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Topologically distinct ultrafilters

Diploma thesis

Topologically distinct ultrafilters

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Study programme: Mathematics
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Univerzita Karlova v Praze
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Diplomová práce

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Katedra teoretické informatiky a matematické logiky

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Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 20. dubna 2007 Jonathan Verner

Contents

Název práce: **Topologically distinct ultrafilters**
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Abstrakt: Od konce padesátých let je známo, že ω^* není homogenní. Z. Frolík ve svém článku [Fro67a] ukázal, že každý prostor, který není pseudokompaktní, má nehomogenní Čech-Stoneův přírůstek. Prvním důkazem v tomto ohledu však byl důkaz W. Rudina ([Rud56]), který za Martinova Axiomu dokázala existenci P-bodu v ω^*. S. Shelah později ukázal, že jejich neexistence je konzistentní s axiomy ZFC a proto nějaké dodatečné množinové axiomy nelze z Rudinova důkazu vypustit úplně. Navazuje na tyto výsledky K. Kunen definoval tzv. dobré body a O.K. body a dokázal jejich existenci v ZFC. Završením těchto snah byl van Millův článek [vM82], kde je definováno 16 topologicky různých ultrafiltrů a je ukázána jejich existence v ZFC. Mezi těmito ultrafiltry je též bod, který je hromadným bodem spočetné podmnožiny ω^*, ale není hromadným bodem diskrétní množiny. K. Kunen ukázal existenci podobného bodu za MA. Jeho bod měl navíc tu vlastnost, že spočetné množiny, v jejichž uzávěru byl, tvořily filtr. Tato práce pokračuje v nastíněném směru. Ukážeme, že pokud existuje spočetný, extremálně nesouvislý, nuldimenzionální OHI prostor se slabým P-bodem, který je zároveň remote bodem, pak lze použití MA obejít. Dále najdeme potřebný prostor se slabým P-bodem a také spočetný, extremálně nesouvislý, T_2 OHI prostor s $\mathfrak{c}$-O.K. bodem. Na konec ukážeme, že v hledaném prostoru nelze získat O.K. bod, který by byl remote bodem.

Klíčová slova: $\beta\omega$, irreducibilita, slabé P-body, remote body, topologický typ

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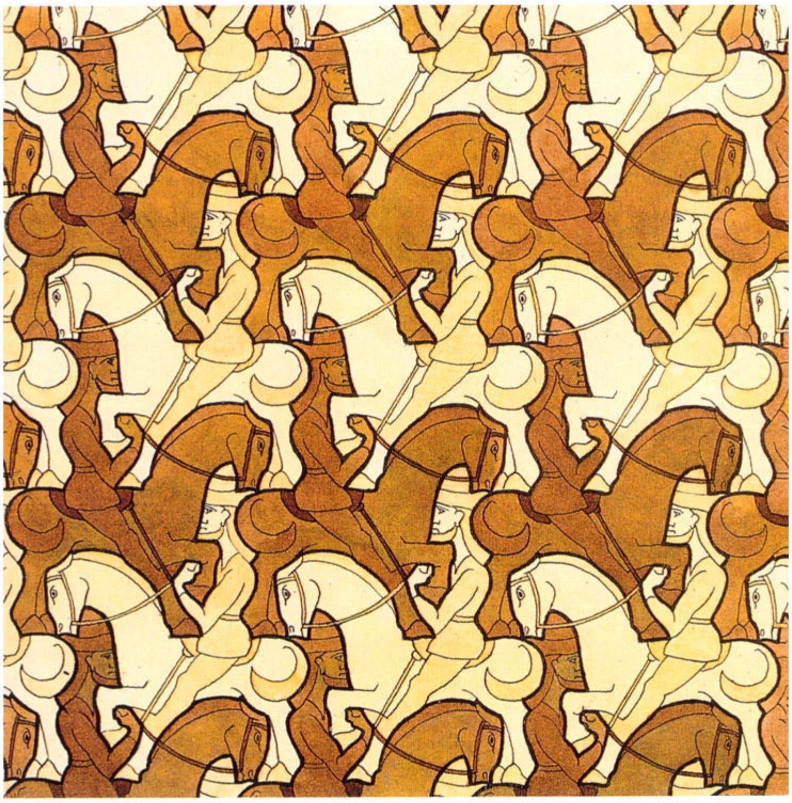
Abstract: It is long known, that ω^* is not homogeneous. In fact, it was proved by Z. Frolík in [Fro67a] that the Čech-Stone growth of any nonpseudocompact space is not homogeneous. Preceding Z. Frolík, W. Rudin has shown (under MA) that ω^* contains a P-point. Later S. Shelah proved, that it is consistent with ZFC, that ω^* has no P-points, so some set-theoretic assumptions beyond ZFC cannot be altogether dropped from W. Rudin's proof. Further work has been done by Kunen, who showed the existence of good points in ZFC and finally Jan van Mill, who has given a *topological* description of 16 distinct types in ω^*. Among these types there was also a point, which is in the closure of a countable set but not of a countable discrete set. K. Kunen has shown (under MA), that there is such a point having the further property, that any two countable sets having it in their closure, must intersect. We investigate along these lines searching for a ZFC result. It is shown, that if we can find a countable OHI, extremally disconnected, zerodimensional space with a remote weak P-point, then such a point exists. We prove the existence of a countable irresolvable, extremally disconnected space with a remote point and a countable, irresolvable, extremally disconnected space with an ω_1-O.K. point.

Keywords: $\beta\omega$, irreducibility, weak P-point, remote point, topological type

“What is the use of a book,”
thought Alice,
“without pictures or conversation?”

Lewis Carroll



M. C. Escher: **Horsemen**

Introduction

THE starting point for the study of the nonhomogeneity of ω^* was W. Rudin's proof ([Rud56]) that, under CH, there are P-points in ω^*. Since there are obviously points in ω^* which are not P-points, this shows, supposing CH, that ω^* is not homogeneous. The continuum hypothesis in his proof can be weakened to Martin's axiom but, by a deep and hard result of Shelah ([Wim82]), it is consistent with ZFC that there are no P-points in ω^*. In 1967 Frolík gave a surprising answer to the (ZFC) question of whether ω^* is or is not homogeneous ([Fro67a], [Fro67b]). It is not; in fact, there are $2^{\mathfrak{c}}$ pairwise "topologically different" points (i.e. there is no homeomorphism taking one to another) in ω^*. (In his paper he showed that X^* is not homogeneous for any non-pseudocompact space X.) The problem with his proof was that it was based on cardinality arguments and did not yield a "topological" description of even two different points. A next step forward was Kunen's proof of the existence of weak P-points in ZFC ([Kun78], actually he found $\mathfrak{c}$-O.K. points). Weak P-points are points in ω^*, which are not limit points of any countable set. Obviously not every point of ω^* is a weak P-point, so this also gives a proof of the nonhomogeneity of ω^*. And it actually shows two concrete topologically distinct points (a weak P-point and a non-weak P-point) witnessing the nonhomogeneity. It is an "effective" proof in the sense of van Douwen [vD81], because it provides a topological property which one class of points has and another does not. The next result and a huge step forward was van Mill's description of sixteen distinct topological properties of points in ω^* ([vM82]). We continue this line of development by looking for a seventeenth property — topological type.

One of the points Mill constructed had the property that it was in the closure of a crowded countable set but not in the closure of any discrete set. In 1976 Kunen published [Kun76], where he proved the consistency of nonexistence of selective ultrafilters, and found, assuming MA, a point which was also in the closure of dense in itself countable set and not in the closure of any discrete countable set. His point, however, satisfied the following: any two countable sets having the point in their closure have to intersect.

Mill's point does not have this property, because it was found in the absolute of 2^ω (and then embedded into ω^*). If such a point is in the closure of a countable set, the set can always be split into two disjoint parts again having the point in their closures. In our work we investigate points of the type Kunen constructed and look for them in ZFC. Ideally we are looking for a point satisfying the following:

Definition 0.1. A point is *uniquely ω-accessible* in a space X if it is in the closure of a countable set, not in the closure of a discrete set, and any two countable sets, whose limit point it is, intersect.

The definition says, that there is, so to say, only a "single" countable set, whose closure contains the point (i.e. the countable sets form a filter base). See [VD93] for a similar notion of accessibility. Unlike in that paper we include the requirement that the point is not in the closure of a discrete set. Such a points were constructed by van Mill ([vM82]).

There are, a priori, two approaches to finding a specific point in ω^*. One can use transfinite induction to construct a filter on ω, at each step ensuring the necessary properties. This process is usually aided by some independent matrix. The other way is to find a space with the needed point and embed it in an appropriate way into ω^*. (The second chapter deals with these embeddings.) We chose this other approach, because it seemed simpler, although it certainly is a matter of taste. A part of the last chapter will be devoted to showing that no generality is lost.

When we look at the definition of a uniquely ω-accessible point, we can see that if p is such a point and X is a countable set with p a limit point of X, then any two dense subsets of X must intersect. Hence the following definition is relevant:

Definition 0.2 ([Hew43])**.** A space is *irresolvable* if any two dense sets intersect.

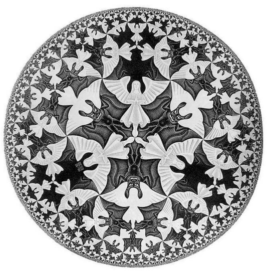
A space with a uniquely ω-accessible point will be irresolvable. The third chapter will study irresolvable spaces.

The idea behind the construction we present is to find a countable irresolvable space, whose compactification contains a remote weak P-point. This point cannot be accessed by a countable set from the growth (since it is a weak P-point) and if the space satisfies a slightly stronger irresolvability condition then it will not be a limit point of two disjoint sets from the space. Thus it will be uniquely ω-accessible and all that will remain to be done is to embed this space into ω^* in such a way as to preserve unique ω-accessibility (this will be possible if the space satisfies certain additional conditions).

The fourth chapter will deal with the existence (and constructions) of weak P-points, the fifth will turn to a discussion of remote points.

The last chapter will summarize the results and unfortunately present a theorem which limits the useability of our methods. It will show, that we would need a method to construct weak P-points which are not ω_1-O.K. Unfortunately no suitable method (in ZFC) is known to date. We will, however, introduce the weaker notion of a relatively ω-uniquely accessible point and prove that such points do in fact exist in ZFC.

The main results of this thesis are Corollary 5.13 and Theorem 4.17.



Chapter 1

Basic definitions

THE reader should be familiar with basic topological and set-theoretical concepts and language. In this chapter we include some of the ones we will be using but its purpose is rather to fix notation then to introduce the reader into the subject. A slight exception to this is the section devoted to Čech-Stone compactifications. A reader looking for introduction to topology is referred to [Eng]. An excellent introduction to set theory is [Balcar] or [Kun80] which also includes a lot of material concerning independence and Martin's axiom, [Jech] can serve as a reference for more advanced results. Boolean algebras are covered in [HBA].

Another purpose of this chapter is to state some standard theorems so that we may refer to them in later chapters. In the cases where we do not provide proofs they may be found in the cited works.

1.1 Set Theory & Topology

First we introduce notation. The Greek letters κ, λ, θ will denote infinite cardinal numbers, α, β ordinal numbers, k, n, m, i, j natural numbers. The first infinite cardinal will be denoted by ω and $\mathfrak{c}$ will be the cardinality of the powerset of ω. For two sets X, Y their *symmetric difference* is denoted by $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$. The symbol $\mathcal{F}R(X)$ will stand for the generalized Fréchet filter on X, that is $\mathcal{F}R(X) = \{F \subseteq X : |X \setminus F| < |X|\}$. A *filter base* for a filter $\mathcal{F}$ is a system of sets from the filter such that any set in the filter contains a set from the filter base. The character of a filter (denoted by $\chi(\mathcal{F})$) is the minimal cardinality of a filter base for $\mathcal{F}$. If X is a set let its powerset be denoted by $\mathcal{P}(X)$. The symbols $[X]^\kappa, [X]^{<\kappa}$ shall denote the set of all subsets of X of cardinality κ and less than κ respectively. ${}^X Y$ shall denote the set of all functions from X to Y. The cardinality of a set X shall be denoted by $|X|$ and $2^{|X|}$ shall be the cardinality of ${}^X 2$. We shall say that a system of sets is *centered* (or, equivalently, has the *finite intersection property*), if any finite subsystem has nonempty intersection. If F is a centered system, denote by $\langle F \rangle$ the filter generated by F.

Turning to topology, we note, that all topological spaces we will consider will be (at least) Hausdorff (i.e. T_2). Other separation properties we will use are T_0, T_1, regularity (T_3) and complete regularity ($T_{3\frac{1}{2}}$). Note that $T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$. We will also need the notion of a *regular open* set, i.e. a set, which is equal to the interior of its closure. Among the separation properties we may also count *zero-dimensionality* (the space has a base consisting of closed-and-open sets, *clopen* for short) and *extremal disconnectedness* (i.e., the closure of any open set is open). Note that any discrete space has all the listed separation properties. A further notion, which proves to be useful, is *extremal disconnectedness at a point.* We say that a space X is *extremally disconnected* at $p \in X$ if p is not in the closure of two disjoint open sets. Let us note a simple lemma:

Lemma 1.1. *A T_0 zero-dimensional space satisfies $T_{3\frac{1}{2}}$*

For a topological space X, denote by $\tau(X)$ the topology of X. Let $\overline{A}^X$ be the closure of A in X and int A the interior of A (the largest open set contained in A). If X is clear from the context, we will drop it. $Clopen(X)$ is the set of closed and open sets of X. A set G is *functionally open* in X if it is the preimage of $(0,1)$ by some continuous map $f : X \to \mathbb{R}$. It is *functionally closed* if it is the preimage of $\{0\}$ by such a map. A subset of a topological space is *dense* if its closure is the whole space or, equivalently, if it meets any nonempty open set. It is called *nowhere dense* (n.w.d. for short) if its closure has empty interior (or, equivalently, if the complement of its closure is dense), and it is called *somewhere dense* otherwise.

Lemma 1.2. *The nowhere dense sets in a space form an ideal.*

A point p in a topological space X is *isolated*, provided that $\{p\}$ is open in X. A topological space is *crowded*, if it has no isolated point. Note that a dense subset of X must contain all its isolated points.

The *weight* of a space (denoted by $w(X)$) is the minimal cardinality of a *base* for the space (i.e., a system of open sets of X such that any open set is a union of sets from the system). A *π-base* for a space is a family of nonempty open sets such that any nonempty open set of the space contains a set from the π-base. The π-weight of a space (denoted $\pi w(X)$) is the minimal cardinality of a π-base. A *local base* at a point x is a system of open sets containing x such that any open set containing x contains a set from the local base. A *local π-base* at x is a system of nonempty open sets such that any open neighborhood of x contains a set from the π-base. Define the character $(\chi(x))$ and π-character $(\pi\chi(x))$ of a point $x \in X$ as the minimal cardinality of a base and π-base respectively at x. The *pseudocharacter* of a point in p is the minimal cardinality of a system of neighbourhoods of p which contain only p in their intersection. It is denoted by $\psi(p)$.

Observation 1.3. *If $p \in X$ then $\psi(p) \leq |X|$.*

Fact 1.4. *If X is T_2 and compact then $\psi(p) = \chi(p)$.*

A space is said to be *κ-cc* if every family of disjoint open sets has cardinality strictly less than κ. Instead of ω_1-cc it is customary to say just *ccc.*

A space is *compact* if any cover of the space by open sets contains a finite subcover or, equivalently, if any centered system of closed sets has nonempty intersection. It is *locally compact* if any point has an (open) neighborhood with compact closure and it is *nowhere locally compact* if the closure of any nonempty open set is noncompact. Note that a subset of a compact, T_2 space is compact iff it is closed and any compact subset of a T_2 space is closed in this space.

A *homeomorphism* between two topological spaces is a continuous bijection which has a continuous inverse. A continuous map (function) is *open*, if the images of open sets are open. It is *quasiopen*, if the images of nonempty open sets have nonempty interiors. It *closed* if the images of closed sets are closed and it is *irreducible*, if the image of a proper closed subspace of the domain is never onto. A closed map is *perfect* if the preimages of points are compact. For a space X we say that EX is its *projective cover* iff it is extremally disconnected and admits an irreducible perfect map onto X. EX (sometimes called the *absolute* of X) can be shown to exist for any completely regular space X (take the space of ultrafilters on $RO(X)$)

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A topological space is *homogeneous* if, for any two points x, y, there is a homeomorphism $f_{x,y}$ from the space onto itself such that $f(x) = y$. A *topological type* is a ("topologically defined") class of points of a topological space, such that no point outside of this class can be mapped to a point inside it via a homeomorphism.

Proposition 1.5. *A countable crowded, T_0 zerodimensional space X is nowhere locally compact.*

Proof. First notice that the space must be T_2. Then fix a clopen set C and enumerate all points of C as $\{x_n : n \in \omega\}$. Now construct a decreasing sequence of clopen sets C_n such that for each $n \in \omega$, $x_n \notin C_{n+1}$: Let $C_0 = C$ and, if all C_i's were constructed for $i \leq n$, split C_n into two parts (X is crowded, T_2). One of them must contain x_n, so let the other be C_{n+1}. This gives us a decreasing system of closed sets with empty intersection, i.e. C is not compact. □

1.2 Čech-Stone compactification

For any completely regular space X there is a compact space βX, such that X embeds densely into βX and any continuous function from X into $[0, 1]$ can be continuously extended to βX. (The Stone theorem says, that this is equivalent to requiring that a continuous function into *any* compact space can be continuously extended.) The space βX is called the Čech-Stone compactification of X. The book [Wal74] is a standard reference for Čech-Stone compactifications. We refer the reader to this book for the proofs in this section which we omit. Čech-Stone compactification can be constructed as a space of maximal filters. The idea is to add a point into the intersection of each closed filter (as required by compactness). First we need to be more precise about which filters we will take:

Definition 1.6. A *z-filter* in a topological space X is a filter which consists of functionally closed sets. If F is a functionally closed set, by $\hat{F}$ we denote the set of maximal z-filters containing F.

Theorem 1.7. *If X is completely regular. Then the set of all maximal z-filters on X with the topology generated by $\{\hat{F} : F$ is functionally closed$\}$ (as a closed subbase) is isomorphic to βX.*

In the case of zerodimensional spaces maximal z-filters coincide with ultrafilters on the algebra of clopen sets. In the case of discrete spaces, maximal z-filters are just ordinary set filters.

Dealing with Čech-Stone compactifications, it is customary that X^* stands for the (Čech-Stone) *growth* of X, i.e. $X^* = \beta X \setminus X$. Let us now give a definition of four concrete topological types relevant to (Čech-Stone) growths:

Definition 1.8. A point $p \in X^*$ is a *remote point* of X iff it is not in the closure (in βX) of any n.w.d. subset of X. A slightly weaker requirement on $p \in X^*$ is that it is not a limit point of a countable discrete subset of X. We call such points *ω-far*.

Definition 1.9. A point p in X^* is a *P-point* if the intersection of countably many neighbourhoods of p is again a neighbourhood of p. Such points are known to exist in ω^* under some additional assumptions beyond ZFC, but their existence is independent of ZFC. We may weaken the condition on neighbourhoods and require instead, that the point is not the limit point of a countable set from X^*. Such points are called *weak P-points*.

Weak P-points are known to exist in ω^* assuming just ZFC. Their existence was first proved by Kunen, who introduced a somewhat stronger property and shown that points having this property do exist:

Definition 1.10. A point $p \in X$ is a *κ-O.K. point* of X iff for any countable sequence $\langle U_n : n \in \omega \rangle$ of neighborhoods of p there is a system $\{V_\alpha : \alpha < \kappa\}$ of neighborhoods of p such that for any finite $K \in [\kappa]^{<\omega}$, the following is true:

$$\bigcap_{\alpha \in K} V_\alpha \subseteq U_{|K|}$$

Note that, if $\kappa < \lambda$, then any λ-O.K. point is also a κ-O.K. point and if $\mathcal{B}$ is a base for the topology of X, then the definition is equivalent if we only consider sequences of neighborhoods from the base. A point $p \in X$ is a *weak P-point* of X if it is not a limit point of any countable set. A closed subset Y of a space X is a *weak P-set* (*κ-O.K. set*) if Y is a weak P-point (κ-O.K. point) of the quotient space X/Y. Note that a weak P-set does not contain a limit point of any countable set disjoint from it.

Proposition 1.11. *If X is a T_1 space and p is an ω_1-O.K. point of X, then p is a weak P-point of X.*

Proof. If $\{x_n : n \in \omega\} \subseteq X \setminus \{p\}$, then because X is T_1 we can choose a descending sequence of neighborhoods U_n of p such that U_n misses x_n. Then, because p is ω_1-O.K., choose $\{V_\alpha : \alpha < \omega_1\}$ neighborhoods of p, so that the intersection of any n of them is contained in U_n. Then each x_n is contained in only finitely many of them, so there is an $\alpha < \omega_1$ which misses all of them, so p is not in the closure of $\{x_n : n \in \omega\}$. □

A similar argument can be used to show the following proposition:

Proposition 1.12. *If X is T_1 and Y is a closed subset of X which is an ω_1-O.K. subset of X, then Y is a weak P-set of X.*

In the following we somewhat haphazardly list facts and theorems which will be referred to later.

Fact 1.13 ([Wal74],1.59). *X^* is compact iff X is locally compact.*

Fact 1.14 ([Wal74]). *X^* is dense in βX iff X is nowhere locally compact.*

Fact 1.15 ([Wal74],21.3). *A space X is extremally disconnected iff βX is.*

Fact 1.16 ([vD81],5.2). *βX is extremally disconnected at each remote point of X, and if X is nowhere locally compact, X^* is also extremally disconnected at each remote point of X.*

Fact 1.17. *Any countable subset of ω^* is extremally disconnected.*

Proposition 1.18. *Let X be extremally disconnected. If $p \in X^*$ is a remote point and $p \in \overline{D_0}^{\beta X} \cap \overline{D_1}^{\beta X}$ for two sets $D_0, D_1 \subseteq X$, then there is an open $G \subseteq X$ such that $G \subseteq \overline{D_0}^{X} \cap \overline{D_1}^{X}$.*

Proof. First observe, that a point in an extremally disconnected space cannot be in the closure of two disjoint open sets. Let $G_i = int(\overline{D_i}^X)$. The set $N_i = G_i \setminus D_i$ is n.w.d. Then, since p is remote and cannot be in the closure of N_i, p is in the closure of both G_0, G_1, hence by our observation $G = G_0 \cap G_1$ is nonempty. □

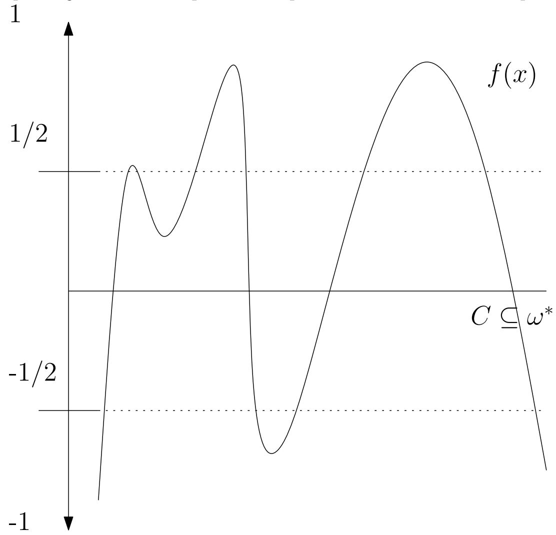
Theorem 1.19. *For $F \subseteq X^*$ we have $\overline{F} \approx \beta F$ iff F is C^*-embedded in X^*.*

Proof. F is certainly dense in the compact set $\overline{F}$. We only need to show that any function from F into $[0,1]$ can be extended to $\overline{F}$, but that immediately follows from the fact that F is C^*-embedded in X^*. □

Theorem 1.20 ([HSTT], 1.5.2)**.** *Any countable subset of ω^* is C^*-embedded in ω^*.*

Proof. The proof is an adaptation of the proof of the Tietze extension theorem.

Suppose $C \in [\omega^*]^\omega$ is countable and $f : C \to [-1,1]$. Fix some small $1/4 > \epsilon > 0$. Since C is countable, the complement of the image of f is dense in all subintervals of $[-1,1]$. Thus we can find $r_0 \in (1/2-\epsilon, 1/2+\epsilon)$ such that both r_0 and $-r_0$ are not in $\mathrm{Im} f$. Then Let $A^0_{-1} := f^{-1\prime\prime}[-1,-r_0]$, $A^0_0 := f^{-1\prime\prime}[-r_0,r_0]$ and $A^0_1 := f^{-1\prime\prime}[r_0,1]$.



We shall use the following claim to find $B^0_i \in \mathcal{P}(\omega)$ such that $B^{0*}_i = A^0_i$:

Claim 1. For any countable $C \in [\omega^*]^\omega$ and a clopen disjoint A_{-1}, A_O, A_1 partition of C, there are pairwise almost disjoint B_{-1}, B_0, B_1 in $[\omega]^\omega$ such that $B^*_i \cap C \subseteq A_i$.

Proof. Enumerate $A_{-1} \cup A_0 \cup A_1$ as $\{x_n : n < \omega\}$. By induction construct disjoint $B^i_n, i = -1,0,1$ subsets of ω such that $x_n \in \bigcup_{j \le n, i=-1,0,1} B^{i*}_j$ and $B^{i*}_n \cap C \subseteq A_i$. If x_n is already covered, let $B^i_n = \emptyset$. Otherwise suppose (WLOG) $x_n \in A_{-1}$. There is a closed subset of ω^* such that $A_0 \cup A_1 = F \cap C$. Because $x_n \notin F \cup \bigcup_{j<n, i=-1,0,1} B^{i*}_j$ there is an open (in ω^*!) U disjoint from $F \cup \bigcup_{j<n,i=-1,0,1} B^{i*}_j$ and containing x_n. Because ω^* is zerodimensional there is a clopen (in ω^*) U' subset of U containing x_n which misses $F \cup \bigcup_{j<n,i=-1,0,1} B^{i*}_j$. Choose $B^0_n \subseteq \omega$ disjoint with $B^i_j, j < n, i = -1,0,1$ and $B^{0*}_n = U'$. This is possible because U' is disjoint from each $B^{i*}_j, j < n, i = -1,0,1$. Since $U' \cap C \subseteq A_{-1}$, $B^{-1}_n \cap C \subseteq A_{-1}$. Let $B^0_n = B^1_n = \emptyset$. Now $B_i = \bigcup_{n<\omega} B^i_n$ are as required. □

Now the (continuous) function $f_0 : (B^0_{-1} \cup B^0_0 \cup B^0_1) \to [-1,1]$ having value $-1/2$ on B^0_{-1}, 0 on B^0_0 and $1/2$ on B^0_1 can be extended to some $F_0 : \beta\omega \to [-1,1]$. Necessarily $F_0 \restriction A^0_{-1} \equiv -1/2$, $F_0 \restriction A^0_0 \equiv 0$, $F_0 \restriction A^0_1 \equiv 1/2$. Then the supremum of $\{|F_0(x) - f(x)| : x \in C\}$ is less than $1/2+\epsilon$. By the same reasoning looking at $g = F_0 - f : C \to [-1/2-\epsilon, 1/2+\epsilon]$ we can inductively construct $F_n : \beta\omega \to [-(1/2+\epsilon)^n, (1/2+\epsilon)^n]$ such that the supremum of $\{|F_n(x) - (f(x) - \sum_{i=0}^{n-1} F_i(x))|\}$ is less than $(1/2+\epsilon)^{n+1}$. Then the sum of the functions F_n converges uniformly on ω^* so if we let F be their sum it the required continuous extension. □

1.3 Boolean algebras

Boolean algebras are only used in chapter 5, although in chapter 3 we use a theorem that is proved in this section. We assume the reader is familiar with the introductory part of [HBA]. We shall use the letters $\mathbb{A}, \mathbb{B}, \mathbb{C}$ for boolean algebras, the symbols $\wedge, \vee, \bigwedge, \bigvee$ to denote the operations and $\mathbf{0}, \mathbf{1}$ to denote the zero and one of a b.a.

The following is the Stone duality:

Theorem 1.21. *The functor which assigns to each compact zerodimensional T_0 space the algebra of clopen sets of this space is adjoint to the functor which assigns to each algebra the space of ultrafilters on this algebra.*

Proposition 1.22. *If Y is a zerodimensional, T_0 space then βY is homeomorphic to $St(Clopen(Y))$, the stone space of the algebra of clopen subsets of Y.*

Corollary 1.23. *For a zerodimensional, T_0 Y the algebra $Clopen(Y)$ is isomorphic to $Clopen(\beta Y)$.*

In chapter 5 we will need to find a quotient algebra of $\mathcal{P}(\omega)$ which has large hereditary independence. The proof is standard and we include it for completeness. A definition is in order:

Definition 1.24. For a Boolean algebra $\mathbb{B}$ and a set $A \subseteq \mathbb{B}$ we define an *elementary meet* over the set A to be

$$\bigwedge_{i=0}^{n} \varepsilon(i)a_i,$$

for any $n < \omega$, $\{a_0, \ldots, a_n\} \subseteq A$ and $\varepsilon : n \to \{-1, 1\}$ (where $-1a_i = -a_i$ and $1a_i = a_i$). A is said to be *independent*, if all elementary meets over A are nonzero. The minimal cardinality of a maximal (with respect to inclusion) independent subset of $\mathbb{B}$ is denoted by $\mathfrak{i}(\mathbb{B})$. We say that $\mathbb{B}$ has *hereditary independence* κ if $(\forall b \in \mathbb{B})(\mathfrak{i}(\mathbb{B} \restriction b) \geq \kappa)$.

Lemma 1.25. *There is an ideal I on ω, extending FIN and such that $\mathcal{P}(\omega)/I$ has hereditary independence $\mathfrak{c}$.*

Proof. Look at $\mathbb{B} = Compl(Clopen(2^{\mathfrak{c}}))$. We shall show that $\mathbb{B}$ has hereditary independence $\mathfrak{c}$. The family $\mathcal{D} = \{[f] : \mathrm{dom} f \in [\mathfrak{c}]^{<\omega}, \mathrm{rng} f \subseteq 2\}$ where $[f] = \{g \in {}^{\mathfrak{c}}2 : f \subseteq g\}$ is the basic clopen subset of $2^{\mathfrak{c}}$ is dense in $\mathbb{B}$. If $A \in [\mathbb{B}]^{<\mathfrak{c}}$ is independent, find for each $a \in A$ an element $[f_a] \in \mathcal{D}$ which is below a. Then $|J = \bigcup_{a\in A} \mathrm{dom}\, f_a| < \mathfrak{c}$, so we can find an $\alpha \in \mathfrak{c} \setminus J$. Then $A \cup \{[\langle \alpha, 0\rangle]\}$ is still independent, so A was not maximal independent. We have proved that the independence of $\mathbb{B}$ is (at least) $\mathfrak{c}$. Since $\mathbb{B}$ is homogeneous, even the hereditary independence of $\mathbb{B}$ is (at least) $\mathfrak{c}$.

Since $\mathbb{B}$ is σ-centered, the following standard fact finishes the proof of the lemma: □

Fact 1.26. *Every σ-centered algebra is isomorphic to a quotient algebra of $\mathcal{P}(\omega)/FIN$*

Lemma 1.27 (Normal form Theorem)**.** *Let $\mathbb{B}$ be a Boolean algebra and $A \subseteq \mathbb{B}$ be a subset. Then every element in $\langle A\rangle$, the algebra generated by A, can be written in the form of a finite join of elementary meets over A.*

As a special case of this lemma, we will be using the following corollary

Corollary 1.28. *Let $\mathbb{C}$ be a Boolean algebra, $\mathbb{B}$ a subalgebra and $c \in \mathbb{C}$. Then every atom of $\langle\{c\} \cup \mathbb{B}\rangle$ can be written in the form $c \wedge b$ or $-c \wedge b$ for some $b \in \mathbb{B}$.*

The following lemmas and definitions will be used in chapter 3. If $\mathbb{B}$ is an algebra of sets, i.e. $\mathbb{B} \leq \mathcal{P}(X)$, then $\mathbb{B}$ is a base for some zero-dimensional topology on X. Call this topology $\tau_{\mathbb{B}}$. (Every zero-dimensional space arises in this way.) The next lemmas (and definitions) describe this topology.

Lemma 1.29. *$\mathbb{B}$ is atomless if $(X, \tau_{\mathbb{B}})$ is T_2 and crowded. If X is T_2 then $\mathbb{B}$ is atomary iff $(X, \tau_{\mathbb{B}})$ is discrete.*

Lemma 1.30. *$\mathbb{B}$ is complete iff $(X, \tau_{\mathbb{B}})$ is extremally disconnected.*

Proof. Let $\mathbb{B}$ be complete and $U = \cup\mathfrak{U}$ for some $\mathfrak{U} \subseteq \mathbb{B}$. Then $\bigvee \mathfrak{U}$ (which is, usually, *different* from $\cup\mathfrak{U}$, even though for any $a, b \in \mathbb{B}$ $a \vee b = a \cup b$) is the closure of U: It is closed and contains U. If $x \notin \bigvee \mathfrak{U}$ then $-\bigvee \mathfrak{U}$ is a neighbourhood of x which is disjoint from U. So we have proved that the closure of an open set is open.

On the other hand if X is extremally disconnected and $\mathfrak{U} \subseteq \mathbb{B}$, then $A = \overline{\cup\mathfrak{U}} = \bigvee \mathfrak{U}$: A is certainly an upper bound of $\mathfrak{U}$, so suppose that some $\mathbf{0} \neq B \leq A - \bigvee \mathfrak{U}$. Then this B is disjoint from all $U \in \mathfrak{U}$, so it is disjoint from A a contradiction. This shows that $\mathbb{B}$ is complete. □

Lemma 1.31. *Let $\mathbb{B}$ be an atomless subalgebra of a complete algebra $\mathbb{C}$. If $D \subseteq \mathbb{B}$ has no supremum in $\mathbb{B}$, then there is a $c \in \mathbb{C} \setminus \mathbb{B}$ such that $\langle \mathbb{B} \cup \{c\} \rangle$ is atomless.*

Proof. Let c' be the supremum of D in $\mathbb{C}$. Define $U = \{u \in \mathbb{B} : u \geq c'\}$. Let c be the infimum of U in $\mathbb{C}$. This c is not in $\mathbb{B}$, because otherwise it would be supremum of D in $\mathbb{B}$. Aiming towards a contradiction suppose a were an atom of $\langle \mathbb{B} \cup \{c\} \rangle$. Then, by Lemma 1.28, there are two cases:

Case 1. For some $b \in \mathbb{B}$, $a = b \wedge c$. Notice that if for some $d \in D$, $d \wedge b \neq \mathbf{0}$ then, because $d \leq c' \leq c$, the nonzero element $d \wedge b$ of $\mathbb{B}$ would be less than the atom $c \wedge b = a$ which cannot be, since $\mathbb{B}$ is atomless. So all $d \in D$ must be below $-b$. This shows that $-b \in U$: because D is below $-b$, $-b$ must be above c', the supremum of D, so it must also be above c. This is a contradiction with a being nonzero.

Case 2. For some $b \in \mathbb{B}$, $a = b \wedge -c$. If for some $u \in U$, u would not be above b, then $-u \wedge b \neq \mathbf{0}$ and the nonzero element $-u \wedge b$ of $\mathbb{B}$ would be less than $-c \wedge b = a$ (notice that $-u \leq -c$), which is impossible since a is an atom and $\mathbb{B}$ is atomless. So all $u \in U$ are above b so b must be below c, the infimum of U. Then $a = -c \wedge b = \mathbf{0}$ a contradiction with a being nonzero. □

Proposition 1.32. *If $\mathbb{B}$ is an atomless subalgebra of a complete algebra $\mathbb{C}$, then there is a complete algebra $\mathbb{B}'$, which is a subalgebra of $\mathbb{C}$ (not necessarily a complete subalgebra of $\mathbb{C}$), which is atomless and contains $\mathbb{B}$.*

Proof. Order the atomless subalgebras of $\mathbb{C}$ by inclusion. Since the union of a chain of atomless algebras is an atomless algebra we can use Zorn's lemma to get a maximal atomless subalgebra of $\mathbb{C}$ containing $\mathbb{B}$. This algebra is necessarily complete, since otherwise we could use Lemma 1.31 to show that it is not complete □

Note an easy corollary to the proof of the previous proposition:

Corollary 1.33. *If $\mathbb{B}$ is a maximal atomless subalgebra of a complete Boolean algebra $\mathbb{C}$, then $\mathbb{B}$ is complete.*

The following topological reformulation of 1.33 shall later be used:

Theorem 1.34. *Every maximal crowded and zerodimensional space is extremally disconnected.*

1.4 Consistency and Independence

Since Gödel's results from the thirties, it is known, that our descriptions of the mathematical objects (whatever they are) cannot be complete. The precise formulation of this fact is the famous Gödel's incompleteness theorem which says, that any first-order, recursively axiomatizable theory which is strong enough to formalize arithmetic does not decide the truth of all sentences expressible in its language. His proof was constructive, but yielded a rather artificial undecidable statement. Thus it was quite a shock when in 1963 Paul Cohen discovered, that the Continuum Hypothesis:

Hypothesis 1.35 (CH). $|\mathcal{P}(\omega)| = |\mathbb{R}| = \omega_1$.

is undecidable from the usual axioms of Set Theory (Cohen actually only showed that CH cannot be proved. The undecidability follows from Gödel's proof that CH cannot be refuted). Cohen came up with the Forcing method, which has become extremely fruitful. Using his method it was soon established that many statements cannot be proved in ZFC. This section will be devoted to three hypotheses which are in the line of CH. Any of them would be enough to guarantee the existence of the points we are looking for, but each of them is undecidable. The first one is CH itself. If we assume CH, then all subsets of ω may be numbered by countable ordinals and that helps enormously in inductive constructions. As an example we give the proof of the existence of P-points in ω^* under CH:

Theorem 1.36 ([Rud56]). *Assume $2^\omega = \omega_1$. Then there is a P-point p in ω^*.*

Proof. Let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of $\mathcal{P}(\omega)$ and let $\langle C_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of ${}^\omega\mathcal{P}(\omega)$ with each sequence listed cofinally often. By induction construct filters $\mathcal{F}_\alpha$ for $\alpha < \omega_1$ satisfying:

(i) For each $\alpha < \omega_1$ either $A_\alpha \in \mathcal{F}_{\alpha+1}$ or $(\omega \setminus A_\alpha) \in \mathcal{F}_{\alpha+1}$.

(ii) For each $\alpha < \omega_1$ if $\operatorname{Im} C_\alpha \subseteq \mathcal{F}_\alpha$ then there is a $B \in \mathcal{F}_{\alpha+1}$ such that $|B \setminus C_\alpha(n)| < \omega$ for all $n \in \omega$.

(iii) For each $\alpha < \omega_1$ the filter $\mathcal{F}_\alpha$ has a countable basis.

Let $\mathcal{F}_0$ be the Fréchet filter on ω. If $\alpha < \omega_1$ is limit, let $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$ and both (i), (ii) and (iii) are satisfied. So suppose $\alpha < \omega_1$ is not limit. If there is an $F \in \mathcal{F}_\alpha$ such that $F \cap A_\alpha = \emptyset$ then let $\mathcal{F}'_\alpha = \mathcal{F}_\alpha \cup \{(\omega \setminus A_\alpha)\}$ otherwise let $\mathcal{F}'_\alpha$ be the filter generated by $\mathcal{F}_\alpha \cup \{A_\alpha\}$. $\mathcal{F}'_\alpha$ is a filter satisfying (i),(iii). Suppose $\operatorname{Im} C_\alpha \subseteq \mathcal{F}'_\alpha$. Let $\{F_n : n < \omega\}$ be an enumeration of the basis of $\mathcal{F}'_\alpha$. Inductively for each $k < \omega$ choose $n_k \in (\bigcap_{i<k} F_i) \setminus \{n_0, \ldots, n_{k-1}\}$. This is possible since $\mathcal{F}'_\alpha$ is centered. Now let $\mathcal{F}_{\alpha+1} = \langle \mathcal{F}_\alpha \cup \{\{n_k : k < \omega\}\} \rangle$. This is a centered system and the set $|\{n_k : k < \omega\} \setminus C_\alpha(n)| < \omega$ (in fact, if $C_\alpha(n) = F_i$, then the cardinality is at most i).

Now let $\mathcal{F} = \bigcup_{\alpha<\omega_1} F_\alpha$. This $\mathcal{F}$ is an ultrafilter (if $A \in \mathcal{P}(\omega)$, then $A = A_\alpha$ for some $\alpha < \omega_1$. Then either $A \in \mathcal{F}_{\alpha+1} \subseteq \mathcal{F}$ or $(\omega \setminus A) \in \mathcal{F}_{\alpha+1} \subseteq \mathcal{F}$ by (i). It is a P-point because if $\langle U_n : n < \omega \rangle$ is a sequence of neighbourhoods of $\mathcal{F}$ with $C_n \in \mathcal{P}(\omega)$ such that $U_n = C_n^*$, then the sequence $\langle C_n : n < \omega \rangle$ is listed as C_α for some $\alpha < \omega_1$ and there is, by (ii), a $B \in \mathcal{F}_{\alpha+1} \subseteq \mathcal{F}$ which is almost contained in all C_n's so $p \in B^* \subseteq \bigcap_{n<\omega} C_n^*$. □

The CH turns out to be very strong and simplifies the study of ω^* considerably. But what if the continuum is larger and hence ω^* is richer? Martin proposed an axiom, which is weaker than CH, allows for the continuum to be arbitrarily large and yet is still strong enough for a lot of constructions to work:

Axiom 1.37 (MA). *If $(P, \leq)$ is a ccc poset of size at most $\mathfrak{c}$, $\lambda < \mathfrak{c}$ and $\langle D_\alpha : \alpha < \lambda \rangle$ is a family of subsets of P each of which is dense, then there is a filter G on $(P, \leq)$ which meets every D_α.*

Martin's axiom can prove the existence of P-points. The proof proceeds similarly as in 1.36 where condition (ii) is guaranteed using the following standard lemma:

Lemma 1.38 (MA). *If $\lambda < \mathfrak{c}$ and $\{A_\alpha : \alpha < \lambda\}$ is a centered system of subsets of ω then they have pseudointersection, that is there is an A which is almost contained in all A_α's.*

Proof. Define $P = \{(F, I) : F \in [\omega]^{<\omega}, I \in [\lambda]^{<\omega}$ $(F, I) \leq (G, J)$ if and only if $F \subseteq G$, $I \subseteq J$ and $G \setminus F \subseteq \bigcap_{\alpha \in I} A_\alpha$. Then $(P, \leq)$ is a ccc poset of size λ. Now consider the sets

$$\mathcal{D}_\alpha = \{(F, I) \in P : \alpha \in I\}, \; \mathcal{D}'_n = \{(F, I) \in P : |F| \geq n\}.$$

Each of them is dense in $(P, \leq)$. So there is a filter $\mathcal{F}$ on $(P, \leq)$ which meets each of them. If we let $A = \bigcup\{F : (\exists I \in [\lambda]^{<\omega})(F, I) \in \mathcal{F}\}$ then A is infinite because $\mathcal{F}$ meets each $\mathcal{D}'_n$ and is almost contained in each A_α because some (F, I) is contained in $\mathcal{D}_\alpha \cap \mathcal{F}$. Then $A \setminus A_\alpha \subseteq F$. □

The previous proofs in fact give filters which are even $P_{<\mathfrak{c}}$-points (i.e. the intersection of less than $\mathfrak{c}$ neighbourhoods is again a neighbourhood). In [Ket76] it is shown that to get only a P-point, much less is needed:

Definition 1.39. A family $\mathcal{F}$ of functions from ω to ω is *dominating* iff for any $g \in {}^\omega\omega$ there is an $f \in \mathcal{F}$ with $g \leq^* f$ (i.e. $g(n) > f(n)$ for only finitely many n's). The *dominating number* $\mathfrak{d}$ is defined to be the least cardinality of a dominating family.

It is easy to see, that $\omega < \mathfrak{d} \leq \mathfrak{c}$ so CH implies $\mathfrak{d} = \mathfrak{c}$.

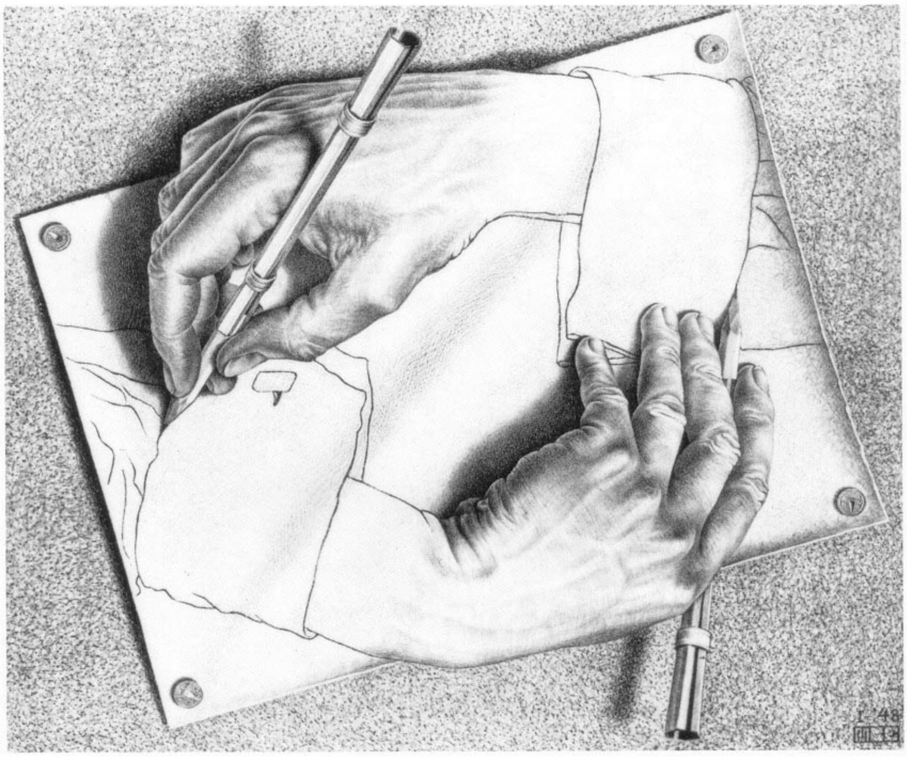
Fact 1.40 (MA). $\mathfrak{d} = \mathfrak{c}$.

Theorem 1.41 ([Ket76],1.3). *Assume $\mathfrak{d} = \mathfrak{c}$. If $\mathcal{F}$ is a centered system of size $< \mathfrak{c}$, and $\langle F_n : n < \omega \rangle$ is a sequence of members of $\mathcal{F}$ then there is an A which is almost contained in each F_n and such that $\mathcal{F} \cup \{A\}$ is centered.*

Proof. We may assume that the sequence is descending mod $\subseteq$. Define for $F \in \mathcal{F}$ a function f_F as follows: $f_F(n) = \min F_n \cap F$. Then the family $\{f_F : F \in \mathcal{F}\}$ has size $< \mathfrak{c}$ so it is not dominating by our assumption, so there is a $g \in {}^\omega\omega$ with $\{n : g(n) > f_F(n)\}$ infinite for each $F \in \mathcal{F}$. Then if we let $A = \bigcup_{i<\omega} F_i \cap g(i)$ we are done. □

Looking at the proof of 1.36, the following corollary is immediate:

Corollary 1.42 ($\mathfrak{d} = \mathfrak{c}$). *There is a P-point in ω^*.*



M. C. Escher: **Drawing hands**

Chapter 2

Embedding spaces into the growth of integers

IN this chapter we give needed definitions and quote theorems which will allow us to embed certain spaces into ω^* in a manner that will preserve unique ω-accessibility of points. We prove this preservation property after giving the theorems. All of the listed results are known. The following theorem due to Simon is the main theorem of the chapter.

Theorem 2.1 ([Sim85]). *The Čech-Stone compactification of any T_3 extremally disconnected space X of weight $\leq \mathfrak{c}$ can be embedded as an ω_1-O.K. set into ω^*.*

The previous theorem has a generalization by Kunen and Baker ([KB02], Theorem 5.6) which we shall present without proof. First we need some rather technical definitions which will not be needed in the other chapters and may be safely skipped.

Definition 2.2. A set function $\hat{}: [\theta]^{<\omega} \longrightarrow [\kappa]^{<\omega}$ is called a (θ, κ)-*hatfunction.* We say that $\hat{}$ is *monotone*, if for any two sets $I \subseteq J$ in the domain of $\hat{}$, $\hat{J} \subseteq \hat{I}$. A set $P \subseteq X$ in a topological space is a $\hat{}$-*set* iff for any sequence $\langle U_K : K \in [\kappa]^{<\omega}\rangle$ of neighborhoods of P there is a sequence $\langle V_\alpha : \alpha < \theta\rangle$ of neighborhoods of P such that for any I in the domain of $\hat{}$ the following is true:

$$\bigcap_{\alpha \in I} V_\alpha \subseteq U_{\hat{I}}$$

The function which assigns to each $I \in [\theta]^{<\omega}$ its cardinality shall be called the θ-O.K. function.

Definition 2.3. A sequence $\langle M_\alpha : \alpha \in I\rangle$ of subsets of $\mathbb{B}$ is a *matrix independent with respect to a filter* $\mathcal{F}$ on $\mathbb{B}$, if for any $I_0 \in [I]^{<\omega}$, $c \in \mathcal{F}$ and $f : I_0 \longrightarrow \cup\{M_\alpha : \alpha \in I\}$ so that $f(\alpha) \in M_\alpha$ the following intersection is nonzero:

$$c \wedge \bigwedge_{\alpha \in I_0} f(\alpha)$$

Definition 2.4. If $\mathcal{G}$ is a filter on $\mathbb{B}$ then $M \subseteq \mathbb{B}$ is a $\hat{}$-step family on $(\mathbb{B}, \mathcal{G})$ iff it is of the form:

$$M = \{e_K : K \in [\kappa]^{<\omega}\} \cup \{a_\alpha : \alpha < \theta\} \cup \left\{ e_K \wedge \bigwedge_{\alpha \in I} a_\alpha : K \in [\kappa]^{<\omega}, I \in [\theta]^{<\omega}, \hat{I} \subseteq K \right\}$$

and satisfies:

(i) $\{e_K : K \in [\kappa]^{<\omega}\}$ is a partition of unity (i.e. it is a set of disjoint elements with supremum $\mathbf{1}$).

(ii) For each $I \in [\theta]^{<\omega}$

$$-\left(\bigwedge_{\alpha\in I} a_\alpha \wedge \bigvee\{e_K : \hat{I} \not\subseteq K\}\right) \in \mathcal{G}$$

(iii) For each $I \in [\theta]^{<\omega}$ and $K \in [\kappa]^{<\omega}$ if $\hat{I} \subseteq K$ then

$$e_K \wedge \bigwedge_{\alpha\in I} a_\alpha \notin \mathcal{G}^*,$$

where $\mathcal{G}^*$ is the dual ideal to the filter $\mathcal{G}$.

Now we can state the theorem of Kunen and Baker. We will not prove it, however, and the interested reader can find it in their survey paper [KB02].

Theorem 2.5 (Kunen, Baker). *Let $\mathbb{B}$ be a complete Boolean algebra of size 2^κ with $\mathcal{G} \subseteq \mathcal{F}$ two filters on $\mathbb{B}$. Let $\hat{}$ be any monotone (θ, κ)-hatfunction. Assume that $\mathbb{M} = \langle \mathcal{M}^i : i \in 2^\kappa \rangle$ is a matrix independent with respect to $\mathcal{F}$ so that each $\mathcal{M}^i$ is a $\hat{}$-step family on $(\mathbb{B}, \mathcal{G})$. Then for every complete Boolean algebra $\mathbb{A}$ of size $\leq 2^\kappa$, there is an $h : \mathbb{B} \twoheadrightarrow \mathbb{A}$ such that $h''\mathcal{F} = \{1\}$ and such that $h^*(st(\mathbb{A})) \subseteq st(\mathbb{B}/\mathcal{F})$ is a $\hat{}$-set in $st(\mathbb{B}/\mathcal{G})$.*

Proof of 2.1. Let $\mathbb{A}$ be a basis of X of size $\leq \mathfrak{c}$ consisting of clopen sets. Since X is extremally disconnected $\mathbb{A}$ is a complete Boolean algebra. Let $\mathbb{B}$ be $\mathcal{P}(\omega)$, $\mathcal{F}, \mathcal{G}$ the Fréchet filter on ω, $\hat{}$ the ω_1-O.K. hatfunction from definition 2.2 and $\mathbb{M}$ the matrix given by Theorem 3.9 of [KB02] (which is proved in [KB01]). Then the preceding theorem gives us an embedding of $st(\mathbb{A}) \approx \beta X$ as an ω_1-O.K. set into $st(\mathcal{P}(\omega)/FIN) \approx \omega^*$. □

Note 2.6. Kunen has developed methods, which work not only for the whole Čech-Stone compactification but, in some cases, also for the growth of the Čech-Stone compactification. He needs the algebra of clopen sets of the growth to be a quotient algebra of the algebra of clopen sets of the whole Čech-Stone compactification. This is equivalent to the original space being locally compact. Unfortunately we will be interested in irresolvable spaces which are far from being locally compact (see 1.5).

We also mention a useful corollary of a theorem of van Mill ([vM82]):

Theorem 2.7 (van Mill). *The projective cover of a continuous image of ω^* can be embedded as a $\mathfrak{c}$-O.K. set in ω^*.*

The following proposition proves that the previous embeddings preserve ω-unique accessibility:

Proposition 2.8. *If $p \in Y \subseteq X$ is an ω-uniquely accessible point of Y and if Y is a closed ω_1-O.K. set in X which is T_3, then p is an ω-uniquely accessible point of X.*

Proof. Suppose $C, D \in [X]^\omega$ are two disjoint sets with $p \in \overline{C} \cap \overline{D}$. Then, since Y is a weak P-set of X by Proposition 1.12, $p \in \overline{C \cap Y} \cap \overline{D \cap Y}$ and, by ω-unique accessibility of p in Y we have, that $\emptyset \neq (C \cap Y) \cap (D \cap Y) \subseteq C \cap D$. □

Chapter 3

Irresolvable spaces

"You know I always thought
unicorns were fabulous creatures too,
although I never saw one alive before."
"Well, now that we have met,"
said the unicorn,
"If you'll believe in me, I'll believe in you."
Lewis Carroll

THIS chapter investigates topologies which have no disjoint dense sets (they are called irresolvable) and ways to construct them. Since we want to embed the resulting spaces into ω^* we will need to look at the situation for zerodimensional spaces. The main result is 3.14.

3.1 Constructions

Irresolvable spaces have been first investigated by E. Hewitt ([Hew43]) and Katětov at the end of the forties. Since any space with an isolated point contains no two disjoint dense sets, it is customary to consider only crowded spaces. The following definition is due to E. Hewitt.

Definition 3.1 (Hewitt). A crowded topological space is *resolvable* if it contains at least two disjoint dense subsets. It is *irresolvable* if it is not resolvable.

On the side of resolvability we state only a single easy observation, which will prove beneficial later.

Observation 3.2. *The union of resolvable spaces is resolvable.*

Proof. Let $\{X_\alpha : \alpha < \kappa\}$ be an increasing chain of resolvable topological spaces and D^i_α, $\alpha < \kappa, i = 0, 1$ be disjoint sets dense in X_α. Define $D'^i_0 = D^i_0$ and

$$D'^i_\alpha = D^i_\alpha \setminus \overline{\bigcup_{\beta<\alpha} D'^i_\alpha}$$

Let $D^i = \bigcup_{\alpha<\kappa} D'^i_\alpha$. Now D^0 is disjoint from D^1 and both are dense in every X_α so also in their union, so their union is resolvable. □

Quite recently the notion of irresolvability (and especially resolvability) has become popular again, see e.g. [CW05], [JSS05] or [Pav05], which is a summary. In some sense the irresolvable spaces are close to discrete spaces as is suggested by the method Hewitt used to construct them. In his 1943 paper ([Hew43]) he considered maximal crowded topologies which turn out to be irresolvable. First we need a definition:

Definition 3.3. If P is a property of a topology (e.g. T_1, T_2, crowded, etc.), we say that τ is *maximal* P if it has P but cannot be refined to a strictly stronger topology having P.

Note 3.4. Originally Hewitt defined maximal topologies to be maximal crowded topologies. In this chapter we will use the modern terminology (as in the previous definition) not to confuse the reader.

Proposition 3.5 (Hewitt). *If τ is a maximal crowded topology on X, then any two τ-dense sets intersect.*

Proof. Suppose D_1, D_2 are disjoint dense. Then D_1 is not open and the topology generated by $\tau \cup \{D_1\}$ is a strictly finer topology which does not have any isolated point. □

The motivation for the following definition comes from the fact, that the irresolvability of a space is not hereditary.

Definition 3.6. A space is said to be *hereditarily irresolvable* iff any crowded subspace is irresolvable. It is said to be *open hereditarily irresolvable* (OHI for short), if every open subspace is irresolvable.

The topology in the previous proposition in fact turns out to be OHI, with virtually the same proof, which we omit:

Proposition 3.7 (Hewitt). *If τ is a maximal crowded topology on X, then (X, τ) is OHI.*

Van Douwen in his paper [vD93] gives the following characterization of OHI spaces:

Proposition 3.8 ([vD93],1.13). *For a crowded space X the following is equivalent:*

(i) X is open hereditarily irresolvable.

(ii) Subsets of X with empty interior are nowhere dense in X.

Another equivalent condition is the following:

Lemma 3.9. *For a crowded space X the following is equivalent:*

(i) X is open hereditarily irresolvable.

(iii) The dense subsets of X form a filter.

Proof. Suppose the dense sets form a filter and that for some open $G \subset X$ there are two disjoint D_0, D_1 dense in G. Let $D_i' = (X \setminus G) \cup D_i$. Then D_i' are dense in X for $i = 0, 1$ and $D_0' \cap D_1'$ is not dense because it is disjoint from G, which is a contradiction.

On the other hand suppose that D_0, D_1 are dense, while $D_0 \cap D_1$ is disjoint from some open G. Then $D_0 \cap G$ and $D_1 \cap G$ are disjoint and both dense in G, so X is not OHI. □

Before considering zerodimensional spaces we look at the main result of section 2 of [vD93]. But first we need some definitions:

Definition 3.10. A topological space is *perfectly disconnected* if no point is a limit point of two disjoint sets. It is *nodec* if every nowhere dense set is closed and is *ultradisconnected* if it is crowded and any two disjoint crowded subsets have disjoint closures.

Theorem 3.11 ([vD93], 2.2). *For a crowded space X the following are equivalent*

(i) X is perfectly disconnected

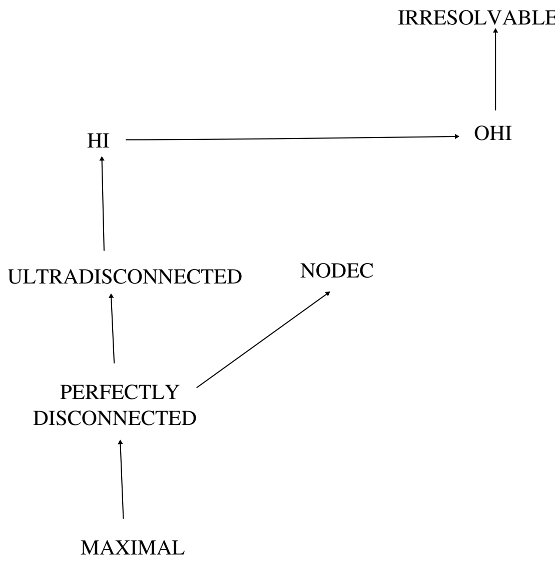
(ii) a subset of X is open iff it is crowded

(iii) X is maximal crowded.

(iv) X is ultradisconnected and nodec

(v) X is extremally disconnected, OHI and nodec

Note, that an ultradisconnected space is hereditarily irresolvable (i.e. any crowded subspace is irresolvable) and extremally disconnected. See below a diagram illustrating some relations between different irresolvability properties:



The following fact summarizing the work of Hewitt gives us a tool to construct OHI (even stronger) spaces:

Fact 3.12. *If the topology of X is the finest crowded T_i topology on X ($i \in \{1, 2, 3\}$), then X is ultradisconnected.*

Before continuing, we state a corollary of 3.2 which will be useful and also a slightly obscure result, which we discovered while refining topologies:

Corollary 3.13. *Any irresolvable, not hereditarily irresolvable space contains a maximal (w.r.t. inclusion) resolvable subspace. Its complement is hereditarily irresolvable.*

Proof. Suppose X is non HI, there is an subset A of X which is resolvable. By 3.2 the union R of all resolvable subspaces of X containing A is a resolvable proper (since X is irresolvable) subspace of X. Notice that R must be closed because the closure of a resolvable space is resolvable. Suppose $B \subseteq X \setminus R$ is resolvable. Then $R \cup B$ is resolvable, a contradiction with the definition of R. □

Next we prove a theorem which is the result of this section that will be used later on along with 3.13:

Theorem 3.14. *Any T_2, zerodimensional crowded topology can be refined to an OHI extremally disconnected, zerodimensional crowded topology.*

Proof. First notice, that the union of zerodimensional crowded topologies again generates a crowded topology: crowdedness is clear, and since the union of the basis for the topologies forms a basis of the resulting topology, we get a clopen base for the resulting topology which shows that it is zerodimensional.

Take a maximal crowded and zerodimensional topology τ and prove, that it must be OHI: suppose otherwise. Then there is a (wlog) clopen G and two disjoint $D_0, D_1 \subseteq G$ such that $\mathrm{cl}_\tau(D_0) = \mathrm{cl}_\tau(D_1) = G$. Then D_i cannot be τ-clopen so $\tau' = \langle \tau \cup \{D_0, X \setminus D_0\}\rangle \neq \tau$. If we can show that τ' is crowded zerodimensional, we are done. It is certainly zerodimensional. Suppose that for some τ-open U the intersection $U \cap D_0$ is nonempty finite. Then so is $U \cap G \cap D_0$ (since $D_0 \subseteq G$) which is a contradiction (D_0 is dense in G and X is T_2, D_0 must intersect any nonempty open subset — e.g. $U \cap G$ — of G in an infinite set). If on the other hand for some τ-open U we had $U \cap (X \setminus D_0)$ nonempty finite, then again necessarily also $U \cap (X \setminus D_0) \cap G$ is nonempty finite (because $U \cap (X \setminus G) \cap (X \setminus D_0) = U \cap (X \setminus G)$ which is open in τ so it is either infinite, or empty). But since $(X \setminus D_0) \cap G = D_1$ which is dense in G the same argument as before applies.

Now we can apply theorem 1.34 which says that a maximal crowded and zerodimensional topology is extremally disconnected.

□

3.2 Irresolvable spaces and compactness

The last part of this chapter will be devoted to theorems investigating compactness in the context of irresolvability. We do not know if the theorems are known. As a remark we should warn the reader that the spaces we get from theorem 3.14 are extremally disconnected and OHI but are not necessarily maximal crowded. In the context of extremally disconnected OHI spaces maximal crowdedness is equivalent to the nodec property by theorem 3.11, so theorem 3.18 need not apply. The following theorem was suggested to us by prof. Simon:

Theorem 3.15. *Any T_2 OHI space is nowhere locally compact.*

Proof. Suppose X is T_2, OHI and there is an open $G \subseteq X$ such that $K = \overline{G}$ is compact. Without loss of generality we may assume that any open subset of K has cardinality $|K|$. Work in K. By theorem 1.4, the character of each point of K is equal to the pseudocharacter, which is at most $|K|$ by 1.3. If $\mathcal{V}(x)$ is a local base at x of cardinality $\chi(x)$, then $\mathcal{B} = \{\mathcal{V}(x) : x \in X\}$ is a base of K of cardinality $|K|$. Enumerate $\mathcal{B}$ as $\{U_\alpha : \alpha < |K|\}$. By induction choose $x^0_\alpha \neq x^1_\alpha \in U_\alpha \setminus \{x^i_\beta : \beta < \alpha, i < 2\}$. Then $D_i = \{x^i_\alpha : \alpha < |K|\}$ for $i = 0, 1$ are two disjoint dense subsets of K a contradiction. □

Corollary 3.16. *Any T_2 irresolvable space is not compact.*

Observation 3.17. *If X is nodec, then all n.w.d. subsets of X are discrete.*

Proof. Suppose A is n.w.d. and $a \in A$. Since $A \setminus \{a\}$ is also n.w.d. and hence closed, a cannot be an accumulation point of A. □

Theorem 3.18. *If X is T_2, OHI and nodec, then it contains only finite compact sets.*

Proof. Suppose $K \subseteq X$ is infinite compact. By the previous observation, K cannot be nowhere dense. Because K is compact and X is T_2, K must be closed. So $\mathrm{int}\, K = U \neq \emptyset$. But by 3.15 X is nowhere locally compact so $\overline{U} \subseteq K$ cannot be compact, a contradiction.

□

Chapter 4

Weak P-points

Sometimes I've believed as many as six
impossible things before breakfast.
Lewis Carroll

NOW we turn our attention to finding weak P-points points in general growths. The main result of this chapter is 4.17. The problem with constructions of weak P-points is that the definition talks about sequences of points. There are usually too many such sequences to deal with in a straightforward induction. For example when constructing a filter in ω^* we have at most $|\mathcal{P}(\omega)| = \mathfrak{c}$ many steps but there are $2^{\mathfrak{c}}$ many sequences in ω^*. There are different methods to overcome this problem. In [Kun76] Kunen noticed, that we can check the property by only considering sequences of clopen sets. There are only $\mathfrak{c}^\omega = \mathfrak{c}$ many such sequences, so there is hope. Unfortunately Kunen needed MA (in fact, only $\mathfrak{b} = \mathfrak{c}$ was needed) at successor stages of the inductive construction. Another method, also due to Kunen, constructs a point having a stronger property:

Definition 4.1. A point p is κ-O.K. provided that for any sequence $\langle V_n : n \in \omega\rangle$ of neighbourhoods of p there is a family $\{U_\alpha : \alpha < \kappa\}$ of neighbourhoods of p such that for any $I \in [\kappa]^{<\omega}$

$$\bigcap_{\alpha \in I} U_\alpha \subseteq V_{|I|}$$

It is easy to see, that if $\kappa < \lambda$ then any λ-O.K. point is also κ-O.K. The fact that any ω_1-O.K. point is a weak P-point follows from the following proposition ([Kun78]):

Proposition 4.2. *If $A \subseteq X \setminus \{x\}$ is κ-cc, cf $\kappa > \omega$, and if x is κ-O.K., then $x \notin \overline{A}$.*

Proof. Suppose, aiming for a contradiction, that $x \in \overline{A}$. Then we can find a descending sequence $\langle V_n : n \in \omega\rangle$ of neighbourhoods of x each of which hits A. We can also choose the V_n's such, that if we let $D_n = (V_n \setminus V_{n+1}) \cap A$, then also int $D_n \neq \emptyset$. Because x is κ-O.K., there are neighbourhoods $\{U_\alpha : \alpha < \kappa\}$ of x such that the intersection of any n of them is contained in V_n. For each $n \in \omega$ define $I_n = \{\alpha < \kappa : U_\alpha \cap D_n \neq \emptyset\}$. Since $\bigcup_{n\in\omega} I_n = \kappa$ there is an $n_0 \in \omega$ such that $|I_{n_0}| = \kappa$ (here we use that cf $\kappa > \omega$). Now each $U_\alpha \cap D_{n_0}$ for $\alpha \in I_{n_0}$ meets at most n_0-many U_β's so we can pick $I' \in [I_{n_0}]^\kappa$ such that $\{U_\alpha \cap D_{n_0} : \alpha \in I'\}$ is a system of disjoint subsets of A. This is a contradiction because A is κ-cc. □

In ([Kun78]) Kunen proved in ZFC that there are $\mathfrak{c}$-O.K. points in ω^*. In the following section we will prove the existence of $\mathfrak{c}$-O.K. points in general growths. The methods are taken from Mill's [vM82], and are generalizations of the methods first introduced by Kunen.

4.1 Existence theorems for $\mathfrak{c}$-O.K. points

The proofs work by constructing the filter in an inductive process going up to $\mathfrak{c}$. At limit steps we can just take the union of the filters constructed so far, so the crucial part is to ensure, that the inductive process does not stop at successor stages before $\mathfrak{c}$. This can be done using an independent family (a tool due to Kunen):

Definition 4.3. A system of closed subsets of a topological space X is called *precisely n-linked* if the intersection of n members of this system is non-compact but the intersection of any $n+1$ members of this system is compact. A system $\{A(\beta, n) : \beta \in J, n \in \omega\}$ is a *linked* system, if

(i) each $\{A(\beta, n) : \beta \in J\}$ is precisely n-linked and

(ii) for each $\beta \in J$, $A(\beta, n) \subseteq A(\beta, n+1)$.

A system $\{A_\alpha(\beta, n) : \alpha \in I, \beta \in J, n \in \omega\}$ is an $|I|$ *by* $|J|$ *independent linked system* with respect to a closed (i.e. consisting of closed sets) filter $\mathcal{F}$ if each $\{A_\alpha(\beta, n) : \beta \in J, n \in \omega\}$ is a linked system and for any $I_0 \in [I]^{<\omega}$, $F \in \mathcal{F}$, $n \in {}^{I_0}\omega$, $\beta \in {}^{I_0}J$ the following intersection is non-compact:

$$F \cap \bigcap_{\alpha \in I_0} A_\alpha(\beta(\alpha), n(\alpha)).$$

A filter $\mathcal{F}$ on a topological sum $\sum X_n$ is called *nice*, provided for each $F \in \mathcal{F}$ the set $\{n \in \omega : F \cap X_n = \emptyset\}$ is finite.

The following theorem showing that independent linked families exist is due to Kunen:

Theorem 4.4 ([Kun78])**.** *There is a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked system on the integers with respect to the Fréchet filter.*

Proof. (Simon) Let $\mathbf{X} = \{\langle k, f\rangle : k \in \omega, f \in {}^{\mathcal{P}(k)}\mathcal{P}(\mathcal{P}(k))\}$ and for $X, Y \in \mathcal{P}(\omega)$, $n \in \omega$ let

$$F(Y, X, n) = \{\langle k, f\rangle : |f(Y \cap k)| \leq n \;\&\; X \cap k \in f(Y \cap k)\}.$$

The family $\{F(Y, X, n) : X, Y \in \mathcal{P}(\omega), n \in \omega\}$ is a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family with respect to the Fréchet filter on $\mathbf{X}$. $\square$

The following theorem is a slight modification of Theorem 2.4 of van Mill [vM82].

Theorem 4.5. *Suppose X is a space of weight $\leq \mathfrak{c}$ admitting a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked system with respect to some filter $\mathcal{C}$. Let $Y = \omega \times X$. Then there is a $\mathfrak{c}$-O.K. point in Y^*, which lies in the intersection $\bigcap\{F^* : F \in \mathcal{F}\}$, where*

$$\mathcal{F} = \left\{ \bigcup_{n \in I} \{n\} \times F(n) : F \in {}^{\omega}\mathcal{C}, I \in \mathcal{F}R(\omega) \right\}$$

Proof. Let $\{A_\alpha(\beta, n) : \alpha, \beta < \mathfrak{c}, n < \omega\}$ be the independent linked system on ω from Theorem 4.4, $\{B_\alpha(\beta, n) : \alpha, \beta < \mathfrak{c}, n < \omega\}$ be the respective independent linked system on X. Note, that $\mathcal{F}$ is a nice filter on Y (and if $\mathcal{C}$ was remote, then so is $\mathcal{F}$), and that the following sets form an independent linked system mod $\mathcal{F}$ on Y:

$$C_\alpha(\beta, n) = \bigcup_{m \in A_\alpha(\beta,n)} \{m\} \times B_\alpha(\beta, n).$$

Let $\mathcal{B}$ be a base of Y of cardinality $\leq \mathfrak{c}$ and $\{D^\alpha = \langle D^\alpha_n : n \in \omega\rangle : \alpha < \mathfrak{c}\}$ be an enumeration of all sequences of closures of sets from $\mathcal{B}$ satisfying $D^\alpha_{n+1} \subseteq \text{int } D^\alpha_n \setminus (n \times X)$. Without loss of generality let each such sequence be listed cofinally many times. By induction on $\alpha < \mathfrak{c}$ we construct $\mathcal{F}_\alpha \supseteq \mathcal{F}$ and $K_\alpha \subseteq \mathfrak{c}$ satisfying

(i) $\{C_\beta(\mu, n) : \beta \in K_\alpha,\ \mu < \mathfrak{c},\ n < \omega\}$ is an independent linked system mod $\mathcal{F}_\alpha$ for all $\alpha < \mathfrak{c}$.

(ii) $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ for all $\alpha < \beta$ are centered systems of closed sets

(iii) $K_\beta \subseteq K_\alpha$ for all $\alpha < \beta$ and $K_\beta \setminus K_{\beta+1}$ is finite.

(iv) If $D^\alpha_n \in \mathcal{F}_\alpha$ for all $n \in \omega$, then there are $\{E^\alpha_\gamma : \gamma < \mathfrak{c}\} \subseteq \mathcal{F}_{\alpha+1}$ witnesses to the O.K. property for D^α.

Let $K_0 = \mathfrak{c}$ and $\mathcal{F}_0 = \mathcal{F}$. If α is limit, then let

$$\mathcal{F}_\alpha = \bigcup\{\mathcal{F}_\beta : \beta < \alpha\} \text{ and } K_\alpha = \bigcap\{K_\beta : \beta < \alpha\}.$$

Now suppose we have constructed $K_\alpha, \mathcal{F}_\alpha$ and that D^α satisfies the assumption in (iv). Choose $\beta \in K_\alpha$ and let $K_{\alpha+1} = K_\alpha \setminus \{\beta\}$. Define

$$E^\alpha_\gamma = \bigcup_{n<\omega} \underbrace{C_\beta(\gamma, n) \cap D^\alpha_n}_{\text{meets any } F \in \mathcal{F}}$$

and let $\mathcal{F}_{\alpha+1}$ be generated by $\mathcal{F}_\alpha$ and $\{E^\alpha_\gamma : \gamma < \mathfrak{c}\}$. Note that, for any $A_0 \in [\mathfrak{c}]^n$

$$\left(\bigcap_{\gamma \in A_0} E^\alpha_\gamma\right) \setminus D^\alpha_n \subseteq \left(\bigcap_{\gamma \in A_0} C_\beta(\gamma, n)\right)$$

and the last term is compact, giving us (iv).

If we let $H = \bigcup\{\mathcal{F}_\alpha : \alpha < \mathfrak{c}\}$ then any $p \in Y^*$ containing H will (by (iv)) be $\mathfrak{c}$-O.K. point of Y^*. □

Note 4.6. If $\mathcal{C}$ was remote, then also the resulting point will be remote.

Some further analysis of the previous proof shows that requiring an independent linked system is, in fact, not needed. We can weaken the conditions to only require for each $n \in \omega$ a precisely n-linked system of closed sets, independent with respect to a remote filter. Finding such a system is easier in theory, but it is not clear, whether it is really easier in practice. Since the proof of the modified theorem is somewhat involved and we do not use it anywhere, we do not state it precisely or prove it.

4.2 A crowded space with an independent linked system

In this section we proceed to construct a T_2 space containing a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family with respect to the filter of cofinite sets. In the following we shall make heavy use of following notation which simplifies work with (finite) intersections:

Notation 4.7. *If $\mathcal{A} = \{A_{\alpha,\beta} : \alpha \in I, \beta \in J\}$ is an indexed family, $J' \subseteq J$ is a set of indices and $\alpha \in I$ let $A_{\alpha,J'}$ stand for the intersection*

$$\bigcap_{\beta \in J'} A_{\alpha,\beta}.$$

Similarly we define $A_{\alpha,J',n}$ for a family indexed by three indices.

We shall need a standard definition:

Definition 4.8. A system $\mathcal{A} = \{\mathcal{A}_\alpha : \alpha \in I\}$ of families of subsets of ω is *weakly independent*, if for $\alpha_0 < \cdots < \alpha_n \in I$, $A_0 \in \mathcal{A}_{\alpha_0}, \ldots, A_n \in \mathcal{A}_{\alpha_n}$ the intersection

$$\bigcap_{i=0}^{n} A_i$$

is infinite. If each $\mathcal{A}_\alpha$ is almost disjoint we say that the system is *independent*. For $J \subseteq I$ and $X \subseteq \omega$ we say that X is $(J, \mathcal{A})$-*big* if X meets any of the previous intersections where each $\alpha_i \notin J$. If for some J each $\mathcal{A}_\alpha$ is an almost disjoint system of the form $\mathcal{A}_\alpha = \{A_{\alpha,\beta} : \beta \in J\}$ we say that $\mathcal{A} = \{A_{\alpha,\beta} : \alpha \in I, \beta \in J\}$ is an *I by J independent matrix on ω*.

A standard result is:

Theorem 4.9 ([Kun78]). *There exists an independent $\mathfrak{c}$ by $\mathfrak{c}$ matrix $\mathcal{A}$ of subsets of ω*

We shall now introduce a slightly modified definition of a linked system, which will be suitable for our purposes:

Definition 4.10. Let $\mathcal{L} = \{L_{\alpha,\beta,n} : \alpha, \beta < \kappa, n < \omega\}$ be a family of sets. Define $\mathcal{B}_\alpha(\mathcal{L})$ as follows:

$$\mathcal{B}_\alpha(\mathcal{L}) = \left\{ L_{\alpha,J_0,|J_0|} \cap \cdots \cap L_{\alpha,J_n,|J_n|} : n < \omega, J_0 \subseteq \cdots \subseteq J_n, J_n \in [\kappa]^{<\omega} \right\}$$

We say that $\mathcal{A}$ is *s-independent linked* of size κ if

(i) The $\mathcal{B}(\mathcal{L}) = \{\mathcal{B}_\alpha(\mathcal{L}) : \alpha < \kappa\}$ is a weakly independent system.

(ii) For any $\alpha < \kappa$, $n < \omega$, $J \in [\kappa]^{n+1}$ the intersection $\bigcap_{\beta \in J} L_{\alpha,\beta,n}$ is finite.

(iii) For any $\alpha \in I$,$\beta \in J$,$n < m < \omega$ the set $L_{\alpha,\beta,n}$ is a subset of $L_{\alpha,\beta,m}$.

In the definition of $\mathcal{B}_\alpha$ if we fixed n to be 0, we would get the usual definition of an independent linked system. It is clear, that any s-independent linked system of size κ is a κ by κ independent linked system. We will now present a technical lemma which shall be used later in the proof of 4.17. Afterwards we will show that s-independent systems exist.

Lemma 4.11. *If $\mathcal{L} = \{L_{\alpha,\beta,n} : \alpha, \beta < \kappa, n < \omega\}$ is an s-independent linked system on ω, $\alpha, \beta_0, \ldots, \beta_n < \kappa$ and $k_0, \ldots, k_n < \omega$, then any set of the form*

$$\bigcap_{i=0}^{n} (\omega \setminus L_{\alpha,\beta_i,k_i})$$

contains a set from $\mathcal{B}_\alpha(\mathcal{L})$ and so is $(\alpha, \mathcal{B}(\mathcal{L}))$-big.

Proof. Notice that for any $\alpha, \beta < \kappa$, $J \in [\kappa]^{<\omega}$ if $\beta \notin J$ then $L_{\alpha,J,|J|}$ is almost contained in $(\omega \setminus L_{\alpha,\beta,|J|})$. Fix an increasing chain of sets $J_0 \subseteq \cdots J_n$ with $|J_i| = k_i$ and $\beta_i \notin J_n$ for each $i = 0, \ldots, n$. Then the intersection

$$L_{\alpha,J_0,k_0} \cap \cdots \cap L_{\alpha,J_n,k_n}$$

is a member of $\mathcal{B}_\alpha(L)$ and is contained in our original intersection as was to be proved. □

Corollary 4.12. *If $\mathcal{L}$ is an s-independent linked system and $\alpha < \kappa$ then*

$$\mathcal{F}_\alpha = \{\omega \setminus L_{\alpha,\beta,n} : \beta < \kappa, n < \omega\}$$

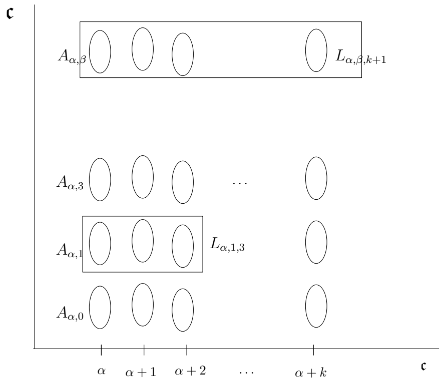
is a centered system of α-big sets.

Using 4.9 we are able to build an s-independent linked system on ω:

Theorem 4.13. *There is an s-independent linked system of size $\mathfrak{c}$ on ω.*

Proof. If $\mathcal{A} = \{A_{\alpha,\beta} : \alpha, \beta < \mathfrak{c}\}$ is an independent matrix of subsets of ω (such a matrix exists by the previous fact) define for each $\beta < \mathfrak{c}$, limit $\alpha < \mathfrak{c}$, $k < \omega$

$$L_{\alpha,\beta,k} = \bigcup_{i<k} A_{\alpha+i,\beta}.$$



Then the system $\mathcal{L} = \{L_{\alpha,\beta,k} : \alpha, \beta < \mathfrak{c}, \alpha \text{ limit}, k < \omega\}$ is an s-independent linked system of size $\mathfrak{c}$.

Condition (iii) is a direct consequence of the definition of $\mathcal{L}$. Before looking at the other conditions, let us introduce some more notation:

Notation 4.14. *If $\alpha < \kappa$ is limit, $J \in [\kappa]^{<\omega}$ and $f : J \to \omega$ denote by $A_{\alpha,f}$ the following intersection*

$$\bigcap_{\beta \in J} A_{\alpha+f(\beta),\beta}$$

Notice that for any $\alpha < \kappa$ limit, $n < \omega$, $J \in [\kappa]^{<\omega}$, the following is true (using the previous notation and notation 4.7):

$$L_{\alpha,J,n} = \bigcup_{f \in {}^J n} A_{\alpha,f} \tag{4.1}$$

Observation 4.15. *The following easily follows from the fact that $\mathcal{A}$ is an independent matrix.*

(a) $A_{\alpha,f}$ is infinite if and only if f is injective.

(b) For $f \subseteq g$ we have that $A_{\alpha,f} \supseteq A_{\alpha,g}$.

(c) For $\alpha \neq \beta$ and any f, g the sets $A_{\alpha,f}$ and $A_{\beta,g}$ have infinite intersection.

From 4.1 and (a) the condition (ii) in the definition of an s-independent linked system immediately follows, since there cannot be an injective function from a set of size $n + 1$ into n.

The fact that $\mathcal{B}(\mathcal{L})$ is weakly independent follows from the previous observation — take an injective function f_0 from J_0 onto $|J_0|$ now extend it to injective functions $f_k : J_k \to |J_k|$ such that $f_k \subseteq f_{k+1}$. This is clearly possible. Now use 4.1 and (a,b,c).

□

We proved that s-independent linked systems exist. The next lemma shows that we can (without loss of generality) assume they have additional properties.

Lemma 4.16. *Suppose $\mathcal{A} = \{A_{\alpha,\beta,n} : \alpha, \beta < \kappa, n \in \omega\}$ is an s-independent linked system of size κ on a countable X. Then there is an s-independent linked system $\mathcal{A}'$ of the same size such that $\{A'_{n,0,0} : n \in \omega\}$ separate the points of X (i.e. for each $\{x, y\} \in [X]^2$ there is an $n < \omega$ such that $|A'_{n,0,0} \cap \{x, y\}| = 1$.*

Now we will put our notion of an s-independent linked system to good use in the following theorem:

Theorem 4.17. *There is a countable crowded T_2 space with a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family w.r.t. the filter of cofinite sets consisting of closed sets.*

Proof. Let $\mathcal{A} = \{A_{\alpha,\beta,n} : \alpha, \beta < \mathfrak{c}, n < \omega\}$ be an s-independent linked family of size $\mathfrak{c}$ on ω. We may (use the previous lemma) assume that the $\{A_{n,0,0} : n < \omega\}$ separate points in ω. Let $K_0 = \{A_{n,0,0}, \omega \setminus A_{n,0,0} : n < \omega\}$ and $K_1 = \{\omega \setminus A_{\alpha,\beta,n} : n < \omega < \alpha, \beta < \mathfrak{c}\}$. Let τ be the topology generated by $K_0 \cup K_1$. This topology is T_2 because K_0 separates points. Moreover $\{A_{\alpha,\beta,n} : n < \omega < \alpha, \beta < \mathfrak{c}\}$ is an independent linked family w.r.t. the finite sets and it consists of τ-closed sets. So it remains to be shown that the topology is crowded. But that follows if we apply 4.12. □

Theorem 4.17 is just a step away from yielding an OHI space with a weak P-point.
Since any finer topology retains the closed, independent linked system, any finer
crowded T_3 topology must (use 4.5) contain a weak P-point. Any such
T_3 topology can be further refined to an OHI topology using 3.14.
Unfortunately the gap between T_2 and T_3 is unbridgeable:
suppose, the topology could be refined to a crowded T_3
topology. Then, using 4.5 we would get a *nowhere*
locally compact space with an ω_1-O.K. point,
which is impossible, as a theorem
in the last chapter shall
shall clearly
show
.

Chapter 5

Remote points

Definition 5.1. A point $p \in X^*$ is a *remote point* of X if for any nowhere dense subset N of X, $p \notin \overline{N}$.

As stated in the introduction, remote points are an essential tool for constructing ω-uniquely accessible points. We will first list some general conditions guaranteeing the existence of remote points in a large class of spaces and then give a concrete construction of a suitable space with a remote point. After modifying this space using the methods of Chapter 3 we will use it in the last chapter. The main result is theorem 5.13

5.1 General theorems

This section will list some conditions under which we can have remote points. It will only be an overview; we do not include any proofs. The notion of a remote point was introduced by Fine and Gillman in [FG62] as a method for studying the nonhomogeneity of βX. The existence of remote points for spaces of countable π-weight was proved independently by van Douwen in [vD81] and Chae and Smith in [CS80]. The assumption of non-pseudocompactness in the theorems is due to the fact that any pseudocompact space of π-weight less than the first measurable cardinal has no remote points (see [Ter79]).

Definition 5.2. A space X is *pseudocompact* if it is completely regular and every continuous realvalued function f on X is bounded. It is *non-pseudocompact* otherwise.

Definition 5.3. A family $\mathfrak{F}$ is *locally finite* in X if for any $x \in X$ there is a neighbourhood G of X such that $\{F \in \mathfrak{F} : F \cap G \neq \emptyset\}$ is finite. A family $\mathfrak{F}$ is *σ-locally finite* if there are $\{\mathfrak{F}_n : n < \omega\}$ each of which is locally finite and $\mathfrak{F} = \bigcup_{n<\omega} \mathfrak{F}_n$.

Theorem 5.4 ([Dow84]). *Any ccc non-pseudocompact space of π-weight ω_1 has a remote point. (A space is ccc if any system of disjoint open subsets of the space is at most countable.)*

Theorem 5.5 ([Dow82]). *Under MA any ccc non-pseudocompact space of π-weight at most $\mathfrak{c}$ has a remote point.*

Theorem 5.6 ([HP88]). *A non-pseudocompact space with a σ-locally finite π-base has a remote point.*

5.2 A crowded space with a strongly remote filter

The aim of this section is to construct a crowded countable space with a closed remote filter which remains remote in any strictly finer topology. Because remoteness is, in general, not retained when refining a topology, we need the following stronger condition:

Definition 5.7. A τ-closed filter $\mathcal{F}$ on X is strongly remote if for any $N \subseteq X$ with $\mathrm{int}_\tau N = \emptyset$, there is an $F \in \mathcal{F}$ such that $F \cap N = \emptyset$.

Observation 5.8. *If a filter $\mathcal{F}$ is strongly remote, then it is remote.*

Proposition 5.9. *A strongly remote filter on (X, τ) is strongly remote in any finer topology.*

Proof. If $\tau \subseteq \tau'$ and $N \subseteq X$ such that $\mathrm{int}_{\tau'} N = \emptyset$ then N has no τ'-open subset so, *a fortiori*, it has no τ-open subset. So there is $F \in \mathcal{F}$ which misses N because $\mathcal{F}$ is strongly remote on (X, τ). □

Theorem 5.10. *There is a crowded, T_2, zerodimensional topology τ on ω and a strongly remote filter on ω with this topology.*

Proof. Throughout this proof we will adopt the following notation:

Notation 5.11. *If $\mathcal{F}$ is a system of subsets of ω and I is an ideal on ω, let $\tau_I(\mathcal{F})$ be the topology on ω generated by $\{F, \omega \setminus F : F \in \mathcal{F}\} \cup \{\omega \setminus S : S \in I\}$.*

The proof of the theorem will come in several steps. First, we state a standard lemma from boolean algebras.

Lemma 5.12. *There is an ideal I on ω, extending FIN and such that $\mathcal{P}(\omega)/I$ has hereditary independence $\mathfrak{c}$.*

Proof. The complete Boolean algebra $\mathbb{B} = Compl(Clopen(2^{\mathfrak{c}}))$ has hereditary independence $\mathfrak{c}$ and is σ-centered so there is an ideal I on ω such that $\mathbb{B}$ is isomorphic to $\mathcal{P}(\omega)/I$. □

By the previous lemma, we can fix an ideal $I \supseteq FIN$ on ω such that $\mathcal{P}(\omega)/I$ has hereditary independence $\mathfrak{c}$. Now let $\langle A_\alpha : \omega < \alpha < \mathfrak{c}\rangle$ be an enumeration of $\mathcal{P}(\omega)$ and $\langle K_n : n < \omega\rangle$ be an enumeration of $[\omega]^2$. Let $\mathcal{F}_0 = \emptyset$ Proceed by induction constructing $\mathcal{F}_\alpha$ for $\alpha < \mathfrak{c}$ such that the following is satisfied:

(i) $|\mathcal{F}_\alpha| \leq \alpha$ for each $\alpha < \mathfrak{c}$.

(ii) For each $\omega < \alpha < \mathfrak{c}$ either $\mathrm{int}_{\tau_I(\mathcal{F}_{\alpha+1})} A_\alpha \neq \emptyset$ or there is $F \in \mathcal{F}_{\alpha+1}$ which misses A_α.

(iii) For each $n < \omega$ there is an $F \in \mathcal{F}_{n+1}$ such that $|F \cap K_n| = 1$

(iv) The family $\{[F]_I : F \in \mathcal{F}_\alpha\}$ is independent in $\mathcal{P}(\omega)/I$.

Suppose, that the construction can indeed be carried out. Then $\mathcal{F} = \bigcup\{\mathcal{F}_\alpha : \alpha < \mathfrak{c}\}$ and $\tau = \tau_I(\mathcal{F})$ satisfy the conclusion of the proof:

The topology is zerodimensional (by virtue of the definition of $\tau_I(\mathcal{F})$).

The topology is also T_2 because if $x \neq y \in \omega$ then there is $n < \omega$, such that $K_n = \{x, y\}$ and by (iii) there is $F \in \mathcal{F}_n \subseteq \mathcal{F}$ such that $|F \cap K_n| = 1$. This F is τ-clopen and separates x from y.

To show that τ is crowded it is sufficient to consider its basis, which consists of elements of the form:

$$\bigcap_{F \in P} F \cap \bigcap_{F \in N} (\omega \setminus F)$$

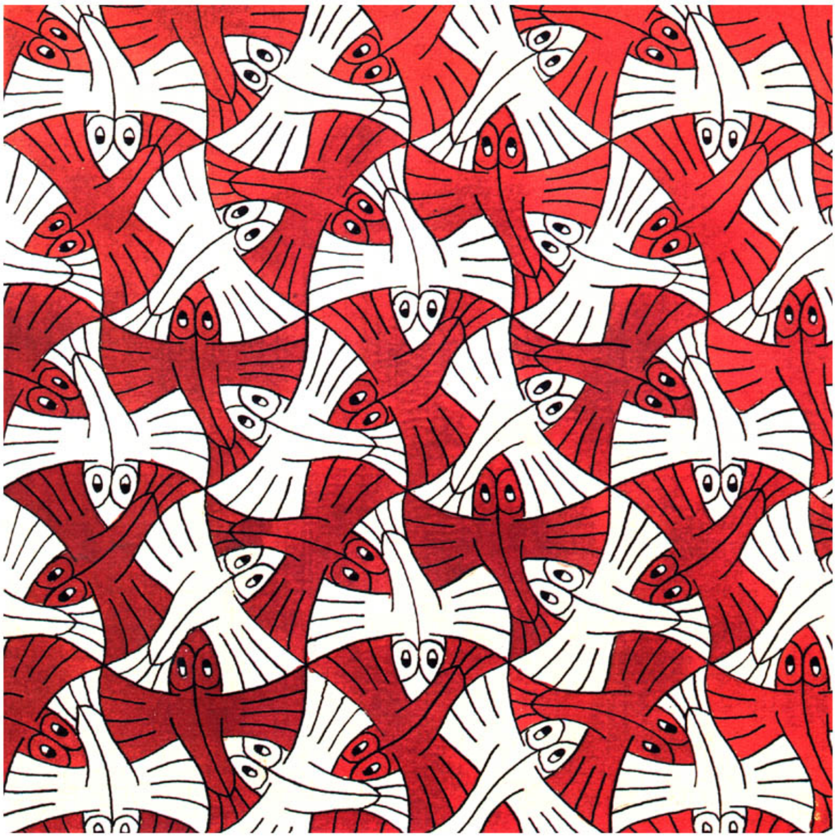
where $P, N \in [\mathcal{F}]^{<\omega}$. Now, by (iv) the family $\{[F]_I : F \in \mathcal{F}\}$ is independent in $\mathcal{P}(\omega)/I$ with $FIN \subseteq I$ so the only finite elements must have some $F \in N \cap P$. Then the element must be a subset of $F \cap (\omega \setminus F)$ so must be empty. Thus the basis does not contain any finite sets beyond the empty set, so it is crowded as is the whole topology.
To prove that $\mathcal{F}$ is strongly remote, choose $O \subseteq \omega$ such that $\text{int}_\tau O = \emptyset$. There is an $\alpha < \mathfrak{c}$, such that $O = A_\alpha$. Then $\text{int}_{\tau(\mathcal{F}_{\alpha+1})} A_\alpha (= O) = \emptyset$, so there is $F \in \mathcal{F}_{\alpha+1} \subseteq \mathcal{F}$ such that $F \cap A_\alpha (= F \cap O) = \emptyset$.
So it remains to be shown that the inductive construction can be carried out all the way up to $\mathfrak{c}$. Suppose that we have $\mathcal{F}_\beta$'s satisfying (i–iv) for $\beta < \alpha$.
If α is limit, we can let $\mathcal{F}_\alpha = \bigcup \{\mathcal{F}_\beta : \beta < \alpha\}$ and the only relevant conditions (i) and (iv) will be satisfied.
Otherwise $\alpha = \beta + 1$. There are two cases:

Case $\beta = n < \omega$. Let $K_n = \{x, y\}$. Then by (iv) the subset $\{[F] : F \in \mathcal{F}_n\}$ of $\mathcal{P}(\omega)/I$ is independent. Since $\mathcal{P}(\omega)/I$ has independence $\mathfrak{c}$ and since $|\mathcal{F}_n| \leq n < \mathfrak{c}$, there is an $F' \in \mathcal{P}(\omega)$ such that $\{[F]_I : F \in \mathcal{F}_n\} \cup \{[F']_I\}$ is still independent. Then let $F = F' \cup \{x\} \setminus \{y\}$. We have that $[F']_I = [F]_I$ so condition's (iii) and (iv) are both satisfied if we let $\mathcal{F}_\alpha = \mathcal{F}_n \cup \{F\}$.

Case $\omega < \beta < \mathfrak{c}$. If there is an $F \in \mathcal{F}_\beta$ such that $F \cap A_\beta = \emptyset$ or if $\text{int}_{\tau_\beta} A_\beta \neq \emptyset$, then we can let $\mathcal{F}_\alpha = \mathcal{F}_\beta$, $\tau_\alpha = \tau_\beta$ and all conditions (i-iv) are satisfied.
If $\{[F]_I : F \in \mathcal{F}_\beta\} \cup \{[\omega \setminus A_\beta]_I\}$ is independent in $\mathcal{P}(\omega)/I$, then we can let $\mathcal{F}_\alpha = \mathcal{F}_\beta \cup \{\omega \setminus A_\beta\}$ and again all conditions (i-iv) are satisfied. So suppose otherwise.
If we let $B = \omega \setminus A_\beta$, necessarily $B \notin I$ (otherwise already $\text{int}_{\tau_0} A_\beta \neq \emptyset$). We claim, that $\{[F \cap B]_I : F \in \mathcal{F}_\beta\}$ is independent in $\mathcal{P}(\omega)/I \restriction [B]_I$: If it were not, then for some elementary meet M over $\mathcal{F}_\beta$ we would have that $M \cap B \in I$ but then, since $M \subseteq_I A_\beta$, so $\text{int}_{\tau_\beta} A_\beta \neq \emptyset$ a contradiction. Now, since $\mathcal{P}(\omega)/I$ has hereditary independence $\mathfrak{c}$, $\{[F \cap B]_I : F \in \mathcal{F}_\beta\}$ is not maximal independent in $\mathcal{P}(\omega)/I \restriction [B]_I$ (by (i) $|F_\beta| \leq \beta < \mathfrak{c}$), so there is $F \subseteq B$ such that $\{[F \cap B]_I : F \in \mathcal{F}_\beta\} \cup \{[F]_I\}$ is independent in $\mathcal{P}(\omega)/I \restriction [B]_I$ so, a fortiori, $\{[F]_I : F \in \mathcal{F}_\beta\} \cup \{F\}$ is independent in $\mathcal{P}(\omega)/I$ and if we let $\mathcal{F}_\alpha = \mathcal{F}_\beta \cup \{F\}$ all conditions (i-iv) are satisfied and we are done. □

Using theorem 3.14 we get the following corollary which is the main result of this chapter:

Corollary 5.13. *There is a countable, zerodimensional, extremally disconnected OHI space X with a remote point.*



M. C. Escher: **Flying fish**

Chapter 6

Conclusion

Everything has got a moral
if you can only find it.
Lewis Carroll

IN the introduction we outlined the method we were investigating: find a suitable OHI space X with a remote point which is a weak P-point in X^* and embed βX into ω^*. Using theorem 2.1 and theorem 2.8 we get:

Theorem 6.1. *If there is a countable, zerodimensional, extremally disconnected space with an ω-uniquely accessible point, then ω^* contains an ω-uniquely accessible point.*

Coupled with

Theorem 6.2. *If $p \in X^*$ is a remote, weak P-point and X is OHI, then p is ω-uniquely accessible in βX.*

Proof. Suppose $D_0, D_1 \subseteq \beta X$ are two countable sets with $p \in \overline{D_0} \cap \overline{D_1}$. Then, because p is a weak P-point of X^* $p \in \overline{D_0 \cap X} \cap \overline{D_1 \cap X}$. Because p is remote, $p \in \overline{\operatorname{int} \overline{D_0 \cap X}} \cap \overline{\operatorname{int} \overline{D_1 \cap X}}$. Again, because p is remote, βX is extremally disconnected at p so p cannot be in the closure of two disjoint open sets, so $\operatorname{int} \overline{D_0 \cap X} \cap \operatorname{int} \overline{D_1 \cap X} = G \neq \emptyset$, but now, since X is OHI and D_0, D_1 are both dense in G, we have that $D_0 \cap D_1 \neq \emptyset$. □

we get:

Theorem 6.3. *If there is a countable, extremally disconnected OHI space X with a remote, weak P-point, then there is an ω-uniquely accessible point in ω^*.*

On the other hand we have:

Theorem 6.4. *If there is an ω-uniquely accessible point in ω^* then there is a countable, zerodimensional, extremally disconnected OHI space X with a remote point p which is a weak P-point of X^*.*

Proof. Let $p \in \omega^*$ be uniquely accessible and $S \in [\omega^* \setminus \{p\}]^\omega$ with $p \in \overline{S}$. Since p is ω-uniquely accessible, S is irresolvable. Using lemma 3.13, we can find $S' \subseteq S$ which is hereditarily irresolvable. Now p cannot be in the closure of $S \setminus S'$ because that is a resolvable space, $p \in \overline{S'}$. So S' is a countable zerodimensional OHI space and $\overline{S'} \approx \beta S'$ because any countable subset of ω^* is C^*-embedded. S' is extremally disconnected by 1.17. The point p is a remote point of S', because if $N \subseteq S'$ is n.w.d. then $p \in \overline{S' \setminus N}$ so $p \notin \overline{N}$. On the other hand, since $p \in \overline{S'}$, p cannot be in the closure of any countable subset of S'^* because it is ω-uniquely accessible, so it is a weak P-point. □

So the original question is equivalent to finding a countable, extremally disconnected OHI space with a remote weak P-point in its Čech-Stone growth. In chapter 5 we found such a space with a remote point. In chapter 4 we were just a step away from finding such a space with a point, which is a weak P-point in the growth. The question is, can we get both? Maybe yes, but unfortunately, as the following theorem shows, such a point cannot be an O.K. point:

Theorem 6.5. *If X is a countable OHI space, than X^* cannot contain an ω_1-O.K. point.*

Proof. Since X is countable, it is ccc as is βX. Since X is OHI, by 3.15, it is nowhere locally compact. By 1.14 if X is nowhere locally compact X^* is dense in βX, so X^* must be ccc. Now we can use 4.2 to show, that X^* cannot contain an ω_1-O.K. point. □

We were only able to find somewhat weaker points, as in the following definition:

Definition 6.6. A point p in X is *relatively uniquely ω-accessible* with respect to $S \subseteq X$ if $p \in \overline{S}$ and any two countable subsets of S having p in their closure intersect.

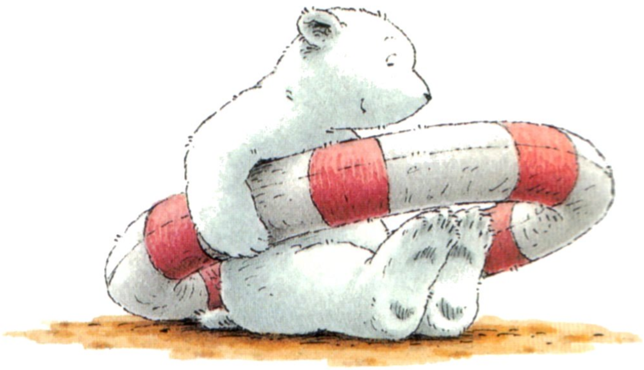
Observation 6.7. *Any remote point of an OHI space X is relatively uniquely ω-accessible in βX with respect to X.*

Proof. Copy a part of the proof of 6.4. □

Theorem 6.8. *There is a countable set $S \subseteq \omega^*$ and a point $p \in \omega^*$ which is relatively uniquely ω-accessible in ω^* with respect to $S \cup (\omega^* \setminus \overline{S})$.*

Proof. Theorem 5.13 gives us a countable extremally disconnected space X with a remote point p. Since X is countable, the weight of X is at most $\mathfrak{c}$ so (by 2.1) βX can be embedded onto some $Y \subseteq \omega^*$ which is a weak P-set in ω^*. Then the remote point p will be mapped onto a relatively uniquely ω-accessible point of ω^* with respect to the image of X. Since Y is a weak P-set of ω^* the point will not be in the closure of *any* countable subset of $\omega^* \setminus X$ and the conclusion follows. □

To summarize our work, we found (in ZFC) a point $p \in \omega^*$ which is relatively ω-uniquely accessible. This point is different from the one constructed by Mill in [vM82], so we have found another type in ω^*. Moreover we showed that to get an ω-uniquely accessible point we would need a new method to construct weak P-points in Čech-Stone compactifications. Such a method is, as far as we know, not known in ZFC.



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Topologically distinct ultrafilters
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