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Gradient polyconvexity and its application to problems of mathematical elasticity and plasticity

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Prague, 10^{th} May 2019 Jiří Zeman

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Abstract: Polyconvexity is a standard assumption on hyperelastic stored energy densities which, together with some growth conditions, ensures the weak lower semicontinuity of the respective energy functional. The present work first reviews known results about gradient polyconvexity, introduced by Benešová, Kružík and Schlömerkemper in 2017. It is an alternative property to polyconvexity, better-suited e.g. for the modelling of shape-memory alloys. The principal result of this thesis is the extension of an elastic material model with gradient polyconvex energy functional to an elastoplastic body and proving the existence of an energetic solution to an associated rate-independent evolution problem, proceeding from previous work of Mielke, Francfort and Mainik.

Keywords: weak convergence, Sobolev spaces, convexity

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Contents

Notation

$\mathbb{A} = (A_{ij})_{i,j=1}^{d} \in \mathbb{R}^{d\times d}$	matrix and its components
$\mathbb{A} : \mathbb{B}$	usual scalar product of matrices [18, p. 9]
$\boldsymbol{a} \otimes \boldsymbol{b}$	tensor product of vectors $\boldsymbol{a}$, $\boldsymbol{b}$ [18, p. 9]
$\mathrm{B}([0,T];\mathcal{A})$	bounded (possibly non-measurable) functions with values in $\mathcal{A} \subset \mathcal{X}$, $\mathcal{X}$ is a normed linear space
$\operatorname{cof} \mathbb{F}$	the cofactor matrix of $\mathbb{F}$ ($\operatorname{cof} \mathbb{F} = (\operatorname{adj} \mathbb{F})^\top$, see [19, p. 356])
$\boldsymbol{e}_1$, $\boldsymbol{e}_2$, ..., $\boldsymbol{e}_d$	canonical basis vectors in $\mathbb{R}^d$ [19, p. 115]
$\mathcal{H}^k$	k-dimensional Hausdorff measure [13, p. 81]
$\mathbb{I}$	the identity matrix
$\operatorname{Int} \mathcal{A}$	interior of a set $\mathcal{A}$ in a metric space [11, p. 14, 248]
$\mathcal{L}^d$	d-dimensional Lebesgue measure [13, p. 34]
$L^1_{loc}(\Omega;\mathbb{R}^k)$	locally integrable functions (i.e. componentwise locally integrable as in [14, p. 39])
$L^p(\Omega;\mathbb{R}^k)$	Lebesgue spaces (of $\mathbb{R}^k$-valued functions) [28, p. 595]
$\boldsymbol{n}$	unit outward normal vector [6, p. 37]
$\boldsymbol{o}$	zero vector in $\mathbb{R}^k$
$\mathbb{R}^{d\times d}$	d by d real matrices
$\mathbb{R}^{d\times d}_+$	$\{\mathbb{A} \in \mathbb{R}^{d\times d};\ \det \mathbb{A} > 0\}$
$\mathbb{R}_\infty$	$\mathbb{R} \cup \{+\infty\}$
$\mathrm{SL}(d)$	$\{\mathbb{P} \in \mathbb{R}^{d\times d};\ \det \mathbb{P} = 1\}$
$\mathrm{SO}(d)$	$\{\mathbb{R} \in \mathrm{SL}(d);\ \mathbb{R}\mathbb{R}^\top = \mathbb{R}^\top\mathbb{R} = \mathbb{I}\}$
$\operatorname{Tr} \mathbb{A}$	the trace of a matrix [19, p. 301]
$\mathcal{X}^*$	dual space to the normed linear space $\mathcal{X}$ [31, p. 108]
$\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$	continuous embedding of normed linear spaces
$\mathcal{X}_1 \hookrightarrow\hookrightarrow \mathcal{X}_2$	compact embedding of normed linear spaces [21, p. 12]
$\delta J(\boldsymbol{y};\boldsymbol{\vartheta})$	the directional derivative of a functional J at the point $\boldsymbol{y}$ in the direction $\boldsymbol{\vartheta}$ [6, p. 10]
$\partial\mathcal{A}$	boundary of a set $\mathcal{A}$ in a metric space [11, p. 24, 248]
$\Omega \in \mathcal{C}^{0,1}$	domain with Lipschitz boundary [6, p. 32]
∇f, $\nabla \boldsymbol{f}$, $\underline{\nabla \mathbb{F}}$	$\nabla_{\boldsymbol{X}} f$, $\nabla_{\boldsymbol{X}} \boldsymbol{f}$, $\underline{\nabla_{\boldsymbol{X}} \mathbb{F}}$ for f, $\boldsymbol{f}$, $\mathbb{F}$ depending on space and time
$\overline{\mathcal{A}}$	closure of a set $\mathcal{A}$ in a metric space [11, p. 13, 248]
$x_n \rightharpoonup x$	weak convergence in a normed linear space [21, p. 14]
$f_n \overset{*}{\rightharpoonup} f$	weak* convergence in the dual space [28, p. 586]
$\lvert\mathbb{A}\rvert$	Frobenius norm of a matrix [28, p. 277]

Bold letters ($\boldsymbol{y}$, $\boldsymbol{X}$, ...) denote vector quantities, blackboard bold letters ($\mathbb{F}$, $\mathbb{P}$, $\mathbb{\sigma}$, $\mathbb{\varepsilon}$, ...) are used for second-order tensor quantities (which are identified with matrices) and an underline denotes a third-order tensor ($\underline{\mathbb{H}}$, $\underline{\nabla\mathbb{P}}$). A double underline forms part of the symbol for the generalised stress $\underline{\underline{\Sigma}}$ and the generalised plastic strain $\underline{\underline{\mathbb{P}}}$. We write ∇f for the gradient of a scalar field, $\nabla \boldsymbol{f} = (\frac{\partial f_i}{\partial X_j})_{i,j=1}^d$ for the gradient of a vector field (i.e. its Jacobian matrix) and $\underline{\nabla\mathbb{A}}$ if $\mathbb{A}$ is a second-order tensor. (For $\mathbb{A} \in W^{1,q}(\Omega,\mathbb{R}^{d\times d})$ ($\Omega \subset \mathbb{R}^d$ open, $1 \le q \le +\infty$) let us define its gradient as

$$[\underline{\nabla\mathbb{A}}]_{ijk} = \frac{\partial A_{ij}}{\partial X_k}, \quad i,j,k \in \{1,2,\ldots,d\}.)$$

The second gradient $\underline{\nabla^2 \boldsymbol{f}} := \underline{\nabla(\nabla \boldsymbol{f})}$ for $\boldsymbol{f}$ smooth enough.

Introduction

The study of deformations of solid bodies brings benefit to diverse areas of human activity. Architecture, mechanical engineering, materials science or biomedical research can all profit from achievements of mathematical modelling in solid-state physics. Numerical simulations on supercomputers efficiently deal with problems whose solution would otherwise remain out of reach. However, the tasks of developing realistic material models and of their mathematical analysis are still topical as well.

A workable approximation of material behaviour in situations where the acting forces are not strong enough to cause irreversible changes (e.g. cracks) is provided by *elasticity theory*. Although mathematical elasticity is commonly used, a fair number of open problems are unsolved as of now [2]. Chapter 1 goes through the basic concepts in mathematical elasticity and continuum mechanics, which is a more general theory in the sense that it also applies to fluids.

If the forces affecting a solid body exceed a specific limit, the body undergoes a *plastic deformation* and the elastic description must be upgraded to an *elastoplastic* one. There exists a rich spectrum of plastic effects that may take place and a comprehensive treatise of those would definitely be long. Chapter 2 contains an introduction to plasticity and the most relevant (by the author's judgement) plastic phenomena are mentioned. At the end of the chapter, the idea of *energetic solutions* (cf. [28]) for rate-independent systems arising in elastoplasticity is presented. This weaker concept of solutions can be advantageous when one needs to overcome the lack of differentiability of the appearing physical quantities.

In Chapter 3, it is explained what *gradient polyconvexity* ([5]) is and how it can be utilised in mathematical elasticity and plasticity. The chapter is mainly based on two recent articles [5] and [20].

The contribution of this thesis is an extension of the gradient polyconvex model from [5] to gradient plasticity at finite strains and the use of methods from [15], [25] to prove the existence of a solution to an associated energetic rate-independent system. Especially in Theorems 6, 7 and 11, the previously published ideas had to be adjusted to fit into the elastoplastic framework. In literature, a generalised Helly's selection theorem from [24] is often used without mentioning the particular assumptions needed to make it applicable. This work devotes more space to this argument (Theorem 14), so that such an important step becomes more comprehensible. Claim 2 in Chapter 3 can also be seen as a moderately interesting new inference, since it specifies the particular Sobolev exponents for which deformations of class $W^{2,p}$ are also regular enough for a gradient polyconvex energy functional.

To facilitate reading of this thesis, a summary of used notation is integrated at the very beginning and an appendix with necessary facts from mathematical analysis can be consulted at the end.

1. Continuum mechanics overview

For the reader's convenience, basics of continuum mechanics are briefly summarised, following [6], occasionally [16].

1.1 Basic quantities in continuum mechanics

Continuum mechanics studies the motion and deformation of solid or fluid bodies. It turns out that it is a workable and useful idealisation to treat the matter which the body consists of as *continuously* spread in space, rather than to focus on its underlying particle structure.

Thus in continuum mechanics, we assume the body $\mathcal{B}$ under consideration to occupy and fill up to the boundary a bounded connected set $\bar{\Omega} \subset \mathbb{R}^d$, where Ω is open, with Lipschitz boundary[1] and usually $d \in \{2, 3\}$.

The body $\mathcal{B}$ lies in $\bar{\Omega}$ when no forces are applied, therefore $\bar{\Omega}$ is called the *reference configuration*. To describe translation, rotation and deformation incurred by $\mathcal{B}$ due to applied forces, we use a mapping $\boldsymbol{y} : \bar{\Omega} \to \mathbb{R}^d$ and imagine that $\mathcal{B}$ is transformed from $\bar{\Omega}$ to a *deformed configuration* $\boldsymbol{y}(\bar{\Omega})$.

In order to have a physically sound movement, $\boldsymbol{y}$ has to satisfy certain properties: it belongs to a suitable smoothness class, preserves orientation and $\boldsymbol{y}|_{\Omega}$ is injective.

We define the *deformation gradient* as $\mathbb{F}(\boldsymbol{X}) = \nabla \boldsymbol{y}(\boldsymbol{X})$, $\boldsymbol{X} \in \Omega$, and extend it by continuity to $\bar{\Omega}$.

1.2 Stress and response of the material

Physical principles of conservation of mass, momentum and energy for the continuous body $\mathcal{B}$ translate into a system of partial differential equations (see e. g. [16, pages 128, 140, 185]) which governs the mechanical and thermodynamic state of $\mathcal{B}$. However, we need to add so-called *constitutive equations* to it which express the material-specific behaviour of $\mathcal{B}$. For example, if $\mathcal{B}$ is a fluid body, in the most simple case we add a relation between the velocity field of imaginary fluid particles and the corresponding mechanical stress.

Stress is generated by contact forces that are exerted on the boundary of Ω (or on some internal surface). The stress might even lead to internal changes in the material such as cracks in buildings after an earthquake.

There are different tensor quantities that describe the stress distribution in the body.

The first of them is the *Cauchy stress tensor* $\mathbb{T} : \boldsymbol{y}(\bar{\Omega}) \to \mathbb{R}^{d \times d}$. For any $\boldsymbol{x} \in \boldsymbol{y}(\bar{\Omega})$, a surface $\mathcal{S}$ that $\boldsymbol{x}$ lies on and $\boldsymbol{n}(\boldsymbol{x})$ being the unit normal to $\mathcal{S}$ at $\boldsymbol{x}$, $\mathbb{T}(\boldsymbol{x})$ gives the surface density $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{n}(\boldsymbol{x}))$ of contact forces acting upon $\mathcal{S}$ at $\boldsymbol{x}$ via

$$\mathbb{T}(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{T}(\boldsymbol{x}, \boldsymbol{n}(\boldsymbol{x})).$$

Another such quantity is the *first Piola–Kirchhoff stress tensor* $\mathbb{T}^{(1)} : \bar{\Omega} \to \mathbb{R}^{d \times d}$, which represents the stress measured per unit area in the reference body [16, p. 173]. The

[1] It means that the boundary can be locally described as a graph of a Lipschitz function, see e.g. [6, p. 32] for the exact definition.

two stress tensors are interlinked by

$$\mathbb{T}^{(1)}(\boldsymbol{X}) = \mathbb{T}(\boldsymbol{y}(\boldsymbol{X}))\mathrm{cof}\,\mathbb{F}(\boldsymbol{X}), \text{ for every } \boldsymbol{X} \in \bar{\Omega}.$$

Let us explore some constitutive equations in the mechanics of solids.

Definition 1 (Cauchy elastic material)**.** *We call a material* elastic *if there exists a response function* $\hat{\mathbb{T}} : \bar{\Omega} \times \mathbb{R}_+^{d\times d} \to \mathbb{R}^{d\times d}$ *such that*

$$\mathbb{T}^{(1)}(\boldsymbol{X}) = \hat{\mathbb{T}}(\boldsymbol{X}, \mathbb{F}(\boldsymbol{X}))$$

for every $\boldsymbol{X} \in \bar{\Omega}$*.*

Definition 2 (hyperelastic material)**.** *An elastic material is said to be* hyperelastic *if there is a function* $W : \bar{\Omega} \times \mathbb{R}_+^{d\times d} \to \mathbb{R}_\infty$ *such that*

$$\hat{\mathbb{T}}(\boldsymbol{X}, \mathbb{F}) = \frac{\partial W(\boldsymbol{X}, \mathbb{F})}{\partial \mathbb{F}}$$

for every $\boldsymbol{X} \in \bar{\Omega}$ *and* $\mathbb{F} \in \mathbb{R}_+^{d\times d}$*. The function* W *is called the* stored energy density*.*

At the microscopic level, it is the energy stored in interatomic links of the atomic lattice that inspires this phenomenological description [28, p. 238]. In the following, we restrict ourselves to *homogeneous* stored energies, i.e. $\frac{\partial W(\boldsymbol{X},\mathbb{F})}{\partial \boldsymbol{X}} = 0$, $\boldsymbol{X} \in \bar{\Omega}$. Thus we may drop the dependence on $\boldsymbol{X}$. We place the following restrictions on W:

$$W : \mathbb{R}_+^{d\times d} \to \mathbb{R} \text{ is continuous,} \tag{1.1a}$$

$$W(\mathbb{F}) = W(\mathbb{R}\mathbb{F}) \text{ for all } \mathbb{R} \in \mathrm{SO}(d) \text{ and all } \mathbb{F} \in \mathbb{R}^{d\times d}$$
$$\textit{(a consequence of material frame-indifference, see [16])}, \tag{1.1b}$$

$$W(\mathbb{F}) \to +\infty \text{ if } \det \mathbb{F} \to 0+ \ \textit{(large strain implies a large stress)}, \tag{1.1c}$$

$$W(\mathbb{F}) = +\infty \text{ if } \det \mathbb{F} \leq 0 \ \textit{(non-physical deformation).} \tag{1.1d}$$

To reflect better some material properties or for mathematical purposes (see later), one can consider more general constitutive equations, involving stored energies $\tilde{W}$ such as

$$\tilde{W} = \tilde{W}(\mathbb{F}, \underline{\mathbb{H}}),$$

where $\underline{\mathbb{H}}$ is a placeholder for $\underline{\nabla^2 \boldsymbol{y}}$. A relation such as the above one was coined by R. Toupin in [32]. Materials where the stored energy density depends on higher gradients of the motion mapping $\boldsymbol{y}$ are now referred to as non-simple materials.

1.3 Minimisation of total energy

As it was said in the beginning of Section 1.2, the motion of a material body is subject to a set of partial differential equations (PDEs).

Namely, the conservation of momentum yields the *equations of equilibrium* (in case there is no acceleration). In the reference configuration, the boundary value problem reads

$$\begin{cases} \operatorname{div} \mathbb{T}^{(1)}(\boldsymbol{X}) + \boldsymbol{f}(\boldsymbol{X}) = \boldsymbol{o}, \ \boldsymbol{X} \in \Omega \\ \boldsymbol{y} = \boldsymbol{y}_{\mathrm{Dir}} \quad \text{on } \Gamma_0 \subset \partial\Omega, \\ \boldsymbol{g} = \mathbb{T}^{(1)}\boldsymbol{n} \quad \text{on } \Gamma_1 \subset \partial\Omega. \end{cases} \tag{1.2}$$

Here $\boldsymbol{f} : \Omega \to \mathbb{R}^3$ denotes the density of applied body force (e.g. gravity), $\boldsymbol{y}_{\mathrm{Dir}} : \Gamma_0 \to \mathbb{R}^3$ a given deformation of one part of the boundary and $\boldsymbol{g} : \Gamma_1 \to \mathbb{R}^3$ the density of applied contact force.

The equations of equilibrium admit a variational formulation. If we assume that the material is hyperelastic and introduce the *total energy functional*

$$J(\boldsymbol{y}) = \int_\Omega W(\nabla \boldsymbol{y}(\boldsymbol{X}))\,\mathrm{d}\boldsymbol{X} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{y}\,\mathrm{d}\boldsymbol{X} - \int_{\Gamma_1} \boldsymbol{g} \cdot \boldsymbol{y}\,\mathrm{d}S, \tag{1.3}$$

we observe that the system is formally equivalent to

$$\delta J(\boldsymbol{y}; \boldsymbol{\vartheta}) = 0 \quad \text{for all } \boldsymbol{\vartheta} : \bar{\Omega} \to \mathbb{R}^3 \text{ smooth that vanish on } \Gamma_0. \tag{1.4}$$

If J attains a local minimum at $\boldsymbol{y}$ and $\boldsymbol{y}$ is smooth enough, then we recover the equilibrium equations from (1.4), being a necessary condition for a local extremum of J. In the other direction, (1.4) does not automatically guarantee that $\boldsymbol{y}$ is a minimiser of J. That being said, we rule out such $\boldsymbol{y}$'s as *metastable states*, which the body $\mathcal{B}$ leaves upon a small perturbation of the external forces [4]. Hence finding minima of J turns out to be a plausible alternative to solving (1.2).

In pursuance of proving the existence of a minimiser of J, we may suppose W to be a *polyconvex* function (Definition 5). A short discussion about other possible assumptions, which can replace polyconvexity, follows in Chapter 3.

1.4 Linear elasticity

In case the difference $\mathbb{F} - \mathbb{I}$ is small (the deformation gradient is not far from identity), we formally arrive at the simplified theory of *linearised elasticity*.

To explain where the theory stems from, let us define two quantities, the *displacement*

$$\boldsymbol{u}(\boldsymbol{X}) = \boldsymbol{y}(\boldsymbol{X}) - \boldsymbol{X}, \quad \boldsymbol{X} \in \overline{\Omega}, \tag{1.5}$$

and the *Green–Saint Venant strain tensor*

$$\mathbb{E}(\boldsymbol{X}) = \frac{1}{2}(\mathbb{F}^\top(\boldsymbol{X})\mathbb{F}(\boldsymbol{X}) - \mathbb{I}), \quad \boldsymbol{X} \in \overline{\Omega}. \tag{1.6}$$

The Green–Saint Venant tensor is one possible quantity which measures *strain*. Its advantage is that it equals zero when $\mathbb{F}$ is a mere rotation, which obviously does not generate strain.

Taking the gradient of (1.5) yields

$$\mathbb{F} = \mathbb{I} + \nabla \boldsymbol{u} \quad \text{in } \Omega$$

and substituting for $\mathbb{F}$ in (1.6) leads to

$$\mathbb{E} = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top + (\nabla \boldsymbol{u})^\top \nabla \boldsymbol{u}) \quad \text{in } \Omega.$$

If $\nabla \boldsymbol{u}$ is 'close to the zero matrix' (the case of so-called *small deformations*), we neglect the second-order term $(\nabla \boldsymbol{u})^\top \nabla \boldsymbol{u}$ in the above equation and instead of $\mathbb{E}$ work with the *infinitesimal strain*

$$\mathbb{\varepsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top).$$

Since for small deformations, the reference and deformed configurations have almost the same shape, we do not distinguish them and use only one *stress tensor*

$$\mathbb{\sigma} \equiv \mathbb{T}^{(1)} \approx \mathbb{T}.$$

2. Plasticity and rate-independent processes

The constitutive equations that we saw in Definitions 1 and 2 had been designated for elastic materials. Elasticity theory can describe the response of rubber-like materials (even for large deformations) or metals provided the strain is very small [16]. But if we increase the strain, different plastic phenomena can occur that are no longer tractable by mere elastic models. In this chapter, an excursion to mathematical *elastoplasticity* is given.

For an exposition on the historical development of plasticity, starting with the first paper by Tresca (1864), see Section 1.2 in [18].

2.1 Internal variables

According to Han and Reddy [18, p. 34], 'plasticity is most conveniently described in the framework of materials with internal variables'. Those material models are not only governed by usual variables, such as temperature or strain, but also incorporate several *internal variables*, which describe e.g. an ongoing chemical reaction or elastoplastic behaviour. Details can be found in [17] or in more recent works on the subject listed in [18, p. 39]. On the basis of this idea, we may borrow the stored energy density W from Definition 2 and let it depend not only on the deformation gradient, but also on κ internal variables $\boldsymbol{\xi}_1$,$\boldsymbol{\xi}_2$, ..., $\boldsymbol{\xi}_\kappa$ [28, p. 244]. Some of them may be scalars and some vectors or tensors:

$$W = W(\mathbb{F}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_\kappa).$$

Such a constitutive equation is insufficient to treat the complexity of plastic effects and must be accompanied with evolution equations of the type

$$\frac{\mathrm{d}\boldsymbol{\xi}_i}{\mathrm{d}t} = \boldsymbol{\eta}_i(\mathbb{F}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_\kappa), \quad 1 \leq i \leq \kappa.$$

2.2 Motivation: a one-dimensional example

Researchers in plasticity have amassed an astonishing amount of abstract concepts (e.g. yield surfaces, the postulate of maximum plastic work...) as well as advanced mathematical tools from convex analysis.

Not to becloud some key characteristics of elastoplastic materials from the very beginning, let us start with a simple example [18, p. 42]. Imagine a thin rod $\Omega \subset \mathbb{R}^3$, with its longest side parallel to the x_1-axis, subject to simple tension $\mathbb{T}(\boldsymbol{x}) = \sigma_{11}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$ for every $\boldsymbol{x} \in \Omega$ (Figure 2.1). Of course, this is an idealisation; in reality, exerting forces at the ends of the rod would create an inhomogeneous stress field. The symbol $\mathbb{T}$ denotes the stress tensor, as in Chapter 1. (For the moment, linearised description is used, see Section 1.4.) Gradually increasing the loading $\sigma \equiv \sigma_{11}$, we can measure the infinitesimal strain $\varepsilon \equiv \varepsilon_{11}$ incurred by the rod and plot σ against ε (Figure 2.2). As long as the stress does not exceed a certain value $\sigma_0 > 0$, the elastoplastic material obeys Hooke's law (portion $0A$ of the curve in the figure). For $\sigma > \sigma_0$, the stress-strain relation is not given by linear dependence any more and the material is affected by irreversible changes due to plastic deformation.

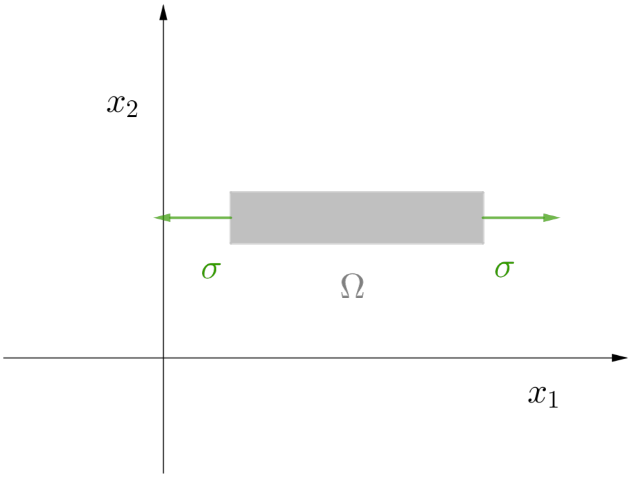

Figure 2.1: An elastoplastic rod undergoing a tension test. (All figures in this chapter were created following [18].)

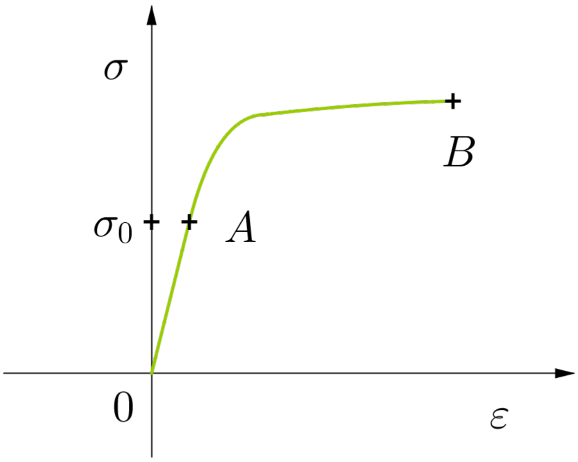

Figure 2.2: A stress-strain curve which exhibits hardening.

The effect depicted in Figure 2.2 is called *hardening*. The stress increases with increasing strain, even though the curve's slope in the segment AB is less than in the linearly elastic region.

Whereas *softening* occurs if the slope becomes negative (Figure 2.3). In reality, stress-strain curves can be more complicated, e.g. with hardening behaviour for a certain range of strains, but showing softening when the strain is larger. To name some examples, this last phenomenon happens in soil or concrete.

Another possibility is that the loaded material fractures on leaving the elastic regime.

In some applications, it is enough to consider an idealised curve with zero hardening (Figure 2.4); the case is known as *perfect plasticity*.

The value σ_0 bears the name *initial yield stress*, where the adjective 'initial' is related to the fact that the whole loading process started in a state with no stress and no strain.

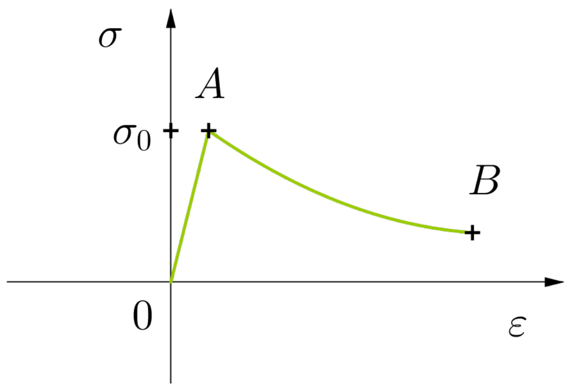

Figure 2.3: A stress-strain curve which exhibits softening.

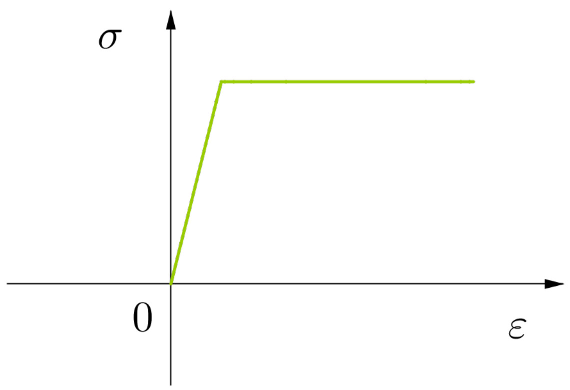

Figure 2.4: The stress-strain curve of a perfectly plastic material.

Rate-independence

To explain the assumption of *rate-independence*, let us now focus on the *rate* at which the force is applied to the body Ω [18, p. 46]. First, we can increase the stress σ slowly all the time, then the corresponding stress-strain curve will be the darkest one in Figure 2.5. Or we can perform the whole loading faster – and we get the lighter curve in the figure. We might still be dissatisfied with the speed and repeat an identical experiment once more, but with an even faster loading, resulting in the lightest curve in Figure 2.5. The message of this series of three experiments is that the material behaviour in the plastic range depended on the rate of the external force.

We may not trouble ourselves about this material property and will simply neglect it. Such *rate-independence* is a reasonable assumption for some particular materials or for the modelling of processes with low rates of external loading.

The theory of *viscoplasticity* deals with rate-dependent situations (more can be found in [23]).

Decomposition of strain

On a microscopic scale, we discern elastic behaviour from plastic effects by the nature of changes they inflict on the crystal lattice. Elastic deformation is responsible for stretch and rotation of the crystal lattice, while plastic phenomena involve the local deformation of material, caused by defects (such as dislocations) flowing through the microscopic structure [16, p. 423]. The overall strain has an elastic and a plastic component. How these components are defined in the mathematical description, depends on the preferred theory. For small deformations, linearised elastoplasticity can be used.

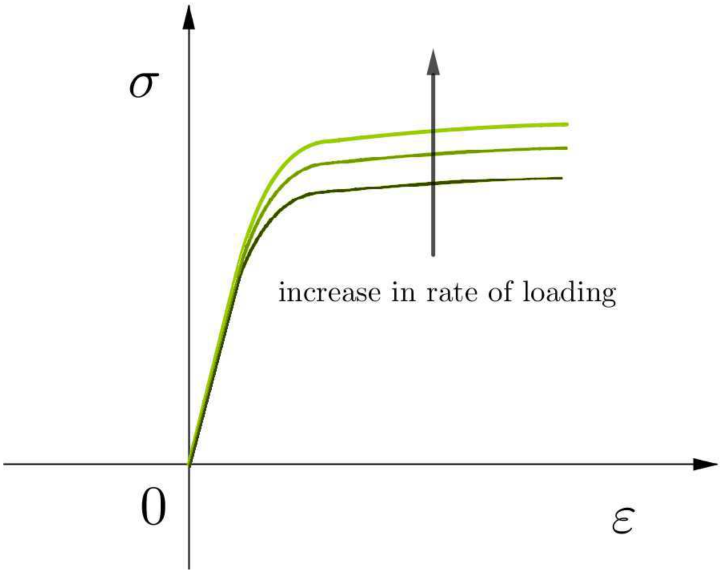

Figure 2.5: The material's response depends on the rate of loading.

There it is assumed that $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\mathrm{el}} + \boldsymbol{\varepsilon}_{\mathrm{p}}$, where $\boldsymbol{\varepsilon}_{\mathrm{el}}$ stands for the elastic component of strain and $\boldsymbol{\varepsilon}_{\mathrm{p}}$ for the plastic one. For the description of large deformations (we use the term *finite strain* or *finite elastoplasticity*), it is the deformation gradient $\mathbb{F}$, not a strain tensor, that is split, and a multiplicative decomposition is usually employed: $\mathbb{F} = \mathbb{F}_{\mathrm{el}}\mathbb{P}$ [25]. We will call $\mathbb{F}_{\mathrm{el}}$, $\mathbb{P}$ the *elastic* and *plastic distortion* (or plastic strain), respectively.

Experiments carried out on metals have shown that changes in volume happen almost exclusively in an elastic manner [18, p. 53]. Thus in finite-strain theory, it is assumed that $\det \mathbb{P} = 1$, which means that the plastic distortion preserves volume. The linearised analogy of this property is $\operatorname{Tr} \boldsymbol{\varepsilon}_{\mathrm{p}} = 0$.

Plastic flow rule

Similarly as in Section 1.2, a mathematical model of an elastoplastic material would be incomplete without constitutive relations, which determine the deformation corresponding to a given stress state. In particular, such a model must be able to recognise whether plastic deformation occurs and if so, describe it.

At the outset, let us explain why even in linearised elastoplasticity, one cannot expect any simple relations of the type $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}_{\mathrm{p}})$ or $\boldsymbol{\varepsilon}_{\mathrm{p}} = \boldsymbol{\varepsilon}_{\mathrm{p}}(\boldsymbol{\sigma})$ [18, p. 46]. In Figure 2.6, we can see what happens if the rod from 2.1 is loaded from the stress-free state into the hardening range and then unloaded on attaining $\sigma = \sigma_1 > \sigma_0$. We observe that the unloading is an elastic process, which does not follow the original path $0AB$ but a straight line segment from the point B with the same slope as $0A$. This shows that the strain induced by given stress does not only depend on this stress but also on the previous loading history.

It is visible from Figure 2.6 that the stress-strain curves can be extended to the lower half-plane; negative values of σ then express compression instead of tension. The interval $(-\sigma_0', \sigma_0)$ represents the *initial elastic range* (starting from zero stress and zero strain, the material behaves elastically for $\sigma \in (-\sigma_0', \sigma_0)$). Plastic deformations change the elastic range of the material, e.g. the deformation along the path $0AB$ enlarged the elastic range to $(-\sigma_1', \sigma_1)$.

Even though direct relations between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}_{\mathrm{p}}$ are unavailable, we can still find a

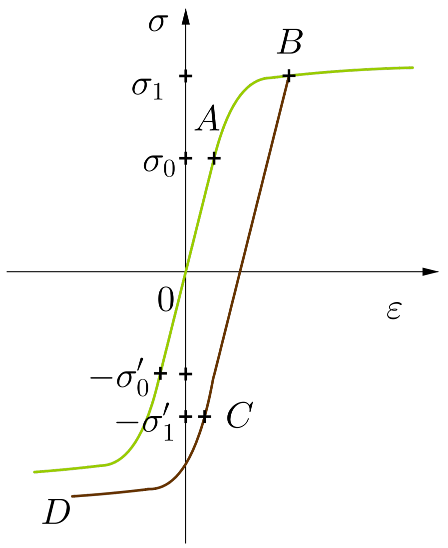

Figure 2.6: Plastic loading and elastic unloading.

law interconnecting their rates (time-derivatives). This can be done in the linearised setting as well as in large deformations. A seed of the idea is already present in our one-dimensional example. For instance, the plastic strain rate $\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_\mathrm{p}$ at the point B in Figure 2.6 depends on the sign of $\frac{\mathrm{d}}{\mathrm{d}t}\sigma$. Here $\varepsilon_\mathrm{p} \equiv (\mathbb{E}_\mathrm{p})_{11}$. If the loading σ is increasing at $\sigma = \sigma_1$, plastic deformation will take place. If $\frac{\mathrm{d}}{\mathrm{d}t}\sigma < 0$, we expect an elastic unloading along the line BC. As put together in [18, p. 47],

$$\frac{\mathrm{d}\varepsilon_\mathrm{p}}{\mathrm{d}t} = 0 \text{ if } \begin{cases} & \sigma \in (-\sigma_1', \sigma_1) \\ \text{or} & \sigma = \sigma_1 \text{ and } \frac{\mathrm{d}\sigma}{\mathrm{d}t} < 0 \\ \text{or} & \sigma = -\sigma_1' \text{ and } \frac{\mathrm{d}\sigma}{\mathrm{d}t} > 0 \end{cases} \tag{2.1}$$

and

$$\frac{\mathrm{d}\varepsilon_\mathrm{p}}{\mathrm{d}t} = \frac{1}{h}\frac{\mathrm{d}\sigma}{\mathrm{d}t} \text{ if } \begin{cases} & \sigma = \sigma_1 \text{ and } \frac{\mathrm{d}\sigma}{\mathrm{d}t} > 0 \\ \text{or} & \sigma = -\sigma_1' \text{ and } \frac{\mathrm{d}\sigma}{\mathrm{d}t} < 0. \end{cases} \tag{2.2}$$

The positive variable h measures the degree of hardening. Such relations as (2.1)–(2.2) (and their generalisations) are called a *plastic flow rule.*

2.3 Three-dimensional plasticity with internal variables

In three dimensions of space, there are many other types of loading than simple tension (shear, torsion...). The caused stress is described by the stress tensor $\mathbb{T}$ in the linearised theory or the Cauchy stress tensor $\mathbb{T}$ at large strains. It is expectable that the plastic flow rule will involve tensorial quantities, too.

We noticed in Section 2.2 that elastoplastic constitutive equations must take into account previous irreversible changes incurred by the material. Therefore we introduce a set of internal variables $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_\kappa$, which are scalars or second-order tensors and characterise e.g. hardening.

Further we accompany the set of internal variables by *internal forces* $\boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \ldots, \boldsymbol{\chi}_\kappa$ that ‘are generated as a result of the internal restructuring that occurs during plastic deformation’ [18, p. 49].

Then the $(\kappa+1)$-tuple $\underline{\underline{\Sigma}} := (\mathbb{\sigma}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_\kappa)^\top$ can be thought of as the *generalised stress* and $\underline{\underline{\mathbb{P}}} := (\mathbb{\varepsilon}_\mathrm{p}, \boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \ldots, \boldsymbol{\chi}_\kappa)^\top$ as the *generalised plastic strain*.

In the one-dimensional example from Section 2.2, the elastic range was simply described by an interval, say $(-\sigma_0', \sigma_0) \subset \mathbb{R}$. If the stress tensor has more nonzero components than one, a more general sort of elastic domain will arise. In particular, we assign to the studied material a closed convex set $\mathcal{K}$ in the space of generalised stresses so that [18, p. 54, 83]:

- for $\underline{\underline{\Sigma}} \in \operatorname{Int} \mathcal{K}$, all effects are purely elastic,
- for $\underline{\underline{\Sigma}} \in \partial \mathcal{K}$, plastic phenomena can occur,
- the complement of the set $\mathcal{K}$ is unreachable.

The interior $\operatorname{Int} \mathcal{K}$ of $\mathcal{K}$ called the *elastic region* and the boundary $\partial \mathcal{K}$ is known as the *yield surface*. These terms are used in finite-strain elastoplasticity as well.

The plastic flow rule can be stated in several forms. One of its versions in linearised plasticity, which resembles (2.1)–(2.2), is (see [18, p. 85])

$$\mathcal{K} \text{ closed, convex, contains } \boldsymbol{o},$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \underline{\underline{\mathbb{P}}} \in \mathrm{N}_{\mathcal{K}}(\underline{\underline{\Sigma}}).$$

A similar flow law can be found in [26, Equation (2.5)], where finite-strain plasticity is considered. By $\mathrm{N}_{\mathcal{K}}(\underline{\underline{\Sigma}})$ we mean the *normal cone* to the convex set $\mathcal{K}$ at $\underline{\underline{\Sigma}} \in \mathcal{K}$. It generalises the notion of a normal vector for sets with nonsmooth boundaries. The precise definition is given e.g. in [18, p. 72], where basic results from convex analysis are also summarised. For $\underline{\underline{\Sigma}} \in \operatorname{Int} \mathcal{K}$, we have $\mathrm{N}_{\mathcal{K}}(\underline{\underline{\Sigma}}) = \{\boldsymbol{o}\}$, so indeed no plastic deformation takes place in the elastic region.

Let us finish our outline of elastoplasticity by briefly mentioning *gradient plasticity*. In those models, gradients of plastic variables (i.e. $\{\boldsymbol{\xi}_i\}_{i=1}^{\kappa}$, $\mathbb{P}$) appear. They provide compactness and prevent formation of microstructures [25]. This idea is common in the engineering literature [28, p. 250].

Not including gradients of plastic variables hinders existence proofs for *energetic solutions* (see below for a definition) [25, Sec. 1, 6].

2.4 Energetic solutions

In finite-strain elastoplastic processes, discontinuities in space or time can develop [26]. Hence it is useful to seek a weaker concept of solution to elastoplastic problems, which could encompass discontinuities, yet remain physically admissible. One contribution in this direction is *energetic solutions* (let us name [29] as an early reference; other related sources are cited in [28]).

We saw in Chapter 1 that a natural way how to sidestep derivatives (whereby, the regularity of solutions) is a variational formulation. Energetic solutions have certain functionals as their cornerstones, too – these are the total energy functional and the dissipation distance. Without specifying their form, let us content ourselves with noting that:

- $\mathcal{E}\colon [0,T] \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}_\infty$ is the total energy functional, similar to J in (1.3), but depending on time $t \in [0,T]$ through a time-evolving loading. The set $\mathcal{Y}$ contains all admissible deformations of the material body (e.g. satisfying some boundary condition) and $\mathcal{Z}$ is the space of attainable plastic deformations, which are characterised by $\mathbb{P}$ and internal variables.

- $\mathcal{D}\colon \mathcal{Z} \times \mathcal{Z} \to [0, +\infty]$ is the dissipation distance, which measures the minimal amount of energy that is dissipated by changing the internal state from one value in the space $\mathcal{Z}$ to another.

A state $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}) \in \mathcal{Y} \times \mathcal{Z} =: \mathcal{Q}$ of the material body $\overline{\Omega}$ supplies information about the deformation $\hat{\boldsymbol{y}}\colon \overline{\Omega} \to \mathbb{R}^d$ and the internal state $\hat{\boldsymbol{z}} \in \mathcal{Z}$ that groups together the plastic distortion $\hat{\mathbb{P}}$ and the values of internal variables $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_\kappa$. It is customary to call the triple $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ an energetic rate-independent system, which indicates that the theory is developed under the assumption of rate-independent behaviour. Both definitory properties (2.S), (2.E) of energetic solutions are compatible with rate-independence, as explained in [28, p. 25].

Definition 3. *A function* $(\boldsymbol{y}, \boldsymbol{z})\colon [0, T] \to \mathcal{Q}$ *is called an* energetic solution *of the energetic rate-independent system* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ *if it satisfies the* stability condition (2.S) *and the* energy balance (2.E) *for all* $t \in [0, T]$

$$\forall\, (\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) \in \mathcal{Y} \times \mathcal{Z}: \quad \mathcal{E}(t, \boldsymbol{y}(t), \boldsymbol{z}(t)) \le \mathcal{E}(t, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) + \mathcal{D}(\boldsymbol{z}(t), \tilde{\boldsymbol{z}}), \tag{2.S}$$

$$\mathcal{E}(t, \boldsymbol{y}(t), \boldsymbol{z}(t)) + \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{z}; [0, t]) = \mathcal{E}(0, \boldsymbol{y}(0), \boldsymbol{z}(0)) + \int_0^t \partial_t \mathcal{E}(\tau, \boldsymbol{y}(\tau), \boldsymbol{z}(\tau))\mathrm{d}\tau. \tag{2.E}$$

The dissipation $\mathrm{Diss}_{\mathcal{D}}(\boldsymbol{z}; [t, t'])$, $t, t' \in [0, T]$, $t < t'$, along a part of the curve $\boldsymbol{z}$ is defined analogously to the total variation (Definition A.2):

$$\mathrm{Diss}_{\mathcal{D}}(\boldsymbol{z}; [t, t']) = \sup\Big\{\sum_{k=1}^{N} \mathcal{D}(\boldsymbol{z}(t_{k-1}), \boldsymbol{z}(t_k));\ N \in \mathbb{N},\, t = t_0 < \cdots < t_N = t'\Big\}.$$

One advantage of the energetic formulation is that it avoids derivatives of constitutive equations and of the solution itself [26]. The usual form of the involved functionals also enables utilising known methods of the calculus of variations, including homogenisation and relaxation (see Chapter 3 for a short note on the latter).

The mechanical idea behind (2.S) is the following: imagine first that $\tilde{\boldsymbol{z}} := \boldsymbol{z}(t)$, then $\mathcal{D}(\boldsymbol{z}(t), \tilde{\boldsymbol{z}})$ vanishes, since no change in the plastic variables implies no dissipation. Consequently, (2.S) simplifies to $\mathcal{E}(t, \boldsymbol{y}(t), \boldsymbol{z}(t)) \le \mathcal{E}(t, \tilde{\boldsymbol{y}}, \boldsymbol{z}(t))$ for all $\tilde{\boldsymbol{y}} \in \mathcal{Y}$ and $\boldsymbol{y}$ is a global minimiser of $\mathcal{E}(t, \cdot, \boldsymbol{z}(t))$ over $\mathcal{Y}$. So we observe that in this case, (2.S) expresses the elastic equilibrium (cf. Chapter 1) [26]. If $\tilde{\boldsymbol{z}} \neq \boldsymbol{z}(t)$, then the amount of dissipated energy between the states $\tilde{\boldsymbol{z}}$ and $\boldsymbol{z}(t)$ must, by (2.S), at least compensate for, if not outweigh the associated loss in the total energy. Such a property is known as the *principle of maximum dissipation* and is nowadays a widely accepted part of theories of elastoplasticity [18, p. 57].

The last term in (2.E) has the meaning of the work done by external forces, hence a physical interpretation of (2.E) is also available.

In [26], it is shown that for sufficiently smooth solutions and with an appropriate choice of $\mathcal{E}$, $\mathcal{D}$, one can derive from (2.S), (2.E) a weak formulation of the elastic equilibrium equations and a plastic flow rule.

It is worth noticing that the dissipation distance $\mathcal{D}$ is not supposed to be symmetric, as it would contradict hardening [25].

Examples of total energy functionals $\mathcal{E}$ and dissipation distances $\mathcal{D}$ in finite-strain elastoplasticity are brought forward e.g. in [26], [25] and for the case of a *gradient polyconvex* energy functional, in Chapter 4.

3. Gradient polyconvexity

In 2018 B. Benešová, M. Kružík and A. Schlömerkemper introduced *gradient polyconvexity.* To provide some motivation for this mathematical property, this short chapter serves as an excerpt from the article [5], where the concept was presented for the first time.

The authors proceed from the variational formulation of equilibrium equations for a hyperelastic material, which is a common starting point in modern nonlinear elasticity. As pointed out in the previous chapter, it is important to have a property of W that would ensure weak sequential lower semicontinuity of the total energy functional, like J in Equation (1.3), so that existence of minimisers can be proved. Usually, the property is a variation on the theme of convexity (e.g. polyconvexity, quasiconvexity. . .).

Gradient polyconvexity is indeed a property of the stored energy that guarantees a successful minimisation process. Roughly speaking, it means 'convexity in the gradient of $\operatorname{cof}\mathbb{F}$ (and optionally, in the gradient of $\det\mathbb{F}$)', where $\mathbb{F} = \nabla\boldsymbol{y}$ is the deformation gradient.

The exact definition in $\mathbb{R}^3$ (the most physically meaningful case) goes as follows:

Definition 4 (gradient polyconvexity)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Let $\hat{W} : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3\times3} \times \mathbb{R}^3 \to \mathbb{R}_\infty$ be a lower semicontinuous function. The functional*

$$I(\boldsymbol{y}) = \int_\Omega \hat{W}(\nabla\boldsymbol{y}(\boldsymbol{X}), \underline{\nabla[\operatorname{cof}\nabla\boldsymbol{y}(\boldsymbol{X})]}, \nabla[\det\nabla\boldsymbol{y}(\boldsymbol{X})])\mathrm{d}\boldsymbol{X}, \tag{3.1}$$

defined for any measurable function $\boldsymbol{y}\colon \Omega \to \mathbb{R}^3$ for which the weak derivatives $\nabla\boldsymbol{y}$, $\underline{\nabla[\operatorname{cof}\nabla\boldsymbol{y}]}$, $\nabla[\det\nabla\boldsymbol{y}]$ exist and are integrable, is called gradient polyconvex *if the function $\hat{W}(\mathbb{F},\cdot,\cdot)$ is convex for every $\mathbb{F} \in \mathbb{R}^{3\times3}$.*

The most mathematically important attribute is the dependence on $\underline{\nabla\operatorname{cof}\nabla\boldsymbol{y}}$ (the $\nabla\det\nabla\boldsymbol{y}$ can be dropped for the purpose of minimisation). Although e.g. $\hat{W} \equiv 1$ satisfies the above definition, in the premises of existence theorems there are coercivity conditions, which exclude such degenerate energies.

The term 'gradient polyconvexity' originates from *polyconvexity*, which J. M. Ball proposed to achieve existence results applicable to various models in nonlinear elasticity [1].

Definition 5 (polyconvexity)**.** [1] *We call a function $W\colon \mathbb{R}^{3\times3} \to \mathbb{R}_\infty$* polyconvex *if there is a convex function $h\colon \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R} \to \mathbb{R}_\infty$ such that*

$$W(\mathbb{F}) = h(\mathbb{F}, \operatorname{cof}\mathbb{F}, \det\mathbb{F}) \textit{ for every } \mathbb{F} \in \mathbb{R}^{3\times3}.$$

Let us mention that gradient polyconvexity could not be defined for the integrand $\hat{W} = \hat{W}(\mathbb{F}, \dots)$ but required a functional, since taking the spatial gradient of $\operatorname{cof}\mathbb{F}$, $\det\mathbb{F}$ needs $\mathbb{F}$ to be a tensor field $\mathbb{F}\colon \Omega \to \mathbb{R}^{3\times3}$, not just a matrix $\mathbb{F} \in \mathbb{R}^{3\times3}$.

In some situations, polyconvexity is not a good assumption, though [20]. To give an example, in the modelling of shape-memory alloys (SMA), multiwell stored energies of the form

$$W(\mathbb{F}) \begin{cases} = 0 & \text{if } \mathbb{F} = \mathbb{R}\mathbb{F}_i \text{ for some } i \in \{1, \dots, m\} \text{ and some } \mathbb{R} \in \mathrm{SO}(3), \\ > 0 & \text{otherwise,} \end{cases}$$

[1] This is not Ball's original definition, but a modified one, in order to enable easy comparison with gradient polyconvexity while retaining the definition's idea.

are encountered [4]. Such energies, in general, fail to be polyconvex. The matrices $\mathbb{F}_1, \ldots, \mathbb{F}_m \in \mathbb{R}^{3\times 3}$ represent different crystallographic variants of the material.

There exist alternative assumptions on W to polyconvexity that still lead to the existence of minimisers of the total energy functional. For instance, if a finite W is polyconvex, then it is *quasiconvex* (see Definition A.7), but not vice versa (quasiconvexity is more general in this case). Quasiconvexity, as well as polyconvexity, ensures the weak seq. lower semicontinuity of the functional to be minimised, as was shown by Morrey, Acerbi, Fusco and others (see [8, p. 369]):

Theorem 1. [2] *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set and $1 \le p < +\infty$. Let $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ be quasiconvex and satisfying the growth condition*

$$0 \le W(\mathbb{F}) \le C(1 + |\mathbb{F}|^p) \text{ for every } \mathbb{F} \in \mathbb{R}^{3\times 3},$$

where $C \ge 0$. Then $I(\boldsymbol{y}) := \int_\Omega W(\nabla \boldsymbol{y}(\boldsymbol{X}))\mathrm{d}\boldsymbol{X}$ is weakly sequentially lower semicontinuous in $W^{1,p}(\Omega; \mathbb{R}^3)$.

However, it is clear from the assumptions that W must not escape to infinity as in (1.1c), not to mention that multiwell energies of SMA are generally not even quasiconvex.

Interestingly, it is unknown up to now if quasiconvex energies with property (1.1d) guarantee existence of minimisers [5].

One can also regularise multiwell energies by adding a dependence on $\|\underline{\nabla^2 \boldsymbol{y}}\|_{L^p}$ (Toupin's model in Section 1.2). This is beneficial from the mathematical point of view, as 'differentiability brings compactness' and it is easier to pass to the limit in terms depending on the deformation gradient thanks to strong convergence. Gradient polyconvexity shows that bounding the whole $\underline{\nabla^2 \boldsymbol{y}}$ in some L^p is not necessary – it is enough to suppose the integrability of the distributional derivatives of certain *polynomials in the components of* $\nabla \boldsymbol{y}$ (the polynomials are $\det \nabla \boldsymbol{y}$ and the components of $\operatorname{cof} \nabla \boldsymbol{y}$). Existence of minimisers, even with so-called *locking constraints*, is obtained, see later sections of this chapter.

To name one last approach to non-quasiconvex energies, we remark that *relaxation* techniques replace the original energy by its quasiconvex envelope, which is a problem on its own, with difficulties and limitations [8].

3.1 Interpretation

The stored energy of a gradient polyconvex functional can depend on $\nabla \det \mathbb{F}$. Together with the dependence on $\underline{\nabla \operatorname{cof} \mathbb{F}}$, it creates a model with a possible physical interpretation. Since $\det \mathbb{F}$ measures the local change in volume between the reference and current configuration[3] and similarly, $\operatorname{cof} \mathbb{F}$ describes the transformation of area [16, p. 78], the stored energy W depending on their gradients offers a control of how abruptly these changes vary in space.

Remarkably, Ball also writes in his paper [1] that $\mathbb{F}$ governs the deformation of line elements [16, p. 65] and is aware of the geometric meaning of minors of $\mathbb{F}$ when defining polyconvexity.

[2] The result is only stated here for illustration and could be strengthened.
[3] Substitution theorem for multiple integrals.

3.2 Advantages

The definition of gradient polyconvexity admits a nonconvex dependence of $\hat{W}$ on the deformation gradient. This is important in applications to nonlinear elasticity[4].

Another advantage of gradient polyconvex stored energies is that they allow for general *locking constraints*. Material locking expresses the physical body's resistance against full compression or large strain in general. This is an idea proposed by W. Prager.

Gradient polyconvex energies are easy to construct, too. For example, the model of the Saint Venant–Kirchhoff material can be adjusted in such a way that it is gradient polyconvex (Example 5.2 in [5]).

3.3 In general dimension

The previous discussion was taking place in the case that $\boldsymbol{y}\colon \Omega \to \mathbb{R}^3$, $\Omega \subset \mathbb{R}^3$. Now let us investigate what changes if we consider $\boldsymbol{y}\colon \Omega \to \mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$.

In the scalar case $d = 1$, we have $y' = \nabla y = \det \nabla y$ and $\operatorname{cof} \nabla y \overset{\mathsf{def}}{=} 1$. Thus $\nabla \det \nabla y = y''$, $\nabla \operatorname{cof} \nabla y \equiv 0$ and $\hat{W}$ is, in fact, convex in y''.

In case $d = 2$, it is pointed out in [5] that

$$\operatorname{cof} \mathbb{F} = \begin{pmatrix} F_{22} & -F_{21} \\ -F_{12} & F_{11} \end{pmatrix},$$

so, as well as if $d = 1$, the functional I from Def. 4 depends on $\underline{\nabla^2 \boldsymbol{y}}$.

For $d > 3$, the authors of [5] suggest maintaining the dependence of I on $\underline{\nabla \operatorname{cof} \nabla \boldsymbol{y}}$. It would be theoretically possible to involve gradients of minors of any order k, $1 \leq k \leq d-1$, were it useful for some purpose.

3.4 Regularity of deformations

We have noticed that if $d = 1$ or $d = 2$, there is no difference between deformations of class $W^{2,p}(\Omega; \mathbb{R}^d)$ (for some $p \in [1, +\infty]$) and elastic deformations y entering a gradient polyconvex functional (with a suitable choice of the Sobolev exponents for the minors).

Nonetheless, for $d > 2$ this is not the case. First, let us observe that $W^{2,p}$-deformations y have Sobolev $\operatorname{cof} \nabla \boldsymbol{y}$ and $\det \nabla \boldsymbol{y}$, provided certain conditions hold.

Claim 2. *Let $d > 2$, $\Omega \subset \mathbb{R}^d$, $\Omega \in \mathcal{C}^{0,1}$ and $\boldsymbol{y} \in W^{2,q_{\mathrm{s}}}(\Omega; \mathbb{R}^d)$, $1 \leq q_{\mathrm{s}} < +\infty$. Further, assume that $\Gamma \subset \partial\Omega$, $\mathcal{H}^{d-1}(\Gamma) > 0$, $\boldsymbol{y}_{\mathrm{Dir}} \in W^{2-\frac{1}{q_{\mathrm{s}}}, q_{\mathrm{s}}}(\Gamma; \mathbb{R}^d)$ and $\boldsymbol{y}\big|_{\Gamma} = \boldsymbol{y}_{\mathrm{Dir}}$ in the trace sense.*

(a) Suppose that $\frac{d^2}{2d-1} \leq q_{\mathrm{s}} < d$, $q_{\mathrm{g}} := \frac{dq_{\mathrm{s}}}{d(d-1) - q_{\mathrm{s}}(d-2)} \geq 1$, $q_{\mathrm{Y}} := \frac{dq_{\mathrm{s}}}{d - q_{\mathrm{s}}}$, $\tilde{q}_{\mathrm{D}} := \frac{dq_{\mathrm{s}}}{d^2 - q_{\mathrm{s}}(d-1)} \geq 1$ and $q_{\mathrm{D}} := \min\{\tilde{q}_{\mathrm{D}}, \frac{1}{d}\frac{dq_{\mathrm{s}} - q_{\mathrm{s}}}{d - q_{\mathrm{s}}}\}$. Then $\boldsymbol{y} \in W^{1,q_{\mathrm{Y}}}(\Omega; \mathbb{R}^d)$, $\operatorname{cof} \nabla \boldsymbol{y} \in W^{1,q_{\mathrm{g}}}(\Omega; \mathbb{R}^d)$ and $\det \nabla \boldsymbol{y} \in W^{1,q_{\mathrm{D}}}(\Omega)$.

(b) If $q_{\mathrm{s}} = d$ and if $q_{\mathrm{Y}} \in [1, +\infty)$, $q_{\mathrm{g}} \in [1, d)$, $q_{\mathrm{D}} \in [1, d)$ are arbitrary, then $\boldsymbol{y} \in W^{1,q_{\mathrm{Y}}}(\Omega; \mathbb{R}^d)$, $\operatorname{cof} \nabla \boldsymbol{y} \in W^{1,q_{\mathrm{g}}}(\Omega; \mathbb{R}^d)$ and $\det \nabla \boldsymbol{y} \in W^{1,q_{\mathrm{D}}}(\Omega)$.

(c) If $q_{\mathrm{s}} > d$, then $\boldsymbol{y} \in W^{1,\infty}(\Omega; \mathbb{R}^d)$, $\operatorname{cof} \nabla \boldsymbol{y} \in W^{1,q_{\mathrm{s}}}(\Omega; \mathbb{R}^d)$ and $\det \nabla \boldsymbol{y} \in W^{1,q_{\mathrm{s}}}(\Omega)$.

---

[4] Unquestionably, convexity in the deformation gradient leads to weakly sequentially lower semicontinuous functionals as well – and it can be proved even more easily than e.g. for quasiconvex integrands. Yet, convexity is excluded by (1.1d) in elasticity.

Proof. Jacobi's formula ([6, p. 20]) says that for any $\mathbb{A} \in \mathrm{I\!R}^{d\times d}$

$$\frac{\partial \det \mathbb{A}}{\partial \mathbb{A}} = \operatorname{cof} \mathbb{A}. \tag{3.2}$$

If $\hat{\mathbb{F}} \in \mathcal{C}^\infty(\Omega; \mathrm{I\!R}^{d\times d})$, then

$$\frac{\partial}{\partial X_i} \det \hat{\mathbb{F}} = \sum_{j,k=1}^{d} (\operatorname{cof} \hat{\mathbb{F}})_{jk} \frac{\partial \hat{F}_{jk}}{\partial X_i}, \quad i = 1, 2, \dots, d \text{ and}$$

$$\frac{\partial}{\partial X_i} (\operatorname{cof} \hat{\mathbb{F}})_{jk} = \sum_{l,m=1}^{d} \frac{\partial (\operatorname{cof} \hat{\mathbb{F}})_{jk}}{\partial F_{lm}} \frac{\partial \hat{F}_{lm}}{\partial X_i}$$

by the chain rule and (3.2). The symbol $\frac{\partial (\operatorname{cof} \mathbb{F})_{jk}}{\partial F_{lm}}$ denotes the partial derivative of the mapping $\mathbb{F} \mapsto (\operatorname{cof} \mathbb{F})_{jk}$ with respect to the (l, m)-th component of $\mathbb{F} \in \mathrm{I\!R}^{d\times d}$. For any $\tilde{\boldsymbol{y}} \in \mathcal{C}^\infty(\Omega; \mathrm{I\!R}^d)$ this gives

$$\frac{\partial}{\partial X_i} \det \nabla \tilde{\boldsymbol{y}} = \sum_{j,k=1}^{d} (\operatorname{cof} \nabla \tilde{\boldsymbol{y}})_{jk} \frac{\partial^2 \tilde{y}_j}{\partial X_k \partial X_i}, \quad i = 1, 2, \dots, d \text{ and} \tag{3.3a}$$

$$\frac{\partial}{\partial X_i} (\operatorname{cof} \nabla \tilde{\boldsymbol{y}})_{jk} = \sum_{l,m=1}^{d} \frac{\partial (\operatorname{cof} \nabla \tilde{\boldsymbol{y}})_{jk}}{\partial F_{lm}} \frac{\partial^2 \tilde{y}_l}{\partial X_m \partial X_i}. \tag{3.3b}$$

We need to extend formulas (3.3a)–(3.3b) to $\tilde{\boldsymbol{y}} \in W^{2,q_\mathrm{s}}(\Omega; \mathrm{I\!R}^d)$ as mentioned (without a proof) in [5]. Let $\boldsymbol{y}_\varepsilon$ be the standard mollification of $\boldsymbol{y}$.[5] Then $\boldsymbol{y}_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon; \mathrm{I\!R}^d)$, $\Omega_\varepsilon := \{\boldsymbol{X} \in \Omega;\ \operatorname{dist}(\boldsymbol{X}, \partial\Omega) > \varepsilon\}$, and $\boldsymbol{y}_\varepsilon \to \boldsymbol{y}$, $\nabla \boldsymbol{y}_\varepsilon \to \nabla \boldsymbol{y}$ a.e. in Ω as $\varepsilon \to 0+$ [12, p. 250, 629–631]. For $\boldsymbol{y}_\varepsilon$ in place of $\tilde{\boldsymbol{y}}$, (3.3a)–(3.3b) are valid. Integration by parts and Lebesgue's dominated convergence theorem prove the equalities for $\boldsymbol{y}$ in the sense of weak derivatives, since

$$\begin{array}{ccccc}
-\displaystyle\int_\Omega (\det \nabla \boldsymbol{y}_\varepsilon) \frac{\partial \varphi}{\partial X_i} \mathrm{d}\boldsymbol{X} & = \displaystyle\int_\Omega \left(\frac{\partial (\det \nabla \boldsymbol{y}_\varepsilon)}{\partial X_i} \right) \varphi \mathrm{d}\boldsymbol{X} & = & \displaystyle\int_\Omega \left(\sum_{j,k=1}^{d} (\operatorname{cof} \nabla \boldsymbol{y}_\varepsilon)_{jk} \frac{\partial^2 (\boldsymbol{y}_\varepsilon)_j}{\partial X_k \partial X_i} \right) \varphi \mathrm{d}\boldsymbol{X} \\
\downarrow & & & \downarrow \\
-\displaystyle\int_\Omega (\det \nabla \boldsymbol{y}) \frac{\partial \varphi}{\partial X_i} \mathrm{d}\boldsymbol{X} & & & \displaystyle\int_\Omega \left(\sum_{j,k=1}^{d} (\operatorname{cof} \nabla \boldsymbol{y})_{jk} \frac{\partial^2 y_j}{\partial X_k \partial X_i} \right) \varphi \mathrm{d}\boldsymbol{X}
\end{array}$$

with $\varepsilon \to 0+$ and for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $i \in \{1, 2, \dots, d\}$. The integrable majorant for the left-hand side could be found by Hadamard's determinant inequality (see e.g. [22]) and the property $||\nabla \boldsymbol{y}_\varepsilon||_d \le C_\mathrm{m} ||\nabla \boldsymbol{y}||_d$ for some constant $C_\mathrm{m} > 0$ independent of ε ($\nabla \boldsymbol{y} \in L^d(\Omega; \mathrm{I\!R}^{d\times d})$ because $q_\mathrm{s} \ge \frac{d^2}{2d-1} \ge \frac{d}{2}$ implies $W^{1,q_\mathrm{s}}(\Omega; \mathrm{I\!R}^{d\times d}) \hookrightarrow L^d(\Omega; \mathrm{I\!R}^{d\times d})$). The integrable majorant for the right-hand side could be found by general Hölder's inequality ([12, p. 623], $q_\mathrm{s} \ge \frac{d^2}{2d-1}$ ensures that $\frac{d-1}{q_\mathrm{Y}} + \frac{1}{q_\mathrm{s}} \le 1$). Analogously for (3.3b) (here, the restrictions on Lebesgue exponents are weaker, since at most $(d-1)$-fold products of partial derivatives of $\boldsymbol{y}$ are involved). In this way, we have shown that the equalities hold for $\boldsymbol{y}$ in lieu of $\tilde{\boldsymbol{y}}$.

[5] $\boldsymbol{y}_\varepsilon(\boldsymbol{X}) := \int_{\mathrm{I\!R}^d} \eta_\varepsilon(\boldsymbol{X} - \boldsymbol{X}') \boldsymbol{y}(\boldsymbol{X}') \mathrm{d}\boldsymbol{X}'$, where $\eta_\varepsilon(\boldsymbol{X}) = 0$ for $|\boldsymbol{X}| \ge \varepsilon$ and $\eta_\varepsilon(\boldsymbol{X}) = C_\varepsilon \exp(-\frac{1}{1-|\boldsymbol{X}|^2/\varepsilon^2})$ otherwise; the positive constant C_ε is chosen so that $||\eta_\varepsilon||_1 = 1$, $\varepsilon > 0$.

Now we focus on the regularity of $\boldsymbol{y}$, $\operatorname{cof}\nabla\boldsymbol{y}$ and $\det\nabla\boldsymbol{y}$. The continuous embedding of $W^{2,q_\mathrm{s}}(\Omega;\mathbb{R}^d)$ into $W^{1,q_\mathrm{Y}}(\Omega;\mathbb{R}^d)$ gives $\boldsymbol{y}\in W^{1,q_\mathrm{Y}}(\Omega;\mathbb{R}^d)$ [3, Th. 3.6–3.8]. By general Hölder's inequality and (3.3b), we get $\underline{\nabla\operatorname{cof}\nabla\boldsymbol{y}}\in L^{q_\mathrm{g}}(\Omega;\mathbb{R}^{d\times d\times d})$ with

$$q_\mathrm{g}\le\frac{q_\mathrm{Y}q_\mathrm{s}}{q_\mathrm{Y}+q_\mathrm{s}(d-2)},\tag{3.4}$$

since the condition to be verified was $\frac{1}{q_\mathrm{s}}+\frac{d-2}{q_\mathrm{Y}}\le\frac{1}{q_\mathrm{g}}$. Note that if $q_\mathrm{s}\ge d$, the property $q_\mathrm{g}\ge 1$ follows automatically, but for $q_\mathrm{s}<d$, this has to be assumed. The exact values of q_g are obtained by substituting for q_Y in (3.4).

Since $\nabla\boldsymbol{y}\in W^{1,q_\mathrm{s}}(\Omega;\mathbb{R}^{d\times d})$ and either $q_\mathrm{s}\ge d$ or $\frac{dq_\mathrm{s}-q_\mathrm{s}}{d-q_\mathrm{s}}\ge(d-1)q_\mathrm{g}$, we have $(\nabla\boldsymbol{y})\big|_\Gamma\in L^{(d-1)q_q}(\Gamma;\mathbb{R}^{d\times d})$ on account of the trace theorem (see [28, p. 601]). Thus $(\operatorname{cof}\nabla\boldsymbol{y})\big|_\Gamma\in L^{q_\mathrm{g}}(\Gamma;\mathbb{R}^{d\times d})$ by Hölder's inequality for surface integrals. Furthermore, Poincaré's inequality

$$||\operatorname{cof}\nabla\boldsymbol{y}||_{W^{1,q_\mathrm{g}}(\Omega;\mathbb{R}^{d\times d})}\le C_\mathrm{Poi}\sqrt[q_\mathrm{g}]{\int_\Gamma|\operatorname{cof}\nabla\boldsymbol{y}_\mathrm{Dir}|^{q_\mathrm{g}}\mathrm{d}S+\int_\Omega\left|\underline{\nabla\operatorname{cof}\nabla\boldsymbol{y}}\right|^{q_\mathrm{g}}\mathrm{d}\boldsymbol{X}}$$

with a constant $C_\mathrm{Poi}>0$ shows that $\operatorname{cof}\nabla\boldsymbol{y}\in W^{1,q_\mathrm{g}}(\Omega;\mathbb{R}^{d\times d})$.

To prove that $\nabla\det\nabla\boldsymbol{y}\in L^{q_\mathrm{D}}(\Omega;\mathbb{R}^d)$, we discuss cases (a), (b), (c) separately.

In (a), we have two possibilities how to apply Hölder's inequality to (3.3a) (its absolute value being raised to the power $\tilde{q}_\mathrm{D}$ and integrated over Ω). We may either regard $(\operatorname{cof}\nabla\boldsymbol{y})_{jk}$, $j,k\in\{1,2,\dots,d\}$, as a product of partial derivatives of $\boldsymbol{y}$, or take into account the continuous embedding of $W^{1,q_\mathrm{g}}(\Omega;\mathbb{R}^{d\times d})$ so that $(\operatorname{cof}\nabla\boldsymbol{y})_{jk}\in L^{q_\mathrm{v}}(\Omega)$, where $q_\mathrm{v}=\frac{dq_\mathrm{g}}{d-q_\mathrm{g}}=\frac{dq_\mathrm{s}}{d(d-1)-q_\mathrm{s}(d-1)}$, as $q_\mathrm{g}<d$. We remark that $q_\mathrm{v}\ge 1$ thanks to $q_\mathrm{g}\ge 1$. In the first situation, we obtain a Hölder exponent $\tilde{q}_{\mathrm{D}1}$ satisfying $\frac{1}{\tilde{q}_{\mathrm{D}1}}=\frac{1}{q_\mathrm{s}}+\frac{d-1}{q_\mathrm{Y}}$, or equivalently, $\tilde{q}_{\mathrm{D}1}=\frac{dq_\mathrm{s}}{d^2-dq_\mathrm{s}+q_\mathrm{s}}\ge 1$. In the second situation, there is another exponent $\tilde{q}_{\mathrm{D}2}$ with $\frac{1}{\tilde{q}_{\mathrm{D}2}}=\frac{1}{q_\mathrm{s}}+\frac{1}{q_\mathrm{v}}$, which means that $\tilde{q}_{\mathrm{D}2}=\frac{dq_\mathrm{s}}{d^2-q_\mathrm{s}(d-1)}$. We observe that the two exponents $\tilde{q}_{\mathrm{D}1}$, $\tilde{q}_{\mathrm{D}2}$ are equal. Altogether, $\nabla\det\nabla\boldsymbol{y}\in L^{\tilde{q}_\mathrm{D}}(\Omega;\mathbb{R}^d)$, where $\tilde{q}_\mathrm{D}=\tilde{q}_{\mathrm{D}1}=\tilde{q}_{\mathrm{D}2}$ is greater than one by assumption.

Case (b) is easier, we similarly use Hölder's inequality in (3.3a), relying on the continuous embedding $W^{1,q_\mathrm{g}}(\Omega;\mathbb{R}^{d\times d})\hookrightarrow L^{q_\mathrm{v}}(\Omega;\mathbb{R}^{d\times d})$, $q_\mathrm{v}=\frac{dq_\mathrm{g}}{d-q_\mathrm{g}}=\frac{q_\mathrm{Y}}{d-2}$. As a result, $\nabla\det\nabla\boldsymbol{y}\in L^{q_\mathrm{D}}(\Omega;\mathbb{R}^d)$ for $\frac{1}{q_\mathrm{D}}=\frac{1}{q_\mathrm{s}}+\frac{1}{q_\mathrm{v}}$, so $q_\mathrm{D}\in[1,q_\mathrm{s})=[1,d)$.

We proceed likewise in case (c), using $\operatorname{cof}\nabla\boldsymbol{y}\in L^\infty(\Omega;\mathbb{R}^{d\times d})$ in (3.3a).

The final piece of the puzzle is to show that $\det\nabla\boldsymbol{y}\in L^{q_\mathrm{D}}(\Omega)$. This is a consequence of $(\nabla\boldsymbol{y})\big|_\Gamma\in L^{dq_\mathrm{D}}(\Gamma;\mathbb{R}^{d\times d})$ (by the trace theorem) and of Poincaré's inequality

$$||\det\nabla\boldsymbol{y}||_{W^{1,q_\mathrm{D}}(\Omega)}\le\tilde{C}_\mathrm{Poi}\sqrt[q_\mathrm{D}]{\int_\Gamma|\det\nabla\boldsymbol{y}_\mathrm{Dir}|^{q_\mathrm{D}}\mathrm{d}S+\int_\Omega|\nabla\det\nabla\boldsymbol{y}|^{q_\mathrm{D}}\mathrm{d}\boldsymbol{X}},\quad\tilde{C}_\mathrm{Poi}>0,$$

analogously to the part where we proved $\operatorname{cof}\nabla\boldsymbol{y}\in W^{1,q_\mathrm{g}}(\Omega;\mathbb{R}^{d\times d})$. □

Example 5.1 from [5] shows that there exist mappings $\boldsymbol{y}\colon\Omega\to\mathbb{R}^d$, $d>2$, such that $\boldsymbol{y}\in W^{1,q_\mathrm{Y}}(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$ for some $q_\mathrm{Y}\ge 1$, $\operatorname{cof}\nabla\boldsymbol{y}\in W^{1,\infty}(\Omega;\mathbb{R}^{d\times d})$ and $\det\nabla\boldsymbol{y}\in W^{1,\infty}(\Omega)$, but $\underline{\nabla^2\boldsymbol{y}}\notin W^{2,1}(\Omega;\mathbb{R}^d)$. This means that the class of admissible deformations for gradient polyconvex energies is wider than the one of second-grade materials. The example is formulated in three dimensions, but generalising it to any $d>2$ could be done with ease. It is also worth remarking that Example 2.1 in [20] offers a hint on how to construct similar mappings $\boldsymbol{y}$ which have Sobolev minors but do not belong to $W^{2,1}(\Omega;\mathbb{R}^d)$.

3.5 Recent existence results

Eventually, two examples of existence results with gradient polyconvex functionals are cited, to give an idea of possible applications.

Theorem 3. *(see [5]) Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, and let $\Gamma_1 \cup \Gamma_2$ be a measurable partition of $\partial\Omega$ with $\mathcal{H}^2(\Gamma_0) > 0$. Let further $\ell \colon W^{1,p}(\Omega;\mathbb{R}^3) \to \mathbb{R}$ be a linear bounded functional and $J(\boldsymbol{y}) := I(\boldsymbol{y}) - \ell(\boldsymbol{y})$ with*

$$I(\boldsymbol{y}) := \int_\Omega \hat{W}(\nabla \boldsymbol{y}, \underline{\nabla[\operatorname{cof} \nabla \boldsymbol{y}]}) \mathrm{d}\boldsymbol{X}$$

being gradient polyconvex and such that the coercivity condition

$$W(\mathbb{F}, \underline{\Delta_1}) \geq \begin{cases} c\big(|\mathbb{F}|^p + |\operatorname{cof} \mathbb{F}|^q + (\det \mathbb{F})^r + (\det \mathbb{F})^{-s} + |\,\underline{\Delta_1}\,|^q\,\big) & \text{if } \det \mathbb{F} > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

holds true. Let $L \colon \mathbb{R}^{3\times3} \to \mathbb{R}$ be lower semicontinuous. Finally, let $p \geq 2$, $q \geq \frac{p}{p-1}$, $r > 1$, $s > 0$ and assume that for some given map $\boldsymbol{y}_{\mathrm{Dir}} \in W^{1,p}(\Omega;\mathbb{R}^3)$, the following set

$$\begin{aligned} \mathcal{A} := \{\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3),\ \operatorname{cof}\nabla\boldsymbol{y} \in W^{1,q}(\Omega;\mathbb{R}^{3\times3}),\ \det\nabla\boldsymbol{y} \in L^r(\Omega),\\ (\det\nabla\boldsymbol{y})^{-s} \in L^1(\Omega),\ \det\nabla\boldsymbol{y} > 0 \ \textit{a.e. in } \Omega,\ L(\nabla\boldsymbol{y}) \leq 0 \ \textit{a.e. in } \Omega,\\ \boldsymbol{y} = \boldsymbol{y}_{\mathrm{Dir}} \ \textit{on } \Gamma_0\} \end{aligned}$$

is nonempty and that $\inf_{\mathcal{A}} J < +\infty$. Then the following holds:

(i) The functional J has a minimiser on $\mathcal{A}$, i.e. $\inf_{\mathcal{A}} J$ is attained.

(ii) Moreover, if $q > 3$ and $s > \frac{6q}{q-3}$, then there is $\varepsilon > 0$ such that for every minimiser $\boldsymbol{y} \in \mathcal{A}$ of J, it holds that $\det\nabla\boldsymbol{y} \geq \varepsilon$ in Ω.

The mapping L in the above theorem realises the locking constraint.

The authors of [20] use gradient polyconvex stored energy in an evolutionary model of SMA. They define a mapping $\boldsymbol{\lambda} \colon \mathbb{R}^{3\times3} \to \mathbb{R}^{m+1}$, $\boldsymbol{\lambda} \colon \mathbb{F} \mapsto \boldsymbol{\lambda}(\mathbb{F})$ called a *volume fraction*, to determine which microstructural variant of the material the deformation gradient $\mathbb{F}$ corresponds to. There are m *martensitic* variants plus a high-temperature one, referred to as the *austenite*. The stored energy functional I in the model takes form (3.1).The growth condition of $\hat{W}$ includes the gradient of the determinant as well. Namely, it is supposed that there are numbers q, $r > 1$ and c, $s > 0$ such that for every $\mathbb{F} \in \mathbb{R}^{3\times3}$, $\underline{\Delta_1} \in \mathbb{R}^{3\times3\times3}$, and every $\Delta_2 \in \mathbb{R}^3$

$$\begin{aligned} &\hat{W}(\mathbb{F}, \underline{\Delta_1}, \Delta_2) \geq \\ &\begin{cases} c\big(|\mathbb{F}|^p + |\operatorname{cof}\mathbb{F}|^q + (\det\mathbb{F})^r + (\det\mathbb{F})^{-s} + |\,\underline{\Delta_1}\,|^q + |\Delta_2|^r\,\big) & \text{if } \det\mathbb{F} > 0, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{3.5}$$

One challenge for elasticity theory is to prevent the modelled moving matter from interpenetration. This translates, mathematically speaking, into the injectivity of $\boldsymbol{y}$. The article [20] settles this issue as well, by prescribing the *Ciarlet-Nečas condition*

$$\int_\Omega \det\nabla\boldsymbol{y}(X)\mathrm{d}\boldsymbol{X} \leq \mathcal{L}^3(\boldsymbol{y}(\Omega)), \tag{3.6}$$

which appeared in [7]. With this in mind, the set of admissible deformations is defined as

$$\begin{aligned} \mathcal{A} = \{\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3),\ \operatorname{cof}\nabla\boldsymbol{y} \in W^{1,q}(\Omega;\mathbb{R}^{3\times3}),\ \det\nabla\boldsymbol{y} \in W^{1,r}(\Omega),\\ (\det\nabla\boldsymbol{y})^{-s} \in L^1(\Omega),\ \det\nabla\boldsymbol{y} > 0 \text{ a.e. in } \Omega,\ \boldsymbol{y} = \boldsymbol{y}_{\mathrm{Dir}} \text{ on } \Gamma_0,\ (3.6) \text{ holds}\} \end{aligned}$$

and assumed nonempty. The given measurable function $\boldsymbol{y}_{\mathrm{Dir}}\colon \Gamma_0 \to \mathbb{R}^3$ describes a boundary displacement.

Now the volume fraction $\boldsymbol{\lambda}$ comes into play again, since the dissipative variable $\boldsymbol{z} \in \mathbb{R}^{m+1}$ is given the meaning of the volume fractions of the $m+1$ material's variants.

$$\mathcal{Q} := \{(\boldsymbol{y}, \boldsymbol{z}) \in \mathcal{A} \times \mathcal{Z}\colon\ \boldsymbol{\lambda}(\nabla \boldsymbol{y}) = \boldsymbol{z} \text{ a.e. in } \Omega\},$$

where $\mathcal{Z} := L^\infty(\Omega; \mathbb{R}^{m+1})$ denotes the set of internal variables. The dissipation distance is defined as

$$\mathcal{D}(\boldsymbol{z}^1, \boldsymbol{z}^2) = \int_\Omega |\boldsymbol{z}^1(\boldsymbol{X}) - \boldsymbol{z}^2(\boldsymbol{X})| \mathrm{d}\boldsymbol{X},\ \boldsymbol{z}^1,\ \boldsymbol{z}^2\colon \Omega \to \mathbb{R}^{m+1} \text{ measurable},$$

and the total energy functional, for $(t, \boldsymbol{y}, \boldsymbol{z}) \in [0, T] \times \mathcal{A} \times \mathcal{Z}$, by

$$\mathcal{E}(t, \boldsymbol{y}, \boldsymbol{z}) := \begin{cases} I(\boldsymbol{y}) - \ell(t, \boldsymbol{y}) & \text{if } \boldsymbol{z} = \boldsymbol{\lambda}(\nabla \boldsymbol{y}) \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\ell(t, \cdot)$ is a functional on deformations, which expresses the loading of the specimen.

For each $t \in [0, T]$, the *stable set* is defined to be

$$\mathcal{S}(t) = \{(\boldsymbol{y}, \boldsymbol{z}) \in \mathcal{Q};\ \mathcal{E}(t, \boldsymbol{y}, \boldsymbol{z}) < +\infty,\ \forall (\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) \in \mathcal{Q}\colon \\ \mathcal{E}(t, \boldsymbol{y}, \boldsymbol{z}) \leq \mathcal{E}(t, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) + \mathcal{D}(\boldsymbol{z}, \tilde{\boldsymbol{z}})\}.$$

The main result of [20] is the next

Theorem 4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, and let $\Gamma = \Gamma_0 \cup \Gamma_1$ be an $\mathcal{H}^2$-measurable partition of $\Gamma = \partial\Omega$ with the area of $\Gamma_0 > 0$. Let I be gradient polyconvex on Ω and such that the stored energy density $\hat{W}$ satisfies* (3.5)*. Further, let $\ell \in \mathcal{C}^1([0, T]; (W^{1,p}(\Omega; \mathbb{R}^3))^*)$ be such that, for some $C > 0$ and $1 \leq \alpha < p$,*

$$\ell(t, \boldsymbol{y}) \leq C||\boldsymbol{y}||^\alpha_{W^{1,p}},\ \textit{for all } t \in [0, T],\ T > 0,$$

and $\boldsymbol{y} \mapsto -\ell(t, \boldsymbol{y})$ is weakly sequentially lower semicontinuous on $W^{1,p}(\Omega; \mathbb{R}^3)$ for all $t \in [0, T]$. For the Lebesgue exponents, assume that $p > 6$, $q \geq \frac{p}{p-1}$, $r > 1$ and $s > \frac{2p}{p-6}$. Finally, let $\inf_{(\boldsymbol{y},\boldsymbol{z}) \in \mathcal{Q}} \mathcal{E}(t, \boldsymbol{y}, \boldsymbol{z}) + \mathcal{D}(\boldsymbol{z}, \hat{\boldsymbol{z}}) < +\infty$ for every $t \in [0, T]$, $\hat{\boldsymbol{z}} \in \mathcal{Z}$, and let the initial condition be stable, i.e. $\boldsymbol{q}^0 := (\boldsymbol{y}^0, \boldsymbol{z}^0) \in \mathcal{S}(0)$.

Then there is an energetic solution to $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ satisfying $\boldsymbol{q}(0) = \boldsymbol{q}^0$ and such that $\boldsymbol{y} \in \mathrm{B}([0, T]; \mathcal{A})$, $\boldsymbol{z} \in \mathrm{BV}\left([0, T]; L^1(\Omega; \mathbb{R}^{m+1})\right) \cap L^\infty(0, T; \mathcal{Z})$, and such that for all $t \in [0, T]$ the identity $\boldsymbol{\lambda}(\nabla \boldsymbol{y}(t, \cdot)) = \boldsymbol{z}(t, \cdot)$ holds a.e. in Ω. Moreover, for all $t \in [0, T]$, the deformation $\boldsymbol{y}(t)$ is injective everywhere in Ω.

In Chapter 4, gradient plasticity is considered in the model of a material with gradient polyconvex stored energy.

4. Existence of solutions in plasticity

This chapter shows how rate-independent finite-strain gradient plasticity can be added to the elastic model from [5] with a gradient polyconvex stored energy functional. The formulation is based on [25] and the existence proof for an energetic solution adheres to the structure from [15].

4.1 Formulation of the problem

Suppose that $\Omega \subset \mathrm{I\!R}^d$, $d > 1$, is a domain with Lipschitz boundary. Its closure $\overline{\Omega}$ represents the reference configuration of a material body, like in Chapter 1. We wish to establish the existence of a motion mapping $\boldsymbol{y} : [0,T] \to \mathcal{Y}$, $T \in (0,+\infty)$, where the set $\mathcal{Y}$ of admissible deformations of Ω will be specified later. Thus, contrary to Chapter 1, we study the whole process of deformation as an evolution problem (not only the initial and final state). We work under the assumption of rate-independent material behaviour (explained in Section 2.2) and consider a so-called *quasistatic evolution*, where acceleration (inertial effects) is neglected. Also we suppose that the evolution is isothermal, i.e. with no flow of heat and no temperature changes.

For notational simplicity, we sometimes write $\boldsymbol{y}(t,\boldsymbol{X})$ instead of $[\boldsymbol{y}(t)](\boldsymbol{X})$, $t \in [0,T]$, $\boldsymbol{X} \in \Omega$ (analogously for other mappings from $[0,T]$ into a space of functions on Ω).

Furthermore, we wish to incorporate *plasticity* in the model. Therefore we split the deformation gradient $\mathbb{F}(t,\boldsymbol{X}) = \nabla \boldsymbol{y}(t,\boldsymbol{X})$ as

$$\mathbb{F}(t,\boldsymbol{X}) = \mathbb{F}_{\mathrm{el}}(t,\boldsymbol{X})\mathbb{P}(t,\boldsymbol{X}), \quad t \in [0,T],\ \boldsymbol{X} \in \Omega,$$

where $\mathbb{F}_{\mathrm{el}} : [0,T] \to \{\hat{\mathbb{F}}_{\mathrm{el}} : \Omega \to \mathrm{I\!R}^{d\times d}_{+}\}$ is the *elastic distortion* and $\mathbb{P} : [0,T] \to \mathcal{Z}$ the *plastic distortion* (or *plastic strain*). This multiplicative decomposition is used in the framework of finite strains (as opposed to an additive split in the linearised setting). The set $\mathcal{Z}$ will be defined in Subsection 4.1.2.

Remark. Sometimes a different notational convention appears in literature: $\mathbb{F}_{\mathrm{p}}$ is the plastic component of the deformation gradient and the plastic tensor $\mathbb{P}$ means $\mathbb{F}_{\mathrm{p}}^{-1}$ (cf. [26]).

We do not have any internal variables in the model, but the addition of some hardening parameters would be feasible (see [25]).

The stored energy density $W : \mathrm{I\!R}^{d\times d} \times \mathrm{I\!R}^{d\times d} \to \mathrm{I\!R}_{\infty}$ is supposed to be lower semicontinuous and $W(\cdot,\mathbb{P}_*)$ to satisfy properties (1.1) for every $\mathbb{P}_* \in \mathrm{SL}(d)$. Moreover, the growth condition below is assumed to hold for any $\mathbb{F}_{\mathrm{e}*} \in \mathrm{I\!R}^{d\times d}$ and $\mathbb{P}_* \in \mathrm{I\!R}^{d\times d}$ with real constants $q_{\mathrm{F}} > q_{\mathrm{Y}}$, $q_{\mathrm{c}} > d$, $q_{\mathrm{d}} > 0$, $q_{\mathrm{P}} > 1$, $c > 0$:

$$W(\mathbb{F}_{\mathrm{e}*},\mathbb{P}_*) \geq \begin{cases} c(|\mathbb{F}_{\mathrm{e}*}|^{q_{\mathrm{F}}} + |\mathrm{cof}\,\mathbb{F}_{\mathrm{e}*}|^{q_{\mathrm{c}}} + |\det \mathbb{F}_{\mathrm{e}*}|^{-q_{\mathrm{d}}} \\ \qquad + |\mathbb{P}_*|^{q_{\mathrm{P}}}) & \text{if } \det \mathbb{F}_{\mathrm{e}*} > 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.1}$$

We notice that the dependence of W on $\mathbb{F}_*$ takes place only through the elastic part $\mathbb{F}_{\mathrm{e}*}$.

Remark (notation). In this chapter, physical quantities such as $\boldsymbol{y}$, $\mathbb{P}$ or $\mathbb{F}_{\mathrm{el}}$ appear

(i) as constant values,

(ii) as spatial fields defined in Ω, but independent of time,

(iii) as mappings defined in a subset of time-space.

One option would be to use the same symbol $\boldsymbol{y}$, $\mathbb{P}$ or $\mathbb{F}_{\mathrm{el}}$ to denote all three versions (i)–(iii) (as it is done in [25]). In the hope of a clearer notation, this convention is not followed here and symbols with an asterisk subscript (e.g. $\mathbb{F}_{e*} \in \mathbb{R}^{d\times d}$) stand for constants, letters with a hat (e.g. $\hat{\boldsymbol{y}} : \Omega \to \mathcal{Y}$) are used in case (ii) and unmarked letters $\boldsymbol{y}$, $\mathbb{P}$ etc. in case (iii).

4.1.1 Boundary conditions

Let the boundary of Ω be decomposed as $\partial\Omega = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}}$ where Γ_{D}, Γ_{N} are open in $\partial\Omega$ and $\mathcal{H}^{d-1}(\Gamma_{\mathrm{D}}) > 0$. For each $t \in [0, T]$, we impose this boundary condition on $\hat{\boldsymbol{y}} := \boldsymbol{y}(t)$:

$$\hat{\boldsymbol{y}} = \hat{\boldsymbol{y}}_{\mathrm{Dir}} \quad \text{on } \Gamma_{\mathrm{D}}.$$

The boundary displacement $\hat{\boldsymbol{y}}_{\mathrm{Dir}} \in W^{1-\frac{1}{q_{\mathrm{Y}}},q_{\mathrm{Y}}}(\Gamma_{\mathrm{D}};\mathbb{R}^d)$ is a given function. The exponent

$$q_{\mathrm{Dir}} \in \begin{cases} [1, \frac{dq_{\mathrm{Y}}-q_{\mathrm{Y}}}{d-q_{\mathrm{Y}}}] & \text{if } q_{\mathrm{Y}} < d, \\ [1,\infty) & \text{if } +\infty > q_{\mathrm{Y}} \geq d \end{cases}$$

so that we can use the continuous embedding (see [28, p. 601])

$$W^{1-\frac{1}{q_{\mathrm{Y}}},q_{\mathrm{Y}}}(\Gamma_{\mathrm{D}};\mathbb{R}^d) \hookrightarrow L^{q_{\mathrm{Dir}}}(\Gamma_{\mathrm{D}};\mathbb{R}^d).$$

On the Neumann part Γ_{N} of the boundary, a surface load $\hat{\boldsymbol{T}}$ can be prescribed, as in [15] but independent of time:

$$\frac{\partial W}{\partial \mathbb{F}_{\mathrm{e}*}}\boldsymbol{n} = \hat{\boldsymbol{T}} \quad \text{on } \Gamma_{\mathrm{N}}.$$

This would lead to the following definition of the functional $\hat{\ell}$ expressing the work of external forces:

$$\langle \hat{\ell}, \hat{\boldsymbol{y}} \rangle = \int_{\Omega} \hat{\boldsymbol{f}} \cdot \hat{\boldsymbol{y}} \, \mathrm{d}\boldsymbol{X} + \int_{\Gamma_{\mathrm{N}}} \hat{\boldsymbol{T}} \cdot \hat{\boldsymbol{y}} \, \mathrm{d}S,$$

where $\hat{\boldsymbol{f}} : \Omega \to \mathbb{R}^d$ is the density of body force as in (1.2).

However, we are not limited to external forces of this form, so later, a more general time-dependent functional ℓ is considered.

4.1.2 Elastoplastic energy functional

Define the set of admissible deformations (cf. Theorem 3)

$$\begin{aligned} \mathcal{Y} = \{\hat{\boldsymbol{y}} \in W^{1,q_{\mathrm{Y}}}(\Omega;\mathbb{R}^d);\ \operatorname{cof}\nabla\hat{\boldsymbol{y}} \in W^{1,q_{\mathrm{g}}}(\Omega;\mathbb{R}^{d\times d}),\ (\det\nabla\hat{\boldsymbol{y}})^{-q_{\mathrm{d}}} \in L^1(\Omega), \\ \det\nabla\hat{\boldsymbol{y}} > 0 \text{ a.e. in } \Omega,\ \hat{\boldsymbol{y}}\big|_{\Gamma_{\mathrm{D}}} = \hat{\boldsymbol{y}}_{\mathrm{Dir}}\}, \quad q_{\mathrm{Y}} > d-1,\ q_{\mathrm{c}} \geq q_{\mathrm{g}} > \frac{d}{2}, \end{aligned}$$

the set of plastic tensors

$$\mathcal{Z} = \{\hat{\mathbb{P}} \in W^{1,q_{\mathrm{P}}}(\Omega;\mathbb{R}^{d\times d});\ \hat{\mathbb{P}}(\boldsymbol{X}) \in \mathrm{SL}(d) \text{ for a.a. } \boldsymbol{X} \in \Omega\},$$

and the elastoplastic energy functional

$$I(\hat{\boldsymbol{y}}, \hat{\mathbb{P}}) = \int_\Omega W(\nabla\hat{\boldsymbol{y}}(\boldsymbol{X})\hat{\mathbb{P}}^{-1}(\boldsymbol{X}), \hat{\mathbb{P}}(\boldsymbol{X}))\mathrm{d}\boldsymbol{X} + \alpha \int_\Omega \left|\underline{\nabla[(\mathrm{cof}\,\nabla\hat{\boldsymbol{y}}(\boldsymbol{X}))\hat{\mathbb{P}}^\top(\boldsymbol{X})]}\right|^{q_\mathrm{c}} \mathrm{d}\boldsymbol{X} + \beta \int_\Omega \left|\underline{\nabla\hat{\mathbb{P}}(\boldsymbol{X})}\right|^{q_\mathrm{P}} \mathrm{d}\boldsymbol{X}, \quad \hat{\boldsymbol{y}} \in \mathcal{Y}, \hat{\mathbb{P}} \in \mathcal{Z}, \alpha, \beta > 0.$$

It is assumed that the following condition is fulfilled:

$$q_\mathrm{P} > d \ \wedge \ q_\mathrm{g} \le q_\mathrm{P}. \tag{4.2}$$

Besides, application of Hölder's inequality to some terms later necessitates that

$$q_\mathrm{z} := \frac{q_\mathrm{c} q_\mathrm{P}}{q_\mathrm{c}(d-1) + q_\mathrm{P}} \ge 1, \quad q_\mathrm{a} := \frac{q_\mathrm{P} q_\mathrm{Y}}{q_\mathrm{P}(d-1) + q_\mathrm{Y}} \ge 1, \tag{4.3a}$$
$$\frac{1}{q_\mathrm{F}} + \frac{1}{q_\mathrm{P}} = \frac{1}{q_\mathrm{Y}}.$$

Throughout this chapter, we suppose that the set $\mathcal{Y}$ is nonempty[1].

Remarks

(i) For an arbitrary $t \in [0, T]$, write $\hat{\boldsymbol{y}} = \boldsymbol{y}(t)$, $\hat{\mathbb{F}}_\mathrm{el} = \mathbb{F}_\mathrm{el}(t)$, $\hat{\mathbb{P}} = \mathbb{P}(t)$. By the product rule for cofactor matrices [6, p. 4], we have

$$\mathrm{cof}\,\hat{\mathbb{F}}_\mathrm{el} = \mathrm{cof}\,(\nabla\hat{\boldsymbol{y}}\hat{\mathbb{P}}^{-1}) = \mathrm{cof}\,(\nabla\hat{\boldsymbol{y}})\mathrm{cof}\,(\hat{\mathbb{P}}^{-1}) \quad \text{a.e. in } \Omega$$

and as $\hat{\mathbb{P}}^{-1}$ is invertible with $\det(\hat{\mathbb{P}}^{-1}) = 1$, we get

$$\mathrm{cof}\,(\mathbb{P}^{-1}) = \det(\mathbb{P}^{-1})(\mathbb{P}^{-1})^{-\top} = \mathbb{P}^\top.$$

Thus $(\mathrm{cof}\,\nabla\hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top = \mathrm{cof}\,\hat{\mathbb{F}}_\mathrm{el}$.

(ii) It was supposed that $q_\mathrm{Y} > d-1$ for weak continuity of cofactor matrices (equation (4.9)).

(iii) The term $\int_\Omega |\underline{\nabla[(\mathrm{cof}\,\nabla\hat{\boldsymbol{y}}(\boldsymbol{X}))\hat{\mathbb{P}}^\top(\boldsymbol{X})]}|^{q_\mathrm{c}}\mathrm{d}\boldsymbol{X}$ is what makes the functional I gradient polyconvex.

Observation 5. *The functional $I(\cdot, \hat{\mathbb{P}}) : \mathcal{Y} \to \mathbb{R}_\infty$ is well-defined for any $\hat{\mathbb{P}} \in \mathcal{Z}$.*

Proof. The function W is lower semicontinuous and thus measurable (by Lemma A.2 – it suffices to show that all sets of the form $W^{-1}((a, +\infty])$, $a \in \mathbb{R}$, are measurable). The growth condition (4.1) then ensures that the respective integral exists (finite or infinite).

Both $\mathrm{cof}\,\nabla\hat{\boldsymbol{y}}$ and $\hat{\mathbb{P}}^\top$ are Sobolev functions. However, we must investigate if their product is also weakly differentiable. Theorem A.11 and the aforementioned assumptions on q_g and q_P imply that $(\mathrm{cof}\,\nabla\hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top \in L^1(\Omega; \mathbb{R}^{d\times d})$. Choose $i \in \{1, 2, \ldots d\}$. Then $(\mathrm{cof}\,\nabla\hat{\boldsymbol{y}})(\frac{\partial}{\partial X_i}\hat{\mathbb{P}}^\top) \in L^{q_\mathrm{a}}(\Omega; \mathbb{R}^{d\times d})$ by Hölder's inequality and (4.3a). Hölder's inequality also gives $(\frac{\partial}{\partial X_i}\mathrm{cof}\,\nabla\hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top \in L^1(\Omega; \mathbb{R}^{d\times d})$ (for $d \ge 3$, this follows from $q_\mathrm{P}^{-1} + q_\mathrm{g}^{-1} \le d^{-1} + 2d^{-1} \le 1$; for $d = 2$, we use the continuous embedding of $W^{1,q_\mathrm{P}}(\Omega; \mathbb{R}^{d\times d})$ into $L^\infty(\Omega; \mathbb{R}^{d\times d})$). Now the function $(\frac{\partial}{\partial X_i}\mathrm{cof}\,\nabla\hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top + (\mathrm{cof}\,\nabla\hat{\boldsymbol{y}})(\frac{\partial}{\partial X_i}\hat{\mathbb{P}}^\top)$ is the weak partial

[1] At present it is unknown (apart from some special cases) which $\hat{\boldsymbol{y}}_\mathrm{Dir} \in W^{1-1/q_\mathrm{Y}, q_\mathrm{Y}}(\Gamma_\mathrm{D}; \mathbb{R}^d)$ are compatible with the assumption $\det\nabla\hat{\boldsymbol{y}} > 0$ a.e. in Ω , so the nontriviality of $\mathcal{Y}$ has to be explicitly assumed [5]. For more details, see also [4].

derivative of $(\operatorname{cof}\nabla\hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top$ with respect to X_i. (This can be shown for any component (i,k) of the matrices by approximation with mollified functions $((\operatorname{cof}\nabla\hat{\boldsymbol{y}})_{ij})_\varepsilon$, $(\hat{P}_{kj})_\varepsilon$, $\varepsilon > 0$, $1 \le i,j,k \le d$, cf. the proof of Claim 2.)

This implies that $\nabla[\underline{(\operatorname{cof}\nabla\hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top}]$ exists and so does (in $\mathbb{R}_\infty$) the integral of the q_c-th power of the Frobenius norm of this term.

Note that if both $q_\mathrm{g} > d$ and $q_\mathrm{P} > d$, we could get at least

$$(\operatorname{cof}\nabla\hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top \in W^{1,1}(\Omega;\mathbb{R}^{d\times d})$$

directly by Theorem A.11, reasoning componentwise. □

4.1.3 Energetic solution and incremental minimisation

The total energy functional is given by

$$\mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) = I(\hat{\boldsymbol{y}},\hat{\mathbb{P}}) - \ell(t,\hat{\boldsymbol{y}}), \quad t\in[0,T],\ \hat{\boldsymbol{y}}\in\mathcal{Y},\ \hat{\mathbb{P}}\in\mathcal{Z}.$$

For every time $t \in [0,T]$, $\ell(t,\cdot)$ is a continuous linear functional on $W^{1,q_\mathrm{Y}}(\Omega;\mathbb{R}^d)$, representing the work of applied forces (compare with the classical elastic case (1.3)). To obtain certain estimates later, it is useful to assume that ℓ is in $\mathcal{C}^1([0,T];(W^{1,q_\mathrm{Y}}(\Omega;\mathbb{R}^d))^*)$.

Remark. In fact, the functionals $I(\cdot,\hat{\mathbb{P}})$ and $\mathcal{E}(t,\cdot,\hat{\mathbb{P}})$, $\hat{\mathbb{P}}\in\mathcal{Z}$ and $t\in[0,T]$ fixed, can be defined (by the same formulas) for a larger class of mappings $\hat{\boldsymbol{y}}$ than $\mathcal{Y}$. Nevertheless, if e.g. $\det\nabla\hat{\boldsymbol{y}} \le 0$ in a nonnull subset of Ω, condition (4.1) causes $I(\hat{\boldsymbol{y}},\hat{\mathbb{P}})$ to be infinite.

Further we introduce the dissipation distance $\mathcal{D}$ as in [25]:

$$\mathcal{D}(\hat{\mathbb{P}}_1,\hat{\mathbb{P}}_2) = \int_\Omega D(\boldsymbol{X},\hat{\mathbb{P}}_1(\boldsymbol{X}),\hat{\mathbb{P}}_2(\boldsymbol{X}))\mathrm{d}\boldsymbol{X}, \quad \hat{\mathbb{P}}_1,\hat{\mathbb{P}}_2\in\mathcal{Z}.$$

We require that the function $D\colon \Omega\times(\mathrm{SL}(d))^2 \to [0,+\infty)$ satisfy:

$$D\colon \Omega\times(\mathrm{SL}(d))^2 \to [0,+\infty) \text{ is a Carathéodory mapping,} \tag{4.4a}$$

$$\forall\,\boldsymbol{X}\in\Omega, \mathbb{P}_{*1},\mathbb{P}_{*2}\in\mathrm{SL}(d) : D(\boldsymbol{X},\mathbb{P}_{*1},\mathbb{P}_{*2}) = 0 \Longleftrightarrow \mathbb{P}_{*1} = \mathbb{P}_{*2}, \tag{4.4b}$$

$$\forall\,\boldsymbol{X}\in\Omega, \mathbb{P}_{*1},\mathbb{P}_{*2},\mathbb{P}_{*3}\in\mathrm{SL}(d) :$$

$$D(\boldsymbol{X},\mathbb{P}_{*1},\mathbb{P}_{*3}) \le D(\boldsymbol{X},\mathbb{P}_{*1},\mathbb{P}_{*2}) + D(\boldsymbol{X},\mathbb{P}_{*2},\mathbb{P}_{*3}), \tag{4.4c}$$

$$D(\boldsymbol{X},\mathbb{P}_{*1},\mathbb{P}_{*2}) \le h(\boldsymbol{X}) + |\mathbb{P}_{*1}|^{q_\mathrm{P}}| + |\mathbb{P}_{*2}|^{q_\mathrm{P}} \text{ for some } h\in L^1(\Omega), \tag{4.4d}$$

$$|\mathbb{P}_{*1}-\mathbb{P}_{*2}| \le D(\boldsymbol{X},\mathbb{P}_{*1},\mathbb{P}_{*2}). \tag{4.4e}$$

Under the conditions stated above, $\mathcal{D}$ is a quasimetric on $\mathcal{Z}$ (we may not have symmetry). Asymmetry is necessary for modelling in elastoplasticity. [28, p. 47]

Next let us define the *sets* $\mathcal{S}(t)$ *of stable states at time* t for every $t\in[0,T]$ by

$$\mathcal{S}(t) = \{(\hat{\boldsymbol{y}},\hat{\mathbb{P}})\in\mathcal{Y}\times\mathcal{Z};\ \mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) < +\infty,\ \forall(\tilde{\boldsymbol{y}},\tilde{\mathbb{P}})\in\mathcal{Y}\times\mathcal{Z} : \\ \mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) \le \mathcal{E}(t,\tilde{\boldsymbol{y}},\tilde{\mathbb{P}}) + \mathcal{D}(\hat{\mathbb{P}},\tilde{\mathbb{P}})\}.$$

We are looking for an energetic solution of our evolutionary problem (definition according to [15]).

Definition 6. *A function* $(\boldsymbol{y},\mathbb{P})\colon [0,T]\to\mathcal{Y}\times\mathcal{Z}$ *is called an* energetic solution *of the rate-independent problem associated with* $\mathcal{E}$ *and* $\mathcal{D}$*, if* $t\mapsto\partial_t\mathcal{E}(t,\boldsymbol{y}(t),\mathbb{P}(t))$ *lies in* $L^1((0,T),\mathbb{R})$ *and if for all* $t\in[0,T]$ *we have*

$$\textit{stability: } (\boldsymbol{y}(t),\mathbb{P}(t))\in\mathcal{S}(t), \tag{S}$$

energy balance:

$$\mathcal{E}(t,\boldsymbol{y}(t),\mathbb{P}(t)) + \mathrm{Diss}_\mathcal{D}(\mathbb{P};[0,t]) = \mathcal{E}(0,\boldsymbol{y}(0),\mathbb{P}(0)) + \int_0^t \partial_t\mathcal{E}(\tau,\boldsymbol{y}(\tau),\mathbb{P}(\tau))\mathrm{d}\tau. \tag{E}$$

The symbol $\mathrm{Diss}_{\mathcal{D}}(\mathbb{P};[t,t'])$, $t,t' \in [0,T]$, $t<t'$, denotes the (total) dissipation along a part of the curve $\mathbb{P}$ and its definition reads

$$\mathrm{Diss}_{\mathcal{D}}(\mathbb{P};[t,t']) = \sup\Big\{\sum_{k=1}^{N} \mathcal{D}(\mathbb{P}(t_{k-1}),\mathbb{P}(t_k));\ N\in\mathbb{N},\ t=t_0<\cdots<t_N=t'\Big\}.$$

Further, the total dissipation is additive [28, p. 47]:

$$\mathrm{Diss}_{\mathcal{D}}(\mathbb{P};[t,t'']) = \mathrm{Diss}_{\mathcal{D}}(\mathbb{P};[t,t']) + \mathrm{Diss}_{\mathcal{D}}(\mathbb{P};[t',t''])\ \text{for all } t<t'<t''. \tag{4.5}$$

To start with the existence proof for an energetic solution, we perform a time-incremental minimisation. Let us break the interval $[0,T]$ into N subintervals by taking a partition $\{t_k\}_{k=1}^{N}$, $0=t_0\le t_1\le\cdots\le t_N=T$. Given an initial plastic tensor $\hat{\mathbb{P}}_0\in\mathcal{Z}$, there exists a minimiser $\hat{\boldsymbol{y}}_0$ of $\mathcal{E}(0,\cdot,\hat{\mathbb{P}}_0)$ over $\mathcal{Y}$, which can be proved analogously to Section 4.5 (only $\mathcal{D}$ is missing). Then for every $k\in\{1,\ 2,\dots N\}$, let us find

$$\begin{aligned}\hat{\boldsymbol{y}}_k\in\mathcal{Y},\hat{\mathbb{P}}_k\in\mathcal{Z}\ \text{such that}\ \mathcal{E}(t_k,\hat{\boldsymbol{y}}_k,\hat{\mathbb{P}}_k)+\mathcal{D}(\hat{\mathbb{P}}_{k-1},\hat{\mathbb{P}}_k) &=\\ \min\{\mathcal{E}(t_k,\hat{\boldsymbol{y}},\hat{\mathbb{P}})+\mathcal{D}(\hat{\mathbb{P}}_{k-1},\hat{\mathbb{P}});\ \hat{\boldsymbol{y}}\in\mathcal{Y},\ \hat{\mathbb{P}}\in\mathcal{Z}\}.&\end{aligned} \tag{4.6}$$

A solution of such an incremental problem exists as we will also see in Section 4.5. Before that, it will be good to have some further properties of the problem, in our hands. To keep track of the particular assumptions on the Lebesgue exponents q_{F}, q_{Y}, q_{c}, q_{P}, q_{g}, q_{z} and q_{a}, the conditions which they must fulfil are restated in the formulations of theorems in Sections 4.2–4.3, where appropriate.

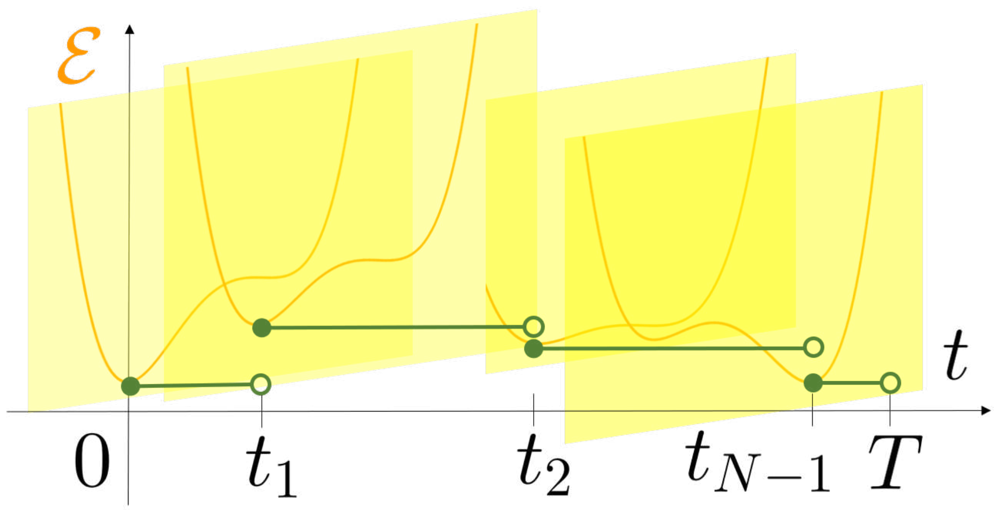

Figure 4.1: An illustration of the incremental minimisation. The figure is not accurate, as pictures in functional analysis can hardly be accurate, but it can still give some insight.

4.2 Compactness and closedness

The proof of Theorem 6 is inspired by Theorem 3 (i.e. Proposition 5.1 in [5]), but here we obtain convergence properties even in our elastoplastic framework.

Theorem 6. *Suppose* $\Omega \in \mathcal{C}^{0,1}$ $d-1 \leq q_{\mathrm{P}} < +\infty$, $1 < q_{\mathrm{c}} < +\infty$, $d-1 < q_{\mathrm{Y}} < +\infty$, $q_{\mathrm{z}} = \frac{q_{\mathrm{c}} q_{\mathrm{P}}}{q_{\mathrm{c}}(d-1)+q_{\mathrm{P}}} \geq 1$, $q_{\mathrm{a}} = \frac{q_{\mathrm{P}} q_{\mathrm{Y}}}{q_{\mathrm{P}}(d-1)+q_{\mathrm{Y}}} \geq 1$. *Let* $\{\hat{\boldsymbol{y}}_n\}_{n=1}^{\infty} \subset \mathcal{Y}$, $\{\hat{\mathbb{P}}_n\}_{n=1}^{\infty} \subset \mathcal{Z}$ *be sequences and* $C_{\mathrm{c}} > 0$ *a constant such that for all* $n \in \mathbb{N}$

$$||\nabla \hat{\boldsymbol{y}}_n||_{q_{\mathrm{Y}}} \leq C_{\mathrm{c}},\ ||\hat{\mathbb{P}}_n||_{W^{1,q_{\mathrm{P}}}(\Omega;\mathbb{R}^{d\times d})} \leq C_{\mathrm{c}},\ ||(\operatorname{cof} \nabla \hat{\boldsymbol{y}}_n)\hat{\mathbb{P}}_n^{\top}||_{W^{1,q_{\mathrm{c}}}(\Omega;\mathbb{R}^{d\times d})} \leq C_{\mathrm{c}},$$
$$||(\det \nabla \hat{\boldsymbol{y}}_n)^{-q_{\mathrm{d}}}||_1 \leq C_{\mathrm{c}}.$$

Then there exist subsequences $\{\hat{\boldsymbol{y}}_{n_j}\}_{j=1}^{\infty} \subset \{\hat{\boldsymbol{y}}_n\}_{n=1}^{\infty}$, $\{\hat{\mathbb{P}}_{n_j}\}_{j=1}^{\infty} \subset \{\hat{\mathbb{P}}_n\}_{n=1}^{\infty}$ *and functions* $\hat{\boldsymbol{y}} \in W^{1,q_{\mathrm{Y}}}(\Omega;\mathbb{R}^d)$, $\hat{\mathbb{P}} \in W^{1,q_{\mathrm{P}}}(\Omega;\mathbb{R}^{d\times d})$ *satisfying for* $j \to +\infty$

$$\hat{\mathbb{P}}_{n_j} \to \hat{\mathbb{P}} \text{ a.e. in } \Omega, \quad \hat{\mathbb{P}}_{n_j}^{-1} \to \hat{\mathbb{P}}^{-1} \text{ a.e. in } \Omega,$$
$$\hat{\mathbb{P}}_{n_j} \to \hat{\mathbb{P}} \text{ in } L^{q_{\mathrm{P}}}(\Omega;\mathbb{R}^{d\times d}), \quad \underline{\nabla \hat{\mathbb{P}}_{n_j}} \rightharpoonup \underline{\nabla \hat{\mathbb{P}}} \text{ in } L^{q_{\mathrm{P}}}(\Omega;\mathbb{R}^{d\times d\times d}),$$
$$\hat{\boldsymbol{y}}_{n_j} \rightharpoonup \hat{\boldsymbol{y}} \text{ in } W^{1,q_{\mathrm{Y}}}(\Omega;\mathbb{R}^d), \quad \nabla \hat{\boldsymbol{y}}_{n_j} \to \nabla \hat{\boldsymbol{y}} \text{ a.e. in } \Omega$$
$$\text{and } (\operatorname{cof} \nabla \hat{\boldsymbol{y}}_{n_j})\hat{\mathbb{P}}_{n_j}^{\top} \rightharpoonup (\operatorname{cof} \nabla \hat{\boldsymbol{y}})\hat{\mathbb{P}}^{\top} \text{ in } W^{1,q_{\mathrm{c}}}(\Omega;\mathbb{R}^{d\times d}).$$

Moreover, it follows that $\det \nabla \hat{\boldsymbol{y}} > 0$ *a.e. in* Ω *and* $(\det \nabla \hat{\boldsymbol{y}})^{-q_{\mathrm{d}}} \in L^1(\Omega)$.

Proof. Invoking the Poincaré inequality (see Theorem A.10), we first observe that $\hat{\boldsymbol{y}}_n|_{\Gamma_{\mathrm{D}}} = \hat{\boldsymbol{y}}_{\mathrm{Dir}}$, $n \in \mathbb{N}$, implies that the whole Sobolev norms $||\hat{\boldsymbol{y}}_n||_{W^{1,q_{\mathrm{Y}}}(\Omega;\mathbb{R}^d)} \leq \tilde{C}_c$ for some $\tilde{C}_c > 0$. Since $W^{1,q_{\mathrm{P}}}(\Omega;\mathbb{R}^{d\times d})$, $W^{1,q_{\mathrm{c}}}(\Omega;\mathbb{R}^{d\times d})$ and $W^{1,q_{\mathrm{Y}}}(\Omega;\mathbb{R}^d)$ are reflexive Banach spaces (see [28, p. 599]) and the sequences $\{\hat{\boldsymbol{y}}_n\}_{n=1}^{\infty}$, $\{(\operatorname{cof} \nabla \hat{\boldsymbol{y}}_n)\hat{\mathbb{P}}_n^{\top}\}_{n=1}^{\infty}$, $\{\hat{\mathbb{P}}_n\}_{n=1}^{\infty}$ are bounded, we can extract subsequences $\{\hat{\boldsymbol{y}}_{n_j}\}_{j=1}^{\infty}$, $\{(\operatorname{cof} \nabla \hat{\boldsymbol{y}}_{n_j})\hat{\mathbb{P}}_{n_j}^{\top}\}_{j=1}^{\infty}$, $\{\hat{\mathbb{P}}_{n_j}\}_{j=1}^{\infty}$ such that

$$\hat{\mathbb{P}}_{n_j} \rightharpoonup \hat{\mathbb{P}} \text{ in } W^{1,q_{\mathrm{P}}}(\Omega;\mathbb{R}^{d\times d}) \text{ for some } \hat{\mathbb{P}} \in W^{1,q_{\mathrm{P}}}(\Omega;\mathbb{R}^{d\times d}),$$
$$(\operatorname{cof} \nabla \hat{\boldsymbol{y}}_{n_j})\hat{\mathbb{P}}_{n_j}^{\top} \rightharpoonup \hat{\mathbb{H}} \text{ in } W^{1,q_{\mathrm{c}}}(\Omega;\mathbb{R}^{d\times d}) \text{ for some } \hat{\mathbb{H}} \in W^{1,q_{\mathrm{c}}}(\Omega;\mathbb{R}^{d\times d}),$$
$$\hat{\boldsymbol{y}}_{n_j} \rightharpoonup \hat{\boldsymbol{y}} \text{ in } W^{1,q_{\mathrm{Y}}}(\Omega;\mathbb{R}^d) \text{ for some } \hat{\boldsymbol{y}} \in W^{1,q_{\mathrm{Y}}}(\Omega;\mathbb{R}^d), \quad j \to +\infty. \tag{4.7a}$$

By compact embedding,

$$\hat{\mathbb{P}}_{n_j} \to \hat{\mathbb{P}} \text{ in } L^{q_{\mathrm{P}}}(\Omega;\mathbb{R}^{d\times d}), \tag{4.8a}$$
$$(\operatorname{cof} \nabla \hat{\boldsymbol{y}}_{n_j})\hat{\mathbb{P}}_{n_j}^{\top} \to \hat{\mathbb{H}} \text{ in } L^{q_{\mathrm{c}}}(\Omega;\mathbb{R}^{d\times d}), \tag{4.8b}$$
$$\hat{\boldsymbol{y}}_{n_j} \to \hat{\boldsymbol{y}} \text{ in } L^{q_{\mathrm{Y}}}(\Omega;\mathbb{R}^d), \quad j \to +\infty. \tag{4.8c}$$

Weak sequential continuity of the minors of gradients ([8], Th. 8.20) implies that

$$\operatorname{cof} \nabla \hat{\boldsymbol{y}}_{n_j} \rightharpoonup \operatorname{cof} \nabla \hat{\boldsymbol{y}} \text{ in } L^{\frac{q_{\mathrm{Y}}}{d-1}}(\Omega;\mathbb{R}^{d\times d}), \tag{4.9}$$

and using the fact that the components of $\operatorname{cof} \hat{\mathbb{P}}_{n_j}$ are linear combinations of $(d-1)$-tuples of components of $\hat{\mathbb{P}}_{n_j}$, we deduce the convergence

$$\operatorname{cof} \hat{\mathbb{P}}_{n_j} \to \operatorname{cof} \hat{\mathbb{P}} \text{ in } L^{\frac{q_{\mathrm{P}}}{d-1}}(\Omega;\mathbb{R}^{d\times d}). \tag{4.10}$$

Multiplying (4.9) with transposed (4.8a), we get

$$(\operatorname{cof} \nabla \hat{\boldsymbol{y}}_{n_j})\hat{\mathbb{P}}_{n_j}^{\top} \rightharpoonup (\operatorname{cof} \nabla \hat{\boldsymbol{y}})\hat{\mathbb{P}}^{\top} \text{ in } L^{q_{\mathrm{a}}}(\Omega;\mathbb{R}^{d\times d}) \tag{4.11}$$

because it is a product of a weakly and a strongly convergent sequence [14, p. 183]. Trivial embedding of Lebesgue spaces on a bounded domain together with (4.8b), (4.11) leads to $\hat{\mathbb{H}} = (\operatorname{cof} \nabla \hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top$. Consequently, (4.11) is even true for $q_\mathrm{m} := \max\{q_\mathrm{c}, q_\mathrm{a}\}$ in place of q_a[2].

In remark (i) in Subsection 4.1.2, we noticed that $\operatorname{cof}(\mathbb{P}^{-1}) = \mathbb{P}^\top$, whenever $\mathbb{P} \in \mathrm{SL}(d)$. Thanks to this, (4.10) can be written as

$$\hat{\mathbb{P}}_{n_j}^{-1} \to \hat{\mathbb{P}}^{-1} \text{ in } L^{\frac{q_\mathrm{P}}{d-1}}(\Omega; \mathrm{I\!R}^{d\times d}). \tag{4.12}$$

Let us multiply (4.8b) with the transpose of (4.12) (we already know that $\hat{\mathbb{H}} = (\operatorname{cof} \nabla \hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top$):

$$\operatorname{cof} \nabla \hat{\boldsymbol{y}}_{n_j} = (\operatorname{cof} \nabla \hat{\boldsymbol{y}}_{n_j})\hat{\mathbb{P}}_{n_j}^\top \hat{\mathbb{P}}_{n_j}^{-\top} \to (\operatorname{cof} \nabla \hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top \hat{\mathbb{P}}^{-\top} = \operatorname{cof} \nabla \hat{\boldsymbol{y}} \text{ in } L^{q_z}(\Omega; \mathrm{I\!R}^{d\times d}). \tag{4.13}$$

This convergence of cofactor matrices is very useful and provides pointwise convergence of $\{\nabla \hat{\boldsymbol{y}}_{n_j}\}$ as we shall see soon. The procedure was presented in [5].

By (4.8a) and (4.13), we pass to subsequences such that

$$\operatorname{cof} \nabla \hat{\boldsymbol{y}}_{n_j} \to \operatorname{cof} \nabla \hat{\boldsymbol{y}} \text{ pointwise a.e. in } \Omega, \tag{4.14a}$$

$$\hat{\mathbb{P}}_{n_j} \to \hat{\mathbb{P}} \text{ pointwise a.e. in } \Omega. \tag{4.14b}$$

Even though it is not entirely correct, we shall not relabel the subsequences, for writing out all the indices would soon become tedious.

As $\det \nabla \hat{\boldsymbol{y}}_{n_j} = \sqrt[d-1]{\det \operatorname{cof} \nabla \hat{\boldsymbol{y}}_{n_j}}$ a.e. [5], we obtain

$$\det \nabla \hat{\boldsymbol{y}}_{n_j} \to \det \nabla \hat{\boldsymbol{y}} \text{ pointwise a.e. in } \Omega. \tag{4.15}$$

From this, we infer that $\det \nabla \hat{\boldsymbol{y}} \geq 0$ a.e. in Ω, since $\hat{\boldsymbol{y}}_{n_j} \in \mathcal{Y}$. However, a strict inequality would be desirable here. To see that it indeed takes place, let us use

$$C_\mathrm{c} \geq \liminf_{j\to+\infty} \int_\Omega \frac{1}{|\det \nabla \hat{\boldsymbol{y}}_{n_j}(\boldsymbol{X})|^{q_\mathrm{d}}} \mathrm{d}\boldsymbol{X}$$

from the assumptions of the theorem. Applying Fatou's lemma (Theorem A.3) to the right-hand side of the above inequality, we deduce that $(\det \nabla \hat{\boldsymbol{y}})^{-q_\mathrm{d}} \in L^1(\Omega)$ and $\det \nabla \hat{\boldsymbol{y}} > 0$ a.e. in Ω (otherwise the integral of $(\det \nabla \hat{\boldsymbol{y}})^{-q_\mathrm{d}}$ could not be convergent).

By the formula for an inverse matrix (cf. [19, p. 356]), we get for almost every $\boldsymbol{X} \in \Omega$

$$(\nabla \hat{\boldsymbol{y}}_{n_j}(\boldsymbol{X}))^{-1} = \frac{(\operatorname{cof} \nabla \hat{\boldsymbol{y}}_{n_j}(\boldsymbol{X}))^\top}{\det \nabla \hat{\boldsymbol{y}}_{n_j}(\boldsymbol{X})} \to \frac{(\operatorname{cof} \nabla \hat{\boldsymbol{y}}(\boldsymbol{X}))^\top}{\det \nabla \hat{\boldsymbol{y}}(\boldsymbol{X})} = (\nabla \hat{\boldsymbol{y}}(\boldsymbol{X}))^{-1}. \tag{4.16}$$

It is known from functional analysis that if $\mathcal{X}$ is a Banach space and $\mathfrak{I}(\mathcal{X})$ denotes the set of all bounded linear invertible operators on $\mathcal{X}$, then the mapping ${}^{-1}\colon \mathfrak{I}(\mathcal{X}) \to \mathfrak{I}(\mathcal{X})$, assigning to any invertible operator its inverse, is continuous [31, p. 353]. We would like to use this result to derive pointwise convergence (in a subset of Ω of full measure) of $\{\nabla \hat{\boldsymbol{y}}_{n_j}\}$ from (4.16). But first we need to check that all matrices involved are in $\mathfrak{I}(\mathrm{I\!R}^{d\times d})$.

Fix $\boldsymbol{X}_0 \in \Omega$ so that $\det \nabla \hat{\boldsymbol{y}}(\boldsymbol{X}_0) > 0$ and pointwise convergence (4.16) holds at $\boldsymbol{X}_0$. Hence $\nabla \hat{\boldsymbol{y}}(\boldsymbol{X}_0) \in \mathfrak{I}(\mathrm{I\!R}^{d\times d})$. We already know that $\nabla \hat{\boldsymbol{y}}_{n_j}(\boldsymbol{X}_0)$, $j \in \mathbb{N}$, have positive

[2]An associated question is whether strong convergence (4.8b) also holds with q_c replaced by q_m. Apparently, the answer is positive when $q_\mathrm{c} \geq q_\mathrm{a}$ and we can claim that q_c can be chosen such, since it is a constant of the model. Another way how to achieve strong convergence in some better (i.e. with a higher Lebesgue exponent q) L^q space is to use compact embedding $W^{1,q_\mathrm{c}} \hookrightarrow\hookrightarrow L^\infty$ since in the end $q_\mathrm{c} > d$.

determinants, therefore they are invertible as well. This implies that $(\nabla\hat{\boldsymbol{y}}_{n_j})^{-1}(\boldsymbol{X}_0) \in \mathfrak{I}(\mathbb{R}^{d\times d})$, $j \in \mathbb{N}$, $(\nabla\hat{\boldsymbol{y}})^{-1}(\boldsymbol{X}_0) \in \mathfrak{I}(\mathbb{R}^{d\times d})$. The necessary result

$$\nabla\hat{\boldsymbol{y}}_{n_j} \to \nabla\hat{\boldsymbol{y}}, \quad j \to +\infty, \quad \text{a.e. in } \Omega$$

then follows. Continuity of the inversion mapping similarly implies pointwise convergence of $\hat{\mathbb{P}}_{n_j}^{-1}$ a.e. □

Theorem 7. *Let $1 < q_{\mathrm{P}} < +\infty$, $1 < q_{\mathrm{Y}} < +\infty$, $d < q_{\mathrm{c}} < +\infty$ and $1 \le q_{\mathrm{g}} \le q_{\mathrm{c}}$. Provided $\hat{\mathbb{P}}_n \in \mathcal{Z}$, $\hat{\mathbb{P}} \in W^{1,q_{\mathrm{P}}}(\Omega;\mathbb{R}^{d\times d})$, $\hat{\boldsymbol{y}}_n \rightharpoonup \hat{\boldsymbol{y}}$ in $W^{1,q_{\mathrm{Y}}}(\Omega;\mathbb{R}^{d\times d})$, $\hat{\boldsymbol{y}}_n|_{\Gamma_{\mathrm{D}}} = \hat{\boldsymbol{y}}_{\mathrm{Dir}}$, $\hat{\boldsymbol{y}}_{\mathrm{Dir}} \in W^{1-\frac{1}{q_{\mathrm{Y}}},q_{\mathrm{Y}}}(\Gamma_{\mathrm{D}};\mathbb{R}^d)$ and*

$$\hat{\mathbb{P}}_n \to \hat{\mathbb{P}} \text{ a.e. in } \Omega, \tag{4.17}$$

then $\hat{\mathbb{P}} \in \mathcal{Z}$ and $\hat{\boldsymbol{y}}|_{\Gamma_{\mathrm{D}}} = \hat{\boldsymbol{y}}_{\mathrm{Dir}}$. If, additionally, $(\operatorname{cof}\nabla\hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top \in W^{1,q_{\mathrm{c}}}(\Omega;\mathbb{R}^{d\times d})$ and (4.2) is true, then $\operatorname{cof}\nabla\hat{\boldsymbol{y}} \in W^{1,q_{\mathrm{g}}}(\Omega;\mathbb{R}^{d\times d})$.

Proof. If (4.2) is applicable, then both q_{c} and q_{P} are greater than d and $W^{1,q_{\mathrm{c}}}(\Omega)$, $W^{1,q_{\mathrm{P}}}(\Omega)$ are algebras (i.e. closed under multiplication of functions belonging to them, see [6, p. 278]). In the first place, this gives $\hat{\mathbb{P}}^{-\top} = \operatorname{cof}\hat{\mathbb{P}} \in W^{1,q_{\mathrm{P}}}(\Omega;\mathbb{R}^{d\times d})$, as the components of $\operatorname{cof}\hat{\mathbb{P}}$ consist of sums of products of components of $\hat{\mathbb{P}}$. In the second place, we get $\operatorname{cof}\nabla\hat{\boldsymbol{y}} = [(\operatorname{cof}\nabla\hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top]\hat{\mathbb{P}}^{-\top} \in W^{1,\min\{q_{\mathrm{c}},q_{\mathrm{P}}\}}(\Omega;\mathbb{R}^{d\times d})$. Since $q_{\mathrm{g}} \le q_{\mathrm{c}}$ and $q_{\mathrm{g}} \le q_{\mathrm{P}}$ here, we see that $\operatorname{cof}\nabla\hat{\boldsymbol{y}} \in W^{1,q_{\mathrm{g}}}(\Omega;\mathbb{R}^{d\times d})$.

Using the trace operator's continuity, we conclude that $\hat{\boldsymbol{y}}|_{\Gamma_{\mathrm{D}}} = \hat{\boldsymbol{y}}_{\mathrm{Dir}}$. To see that $\hat{\mathbb{P}} \in \mathcal{Z}$, apply the continuity of the determinant function to (4.17). □

4.3 Coercivity and weak (semi)continuity

First we need a consequence of Young's inequality, mentioned in [25] at the beginning of Subsection 5.1.

Lemma 8. *Suppose that $a > 0$, $b > 0$, $\delta > 0$, $r > 1$. Then*

$$\frac{a}{b} \ge r\delta^{\frac{r}{r-1}}a^{\frac{1}{r}} - (r-1)\delta^{\frac{r^2}{(r-1)^2}}b^{\frac{1}{r-1}}.$$

Proof. Young's inequality (cf. [12, p. 622]) states that given a pair of positive numbers f, g and $1 < p, q < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$fg \le \frac{f^p}{p} + \frac{g^q}{q}. \tag{4.18}$$

Set $p = r$, $f = a^{\frac{1}{r}}$, $g = \delta^{\frac{r}{r-1}}b$ in (4.18). Then $q = \frac{r}{r-1}$ and Young's inequality yields

$$\frac{r-1}{r}\delta^{\frac{r^2}{(r-1)^2}}b^{\frac{r}{r-1}} + \frac{1}{r}a \ge \delta^{\frac{r}{r-1}}a^{\frac{1}{r}}b,$$

which after multiplying by $\frac{r}{b}$ implies the desired result. □

Lemma 9. *Let $t \in [0,T]$ and $\tilde{\mathbb{P}} \in \mathcal{Z}$ be given. Additionally, let q_{F}, q_{P}, $q_{\mathrm{Y}} \in [1,+\infty)$, $\frac{1}{q_{\mathrm{F}}} + \frac{1}{q_{\mathrm{P}}} = \frac{1}{q_{\mathrm{Y}}}$, $q_{\mathrm{F}} > q_{\mathrm{Y}}$ and $q_{\mathrm{Y}} > 1$. Then the functional $(\hat{\boldsymbol{y}},\hat{\mathbb{P}}) \mapsto \mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) + \mathcal{D}(\tilde{\mathbb{P}},\hat{\mathbb{P}})$ is bounded from below and weakly coercive with respect to $\mathcal{Y} \times \mathcal{Z}$.*

Proof. Choose $\hat{\boldsymbol{y}} \in \mathcal{Y}$ and $\hat{\mathbb{P}} \in \mathcal{Z}$. The growth condition on W (4.1) gives

$$\int_\Omega W(\nabla\hat{\boldsymbol{y}}(\boldsymbol{X})\hat{\mathbb{P}}^{-1}(\boldsymbol{X}), \hat{\mathbb{P}}(\boldsymbol{X}))\mathrm{d}\boldsymbol{X} \geq c\Big(\int_\Omega |\nabla\hat{\boldsymbol{y}}(\boldsymbol{X})\hat{\mathbb{P}}^{-1}|^{q_\mathrm{F}} + \big|(\mathrm{cof}\,\nabla\hat{\boldsymbol{y}}(\boldsymbol{X}))\hat{\mathbb{P}}^\top(\boldsymbol{X})\big|^{q_\mathrm{c}} + \Big|(\det\nabla\hat{\boldsymbol{y}}(\boldsymbol{X}))\,(\det(\hat{\mathbb{P}}^{-1}(\boldsymbol{X})))\Big|^{-q_\mathrm{d}} + |\hat{\mathbb{P}}(\boldsymbol{X})|^{q_\mathrm{P}}\mathrm{d}\boldsymbol{X}\Big). \tag{4.19}$$

The stated assumption ensures that $\frac{q_\mathrm{Y}}{q_\mathrm{F}} + \frac{q_\mathrm{Y}}{q_\mathrm{P}} = 1$. Thus by Hölder's inequality and multiplicativity of the Frobenius norm (see [25]), it follows that

$$\int_\Omega |\nabla\hat{\boldsymbol{y}}(\boldsymbol{X})\hat{\mathbb{P}}^{-1}|^{q_\mathrm{F}}\mathrm{d}\boldsymbol{X} \geq \frac{(\int_\Omega |\nabla\hat{\boldsymbol{y}}|^{q_\mathrm{Y}}\mathrm{d}\boldsymbol{X})^{\frac{q_\mathrm{F}}{q_\mathrm{Y}}}}{(\int_\Omega |\hat{\mathbb{P}}(\boldsymbol{X})|^{q_\mathrm{P}}\mathrm{d}\boldsymbol{X})^{\frac{q_\mathrm{F}}{q_\mathrm{P}}}}. \tag{4.20}$$

If $\nabla\hat{\boldsymbol{y}}\mathbb{P}^{-1} \notin L^{q_\mathrm{F}}$, the corresponding integral is infinite, but it does not invalidate (4.20).

Taking $r = \frac{q_\mathrm{F}}{q_\mathrm{Y}} > 1$, $a = ||\nabla\hat{\boldsymbol{y}}||_{q_\mathrm{Y}}^{q_\mathrm{F}}$, $b = ||\hat{\mathbb{P}}||_{q_\mathrm{P}}^{q_\mathrm{F}}$ in Lemma 8, we get for any $\delta > 0$

$$\frac{(\int_\Omega |\nabla\hat{\boldsymbol{y}}|^{q_\mathrm{Y}}\mathrm{d}\boldsymbol{X})^{\frac{q_\mathrm{F}}{q_\mathrm{Y}}}}{(\int_\Omega |\hat{\mathbb{P}}(\boldsymbol{X})|^{q_\mathrm{P}}\mathrm{d}\boldsymbol{X})^{\frac{q_\mathrm{F}}{q_\mathrm{P}}}} \geq \frac{q_\mathrm{F}}{q_\mathrm{Y}}\delta^{q_\mathrm{F}/(q_\mathrm{F}-q_\mathrm{Y})}||\nabla\hat{\boldsymbol{y}}||_{q_\mathrm{Y}}^{q_\mathrm{Y}} - \frac{q_\mathrm{F}-q_\mathrm{Y}}{q_\mathrm{Y}}\delta^{q_\mathrm{F}^2/(q_\mathrm{F}-q_\mathrm{Y})^2}||\hat{\mathbb{P}}||_{q_\mathrm{P}}^{q_\mathrm{P}}. \tag{4.21}$$

From the definition of the dissipation distance, we have $\mathcal{D}(\tilde{\mathbb{P}}, \hat{\mathbb{P}}) \geq 0$.

Thus, using (4.19) and the property $\det(\hat{\mathbb{P}}^{-1}) = 1$ a.e.,

$$\begin{aligned} \mathcal{E}(t, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) + \mathcal{D}(\tilde{\mathbb{P}}, \hat{\mathbb{P}}) \geq I(t, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) - \ell(t, \hat{\boldsymbol{y}}) \geq \\ c_1(||\nabla\hat{\boldsymbol{y}}\hat{\mathbb{P}}^{-1}||_{q_\mathrm{F}}^{q_\mathrm{F}} + ||(\mathrm{cof}\,\nabla\hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top||_{q_\mathrm{c}}^{q_\mathrm{c}} + ||(\det\nabla\hat{\boldsymbol{y}})^{-q_\mathrm{d}}||_1 + ||\hat{\mathbb{P}}||_{q_\mathrm{P}}^{q_\mathrm{P}} + \\ ||\underline{\nabla[(\mathrm{cof}\,\nabla\hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top]}||_{q_\mathrm{c}}^{q_\mathrm{c}} + ||\underline{\nabla\hat{\mathbb{P}}}||_{q_\mathrm{P}}^{q_\mathrm{P}}) - c_2||\hat{\boldsymbol{y}}||_{W^{1,q_\mathrm{Y}}}, \end{aligned} \tag{4.22}$$

where $c_1 = \min\{c, \alpha, \beta\}$, $c_2 = ||\ell(t, \cdot)||_{(W^{1,q_\mathrm{Y}}(\Omega;\mathbb{R}^d))^*}$.

Poincaré's inequality implies that

$$||\hat{\boldsymbol{y}}||_{W^{1,q_\mathrm{Y}}} \leq c_\mathrm{Pc}(||\nabla\hat{\boldsymbol{y}}||_{q_\mathrm{Y}} + ||\hat{\boldsymbol{y}}_\mathrm{Dir}||_{L^{q_\mathrm{Y}}(\Gamma_\mathrm{D};\mathbb{R}^d)}) \text{ for a constant } c_\mathrm{Pc} > 0.$$

We can combine (4.20) with (4.21) and take δ small enough to achieve

$$1 - \frac{q_\mathrm{F}-q_\mathrm{Y}}{q_\mathrm{Y}}\delta^{q_\mathrm{F}^2/(q_\mathrm{F}-q_\mathrm{Y})^2} =: c_\mathrm{P} > 0$$

and infer from (4.22)

$$\begin{aligned} \mathcal{E}(t, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) + \mathcal{D}(\tilde{\mathbb{P}}, \hat{\mathbb{P}}) \geq c_1\frac{q_\mathrm{F}}{q_\mathrm{Y}}\delta^{\frac{q_\mathrm{F}}{q_\mathrm{F}-q_\mathrm{Y}}}||\nabla\hat{\boldsymbol{y}}||_{q_\mathrm{Y}}^{q_\mathrm{Y}} + c_\mathrm{P}||\hat{\mathbb{P}}||_{q_\mathrm{P}}^{q_\mathrm{P}} + c_1||\underline{\nabla\hat{\mathbb{P}}}||_{q_\mathrm{P}}^{q_\mathrm{P}} \\ -c_2c_\mathrm{Pc}||\nabla\hat{\boldsymbol{y}}||_{q_\mathrm{Y}} - c_0, \end{aligned} \tag{4.23}$$

where $c_0 = c_2c_\mathrm{Pc}||\hat{\boldsymbol{y}}_\mathrm{Dir}||_{L^{q_\mathrm{Y}}(\Gamma_\mathrm{D};\mathbb{R}^d)}$. Hence the examined functional is bounded from below.

Moreover, if $||\hat{\boldsymbol{y}}||_{W^{1,q_\mathrm{Y}}(\Omega;\mathbb{R}^d)} \to +\infty$, $||\hat{\mathbb{P}}||_{W^{1,q_\mathrm{P}}(\Omega;\mathbb{R}^{d\times d})} \to +\infty$, $(\hat{\boldsymbol{y}}, \hat{\mathbb{P}}) \in \mathcal{Y} \times \mathcal{Z}$, then the left-hand side in (4.23) also tends to infinity, which proves that the functional is weakly coercive with respect to $\mathcal{Y} \times \mathcal{Z}$ (see Definition A.6). □

Lemma 10. *Under assumptions* (4.4)*, the mapping* $\mathcal{D}(\cdot, \tilde{\mathbb{P}})$ *is weakly sequentially continuous on* $\mathcal{Z}$ *for any* $\tilde{\mathbb{P}} \in \mathcal{Z}$.

Proof. Let $\hat{\mathbb{P}}_k \rightharpoonup \hat{\mathbb{P}}$ in $W^{1,q_\mathrm{P}}(\Omega;\mathbb{R}^{d\times d})$, then by compact embedding $\hat{\mathbb{P}}_k \to \hat{\mathbb{P}}$ in L^{q_P} and there is a subsequence that converges a.e. in Ω. Assumptions (4.4a), (4.4d) ensure that we can use the dominated convergence theorem. The limit $\mathcal{D}(\hat{\mathbb{P}},\tilde{\mathbb{P}})$ is the same for any subsequence, thus we draw the conclusion for the whole sequence. □

Theorem 11. *Under the hypotheses on Lebesgue exponents from the beginning of the chapter, from Theorem 6 and from Lemma 9, assume W to be measurable, non-negative and lower semicontinuous, satisfying the growth condition* (4.1) *and let $\ell \in \mathcal{C}([0,T];(W^{1,q_\mathrm{Y}}(\Omega;\mathbb{R}^d)^*)$. Further, let $t_n \in [0,T]$, $\hat{\boldsymbol{y}}_n \in \mathcal{Y}$, $\hat{\mathbb{P}}_n \in \mathcal{Z}$ and assume there is a constant $E_* > 0$ such that for every $n \in \mathbb{N}$, we have $\mathcal{E}(t_n,\hat{\boldsymbol{y}}_n,\hat{\mathbb{P}}_n) \leq E_*$. If $t_n \to t$ in $\mathbb{R}$, $\hat{\boldsymbol{y}}_n \rightharpoonup \hat{\boldsymbol{y}}$ in $W^{1,q_\mathrm{Y}}(\Omega;\mathbb{R}^d)$ and $\hat{\mathbb{P}}_n \rightharpoonup \hat{\mathbb{P}}$ in $W^{1,q_\mathrm{P}}(\Omega;\mathbb{R}^{d\times d})$, then*

$$\mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) \leq \liminf_{n\to+\infty} \mathcal{E}(t_n,\hat{\boldsymbol{y}}_n,\hat{\mathbb{P}}_n).$$

Proof. In the beginning, let us find a subsequence $\{(t_{n_j},\hat{\boldsymbol{y}}_{n_j},\hat{\mathbb{P}}_{n_j})\}$ such that

$$\liminf_{n\to+\infty} \mathcal{E}(t_n,\hat{\boldsymbol{y}}_n,\hat{\mathbb{P}}_n) = \lim_{j\to+\infty} \mathcal{E}(t_{n_j},\hat{\boldsymbol{y}}_{n_j},\hat{\mathbb{P}}_{n_j}).$$

All t_n, $\hat{\boldsymbol{y}}_n$, $\hat{\mathbb{P}}_n$ lie in a sublevel set of $\mathcal{E}$ (i.e. $\mathcal{E}(t_n,\hat{\boldsymbol{y}}_n,\hat{\mathbb{P}}_n) \leq E_*$). Like in the proof of Lemma 9, we get from this fact the estimate

$$\sup_{n\in\mathbb{N}} ||\nabla\hat{\boldsymbol{y}}_n||_{q_\mathrm{Y}} + ||\hat{\mathbb{P}}_n||_{q_\mathrm{P}} + ||\nabla\hat{\mathbb{P}}_n||_{q_\mathrm{P}} + ||(\operatorname{cof}\nabla\hat{\boldsymbol{y}}_n)\hat{\mathbb{P}}_n^\top||_{q_\mathrm{c}} + ||\underline{\nabla[(\operatorname{cof}\nabla\hat{\boldsymbol{y}}_n)\hat{\mathbb{P}}_n^\top}||_{q_\mathrm{c}} < +\infty.$$

Hence Theorem 6 gives subsequences (for which we do not explicitly change notation) $\{\hat{\mathbb{P}}_{n_j}\}$, $\{\hat{\boldsymbol{y}}_{n_j}\}$ with

$$\hat{\mathbb{P}}_{n_j} \to \hat{\mathbb{P}} \text{ a.e. in } \Omega, \tag{4.24}$$
$$\hat{\mathbb{P}}_{n_j}^{-1} \to \hat{\mathbb{P}}^{-1} \text{ a.e. in } \Omega, \tag{4.25}$$
$$\underline{\nabla\hat{\mathbb{P}}_{n_j}} \rightharpoonup \underline{\nabla\hat{\mathbb{P}}} \text{ in } L^{q_\mathrm{P}}(\Omega;\mathbb{R}^{d\times d\times d}), \tag{4.26}$$
$$\hat{\boldsymbol{y}}_{n_j} \rightharpoonup \hat{\boldsymbol{y}} \text{ in } W^{1,q_\mathrm{Y}}(\Omega;\mathbb{R}^d), \tag{4.27}$$
$$\nabla\hat{\boldsymbol{y}}_{n_j} \to \nabla\hat{\boldsymbol{y}} \text{ a.e. in } \Omega, \tag{4.28}$$
$$(\operatorname{cof}\nabla\hat{\boldsymbol{y}}_{n_j})\hat{\mathbb{P}}_{n_j}^\top \rightharpoonup (\operatorname{cof}\nabla\hat{\boldsymbol{y}})\hat{\mathbb{P}}^\top \text{ in } W^{1,q_\mathrm{c}}(\Omega;\mathbb{R}^{d\times d}), \quad j \to +\infty. \tag{4.29}$$

Now we proceed term by term.

Firstly, pointwise convergences (4.24), (4.28) and (4.25), non-negativity and lower semicontinuity of W, and Fatou's lemma yield

$$\int_\Omega W\left(\nabla\hat{\boldsymbol{y}}(\boldsymbol{X})\hat{\mathbb{P}}^{-1}(\boldsymbol{X}),\hat{\mathbb{P}}(\boldsymbol{X})\right)\mathrm{d}\boldsymbol{X} \leq \liminf_{j\to+\infty}\int_\Omega W\left((\nabla\hat{\boldsymbol{y}}_{n_j}(\boldsymbol{X})\hat{\mathbb{P}}_{n_j}^{-1}(\boldsymbol{X}),\hat{\mathbb{P}}_{n_j}(\boldsymbol{X})\right)\mathrm{d}\boldsymbol{X}.$$

Secondly, we obtain

$$\int_\Omega \left|\underline{\nabla[(\operatorname{cof}\nabla\hat{\boldsymbol{y}}(\boldsymbol{X}))\hat{\mathbb{P}}^\top(\boldsymbol{X})]}\right|^{q_\mathrm{c}}\mathrm{d}\boldsymbol{X} \leq \liminf_{j\to\infty}\int_\Omega \left|\underline{\nabla[(\operatorname{cof}\nabla\hat{\boldsymbol{y}}_{n_j}(\boldsymbol{X}))\hat{\mathbb{P}}_{n_j}^\top(\boldsymbol{X})]}\right|^{q_\mathrm{c}}\mathrm{d}\boldsymbol{X},$$
$$\int_\Omega \left|\underline{\nabla\hat{\mathbb{P}}(\boldsymbol{X})}\right|^{q_\mathrm{P}}\mathrm{d}\boldsymbol{X} \leq \liminf_{j\to\infty}\int_\Omega \left|\underline{\nabla\hat{\mathbb{P}}_{n_j}(\boldsymbol{X})}\right|^{q_\mathrm{P}}\mathrm{d}\boldsymbol{X}$$

by (4.29), (4.26) and weak sequential lower semicontinuity of norm on a Banach space.

Thirdly, weak convergence (4.27) and the continuity of ℓ in time lead to

$$\lim_{j\to+\infty} \ell(t_{n_j},\hat{\boldsymbol{y}}_{n_j}) = \ell(t,\hat{\boldsymbol{y}}).$$

In total,

$$\mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) \le \liminf_{j\to+\infty} \mathcal{E}(t_{n_j},\hat{\boldsymbol{y}}_{n_j},\hat{\mathbb{P}}_{n_j}) = \lim_{j\to+\infty} \mathcal{E}(t_{n_j},\hat{\boldsymbol{y}}_{n_j},\hat{\mathbb{P}}_{n_j}) = \liminf_{n\to+\infty} \mathcal{E}(t_n,\hat{\boldsymbol{y}}_n,\hat{\mathbb{P}}_n).$$

□

4.4 Power – the time-derivative of energy

In physics, (instantaneous) power is a quantity that describes how energy of a physical system changes in time (this can be done by work or heat transfer). Therefore we will refer to the partial time-derivative $\partial_t\mathcal{E}$ of $\mathcal{E}$ as the *power*.

We will need several mathematical properties of this quantity (i.e. an analogy of Theorem 5.3 in [25], which, in the case of a time-independent Dirichlet boundary condition, will be easier to show). The main idea of the proof is taken from [27], Section 2.2.

Theorem 12. *There exist constants $c_{\mathrm{E}}^{(0)} \in \mathbb{R}$ and $c_{\mathrm{E}}^{(1)} > 0$ such that for $(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) \in [0,T]\times\mathcal{Y}\times\mathcal{Z}$ with $\mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) < +\infty$, we have $\mathcal{E}(\cdot,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) \in \mathcal{C}^1([0,T])$ and the estimate*

$$|\partial_t\mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}})| \le c_{\mathrm{E}}^{(1)}(\mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) + c_{\mathrm{E}}^{(0)}) \tag{4.30}$$

holds. The constant $c_{\mathrm{E}}^{(0)}$ can be found such that $\forall\,(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) \in [0,T]\times\mathcal{Y}\times\mathcal{Z}:\ \mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) \ge -c_{\mathrm{E}}^{(0)}$.

Proof. Let $(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) \in [0,T]\times\mathcal{Y}\times\mathcal{Z}$ be such that $\mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) < +\infty$. Then $I(\hat{\boldsymbol{y}},\hat{\mathbb{P}}) < +\infty$ all the more.

As $\mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) = I(\hat{\boldsymbol{y}},\hat{\mathbb{P}}) - \ell(t,\hat{\boldsymbol{y}})$ and $\ell \in \mathcal{C}^1([0,T];(W^{1,q_{\mathrm{Y}}}(\Omega;\mathbb{R}^d))^*)$, differentiation gives $\partial_t\mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) = -\partial_t\ell(t,\hat{\boldsymbol{y}})$. Moreover, we see that we can take any $t \in [0,T]$ for the differentiation to be still correct (with one-sided derivatives considered at the endpoints of the interval). This shows that $\mathcal{E}(\cdot,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) \in \mathcal{C}^1([0,T])$.

We have already observed in the proof of Lemma 9 that

$$\mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) \ge C_1||\nabla\hat{\boldsymbol{y}}||_{q_{\mathrm{Y}}}^{q_{\mathrm{Y}}} - C_2||\ell(t,\cdot)||_{(W^{1,q_{\mathrm{Y}}}(\Omega;\mathbb{R}^d))^*}||\nabla\hat{\boldsymbol{y}}||_{q_{\mathrm{Y}}} - C_0,$$

which yields, by a subsequent estimate from below

$$\mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) \ge \tilde{C}_1||\nabla\hat{\boldsymbol{y}}||_{q_{\mathrm{Y}}} - \tilde{C}_0$$

for certain non-negative constants C_2, C_0, $\tilde{C}_0$ and $C_1 > 0$, $\tilde{C}_1 > 0$, independent of t. We used the fact that $\ell \in L^\infty((0,T);(W^{1,q_{\mathrm{Y}}}(\Omega;\mathbb{R}^d))^*)$.

Hence, with $c_{\mathrm{Po}} > 0$ from the Poincaré inequality and $C_3 = ||\hat{\boldsymbol{y}}_{\mathrm{Dir}}||_{L^{q_{\mathrm{Y}}}(\Gamma_{\mathrm{D}};\mathbb{R}^d)}$

$$\begin{aligned}|\partial_t\mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}})| = |\langle\partial_t\ell(t),\hat{\boldsymbol{y}}\rangle| \le ||\partial_t\ell(t,\cdot)||_{(W^{1,q_{\mathrm{Y}}})^*}||\hat{\boldsymbol{y}}||_{W^{1,q_{\mathrm{Y}}}} \le \\ c_{\mathrm{Po}}||\partial_t\ell||_{L^\infty((0,T);(W^{1,q_{\mathrm{Y}}})^*)}(||\nabla\hat{\boldsymbol{y}}||_{q_{\mathrm{Y}}} + C_3) \le \\ c_{\mathrm{Po}}||\partial_t\ell||_\infty(\frac{1}{\tilde{C}_1}\mathcal{E}(t,\hat{\boldsymbol{y}},\hat{\mathbb{P}}) + \frac{\tilde{C}_0}{\tilde{C}_1} + C_3).\end{aligned}$$

This shows estimate (4.30). The last assertion is a direct consequence of Lemma 9.

□

Whenever $t_1 \le s_0 \le t_2$ and $\mathcal{E}(s_0, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) < +\infty$, then from the previous theorem we get $\mathcal{E}(\cdot, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) \in \mathcal{C}^1([0,T])$ and by the fundamental theorem of calculus $\mathcal{E}(t_1, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) = \mathcal{E}(t_2, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) + \int_{t_1}^{t_2} \partial_s \mathcal{E}(s, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) \mathrm{d}s$. Adding $c_{\mathrm{E}}^{(0)}$ to both sides and estimating the power by (4.30), we obtain

$$\mathcal{E}(t_2, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) + c_{\mathrm{E}}^{(0)} \le \mathcal{E}(t_1, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) + c_{\mathrm{E}}^{(0)} + \int_{t_1}^{t_2} c_{\mathrm{E}}^{(1)} (c_{\mathrm{E}}^{(0)} + \mathcal{E}(s, \hat{\boldsymbol{y}}, \hat{\mathbb{P}})) \mathrm{d}s.$$

Grönwall's lemma (Theorem A.7) then provides (cf. [15])

$$\mathcal{E}(t_2, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) \le (c_{\mathrm{E}}^{(0)} + \mathcal{E}(t_1, \hat{\boldsymbol{y}}, \hat{\mathbb{P}})) \mathrm{e}^{c_{\mathrm{E}}^{(1)} |t_2 - t_1|} - c_{\mathrm{E}}^{(0)}. \tag{4.31}$$

4.5 Existence of a solution to the incremental problem

In this section, we prove the existence of a solution to incremental problem (4.6) by the direct method of the calculus of variations.

From now on, we shall assume that

$$M_t := \inf_{(\hat{y}, \hat{\mathbb{P}}) \in \mathcal{Y} \times \mathcal{Z}} [\mathcal{E}(t, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) + \mathcal{D}(\tilde{\mathbb{P}}, \hat{\mathbb{P}})] < +\infty \text{ for all } t \text{ in } [0,T] \text{ and } \tilde{\mathbb{P}} \in \mathcal{Z}. \tag{4.32}$$

Fix $\tilde{\mathbb{P}} \in \mathcal{Z}$ and $t \in [0,T]$. Having assumed that the infimum M_t is not $+\infty$ in (4.32), we deduce from the boundedness from below (Lemma 9) that $M_t \in \mathbb{R}$. We may take a minimising sequence $\{(\hat{\boldsymbol{y}}_n, \hat{\mathbb{P}}_n)\}_{n=1}^{\infty}$ in $\mathcal{Y} \times \mathcal{Z}$ such that

$$\lim_{n \to \infty} \mathcal{E}(t, \hat{\boldsymbol{y}}_n, \hat{\mathbb{P}}_n) + \mathcal{D}(\tilde{\mathbb{P}}, \hat{\mathbb{P}}_n) = M_t \tag{4.33}$$

by the definition of infimum. In this section, the symbols $\hat{\boldsymbol{y}}$, $\hat{\mathbb{P}}$ with numeral subscripts denote elements of the minimising sequence (the time being fixed), whereas after this section, we use the same notation for minimisers at different discrete time levels.

Our goal now is to find a subsequence with limit $(\hat{\boldsymbol{y}}, \hat{\mathbb{P}})$ and show that the functional attains a minimum at $(\hat{\boldsymbol{y}}, \hat{\mathbb{P}})$, i.e. $\mathcal{E}(t, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) + \mathcal{D}(\tilde{\mathbb{P}}, \hat{\mathbb{P}}) = M_t$.

Similarly as in the proof of Lemma 9, we derive that

$$\sup_{n \in \mathbb{N}} ||\nabla \hat{\boldsymbol{y}}_n||_{q_{\mathrm{Y}}} + ||\hat{\mathbb{P}}_n||_{q_{\mathrm{P}}} + ||\underline{\nabla \hat{\mathbb{P}}_n}||_{q_{\mathrm{P}}} + ||(\operatorname{cof} \nabla \hat{\boldsymbol{y}}_n) \hat{\mathbb{P}}_n^{\top}||_{q_{\mathrm{c}}} + ||\underline{\nabla[(\operatorname{cof} \nabla \hat{\boldsymbol{y}}_n) \hat{\mathbb{P}}_n^{\top}}||_{q_{\mathrm{c}}} < +\infty.$$

(The upper bound follows from the boundedness of the (convergent) sequence $\{\mathcal{E}(t, \hat{\boldsymbol{y}}_n, \hat{\mathbb{P}}_n) + \mathcal{D}(\tilde{\mathbb{P}}, \hat{\mathbb{P}}_n)\}$.)

In this situation, we would like to use Theorem 6.

Let us return to the growth condition (4.1), which yields

$$\sup_{n \ge n_0} \mathcal{E}(t, \hat{\boldsymbol{y}}_n, \hat{\mathbb{P}}_n) + \mathcal{D}(\tilde{\mathbb{P}}, \hat{\mathbb{P}}_n) + \ell(t, \hat{\boldsymbol{y}}_n) \ge$$
$$\sup_{n \ge n_0} \int_{\Omega} \frac{1}{|\det (\nabla \hat{\boldsymbol{y}}_n(\boldsymbol{X}) \hat{\mathbb{P}}_n^{-1}(\boldsymbol{X}))|^{q_{\mathrm{d}}}} \mathrm{d}\boldsymbol{X} = \sup_{n \ge n_0} \int_{\Omega} \frac{1}{|\det \nabla \hat{\boldsymbol{y}}_n(\boldsymbol{X})|^{q_{\mathrm{d}}}} \mathrm{d}\boldsymbol{X}$$

for a certain $n_0 \in \mathbb{N}$. The left-hand side of the above inequality is finite because of (4.33) and the boundedness of $\{\hat{\boldsymbol{y}}_n\}_{n=1}^{\infty}$ in $W^{1,q_{\mathrm{Y}}}(\Omega; \mathbb{R}^d)$.

This means that the assumptions of Theorem 6 are satisfied and we get convergent subsequences $\{\hat{\boldsymbol{y}}_{n_j}\}_{j=1}^{\infty}$, $\{\hat{\mathbb{P}}_{n_j}\}_{j=1}^{\infty}$ with respective limits $\hat{\boldsymbol{y}}$, $\hat{\mathbb{P}}$ in the sense of Theorem

6. The results of this theorem and of Theorem 7 together assert that $\hat{\boldsymbol{y}}$ is an element of $\mathcal{Y}$ and $\hat{\mathbb{P}} \in \mathcal{Z}$.

It remains to prove that the functional under consideration attains a minimum at $(\hat{\boldsymbol{y}}, \hat{\mathbb{P}})$. As $\hat{\boldsymbol{y}} \in \mathcal{Y}$, $\hat{\mathbb{P}} \in \mathcal{Z}$, it is clearly true that $\mathcal{E}(t, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) + \mathcal{D}(\tilde{\mathbb{P}}, \hat{\mathbb{P}}) \geq M_t$.

In order to obtain the opposite inequality, we argue by Theorem 11 (with a constant sequence of time values).

Also, property (4.4a) of D combined with (4.14b) and Fatou's lemma ensure that

$$\mathcal{D}(\tilde{\mathbb{P}}, \hat{\mathbb{P}}) \leq \liminf_{j \to +\infty} \mathcal{D}(\tilde{\mathbb{P}}, \hat{\mathbb{P}}_{n_j}).$$

Recalling (4.33), we combine the conclusion of Theorem 11 with the above inequality:

$$\mathcal{E}(t, \hat{\boldsymbol{y}}, \hat{\mathbb{P}}) + \mathcal{D}(\tilde{\mathbb{P}}, \hat{\mathbb{P}}) \leq \liminf_{j \to +\infty} \mathcal{E}(t, \hat{\boldsymbol{y}}_{n_j}, \hat{\mathbb{P}}_{n_j}) + \mathcal{D}(\tilde{\mathbb{P}}, \hat{\mathbb{P}}_{n_j}) = M_t,$$

which finishes the proof for incremental problem (4.6).

Therefore let us continue searching for an energetic solution of the introduced rate-independent system. Throughout the remaining part of the proof, which is presented in the rest of this chapter, we follow Section 3 in [15].

4.6 A priori estimates

The result below was established by A. Mielke and F. Theil and its proof can be found in [15] (Theorem 3.2).

Theorem 13. *Assume $(\hat{\boldsymbol{y}}_0, \hat{\mathbb{P}}_0) \in \mathcal{S}(0)$, then every solution $(\hat{\boldsymbol{y}}_k, \hat{\mathbb{P}}_k)$ of* (4.6) *satisfies the discrete versions* (S_d) *and* (E_d) *of stability* (S) *and energy equality* (E)*, namely for all $k \in \{1,\ 2, \dots N\}$ we have*

$$(\hat{\boldsymbol{y}}_k, \hat{\mathbb{P}}_k) \in \mathcal{S}(t_k), \tag{S_d}$$

$$\begin{aligned} \mathcal{E}(t_k, \hat{\boldsymbol{y}}_k, \hat{\mathbb{P}}_k) - \mathcal{E}(t_{k-1}, \hat{\boldsymbol{y}}_{k-1}, \hat{\mathbb{P}}_{k-1}) + \mathcal{D}(\hat{\mathbb{P}}_{k-1}, \hat{\mathbb{P}}_k) \in \\ \Big[\int_{t_{k-1}}^{t_k} \partial_t \mathcal{E}(s, \hat{\boldsymbol{y}}_k, \hat{\mathbb{P}}_k) \mathrm{d}s,\ \int_{t_{k-1}}^{t_k} \partial_t \mathcal{E}(s, \hat{\boldsymbol{y}}_{k-1}, \hat{\mathbb{P}}_{k-1}) \mathrm{d}s\Big]. \end{aligned} \tag{E_d}$$

Moreover, we have the a priori estimates

$$\begin{aligned} \mathcal{E}(t_k, \hat{\boldsymbol{y}}_k, \hat{\mathbb{P}}_k) &\leq (\mathcal{E}(0, \hat{\boldsymbol{y}}_0, \hat{\mathbb{P}}_0) + c_{\mathrm{E}}^{(0)}) \mathrm{e}^{c_{\mathrm{E}}^{(1)} t_k} - c_{\mathrm{E}}^{(0)} \quad \textit{and} \\ \sum_{j=1}^{N} \mathcal{D}(\hat{\mathbb{P}}_{j-1}, \hat{\mathbb{P}}_j) &\leq (\mathcal{E}(0, \hat{\boldsymbol{y}}_0, \hat{\mathbb{P}}_0) + c_{\mathrm{E}}^{(0)}) \mathrm{e}^{c_{\mathrm{E}}^{(1)} T}. \end{aligned}$$

Properties (S_d) and (E_d) are consequences of minimisation and triangle inequality for $\mathcal{D}$. To show the a priori estimates, it is advantageous to use results from Section 4.4.

So far, we only have minimisers at discrete time levels. The next step is to define piecewise constant interpolants $(\boldsymbol{y}^N, \mathbb{P}^N) \colon [0, T] \to \mathcal{Y} \times \mathcal{Z}$ via

$$(\boldsymbol{y}^N(t), \mathbb{P}^N(t)) = \begin{cases} (\hat{\boldsymbol{y}}_k, \hat{\mathbb{P}}_k) & \text{for } t \in [t_k, t_{k+1}) \text{ with } k \in \{0, 1, \dots, N-1\} \\ (\hat{\boldsymbol{y}}_N, \hat{\mathbb{P}}_N) & \text{for } t = t_N = T. \end{cases} \tag{4.34}$$

These step functions are uniformly bounded in a way and this will help us find convergent subsequences as the norm of the partition of $[0, T]$ approaches zero. More precisely, choose $t \in [0, T)$, then there exists a $j \in \{0, 1, \dots, N-1\}$ such that $t \in$

$[t_j, t_{j+1})$. The definition of $(\boldsymbol{y}^N, \mathbb{P}^N)$ and the fundamental theorem of calculus then imply

$$\mathcal{E}(t, \boldsymbol{y}^N(t), \mathbb{P}^N(t)) = \mathcal{E}(t, \hat{\boldsymbol{y}}_j, \hat{\mathbb{P}}_j) = \mathcal{E}(t_j, \hat{\boldsymbol{y}}_j, \hat{\mathbb{P}}_j) + \int_{t_j}^{t} \partial_s \mathcal{E}(s, \hat{\boldsymbol{y}}_j, \hat{\mathbb{P}}_j) \mathrm{d}s. \tag{4.35}$$

By virtue of (4.30) and (4.31), let us write

$$\int_{t_j}^{t} \partial_s \mathcal{E}(s, \hat{\boldsymbol{y}}_j, \hat{\mathbb{P}}_j) \mathrm{d}s \leq \int_{t_j}^{t} c_{\mathrm{E}}^{(1)} (\mathcal{E}(s, \hat{\boldsymbol{y}}_j, \hat{\mathbb{P}}_j) + c_{\mathrm{E}}^{(0)}) \mathrm{d}s \leq$$
$$\int_{t_j}^{t} c_{\mathrm{E}}^{(1)} (\mathcal{E}(0, \hat{\boldsymbol{y}}_0, \hat{\mathbb{P}}_0) + c_{\mathrm{E}}^{(0)}) \mathrm{e}^{c_{\mathrm{E}}^{(1)} s} \mathrm{d}s = c_{\mathrm{E}}^{(1)} (\mathcal{E}(0, \hat{\boldsymbol{y}}_0, \hat{\mathbb{P}}_0) + c_{\mathrm{E}}^{(0)}) \Big[\frac{1}{c_{\mathrm{E}}^{(1)}} \mathrm{e}^{c_{\mathrm{E}}^{(1)} s} \Big]_{s=t_j}^{t} =$$
$$(\mathcal{E}(0, \hat{\boldsymbol{y}}_0, \hat{\mathbb{P}}_0) + c_{\mathrm{E}}^{(0)}) (\mathrm{e}^{c_{\mathrm{E}}^{(1)} t} - \mathrm{e}^{c_{\mathrm{E}}^{(1)} t_j}).$$

Inserting this back into (4.35), where we replace $\mathcal{E}(t_j, \hat{\boldsymbol{y}}_j, \hat{\mathbb{P}}_j)$ using the a priori estimate from Theorem 13, we see that we have derived the bound

$$\mathcal{E}(t, \boldsymbol{y}^N(t), \mathbb{P}^N(t)) \leq (\mathcal{E}(0, \hat{\boldsymbol{y}}_0, \hat{\mathbb{P}}_0) + c_{\mathrm{E}}^{(0)}) \mathrm{e}^{c_{\mathrm{E}}^{(1)} t} - c_{\mathrm{E}}^{(0)}. \tag{4.36}$$

For the total dissipation along $\mathbb{P}^N$, the a priori bound from Theorem 13 shows that

$$\mathrm{Diss}_{\mathcal{D}}(\mathbb{P}^N; [0,T]) = \sum_{j=1}^{N} \mathcal{D}(\mathbb{P}_{j-1}, \mathbb{P}_j) \leq (\mathcal{E}(0, \hat{\boldsymbol{y}}_0, \hat{\mathbb{P}}_0) + c_{\mathrm{E}}^{(0)}) \mathrm{e}^{c_{\mathrm{E}}^{(1)} T}. \tag{4.37}$$

The weak coercivity of $\mathcal{E}$ (Lemma 9) guarantees the boundedness of its sublevel sets. Owing to this, we infer from (4.36) that

$$||\boldsymbol{y}^N||_{L^\infty((0,T);W^{1,q_{\mathrm{Y}}}(\Omega;\mathbb{R}^d))} \leq C_*, \quad ||\mathbb{P}^N||_{L^\infty((0,T);W^{1,q_{\mathrm{P}}}(\Omega;\mathbb{R}^{d\times d}))} \leq C_*,$$
$$||(\mathrm{cof}\, \nabla \boldsymbol{y}^N)(\mathbb{P}^N)^\top||_{L^\infty((0,T);W^{1,q_{\mathrm{C}}}(\Omega;\mathbb{R}^{d\times d}))} \leq C_*, \; ||(\det \nabla \boldsymbol{y}^N)^{-q_{\mathrm{d}}}||_{L^\infty((0,T);L^1(\Omega))} \leq C_*, \tag{4.38}$$

where C_* is independent of N (see the proof of Lemma 9 for the last two inequalities).

Let us note that if we write (E_d) for all indices k between $n{+}1$ and j, $0 \leq n < j \leq N$, and add up the resulting inequalities, we obtain a discrete upper energy estimate

$$\begin{aligned} &\mathcal{E}(t_j, \boldsymbol{y}^N(t_j), \mathbb{P}^N(t_j)) + \mathrm{Diss}_{\mathcal{D}}(\mathbb{P}^N; [t_n, t_j]) \leq \\ &\qquad \mathcal{E}(t_n, \boldsymbol{y}^N(t_n), \mathbb{P}^N(t_n)) + \int_{t_n}^{t_j} \theta^N(s) \mathrm{d}s, \end{aligned} \tag{4.39}$$

where $\theta^N(s) = \partial_t \mathcal{E}(s, \boldsymbol{y}^N(s), \mathbb{P}^N(s))$.

4.7 Selection of subsequences

Take a sequence $\{\Pi^N\}_{N=1}^{\infty}$ of nested partitions[3] of the interval $[0,T]$, $\Pi^N = \{t_k^N\}_{k=0}^{N}$, such that $\lim_{N\to+\infty} \nu(\Pi^N) = 0$. The piecewise constant approximation on Π^N is denoted by $(\boldsymbol{y}^N, \mathbb{P}^N)$, $N \in \mathbb{N}$, as in (4.34).

Let us cite a generalisation of Helly's theorem from [24] (Theorem 3.2) and slightly adapt the statement to our setting. For the sake of completeness, we prove it again here in our more specific case.

[3] See Definition A.1.

Theorem 14. *Let* $\mathcal{Z} \subset W^{1,q_\mathrm{P}}(\Omega; \mathbb{R}^{d\times d})$ *and let* $\mathcal{Z}' \subset \mathcal{Z}$ *be a weakly sequentially compact subset. Assume that* $\mathcal{D}$ *and* $\mathcal{Z}$ *satisfy the following compatibility conditions:*

$$\text{The functional } \mathcal{D}(\cdot,\cdot)\colon \mathcal{Z}\times\mathcal{Z} \to [0,+\infty] \text{ is weakly sequentially lower semicontinuous.} \tag{C1}$$

$$\text{If } (t_k, \hat{\mathbb{P}}_k) \in [0,T]\times\mathcal{Z}',\ k\in\mathbb{N},\ \text{with } t_k \to t \text{ and } \min\{\mathcal{D}(\hat{\mathbb{P}}_k,\hat{\mathbb{P}}), \mathcal{D}(\hat{\mathbb{P}},\hat{\mathbb{P}}_k)\} \to 0, \text{ then } \hat{\mathbb{P}}_k \rightharpoonup \hat{\mathbb{P}},\ k\to+\infty. \tag{C2}$$

Consider a sequence of functions $\mathbb{P}^N\colon [0,T] \to \mathcal{Z}'$ *such that there exists a constant* $C>0$ *such that* $\mathrm{Diss}_\mathcal{D}(\mathbb{P}^N;[0,T]) \le C$ *for all* $N\in\mathbb{N}$. *Then, there exists a subsequence* $\{\mathbb{P}^{N_k}\}_{k\in\mathbb{N}}$ *and functions* $\delta_\infty \in \mathrm{BV}([0,T],\mathbb{R})$, $\mathbb{P}\in \mathrm{BV}_\mathcal{D}([0,T],\mathcal{Z})$ *such that the following holds:*

(a) $\delta_{N_k}(t) := \mathrm{Diss}_\mathcal{D}(\mathbb{P}^{N_k};[0,t]) \to \delta_\infty(t)$ *for all* $t\in[0,T]$ *and* $k\to+\infty$,

(b) $\mathbb{P}^{N_k}(t) \rightharpoonup \mathbb{P}(t)$ *for all* $t\in[0,T]$, $k\to+\infty$.

(c) $\mathrm{Diss}_\mathcal{D}(\mathbb{P};[t,t']) \le \delta_\infty(t') - \delta_\infty(t)$ *for all* $0\le t<t'\le T$.

Proof. By additivity of total dissipation (4.5), the functions

$$\delta_N(t) = \mathrm{Diss}_\mathcal{D}(\mathbb{P}^N;[0,t])$$

are nondecreasing with values in $[0,C]$. Helly's selection theorem A.1 provides a subsequence $\{\delta_{N_l}\}_{l=1}^\infty$ which converges to a function δ_∞ pointwise in $[0,T]$. Hence we have shown *(a)*.

The function δ_∞ is monotone and bounded, hence the set $J\subset[0,T]$ of all its points of discontinuity is countable [31, Th. 8.13]. Write $Q=[0,T]\cap\mathbb{Q}$, $R=J\cup Q$, then the set R is still countable and dense in $[0,T]$. So we can arrange all elements of R in a sequence $\{t_m\}_{m=1}^\infty$. We first need to get convergence of the plastic tensors in R, by a diagonal argument.

Since $\{\mathbb{P}^N(t_1)\}_{N\in\mathbb{N}}$ lie in the weakly sequentially compact set $\mathcal{Z}'\subset\mathcal{Z}$, there is an index set $I_1\subset\mathbb{N}$ such that the subsequence $\{\mathbb{P}^N(t_1)\}_{N\in I_1}$ is weakly convergent in $W^{1,q_\mathrm{P}}(\Omega;\mathbb{R}^{d\times d})$. Let us denote by $\mathbb{P}(t_1)$ its limit. Moving on to t_2, we similarly obtain an index set $I_2\subset I_1$ such that $\mathbb{P}^N(t_2)$ converge weakly to some limit $\mathbb{P}(t_2)$ for $N\in I_2$. Gradually we construct infinite index sets I_m, $m=1,2,3,\dots$, with the property that $I_{m+1}\subset I_m$, and a mapping $\mathbb{P}\colon R\to\mathcal{Z}$ whose values are the corresponding pointwise weak limits, i.e. $\mathbb{P}^N(t_m)\rightharpoonup\mathbb{P}(t_m)$ for $N\in I_m$. Picking an $N_k\in I_k$ for every $k\in\mathbb{N}$, we find a diagonal sequence with $\mathbb{P}^{N_k}(t)\rightharpoonup\mathbb{P}(t)$ for every $t\in R$.

Now we wish to extend $\mathbb{P}$ to $[0,T]\backslash R$. Let $t\in[0,T]\backslash R$. As $\{\mathbb{P}^{N_k}(t)\}_{k=1}^\infty\subset\mathcal{Z}'$ and $\mathcal{Z}'$ is a compact Hausdorff topological space, there is a cluster point $\mathbb{P}(t)$ of $\{\mathbb{P}^{N_k}(t)\}_{k=1}^\infty$ [11, Th. 3.1.23].

If this cluster point were unique, we would have $\mathbb{P}^{N_k}(t)\rightharpoonup\mathbb{P}(t)$ and *(b)* would follow.

To establish the uniqueness of the cluster point, we notice that even though $\mathbb{P}(t)$ was chosen arbitrarily among all cluster points of $\{\mathbb{P}^{N_k}(t)\}_{k=1}^\infty$, any sequence $\{\mathbb{P}(\tilde t_i)\}_{i=1}^\infty$, with $\tilde t_i\in R$, $\tilde t_i\to t$, converges to it. If there were two distinct clusters points, then by the previous property, there would exist a sequence converging to both of them, which is impossible in the weak topology (being a Hausdorff topology).

It remains to show that property. Given $\tilde t_i\in R$, $\lim_{i\to+\infty}\tilde t_i = t$, we have to justify that $\mathbb{P}(\tilde t_i)\rightharpoonup\mathbb{P}(t)$. By a property of cluster points, there is a subsequence $\{\tilde N_\gamma\}_{\gamma=1}^\infty\subset\{N_k\}_{k=1}^\infty$ such that $\mathbb{P}^{\tilde N_\gamma}(t)\rightharpoonup\mathbb{P}(t)$ as γ tends to infinity.

Choose some $i\in\mathbb{N}$, then either $\tilde t_i<t$ or $\tilde t_i>t$.

In the case $\tilde{t}_i < t$, (C1), additivity (4.5) of $\mathrm{Diss}_{\mathcal{D}}$ and (a) imply that

$$0 \le \mathcal{D}(\mathbb{P}(\tilde{t}_i), \mathbb{P}(t)) \le \liminf_{\gamma\to+\infty} \mathcal{D}(\mathbb{P}^{\tilde{N}_\gamma}(\tilde{t}_i), \mathbb{P}^{\tilde{N}_\gamma}(t)) \le$$
$$\liminf_{\gamma\to+\infty} \mathrm{Diss}_{\mathcal{D}}(\mathbb{P}^{\tilde{N}_\gamma}; [\tilde{t}_i, t]) = \liminf_{\gamma\to+\infty} \delta_{\tilde{N}_\gamma}(t) - \delta_{\tilde{N}_\gamma}(\tilde{t}_i) = \delta_\infty(t) - \delta_\infty(\tilde{t}_i).$$

Analogously, if $\tilde{t}_i > t$, it can be derived that $0 \le \mathcal{D}(\mathbb{P}(t), \mathbb{P}(\tilde{t}_i)) \le \delta_\infty(\tilde{t}_i) - \delta_\infty(t)$. So whichever case we are in, it is true that

$$0 \le \min\{\mathcal{D}(\mathbb{P}(\tilde{t}_i), \mathbb{P}(t)), \mathcal{D}(\mathbb{P}(t), \mathbb{P}(\tilde{t}_i))\} \le |\delta_\infty(t) - \delta_\infty(\tilde{t}_i)| \to 0 \text{ for } i \to +\infty$$

by continuity of δ_∞ at t. (The points of discontinuity of δ_∞ were cleverly moved into the set R and treated separately.)

It is now clear that it suffices to apply (C2), whereby we get $\mathbb{P}(\tilde{t}_i) \rightharpoonup \mathbb{P}(t)$, which was to be shown to prove *(b)*.

Assertion *(c)* has $\mathrm{Diss}_{\mathcal{D}}(\mathbb{P}; [t, t'])$ on the left-hand side of the inequality and this term is a supremum of certain sums. So it is enough to show the inequality for one such sum – take $L \in \mathbb{N}$ and any partition $\{\hat{t}_j\}_{j=0}^{L}$ of $[t, t']$, then by (C1), (4.5) and *(a)*,

$$\sum_{j=0}^{L-1} \mathcal{D}(\mathbb{P}(\hat{t}_j), \mathbb{P}(\hat{t}_{j+1})) \le \liminf_{k\to+\infty} \sum_{j=0}^{L-1} \mathcal{D}(\mathbb{P}^{N_k}(\hat{t}_j), \mathbb{P}^{N_k}(\hat{t}_{j+1})) \le$$
$$\liminf_{k\to+\infty} \mathrm{Diss}_{\mathcal{D}}(\mathbb{P}^{N_k}; [t, t']) = \liminf_{k\to+\infty} \delta^{N_k}(t') - \delta^{N_k}(t) = \delta_\infty(t') - \delta_\infty(t)$$

and *(c)* is proved. □

Let us examine whether Theorem 14 can help us find a convergent subsequence of the piecewise constant approximations.

The set of plastic tensors $\mathcal{Z}$ is weakly sequentially closed (if $\hat{\mathbb{P}}_k \in \mathcal{Z}$, $\hat{\mathbb{P}}_k \rightharpoonup \mathbb{P}$ in $W^{1,q_{\mathrm{P}}}(\Omega; \mathbb{R}^{d\times d})$, then by compact embedding into $L^{q_{\mathrm{P}}}(\Omega; \mathbb{R}^{d\times d})$ we get strong convergence and pointwise convergence a.e. for a subsequence $\hat{\mathbb{P}}_{k_j}$, which implies that $1 = \det \hat{\mathbb{P}}_{k_j} \to \det \hat{\mathbb{P}}$ a.e. in Ω). However, to apply Theorem 14, we need a smaller set $\mathcal{Z}' \subset \mathcal{Z}$ that is weakly sequentially *compact*. The existence of such a set is a consequence of a priori bound (4.38) and the reflexivity of $W^{1,q_{\mathrm{P}}}(\Omega; \mathbb{R}^{d\times d})$.

Property (C1) holds (one can proceed similarly to Lemma 10). To verify (C2), we use Lemma 4.1 from [25] (the properties of D easily ensure those of $\mathcal{D}$):

Lemma 15. *If* (4.4) *and* (C1) *hold, then we also have the following: if* $\{\hat{\mathbb{P}}_k\}_{k\in\mathbb{N}}$ *is bounded and if* $\min\{\mathcal{D}(\hat{\mathbb{P}}_k, \hat{\mathbb{P}}), \mathcal{D}(\hat{\mathbb{P}}, \hat{\mathbb{P}}_k)\} \to 0$, *then* $\hat{\mathbb{P}}_k \rightharpoonup \hat{\mathbb{P}}$.

Proof. For a proof, see [25]. □

The boundedness of $\mathrm{Diss}_{\mathcal{D}}(\mathbb{P}^N; [0, T])$ is a consequence of (4.37). So the assumptions of Theorem 14 are satisfied.

Theorem 14 gives rise to a subsequence $\{\mathbb{P}^{N_k}\}_{k\in\mathbb{N}}$ and a limit function

$$\mathbb{P} \in L^\infty((0, T); W^{1,q_{\mathrm{P}}}(\Omega; \mathbb{R}^{d\times d})) \cap \mathrm{BV}([0, T]; L^1(\Omega; \mathbb{R}^{d\times d}))$$

such that

$$\forall t \in [0, T] : \quad \mathbb{P}^{N_k}(t) \to \mathbb{P}(t) \text{ strongly in } L^{q_{\mathrm{P}}} \text{ and weakly in } W^{1,q_{\mathrm{P}}}.$$

Besides, another part of that theorem establishes the existence of

$$\lim_{k\to+\infty} \mathrm{Diss}_{\mathcal{D}}(\mathbb{P}^{N_k}; [0, t]) =: \delta_\infty(t) \tag{4.40}$$

for all $t \in [0,T]$. Let us point out that $\mathbb{P} \in L^\infty((0,T); W^{1,q_{\mathrm{P}}}(\Omega; \mathbb{R}^{d\times d}))$ was obtained by (4.38) and the weak lower semicontinuity of $||\cdot||_{W^{1,q_{\mathrm{P}}}}$; whereas

$$\mathbb{P} \in \mathrm{BV}([0,T]; L^1(\Omega; \mathbb{R}^{d\times d}))$$

is due to (4.4e).

The functions θ^{N_k}, $k \in \mathbb{N}$, are contained in a bounded subset of $L^\infty((0,T))$, thanks to (4.30) and (4.36). Since $L^\infty((0,T))$ has a separable predual, we deduce a (non-relabelled) subsequence $\{\theta^{N_k}\}_{k=1}^\infty$ such that

$$\theta^{N_k} \overset{*}{\rightharpoonup} \theta_* \text{ in } L^\infty((0,T)) \tag{4.41}$$

by Claim A.8. In quest of a pointwise limit, let us define for every $t \in [0,T]$

$$\theta(t) = \limsup_{k\to+\infty} \theta^{N_k}(t) \in \mathbb{R}.$$

Applying reverse Fatou's lemma A.4 and the lemma of Du Bois-Reymond, we obtain $\theta_*(t) \le \theta(t)$ a.e.

The limit superior is a cluster point of a sequence. Thus t-dependent subsequences $\{N_l^t\}_{l=1}^\infty$ of $\{N_k\}_{k=1}^\infty$ can be chosen so that

$$\theta^{N_l^t}(t) \to \theta(t) \quad \text{for all } t \in [0,T] \text{ and } l \to +\infty. \tag{4.42}$$

On the basis of (4.38) and Theorem 6, the t-dependent subsequence $\{N_l^t\}_{l=1}^\infty$ could be chosen so that additionally

$$\boldsymbol{y}^{N_l^t}(t) \rightharpoonup \boldsymbol{y}(t) \quad \text{in } W^{1,q_{\mathrm{Y}}}(\Omega; \mathbb{R}^d),\ l \to +\infty \text{ for all } t \in [0,T] \text{ and some } \boldsymbol{y}(t)$$

and $\boldsymbol{y}(t) \in \mathcal{Y}$ by Theorems 6 and 7 (it may involve returning to (4.38) to extract auxiliary subsequences). Likewise, $\mathbb{P}(t) \in \mathcal{Z}$, since $\mathcal{Z}$ is weakly sequentially closed. The function $\boldsymbol{y}\colon [0,T] \to \mathcal{Y}$ is bounded but possibly non-measurable. It remains to check that $(\boldsymbol{y}, \mathbb{P})$ is an energetic solution.

4.8 Stability of the limit process

One property of energetic solutions is (S). To prove that $(\boldsymbol{y}(t), \mathbb{P}(t))$ fulfils it for every $t \in [0,T]$, consider another pair $(\tilde{\boldsymbol{y}}, \tilde{\mathbb{P}}) \in \mathcal{Y} \times \mathcal{Z}$.

Set $\tau_l^t = \max\{\tilde{t} \in \Pi^{N_l^t};\ \tilde{t} \le t\}$, then $\tau_l^t \nearrow t$ for $l \to +\infty$. From (S$_d$) and the fact that $\boldsymbol{y}^{N_l^t}(t) = \boldsymbol{y}^{N_l^t}(\tau_l^t)$, $\mathbb{P}^{N_l^t}(t) = \mathbb{P}^{N_l^t}(\tau_l^t)$, it is easily seen that

$$(\boldsymbol{y}^{N_l^t}(t), \mathbb{P}^{N_l^t}(t)) \in \mathcal{S}(\tau_l^t). \tag{4.43}$$

Using Theorem 11, $(\boldsymbol{y}^{N_l^t}(t), \mathbb{P}^{N_l^t}(t)) \in \mathcal{S}(\tau_l^t)$, continuity of $\mathcal{E}$ in t and Lemma 10, we get

$$\mathcal{E}(t, \boldsymbol{y}(t), \mathbb{P}(t)) \le \liminf_{l\to+\infty} \mathcal{E}(\tau_l^t, \boldsymbol{y}^{N_l^t}(t), \mathbb{P}^{N_l^t}(t)) \le$$
$$\liminf_{j\to+\infty} \mathcal{E}(\tau_l^t, \tilde{\boldsymbol{y}}, \tilde{\mathbb{P}}) + \mathcal{D}(\mathbb{P}^{N_l^t}(t), \tilde{\mathbb{P}}) = \mathcal{E}(t, \tilde{\boldsymbol{y}}, \tilde{\mathbb{P}}) + \mathcal{D}(\mathbb{P}(t), \tilde{\mathbb{P}})$$

and this is what (S) requires. Note that the first inequality above, together with (4.36), implies that $\mathcal{E}(t, \boldsymbol{y}(t), \mathbb{P}(t)) < +\infty$.

4.9 Upper energy estimate

The second property of energetic solutions is (E) and we show it as a conjunction of two inequalities.

From the weak sequential lower semicontinuity of $\mathcal{E}(t,\cdot,\cdot)$, $t \in [0,T]$ arbitrary, we already know that

$$\mathcal{E}(t,\boldsymbol{y}(t),\mathbb{P}(t)) \le \liminf_{l\to+\infty} \mathcal{E}(t,\boldsymbol{y}^{N_l^t}(t),\mathbb{P}^{N_l^t}(t)).$$

A question is whether we can strengthen this to an equality and replace the limit inferior by a limit. The answer is yes, if we employ the continuity of $\mathcal{E}$ in time, weak sequential continuity of $\mathcal{D}$ (Lemma 10) and (4.43):

$$\mathcal{E}(t,\boldsymbol{y}(t),\mathbb{P}(t)) = \lim_{l\to+\infty} \mathcal{E}(\tau_l^t,\boldsymbol{y}(t),\mathbb{P}(t)) + \mathcal{D}(\mathbb{P}^{N_l^t}(t),\mathbb{P}(t)) \ge$$
$$\limsup_{l\to+\infty} \mathcal{E}(\tau_l^t,\boldsymbol{y}^{N_l^t}(t),\mathbb{P}^{N_l^t}(t)) = \limsup_{l\to+\infty} \mathcal{E}(t,\boldsymbol{y}^{N_l^t}(t),\mathbb{P}^{N_l^t}(t)).$$

The last equality was due to the properties of ℓ. Proving the convergence of the power,

$$\begin{aligned}\partial_t\mathcal{E}(t,\boldsymbol{y}(t),\mathbb{P}(t)) = -\langle\partial_t\ell(t),\boldsymbol{y}(t)\rangle = -\lim_{l\to+\infty}\langle\partial_t\ell(t),\boldsymbol{y}^{N_l^t}(t)\rangle = \\ \lim_{l\to+\infty}\partial_t\mathcal{E}(t,\boldsymbol{y}^{N_l^t}(t),\mathbb{P}^{N_l^t}(t))\end{aligned} \tag{4.44}$$

is even easier (in our case of time-independent Dirichlet data).

Hence pointwise convergence (4.42) gives

$$\theta(t) = \lim_{l\to+\infty}\theta^{N_l^t}(t) \overset{\mathsf{def}}{=} \lim_{l\to+\infty}\partial_t\mathcal{E}(t,\boldsymbol{y}^{N_l^t}(t),\mathbb{P}^{N_l^t}(t)) = \partial_t\mathcal{E}(t,\boldsymbol{y}(t),\mathbb{P}(t)).$$

Note that we have also shown that the mapping $t\mapsto\partial_t\mathcal{E}(t,\boldsymbol{y}(t),\mathbb{P}(t))$ is measurable, for θ was defined as a pointwise limit superior of measurable functions.

Lemma 16. *Provided $t\in[0,T]$, there is a constant $C>0$ such that for all $l\in\mathbb{N}$,*

$$\begin{aligned}\mathcal{E}(t,\boldsymbol{y}^{N_l^t}(t),\mathbb{P}^{N_l^t}(t)) + \mathrm{Diss}_{\mathcal{D}}(\mathbb{P}^{N_l^t};[0,t]) \le \\ \mathcal{E}(\tau_l^t,\boldsymbol{y}^{N_l^t}(\tau_l^t),\mathbb{P}^{N_l^t}(\tau_l^t)) + \mathrm{Diss}_{\mathcal{D}}(\mathbb{P}^{N_l^t};[0,\tau_l^t]) + C\,\nu(\Pi^{N_l^t}).\end{aligned} \tag{4.45}$$

Proof. First let us note that $\mathrm{Diss}_{\mathcal{D}}(\mathbb{P}^{N_l^t};[0,t]) = \mathrm{Diss}_{\mathcal{D}}(\mathbb{P}^{N_l^t};[0,\tau_l^t])$ since $\mathbb{P}^{N_l^t}\big|_{[\tau_l^t,t]}$ is constant.

Then we compute, with (4.30), (4.31) in mind,

$$\mathcal{E}(t,\boldsymbol{y}^{N_l^t}(t),\mathbb{P}^{N_l^t}(t)) - \mathcal{E}(\tau_l^t,\boldsymbol{y}^{N_l^t}(\tau_l^t),\mathbb{P}^{N_l^t}(\tau_l^t)) = \int_{\tau_l^t}^t \partial_s\mathcal{E}(s,\boldsymbol{y}^{N_l^t}(s),\mathbb{P}^{N_l^t}(s))\mathrm{d}s \le$$
$$\int_{\tau_l^t}^t c_{\mathrm{E}}^{(1)}(c_{\mathrm{E}}^{(0)} + \mathcal{E}(s,\boldsymbol{y}^{N_l^t}(s),\mathbb{P}^{N_l^t}(s))\mathrm{d}s \le \int_{\tau_l^t}^t c_{\mathrm{E}}^{(1)}(c_{\mathrm{E}}^{(0)} + \mathcal{E}(0,\hat{\boldsymbol{y}}_0,\hat{\mathbb{P}}_0))\mathrm{e}^{c_{\mathrm{E}}^{(1)}s}\mathrm{d}s =$$
$$(c_{\mathrm{E}}^{(0)} + \mathcal{E}(0,\hat{\boldsymbol{y}}_0,\hat{\mathbb{P}}_0))(\mathrm{e}^{c_{\mathrm{E}}^{(1)}t} - \mathrm{e}^{c_{\mathrm{E}}^{(1)}\tau_l^t}) \le C(t-\tau_l^t).$$

□

In view of (4.39) and the boundedness of $\partial_t\mathcal{E}$ by some $C'>0$ on the corresponding interval, we continue estimating from above:

$$\mathcal{E}(\tau_l^t,\boldsymbol{y}^{N_l^t}(\tau_l^t),\mathbb{P}^{N_l^t}(\tau_l^t)) + \mathrm{Diss}_{\mathcal{D}}(\mathbb{P}^{N_l^t};[0,\tau_l^t]) + C\nu(\Pi^{N_l^t}) \le$$
$$\mathcal{E}(0,\boldsymbol{y}(0),\mathbb{P}(0)) + \int_0^{\tau_l^t}\theta^{N_l^t}(s)\mathrm{d}s + C\nu(\Pi^{N_l^t}) \le$$
$$\mathcal{E}(0,\boldsymbol{y}(0),\mathbb{P}(0)) + \int_0^t\theta^{N_l^t}(s)\mathrm{d}s + (C+C')\nu(\Pi^{N_l^t}).$$

For a fixed t, let $l \to +\infty$ in inequality (4.45) combined with the previous one. The limit inequality so obtained reads

$$\mathcal{E}(t, \boldsymbol{y}(t), \mathbb{P}(t)) + \delta_\infty(t) \leq \mathcal{E}(0, \boldsymbol{y}(0), \mathbb{P}(0)) + \int_0^t \theta_*(s)\mathrm{d}s,$$

because of (4.41) and (4.40). The weak seq. lower semicontinuity of the dissipation results in $\mathrm{Diss}_{\mathcal{D}}(\mathbb{P}; [0,t]) \leq \delta_\infty(t)$ (Theorem 14(c)). Applying it, alongside the inequality $\theta_*(t) \leq \theta(t) = \partial_t \mathcal{E}(t, \boldsymbol{y}(t), \mathbb{P}(t))$, which belongs to fruit of Section 4.7 and to that of (4.44), we finally get the upper energy estimate

$$\mathcal{E}(t, \boldsymbol{y}(t), \mathbb{P}(t)) + \mathrm{Diss}_{\mathcal{D}}(\mathbb{P}; [0,t]) \leq \mathcal{E}(0, \boldsymbol{y}(0), \mathbb{P}(0)) + \int_0^t \partial_t \mathcal{E}(\tau, \boldsymbol{y}(\tau), \mathbb{P}(\tau))\mathrm{d}\tau.$$

4.10 Lower energy estimate

To obtain the lower energy estimate

$$\mathcal{E}(t', \boldsymbol{y}(t), \mathbb{P}(t')) + \mathrm{Diss}_{\mathcal{D}}(\mathbb{P}; [t,t']) \geq \mathcal{E}(t, \boldsymbol{y}(t), \mathbb{P}(t)) + \int_t^{t'} \partial_t \mathcal{E}(\tau, \boldsymbol{y}(\tau), \mathbb{P}(\tau))\mathrm{d}\tau \quad (4.46)$$

for arbitrary $0 \leq t < t' \leq T$ (in particular for $t := 0$, $t' := T$), we use approximations by Riemann sums as in [15]. For this purpose, consider any partition $\Pi_M = \{\tau_j^M\}_{j=0}^{j_M}$ of $[0,T]$ (which is in no relation with the partitions used in Section 4.7). By stability of $(\boldsymbol{y}(\tau_{j-1}^M), \mathbb{P}(\tau_{j-1}^M))$, derived in Section 4.8, and the fundamental theorem of calculus

$$\mathcal{E}(\tau_{j-1}^M, \boldsymbol{y}(\tau_{j-1}^M), \mathbb{P}(\tau_{j-1}^M)) \leq \mathcal{E}(\tau_{j-1}^M, \boldsymbol{y}(\tau_j^M), \mathbb{P}(\tau_j^M)) + \mathcal{D}(\mathbb{P}(\tau_{j-1}^M), \mathbb{P}(\tau_j^M)) =$$

$$\mathcal{E}(\tau_j^M, \boldsymbol{y}(\tau_j^M), \mathbb{P}(\tau_j^M)) + \mathcal{D}(\mathbb{P}(\tau_{j-1}^M), \mathbb{P}(\tau_j^M)) - \int\limits_{\tau_{j-1}^M}^{\tau_j^M} \partial_t \mathcal{E}(s, \boldsymbol{y}(\tau_j^M), \mathbb{P}(\tau_j^M))\mathrm{d}s, \ 1 \leq j \leq j_M.$$

If we sum over $j = 1, 2, \ldots, j_M$, the values of energy at intermediate time levels mutually cancel out and $\sum_{j=1}^{j_M} \mathcal{D}(\mathbb{P}(\tau_{j-1}^M), \mathbb{P}(\tau_j^M))$ can be estimated from above by the total dissipation:

$$\mathcal{E}(t', \boldsymbol{y}(t'), \mathbb{P}(t')) + \mathrm{Diss}_{\mathcal{D}}(\mathbb{P}; [t,t']) - \mathcal{E}(t, \boldsymbol{y}(t), \mathbb{P}(t)) \geq$$

$$\sum_{j=1}^{j_M} \int\limits_{\tau_{j-1}^M}^{\tau_j^M} \partial_t \mathcal{E}(s, \boldsymbol{y}(\tau_j^M), \mathbb{P}(\tau_j^M))\mathrm{d}s =$$

$$\sum_{j=1}^{j_M} \partial_t \mathcal{E}(\tau_j^M, \boldsymbol{y}(\tau_j^M), \mathbb{P}(\tau_j^M))(\tau_j^M - \tau_{j-1}^M) + \sum_{j=1}^{j_M} (\tau_j^M - \tau_{j-1}^M)\rho_j^M. \quad (4.47)$$

The values ρ_j^M are defined as

$$\rho_j^M = \frac{1}{\tau_j^M - \tau_{j-1}^M} \int_{\tau_{j-1}^M}^{\tau_j^M} [\partial_t \mathcal{E}(s, \boldsymbol{y}(\tau_j^M), \mathbb{P}(\tau_j^M)) - \partial_t \mathcal{E}(\tau_j^M, \boldsymbol{y}(\tau_j^M), \mathbb{P}(\tau_j^M))]\mathrm{d}s, \ 1 \leq j \leq j_M,$$

and satisfy for all $j \in \{1, 2, \ldots, j_M\}$

$$\left|\rho_j^M\right| \leq \frac{1}{\tau_j^M - \tau_{j-1}^M} \int_{\tau_{j-1}^M}^{\tau_j^M} |\partial_t \ell(s, \boldsymbol{y}(\tau_j^M)) - \partial_t \ell(\tau_j^M, \boldsymbol{y}(\tau_j^M))|\mathrm{d}s \leq$$

$$\frac{1}{\tau_j^M - \tau_{j-1}^M} \int_{\tau_{j-1}^M}^{\tau_j^M} ||\partial_t \ell(s) - \partial_t \ell(\tau_j^M)||_{(W^{1,q_Y}(\Omega;\mathbb{R}^d))^*} ||\boldsymbol{y}(\tau_j^M)||_{W^{1,q_Y}(\Omega;\mathbb{R}^d)}\mathrm{d}s.$$

Given an $\varepsilon > 0$, there exists a $\delta > 0$ such that if $s_1, s_2 \in [0,T]$, $|s_1 - s_2| < \delta$, we have $||\partial_t \ell(s_1) - \partial_t \ell(s_2)||_{(W^{1,q_Y}(\Omega;\mathbb{R}^d))^*} < \varepsilon$ (this is the uniform continuity of $\partial_t \ell \colon [0,T] \to (W^{1,q_Y}(\Omega;\mathbb{R}^d))^*$). Therefore, if the partition Π_M has norm $\nu(\Pi_M) < \delta$, then

$$\left|\sum_{j=1}^{j_M}(\tau_j^M - \tau_{j-1}^M)\rho_j^M\right| \leq \sum_{j=1}^{j_M} \frac{\tau_j^M - \tau_{j-1}^M}{\tau_j^M - \tau_{j-1}^M} \int_{\tau_{j-1}^M}^{\tau_j^M} \varepsilon ||\boldsymbol{y}(\tau_j^M)||_{W^{1,q_Y}(\Omega;\mathbb{R}^d)} \mathrm{d}s \leq C_* \varepsilon T.$$

Since $\varepsilon > 0$ was arbitrary, we infer that the last sum in (4.47) goes to zero if $\nu(\Pi_M)$ tends to 0 with $M \to +\infty$.

Now we turn to the first sum in (4.47). We know from Section 4.9 that the mapping $\tau \mapsto -\partial_t \ell(\tau, \boldsymbol{y}(\tau))$ is measurable and by

$$\ell \in \mathcal{C}([0,T];(W^{1,q_Y}(\Omega;\mathbb{R}^d))^*), \quad \boldsymbol{y} \in \mathrm{B}([0,T];W^{1,q_Y}(\Omega;\mathbb{R}^d))$$

it follows that it is even in $L^1((0,T))$.

Hence Theorem A.5 on approximation of (here Lebesgue) integrals via Riemann sums establishes the existence of a sequence $\{\Pi_M\}_{M=1}^{\infty}$ of partitions of $[t,t']$, $\Pi_M = \{\tau_j\}_{j=0}^{j_M}$, $M \in \mathbb{N}$, with $\lim_{M\to+\infty} \nu(\Pi_M) = 0$ for which

$$\lim_{M\to+\infty} \left| \sum_{j=1}^{j_M} -\partial_t \ell(\tau_j^M, \boldsymbol{y}(\tau_j^M))(\tau_j^M - \tau_{j-1}^M) - \int_{\tau_{j-1}^M}^{\tau_j^M} -\partial_t \ell(\tau, \boldsymbol{y}(\tau)) \mathrm{d}\tau \right| = 0.$$

In consequence, (4.47) yields lower energy estimate (4.46) in the limit.

4.11 Improved convergence and summary

In [15], it is pointed out that it is possible to get stronger convergence and dispose of the t-dependence of index subsequences, at least for some of the previously introduced sequences of functions.

The result that has been proved in this chapter can be summarised in

Theorem 17. *Suppose that $\Omega \subset \mathbb{R}^d$, $d > 1$, $\Omega \in \mathcal{C}^{0,1}$ and $T > 0$. Let W be measurable, nonnegative, lower semicontinuous and satisfying* (4.1)*; $W(\cdot, \mathbb{P}_*)$ satisfying* (1.1) *for every $\mathbb{P}_* \in \mathrm{SL}(d)$. Further, assume that $\ell \in \mathcal{C}^1([0,T];(W^{1,q_Y}(\Omega;\mathbb{R}^d))^*)$, D has properties* (4.4)*, hypothesis* (4.32) *holds, all Lebesgue exponents are as above, $\hat{\boldsymbol{y}}_{\mathrm{Dir}} \in W^{1-\frac{1}{q_Y},q_Y}(\Gamma_D;\mathbb{R}^d)$ and $\mathcal{Y} \neq \emptyset$.*

Then, for each $(\hat{\boldsymbol{y}}_0, \hat{\mathbb{P}}_0) \in \mathcal{S}(0)$ there exists an energetic solution $(\boldsymbol{y}, \mathbb{P}) \colon [0,T] \to \mathcal{Y} \times \mathcal{Z}$ of (S) *and* (E) *with $(\boldsymbol{y}(0), \mathbb{P}(0)) = (\hat{\boldsymbol{y}}_0, \hat{\mathbb{P}}_0)$ and*

$$\boldsymbol{y} \in \mathrm{B}([0,T];W^{1,q_Y}(\Omega;\mathbb{R}^d)) \text{ and}$$
$$\mathbb{P} \in L^\infty((0,T);W^{1,q_P}(\Omega;\mathbb{R}^{d\times d})) \cap \mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^{d\times d})).$$

Conclusion

13. Looking back. (...) There remains always something to do; with sufficient study and penetration, we could improve any solution, and, in any case, we can always improve our understanding of the solution.

George Polya, How to Solve It (1945)

Finite-strain elastoplasticity with a multiplicative decomposition of the deformation gradient $\mathbb{F}$ into the elastic part $\mathbb{F}_{\text{el}}$ and the plastic part $\mathbb{P}$

$$\mathbb{F} = \mathbb{F}_{\text{el}}\mathbb{P}$$

has been around for a few decades, but the field is still a 'work in progress' [10]. Nowadays there exist a number of approaches and many proposed ideas are a matter of ongoing discussion.

This thesis falls into the theory of variational rate-independent evolutions, which has been developed by Mielke [26], Francfort [15] and others, and has been applied to brittle fracture, damage etc. [10].

When one elaborates on minimisation of energy functionals, sufficient conditions for the existence of minimisers are crucial. A widespread sufficient condition is Ball's polyconvexity from [1], but e.g. for the modelling of shape-memory alloys, this property is not appropriate. Benešová, Kružík and Schlömerkemper [5] set up gradient polyconvexity which can better accommodate multiwell stored energies and has a physical interpretation of limiting strong spatial variation in area/volume changes.

The novelty of this master's thesis is the extension of a material model with gradient polyconvex stored energy from [5] to an elastoplastic body and proving the existence of an energetic solution to a rate-independent evolution in the material. Perfect finite-strain gradient elastoplasticity is considered, but as [25] indicates, addition of hardening variables to the model would not require many changes and the existence result would still hold in a suitable version. The same would apply to stored energy densities W, which also depend on $\mathbb{F}^{-1}$ and the spatial variable $\boldsymbol{X} \in \Omega$ (see [5]), and incorporation of locking constraints is likewise possible. Further work would be necessary to allow a time-dependent Dirichlet boundary condition as in [25].

Energetic solutions, as presented in Chapter 2, are not the only possibility in rate-independent systems. In particular, S. Schwarzacher (Department of Mathematical Analysis of Charles University) remarked that using *balanced-viscosity solutions* (cf. [28, p. 222]) would be desirable, as they 'do not overlook local minima' of the total energy functional.

A. Appendix – facts from mathematical analysis

Definition A.1. *We call a finite sequence of points* $\{t_k\}_{k=0}^N$, $t_k \in \mathbb{R}$, $N \in \mathbb{N}$ *a* partition *of the interval* $[a, b]$ *if*

$$a = t_0 < t_1 < \cdots < t_N = b.$$

By the norm *of the partition* $\Pi = \{t_k\}_{k=0}^N$, *we mean the number*

$$\nu(\Pi) = \max\{t_{k+1} - t_k; k = 0, 1, \ldots, N-1\}.$$

Let Π^N, $N \in \mathbb{N}$, *be partitions of* $[a, b]$. *We say that* $\{\Pi^N\}_{N=1}^\infty$ *is a* sequence of nested partitions *if for every* $N \in \mathbb{N}$, *we have* $\Pi^N \subset \Pi^{N+1}$.

Further we recall a selection theorem known e.g. from courses on nonlinear analysis.

Theorem A.1 (Helly)**.** *Let* $C > 0$ *and suppose* $f_n : \mathbb{R} \to [0, C]$, $n \in \mathbb{N}$, *are nondecreasing functions. Then there is a subsequence* $\{f_{n_k}\}_{k=1}^\infty \subset \{f_n\}_{n=1}^\infty$ *and a nondecreasing function* $f : \mathbb{R} \to [0, C]$ *such that* $f_{n_k} \to f$ *pointwise in* $\mathbb{R}$.

We also need two generalisations of scalar-valued functions of bounded variation.

Definition A.2. *[28, p. 603] Let* $T > 0$ *and let* $\mathcal{X}$ *be a Banach space. The* variation *of* $\mathbb{P} : [0, T] \to \mathcal{X}$ *with respect to the norm of* $\mathcal{X}$ *is defined as*

$$\mathrm{Var}_{\mathcal{X}}(\mathbb{P}, [0, T]) = \sup\Big\{ \sum_{i=1}^{N-1} ||\mathbb{P}(t_{i+1}) - \mathbb{P}(t_i)||_{\mathcal{X}};\ N \in \mathbb{N},$$
$$t_1, t_2, \ldots, t_N \in [0, T],\ t_1 < t_2 < \cdots < t_N \Big\}.$$

A subspace of mappings $\mathbb{P} \in \mathrm{B}([0, T], \mathcal{X})$ *with* bounded variation $\mathrm{Var}_{\mathcal{X}}(\mathbb{P}, [0, T]) < +\infty$, *endowed with the norm* $||\mathbb{P}||_{\mathrm{BV}([0,T];\mathcal{X})} = \sup_{0 \le t \le T} ||\mathbb{P}(t)||_{\mathcal{X}} + \mathrm{Var}_{\mathcal{X}}(\mathbb{P}, [0, T])$, *is denoted by* $\mathrm{BV}([0, T]; \mathcal{X})$.

Definition A.3. *[24, p. 4] Let* $\mathcal{Z}$ *be a Hausdorff topological space,* $\mathcal{D}$ *a dissipation distance as in Chapter 4 and* $T > 0$.

We write $\mathrm{BV}_{\mathcal{D}}([0, T]; \mathcal{Z}) = \{\mathbb{P} : [0, T] \to \mathcal{Z};\ \mathrm{Diss}_{\mathcal{D}}(\mathbb{P}; [0, T]) < +\infty\}$.

For the definition of other function spaces, where the functions take values in some Banach space $\mathcal{X}$, e.g. $\mathcal{C}^1([0, T]; \mathcal{X})$, $T > 0$, the reader is referred to [21, p. 40].

A.1 Measure and integration theory

Lemma A.2. *Let* $(\mathcal{X}, \rho)$ *be a metric space and* $f : \mathcal{X} \to \mathbb{R}_\infty$ *a lower semicontinuous function. Then the set* $f^{-1}((a, +\infty]) \subset \mathcal{X}$ *is open for any* $a \in \mathbb{R}$.

Proof. Given any $a \in \mathbb{R}$, it is enough to prove that the set

$$S := \mathcal{X} \backslash f^{-1}((a, +\infty]) = \{x \in \mathcal{X};\ f(x) \le a\}$$

is closed. To do this, consider any sequence $\{x_n\}_{n=1}^\infty \subset S$ such that $x_n \xrightarrow{\rho} x \in \mathcal{X}$, $n \to +\infty$. Then $f(x) \le \liminf_{n \to +\infty} f(x_n) \le a$, so $x \in S$. □

In the below versions of Fatou's lemma, measurability of sets or functions is meant with respect to Borel σ-algebras in the respective measure spaces.

Theorem A.3 (Fatou's lemma)**.** *[14, p. 42] Suppose $d \in \mathbb{N}$ and $E \subset \mathbb{R}^d$ is a measurable set. If $f_n : E \to [0, +\infty]$ is a sequence of measurable functions, then $f := \liminf_{n\to+\infty} f_n$ is a measurable function and*

$$\int_E f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \leq \liminf_{n\to+\infty} \int_E f_n(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

Theorem A.4 (reverse Fatou's lemma)**.** *[14, p. 42] Under the assumptions of the Fatou lemma above, let $\tilde{f}_n : E \to \mathbb{R}_\infty$ be a sequence of measurable functions such that $\tilde{f}_n \leq g$ for some measurable function $g : E \to [0, +\infty]$ with $\int_E g(\boldsymbol{x})\mathrm{d}\boldsymbol{x} < +\infty$. Then $\tilde{f} := \limsup_{n\to+\infty} \tilde{f}_n$ is a measurable function and*

$$\int_E \tilde{f}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \geq \limsup_{n\to+\infty} \int_E \tilde{f}_n(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

The theorem to be stated now enables approximation of Bochner (all the more, Lebesgue) integrals by Riemann sums and is cited e.g. in [9, Section 4.4].

Theorem A.5. *Let $[a, b]$ be a closed bounded interval, let $\mathcal{X}$ be a Banach space, and let $f : [a, b] \to \mathcal{X}$ be a Bochner-integrable function. Then there exists a sequence $\{\Pi_M\}_{M=1}^{\infty}$ of partitions of the interval $[a, b]$, with $\Pi_M = \{\tau_j^M\}_{j=0}^{j_M}$,*

$$\lim_{M\to+\infty} \nu(\Pi_M) = 0,$$

such that

$$\lim_{M\to+\infty} \left\| \sum_{j=1}^{j_M} (\tau_j^M - \tau_{j-1}^M) f(\tau_j^M) - \int_a^b f(t)\mathrm{d}t \right\| = 0.$$

A.2 Calculus of variations

In (4.4a), D was supposed to be a Carathéodory mapping. A generalisation of the model presented in Chapter 4, which suggests itself, would merely require D to be a *normal integrand* (with some additional assumptions as in [25]), so both terms are defined here subsequently.

Definition A.4 (normal integrand)**.** *[25] If $E \subset \mathbb{R}^d$ is Lebesgue-measurable, U is a topological space and $\mathcal{B}(U)$ its Borel σ-algebra, then a function $f : E \times U \to \mathbb{R}_\infty$ is called a* normal integrand *if*

$$f \text{ is } \mathfrak{L}_E \times \mathcal{B}(U)\text{-measurable and} \tag{NI1}$$

$$\text{for a.a. } \boldsymbol{x} \in E: \quad f(\boldsymbol{x}, \cdot) : U \to \mathbb{R}_\infty \text{ is lower semicontinuous,} \tag{NI2}$$

where $\mathfrak{L}_E$ denotes the Lebesgue-measurable subsets of E.

Definition A.5 (Carathéodory mapping)**.** *[28, p. 598] Under the hypotheses of Definition A.4, assume that $U := \mathbb{R}^{d_c}$, where $d_c \geq 1$. A special case of a normal integrand $f : E \times \mathbb{R}^{d_c} \to \mathbb{R}_\infty$ with $f(\boldsymbol{x}, \cdot) : \mathbb{R}^{d_c} \to \mathbb{R}$ continuous for a.a. $\boldsymbol{x} \in E$ is called a* Carathéodory mapping.

Definition A.6 (weak coercivity)**.** *Let $\mathcal{X}$ be a Banach space. We say that $F : \mathcal{X} \to \mathbb{R}_\infty$ is* weakly coercive *with respect to $\mathcal{A} \subset \mathcal{X}$ if*

$$\lim_{\substack{x\in\mathcal{A} \\ ||x||\to+\infty}} F(x) = +\infty.$$

Strong coercivity could be defined by a similar condition, requiring a faster growth to infinity, but we do not need the term.

Claim A.6. *[14, p. 563] Let $\mathcal{X}$ be a Banach space. If a sequence $\{x_n\} \subset \mathcal{X}$ converges weakly to $x \in \mathcal{X}$, then it is bounded and*

$$||x|| \leq \liminf_{n\to+\infty} ||x_n||.$$

Definition A.7 (quasiconvexity)**.** *[8, p. 156] A Borel-measurable and locally bounded function $W : \mathbb{R}^{d\times d} \to \mathbb{R}$ is said to be* quasiconvex *if*

$$W(\mathbb{A}) \leq \frac{1}{\mathcal{L}^d(E)} \int_E W(\mathbb{A} + \nabla\boldsymbol{\varphi}(\boldsymbol{X}))\mathrm{d}\boldsymbol{X}$$

for every bounded open set $E \subset \mathbb{R}^d$, every $\mathbb{A} \in \mathbb{R}^{d\times d}$ and every $\boldsymbol{\varphi} \in W_0^{1,\infty}(E;\mathbb{R}^d)$.

A.3 Partial differential equations

Next we state Grönwall's lemma, in a form due to Bellman [30, p. 11].

Theorem A.7 (Grönwall's lemma)**.** *Let u and f be continuous and nonnegative functions defined on $[a, b]$, a, $b \in \mathbb{R}$, $a < b$, and let c be a nonnegative constant. Then the inequality*

$$u(t) \leq c + \int_a^t f(s)u(s)\mathrm{d}s, \quad t \in [a, b],$$

implies that

$$u(t) \leq c \exp(\int_a^t f(s)\mathrm{d}s), \quad t \in [a, b].$$

It is known from functional analysis that norm-bounded sets in reflexive normed linear spaces are relatively weakly sequentially compact. The following theorem can be utilised to get weak* limits in non-reflexive Banach spaces.

Claim A.8. *[28, p. 586] In a Banach space with a separable predual, every bounded sequence contains a weakly* convergent subsequence.*

It is supposed that the reader is acquainted with the usual Sobolev spaces $W^{k,p}(\Omega)$, $k \in \mathbb{N}$, $1 \leq p \leq +\infty$, $\Omega \subset \mathbb{R}^d$ open (defined e.g. in [12]), as well as their vector-valued counterparts $W^{k,p}(\Omega;\mathbb{R}^d)$ (see [6, p. 282]). More facts about Sobolev–Slobodeckiĭ spaces $W^{s,p}(\partial\Omega)$, $s > 0$, $s \notin \mathbb{N}$, can be found in [21, p. 330–332].

For the compact embedding below, which is heavily used in the modern theory of PDEs, Ω does not need to be a Lipschitz domain; the class $\mathcal{C}^0$ is sufficient.

Theorem A.9 (compact embedding of Sobolev spaces)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $\mathcal{C}^0$ and $1 \leq q < +\infty$. Then $W^{1,q}(\Omega)$ is compactly embedded into $L^q(\Omega)$.*

In [6], the next inequality can be found as Theorem 6.1-8(b). It would not be difficult to derive a variant valid for vector-valued functions.

Theorem A.10 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and let $1 \leq q < +\infty$. Let Γ be a $\mathcal{H}^{d-1}$-measurable subset of $\partial\Omega$ with $\mathcal{H}^{d-1}(\Gamma) > 0$. Then there exists a constant C_P such that*

$$\int_\Omega |u|^q \mathrm{d}\boldsymbol{X} \leq C_\mathrm{P} \left(\int_\Omega |\nabla u|^q \mathrm{d}\boldsymbol{X} + \left| \int_\Gamma u \,\mathrm{d}S \right|^q \right)$$

for all $u \in W^{1,q}(\Omega)$.

The following theorem on multiplication of Sobolev functions (Theorem 7.5) from [3] gives sufficient conditions on the exponents to ensure that the product lies in a Sobolev space as well.

Theorem A.11. *Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. Assume s_i, s and $1 \leq p_i, p < +\infty$ $(i = 1, 2)$ are real numbers satisfying*

(i) $s_i \geq s$,

(ii) $s \geq 0$,

(iii) $s_i - s \geq d(\frac{1}{p_i} - \frac{1}{p})$,

(iv) $s_1 + s_2 - s > d(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p})$.

In the case where $\max\{p_1, p_2\} > p$ *instead of* (iv) *assume that* $s_1 + s_2 - s > \frac{d}{\min\{p_1,p_2\}}$.

Claim: If $u \in W^{s_1,p_1}(\Omega)$ and $v \in W^{s_2,p_2}(\Omega)$, then $uv \in W^{s,p}(\Omega)$ and moreover the pointwise multiplication of functions is a continuous bilinear map

$$W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \to W^{s,p}(\Omega).$$

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