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Review of “Metric and analytic methods” by Vojtěch Kaluža

This thesis is comprised of two parts: the solution to a problem in metric geometry which was posed by the computer scientist Feige, and a new proof of the Strong Hanani-Tutte Theorem for the projective plane. I will comment only on the solution to Feige's problem, because the second part is further from my expertise. This is a a spectacular result for a thesis, as it resolves a natural geometric problem using a beautiful approach that invokes a range of techniques from analysis and geometry. I understand that this portion of the thesis follows a joint paper with Dymond and Kopecká closely, but note that the author's contribution to this result was clearly substantial.

Feige had asked if there is a constant $L = L(d)$ such that for every subset $S \subset \mathbb{Z}^d$ of cardinality n^d, there exists a bijection from S onto the discrete cube $\{1, \ldots, n\}^d \subset \mathbb{Z}^d$, which is L-Lipschitz with respect to the Euclidean distance. The author shows that no such constant L exists.

The proof has two steps.

The first step of the proof is to reduce (the negative answer to) this discrete mapping problem to a negative answer to a continuous mapping problem involving Lipschitz regular mappings. A Lipschitz map $f : X \to Y$ between metric spaces is *regular* if there is a $C < \infty$ such that for every $y \in Y$, and every $r < \infty$, the inverse image $f^{-1}(B(y, r))$ may be covered by at most C balls of radius Cr in X. Lipschitz regular mappings may be regarded as a generalization of bilipschitz mappings; they were introduced by David-Semmes in the context of geometric measure theory, and have since arisen in a variety of other contexts. The author shows that a negative answer to a certain prescribed Jacobian problem for Lipschitz regular mappings implies a negative answer to Feige's question. Specifically, he shows that it suffices to find a measurable function $\rho : [0, 1]^d \to \mathbb{R}$ such that $0 < \operatorname{ess\,inf} \rho < \operatorname{ess\,sup} \rho < \infty$, and there is no Lipschitz regular mapping $f : [0, 1]^d \to \mathbb{R}^d$ such that the pushforward of the measure $\rho\mathcal{L}$ by f agrees with Lebesgue measure on the image of f: $f_*(\rho\mathcal{L}) = \mathcal{L}|_{\operatorname{Im} f}$. The idea of this reduction is to discretize the measure $\rho\mathcal{L}$, by taking a sequence of subsets $S_j \subset \mathbb{Z}^d$ such that the suitably normalized and rescaled counting measures on S_j weakly converge to $\rho\mathcal{L}$. Arguing by contradiction, one then shows that if Feige's problem had an affirmative answer, then one would obtain a sequence of uniformly Lipschitz mappings $f_j : S_j \to \mathbb{R}^d$, and by passing to a subsequential limit of (suitably rescaled copies of f_j) one would arrive at a Lipschitz regular mapping $f : [0, 1]^d \to \mathbb{R}^d$ such that

$f_*(\rho\mathcal{L}) = \mathcal{L}|_{\operatorname{Im} f}$, a contradiction. This reduction was inspired by a similar step in a theorem of McMullen and Burago and myself on bilipschitz equivalence of separated nets.

The second step of the proof is the construction of the function $\rho : [0,1]^d \to \mathbb{R}$ as above. (In fact, it is shown that "most" continuous function $[0,1]^d \to (0,\infty)$ would work, i.e. all but a σ-porous set of functions will do.) This step is inspired by the construction (due to McMullen-Burago-Kleiner) of a continuous function $u : [0,1]^2 \to (0,\infty)$ which is not the Jacobian of any bilipschitz embedding $[0,1]^2 \to \mathbb{R}^2$; however, it is is far more complicated. One key ingredient is a structure theorem for Lipschitz regular mappings $\mathbb{R}^d \supset U \to \mathbb{R}^d$, which implies that such mappings are actually bilipschitz on countable dense collection of open subsets.

As stated above, this is an excellent thesis result, which made the first real progress on Feige's question. This work leads to many follow-up questions, the most obvious being: What are asymptotics of the optimal constant $L = L(n,d)$ such that every subset $S \subset \mathbb{Z}^d$ of cardinality n^d has an L-Lipschitz bijection to the d-cube $\{1,\ldots,n\}^d \subset \mathbb{R}^d$? I expect this to be the beginning of an interesting line of research in asymptotic metric geometry.

Sincerely,

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