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**Stochastic dominance in portfolio
optimization**

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I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Abstract: The main topic of this thesis is the application of stochastic dominance constraints to portfolio optimization problems. First we recall Markowitz model. Then we present portfolio selection problems with stochastic dominance constraints. Finally we compare performance of these two approaches in an empirical study presented in the last chapter.

Keywords: Stochastic Dominance, Markowitz model, portfolio optimization,

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Introduction

Every investor meets with situation in which he/she has to make decision about selecting some of the available investment opportunities. This means that this decision has to be somehow optimal and has to describe investor's attitude towards risk and his/her return expectation. In literature we can find a lot of different approaches which try to optimize profitability of the investments which returns are random variables.

In the last decades several portfolio selection models were developed. The basics of the modern portfolio theory was introduced in 1952 by Harry Markowitz in Markowitz [1952]. This model maximizes expected return and minimizes variance of the portfolio. There are certain shortcomings of this model. Markowitz approach comprises historical variance as measure of risk and doesn't take into account higher moments of return distribution. Later, risk measurement became very important and alternative measures as semi-variance or mean absolute deviation were introduced. An alternative approach uses utility functions introduced in von Neumann and Morgenstern [1944]. In the 21st century *VaR* and *CVaR* became the most popular risk measures because of their economical meaning. In this work we will show portfolio optimization problem with usage of stochastic dominance constrains. It is usually very complicated to exactly specify investor's utility function. In portfolio selection using stochastic dominance constrains we consider whole classes of utility functions representing groups of investors with the same risk/return attitude. Stochastic dominance also takes into account whole distribution of returns rather than just some of the moments.

In the first chapter we present assumptions for efficient market needed for Markowitz model. We provide the general solution for optimal portfolio construction in case of short sales allowed. We include both, case without riskfree asset and with riskfree included. Then we introduce alternative definitions of risk including axioms of coherent risk measures.

In the second chapter we introduce the concept of stochastic dominance and the formulation of portfolio optimization problem with stochastic dominance constrains. We present the formulation proposed in Dentcheva and Ruszczyński [2006]. In the end of the chapter we mention portfolio efficiency tests introduced by Kuosmanen [2004].

In the last chapter we apply both of the approaches on real data. We make a comparison of performances between the obtained portfolio and Dow Jones Industrial Average index.

1. Markowitz model

In this chapter we introduce basic model for portfolio optimization presented by H. Markowitz. All of the information in this chapter originate from Dupačová et al. [2002].

1.1 Efficient Market

First we provide assumptions needed for the Markowitz model to work well.

(1) The investors have homogeneous expectations. Which means that investors make decisions on their portfolios based only on the expected returns and covariances. All of the information about means and covariances are equally available to all investors at the same time.

(2) The investors are risk averse and behave rationally. They prefer portfolios with the highest expected return among portfolios with the same level of risk or they prefer portfolios with the smallest risk among portfolios with the same expected return.

(3) All investors invest on the same period of time on the market without transaction fees and no taxes.

(4) All assets are marketable and infinitely divisible.

(5) Short sales are allowed.

(6) There is only one risk free interest rate. All investors are able to lend or borrow any amount of funds at this interest rate.

(7) On the market are just small investors without ability to affect returns of the individual assets.

Market fulfilling assumptions above is called efficient. Market equilibrium occurs because investors are risk averse, behave rationally and have perfect information about the market.

1.2 Portfolio Theory

Let us assume that we want to invest our wealth (divisible money equal to 1) to portfolio consisting of N assets, $n = 1, \dots, N$. Vector $x = (x_1, \dots, x_N)^T$ is a vector of individual weights of the assets in portfolio. Variable x_n represents the selection of investors wealth invested in the n th asset of the portfolio, $n = 1, \dots, N$, while the following holds: $\mathbf{1}^T x = 1$.

There are two possible assumptions for variable x_n . If we assume that $x_n \geq 0$

that means that short sales are forbidden. If investor can sell short it implies that he can sell an asset or stock that he does not own. It is a transaction in which an investor sells borrowed securities in expectation of a price drop. For the moment we suppose no restrictions for x_n . So we define a set of possible weights as $\chi = \{\mathbf{x} \in \mathbb{R}^n : x_1 + x_2 + \dots + x_n = 1\}$.

Definition 1. We define expected returns of N assets of a portfolio as $r = ER = (r_1, \dots, r_N)^T$ where rate of returns $R = (R_1, \dots, R_N)^T$ is a random vector.

We will denote covariance matrix as $\mathbf{V} = (\sigma_{ij})$ where $\sigma_{ij} = \text{cov}(\rho_i, \rho_j)$, $i, j = 1, \dots, N$. We will denote standard deviation of the returns as: $\sigma_i = \sqrt{\sigma_{ii}}$.

Expected return on the portfolio p composed of weights x is

$$r_p = \mathbf{r}^T \mathbf{x} \quad (1.1)$$

and the variance of the portfolio return p is

$$\sigma_p^2 = \mathbf{x}^T \mathbf{V} \mathbf{x}. \quad (1.2)$$

Standard deviation of the returns of the portfolio p is

$$\sigma_p = \sqrt{\sigma_p^2}. \quad (1.3)$$

Definition 2. Any portfolio consisting of the assets in the same ratio as they show on the capital market, stated by their capitalization, is called market portfolio.

Definition 3. A portfolio $\mathbf{x}^* \in \chi$ is called mean - variance efficient if there is no other portfolio $\mathbf{x} \in \chi$ that satisfies

$$(\mathbf{r}^T \mathbf{x}^* < \mathbf{r}^T \mathbf{x} \wedge \mathbf{x}^{*T} \mathbf{V} \mathbf{x}^* \geq \mathbf{x}^T \mathbf{V} \mathbf{x}) \vee (\mathbf{r}^T \mathbf{x}^* = \mathbf{r}^T \mathbf{x} \wedge \mathbf{x}^{*T} \mathbf{V} \mathbf{x}^* > \mathbf{x}^T \mathbf{V} \mathbf{x}).$$

1.2.1 Construction of Optimal Portfolio

As mentioned above, investors want to maximize possible returns and simultaneously minimize possible risks. Investors are looking for an efficient portfolio. There are more possible formulations of optimization problems for finding mean-variance efficient portfolios.

We can find a mean-variance efficient portfolio by solving:

$$\max_{\mathbf{x} \in \chi} \lambda \mathbf{r}^T \mathbf{x} - \frac{1}{2} \mathbf{x}^T \mathbf{V} \mathbf{x} \quad (1.4)$$

parameter $\lambda \geq 0$ reflects investor's risk attitude. Large values of λ are related to investors who are more likely to invest to riskier investment, small values are associated with risk averse investors.

Another formulation of the problem is by setting parameter μ expressing minimal acceptable return

$$\min_{\mathbf{x} \in \chi} \mathbf{x}^T \mathbf{V} \mathbf{x} \quad (1.5)$$

subject to

$$\mathbf{r}^T \mathbf{x} \geq \mu.$$

In both of the formulations above may occur following instances: $\mathbf{x} \in \mathbb{R}^n$ (short sales allowed), $\mathbf{x} \geq 0$ (short sales are forbidden). Matrix \mathbf{V} is positive semidefinite which implies that (1.4) and (1.5) are problems of convex optimization.

Remark 1.2.1 There is no difference between efficient portfolios acquired when we quantify risk by the variance of the portfolio return or its standard deviation. This holds because $\sqrt{\mathbf{x}^T \mathbf{V} \mathbf{x}}$ is strictly increasing transformation of $\mathbf{x}^T \mathbf{V} \mathbf{x}$.

Remark 1.2.2 Another example of risk-adjust return of the portfolio is Sharpe Ratio:

$$\frac{\mathbf{r}^T \mathbf{x}}{\sqrt{\mathbf{x}^T \mathbf{V} \mathbf{x}}}.$$

1.2.2 General Solution

Risky Assets

Now we will focus on case of risky assets and short sales allowed. We will find solution for the problem

$$\min \frac{1}{2} \mathbf{x}^T \mathbf{V} \mathbf{x} \tag{1.6}$$

subject to

$$\mathbf{1}^T \mathbf{x} = 1, \mathbf{r}^T \mathbf{x} \geq \mu$$

where μ is predetermined minimal acceptable return.

Assume that V is positive definite. Primary we exclude the trivial case where $\mathbf{r} = c\mathbf{1}$ for some constant c . We may easily solve this case by selecting asset n_0 for which following holds: $n_0 = \arg \min_{1 \leq n \leq N} \sigma_n^2$. Next we will use the method of Lagrange multipliers.

The Lagrange function is

$$L(\mathbf{x}, \lambda_1, \lambda_2) = \frac{1}{2} \mathbf{x}^T \mathbf{V} \mathbf{x} + \lambda_1 (1 - \mathbf{1}^T \mathbf{x}) + \lambda_2 (\mu - \mathbf{r}^T \mathbf{x})$$

after derivative

$$\frac{\partial}{\partial \mathbf{x}} L = \mathbf{V} \mathbf{x} - \lambda_1 \mathbf{1} - \lambda_2 \mathbf{r} = 0$$

we obtain the optimal solution

$$\mathbf{x}^* = \lambda_1 \mathbf{V}^{-1} \mathbf{1} + \lambda_2 \mathbf{V}^{-1} \mathbf{r}. \tag{1.7}$$

To determine λ_1 and λ_2 , we apply the two constraints

$$1 = \mathbf{1}^T \mathbf{x} = \lambda_1 \mathbf{1}^T \mathbf{V}^{-1} \mathbf{1} + \lambda_2 \mathbf{1}^T \mathbf{V}^{-1} \mathbf{r}$$

$$\mu \leq \mathbf{r}^T \mathbf{x} = \lambda_1 \mathbf{r}^T \mathbf{V}^{-1} \mathbf{1} + \lambda_2 \mathbf{r}^T \mathbf{V}^{-1} \mathbf{r}.$$

Put $A := \mathbf{1}^T \mathbf{V}^{-1} \mathbf{1}$, $B := \mathbf{1}^T \mathbf{V}^{-1} \mathbf{r}$, $C := \mathbf{r}^T \mathbf{V}^{-1} \mathbf{r}$ and $\Delta := AC - B^2$.

Since \mathbf{V} is positive definite and $\mathbf{1}$ and \mathbf{r} are linearly independent, then $A > 0$ and $C > 0$, $\Delta > 0$ results from Cauchy-Schwarz inequality.

If $\lambda_2 = 0$ then

$$1 = \mathbf{1}^T \mathbf{x} = \lambda_1 A$$

implies

$$\lambda_1 = \frac{1}{A}.$$

Otherwise, we obtain constants λ_1 and λ_2 from initial conditions:

$$1 = \mathbf{1}^T \mathbf{x} = \lambda_1 A + \lambda_2 B$$

$$\mu = \mathbf{r}^T \mathbf{x} = \lambda_1 B + \lambda_2 C$$

so that

$$\lambda_1 = \frac{C - \mu B}{\Delta}, \quad \lambda_2 = \frac{\mu A - B}{\Delta}.$$

There are two possible cases for B :

(a) $\mathbf{1}^T \mathbf{V}^{-1} \mathbf{r} = 0$

It is not probable that this could happen in practise however, it is theoretically possible.

Thus

$$\lambda_1 = \frac{1}{\mathbf{1}^T \mathbf{V}^{-1} \mathbf{1}}, \quad \lambda_2 = \frac{\mu}{\mathbf{r}^T \mathbf{V}^{-1} \mathbf{r}},$$

than the optimal portfolio is

$$\mathbf{x}^* = \frac{\mathbf{V}^{-1} \mathbf{1}}{\mathbf{1}^T \mathbf{V}^{-1} \mathbf{1}} + \frac{\mu \mathbf{V}^{-1} \mathbf{r}}{\mathbf{r}^T \mathbf{V}^{-1} \mathbf{r}}.$$

(b) $\mathbf{1}^T \mathbf{V}^{-1} \mathbf{r} \neq 0$

Put $\mathbf{x}^1 = \frac{\mathbf{V}^{-1} \mathbf{1}}{\mathbf{1}^T \mathbf{V}^{-1} \mathbf{1}}$, $\mathbf{x}^2 = \frac{\mathbf{V}^{-1} \mathbf{r}}{\mathbf{r}^T \mathbf{V}^{-1} \mathbf{r}}$.

Solution for this case is

$$\mathbf{x}^* = \delta(\mu) \mathbf{x}^1 + (1 - \delta(\mu)) \mathbf{x}^2 \tag{1.8}$$

where

$$\delta(\mu) := \frac{A(C - \mu B)}{\Delta}.$$

Remark 1.2.3 Note that portfolios \mathbf{x}^1 and \mathbf{x}^2 are independent of given μ however, $\delta(\mu)$ depends on μ . Moreover, portfolio \mathbf{x}^1 is called global minimum variance portfolio that we get when minimizing the variance without the expected return constraint. Finally, portfolio \mathbf{x}^2 maximizes the Sharpe's Ratio.

Riskfree Asset

Remark 1.2.4 By a riskfree asset we mean an asset with a certain future return. In practise we consider government bonds as riskfree.

If we include a riskfree asset into our problem then matrix V becomes singular and we have to find different solution. Same as above we assume N risky assets $1, \dots, N$ with expected returns r where $r \neq c\mathbf{1}$ and c is a constant. Furthermore, we suppose a riskfree asset denoted by index 0 with return r_0 . Vector of returns now has dimension $(N + 1) \times 1$ and we define it as $\tilde{\boldsymbol{\rho}} = (r_0, \boldsymbol{\rho}^T)^T$. Now investor divides his wealth between $N + 1$ assets $0, 1, \dots, N$ with weights x_0, x_1, \dots, x_N . We are looking for a portfolio p expressed as $\tilde{\boldsymbol{x}} = (x_0, \boldsymbol{x}^T)^T$ which means finding solution for the problem

$$\min \frac{1}{2} \boldsymbol{x}^T \mathbf{V} \boldsymbol{x} \quad (1.9)$$

subject to

$$\mathbf{1}^T \tilde{\boldsymbol{x}} = 1, \quad x_0 r_0 + \boldsymbol{r}^T \boldsymbol{x} \geq \mu$$

where μ is predetermined minimal acceptable return and covariance matrix of returns of risky assets V is positive definite. Since $x_0 = 1 - \mathbf{1}^T \boldsymbol{x}$, following holds: $(\boldsymbol{r} - r_0 \mathbf{1})^T \boldsymbol{x} = \mu - r_0 := \mu_e$.

Now we have to solve the problem

$$\min \frac{1}{2} \boldsymbol{x}^T \mathbf{V} \boldsymbol{x}$$

subject to

$$(\boldsymbol{r} - r_0 \mathbf{1})^T \boldsymbol{x} \geq \mu_e.$$

Same as before we will use the method of Lagrange multipliers. The Lagrange function is

$$L(\boldsymbol{x}, \gamma) = \frac{1}{2} \boldsymbol{x}^T \mathbf{V} \boldsymbol{x} + \gamma(\mu_e - (\boldsymbol{r} - r_0 \mathbf{1})^T \boldsymbol{x})$$

after derivative

$$\frac{\partial L}{\partial \boldsymbol{x}} = \mathbf{V} \boldsymbol{x} + \gamma(r_0 \mathbf{1} - \boldsymbol{r}) = 0$$

we get the optimal solution

$$\boldsymbol{x}^* = \gamma \mathbf{V}^{-1}(\boldsymbol{r} - r_0 \mathbf{1}), \quad x_0^* = 1 - \mathbf{1}^T \boldsymbol{x}^*. \quad (1.10)$$

If $(\boldsymbol{r} - r_0 \mathbf{1})^T \gamma \mathbf{V}^{-1}(\boldsymbol{r} - r_0 \mathbf{1}) > \mu_e$ then optimal solution is investing our wealth into riskfree asset $x_0 = 1$ and $\boldsymbol{x} = 0$.

Otherwise following holds

$$(\boldsymbol{r} - r_0 \mathbf{1})^T \gamma \mathbf{V}^{-1}(\boldsymbol{r} - r_0 \mathbf{1}) = \mu_e$$

hence

$$\gamma = \frac{\mu_e}{A r_0^2 - 2B r_0 + C}.$$

We define portfolio consisting only of riskfree asset as $\tilde{\boldsymbol{x}}^1 := (1, 0, \dots, 0)^T$ and by

$$\boldsymbol{x}^t = \frac{\mathbf{V}^{-1}(\boldsymbol{r} - r_0 \mathbf{1})}{B - A r_0}$$

we define tangency portfolio as $\tilde{\boldsymbol{x}}^2 := (0, \boldsymbol{x}^{tT})^T$.

Theorem 1. Every mean-variance efficient portfolio can be expressed as

$$\tilde{\boldsymbol{x}}^* = \delta \tilde{\boldsymbol{x}}^1 + (1 - \delta) \tilde{\boldsymbol{x}}^2.$$

Proof can be found in Dupačová et al. [2002]

Short Sales not Allowed

If we add this restriction to our problem we are not able to express the solution explicitly anymore. In this case selection of optimal portfolio leads to quadratic optimization problem

$$\min \frac{1}{2} \mathbf{x}^T \mathbf{V} \mathbf{x}$$

under the conditions

$$\mathbf{x} \in \mathbb{R}_+^n, \mathbf{1}^T \mathbf{x} = 1, \mathbf{r}^T \mathbf{x} \geq \mu.$$

Alternatively we can formulate this problem as maximization

$$\max \mathbf{r}^T \mathbf{x}$$

under the conditions

$$\mathbf{x} \in \mathbb{R}_+^n, \mathbf{1}^T \mathbf{x} = 1, \mathbf{x}^T \mathbf{V} \mathbf{x} \leq \sigma_0^2$$

1.2.3 Alternative definitions of Risk

Harry Markowitz was the first one to invent consistent framework for portfolio risk measurement and diversification. Later on more measures of risk were defined. For instance, it is also possible to measure a risk as the mean absolute deviation

Definition 4. *Mean absolute deviation is defined as follows*

$$m(\mathbf{x}) := E \left| \sum_j \rho_j x_j - \sum_j r_j x_j \right| \quad (1.11)$$

In this case, finding an efficient portfolio means to solve

$$\min_{\mathbf{x} \in \mathcal{X}} E \left| \sum_j \rho_j x_j - \sum_j r_j x_j \right| \quad (1.12)$$

subject to

$$\sum_j r_j x_j \geq \mu.$$

Value at Risk

All of the models above are based on risk and return relations. Value at Risk (VaR) is a holistic approach meaning that it considers all important information about risk. In this case, risk measurement is expressed by losses rather than expected return.

Definition 5. *We define VaR as*

$$VaR_\alpha(Y) = F_Y^{-1}(\alpha) \quad (1.13)$$

where

Y is a random loss variable, $E|Y| < \infty$ and F_Y is a cumulative distribution function of Y .

Later on axioms of risk measurement were introduced.

Definition 6. We have two portfolios with future losses Y and Z . We define four axioms of risk measurement

1. if $Y \leq Z$ then $\rho(Y) \leq \rho(Z)$ (risk is monotonic);
2. $\rho(\lambda Y) = \lambda \rho(Y)$ for $\lambda > 0$ (risk is homogeneous);
3. $\rho(Y + \Gamma) = \rho(Y) + \Gamma$,

where Γ is a loss of a riskless bond (riskless translation invariance);

4. $\rho(Y + Z) \leq \rho(Y) + \rho(Z)$ (risk is sub-additive).

When risk measure $\rho(\cdot)$ satisfies all of the assumptions above it is called coherent risk measure.

Last of the axioms (subadditivity) says that it is possible to effectively diversify our portfolio. We achieve lower risk when investing in Y and Z together. VaR does not fulfill subadditivity axiom therefore it is not coherent risk measure.

Conditional Value at Risk

After introduction of risk measurement axioms new measures were invented. Conditional Value at Risk (CVaR) adheres these axioms and keeps the features of VaR.

Definition 7. CVaR is the expected loss when the VaR loss is exceeded,

$$CVaR_\alpha(Y) = E[Y|Y > VaR_\alpha(Y)]. \quad (1.14)$$

Remark 1.2.5 In contrast with VaR, coherent risk measures are convex. More about risk measures in quantitative finance can be found in Mitra [2009].

2. Stochastic Dominance

Throughout the years many portfolio selection models were introduced. Markowitz model introduced in the previous chapter takes into account distribution characteristics as mean and variance but ignores higher moments of return distribution. On the other hand, with usage of Stochastic Dominance (SD) constrains it is possible to develop a more universal concept. The principle advantage of using SD constrains when constructing optimal portfolio is, that it considers entire probability distribution of returns rather than just some particular moments and does not assume any explicit investor's preferences. In SD approach we don't have to specify exact utility function for an investor, SD requires a class of utility functions representing whole group of investors with same preferences. According to investor's different preferences relevant orders of SD were created. Note, that Stochastic Dominance gives a partial order. Source for information in sections 2.1 - 2.2 was Dentcheva and Ruszczyński [2006].

2.1 Stochastic Dominance Orders

Stochastic Dominance is based on comparing cumulative distribution functions (CDF) of random variables.

Definition 8. We define that a random variable K stochastically dominates random variable S in the first order if

$$F(K; \mu) \leq F(S; \mu) \quad \forall \mu \in \mathbb{R},$$

where

$$F(K; \mu) = \mathbb{P}\{K \leq \mu\} \text{ for } \mu \in \mathbb{R}.$$

We say that K dominates S in FSD sense, denoted $K \succeq_{(1)} S$.

If moreover exists at least one μ for which $F(K; \mu) < F(S; \mu)$ then we say that K strictly dominates S and we denote it as $K \succ_{(1)} S$.

In addition, K FSD dominates S if and only if

$$E[u(K)] \geq E[u(S)] \quad \forall u \in \mathcal{U}_1,$$

where \mathcal{U}_1 is set of all non-decreasing functions for which these expected values are finite.

First order Stochastic Dominance (FSD) corresponds to decision makers who prefer more than less (non-satiation) without giving any assumptions about risk aversion.

Definition 9. We define that a random variable K stochastically dominates random variable S in the second order if

$$F_2(K; \mu) \leq F_2(S; \mu) \quad \forall \mu \in \mathbb{R},$$

where

$$F_2(K; \mu) = \int_{-\infty}^{\mu} F(K; \xi) d\xi \text{ for } \mu \in \mathbb{R}.$$

We say that K dominates S in SSD sense, denoted $K \succeq_{(2)} S$. If moreover exists at least one μ for which $F_2(K; \mu) < F_2(S; \mu)$ then we say that K strictly dominates S and we denote it as $K \succ_{(2)} S$.

Furthemore, K SSD dominates S if and only if

$$E[u(K)] \geq E[u(S)] \quad \forall u \in \mathcal{U}_2,$$

where \mathcal{U}_2 is set of all non-decreasing and concave functions for which these expected values are finite.

Second order Stochastic Dominance (SSD) is more appealing than FSD in problem of portfolio optimization. It is because of the added assumption of non-decreasing concave utility functions which reflect investor's risk aversion.

Remark 2.1.1 It is possible to denote the function $F_2(K; \cdot)$ as the expected shortfall

$$F_2(K; \mu) = E[(\mu - K)_+], \quad (2.1)$$

where $(\mu - K)_+ = \max(\mu - K, 0)$.

Similarly to the case with random variables we say that portfolio x FSD dominates portfolio y if

$$F(R(x); \mu) \leq F(R(y); \mu) \quad \forall \mu \in \mathbb{R}$$

and portfolio x SSD dominates portfolio y if

$$F_2(R(x); \mu) \leq F_2(R(y); \mu) \quad \forall \mu \in \mathbb{R},$$

where we assume that $E[|R_j|] < \infty \quad \forall j = 1, \dots, N$ and $R(x) = R_1x_1 + R_2x_2 + \dots + R_nx_n$ denotes the return rate of whole portfolio. For now we do not allow short sales therefore, we define the set of feasible weights as

$$X = \{\mathbf{x} \in \mathbb{R}^n : x_1 + x_2 + \dots + x_n = 1, x_j \geq 0, j = 1, 2, \dots, N\}.$$

Definition 10. We define portfolio x as SSD-efficient (or FSD-efficient) in a set of portfolios X if there exists no portfolio $y \in X$ satisfying $R(y) \succ_{(2)} R(x)$ (or $R(y) \succ_{(1)} R(x)$).

It is important to mention that it is possible to formulate dominance with CVaR constraints equivalent to SSD restriction. In terms of return rates we define Conditional Value at Risk as

$$CVaR_\alpha(R(x)) = E[R(x) | R(x) \leq \xi_\alpha(R(x))]$$

where $\xi_\alpha(R(x))$ is the right α -quantile of random return rate $R(x)$.

Theorem 2. The SSD constraint, $R(x) \succeq_{(2)} Y$, is equivalent to the continuum of CVaR restrictions

$$CVaR_\alpha(R(x)) \geq CVaR_\alpha(Y) \quad \text{for all } \alpha \in (0, 1]. \quad (2.2)$$

Proof can be found in Dentcheva and Ruszczyński [2006].

2.2 Construction of Optimal Portfolio

Now we will focus on theory regarding portfolio optimization with second order stochastic dominance constrains which refers to risk averse investors. In this case we include benchmark which return rate we want to dominate in terms of SSD. This benchmark could be a stock market index or our current portfolio. We work with non-decreasing and concave utility functions. In mean-variance approach we don't work with whole distribution of returns.

2.2.1 Portfolio selection with dominance constrains

We want to solve following optimization problem :

$$\max f(x) \tag{2.3}$$

subject to

$$R(x) \succeq_{(2)} Y \tag{2.4}$$

$$x \in X \tag{2.5}$$

where $f(x) = E[R(x)]$ or another concave continuous function.

Now we will try to simplify our optimization problem. We propose that in case when assets included in our benchmark portfolio Y have return rates with discrete joint distribution and realizations $y_i, i = 1, \dots, m$ then constraint $R(x) \succeq_{(2)} Y$ is equivalent to

$$E[(y_i - R(x))_+] \leq E[(y_i - Y)_+], i = 1, \dots, m. \tag{2.6}$$

From now on, we assume that return rates of the assets are random variables with discrete joint distribution given by T realizations (scenarios) $r_{jt}, t = 1, \dots, T, j = 1, \dots, N$ which occur with probabilities $p_t, t = 1, \dots, T$. We denote shortfall of $R(x)$ below y_i as s_{it} where

$$s_{it} = \max(0, y_i - \sum_{j=1}^n x_j r_{jt}), i = 1, \dots, m, t = 1, \dots, T.$$

With these assumptions we can formulate our optimization problem as

$$\max f(x) \tag{2.7}$$

$$\sum_{j=1}^n x_j r_{jt} + s_{it} \geq y_i, i = 1, \dots, m, t = 1, \dots, T; \tag{2.8}$$

$$\sum_{t=1}^T p_t s_{it} \leq F_2(Y; y_i), i = 1, \dots, m; \tag{2.9}$$

$$s_{it} \geq 0, i = 1, \dots, m, t = 1, \dots, T; \tag{2.10}$$

$$x \in X. \tag{2.11}$$

For every possible x fulfilling (2.3) - (2.5) we get a feasible pair (x, s) for (2.7) - (2.11). Contrariwise, for any pair (x, s) fulfilling (2.7) - (2.11), constraints (2.8) and (2.10) imply that

$$s_{it} \geq \max(0, y_i - \sum_{j=1}^n x_j r_{jt}), \quad i = 1, \dots, m, \quad t = 1, \dots, T.$$

Now we can modify this inequality by using expected value of both sides, then constraint (2.9) gives us following result

$$F_2(R(x); y_i) \leq F_2(Y; y_i), \quad i = 1, \dots, T.$$

We proposed above that $R(x) \succeq_{(2)} Y$ is equivalent to $E[(y_i - R(x))_+] \leq E[(y_i - Y)_+]$, $i = 1, \dots, m$ which implies that x is feasible for optimization problem (2.3) - (2.5) which results in the following statement

Proposition 1 If R_j , $j = 1, \dots, N$ and Y have discrete distributions then optimization problem (2.3) - (2.5) is equivalent to problem (2.7) - (2.11).

Proof can be found in Dentcheva and Ruszczyński [2006].

Optimality

As above we assume that there are finitely many outcomes of benchmark Y and they have discrete probability distribution, same holds for return rates. Furthermore, we assume that realizations of Y are ordered: $y_1 < y_2 < \dots < y_m$. Realizations occur with probabilities π_i , $i = 1, \dots, m$.

We define set \mathcal{U} of functions $u : \mathbb{R} \rightarrow \mathbb{R}$ which adhere following conditions

- $u(\cdot)$ is non-decreasing concave function;
- $u(\cdot)$ is piecewise linear function with break points y_i , $i = 1, \dots, m$;
- $u(t) = 0 \forall t \geq y_m$.

It is obvious that \mathcal{U} is a convex cone.

Now we will define the Lagrangian function of (2.3) - (2.5), $L : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}$,

$$L(x, u) = f(x) + E[u(R(x))] - E[u(Y)]. \quad (2.12)$$

Theorem 3. If \hat{x} is an optimal solution of (2.3) - (2.5) then there exists a function $\hat{u} \in \mathcal{U}$ such that following holds:

$$L(\hat{x}, \hat{u}) = \max_{x \in X} L(x, \hat{u}) \quad (2.13)$$

and

$$E[\hat{u}(R(\hat{x}))] = E[\hat{u}(Y)]. \quad (2.14)$$

On the contrary, if for some function $\hat{u} \in \mathcal{U}$ an optimal solution \hat{x} of (2.13) meets (2.4) and (2.14), then \hat{x} is an optimal solution of (2.3) - (2.5).

Proof can be found in Dentcheva and Ruszczyński [2006].

2.3 Portfolio efficiency tests

In this section we will focus on tests analyzing portfolio efficiency in terms of FSD and SSD. These tests provide us with information about efficiency of given portfolio due to all possible portfolios composed of a set of base assets.

We assume that we have N assets and return rate of each of these assets has T possible scenarios $\tau \equiv \{1, \dots, T\}$ which occur with the same probabilities. We represent this data by matrix $G \equiv (G_1, \dots, G_N)$ where $G_j \equiv (r_{j1}, \dots, r_{jT})$.

2.3.1 Kuosmanen criteria

Now we will introduce efficiency tests based on FSD and SSD introduced in Kuosmanen [2004]. Note that it is possible to represent portfolios by portfolio weights or by portfolio return profiles.

FSD Test

$$\begin{aligned} \theta_1(y_0) &= \max_{\mathbf{x}^0, P} \left(\sum_{t=1}^T \sum_{i=1}^{N+1} G_{it}^0 x_i^0 - \sum_{t=1}^T y_{0t} \right) / T \\ \text{s.t.} \quad & \sum_{i=1}^{N+1} G_{it}^0 x_i^0 \geq \sum_{j=1}^T P_{tj} y_{0j} \quad \forall t \in \tau \\ & P \in \Pi \\ & \mathbf{x}^0 \in X^0, \end{aligned}$$

where P is permutation matrix and

$$\Pi \equiv \left\{ [P_{ij}]_{T \times T} \mid P_{ij} \in \{0, 1\}, \sum_{i=1}^T P_{ij} = \sum_{j=1}^T P_{ij} = 1 \quad \forall i, j \in \tau \right\},$$

matrix $G^0 = (G, y_0)$ is our data matrix augmented by benchmark portfolio return profile and $X^0 = \{(\mathbf{x}^T, x_b)^T \in \mathbb{R}^{N+1}; x_b \geq 0, \mathbf{x} \in X\}$.

Theorem 4. $\theta_1(y_0) = 0$ is both necessary and sufficient condition for FSD efficiency of the benchmark.

Proof can be found in Kuosmanen [2004].

Test statistic $\theta_1(y_0)$ reflects inefficiency of y_0 in terms of expected return. If $\theta_1(y_0) > 0$ it is possible to select portfolio with higher expected return which dominates the benchmark.

Necessary SSD Test

$$\begin{aligned} \theta_2^N(y_0) &= \max_{\mathbf{x}^0, W} \left(\sum_{t=1}^T \sum_{i=1}^{N+1} G_{it}^0 x_i^0 - \sum_{t=1}^T y_{0t} \right) / T \\ \text{s.t.} \quad & \sum_{i=1}^{N+1} G_{it}^0 x_i^0 \geq \sum_{j=1}^T W_{tj} y_{0j} \quad \forall t \in \tau \\ & W \in \Xi \\ & \mathbf{x}^0 \in X^0 \end{aligned}$$

where matrix W is called doubly stochastic and

$$\Xi \equiv \left\{ [W_{ij}]_{T \times T} \mid 0 \leq W_{ij} \leq 1; \sum_{i=1}^T W_{ij} = \sum_{j=1}^T W_{ij} = 1 \forall i, j \in \tau \right\}.$$

Theorem 5. Necessary condition for SSD efficiency of benchmark portfolio is $\theta_2^N(y_0) = 0$.

Proof can be found in Kuosmanen [2004].

It is possible to interpret the test statistic $\theta_2^N(y_0)$ as inefficiency measure of the benchmark portfolio. It expresses the maximum possible increase of expected return we would get by selecting another portfolio which SSD dominates the benchmark. Therefore if this statistic equals zero there is no other SSD dominating portfolio with higher expected return.

Sufficient SSD Test

$$\begin{aligned} \theta_2^S(y_0) &= \min_{W, x^0, s^+, s^-} \sum_{j=1}^T \sum_{i=1}^T (s_{ij}^+ + s_{ij}^-) \\ \text{s.t. } \sum_{n=1}^{N+1} G_{tn}^0 x_i^0 &= \sum_{j=1}^T W_{tj} y_{0j} \quad \forall t \in \tau \\ s_{ij}^+ - s_{ij}^- &= W_{ij} - \frac{1}{2} \quad \forall i, j \in \tau \\ s_{ij}^+, s_{ij}^- &\geq 0 \quad \forall i, j \in \tau \\ W &\in \Xi \\ x^0 &\in X^0 \end{aligned}$$

where $S^+ = \{s_{ij}^+\}_{i,j=1}^T$, $S^- = \{s_{ij}^-\}_{i,j=1}^T$ and $W = \{w_{ij}\}_{i,j=1}^T$.

In Kuosmanen [2004] we can find theoretical maximum and minimum of $\theta_2^S(y_0)$. When we set $W_{ij} = 1/T \forall i, j$ we obtain minimal value. Maximum depends on number of repeating values in vector of returns of the benchmark. We introduce variable d_{0k} which reflects number of k -way ties in y_0 .

$$\theta_2^S(y_0) \in \left[\frac{1}{2}T^2 - T, \frac{1}{2}T^2 - \sum_{k=1}^T k d_{0k} \right]$$

Remark 2.3.1.1 We say it is a 3-way tie if $y_{0i} = y_{0j} = y_{0k}$ etc.

Theorem 6. The benchmark portfolio is SSD efficient if satisfies necessary and sufficient condition:

$$\theta_2^N(y_0) = 0 \wedge \theta_2^S(y_0) = \frac{T^2}{2} - \sum_{k=1}^T k d_{0k}.$$

Proof can be found in Kuosmanen [2004].

If the necessary condition isn't satisfied there is no need to evaluate sufficient test statistic.

Construction using the Kuosmanen's Test

It is possible to find weights \mathbf{x}^0 of optimal portfolio solving optimization problem based on the Kuosmanen's necessary SSD test.

$$\begin{aligned} & \max_{\mathbf{x}^0, W} \left(\sum_{t=1}^T \sum_{i=1}^{N+1} G_{it}^0 x_i^0 \right) / T \\ \text{s.t. } & \sum_{i=1}^{N+1} G_{it}^0 x_i^0 \geq \sum_{j=1}^T W_{tj} y_{0j} \quad \forall t \in \tau \\ & W \in \Xi \\ & \mathbf{x}^0 \in X^0. \end{aligned}$$

This approach is easier to implement than (2.7) – (2.11), therefore we will use it in practical section.

3. Real Data Application

In this chapter we will provide empirical study in which we will compare performance of the portfolio optimization approaches stated in the first and second chapter. We will formulate this problem in both cases as a maximization of return. As a benchmark we chose Dow Jones Industrial Average (DJIA) index that indicates the value of 30 publicly owned companies based in the United States.

We formulated optimization problem using Markowitz approach as

$$\max \mathbf{r}^T \mathbf{x}$$

under the conditions

$$\mathbf{x} \in \mathbb{R}_+^n, \mathbf{1}^T \mathbf{x} = 1, \mathbf{x}^T \mathbf{V} \mathbf{x} \leq \sigma_0^2$$

where σ_0^2 is variance of the DJIA index.

For SSD constraint approach we used formulation based on the Kuosmanen's necessary SSD test

$$\begin{aligned} & \max_{\mathbf{x}^0, W} \left(\sum_{t=1}^T \sum_{i=1}^{N+1} G_{it}^0 x_i^0 \right) / T \\ \text{s.t. } & \sum_{i=1}^{N+1} G_{it}^0 x_i^0 \geq \sum_{j=1}^T W_{tj} y_{0j} \forall t \in \tau \\ & W \in \Xi \\ & \mathbf{x}^0 \in X^0. \end{aligned}$$

Remark 3.1.1 In general, \mathbf{x}^0 includes the weight of investment to the benchmark portfolio. However, it's equal to 0 if a dominating portfolio exists. Therefore we can compare optimal portfolios of SSD constrained problems with those of the Markowitz model.

We downloaded daily returns for DJIA index and it's components from yahoo.finance.com. For our analysis we chose five years period starting 15/10/2013. During this period happened two significant changes in DJIA index. First change of the components occurred in 2015 when Apple replaced AT&T, second change happened in 2018 when Walgreen Boots Alliance replaced General Electric. Because of these changes we decided to include 2 sets of assets that we used in our analysis. We present intersection set, consisting of assets which were part of the DJIA index during whole analyzed period. Second set is union set representing all of the assets which were present in the DJIA index during our 5 year period.

Company	Symbol	Intersection
3M	MMM	Yes
American Express	AXP	Yes
Apple	AAPL	No
AT&T	T	No
Boeing	BA	Yes
Caterpillar	CAT	Yes
Chevron	CVX	Yes
Cisco Systems	CSCO	Yes
Coca-Cola	KO	Yes
DowDuPont	DWDP	Yes
ExxonMobil	XOM	Yes
General Electrics	GE	No
Goldman Sachs	GS	Yes
The Home Depot	HD	Yes
IBM	IBM	Yes
Intel	INTC	Yes
Johnson & Johnson	JNJ	Yes
JPMorgan Chase	JPM	Yes
McDonald's	MCD	Yes
Merck & Company	MRK	Yes
Microsoft	MSFT	Yes
Nike	NKE	Yes
Pfizer	PFE	Yes
Procter & Gamble	PG	Yes
Travelers	TRV	Yes
UnitedHealth Group	UNH	Yes
United Technologies	UTX	Yes
Verizon	VZ	Yes
Visa	V	Yes
Walmart	WMT	Yes
Walgreens Boots Alliance	WBA	No
Walt Disney	DIS	Yes

Table 3.1: Assets included in Analysis

As mentioned before, SSD is most appealing from the stochastic dominance orders because it represents non-satiable, risk averse decision makers. We wanted our portfolio to dominate DJIA index in terms of SSD which leads to linear programming problem.

We presented general solutions for Markowitz approach when we constructed optimal portfolio in case of short sales in first chapter. In the practical application we formulated our problem as maximization of return and we didn't allow short sales which leads to quadratic optimization problem.

For our analysis we chose three different periods during which we optimized our weights.

	Optimization Period	Set
SSD	7, 14, 60	Intersection, Union
Markowitz	7, 14, 60	Intersection, Union

Table 3.2: All of the cases analysed in empirical study

We divide our empirical study into 2 parts, out of sample analysis and in sample analysis. First we will present in sample analysis where we will illustrate FSD dominance graphically. In sample means that for analysis we are using data which were used in the sample for optimization. Then we will compare ratio of portfolios dominating in both FSD and SSD sense for different optimization periods. After that we will check if portfolios obtained by Markowitz approach show signs of SSD dominance. Out of sample analysis means that we will try to use our results from optimization on the data which were not part of the optimization window. In all cases we use one week right after the end of the optimization period for out of sample analysis.

Software implementation

We solved all of our optimization problems in GAMS from which we obtained optimal weights for given periods. Weights for each week for every case were exported from GAMS to text files. After that we loaded the text files into R which we used for evaluating and plotting our out of sample and in sample analysis.

3.1 In Sample Analysis

In first part of our in sample analysis we will focus on answering the question how many of our SSD dominating portfolios dominate the benchmark also in the FSD sense.

Now we will provide illustration graphs showing SSD portfolio dominating also in the FSD sense and portfolio which dominates just in SSD sense.

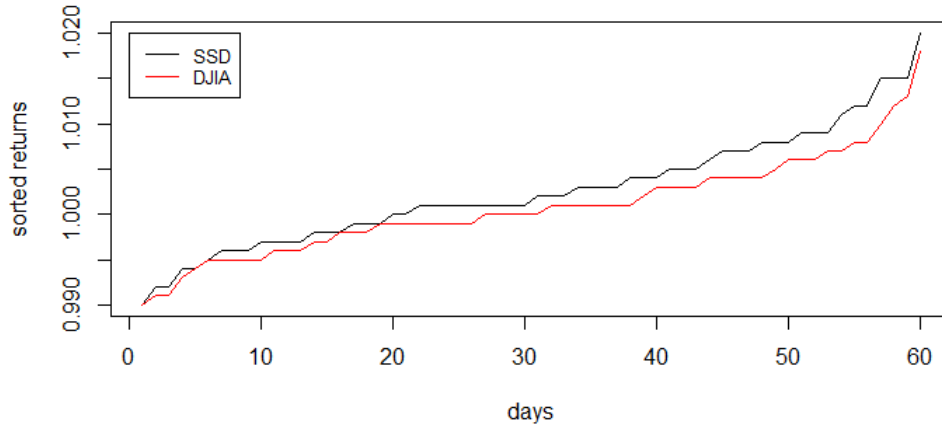


Figure 3.1: Graphic illustration of SSD portfolio dominating the DJIA index also in the FSD sense

On the graph we can see plotted sorted returns of SSD portfolio and benchmark during first 60 days of our data. SSD portfolio was optimized using the same 60 days. We can observe that sorted returns of the DJIA index are always lower or equal to sorted returns of our SSD dominating portfolio which implies that our portfolio dominates the benchmark also in the FSD sense.

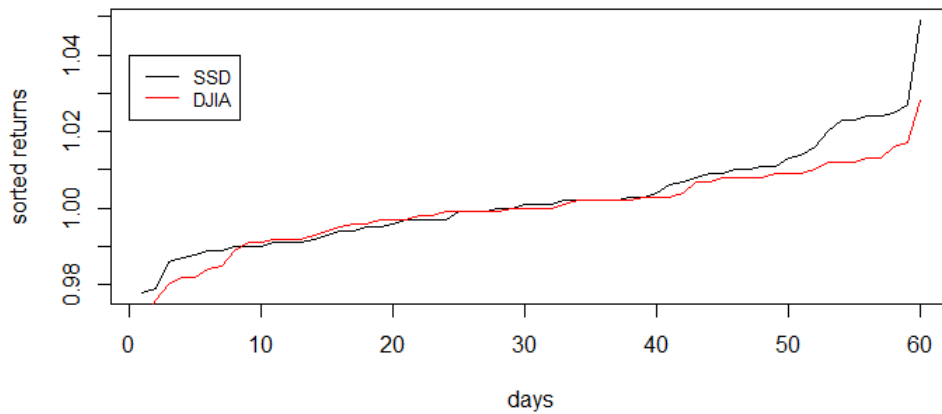


Figure 3.2: Graphic illustration of SSD portfolio not dominating DJIA in FSD sense

In this graph we are showing sorted SSD and benchmark portfolio returns during 160th sixty day window. There are 15 cases when DJIA returns are higher which means that our portfolio doesn't dominate the benchmark in FSD sense.

	Intersection	Union
7 days	78.8	81
14 days	62.9	63.5
60 days	31.4	26.7

Table 3.3: Percentage of SSD portfolios dominating the benchmark also in the FSD sense

We can observe that number of portfolios dominating in both FSD and SSD sense is decreasing when we increase number of days included in optimization period.

Now for the comparison we will check how many of portfolios created by Markowitz approach also satisfy constrains of SSD dominance.

	Intersection	Union
7 days	90.1	89.9
14 days	80.3	83.7
60 days	59.9	51.7

Table 3.4: Percentage of Markowitz portfolios dominating the benchmark in the SSD sense

With optimization period 7 days long there is very high number of Markowitz portfolios which dominate the benchmark in SSD sense but we can see that numbers for 60 days optimization period are much lower.

3.2 Out of Sample Analysis

Now we will provide observations and conclusions we obtained from out of sample analysis. We decided to use weights obtained from our optimization and hold these portfolios always for a week beginning right after the end of optimization period.

7 days optimization period

First we will present results obtained by investing in portfolios which we obtained from 7 days optimization period.

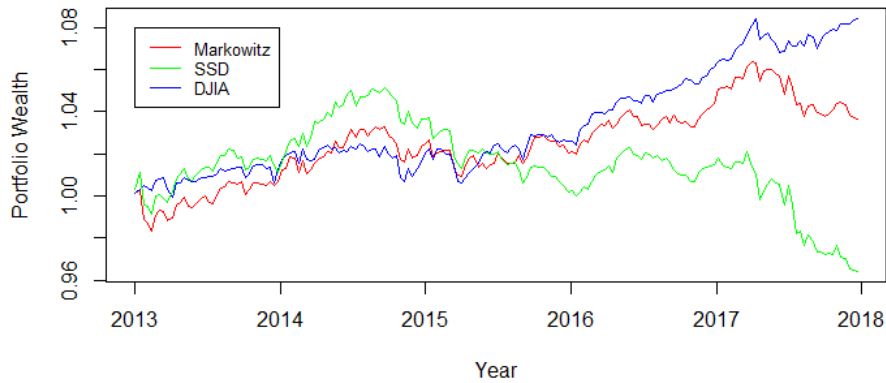


Figure 3.3: Performance of Markowitz and SSD portfolio consisting of assets from intersection set against performance of DJIA index during a 5 years period with usage of historical data from 7 days

Both Markowitz portfolio and portfolio obtained from optimization with SSD constraints had outperformed DJIA index in 48.9% of weeks.

When we compare cumulative product of returns after 5 years we can see that we didn't manage to beat the index. However, Markowitz portfolio at least managed to keep the track with DJIA till the end but SSD portfolio didn't do so well.

If we examine the 2013 - 2015 period we can see that SSD was performing much better than in later years. We can observe that SSD portfolio didn't manage to follow the sharp increase of the DJIA index.

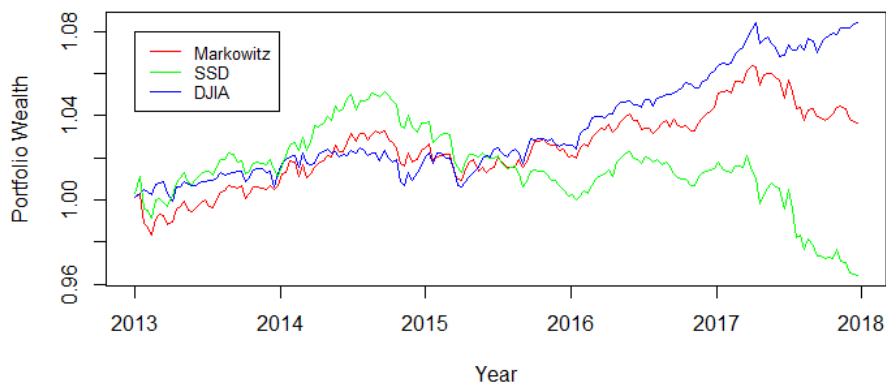


Figure 3.4: Performance of Markowitz and SSD portfolio consisting of assets from union set against performance of DJIA index during a 5 years period with usage of historical data from 7 days

SSD portfolio outperformed DJIA in 44.9% of weeks while Markowitz portfolio in 47.2% weeks. In this case we also didn't beat the index and in the table below we can see that with inclusion of more assets we actually obtained lower performance after 5 years.

	Intersection	Union
Markowitz	1.066104	1.036665
SSD	1.008157	0.9643255
DJIA	1.084231	1.084231

Table 3.5: Cumulative product of return after 5 years, 7 days optimization period

14 days optimization period

Now we will see if including more historical data to our optimization helps us to achieve higher returns.

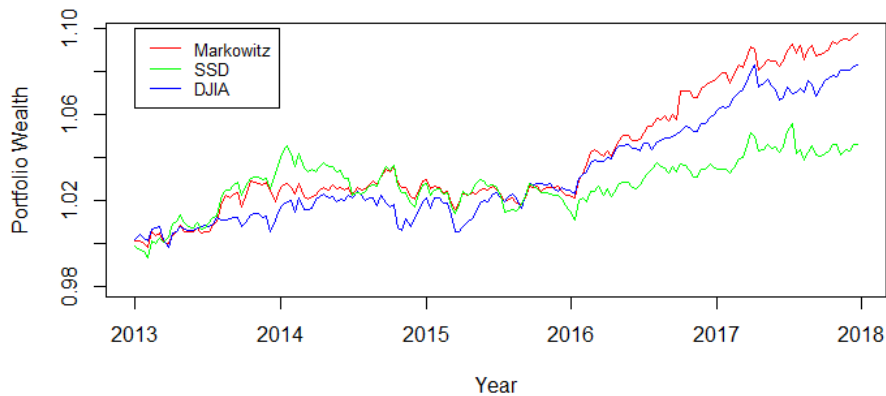


Figure 3.5: Performance of Markowitz and SSD portfolio consisting of assets from intersection set against performance of DJIA index during a 5 years period with usage of historical data from 14 days

This is the first case where one of our portfolios managed to perform better in more than half of instances. Markowitz portfolio had better performance in 53.1 % of weeks, SSD portfolio performed better in 47.5% cases.

This is also first time we accomplished better return after 5 years with Markowitz portfolio but still didn't beat the DJIA index with SSD portfolio. But we can see that adding more historical data to our analysis helped SSD to better results in years 2015 - 2018.

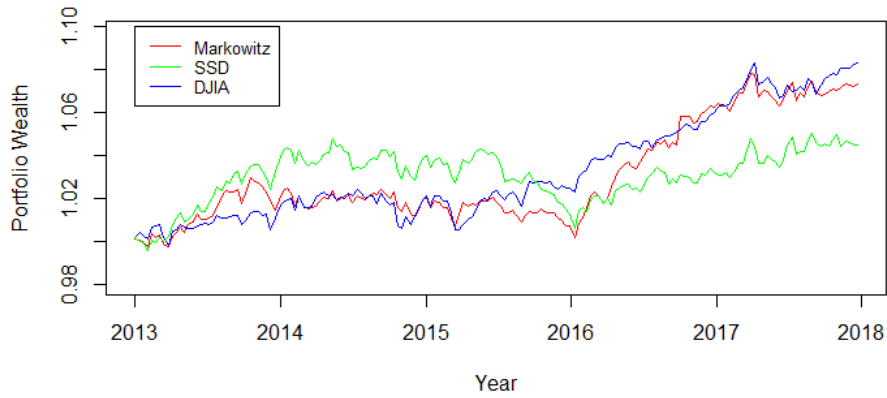


Figure 3.6: Performance of Markowitz and SSD portfolio consisting of assets from union set against performance of DJIA index during a 5 years period with usage of historical data from 14 days

Similarly as in 7 days optimization window with more assets in our set our results were worse than in case of an intersection but this time Markowitz portfolio has been more affected by this change than SSD portfolio which could be observed in the following table.

	Intersection	Union
Markowitz	1.097468	1.073082
SSD	1.046023	1.044652
DJIA	1.082931	1.082931

Table 3.6: Cumulative product of return after 5 years, 14 days optimization period

60 days optimization period

Finally we get to the longest optimization period when we used 60 days of historical data for our optimization.

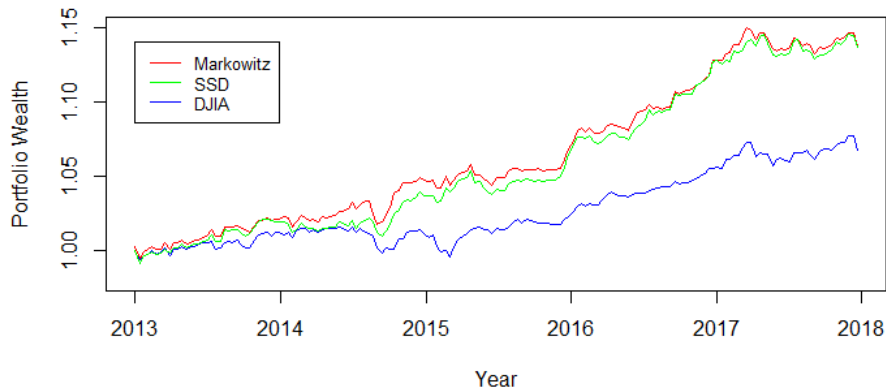


Figure 3.7: Performance of Markowitz and SSD portfolio consisting of assets from intersection set against performance of DJIA index during a 5 years period with usage of historical data from 60 days

Here we can observe our portfolios doing very well in comparison with DJIA index which is beaten by both of them. We can see both portfolios having higher cumulative product of return than DJIA index during the whole period after first few weeks. Final return for Markowitz and SSD portfolio is almost the same.

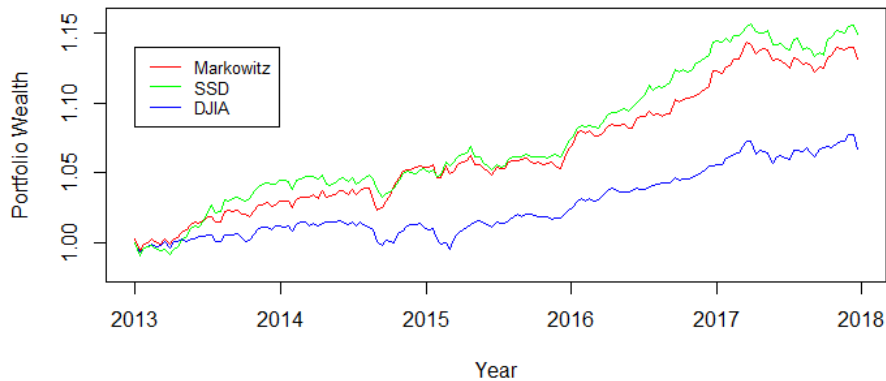


Figure 3.8: Performance of Markowitz and SSD portfolio consisting of assets from union set against performance of DJIA index during a 5 years period with usage of historical data from 60 days

In the last case our SSD portfolio performed better than DJIA index and also beat Markowitz portfolio. In contrast with previous optimization periods, including more assets to our consideration lead to better results.

	Intersection	Union
Markowitz	1.137511	1.131316
SSD	1.136667	1.148797
DJIA	1.067128	1.067128

Table 3.7: Cumulative product of return after 5 years, 60 days optimization period

In our out of sample analysis we showed that it is possible to beat the DJIA index using both of the approaches and our chances of performing better seem to be positively correlated with length of the optimization window. Only with 60 days optimization period our SSD portfolio managed to catch the increasing trend of DJIA index in years 2015 - 2018.

Conclusion

In this work we focused on problem of portfolio optimization considering two different approaches. In the first chapter we included basic assumptions for efficient market and portfolio theory definitions that we needed for introduction of Markowitz model. We introduced mean-variance framework and provided basic formulations of the optimization problem. Then we present general solutions for portfolios with short sales allowed and we included the case of risky assets and also case with inclusion of risk free asset. In the end of the first chapter we present alternative definitions of risk.

In the second part of the work we discussed differences between Markowitz mean-variance model and portfolio optimization with stochastic dominance restrictions. We introduced definitions of different orders of stochastic dominance, presented their attributes and discussed differences among them. We mentioned important relationship between $CVaR$ and second order stochastic dominance constraint. Then we focused on simplification of our problem with assumption of discrete distribution for return rates. Later we introduced portfolio efficiency test based on both FSD and SSD.

In the last chapter of the work we used both of the approaches defined before and tried to beat the performance of Dow Jones Industrial Average index. We used 5 years of historical data during which we optimized with different optimization periods and different sets of assets. In our out of sample analysis we provide for comparison graphs and tables with evaluated performances. We can conclude that with longest optimization window we managed to beat the DJIA index with both approaches. In our in sample analysis we illustrate ratio of SSD dominating portfolios that actually dominate also in the FSD sense and ratio of Markowitz portfolios satisfying restrictions of SSD dominance.

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