Report on the doctoral thesis of Vojtěch Kaluža entitled "Metric and analytic methods"

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The thesis consists of two parts.

The first part is based on the author's joint work with Dymond and his doctoral co-advisor Eva Kopecká, and contains a solution to a well-known question of Feige laying at the interface of computing science, discrete geometry, and mathematical analysis. The question asks if there exists a universal constant $L > 0$ such that every set of n^2 points in $\mathbb{R}^2$ admits an L-Lipschitz bijection into the n by n integer grid. Its positive answer would imply the existence of an efficient approximation algorithm for a natural generalization of the well-known Graph Bandwidth Problem. However, the answer to the question is negative.

The second part is based on the author's joint work with E. Colin de Verdière, Paták, Patáková and his doctoral advisor Martin Tancer, and contains a constructive proof of the variant of the (strong) Hanani-Tutte theorem on the projective plane. Prior to this work, only a non-constructive proof was known to the best of my knowledge.

Since the two parts of the thesis are completely independent below I discuss the content of each part separately, and finish my report with a concluding paragraph.

Feige's quesion. The general strategy in the proof was inspired by a work of Burago and Kleiner, and independently McMullen, showing the existence of separated nets in $\mathbb{R}^2$ that do not admit a bilipschitz bijection into the lattice $\mathbb{Z}^2$. In fact, a result from the thesis can be aparently regarded as a strenghtening of a result of Burago and Kleiner. The proof consists of the two main steps. First, finding a continuous variant of the Feige's question the negative answer to which implies the negative answer to the Feige's question, and second, answering the continuous variant of the question in negative.

The question of Feige appears to be an important one, and the research paper that this part of the thesis is based on was accepted to a highly ranked mathematical journal. Though, it is not mentioned if its importance stems also from something else besides the Graph Bandwidth Problem, since its significance and possible applications are not discussed.

Given that proofs in this part of the thesis are highly technical, especially Section 1.3 and 1.4 which are concerned with the previously mentioned continuous variant of Feige's quesion, I assume that the structuring and writting up the proofs were probably considerable challenges throughout this work. The author and his co-authors rose to the challenge quite well. The structure of the proofs and their division into auxilliary claims seems logical and helpful in following what is going on. This part also seems to be written very carefully. I could spot only a few typos.

I was perhaps only missing a high level overview of longer arguments aimed also at non-experts in the field, concrete examples illustrating the problem, and a better treatment of the quantitative and algorithmic version of the problem.

The strong Hanani–Tutte theorem on the projective plane. The (strong) Hanani-Tutte theorem is a very remarkable classical result that found many applications in discrete geometry and computing science. The theorem and its high dimensional analog known as Van Kampen-Shapiro-Wu theorem is also the starting point of the deep theory of van Kampen obstructions. The theorem characterizes planar graphs as follows. A graph is planar if and only if it can be drawn in the plane so that no pair of its non-adjacent edges cross an odd number of times. Various variants and generalizations of the theorem were discovered in the plane, but for the surfaces of higher genus the analog of the Hanani-Tutte theorem is known to hold besides the plane only on the projective plane. Very recently, it was announced that a counterexample to such an analog exists on any closed orientable surface of orientable genus at least 4.

The thesis contains a constructive proof of the variant of the Hanani-Tutte theorem on the projective plane. The only previously known proof uses the characterization by minimal forbidden minors. Therefore the new proof is an important step in the study of analogous variants on other closed two-dimensional surfaces, where the complete list of minimal forbidden minors is not known, which includes every closed two-dimensional surface besides the plane and the projective plane.

Instead of the forbidden minor characterization the new proof relies on a substantial amount of machinery that allows to identify forbidden and redrawable configurations in terms of non-trivial bridges of a trivial cycle in a drawing of a graph. A relatively large number of considered configutions is perhaps the main drawback of the proof and suggests that this approach cannot be pushed very far in terms of the genus of the surface without significantly novel ideas. On the positive sides, the developed toolbox has a potential to be useful in the study of the Hanani-Tutte variants on other surfaces, and the result might lead to possible counterexamples to HT variants on the non-orientable surfaces and a few remaining orientable surfaces, where the existence of such a variant is an open problem.

This part of the thesis is also well-written.

Conclusion. While most of the material in the thesis is copied from the above mentioned author's joint papers with his thesis advisors and other researchers, the author also interspersed the thesis with additional discussions, high level explanations and suggestions for new research directions.

This together the author's research track record, led me to believe that the author's involvement in both research projects was substantial and that he has a high potential for becoming a strong independent researcher.