\beta})\\
=& \frac{1}{2\beta^2}\int \mathrm{d}^2x\,\delta\gamma^{\alpha\beta}\left[g_{\beta\lambda}\nabla_\alpha\left(\frac{R^\lambda}{\sqrt{-g}}\right)+g_{\alpha\lambda}\nabla_\beta\left(\frac{R^\lambda}{\sqrt{-g}}\right)\right]\\
\delta\Delta I^3 =& -\frac{1}{\beta^2}\int \mathrm{d}^2x\,R^\mu\partial_\mu\delta\sigma=-\frac{1}{2\beta^2}\int \mathrm{d}^2x\,\sqrt{-g}Rg_{\alpha\beta}\delta g^{\alpha\beta}\\
\delta\Delta I^4 =& \frac{1}{\beta^2}\int \mathrm{d}^2x\,R^\mu\gamma_{\mu\nu}\{\partial_\lambda[\epsilon^{\nu\lambda}(\cosh\omega-1)\delta\gamma]+\epsilon^{\nu\lambda}\partial_\lambda\delta r\}\\
=& -\frac{1}{\beta^2}\int \mathrm{d}^2x\,\partial_\lambda(R^\mu\gamma_{\mu\nu})\epsilon^{\nu\lambda}\delta g_{\alpha\beta}[(\cosh\omega-1)\Gamma^{\alpha\beta}+r^{\alpha\beta}].
\end{aligned}$$

While deriving the second form of $\delta\Delta I^2$ we used

$$\partial_\mu T^{\alpha\dots}_{\beta\dots}+\partial_\mu\sigma T^{\alpha\dots}_{\beta\dots}=\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}T^{\alpha\dots}_{\beta\dots})$$

$$g_{\alpha\lambda}\nabla_\beta V^\lambda+g_{\beta\lambda}\nabla_\alpha V^\lambda=g_{\alpha\lambda}\partial_\beta V^\lambda+g_{\beta\lambda}\partial_\alpha V^\lambda+V^\lambda\partial_\lambda g_{\alpha\beta}$$
$$\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta}=0.$$

While deriving $\delta\Delta I^3$ we used the fact that R^μ solves

$$\sqrt{-g}R=\partial_\alpha R^\alpha.$$

In rewriting $\delta\Delta I^4$ we defined

$$\begin{aligned}
\delta\gamma =& \Gamma^{\mu\nu}\delta g_{\mu\nu}\\
\delta r =& r^{\mu\nu}\delta g_{\mu\nu}.
\end{aligned}$$

From previous computation we know

$$\Gamma^{\mu\nu}=\frac{1}{2\sqrt{g_{--}g_{++}}}\begin{pmatrix}-\sinh\gamma & \cosh\gamma\\ \cosh\gamma & -\sinh\gamma\end{pmatrix}. \tag{3.65}$$

Doing this preparations, we divide the computation of the energy-momentum tensor into four parts

$$T^i_{\mu\nu}=\frac{1}{\sqrt{-g}}\frac{\delta\Delta I^i}{\delta g^{\mu\nu}}$$

with $i=1,2,3,4$.

Results are then

$$T^1_{\mu\nu}=\frac{1}{\beta^2}\left(\frac{1}{2}\gamma_{\mu\nu}\gamma_{\alpha\beta}R^\alpha R^\beta-\gamma_{\mu\alpha}\gamma_{\nu\beta}R^\alpha R^\beta\right) \tag{3.66}$$

$$T^2_{\mu\nu}=\frac{1}{\beta^2}\left(g_{\mu\lambda}\nabla_\nu\left(\frac{R^\lambda}{\sqrt{-g}}\right)+g_{\nu\lambda}\nabla_\mu\left(\frac{R^\lambda}{\sqrt{-g}}\right)\right)-\frac{1}{\beta^2}Rg_{\mu\nu} \tag{3.67}$$

$$T^3_{\mu\nu}=-\frac{1}{\beta^2}Rg_{\mu\nu} \tag{3.68}$$

$$T^4_{\mu\nu} = \frac{2}{\sqrt{-g}\beta^2}\partial_\beta\left(\frac{R^\lambda g_{\alpha\lambda}}{\sqrt{-g}}\right)\epsilon^{\alpha\beta}[(\cosh\omega - 1)\Gamma_{\mu\nu} + r_{\mu\nu}]. \tag{3.69}$$

Adding these computed parts of the energy-momentum tensor together we have

$$\begin{aligned}
\beta^2 T^{\mu\nu} =& \frac{1}{g}\left(R^\mu R^\nu - \frac{1}{2}g^{\mu\nu}R\cdot R\right) - 2Rg^{\mu\nu} \\
&+ g^{\mu\alpha}\nabla_\alpha\left(\frac{R^\nu}{\sqrt{-g}}\right) + g^{\nu\alpha}\nabla_\alpha\left(\frac{R^\mu}{\sqrt{-g}}\right) \\
&+ \frac{2}{\sqrt{-g}}\partial_\beta\left(\frac{R^\lambda g_{\alpha\lambda}}{\sqrt{-g}}\right)\epsilon^{\alpha\beta}[(\cosh\omega - 1)\Gamma^{\mu\nu} + r^{\mu\nu}] \\
=& 2g^{\mu\nu}W^\alpha W^\beta g_{\alpha\beta} - 4W^\mu W^\nu - 2Rg^{\mu\nu} \\
&+ 2g^{\mu\alpha}\nabla_\alpha W^\nu + 2g^{\nu\alpha}\nabla_\alpha W^\mu \\
&+ \frac{4}{\sqrt{-g}}\partial_\beta\left(W^\lambda g_{\alpha\lambda}\right)\epsilon^{\alpha\beta}[(\cosh\omega - 1)\Gamma^{\mu\nu} + r^{\mu\nu}].
\end{aligned} \tag{3.70}$$

Since this improvement term should provide the cancellation of the nonzero value of the energy-momentum tensor trace we want to compute the trace of this improvement term

$$\beta^2 T^\mu_\mu = -2R + \frac{4}{\sqrt{-g}}\partial_\beta\left(W^\lambda g_{\alpha\lambda}\right)\epsilon^{\alpha\beta}[(\cosh\omega - 1)\Gamma^{\mu\nu} + r^{\mu\nu}]g_{\mu\nu}. \tag{3.71}$$

Recalling $\Gamma^{\mu\nu}$

$$\Gamma^{\mu\nu} = \frac{2}{(g_{00} + g_{11})^2 - 4g_{01}^2}\left[\frac{1}{2}(\delta^\mu_0\delta^\nu_1 + \delta^\mu_1\delta^\nu_0)(g_{00} + g_{11}) - g_{01}(\delta^\mu_0\delta^\nu_0 + \delta^\nu_1\delta^\mu_1)\right],$$

we see that

$$g_{\mu\nu}\Gamma^{\mu\nu} = 0. \tag{3.72}$$

Therefore

$$T^\mu_\mu = -\frac{2}{\beta^2}R + \frac{4}{\sqrt{-g}}\partial_\beta\left(W^\lambda g_{\alpha\lambda}\right)\epsilon^{\alpha\beta}r^{\mu\nu}g_{\mu\nu}. \tag{3.73}$$

We see then another condition on r

$$g_{\mu\nu}r^{\mu\nu} = 0. \tag{3.74}$$

With this the improvement of the energy-momentum tensor cancels the anomalous trace of (3.2)

$$T^\mu_\mu = \frac{2}{\beta^2}R.$$

This proves the Weyl invariance of the improved Liouville action.

3.6 Transformation of W^μ

Looking for the improved Lagrangian for Liouville theory, and therefore the traceless energy-momentum tensor, we encountered the improvement term for Lagrangian

$$\frac{2}{\beta^2}\int \mathrm{d}^2x\,\sqrt{-g}g_{\mu\nu}W^\nu W^\mu.$$

Since the action is a scalar we want to find transformation properties of W^μ

$$W^\mu = \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}} \left(\frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} \partial_\alpha g_{\beta\nu} + (\cosh\omega - 1)\partial_\nu \gamma + \partial_\nu r \right).$$

From transformation of metric tensor under general coordinate transformation

$$g'_{\mu\nu} = \frac{\partial x}{\partial x'} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta},$$

we see that $\sqrt{-g}$ transforms as scalar density of rank 1

$$\sqrt{-g'} = \left| \det \left(\frac{\partial x}{\partial x'} \right) \right| \sqrt{-g}.$$

From the formula for Levi-Civita symbols (3.34) we can see that single Levi-Civita symbol $\epsilon^{\mu\nu}$ transforms as a tensor density of rank 1. Therefore $\epsilon^{\alpha\beta}/\sqrt{-g}$ transforms as a regular tensor.

Transformation of $\partial_\alpha g_{\mu\nu}$ is

$$\partial'_\alpha g'_{\mu\nu} = \partial'_\alpha \left(\frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma} \right) = \frac{\partial x^\lambda}{\partial x'^\alpha} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \partial_\lambda g_{\rho\sigma} + g_{\rho\sigma} \frac{\partial}{\partial x'^\alpha} \left(\frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \right). \tag{3.75}$$

Transformation of $\gamma_{\mu\nu}$ and its light cone elements is clear from its definition and it transforms as a tensor density of rank -1

$$\gamma'_{\mu\nu} = \left| \det \left(\frac{\partial x'}{\partial x} \right) \right| \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \gamma_{\alpha\beta}. \tag{3.76}$$

We will assume only infinitesimal diffeomorphism transformations i.e.

$$\delta x^\mu = -f^\mu(x),$$

for infinitesimal transformations it holds

$$\frac{\partial x^\rho}{\partial x'^\mu} = \delta^\rho_\mu + \partial_\mu f^\rho. \tag{3.77}$$

So let us compute the quantity

$$\Delta W^\mu \equiv W'\mu(x') - \frac{\partial x'^\mu}{\partial x^\nu} W^\nu(x). \tag{3.78}$$

We are interested in the light-cone components, or better, since we have light-cone terms in W^μ we expect them to be easier to calculate. Therefore, the following calculations are done using these coordinates.

The Jacobian of the coordinate transformation at first order, expressed in light-cone coordinates is

$$\begin{aligned} \left| \det \left(\frac{\partial x'}{\partial x} \right) \right| = \begin{pmatrix} \frac{\partial x'^+}{\partial x^+} & \frac{\partial x'^+}{\partial x^-} \\ \frac{\partial x'^-}{\partial x^+} & \frac{\partial x'^-}{\partial x^-} \end{pmatrix} = \begin{pmatrix} 1 - \partial_+ f^+ & -\partial_- f^+ \\ -\partial_+ f^- & 1 - \partial_- f^- \end{pmatrix} \\ = 1 - \partial_+ f^+ - \partial_- f^-. \end{aligned} \tag{3.79}$$

Since we restricted ourselves only to infinitesimal transformations we get

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu}(1 - \partial_+ f^+ - \partial_- f^-) + \partial_\mu f^\alpha \gamma_{\alpha\nu} + \partial_\nu f^\alpha \gamma_{\mu\alpha}. \tag{3.80}$$

From this we can write

$$\begin{aligned}\gamma'_{++} =& \gamma_{++}(1 + \partial_+ f^+ - \partial_- f^-) + 2\partial_+ f^- \gamma_{+-} \\ \gamma'_{+-} =& \gamma_{+-} + \partial_+ f^- \gamma_{--} + \partial_- f^+ \gamma_{++} \\ \gamma'_{--} =& \gamma_{--}(1 - \partial_+ f^+ + \partial_- f^-) + 2\partial_- f^+ \gamma_{+-}.\end{aligned} \tag{3.81}$$

So far we have

$$\Delta \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}} \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} \partial_\alpha g_{\beta\nu} = \frac{\epsilon^{\mu\nu}\epsilon^{\alpha\beta}}{2\sqrt{-g}^2} g_{\beta\tau} \partial_\alpha \partial_\nu f^\tau \tag{3.82}$$

and

$$\Delta \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}} (\cosh\omega - 1)\partial_\nu \gamma = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\epsilon^{\alpha\beta}}{2\sqrt{-g}} [(\cosh\omega' - 1)\partial_\beta \gamma' - (\cosh\omega - 1)\partial_\beta \gamma]. \tag{3.83}$$

The only unknown is the transformation of γ'

$$\begin{aligned}\gamma' =& \frac{1}{2} \ln \frac{\gamma'_{++}}{\gamma'_{--}} \\ =& \frac{1}{2} \ln \left(\frac{\gamma_{++}}{\gamma_{--}} \frac{1 + \partial_+ f^+ - \partial_- f^- + 2\partial_+ f^- \gamma_{+-}/\gamma_{++}}{1 - \partial_+ f^+ + \partial_- f^- + 2\partial_- f^+ \gamma_{+-}/\gamma_{--}} \right) \\ =& \gamma + \partial_+ f^+ - \partial_- f^- + \partial_+ f^- \frac{\gamma_{+-}}{\gamma_{++}} - \partial_- f^+ \frac{\gamma_{+-}}{\gamma_{--}},\end{aligned} \tag{3.84}$$

where we used Taylor expansion of logarithm and assumed γ_{+-}/γ_{--} is finite (this can be problematic when thinking of flat space limit, but it is intrinsic problem of γ itself).

From this we have

$$\begin{aligned}\Delta \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}} (\cosh\omega - 1)\partial_\nu \gamma =& \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}} [\partial_+ f^- \gamma_{--} \partial_\nu \gamma + \partial_- f^+ \gamma_{++} \partial_\nu \gamma \\ &+ (\cosh\omega - 1)\partial_\nu (\partial_+ f^+ - \partial_- f^-) \\ &+ (\cosh\omega - 1)\partial_\nu (\partial_+ f^- \frac{\gamma_{+-}}{\gamma_{++}} - \partial_- f^+ \frac{\gamma_{+-}}{\gamma_{--}})].\end{aligned} \tag{3.85}$$

The Levi-Civita symbol in light-cone coordinates are discussed in the appendix and it holds $\epsilon^{-+} = -\epsilon^{+-} = 1$ and $\epsilon^{--} = \epsilon^{++} = 0$. This gives us

$$\begin{aligned}\Delta \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}} \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} \partial_\alpha g_{\beta\nu} =& \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}} (\gamma_{++} \partial_\nu \partial_- f^+ + \gamma_{+-} \partial_\nu \partial_- f^- \\ &- \gamma_{+-} \partial_\nu \partial_+ f^+ - \gamma_{--} \partial_\nu \partial_+ f^-).\end{aligned} \tag{3.86}$$

If we forget about the $\partial_\nu r$ part of W^μ for a moment, we have

$$\begin{aligned}
\Delta W^\mu =& \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}}\Bigg[\gamma_{++}\partial_\nu\partial_- f^+ + \gamma_{+-}\partial_\nu\partial_- f^- \\
&- \gamma_{+-}\partial_\nu\partial_+ f^+ - \gamma_{--}\partial_\nu\partial_+ f^- \\
&+ \partial_+ f^-\gamma_{--}\partial_\nu\gamma + \partial_- f^+\gamma_{++}\partial_\nu\gamma \\
&+ (\gamma_{+-}-1)\partial_\nu(\partial_+ f^+ - \partial_- f^-) \\
&+ (\gamma_{+-}-1)\partial_\nu\left(\partial_+ f^-\frac{\gamma_{+-}}{\gamma_{++}} - \partial_- f^+\frac{\gamma_{+-}}{\gamma_{--}}\right)\Bigg] \\
=& \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}}\Bigg[\gamma_{++}\partial_\nu\partial_- f^+ + \partial_\nu\partial_- f^- \\
&- \partial_\nu\partial_+ f^+ - \gamma_{--}\partial_\nu\partial_+ f^- \\
&+ \partial_+ f^-\gamma_{--}\partial_\nu\gamma + \partial_- f^+\gamma_{++}\partial_\nu\gamma \\
&+ (\gamma_{+-}-1)\partial_\nu\left(\partial_+ f^-\frac{\gamma_{+-}}{\gamma_{++}} - \partial_- f^+\frac{\gamma_{+-}}{\gamma_{--}}\right)\Bigg].
\end{aligned} \tag{3.87}$$

Recalling now previously derived formulas

$$\begin{aligned}
\partial_\nu\gamma =& \frac{1}{2}\left(\frac{1}{\gamma_{++}}\partial_\nu\gamma_{++} - \frac{1}{\gamma_{--}}\partial_\nu\gamma_{--}\right) \\
\det\gamma_{\mu\nu} =& -1 = \gamma_{++}\gamma_{--} - \gamma_{+-}^2 \\
\partial\gamma_{+-}\gamma_{+-} =& \frac{1}{2}(\partial_\nu\gamma_{++}\gamma_{--} + \partial_\nu\gamma_{--}\gamma_{++}) \\
(\gamma_{+-}-1)\gamma_{+-} =& \gamma_{++}\gamma_{--} - \gamma_{+-} + 1,
\end{aligned}$$

we can simplify the result

$$\begin{aligned}
\Delta W^\mu =& \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}}\partial_\nu[\gamma_{++}\partial_- f^+ - \gamma_{--}\partial_+ f^- + \partial_- f^- - \partial_+ f^+ \\
&+ (\gamma_{+-}-1)(\partial_+ f^-\frac{\gamma_{+-}}{\gamma_{++}} - \partial_- f^+\frac{\gamma_{+-}}{\gamma_{--}})] \\
=& \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}}\partial_\nu[\partial_- f^- - \partial_+ f^+ - \partial_+ f^-\frac{\gamma_{+-}-1}{\gamma_{++}} + \partial_- f^+\frac{\gamma_{+-}-1}{\gamma_{--}}].
\end{aligned} \tag{3.88}$$

This can be written in a form provided in paper [4]

$$\Delta W^\mu = \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}}\partial_\nu\left[\left(\partial_- - e^{-\gamma}\tanh\frac{\omega}{2}\partial_+\right)f^- - \left(\partial_+ - e^{\gamma}\tanh\frac{\omega}{2}\partial_-\right)f^+\right]. \tag{3.89}$$

If f is generated by Lorentz transformation we get

$$\begin{aligned}
\Delta W^\mu =& \frac{\epsilon^{\mu\nu}}{2\sqrt{-g}}[\partial_\nu\left(e^{\gamma}\tanh\frac{\omega}{2}\right)\partial_- f^+ - \partial_\nu\left(e^{-\gamma}\tanh\frac{\omega}{2}\right)\partial_+ f^-] \\
=& \frac{\epsilon^{\mu\nu}}{\sqrt{-g}}\partial_\nu[\cosh\gamma\tanh\frac{\omega}{2}]\Lambda^+{}_-,
\end{aligned} \tag{3.90}$$

where $\Lambda^+{}_-$ is the Lorentz transformation matrix. The term in parenthesis is not zero in general. ΔW^μ is therefore not zero in general and, the improvement term in Lagrangian does not transform as a scalar. What makes its covariant divergence a scalar is the presence of Levi-Civita symbol.

This proves the remarkable result posited in [9], i.e., that W^μ *is not a contravariant vector.*

A more direct way to prove the lack of diffeomorphic invariance is to compute $\nabla_\mu T^{\mu\nu}$ and see whether it is non-zero (and why). This computation proved to be quite involved and we could carry it on up to a point, that we illustrate in the appendix C.

Conclusions

In this thesis we studied Weyl and conformal symmetries, for classical field theories. These two symmetries are tightly related, but different, as the first applies to diffeomorphic-covariant (e.g., curved space) contexts, while the second lives in flat space. In fact, the necessary and sufficient conditions, for flat space scale invariant theories to be fully conformally invariant, were established in [7], and they heavily involve Weyl symmetry of the diffeomorphic-covariant 'version' of the flat space theory. That discussion applies to fields of arbitrary spin, and to any number of dimensions n, including the special case of $n = 2$.

Within that approach, we focused here on the special case of the two-dimensional Liouville scalar field theory. This theory enjoys full conformal symmetry in flat space, even though it has a macroscopic scale (the mass) and it is selfinteracting. This is possible because conformal symmetry, at least in two dimensions, can be "augmented" by the insertion of a so-called "central extension". That is, the infinite charges of conformal symmetry, l_n, get one more, the central charge c. As well known, in the quantum regime, and in presence of curvature, such central charge is related to an obstruction of Weyl symmetry, namely the vacuum expectation of the trace of the energy momentum tensor is not zero, but proportional to the curvature, to $\hbar$, and to c.

This is an instance of an *anomaly* (trace or Weyl anomaly, in this case), that is a soft breaking of the symmetry, induced by *quantum* effects. There are other instances of anomalies in physics, the most famous (and historically the first example) being the chiral anomaly of Adler, Bell and Jackiw. One of interest here is the 'gravitational anomaly', or lack of diffeomorphic-invariance, for instance signalled by the lack of covariant conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} \neq 0$. In all cases, anomalies here are *quantum* effects.

Interestingly, in [9] Jackiw raised the question of whether, for Liouville theory, we are in the presence of a classical instance, either of Weyl anomaly, or of gravitational anomaly. In other words, in [9] it is conjectured that, for Liouville theory, one is not entitled to have at once diffeomorphic and Weyl symmetries, so either of them need be "anomalous". Both instances would be fascinating, because they would provide examples of soft breaking of the symmetry not due to quantum effects.

In this thesis, we have proved that indeed this is the case. We did so by, on the one hand, explicitly computing the improvement terms of the energy-momentum tensor (whose role is to make the latter traceless). These terms involve the Weyl potential W^μ, as required by the general procedure of [7] (such calculations are not trivial, and are presented in Section 3.5). On the other hand, we show explicitly that W^μ is not a contravariant vector under diffeomorphisms (such calculations are not trivial, and are presented in Section 3.6). This means that indeed a gravitational anomaly is taking place. Notice that the statement of non covariance of W^μ is in [9], but that was not explicitly proved there.

We also performed the difficult computation of $\nabla_\mu T^{\mu\nu}$, to have an explicit expression for the gravitational anomaly. Such computations are reported in the Appendix C, and their outcome is not conclusive in this respect, but can be useful for later investigations.

Bibliography

[1] N. D. Birrell and P. C. W. Davies. *Quantum Fields in Curved Space*. Cambridge University Press, 1986.

[2] R. Blumenhagen and E. Plauschinn. *Introduction to Conformal Field Theory: With Applications to String Theory*. Springer, 2009.

[3] S. Carroll. *Spacetime and Geometry: An introduction to General Relativity*. Pearson, 2003.

[4] S. Deser and R. Jackiw. Energy-momentum tensor improvements in two dimensions. *Int. J. Mod. Phys. B*, 495:1499–1506, 1996.

[5] P. Di Francesco, P. Mathieu, and D. Sénéchal. *Conformal Field Theory*. Springer-Verlag, 1997.

[6] A. Iorio. Weyl-gauge symmetry of graphene. *Ann. of Phys.*, 326, 2010.

[7] A. Iorio, L. O'Raifeartaigh, I. Sachs, and C. Wiesendanger. Weyl gauging and conformal invariance. *Nucl. Phys. B*, 495:433–450, 1997.

[8] R. Jackiw. *Diverse Topics in Theoretical and Mathematical Physics*. World Scientific Publishing Company, 1995.

[9] R. Jackiw. Weyl symmetry and the Liouville theory. *Theor. and Math. Phys.*, 148:941–947, 2006.

[10] J. Maldacena. Einstein gravity from conformal gravity. *arXiv preprint arXiv:1105.5632*, 2011.

[11] P. D. Mannheim. Making the case for conformal gravity. *Found. Phys.*, 42: 388–420, 2012.

[12] Yu Nakayama. Scale invariance vs conformal invariance. *Phys. Rep.*, 569: 1–93, 2015.

[13] L. O'Raifeartaigh. *The Dawning of Gauge Theory*. Princeton University Press, 1997.

[14] H. Weyl. Gravitation und elektricität. *Sitzungsber. Akademie der Wissenschaften Berlin*, pages 465–480, 1918.

A. Weyl transformations

In this appendix we go through the computations of Weyl variations of several important objects. First we look at Christoffel symbols and covariant derivates, then we scrutinize the the Weyl variations of geometrical objects, Ricci tensor and Ricci scalar.

First we recall how the metric transforms under Weyl transformation

$$g_{\mu\nu} \to e^{2\omega} g_{\mu\nu}. \tag{A.1}$$

From this it is easy to find the Weyl transformation of Christoffel symbols (notation $\partial_\mu \omega = \omega_\mu$)

$$\begin{aligned}
\Gamma^\rho{}_{\mu\nu} &= \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \\
&\to \frac{1}{2} e^{-2\omega} g^{\rho\sigma} \left(\partial_\mu \left(e^{2\omega} g_{\sigma\nu}\right) + \partial_\nu \left(e^{2\omega} g_{\mu\sigma}\right) - \partial_\sigma \left(e^{2\omega} g_{\mu\nu}\right)\right) \\
&= \Gamma^\rho{}_{\mu\nu} + \frac{1}{2} g^{\rho\sigma} \left(2\omega_\mu g_{\sigma\nu} + 2\omega_\nu g_{\mu\sigma} - 2\omega_\sigma g_{\mu\nu}\right).
\end{aligned} \tag{A.2}$$

Therefore the Weyl variation is

$$\Delta \left(\Gamma^\rho{}_{\mu\nu}\right) = g^{\rho\sigma} \left(\omega_\mu g_{\sigma\nu} + \omega_\nu g_{\mu\sigma} - \omega_\sigma g_{\mu\nu}\right). \tag{A.3}$$

Our aim is to study general cases. Therefore, we have to include spin connection in covariant derivatives as in (1.60). Knowing the Weyl variation of spin connection $s_\mu{}^a{}_b$

$$\begin{aligned}
s_\mu{}^a{}_b \equiv e^a_\lambda (\delta^\lambda_\nu \partial_\mu + \Gamma^\lambda{}_{\mu\nu}) E^\nu_b \to & e^\omega e^a_\lambda (\delta^\lambda_\nu \partial_\mu + \Gamma^\lambda{}_{\mu\nu} + \Delta\Gamma^\lambda{}_{\mu\nu}) e^{-\omega} E^\nu_b \\
= & e^a_\lambda (\delta^\lambda_\nu \partial_\mu + \Gamma^\lambda{}_{\mu\nu}) E^\nu_b - e^a_\lambda \delta^\lambda_\nu \omega_\mu E^\nu_b \\
& + e^a_\lambda g^{\lambda\alpha} (\omega_\mu g_{\nu\alpha} + \omega_\nu g_{\mu\alpha} - \omega_\alpha g_{\mu\nu}) E^\nu_b \\
= & e^a_\lambda (\delta^\lambda_\nu \partial_\mu + \Gamma^\lambda{}_{\mu\nu}) E^\nu_b + e^a_\mu \omega_\nu E^\nu_b \\
& - e^a_\lambda g^{\lambda\alpha} \omega_\alpha g_{\mu\nu} e^\nu_b
\end{aligned} \tag{A.4}$$

and by using antisymmetry of Lorentz generators we get

$$(S_\mu)^i{}_j \to (S_\mu)^i{}_j - 2\omega_\nu (\Sigma_\mu{}^\nu)^i{}_j. \tag{A.5}$$

The next object of our interest is $\nabla_\mu W_\nu$. Its Weyl transformation is

$$\begin{aligned}
\nabla_\mu W_\nu = & \partial_\mu W_\nu - \Gamma^\rho{}_{\mu\nu} W_\rho \\
\to & \partial_\mu \left(W_\nu - \omega_\nu\right) - \left(\Gamma^\rho{}_{\mu\nu} + \Delta\Gamma^\rho{}_{\mu\nu}\right) \left(W_\rho - \omega_\rho\right) \\
= & \nabla_\mu W_\nu - \nabla_\mu \omega_\nu - \left(\omega_\mu g_{\sigma\nu} + \omega_\nu g_{\mu\sigma} - \omega_\sigma g_{\mu\nu}\right) \left(W^\sigma - \omega^\sigma\right) \\
= & \nabla_\mu W_\nu - \nabla_\mu \omega_\nu + \left(2\omega_\mu \omega_\nu - g_{\mu\nu} \omega_\sigma \omega^\sigma\right) \\
& - \left(\omega_\mu W_\nu + \omega_\nu W_\mu - g_{\mu\nu} \omega_\sigma W^\sigma\right).
\end{aligned} \tag{A.6}$$

We compute the Weyl variation of Ricci tensor $R_{\mu\nu}$

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \Gamma^\rho{}_{\rho\sigma} \Gamma^\sigma{}_{\mu\nu} - \Gamma^\rho{}_{\nu\sigma} \Gamma^\sigma{}_{\mu\rho},$$

by transforming it part by part. So the terms with derivatives transform as

$$\begin{aligned}\partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} \rightarrow & \partial_\rho \left(\Gamma^\rho{}_{\mu\nu} + \Delta\Gamma^\rho{}_{\mu\nu}\right) - \left(\partial_\nu \Gamma^\rho{}_{\mu\rho} + \Delta\partial_\nu\Gamma^\rho{}_{\mu\rho}\right) \\ =&\partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \partial_\rho \left(\omega_\mu g^\rho_\nu + \omega_\nu g^\rho_\mu - \omega^\rho g_{\mu\nu}\right) \\ & - \partial_\nu \left(\omega_\mu g^\rho_\rho + \omega_\rho g^\rho_\mu - \omega^\rho g_{\mu\rho}\right) \\ =&\partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \partial_\mu\omega_\nu + \partial_\nu\omega_\mu - \partial_\rho\left(\omega^\rho g_{\mu\nu}\right) \\ & - n\partial_\nu\omega_\mu - \partial_\nu\omega_\mu + \partial_\nu\omega_\mu,\end{aligned} \tag{A.7}$$

where n is the dimension. Using the definition of ω_μ and commutation of partial derivatives

$$\partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} \rightarrow \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + (2-n)\partial_\mu\omega_\nu - \partial_\rho\left(\omega^\rho g_{\mu\nu}\right). \tag{A.8}$$

The third term of Ricci tensor transforms as

$$\begin{aligned}\Gamma^\rho{}_{\rho\sigma}\Gamma^\sigma{}_{\mu\nu} \rightarrow & \left(\Gamma^\rho{}_{\rho\sigma} + \omega_\rho g^\rho_\sigma + \omega_\sigma g^\rho_\rho - \omega^\rho g_{\rho\sigma}\right)\cdot \\ & \cdot\left(\Gamma^\sigma{}_{\mu\nu} + \omega_\mu g^\sigma_\nu + \omega_\nu g^\sigma_\mu - \omega^\sigma g_{\mu\nu}\right) \\ =& \left(\Gamma^\rho{}_{\rho\sigma} + n\omega_\sigma\right)\left(\Gamma^\sigma{}_{\mu\nu} + \omega_\mu g^\sigma_\nu + \omega_\nu g^\sigma_\mu - \omega^\sigma g_{\mu\nu}\right) \\ =&\Gamma^\rho{}_{\rho\sigma}\Gamma^\sigma{}_{\mu\nu} + \Gamma^\rho{}_{\rho\nu}\omega_\mu + \Gamma^\rho{}_{\rho\mu}\omega_\nu - \Gamma^\rho{}_{\rho\sigma}\omega^\sigma g_{\mu\nu} + \\ & + n\omega_\sigma\Gamma^\sigma{}_{\mu\nu} + 2n\omega_\nu\omega_\mu - n\omega_\sigma\omega^\sigma g_{\mu\nu}.\end{aligned} \tag{A.9}$$

The fourth term of Ricci tensor transforms as

$$\begin{aligned}\Gamma^\rho{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\rho} \rightarrow & \left(\Gamma^\rho{}_{\nu\sigma} + \omega_\nu g^\rho_\sigma + \omega_\sigma g^\rho_\nu - \omega^\rho g_{\nu\sigma}\right)\cdot \\ & \cdot\left(\Gamma^\sigma{}_{\mu\rho} + \omega_\mu g^\sigma_\rho + \omega_\rho g^\sigma_\mu - \omega^\sigma g_{\mu\rho}\right) \\ =&\Gamma^\rho{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\rho} + \Gamma^\rho{}_{\nu\rho}\omega_\mu + \Gamma^\rho{}_{\nu\mu}\omega_\rho - \Gamma^\rho{}_{\nu\sigma}\omega^\sigma g_{\mu\rho} + \\ & + \omega_\nu\Gamma^\rho{}_{\mu\rho} + n\omega_\nu\omega_\mu \\ & + \omega_\sigma\Gamma^\sigma{}_{\mu\nu} + 2\omega_\nu\omega_\mu - \omega_\sigma\omega^\sigma g_{\mu\nu} - \\ & - \omega^\rho g_{\nu\sigma}\Gamma^\sigma{}_{\mu\rho} - \omega^\sigma\omega_\sigma g_{\nu\mu} \\ =&\Gamma^\rho{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\rho} + \Gamma^\rho{}_{\nu\rho}\omega_\mu + \Gamma^\rho{}_{\mu\rho}\omega_\nu + \Gamma^\rho{}_{\nu\mu}\omega_\rho + \Gamma^\rho{}_{\mu\nu}\omega_\rho - \\ & - \Gamma^\rho{}_{\nu\sigma}\omega^\sigma g_{\mu\rho} - \Gamma^\rho{}_{\mu\sigma}\omega^\sigma g_{\nu\rho} + (n+2)\omega_\nu\omega_\mu - 2\omega_\sigma\omega^\sigma g_{\mu\nu}.\end{aligned} \tag{A.10}$$

Now putting everything together

$$\begin{aligned}R_{\mu\nu} \rightarrow & \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + (2-n)\partial_\mu\omega_\nu - \partial_\rho\left(\omega^\rho g_{\mu\nu}\right) + \\ & + \Gamma^\rho{}_{\rho\sigma}\Gamma^\sigma{}_{\mu\nu} + \Gamma^\rho{}_{\rho\nu}\omega_\mu + \Gamma^\rho{}_{\rho\mu}\omega_\nu - \Gamma^\rho{}_{\rho\sigma}\omega^\sigma g_{\mu\nu} + \\ & + n\omega_\sigma\Gamma^\sigma{}_{\mu\nu} + 2n\omega_\nu\omega_\mu - n\omega_\sigma\omega^\sigma g_{\mu\nu} - \\ & - \Gamma^\rho{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\rho} - \Gamma^\rho{}_{\nu\rho}\omega_\mu - \Gamma^\rho{}_{\mu\rho}\omega_\nu - \Gamma^\rho{}_{\nu\mu}\omega_\rho - \Gamma^\rho{}_{\mu\nu}\omega_\rho + \\ & + \Gamma^\rho{}_{\nu\sigma}\omega^\sigma g_{\mu\rho} + \Gamma^\rho{}_{\mu\sigma}\omega^\sigma g_{\nu\rho} - (n+2)\omega_\nu\omega_\mu + 2\omega_\sigma\omega^\sigma g_{\mu\nu} \\ =&R_{\mu\nu} + (2-n)\left(\partial_\mu\omega_\nu - \omega_\sigma\Gamma^\sigma{}_{\mu\nu} - \omega_\nu\omega_\mu + \omega_\sigma\omega^\sigma g_{\mu\nu}\right) - \\ & - \partial_\rho\left(\omega^\rho g_{\mu\nu}\right) - \Gamma^\rho{}_{\rho\sigma}\omega^\sigma g_{\mu\nu} + \Gamma^\rho{}_{\nu\sigma}\omega^\sigma g_{\mu\rho} + \Gamma^\rho{}_{\mu\sigma}\omega^\sigma g_{\nu\rho} \\ =&R_{\mu\nu} - (n-2)\left(\nabla_\mu\nabla_\nu\omega - \omega_\mu\omega_\nu + \omega_\sigma\omega^\sigma g_{\mu\nu}\right) - \partial_\rho\omega^\rho g_{\mu\nu} - \\ & - \omega^\rho\partial_\rho g_{\mu\nu} - \Gamma^\rho{}_{\rho\sigma}\omega^\sigma g_{\mu\nu} + \Gamma^\rho{}_{\nu\sigma}\omega^\sigma g_{\mu\rho} + \Gamma^\rho{}_{\mu\sigma}\omega^\sigma g_{\nu\rho} \\ =&R_{\mu\nu} - g_{\mu\nu}\nabla^2\omega - (n-2)\left(\nabla_\mu\nabla_\nu\omega - \omega_\mu\omega_\nu + \omega_\sigma\omega^\sigma g_{\mu\nu}\right) + \\ & + \omega^\sigma\frac{1}{2}\left(\left(\partial_\sigma g_{\mu\nu} + \partial_\nu g_{\sigma\mu} - \partial_\mu g_{\nu\sigma}\right) + \left(\partial_\sigma g_{\nu\mu} + \partial_\mu g_{\sigma\nu} - \partial_\nu g_{\mu\sigma}\right)\right) \\ & - \omega^\sigma\partial_\sigma g_{\mu\nu} \\ =&R_{\mu\nu} - g_{\mu\nu}\nabla^2\omega - (n-2)\left(\nabla_\mu\nabla_\nu\omega - \omega_\mu\omega_\nu + \omega_\sigma\omega^\sigma g_{\mu\nu}\right).\end{aligned} \tag{A.11}$$

From this we should derive the Weyl transformation of Ricci scalar without any problems

$$
\begin{aligned}
R = g^{\mu\nu} R_{\mu\nu} \to & e^{-2\omega} g^{\mu\nu} \left(R_{\mu\nu} - g_{\mu\nu} \nabla^2 \omega \right) \\
& - e^{-2\omega} g^{\mu\nu} (n-2) \left(\nabla_\mu \nabla_\nu \omega - \omega_\mu \omega_\nu + \omega_\sigma \omega^\sigma g_{\mu\nu} \right) \\
= & e^{-2\omega} \left(R - n \nabla^2 \omega - (n-2) \left(\nabla^2 \omega - \omega_\mu \omega^\mu + n \omega_\mu \omega^\mu \right) \right) \\
= & e^{-2\omega} \left(R - 2(n-1) \nabla^2 \omega - (n-2)(n-1) \omega_\mu \omega^\mu \right).
\end{aligned}
\tag{A.12}
$$

B. Light cone coordinates

The light-cone coordinates in two dimensions are defined as

$$x^{\pm} = \frac{1}{\sqrt{2}}\left(x^0 \pm x^1\right). \tag{B.1}$$

The coordinate transformation $(x^0, x^1) \to (x^+, x^-)$ can be obtained by acting with the matrix

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{B.2}$$

The derivatives in ligh-cone coordinates are

$$\begin{aligned} \partial_+ &= \frac{1}{\sqrt{2}}(\partial_0 + \partial_1) \\ \partial_- &= \frac{1}{\sqrt{2}}(\partial_0 - \partial_1). \end{aligned} \tag{B.3}$$

We would like to find components of metric tensor in these coordinates

$$x^2 = g_{\mu\nu}x^\mu x^\nu = g_{00}\left(x^0\right)^2 + 2g_{01}x^0x^1 + g_{11}\left(x^1\right)^2 \tag{B.4}$$

$$x^2 = g_{\mu\nu}x^\mu x^\nu = g_{++}\left(x^+\right)^2 + 2g_{+-}x^+x^- + g_{--}\left(x^-\right)^2. \tag{B.5}$$

By comparing those two equations and using the definition of light-cone coordinates we get

$$g_{++} = \frac{1}{2}\left(g_{00} + 2g_{01} + g_{11}\right) \tag{B.6}$$

$$g_{+-} = \frac{1}{2}\left(g_{00} - g_{11}\right) \tag{B.7}$$

$$g_{--} = \frac{1}{2}\left(g_{00} - 2g_{01} + g_{11}\right) \tag{B.8}$$

and the components of inverse metric $g^{\mu\nu}$ are

$$\begin{aligned} g^{++} &= \frac{g_{--}}{g} \\ g^{+-} &= -\frac{g_{+-}}{g} \\ g^{--} &= \frac{g_{++}}{g}. \end{aligned} \tag{B.9}$$

For the case of Minkowski space with metric $\eta_{\mu\nu}$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{B.10}$$

we get new metric

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{B.11}$$

with $\mu, \nu \in \{+, -\}$.

In this case the scalar product is

$$x^2 = -2x^+x^-. \tag{B.12}$$

Introducing $\gamma_{\mu\nu}$ and its inverse $\gamma^{\mu\nu}$

$$\gamma_{\mu\nu} = \frac{g_{\mu\nu}}{\sqrt{-g}} \qquad \gamma^{\mu\nu} = \sqrt{-g}g^{\mu\nu} \tag{B.13}$$

we can parametrize the light-cone components as

$$\begin{aligned}\gamma_{++} &= e^{\gamma} \sinh \omega \\ \gamma_{+-} &= \cosh \omega \\ \gamma_{--} &= e^{-}\gamma \sinh \omega.\end{aligned} \tag{B.14}$$

The last object we want to study in light-cone coordinates is Levi-Civita symbol. To transform it we recall its matrix representation

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{B.15}$$

To transform it into light-cone coordinates the easiest way is to act on it with matrix S. Written as a symbolic matrix multiplication we want to compute

$$S\epsilon S^{-1}. \tag{B.16}$$

The result is

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{B.17}$$

In these coordinates we have $\epsilon^{+-} = -\epsilon^{-+} = -1$

C. Covariant divergence

In this appendix we illustrate the computation of covariant divergence of the computed energy-momentum improvement (3.70)

$$\begin{aligned}\beta^2\nabla_\mu T^{\mu\nu} =& 2g^{\mu\nu}\partial_\mu\left(W^\alpha W^\beta g_{\alpha\beta}\right) - 2RW^\nu - 4W^\mu\nabla_\mu W^\nu - 2g^{\mu\nu}\partial_\mu R \\ &+ 2g^{\alpha\beta}\nabla_\alpha\nabla_\beta W^\nu + 2g^{\nu\alpha}\nabla_\mu\nabla_\alpha W^\mu \\ &+ \nabla_\mu\left[\frac{4\epsilon^{\alpha\beta}}{\sqrt{-g}}\partial_\beta(g_{\alpha\rho}W^\rho)[(\cosh\omega - 1)\Gamma^{\mu\nu} + r^{\mu\nu}]\right].\end{aligned} \tag{C.1}$$

In two dimensions Riemman tensor can be written as

$$R_{\mu\nu\alpha\beta} = \frac{R}{2}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}). \tag{C.2}$$

Using this we can easily commute covaraint derivatives

$$2g^{\nu\alpha}\nabla_\mu\nabla_\alpha W^\mu = 2g^{\nu\alpha}g^{\mu\rho}R_{\sigma\rho\alpha\mu}W^\sigma + 2g^{\nu\alpha}\nabla_\alpha\nabla_\mu W^\mu = RW^\nu + g^{\mu\nu}\partial_\mu R. \tag{C.3}$$

With this we simplify the expression for $\nabla_\mu T^{\mu\nu}$

$$\begin{aligned}\beta^2\nabla_\mu T^{\mu\nu} =& 2g^{\mu\nu}\partial_\mu\left(W^\alpha W^\beta g_{\alpha\beta}\right) - 4W^\mu\nabla_\mu W^\nu \\ &+ 2g^{\alpha\beta}\nabla_\alpha\nabla_\beta W^\nu - RW^\nu - g^{\mu\nu}\partial_\mu R \\ &+ \nabla_\mu\left[\frac{4\epsilon^{\alpha\beta}}{\sqrt{-g}}\partial_\beta(g_{\alpha\rho}W^\rho)[(\cosh\omega - 1)\Gamma^{\mu\nu} + r^{\mu\nu}]\right]\end{aligned} \tag{C.4}$$

Before proceeding the computation we derive a useful formulation of covariant divergence

$$\begin{aligned}\nabla_\mu T^{\mu\nu} =& \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\alpha}T^{\alpha\nu} + \Gamma^\nu_{\mu\alpha}T^{\mu\alpha} \\ \Gamma^\mu_{\mu\alpha}T^{\alpha\nu} =& \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}T^{\mu\nu} \\ \Gamma^\nu_{\mu\alpha}T^{\mu\alpha} =& g^{\nu\lambda}\partial_\alpha g_{\lambda\mu}T^{\mu\alpha} - \frac{1}{2}g^{\nu\lambda}\partial_\lambda g_{\mu\alpha}T^{\mu\alpha} \\ \nabla_\mu T^{\mu\nu} =& \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}T^{\mu\nu}) + g^{\mu\nu}\partial_\alpha g_{\beta\mu}T^{\alpha\beta} - \frac{1}{2}g^{\mu\nu}\partial_\mu g_{\alpha\beta}T^{\alpha\beta}.\end{aligned} \tag{C.5}$$

Using this we can rewrite the last term in $\nabla_\mu T^{\mu\nu}$

$$\begin{aligned}\beta^2\nabla_\mu T^{\mu\nu}_{last} =& \nabla_\mu\left[\frac{4\epsilon^{\alpha\beta}}{\sqrt{-g}}\partial_\beta(g_{\alpha\rho}W^\rho)[(\cosh\omega - 1)\Gamma^{\mu\nu} + r^{\mu\nu}]\right] \\ =& + \frac{4}{\sqrt{-g}}\partial_\mu\left[\epsilon^{\alpha\beta}\partial_\beta(g_{\alpha\rho}W^\rho)[(\cosh\omega - 1)\Gamma^{\mu\nu} + r^{\mu\nu}]\right] \\ &+ g^{\mu\nu}\partial_\alpha g_{\beta\mu}\frac{4\epsilon^{\sigma\lambda}}{\sqrt{-g}}\partial_\lambda(g_{\sigma\rho}W^\rho)[(\cosh\omega - 1)\Gamma^{\alpha\beta} + r^{\alpha\beta}] \\ &- \frac{2\epsilon^{\alpha\beta}g^{\mu\nu}}{\sqrt{-g}}\partial_\beta(g_{\alpha\rho}W^\rho)[(\cosh\omega - 1)\partial_\mu\gamma + \partial_\mu r].\end{aligned} \tag{C.6}$$

We used here the definition of $\Gamma^{\mu\nu}$ and $r^{\mu\nu}$

$$\delta\gamma = \delta g_{\mu\nu}\Gamma^{\mu\nu} \qquad \delta r = \delta g_{\mu\nu} r^{\mu\nu}.$$

By using (3.35) we can improve the last term

$$\begin{aligned}-\epsilon^{\alpha\beta}g^{\mu\nu}[(\cosh\omega-1)\partial_\mu\gamma+\partial_\mu r] =&\epsilon^{\beta\mu}g^{\alpha\nu}[(\cosh\omega-1)\partial_\mu\gamma+\partial_\mu r]\\ &-\epsilon^{\alpha\mu}g^{\beta\nu}[(\cosh\omega-1)\partial_\mu\gamma+\partial_\mu r]\\ =&2\sqrt{-g}(g^{\alpha\nu}W^\beta-g^{\beta\nu}W^\alpha)\\ &+\frac{\partial_\sigma g_{\rho\mu}}{\sqrt{-g}}g^{\mu\nu}\epsilon^{\sigma\rho}\epsilon^{\alpha\beta}\\ =&2\sqrt{-g}(g^{\alpha\nu}W^\beta-g^{\beta\nu}W^\alpha)\\ &+\sqrt{-g}(g^{\lambda\alpha}\partial_\lambda g^{\beta\nu}-g^{\lambda\beta}\partial_\lambda g^{\alpha\nu})\end{aligned} \tag{C.7}$$

in the second equality we used the definition of W^μ and in the last one we used (3.34).

Therefore we can rewrite the last term as

$$\begin{aligned}\beta^2\nabla_\mu T^{\mu\nu}_{last} =&\frac{4}{\sqrt{-g}}\partial_\mu\left[\epsilon^{\alpha\beta}\partial_\beta(g_{\alpha\rho}W^\rho)[(\cosh\omega-1)\Gamma^{\mu\nu}+r^{\mu\nu}]\right]\\ &+g^{\mu\nu}\partial_\alpha g_{\beta\mu}\frac{4\epsilon^{\sigma\lambda}}{\sqrt{-g}}\partial_\lambda(g_{\sigma\rho}W^\rho)[(\cosh\omega-1)\Gamma^{\alpha\beta}+r^{\alpha\beta}]\\ &+4\partial_\beta(g_{\alpha\rho}W^\rho)\left(g^{\nu\alpha}W^\beta-g^{\nu\beta}W^\alpha\right)\\ &+2\partial_\beta(g_{\alpha\rho}W^\rho)\left(g^{\alpha\lambda}\partial_\lambda g^{\nu\beta}-g^{\beta\lambda}\partial_\lambda g^{\nu\alpha}\right).\end{aligned} \tag{C.8}$$

The full improvement term is then

$$\begin{aligned}\beta^2\nabla_\mu T^{\mu\nu} =&2g^{\mu\nu}\partial_\mu\left(W^\alpha W^\beta g_{\alpha\beta}\right)-4W^\mu\nabla_\mu W^\nu\\ &+2g^{\alpha\beta}\nabla_\alpha\nabla_\beta W^\nu-RW^\nu-g^{\mu\nu}\partial_\mu R\\ &+\frac{4}{\sqrt{-g}}\partial_\mu\left[\epsilon^{\alpha\beta}\partial_\beta(g_{\alpha\rho}W^\rho)[(\cosh\omega-1)\Gamma^{\mu\nu}+r^{\mu\nu}]\right]\\ &+g^{\mu\nu}\partial_\alpha g_{\beta\mu}\frac{4\epsilon^{\sigma\lambda}}{\sqrt{-g}}\partial_\lambda(g_{\sigma\rho}W^\rho)[(\cosh\omega-1)\Gamma^{\alpha\beta}+r^{\alpha\beta}]\\ &+4\partial_\beta(g_{\alpha\rho}W^\rho)\left(g^{\nu\alpha}W^\beta-g^{\nu\beta}W^\alpha\right)\\ &+2\partial_\beta(g_{\alpha\rho}W^\rho)\left(g^{\alpha\lambda}\partial_\lambda g^{\nu\beta}-g^{\beta\lambda}\partial_\lambda g^{\nu\alpha}\right).\end{aligned} \tag{C.9}$$

Let us expand terms quadratic in W^μ

$$2g^{\mu\nu}\partial_\mu\left(W^\alpha W^\beta g_{\alpha\beta}\right)=4g^{\mu\nu}\partial_\mu W^\alpha W^\beta g_{\alpha\beta}+2g^{\mu\nu}W^\alpha W^\beta\partial_\mu g_{\alpha\beta}, \tag{C.10}$$

$$-4W^\mu\nabla_\mu W^\nu=-4W^\mu\partial_\mu W^\nu-4g^{\mu\nu}W^\alpha W^\beta\partial_a g_{\beta\mu}+2W^\alpha W^\beta g^{\mu\nu}\partial_\mu g_{\alpha\beta}, \tag{C.11}$$

$$\begin{aligned}4\partial_\beta(g_{\alpha\rho}W^\rho)\left(g^{\nu\alpha}W^\beta-g^{\nu\beta}W^\alpha\right) =&4g^{\mu\nu}W^\alpha W^\beta\partial_\alpha g_{\beta\mu}-4g^{\mu\nu}\partial_\mu g_{\alpha\beta}W^\alpha W^\beta\\ &-4g_{\alpha\beta}\partial_\mu W^\alpha W^\beta g_{\alpha\beta}+4W^\mu\partial_\mu W^\nu.\end{aligned} \tag{C.12}$$

Putting together these terms we see

$$\begin{aligned} 0 =&2g^{\mu\nu}\partial_\mu\left(W^\alpha W^\beta g_{\alpha\beta}\right) - 4W^\mu\nabla_\mu W^\nu \\ &+ 4\partial_\beta(g_{\alpha\rho}W^\rho)\left(g^{\nu\alpha}W^\beta - g^{\nu\beta}W^\alpha\right) \end{aligned} \tag{C.13}$$

This simplifies the divergence of our improvement to

$$\begin{aligned} \beta^2\nabla_\mu T^{\mu\nu} =&2g^{\alpha\beta}\nabla_\alpha\nabla_\beta W^\nu - RW^\nu - g^{\mu\nu}\partial_\mu R \\ &+ \frac{4}{\sqrt{-g}}\partial_\mu\left[\epsilon^{\alpha\beta}\partial_\beta(g_{\alpha\rho}W^\rho)[(\cosh\omega - 1)\Gamma^{\mu\nu} + r^{\mu\nu}]\right] \\ &+ g^{\mu\nu}\partial_\alpha g_{\beta\mu}\frac{4\epsilon^{\sigma\lambda}}{\sqrt{-g}}\partial_\lambda(g_{\sigma\rho}W^\rho)[(\cosh\omega - 1)\Gamma^{\alpha\beta} + r^{\alpha\beta}] \\ &+ 2\partial_\beta(g_{\alpha\rho}W^\rho)\left(g^{\alpha\lambda}\partial_\lambda g^{\nu\beta} - g^{\beta\lambda}\partial_\lambda g^{\nu\alpha}\right). \end{aligned} \tag{C.14}$$

Using (3.34) we can rewrite the last term as

$$2\partial_\beta(g_{\alpha\rho}W^\rho)\left(g^{\alpha\lambda}\partial_\lambda g^{\nu\beta} - g^{\beta\lambda}\partial_\lambda g^{\nu\alpha}\right) = -2\epsilon^{\alpha\beta}\partial_\beta(g_{\alpha\rho}W^\rho)g^{\nu\sigma}\partial_\lambda g_{\mu\sigma}\frac{\epsilon^{\mu\lambda}}{\sqrt{-g}^2}. \tag{C.15}$$

The term $\nabla\nabla W$ can be written as

$$\begin{aligned} 2g^{\alpha\beta}\nabla_\alpha\nabla_\beta W^\nu =&2g^{\alpha\beta}(\partial_\alpha\partial_\beta W^\nu + \partial_\alpha\Gamma^\nu_{\beta\lambda}W^\lambda + \Gamma^\nu_{\beta\lambda}\partial_\alpha W^\lambda - \Gamma^\sigma_{\alpha\beta}\partial_\sigma W^\nu \\ &- \Gamma^\sigma_{\alpha\beta}\Gamma^\nu_{\sigma\lambda}W^\lambda + \Gamma^\nu_{\sigma\alpha}\partial_\beta W^\sigma + \Gamma^\nu_{\sigma\alpha}\Gamma^\sigma_{\beta\lambda}W^\lambda) \\ =&2g^{\alpha\beta}[\partial_\alpha\partial_\beta W^\nu + 2\Gamma^\nu_{\beta\lambda}\partial_\alpha W^\lambda - \Gamma^\sigma_{\alpha\beta}\partial_\sigma W^\nu \\ &+ W^\lambda({R^\nu}_{\beta\alpha\lambda} + \partial_\lambda\Gamma^\nu_{\alpha\beta})] \\ =&2g^{\alpha\beta}(\partial_\alpha\partial_\beta W^\nu + 2\Gamma^\nu_{\beta\lambda}\partial_\alpha W^\lambda - \Gamma^\sigma_{\alpha\beta}\partial_\sigma W^\nu + W^\lambda\partial_\lambda\Gamma^\nu_{\alpha\beta}) \\ &- RW^\nu. \end{aligned} \tag{C.16}$$

Unfortunately this does not lead to any cancellations and the most pleasant form we could derive is

$$\begin{aligned} \beta^2\nabla_\mu T^{\mu\nu} =&2g^{\alpha\beta}\nabla_\alpha\nabla_\beta W^\nu - RW^\nu - g^{\mu\nu}\partial_\mu R \\ &+ \frac{4}{\sqrt{-g}}\partial_\mu\left[\epsilon^{\alpha\beta}\partial_\beta(g_{\alpha\rho}W^\rho)\right][(\cosh\omega - 1)\Gamma^{\mu\nu} + r^{\mu\nu}] \\ &+ \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}}\partial_\beta(g_{\alpha\rho}W^\rho)\Bigg(4\partial_\mu[(\cosh\omega - 1)\Gamma^{\mu\nu} + r^{\mu\nu}] \\ &+ 4g^{\mu\nu}\partial_\sigma g_{\lambda\mu}[(\cosh\omega - 1)\Gamma^{\sigma\lambda} + r^{\sigma\lambda}] - 2g^{\nu\mu}\frac{\epsilon^{\sigma\lambda}}{\sqrt{-g}}\partial_\lambda g_{\sigma\mu}\Bigg), \end{aligned} \tag{C.17}$$

with

$$\begin{aligned} \Gamma^{\mu\nu} =&\frac{1}{4g_{++}g_{--}}\begin{pmatrix} g_{--} - g_{++} & g_{--} + g_{++} \\ g_{--} + g_{++} & g_{--} - g_{++} \end{pmatrix} \\ =&\frac{1}{2\sqrt{g_{--}g_{++}}}\begin{pmatrix} -\sinh\gamma & \cosh\gamma \\ \cosh\gamma & -\sinh\gamma \end{pmatrix}. \end{aligned} \tag{C.18}$$

We cannot conclude whether the RHS of (C.17) is zero or nonzero, but for consistency with the results of this thesis on the lack of covariance of W^μ, it should not be zero.