FACULTY
OF MATHEMATICS
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DOCTORAL THESIS

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Classical operators of harmonic analysis in Orlicz spaces

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Study programme:	Mathematics
Study branch:	Mathematical Analysis

Prague 2018

To my beloved family

Title: Classical operators of harmonic analysis in Orlicz spaces

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Abstract: We deal with classical operators of harmonic analysis in Orlicz spaces such as the Hardy-Littlewood maximal operator, the Hardy-type integral operators, the maximal operator of fractional order, the Riesz potential, the Laplace transform, and also with Sobolev-type embeddings on open subsets of $\mathbb{R}^n$ or with respect to Frostman measures and, in particular, trace embeddings on the boundary. For each operator (in case of embeddings we consider the identity operator) we investigate the question of its boundedness from an Orlicz space into another. Particular attention is paid to the sharpness of the results. We further study the question of the existence of optimal Orlicz domain and target spaces and their description. The work consists of author's published and unpublished results compiled together with material appearing in the literature.

Keywords: Orlicz space, Sobolev embeddings, Hardy operator, Maximal operator, Riesz potential, Laplace transform, Optimality.

Contents

1. Introduction

In harmonic analysis and partial differential equations we often measure quality or size of a function f living on some domain $\Omega \subset \mathbb{R}^n$ by its affiliation to some function space. The notion of size differs from task to task resulting in a whole zoo of various function spaces. A basic example goes to Lebesgue spaces as a measurement for the integrability of an absolute value of a function raised to a power. The importance of Lebesgue spaces in functional analysis is unquestionable and they stand for one of the main items in the toolbox of functional analysis. Countless of properties of functions and operators have been obtained by the use of such simple-to-understand spaces, however the fine attributes cannot be often captured. For all, let us mention several, probably folklore, examples.

It is very well known that the Hardy-Littlewood maximal operator M, given for locally integrable f by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \,\mathrm{d}y \quad \text{for } x \in \Omega,$$

where the supremum is taken over all cubes $Q \subseteq \Omega$ containing x, acts boundedly from $L^p(\Omega)$ to $L^p(\Omega)$ if and only if $p > 1$. If the measure of Ω is infinite, Mf is never integrable unless f vanishes almost everywhere. However, for the finite measure sets Ω, the integrability of Mf on Ω is achievable and it cannot be governed by Lebesgue spaces. Turning just several pages in a function spaces cookbook may help us to remedy this issue by the use of Orlicz spaces. They refine the scale of Lebesgue spaces by considering wider class of operations applied to f before the integrability is tested. Thus, by the result of [64], $Mf \in L^1(\Omega)$ if and only if

$$\int_\Omega |f| \log_+ |f| < \infty,$$

which we rather write in the form of an embedding

$$M \colon L \log L(\Omega) \to L^1(\Omega).$$

Our second example concerns a famous phenomenon – the gain of integrability. Let Ω be a subset of $\mathbb{R}^n$ having finite measure and let u be a function defined on Ω and vanishing on the boundary $\partial\Omega$. Suppose that u is weakly differentiable and u, $|\nabla u| \in L^p(\Omega)$. The famous Sobolev inequality tells us that if $1 \leq p < n$, then

$$u \in L^{\frac{np}{n-p}}(\Omega),$$

revealing the improvement of integrability, since $\frac{np}{n-p}$ enlarges p. We usually write

$$W^1_0 L^p(\Omega) \to L^{\frac{np}{n-p}}(\Omega)$$

instead. If $p > n$ then even $u \in C(\Omega)$ and we are above the integrability scale. What remains is the most interesting case $p = n$. Working with Lebesgue spaces only, Sobolev inequality gives that $u \in L^q(\Omega)$ for every $q < \infty$, however u might be unbounded, leaving us with an open set of Lebesgue spaces with no best possible choice. Nevertheless, Orlicz spaces save the day. With the help of an exponential space, one has

$$W^1_0 L^n(\Omega) \to \exp L^{\frac{n}{n-1}}(\Omega)$$

and it turns out that the target space cannot be improved to any other Orlicz space.

These examples indicate that Orlicz and Orlicz-Sobolev spaces provide eligible framework for fine description of operators and embeddings. When we add that the Orlicz spaces are fairly easy to understand, it is no surprise that they had found its immediate applications in PDEs and variational tasks.

In this thesis we deal with some classical operators of harmonic analysis in Orlicz spaces. Apart from the maximal operator we investigate the Hardy-type integral operators, Sobolev-type embeddings, maximal operator of fractional order, Riesz potential and Laplace transform. Each operator is treated in its separate chapter. For each operator T from our list we ask the same basic questions:

- For which pairs of Orlicz spaces $L^A(\Omega)$ and $L^B(\Omega)$ the embedding

$$T\colon L^A(\Omega) \to L^B(\Omega) \tag{1}$$

  holds?

- Given an Orlicz space $L^A(\Omega)$, is there a best possible $L^B(\Omega)$ that renders (1) true? How does it look like?

- Given an Orlicz space $L^B(\Omega)$, is there an optimal $L^A(\Omega)$ such that (1) holds? What is $L^A(\Omega)$ eventually?

Some answers were given earlier in the literature and became part of a folklore knowledge, while others remain without satisfactory solution. We would like to collect as much material as possible to present a comprehensive exposition about these classical operators in Orlicz spaces. We explain all the methods used, we uncover weak spots in this theory and we suggest several directions in which the research might continue.

It should be mentioned that many results in this work are not due to the author. We compile known results available in the literature together with either published or unpublished results of the author. The details about the origins are mentioned at the relevant places and the brief history of the objectives in question is appended.

2. Background and preliminaries

2.1 Young functions

We call $A\colon [0,\infty) \to [0,\infty]$ a Young function if it is convex, left-continuous, and $A(0) = 0$. Any function of this kind can be expressed in the form

$$A(t) = \int_0^t a(s)\,\mathrm{d}s \quad \text{for } t \geq 0 \tag{2.1.1}$$

in which $a\colon [0,\infty) \to [0,\infty]$ is a non-decreasing function obeying

$$A(t) \leq t\,a(t) \leq A(2t) \quad \text{for } t \geq 0. \tag{2.1.2}$$

Every Young function satisfies, in particular,

$$kA(t) \leq A(kt) \quad \text{if } k \geq 1 \text{ and } t \geq 0. \tag{2.1.3}$$

The Young conjugate $\widetilde{A}$ of A is given by

$$\widetilde{A}(t) = \sup\{st - A(s) : s \geq 0\} \quad \text{for } t \geq 0$$

and satisfies

$$\widetilde{A}(t) = \int_0^t a^{-1}(s)\,\mathrm{d}s \quad \text{for } t \geq 0, \tag{2.1.4}$$

where a^{-1} is the generalised left-continuous inverse of a. The function $\widetilde{A}$ is a Young function as well, and its Young conjugate is again A. One has that

$$t \leq A^{-1}(t)\,\widetilde{A}^{-1}(t) \leq 2t \quad \text{for } t \geq 0, \tag{2.1.5}$$

where A^{-1} denotes the generalized right-continuous inverse of A. The function B, defined as $B(t) = cA(bt)$, where b, c are positive constants, is also a Young function and

$$\widetilde{B}(t) = c\widetilde{A}\Big(\tfrac{t}{bc}\Big) \quad \text{for } t \geq 0. \tag{2.1.6}$$

A Young function A is said to satisfy the Δ_2-condition near infinity [resp. near zero] [resp. globally] if it is finite-valued and there exist constants $c > 0$ and $t_0 > 0$ such that

$$A(2t) \leq cA(t) \quad \text{for } t \geq t_0 \quad [0 \leq t \leq t_0] \quad [t \geq 0].$$

A Young function A is said to dominate another Young function B near infinity [near zero] [globally] if there exist constants $c > 0$ and $t_0 > 0$ such that

$$B(t) \leq A(ct) \quad \text{for } t \geq t_0 \quad [0 \leq t \leq t_0] \quad [t \geq 0].$$

The functions A and B are called equivalent near infinity [near zero] [globally] if they dominate each other near infinity [near zero] [globally]. We write $A \approx B$.

More generally, the terminology "near infinity", "near zero", "globally" will be adopted to indicate that some property of a function of t holds for $t \geq t_0$, for $0 \leq t \leq t_0$ or for $t \geq 0$, respectively.

2.2 Boyd indices

Given a Young function A, we define the function $h_A^\infty\colon (0,\infty) \to [0,\infty)$ as

$$h_A^\infty(t) = \sup_{s>0} \frac{A^{-1}(st)}{A^{-1}(s)} \quad \text{for } t > 0.$$

The global lower and upper Boyd indices of A are then defined as

$$i_A^\infty = \sup_{1<t<\infty} \frac{\log t}{\log h_A^\infty(t)} \quad \text{and} \quad I_A^\infty = \inf_{0<t<1} \frac{\log t}{\log h_A^\infty(t)}\,, \tag{2.2.1}$$

respectively. One has that

$$1 \le i_A^\infty \le I_A^\infty \le \infty. \tag{2.2.2}$$

It can also be shown that

$$i_A^\infty = \lim_{t\to\infty} \frac{\log t}{\log h_A^\infty(t)} \quad \text{and} \quad I_A^\infty = \lim_{t\to 0^+} \frac{\log t}{\log h_A^\infty(t)}. \tag{2.2.3}$$

The Boyd indices of A admit an alternate expression, that does not call into play A^{-1}, provided that A is finite-valued. Define $\widehat{h}_A^\infty\colon (0,\infty) \to [0,\infty)$ as

$$\widehat{h}_A^\infty(t) = \sup_{s>0} \frac{A(st)}{A(s)} \quad \text{for } t > 0.$$

Then,

$$i_A^\infty = \sup_{0<t<1} \frac{\log \widehat{h}_A^\infty(t)}{\log t} \quad \text{and} \quad I_A^\infty = \inf_{1<t<\infty} \frac{\log \widehat{h}_A^\infty(t)}{\log t}. \tag{2.2.4}$$

Furthermore, the supremum and infimum in (2.2.4) can be replaced with the limits as $t \to 0^+$ and $t \to \infty$, respectively.

The local lower and upper Boyd indices i_A and I_A of A are defined as in (2.2.1), with h_A^∞ replaced by the function $h_A\colon (0,\infty) \to [0,\infty]$ given by

$$h_A(t) = \limsup_{s\to\infty} \frac{A^{-1}(st)}{A^{-1}(s)} \quad \text{for } t > 0.$$

Properties parallel to (2.2.2) and (2.2.3) hold, with i_A^∞ and I_A^∞ replaced by i_A and I_A. Moreover, on defining $\widehat{h}_A\colon (0,\infty) \to [0,\infty)$ as

$$\widehat{h}_A(t) = \limsup_{s\to\infty} \frac{A(st)}{A(s)} \quad \text{for } t > 0,$$

a version of equation (2.2.4) holds for i_A and I_A, with proper replacements, namely

$$i_A = \sup_{0<t<1} \frac{\log \widehat{h}_A(t)}{\log t} \quad \text{and} \quad I_A = \inf_{1<t<\infty} \frac{\log \widehat{h}_A(t)}{\log t}. \tag{2.2.5}$$

Observe that if the function $A^{-1}(t)\,t^{-\sigma}$ is equivalent globally [near infinity], up to multiplicative positive constants, to a non-decreasing function, for some $\sigma \in (0,1)$, then $I_A^\infty \le 1/\sigma$ [$I_A \le 1/\sigma$]. Similarly, if the function $A^{-1}(t)\,t^{-\sigma}$ is equivalent globally [near infinity] to a non-increasing function, then $i_A^\infty \ge 1/\sigma$ [$i_A \ge 1/\sigma$].

In the special case when $A(t) = t^p$ for some $p \ge 1$, one has that $i_A^\infty = I_A^\infty = p$; furthermore, if $A(t) = \infty$ for large t, then $i_A = I_A = \infty$.

We refer the reader to [10] for more details on the material of this section.

2.3 Orlicz spaces

Let $\mathcal{R}$ be a sigma-finite, non-atomic, measure space endowed with a measure ν. Denote by $\mathcal{M}(\mathcal{R})$ the space of real-valued ν-measurable functions in $\mathcal{R}$, and by $\mathcal{M}_+(\mathcal{R})$ the set of non-negative functions in $\mathcal{M}(\mathcal{R})$. Given a Young function A, the Orlicz space $L^A(\mathcal{R})$ is the collection of all functions $f \in \mathcal{M}(\mathcal{R})$ such that

$$\int_{\mathcal{R}} A\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}\nu(x) < \infty$$

for some $\lambda > 0$. The Orlicz space $L^A(\mathcal{R})$ is a Banach space endowed with the Luxemburg norm defined as

$$\|f\|_{L^A(\mathcal{R})} = \inf\left\{\lambda > 0 : \int_{\mathcal{R}} A\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}\nu(x) \le 1\right\}$$

for $f \in \mathcal{M}(\mathcal{R})$. The choice $A(t) = t^p$, with $1 \le p < \infty$, yields $L^A(\mathcal{R}) = L^p(\mathcal{R})$, the customary Lebesgue space. When $A(t) = 0$ for $t \in [0,1]$ and $A(t) = \infty$ for $t \in (1,\infty)$, one has that $L^A(\mathcal{R}) = L^\infty(\mathcal{R})$.

Let E be a non-negligible measurable subset of $\mathcal{R}$, and let χ_E denote its characteristic function. Then

$$\|\chi_E\|_{L^A(\mathcal{R})} = \frac{1}{A^{-1}\left(\frac{1}{\nu(E)}\right)}. \tag{2.3.1}$$

The fundamental function φ_A of $L^A(\mathcal{R})$ is defined as

$$\varphi_A(s) = \frac{1}{A^{-1}\left(\frac{1}{s}\right)} \quad \text{for } 0 < s < \nu(\mathcal{R}), \tag{2.3.2}$$

and $\varphi_A(0) = 0$. Owing to (2.3.1),

$$\varphi_A(s) = \|\chi_E\|_{L^A(\mathcal{R})}$$

for every set $E \subset \mathcal{R}$ such that $\nu(E) = s$. A Hölder type inequality in Orlicz spaces asserts that

$$\|g\|_{L^{\widetilde{A}}(\mathcal{R})} \le \sup_{f \in L^A(\mathcal{R})} \frac{\int_{\mathcal{R}} f(x)g(x)\,\mathrm{d}\nu(x)}{\|f\|_{L^A(\mathcal{R})}} \le 2\|g\|_{L^{\widetilde{A}}(\mathcal{R})} \tag{2.3.3}$$

for every $g \in L^{\widetilde{A}}(\mathcal{R})$.

The inclusion relations between Orlicz spaces can be characterized in terms of the notion of domination between Young functions. Assume that $\nu(\mathcal{R}) < \infty$ [$\nu(\mathcal{R}) = \infty$], and let A and B be Young functions. Then

$$L^A(\mathcal{R}) \to L^B(\mathcal{R}) \text{ if and only if } A \text{ dominates } B \text{ near infinity [globally].} \tag{2.3.4}$$

The alternate notation $A(L)(\mathcal{R})$ for the Orlicz space $L^A(\mathcal{R})$ will be adopted when convenient. In particular, if $\nu(\mathcal{R}) < \infty$, and $A(t)$ is equivalent to $t^p(\log(1+t))^\alpha$ near infinity, where either $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \ge 0$, then the Orlicz space $L^A(\mathcal{R})$ is the so-called Zygmund space denoted by $L^p(\log L)^\alpha(\mathcal{R})$.

It holds that $i_A = I_A = p$. Orlicz spaces of exponential type are denoted by $\exp L^\beta(\mathcal{R})$, and are built upon the Young function $A(t) = e^{t^\beta} - 1$, with $\beta > 0$. We have $i_A = I_A = \infty$.

If $\nu(\mathcal{R}) = \infty$, and $A(t)$ is equivalent to $t^p\, \ell(t)^{\alpha_0}$ near zero and to $t^p\, \ell(t)^{\alpha_\infty}$ near infinity where $\ell(t) = 1 + |\log t|$, $t > 0$, and either $p > 1$ and α_0, $\alpha_\infty \in \mathbb{R}$ or $p = 1$ and $\alpha_0 \leq 0$ and $\alpha_\infty \geq 0$, then the Orlicz space $L^A(\mathbb{R}^n)$, also called Zygmund space, is denoted by $L^p(\log L)^{\mathbb{A}}(\mathbb{R}^n)$, where $\mathbb{A} = [\alpha_0, \alpha_\infty]$. Observe that $i_A^\infty = I_A^\infty = p$. If

$$A(t) \text{ is equivalent to } \begin{cases} \exp(-t^{\beta_0}) & \text{near zero,} \\ \exp(t^{\beta_\infty}) & \text{near infinity,} \end{cases}$$

with $\beta_0 < 0 < \beta_\infty$, then Orlicz space $L^A(\mathbb{R}^n)$ of exponential type is now denoted by $\exp L^{\mathbb{B}}(\mathbb{R}^n)$ where $\mathbb{B} = [\beta_0, \beta_\infty]$. We have $i_A^\infty = I_A^\infty = \infty$ in this case.

2.4 Endpoint spaces

Let $f \in \mathcal{M}(\mathcal{R})$. The distribution function of f is denoted by μ_f and is defined by

$$\mu_f(t) = \nu(\{x \in \mathcal{R} : |f(x)| > t\}) \quad \text{for } t > 0.$$

Then $f^*\colon [0,\infty) \to [0,\infty]$ denotes the decreasing rearrangement of f given by

$$f^*(s) = \inf\{t > 0 : \mu_f(t) \leq s\} \quad \text{for } s \geq 0.$$

An important property of rearrangements is the Hardy-Littlewood inequality [9, Chapter 2, Theorem 2.2], which asserts that, if $f, g \in \mathcal{M}(\mathcal{R})$, then

$$\int_{\mathcal{R}} fg \,\mathrm{d}\nu \leq \int_0^\infty f^*(t) g^*(t) \,\mathrm{d}t. \tag{2.4.1}$$

We denote by $M^A(\mathcal{R})$ the weak Orlicz space associated with A, namely the Marcinkiewicz type space endowed with the norm obeying

$$\|f\|_{M^A(\mathcal{R})} = \sup_{0<t<\nu(\mathcal{R})} \frac{f^{**}(t)}{A^{-1}(\frac{1}{t})}$$

for $f \in \mathcal{M}(\mathcal{R})$. Here, $f^{**}\colon (0,\infty) \to [0,\infty]$ is the function defined as

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \,\mathrm{d}s \quad \text{for } t > 0.$$

Since $f^*(t) \leq f^{**}(t)$ for $t > 0$,

$$\sup_{0<t<\nu(\mathcal{R})} \frac{f^*(t)}{A^{-1}(\frac{1}{t})} \leq \|f\|_{M^A(\mathcal{R})} \tag{2.4.2}$$

for every $f \in \mathcal{R}$.

We denote by $\Lambda^A(\mathcal{R})$ the Lorentz endpoint space associated to A, i.e., the space endowed with the norm

$$\|f\|_{\Lambda^A(\mathcal{R})} = \int_0^{\nu(\mathcal{R})} f^*(t) \,\mathrm{d}\varphi_A(t). \tag{2.4.3}$$

Here, the integral stands for a Lebesgue-Stieltjes integral associated with the fundamental function φ_A given by (2.3.2).

The Orlicz space built upon a Young function A obeys

$$\Lambda^A \to L^A(\mathcal{R}) \to M^A(\mathcal{R}) \tag{2.4.4}$$

for whatever A is. Moreover,

$$\|\chi_E\|_{\Lambda^A(\mathcal{R})} = \|\chi_E\|_{M^A(\mathcal{R})} = \varphi_A(\nu(E)) \tag{2.4.5}$$

for every measurable set $E \subset \mathcal{R}$.

The associate space to Marcinkiewicz space $M^A(\mathcal{R})$ corresponds to the Lorentz space $\Lambda^{\widetilde{A}}$ in which $\widetilde{A}$ is the Young conjugate function to A from (2.1.6). More specifically,

$$\|f\|_{\Lambda^{\widetilde{A}}(\mathcal{R})} \simeq \sup_{g \in M^A(\mathcal{R})} \frac{\int_{\mathcal{R}} f(x)g(x)\,\mathrm{d}\nu(x)}{\|g\|_{M^A(\mathcal{R})}} \tag{2.4.6}$$

for $f \in \mathcal{M}(\mathcal{R})$. Here, and in what follows, the relation $\simeq$ between two expressions means that they are bounded by each other, up to multiplicative positive constants independent of the involved relevant variables.

Analogously, $M^{\widetilde{A}}(\mathcal{R})$ represents the associate space to $\Lambda^A(\mathcal{R})$ and

$$\|f\|_{M^{\widetilde{A}}(\mathcal{R})} \simeq \sup_{g \in \Lambda^A(\mathcal{R})} \frac{\int_{\mathcal{R}} f(x)g(x)\,\mathrm{d}\nu(x)}{\|g\|_{\Lambda^A(\mathcal{R})}}. \tag{2.4.7}$$

2.5 Lorentz spaces

Let $1 \leq p, q \leq \infty$. The Lorentz $L^{p,q}(\mathcal{R})$ space is the set of all functions $f \in \mathcal{M}(\mathcal{R})$ such that the functional

$$\|f\|_{L^{p,q}(\mathcal{R})} = \left\| s^{\frac{1}{p}-\frac{1}{q}} f^*(s) \right\|_{L^q(0,\infty)}$$

is finite. If either $1 < p < \infty$ and $1 \leq q \leq \infty$ or $p = q = 1$ or $p = q = \infty$, then $\|\cdot\|_{L^{p,q}(\mathcal{R})}$ is equivalent to a norm. If $p = \infty$ and $q < \infty$, then $L^{p,q}(\mathcal{R}) = \{0\}$ and if $p = 1$ and $q > 1$, $L^{p,q}(\mathcal{R})$ is a quasi-normed space.

The Lorentz spaces generalize the Lebesgue spaces in a sense that $L^{p,p}(\mathcal{R}) = L^p(\mathcal{R})$ For every $p \in [1, \infty]$. A simple nesting property holds in Lorentz spaces. If $p, q, r \in [1, \infty]$ and $q \leq r$, then

$$L^{p,q}(\mathcal{R}) \to L^{p,r}(\mathcal{R}) \tag{2.5.1}$$

and all spaces on this scale share the one fundamental function, namely we have

$$\|\chi_E\|_{L^{p,q}(\mathcal{R})} \simeq \nu(E)^{\frac{1}{p}} \quad \text{for all } 1 \leq q \leq \infty \tag{2.5.2}$$

and every measurable $E \subseteq \mathcal{R}$. The spaces $L^{p,1}(\mathcal{R})$ and $L^{p,\infty}(\mathcal{R})$ coincide with the endpoint spaces $\Lambda^A(\mathcal{R})$ and $M^A(\mathcal{R})$, respectively, in which $A(t) = t^p$.

2.6 Rearrangement-invariant spaces

Let $\mathcal{R}$ and $\mathcal{S}$ be two σ-finite non-atomic measure spaces equipped with norms ν and μ, respectively. We say that functions $f \in \mathcal{M}(\mathcal{R})$ and $g \in \mathcal{M}(\mathcal{S})$ are equimeasurable, and write $f \sim g$, if $f^* = g^*$ on $(0, \infty)$.

A functional $\varrho \colon \mathcal{M}_+(\mathcal{R}) \to [0, \infty]$ is called a Banach function norm if, for all f, g and $\{f_j\}_{j\in\mathbb{N}}$ in $\mathcal{M}_+(\mathcal{R})$, and every $\lambda \geq 0$, the following properties hold:

(P1) $\varrho(f) = 0$ if and only if $f = 0$; $\varrho(\lambda f) = \lambda\varrho(f)$; $\varrho(f + g) \leq \varrho(f) + \varrho(g)$;

(P2) $f \leq g$ a.e. implies $\varrho(f) \leq \varrho(g)$;

(P3) $f_j \nearrow f$ a.e. implies $\varrho(f_j) \nearrow \varrho(f)$;

(P4) $\varrho(\chi_E) < \infty$ for every $E \subseteq \mathcal{R}$ of finite measure;

(P5) If E is a subset of $\mathcal{R}$ of finite measure, then $\int_E f \,\mathrm{d}\mu \leq C_E \varrho(f)$ for some constant C_E, $0 < C_E < \infty$, depending on E and ϱ but independent of f.

If, in addition, ϱ satisfies

(P6) $\varrho(f) = \varrho(g)$ whenever $f^* = g^*$,

then we say that ϱ is a rearrangement-invariant (or r.i., for short) norm.

If ϱ is a rearrangement-invariant function norm, then the collection

$$X(\mathcal{R}) = X_\varrho(\mathcal{R}) = \{f \in \mathcal{M}(\mathcal{R}) : \varrho(|f|) < \infty\}$$

is called a rearrangement-invariant space corresponding to the norm ϱ. We shall write $\|f\|_X$ instead of $\varrho(|f|)$.

With any rearrangement-invariant function norm ϱ is associated another functional, ϱ', defined for $g \in \mathcal{M}_+(\mathcal{R})$ as

$$\varrho'(g) = \sup\left\{\int_{\mathcal{R}} fg \,\mathrm{d}\nu : f \in \mathcal{M}_+(\mathcal{R}),\ \varrho(f) \leq 1\right\}.$$

It turns out that ϱ' is also a rearrangement-invariant norm, which is called the associate norm of ϱ. The function space $X'(\mathcal{R}) = X_{\varrho'}(\mathcal{R})$ determined by ϱ' is called the associate space of X Note that $(X')' = X$ always holds.

The Hölder inequality in r.i. spaces takes the form

$$\int_{\mathcal{R}} fg \,\mathrm{d}\nu \leq \|f\|_{X(\mathcal{R})}\|g\|_{X'(\mathcal{R})}$$

for every $f, g \in \mathcal{M}(\mathcal{R})$. Such inequality is sharp in a sense that

$$\|f\|_{X(\mathcal{R})} = \sup\left\{\int_{\mathcal{R}} fg \,\mathrm{d}\nu : \|f\|_{X'(\mathcal{R})} \leq 1\right\}. \tag{2.6.1}$$

as follows from [9, Chapter 1, Theorem 2.9].

For every r.i. space $X(\mathcal{R})$ there exists a unique r.i. space $X(0, \nu(\mathcal{R}))$ over the interval $(0, \nu(\mathcal{R}))$ endowed with the one-dimensional Lebesgue measure such that

$$\|f\|_{X(\mathcal{R})} = \|f^*\|_{X(0,\nu(\mathcal{R}))}. \tag{2.6.2}$$

This space is called the representation space of X. This follows from the Luxemburg representation theorem [9, Chapter 2, Theorem 4.10]. If X and Y are two r.i. spaces over $\mathcal{R}$, then, by [9, Theorem 1.8, Chapter 1],

$$X(\mathcal{R}) \subseteq Y(\mathcal{R}) \quad \text{implies} \quad X(\mathcal{R}) \to Y(\mathcal{R}). \tag{2.6.3}$$

It is easy to observe that Orlicz, Marcinkiewicz and Lorentz spaces are r.i. spaces.

2.7 Orlicz-Sobolev spaces

Let $n \in \mathbb{N}$, $n \geq 2$, and let Ω be an open subset of $\mathbb{R}^n$. Given $m \in \mathbb{N}$ and a Young function A, the m-th order Orlicz-Sobolev space built upon A is defined by

$$W^{m,A}(\Omega) = \Big\{u \in \mathcal{M}(\Omega) : u \text{ is } m\text{-times weakly differentiable in } \Omega, \text{ and } \\ |\nabla^k u| \in L^A(\Omega),\ k = 0, 1, \dots, m\Big\}.$$

Here, $\nabla^k u$ denotes the vector of all k-th order weak derivatives of u and $\nabla^0 u = u$. One has that $W^{m,A}(\Omega)$ is a Banach space equipped with the norm defined as

$$\|u\|_{W^{m,A}(\Omega)} = \sum_{k=0}^{m} \|\nabla^k u\|_{L^A(\Omega)}$$

for $u \in W^{m,A}(\Omega)$. By $W_0^{m,A}(\Omega)$ we denote the subspace of $W^{m,A}(\Omega)$ of those functions u in Ω whose continuation by 0 outside Ω belongs to $W^{m,A}(\mathbb{R}^n)$. The notations $W^m L^A(\Omega)$ and $W^m A(L)(\Omega)$ will also be occasionally adopted instead of $W^{m,A}(\Omega)$; analogous alternate notations will be used for $W_0^{m,A}(\Omega)$.

If $|\Omega| < \infty$, an iterated use of a Poincaré type inequality in Orlicz spaces [69, Lemma 3] ensures that the functional

$$\|\nabla^m u\|_{L^A(\Omega)}$$

defines a norm on $W_0^{m,A}(\Omega)$ equivalent to $\|u\|_{W^{m,A}(\Omega)}$.

As in the case of Orlicz spaces, inclusion relations between Orlicz-Sobolev spaces can be described in terms of domination between the defining Young functions A and B. If $|\Omega| < \infty$, then

$$W^{m,A}(\Omega) \to W^{m,B}(\Omega) \quad \Big[W_0^{m,A}(\Omega) \to W_0^{m,B}(\Omega)\Big] \\ \text{if and only if } A \text{ dominates } B \text{ near infinity.} \tag{2.7.1}$$

On the other hand,

$$W^{m,A}(\mathbb{R}^n) \to W^{m,B}(\mathbb{R}^n) \quad \text{if and only if } A \text{ dominates } B \text{ globally.} \tag{2.7.2}$$

A proof of assertions (2.7.1) and (2.7.2) seems not to be available in the literature. We sketch a proof in Proposition 2.7.1 at the end of this section

Sobolev and trace embeddings for functions with unrestricted boundary values require some regularity on the ground domain. The class of John domains is known to be essentially the largest where Sobolev type embeddings hold in their strongest form. A bounded open set $\Omega \subset \mathbb{R}^n$ is called a John domain if there exist a constant $c \in (0, 1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi \colon [0, l] \to \Omega$, with $l > 0$, parametrized by arclength, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\operatorname{dist}\Big(\varpi(r), \partial\Omega\Big) \geq cr \quad \text{for } r \in [0, l].$$

The class of John domains includes classical families of open sets, such as that of bounded Lipschitz domains, and that of domains with the cone property. Recall that a bounded open set Ω is said to have the cone property if there exists a finite circular cone Λ such that each point in Ω is the vertex of a finite cone contained in Ω and congruent to Λ.

Proposition 2.7.1. *Assume that $m, n \in \mathbb{N}$. Let A and B be Young functions.*
(i) *If Ω is an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$, then*

$$W^{m,A}(\Omega) \to W^{m,B}(\Omega) \quad \textit{if and only if } A \textit{ dominates } B \textit{ near infinity,} \tag{2.7.3}$$

and

$$W_0^{m,A}(\Omega) \to W_0^{m,B}(\Omega) \quad \textit{if and only if } A \textit{ dominates } B \textit{ near infinity.} \tag{2.7.4}$$

(ii)

$$W^{m,A}(\mathbb{R}^n) \to W^{m,B}(\mathbb{R}^n) \quad \textit{if and only if } A \textit{ dominates } B \textit{ globally.} \tag{2.7.5}$$

Proof. The "if" parts of assertions (2.7.3)–(2.7.5) are straightforward consequences of (2.3.4). The reverse implications in (2.7.3) and (2.7.4) can be verified as follows. Assume that $|\Omega| < \infty$, and

$$W^{m,A}(\Omega) \to W^{m,B}(\Omega) \quad \text{or} \quad W_0^{m,A}(\Omega) \to W_0^{m,B}(\Omega). \tag{2.7.6}$$

Suppose, without loss of generality, that $0 \in \Omega$. Let $\delta > 0$ be so small that the cube Q centered at 0, whose sides are parallel to the coordinates axes and have length 2δ, is contained in Ω. Given any function $f \in L^A(-\delta, \delta)$, define the function $v\colon Q \to \mathbb{R}$ as

$$v(x) = \int_0^{x_1} \int_0^{s_1} \cdots \int_0^{s_{m-1}} f(s_m)\, \mathrm{d}s_m\, \mathrm{d}s_{m-1} \dots \mathrm{d}s_1 \quad \text{for } x \in Q,$$

where we have adopted the notation $x = (x_1, x_2, \dots, x_n)$. The function v is m-times weakly differentiable in Q. Moreover, $\frac{\partial^k v}{\partial x_1^k} \in L^\infty(Q)$ if $1 \le k \le m-1$, $\frac{\partial^m v}{\partial x_1^m}(x) = f(x_1)$ for $x \in Q$, and any other derivative vanishes identically. Hence, $v \in W^{m,A}(Q)$. By [33, Theorem 4.1], there exists a bounded linear extension operator $\mathcal{E}\colon W^{m,A}(Q) \to W^{m,A}(\mathbb{R}^n)$. Fix any function $\eta \in C_0^\infty(\Omega)$ such that $\eta = 1$ in Q. Define $u\colon \Omega \to \mathbb{R}$ as

$$u = \eta\, \mathcal{E}(v). \tag{2.7.7}$$

Then $u \in W^{m,A}(\Omega)$, and, in fact, $u \in W_0^{m,A}(\Omega)$. By either of embeddings (2.7.6), $u \in W^{m,B}(\Omega)$ as well, and hence $f \in L^B(-\delta, \delta)$. Owing to the arbitrariness of f, this implies that $L^A(-\delta, \delta) \subset L^B(-\delta, \delta)$, and by [9, Theorem 1.8, Chapter 1], in fact $L^A(-\delta, \delta) \to L^B(-\delta, \delta)$. Hence, by (2.3.4), A dominates B near infinity.

As far as the "only if" part of assertion (2.7.5) is concerned, the choice of trial functions u as in (2.7.7) implies that A dominates B near infinity also when $\Omega = \mathbb{R}^n$. On the other hand, if embedding (2.7.5) is in force, then, in particular,

$$W^{m,A}(\mathbb{R}^n) \to L^B(\mathbb{R}^n),$$

whence A dominates B also near zero, by (4.2.6). Therefore, A dominates B globally. $\square$

2.8 Operator properties

Let $X(\mathcal{R})$ and $Y(\mathcal{S})$ be Banach function spaces over measure spaces $\mathcal{R}$ and $\mathcal{S}$ endowed with measures ν and μ, respectively. Given a linear operator $T\colon X(\mathcal{R}) \to Y(\mathcal{S})$, its adjoint operator $T'\colon Y'(\mathcal{S}) \to X'(\mathcal{R})$ is defined by the identity

$$\int_{\mathcal{S}} (Tf)g \,\mathrm{d}\mu = \int_{\mathcal{R}} (T'g)f \,\mathrm{d}\nu \tag{2.8.1}$$

for every $f \in X(\mathcal{R})$ and $g \in Y'(\mathcal{S})$, whenever the integrals converge. Moreover, by (2.6.1),

$$\begin{aligned} \|T'\| &= \sup_{\|g\|_{Y'(\mathcal{S})}\le 1} \|T'g\|_{X'(\mathcal{R})} = \sup_{\|g\|_{Y'(\mathcal{S})}\le 1} \sup_{\|f\|_{X(\mathcal{R})}\le 1} \int_{\mathcal{R}} (T'g)f \,\mathrm{d}\nu \\ &= \sup_{\|f\|_{X(\mathcal{R})}\le 1} \sup_{\|g\|_{Y'(\mathcal{S})}\le 1} \int_{\mathcal{S}} (Tf)g \,\mathrm{d}\mu = \sup_{\|f\|_{X(\mathcal{R})}\le 1} \|Tf\|_{Y(\mathcal{S})} = \|T\|. \end{aligned} \tag{2.8.2}$$

Let E_s denote the dilation operator defined for each $s \in (0,\infty)$ and every non-negative measurable function f on $(0,\infty)$ by

$$E_s f(t) = f(st) \quad \text{for } t \ge 0.$$

Then the operator E_s is bounded on every r.i. space over $(0,\infty)$, and

$$\|E_s f\|_{X(0,\infty)} \le \max\{1, \tfrac{1}{s}\} \, \|f\|_{X(0,\infty)} \quad \text{for } s > 0 \tag{2.8.3}$$

and for every $f \in X(0,\infty)$. See [9, Chapter 3, Proposition 5.11].

The following proposition enables us to reduce an embedding to a Lorentz endpoint spaces only to testing on characteristic functions. The idea of this statement is based on [17, Theorem 7], where the Lorentz space $L^{p,1}$ occurs as a target space, nonetheless the proof also works for any Lorentz endpoint space. For the sake of completeness, we show also the proof here.

Proposition 2.8.1. *Let $Y(\mathcal{R})$ be a Banach function space and $\Lambda(\mathcal{R})$ be a Lorentz endpoint space over $\mathcal{R}$ and let T be a sublinear operator mapping $\Lambda(\mathcal{R})$ to $Y(\mathcal{R})$. Suppose that there is a $C > 0$ such that*

$$\|T\chi_E\|_{Y(\mathcal{R})} \le C\|\chi_E\|_{\Lambda(\mathcal{R})} \tag{2.8.4}$$

for every measurable set $E \subseteq \mathcal{R}$. Then

$$\|Tf\|_{Y(\mathcal{R})} \le C\|f\|_{\Lambda(\mathcal{R})}$$

for every $f \in \Lambda(\mathcal{R})$.

Proof. Let f be a simple non-negative function on $\mathcal{R}$. Thus f can be written as a finite sum $f = \sum_j \lambda_j \chi_{E_j}$, where λ_j are positive real numbers and the sets E_j are measurable subsets of $\mathcal{R}$ satisfying $E_1 \subseteq E_2 \subseteq \cdots$. Then, as readily seen, we have $f^* = \sum_j \lambda_j \chi^*_{E_j}$. Let φ be a fundamental function of $\Lambda(\mathcal{R})$. By the definition of the Lorentz norm we have

$$\|f\|_{\Lambda(\mathcal{R})} = \int_0^\infty f^* \,\mathrm{d}\varphi = \int_0^\infty \sum_j \lambda_j \chi^*_{E_j} \,\mathrm{d}\varphi = \sum_j \lambda_j \int_0^\infty \chi^*_{E_j} \,\mathrm{d}\varphi = \sum_j \lambda_j \|\chi_{E_j}\|_{\Lambda(\mathcal{R})}.$$

On account of the sublinearity of T we have $|Tf| \leq \sum_j \lambda_j |T\chi_{E_j}|$, and consequently by (2.8.4) and by axioms (P1) and (P2) we obtain

$$\|Tf\|_{Y(\mathcal{R})} \leq \sum_j \lambda_j \|T\chi_{E_j}\|_{Y(\mathcal{R})} \leq C \sum_j \lambda_j \|\chi_{E_j}\|_{\Lambda(\mathcal{R})} = \|f\|_{\Lambda(\mathcal{R})}.$$

Now if f is simple but no longer nonnegative, we use the same for the positive part of f and for the negative part of f.

Suppose that f is an arbitrary function in $\Lambda(\mathcal{R})$ and let f_n be a sequence of simple integrable functions converging to f in $\Lambda(\mathcal{R})$. Then

$$\|T(f_n) - T(f_m)\|_{Y(\mathcal{R})} \leq \|T(f_n - f_m)\|_{Y(\mathcal{R})} \leq C\|f_n - f_m\|_{\Lambda(\mathcal{R})},$$

and Tf_n is Cauchy, hence convergent in $Y(\mathcal{R})$. Since limits are unique in $Y(\mathcal{R})$, it follows that $\lim Tf_n = Tf$ and

$$\|Tf\|_{Y(\mathcal{R})} = \lim \|Tf_n\|_{Y(\mathcal{R})} \leq C \lim \|f_n\|_{\Lambda(\mathcal{R})} = \|f\|_{\Lambda(\mathcal{R})}$$

as we wished to show. □

3. Hardy operator

3.1 Introduction

This entire chapter is devoted to the study of the one-dimensional Hardy-type operator defined by

$$f \mapsto \int_{t^\beta}^{R} f(s)\, s^{\alpha-1}\,\mathrm{d}s, \quad (0 < t < R^{1/\beta}), \tag{3.1.1}$$

for any function $f \in \mathcal{M}(0,R)$ whenever the integral in (3.1.1) converges. Here $0 < \alpha < 1$, $\beta > 0$ and $\alpha + 1/\beta \geq 1$ and $R \in (0,\infty]$ is arbitrary.

Our aim is to study the action of the operator (3.1.1) between Orlicz spaces with the special attention to the sharpness of the results. More specifically, we first give a necessary and sufficient condition to Orlicz spaces L^A and L^B for which

$$\left\| \int_{t^\beta}^{R} f(s)\, s^{\alpha-1}\,\mathrm{d}s \right\|_{L^B(0,R^{1/\beta})} \leq C \|f\|_{L^A(0,R)} \tag{3.1.2}$$

holds true with a constant C independent of $f \in L^A(0,R)$. Secondly, we are concerned with the optimal form of (3.1.2) This task consists of two separate subtasks.

Suppose that L^A is given. One may to ask what is the smallest Orlicz target space L^B for which (3.1.2) holds. By "smallest" we mean such an Orlicz space L^B that if (3.1.2) holds with L^B replaced by $L^{\widehat{B}}$, then $L^B \to L^{\widehat{B}}$. On the other hand, assume that L^B is given and we are asked to find the largest Orlicz domain space L^A so that (3.1.2) is satisfied. Here, "largest" means that if (3.1.2) is true with $L^{\widehat{A}}$ in place of L^A, then $L^{\widehat{A}} \to L^A$. To simplify the notation, we often use a word "optimal" for both the above-mentioned notions.

The techniques for handling such operator may differ at some stage if R is ether finite or equal to infinity. Also, in the former, case the value of R is immaterial due to the use of the dilation operator. The details will be discussed later. For the sake of comprehensive notation, let us denote by $H_{\alpha,\beta}$ the operator (3.1.1) for $R = 1$ and by $H^\infty_{\alpha,\beta}$ the same operator with $R = \infty$. We then refer to

$$H^\infty_{\alpha,\beta}\colon L^A(0,\infty) \to L^B(0,\infty) \tag{3.1.3}$$

and

$$H_{\alpha,\beta}\colon L^A(0,1) \to L^B(0,1) \tag{3.1.4}$$

instead of (3.1.2).

The chapter is structured as follows. The preliminary section is devoted to the analysis of certain properties of Young functions and their connection with Boyd indices. We then introduce a characterization of the inequality (3.1.2) and we describe the optimal Orlicz target spaces. After that, we prove another variant of necessary and sufficient conditions to A and B to fulfill (3.1.2) and then we investigate the optimal Orlicz domain spaces.

3.2 Preliminaries

Let us slightly start with an easy calculation of Luxemburg norms of various truncated power functions.

Lemma 3.2.1. *Let E be a Young function and let ξ be a non-zero real number. Assuming*

$$\int_0 E(s)\, s^{\frac{1}{\xi}-1}\,\mathrm{d}s < \infty, \tag{3.2.1}$$

we define

$$F_\xi(t) = t^{-\frac{1}{\xi}} \int_0^t E(s)\, s^{\frac{1}{\xi}-1}\,\mathrm{d}s \quad \textit{for } t > 0.$$

Such F_ξ is a non-decreasing mapping of $(0,\infty)$ onto itself. Moreover, if $R \in (0,\infty]$, then the following relations hold.

$$\|t^\xi \chi_{(0,r)}(t)\|_{L^E(0,R)} = \frac{r^\xi}{F_\xi^{-1}\left(\frac{|\xi|}{r}\right)}, \quad r \in (0,R),\ \xi > 0, \tag{3.2.2}$$

$$\|t^\xi \chi_{(r,\infty)}(t)\|_{L^E(0,\infty)} = \frac{r^\xi}{F_\xi^{-1}\left(\frac{|\xi|}{r}\right)}, \quad r \in (0,\infty),\ \xi < 0. \tag{3.2.3}$$

Here F_ξ^{-1} denotes the generalised right-continuous inverse of F_ξ. If, in addition, $\varepsilon \in (0,R)$ and if $\xi < 0$ then

$$\|t^\xi \chi_{(r,R)}(t)\|_{L^E(0,R)} \simeq \|t^\xi \chi_{(r,\infty)}(t)\|_{L^E(0,\infty)}, \quad r \in (0, R-\varepsilon), \tag{3.2.4}$$

where the equivalence constant depends on E, ξ and ε.

Proof. Assume (3.2.1). By the change of variables $s \mapsto ts$ we have

$$F_\xi(t) = \int_0^1 E(ts)\, s^{\frac{1}{\xi}-1}\,\mathrm{d}s \quad \text{for } t > 0,$$

hence F_ξ is non-decreasing. By definition of the Luxemburg norm, we have

$$\|t^\xi \chi_{(0,r)}(t)\|_{L^E(0,R)} = \inf\left\{\lambda > 0 : \int_0^r E\left(\tfrac{t^\xi}{\lambda}\right)\,\mathrm{d}t \le 1\right\}.$$

If $\xi > 0$, we get, by the change of variables,

$$\begin{aligned}\|t^\xi \chi_{(0,r)}(t)\|_{L^E(0,R)} &= \inf\left\{\lambda > 0 : \frac{\lambda^{\frac{1}{\xi}}}{\xi} \int_0^{\frac{r^\xi}{\lambda}} E(s)\, s^{\frac{1}{\xi}-1}\,\mathrm{d}s \le 1\right\} \\ &= \inf\left\{\lambda > 0 : \frac{r}{\xi} F_\xi\left(\tfrac{r^\xi}{\lambda}\right) \le 1\right\} = \frac{r^\xi}{F_\xi^{-1}\left(\frac{\xi}{r}\right)}\end{aligned}$$

which proves (3.2.2). The proof of the relation (3.2.3) can be done in an analogous way and we omit it.

It remains to prove the (3.2.4). The inequality "$\le$" is obvious by the monotonicity of the norm. On the other hand, we have, by the triangle inequality,

$$\|t^\xi \chi_{(r,\infty)}(t)\|_{L^E(0,\infty)} \le \|t^\xi \chi_{(r,R)}(t)\|_{L^E(0,R)} + \|t^\xi \chi_{(R,\infty)}(t)\|_{L^E(0,\infty)}.$$

Using (3.2.3), the latter term $\|t^\xi \chi_{(R,\infty)}(t)\|_{L^E(0,\infty)}$ equals $R^\xi / E_\xi^{-1}(|\xi|/R)$ since $\xi < 0$. Thanks to the assumptions, this quantity is finite, say K. The former term $\|t^\xi \chi_{(r,R)}(t)\|_{L^E(0,R)}$ is a decreasing function of the variable r, positive on $(0, R)$ and vanishing at R. Hence for every $\varepsilon \in (0, R)$ there exists a constant C such that

$$K \le C\, \|t^\xi \chi_{(r,R)}(t)\|_{L^E(0,R)}, \quad r \in (0, R - \varepsilon).$$

and (3.2.4) follows. □

We continue with the following proposition, that collects various characterizations of pointwise and integral growth conditions of a Young function, and of its conjugate, in terms of their Boyd indices.

Proposition 3.2.2. *Let E be a finite-valued Young function, and let $0 < \alpha < 1$. The following conditions are equivalent.*

(i) *There exists a constant $k > 1$ such that*

$$\int_t^\infty \frac{E(s)}{s^{1/\alpha+1}}\, \mathrm{d}s \le \frac{E(kt)}{t^{1/\alpha}} \quad \textit{globally} \quad \textit{[near infinity]}.$$

(ii) *There exists a constant $k > 1$ such that*

$$\int_0^t \frac{\widetilde{E}(s)}{s^{1/(1-\alpha)+1}}\, \mathrm{d}s \le \frac{\widetilde{E}(kt)}{t^{1/(1-\alpha)}} \quad \textit{globally}$$

$$\left[\int_1^t \frac{\widetilde{E}(s)}{s^{1/(1-\alpha)+1}}\, \mathrm{d}s \le \frac{\widetilde{E}(kt)}{t^{1/(1-\alpha)}} \quad \textit{near infinity}\right].$$

(iii) *There exist constants $\sigma > 1$ and $c \in (0, 1)$ such that*

$$E(\sigma t) \le c\sigma^{\frac{1}{\alpha}} E(t) \quad \textit{globally} \quad \textit{[near infinity]}.$$

(iv) *There exist constants $\sigma > 1$ and $c > 1$ such that*

$$\widetilde{E}(\sigma t) \ge c\sigma^{\frac{1}{1-\alpha}} \widetilde{E}(t) \quad \textit{globally} \quad \textit{[near infinity]}.$$

(v) *The global [local] upper Boyd index of E satisfies*

$$I_E^\infty < 1/\alpha \qquad \left[I_E < 1/\alpha\right].$$

(vi) *The global [local] lower Boyd index of $\widetilde{E}$ satisfies*

$$i_{\widetilde{E}}^\infty > 1/(1-\alpha) \qquad \left[i_{\widetilde{E}} > 1/(1-\alpha)\right].$$

Proof. We shall prove the statement in the form “near infinity”. The proof of the global version is analogous - even simpler in fact - and will be omitted.

(i) is equivalent to (iii) This equivalence is stated in [67, Lemma 2.3. (ii)], without proof. We provide a proof here, for completeness. Assume that there

exist $k > 1$ and $t_0 > 0$ such that inequality (i) is fulfilled for every $t > t_0$. Fix $\sigma > 1$ and $t > t_0 k$, and let $\rho \in [1, \sigma]$ be such that

$$E(\rho t)\,(\rho t)^{-\frac{1}{\alpha}} = \inf_{t \le r \le \sigma t} E(r)\, r^{-\frac{1}{\alpha}}. \tag{3.2.5}$$

We claim that

$$\sigma \ge \rho \ge e^{-k^{\frac{1}{\alpha}}}\sigma. \tag{3.2.6}$$

The former inequality is part of the definition of ρ. As for the latter, we have that

$$E(\rho t)\,(\rho t)^{-\frac{1}{\alpha}} k^{\frac{1}{\alpha}} \ge \int_{\rho t/k}^{\infty} \frac{E(s)}{s^{1/\alpha+1}}\mathrm{d}s \ge \int_{\rho t}^{\sigma t} \frac{E(s)}{s^{1/\alpha+1}}\mathrm{d}s \ge E(\rho t)\,(\rho t)^{-\frac{1}{\alpha}} \log\Big(\frac{\sigma}{\rho}\Big),$$

whence the claim follows.

Next, we show that E satisfies the Δ_2-condition near infinity. Suppose, by contradiction, that for every $j \in \mathbb{N}$ there exists $t > t_0 k$ such that $E(2t) > jE(t)$. Choosing $\sigma = 2e^{k^{\frac{1}{\alpha}}}$, and ρ defined by (3.2.5), ensures that

$$E(t)t^{-\frac{1}{\alpha}} k^{\frac{1}{\alpha}} \ge \int_{t/k}^{\infty} \frac{E(s)}{s^{1/\alpha+1}}\mathrm{d}s \ge \int_{t}^{\sigma t} \frac{E(s)}{s^{1/\alpha+1}}\mathrm{d}s \ge E(\rho t)\,(\rho t)^{-\frac{1}{\alpha}} \log\sigma\,. \tag{3.2.7}$$

Hence,

$$E(2t)\, k^{\frac{1}{\alpha}} \ge jE(\rho t)\,\rho^{-\frac{1}{\alpha}} \log\sigma \ge jE(2t)\,\sigma^{-\frac{1}{\alpha}} \log\sigma,$$

since $\sigma \ge \rho \ge 2$, by (3.2.6). Therefore,

$$k^{\frac{1}{\alpha}} \ge j\sigma^{-\frac{1}{\alpha}} \log\sigma$$

for all $j \in \mathbb{N}$, which is impossible.

Now suppose that (iii) does not hold. Thus, for every $\sigma > 1$ and $c \in (0,1)$ there exists a sequence $\{t_j\}$ such that $t_j \to \infty$, and

$$E(\sigma t_j) > c\sigma^{\frac{1}{\alpha}} E(t_j) \tag{3.2.8}$$

for $j \in \mathbb{N}$. Let ρ be as in (3.2.5). By (3.2.8) and (3.2.7),

$$E(\sigma t_j)\,(\sigma t_j)^{-\frac{1}{\alpha}} k^{\frac{1}{\alpha}} > cE(t_j)\, t_j^{-\frac{1}{\alpha}} k^{\frac{1}{\alpha}} > cE(\rho t_j)\,(\rho t_j)^{-\frac{1}{\alpha}} \log\sigma. \tag{3.2.9}$$

From (3.2.9), (3.2.6) and the Δ_2-condition near infinity for E, we conclude that there exists a positive constant c_1 such that

$$E(\sigma t_j)\, k^{\frac{1}{\alpha}} > cE\Big(e^{-k^{\frac{1}{\alpha}}}\sigma t_j\Big) \log\sigma > c_1 E(\sigma t_j) \log\sigma$$

for sufficiently large j. Hence,

$$k^{\frac{1}{\alpha}} > c_1 \log\sigma$$

for arbitrarily large σ, a contradiction.

(iii) implies (i). Let $t_0 > 0$ be such that inequality (iii) holds for $t \ge t_0$. Let $j \in \mathbb{N}$. An iterative use of assumption (iii) ensures that

$$E(s) \le c^j \sigma^{\frac{j}{\alpha}} E(s\sigma^{-j}) \quad \text{for } s \ge \sigma^j t_0. \tag{3.2.10}$$

By (3.2.10), if $t \geq t_0$, then

$$\begin{aligned}
\int_t^\infty \frac{E(s)}{s^{1/\alpha+1}}\,\mathrm{d}s &= \sum_{j=0}^\infty \int_{t\sigma^j}^{t\sigma^{j+1}} \frac{E(s)}{s^{1/\alpha+1}}\,\mathrm{d}s \leq \sum_{j=0}^\infty c^j \int_{t\sigma^j}^{t\sigma^{j+1}} \sigma^{j/\alpha}\frac{E(s\sigma^{-j})}{s^{1/\alpha+1}}\mathrm{d}s \\
&= \sum_{j=0}^\infty c^j \int_t^{\sigma t} \frac{E(r)}{r^{1/\alpha+1}}\,\mathrm{d}r \leq \frac{1}{1-c}E(\sigma t)\int_t^{\sigma t} \frac{\mathrm{d}r}{r^{1/\alpha+1}} \\
&= \alpha\frac{1-\sigma^{1/\alpha}}{1-c}\frac{E(\sigma t)}{t^{1/\alpha}}\,.
\end{aligned}$$

Hence, (i) follows via property (2.1.3).

(iii) implies (v). Assume that (v) does not hold, i.e. $I_E \geq 1/\alpha$. By equation (2.2.5),

$$\frac{1}{\alpha} \leq \inf_{1<\sigma<\infty} \frac{\log \widehat{h}_E(\sigma)}{\log \sigma},$$

and hence $\sigma^{1/\alpha} \leq \widehat{h}_E(\sigma)$ for every $\sigma \geq 1$. Owing to the very definition of $\widehat{h}_E$,

$$\sigma^{\frac{1}{\alpha}} \leq \limsup_{t\to\infty} \frac{E(\sigma t)}{E(t)}.$$

Hence, for every $c \in (0,1)$ and $t_0 > 0$, there exists $t > t_0$ such that

$$c\sigma^{\frac{1}{\alpha}} < \frac{E(\sigma t)}{E(t)},$$

and this contradicts (iii).

(v) implies (iii). Assume, by contradiction, that (iii) fails. Thereby, for every $\sigma > 1$ and $c \in (0,1)$ there exists a sequence $t_j \to \infty$ satisfying

$$c\sigma^{\frac{1}{\alpha}}E(t_j) < E(\sigma t_j) \quad \text{for } j \in \mathbb{N}.$$

Thus

$$c\sigma^{\frac{1}{\alpha}} \leq \limsup_{j\to\infty} \frac{E(\sigma t_j)}{E(t_j)} \leq \limsup_{t\to\infty} \frac{E(\sigma t)}{E(t)} = \widehat{h}_E(\sigma)\,,$$

whence

$$\frac{\log\left(c\sigma^{1/\alpha}\right)}{\log\sigma} \leq \frac{\log \widehat{h}_E(\sigma)}{\log\sigma}\,.$$

Thanks to (2.2.5), passing to the limit as $\sigma \to \infty$ yields $1/\alpha \leq I_E$, thus contradicting (v).

(iii) is equivalent to (iv). Condition (iii) is equivalent to

$$E(\sigma t) \leq (c\sigma)^{\frac{1}{\alpha}}E(t) \tag{3.2.11}$$

for some constants $c \in (0,1)$ and $\sigma > 1$, and for sufficiently large t. Taking the Young conjugate of both sides, and making use of (2.1.6) tell us that (3.2.11) is in turn equivalent to

$$\widetilde{E}\left(t\sigma^{-1}\right) \geq (c\sigma)^{\frac{1}{\alpha}}\widetilde{E}\left(t(c\sigma)^{-\frac{1}{\alpha}}\right) \tag{3.2.12}$$

for large t. Setting $\varrho = c^{\frac{1}{\alpha}}\sigma^{\frac{1}{\alpha}-1}$, and changing variables, equation (3.2.12) reads

$$\widetilde{E}(\varrho t) \geq c^{\frac{1}{\alpha-1}}\varrho^{\frac{1}{1-\alpha}}\widetilde{E}(t)$$

for large t. Thus, it suffices to show that $\varrho > 1$. Combining (2.1.3) and (3.2.11) yields

$$\sigma E(t) \leq E(\sigma t) \leq (c\sigma)^{\frac{1}{\alpha}} E(t)$$

for large t, whence $\varrho \geq 1$. If $\varrho > 1$ we are done. On the other hand, $\varrho = 1$ if and only if $E(t) = t$ for large t, and the latter condition implies that $\widetilde{E} = \infty$ near infinity, so that (iv) is trivially satisfied.

The proof of the reverse implication is similar.

(ii) is equivalent to (iv). This is established in [67, Lemma 2.3 (i)].

(iv) is equivalent to (vi). The proof of this fact follows along the same lines as that of the equivalence of (iii) and (v), and will be omitted, for brevity. □

We finis this section by analysis of convergence of the integral of type (3.2.1). Our proof is based on [22, Lemma 2.3].

Proposition 3.2.3. *Let E be a Young function and let $0 < \alpha < 1$. Then*

$$\int^{\infty} \frac{\widetilde{E}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s < \infty \quad \textit{if and only if} \quad \int^{\infty}\left(\frac{s}{E(s)}\right)^{\frac{\alpha}{1-\alpha}}\mathrm{d}s < \infty \tag{3.2.13}$$

and

$$\int_0 \frac{\widetilde{E}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s < \infty \quad \textit{if and only if} \quad \int_0\left(\frac{s}{E(s)}\right)^{\frac{\alpha}{1-\alpha}}\mathrm{d}s < \infty \tag{3.2.14}$$

Proof. Let E and α be given and assume that $E(t) = \int_0^t e(s)\,\mathrm{d}s$ for $t \geq 0$. Then, by Fubini's theorem,

$$\begin{aligned}\int_1^{\infty} \frac{e^{-1}(s)}{s^{1/(1-\alpha)}}\,\mathrm{d}s &= \int_0^{e^{-1}(1)}\int_1^{\infty}\frac{\mathrm{d}s}{s^{1/(1-\alpha)}}\,\mathrm{d}r + \int_{e^{-1}(1)}^{\infty}\int_{e(r)}^{\infty}\frac{\mathrm{d}s}{s^{1/(1-\alpha)}}\,\mathrm{d}r\\ &= e^{-1}(1)\frac{1-\alpha}{\alpha} + \frac{1-\alpha}{\alpha}\int_{e^{-1}(1)}^{\infty}\left(\frac{1}{e(r)}\right)^{\frac{\alpha}{1-\alpha}}\mathrm{d}r.\end{aligned}$$

Now, by (2.1.1) and (2.1.4) with (2.1.2),

$$\int^{\infty}\frac{e^{-1}(s)}{s^{1/(1-\alpha)}}\,\mathrm{d}s \simeq \int_1^{\infty}\frac{\widetilde{E}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s$$

while

$$\int_{e^{-1}(1)}^{\infty}\left(\frac{1}{e(r)}\right)^{\frac{\alpha}{1-\alpha}}\mathrm{d}r \simeq \int^{\infty}\left(\frac{s}{E(s)}\right)^{\frac{\alpha}{1-\alpha}}\mathrm{d}s,$$

whence (3.2.13) follows. The proof of (3.2.14) is analogous. □

3.3 First reduction principle in Orlicz spaces

Suppose that $0 < \alpha < 1$, $\beta > 0$ and $\alpha + 1/\beta \geq 1$ and let A be a given Young function. Denote by $t_0 \in [0, \infty)$ a point such that the integral

$$\int_{t_0} \left(\frac{t}{A(t)} \right)^{\frac{\alpha}{1-\alpha}} \mathrm{d}t$$

converges. Note that such t_0 has to always exist since A is a Young function. Let then $H_\alpha \colon [t_0, \infty) \to [0, \infty)$ be defined by

$$H_\alpha(t) = \left(\int_{t_0}^{t} \left(\frac{s}{A(s)} \right)^{\frac{\alpha}{1-\alpha}} \mathrm{d}s \right)^{1-\alpha} \quad t \geq t_0.$$

Suppose that

$$\int^{\infty} \left(\frac{t}{A(t)} \right)^{\frac{\alpha}{1-\alpha}} \mathrm{d}t = \infty. \tag{3.3.1}$$

then H_α is increasing, absolutely continuous and onto $[0, \infty)$ and hence H_α^{-1} is well defined on $[0, \infty)$.

Let us now define $A_{\alpha,\beta} \colon [0, \infty) \to [0, \infty)$ by

$$A_{\alpha,\beta}(t) = \int_0^t \frac{D_{\alpha,\beta}(s)}{s} \, \mathrm{d}s \quad \text{for } t \geq 0, \tag{3.3.2}$$

where $D_{\alpha,\beta} \colon [0, \infty) \to [0, \infty]$ is given by

$$D_{\alpha,\beta}(s) = \begin{cases} \left(s \, \dfrac{A(H_\alpha^{-1}(s))}{H_\alpha^{-1}(s)} \right)^{\frac{1}{\beta(1-\alpha)}} & \text{if (3.3.1) holds,} \\ \infty & \text{otherwise} \end{cases}$$

for $s > 1$ and $D_{\alpha,\beta}(s) = 0$ for $s \in [0, 1]$.

Remark 3.3.1. Observe that $D_{\alpha,\beta}(s)/s$ is a non-decreasing function since $A(s)/s$ is nondecreasing, H_α and hence H_α^{-1} is increasing and $\frac{1}{\beta(1-\alpha)} \geq 1$. Therefore $A_{\alpha,\beta}$ is a well-defined Young function equivalent to $D_{\alpha,\beta}$, thanks to (2.1.2).

Under the additional assumption

$$\int_0 \left(\frac{t}{A(t)} \right)^{\frac{\alpha}{1-\alpha}} \mathrm{d}t < \infty \tag{3.3.3}$$

we may also define a variant of the function $A_{\alpha,\beta}$ which takes the values of A near zero into account. So, assume (3.3.3) and define $H_\alpha^\infty \colon [0, \infty) \to [0, \infty)$ by

$$H_\alpha^\infty(t) = \left(\int_0^t \left(\frac{s}{A(s)} \right)^{\frac{\alpha}{1-\alpha}} \mathrm{d}s \right)^{1-\alpha} \quad \text{for } t \geq 0. \tag{3.3.4}$$

Suppose that (3.3.1) holds. Then H_α^∞ is increasing, absolutely continuous and onto $[0, \infty)$ and hence $H_\alpha^{\infty-1}$ is well defined on $[0, \infty)$. Observe that the condition (3.3.1) may fail only if A jumps to $+\infty$ at a finite point t_∞ or A is finite-valued and the relevant integral is convergent. In the former case, H_α^∞ is constant (H_∞, say) on $[t_\infty, \infty)$ and, in the latter one, H_α increases to $H_\infty = \lim_{t\to\infty} H_\alpha^\infty(t) < \infty$. If (3.3.1) holds, set also $H_\infty = \infty$.

Let us now define $A_{\alpha,\beta}^\infty \colon [0,\infty) \to [0,\infty)$ by

$$A_{\alpha,\beta}^\infty(t) = \int_0^t \frac{D_{\alpha,\beta}^\infty(s)}{s} \, \mathrm{d}s \quad \text{for } t \geq 0, \tag{3.3.5}$$

where $D_{\alpha,\beta}^\infty \colon [0,\infty) \to [0,\infty]$ is given by

$$D_{\alpha,\beta}^\infty(s) = \begin{cases} \left(s \dfrac{A(H_\alpha^{\infty-1}(s))}{H_\alpha^{\infty-1}(s)} \right)^{\frac{1}{\beta(1-\alpha)}}, & 0 \leq s < H_\infty, \\ \infty, & H_\infty \leq s < \infty. \end{cases}$$

By the same argument as in Remark 3.3.1, one can observe that $A_{\alpha,\beta}^\infty$ is a Young function and

$$A_{\alpha,\beta}^\infty(t) \approx D_{\alpha,\beta}^\infty(t) \quad \text{for } t \geq 0. \tag{3.3.6}$$

We also clearly have that

$$A_{\alpha,\beta}^\infty(t) \approx A_{\alpha,\beta}(t) \quad \text{near infinity.} \tag{3.3.7}$$

To simplify the notation, denote also $t_\infty = \infty$, in the cases when H_α^∞ is increasing on $[0,\infty)$. Consequently, H_α^∞ is increasing onto $[0, H_\infty)$, absolutely continuous and concave on $[0, t_\infty)$ as follows by the calculation

$$H_\alpha^{\infty\prime}(t) = (1-\alpha) H_\alpha^\infty(t)^{-\frac{\alpha}{1-\alpha}} \left(\frac{A(t)}{t} \right)^{-\frac{\alpha}{1-\alpha}} \quad \text{for } t > 0. \tag{3.3.8}$$

We have

$$\begin{aligned} A_{\alpha,\beta}^\infty(t) &= \int_0^t \frac{D_{\alpha,\beta}^\infty(s)}{s} \, \mathrm{d}s \\ &= \int_0^t \left(\frac{A(H_\alpha^{\infty-1}(s))}{H_\alpha^{\infty-1}(s)} \right)^{\frac{1}{\beta(1-\alpha)}} s^{\frac{1+\beta(\alpha-1)}{\beta(1-\alpha)}} \, \mathrm{d}s \\ &= \int_0^{H_\alpha^{\infty-1}(t)} \left(\frac{A(y)}{y} \right)^{\frac{1}{\beta(1-\alpha)}} H_\alpha^\infty(y)^{\frac{1+\beta(\alpha-1)}{\beta(1-\alpha)}} H_\alpha^{\infty\prime}(y) \, \mathrm{d}y \\ &= (1-\alpha) \int_0^{H_\alpha^{\infty-1}(t)} \left(\frac{A(y)}{y} \right)^{\frac{1-\alpha\beta}{\beta(1-\alpha)}} H_\alpha^\infty(y)^{\frac{1-\beta}{\beta(1-\alpha)}} \, \mathrm{d}y, \quad 0 \leq t \leq H_\infty, \end{aligned} \tag{3.3.9}$$

where we used the change of variables $s \mapsto H_\alpha(y)$ with the equation (3.3.8).

In the special case when $\beta = 1$, we obtain from (3.3.9) that

$$A_{\alpha,\beta}^\infty(t) = (1-\alpha) \int_0^{H_\alpha^{\infty-1}(t)} \frac{A(s)}{s} \, \mathrm{d}s, \quad 0 \leq t \leq H_\infty.$$

Next,

$$A(H_\alpha^{\infty-1}(t)) \leq \int_0^{H_\alpha^{\infty-1}(t)} \frac{A(s)}{s} \, \mathrm{d}s \leq A(2H_\alpha^{\infty-1}(t)) \leq A(H_\alpha^{\infty-1}(2t)),$$

where the last inequality follows by the convexity of $H_\alpha^{\infty-1}$. Consequently, we infer that

$$A_{\alpha,\beta}^\infty(t) \approx A(H_\alpha^{\infty-1}(t)) \tag{3.3.10}$$

globally provided that (3.3.1) holds and, conversely, (3.3.10) holds near zero and $A_{\alpha,\beta}^\infty = \infty$ near infinity if (3.3.1) fails.

The Reduction principle for Hardy operator in Orlicz spaces reads as follows.

Theorem 3.3.2. *Let $0 < \alpha < 1$, $\beta > 0$ and $\alpha + 1/\beta \geq 1$, and let A and B be Young functions. Then there exists a constant $C_1 > 0$ such that*

$$\left\| \int_{t^\beta}^{R} f(s)\, s^{\alpha-1}\, \mathrm{d}s \right\|_{L^B(0,R^{1/\beta})} \leq C_1 \|f\|_{L^A(0,R)} \tag{3.3.11}$$

for every $f \in L^A(0,R)$ if and only if either $R < \infty$ and there is a constant $C_2 > 0$ such that

$$B(t) \leq A^\infty_{\alpha,\beta}(C_2 t) \quad \text{near infinity,}$$

where $A_{\alpha,\beta}$ is the Young function defined by (3.3.2), or $R = \infty$, the condition (3.3.3) holds and there is a C_3 such that

$$B(t) \leq A_{\alpha,\beta}(C_3 t) \quad \text{for } t \geq 0.$$

Here the Young function $A^\infty_{\alpha,\beta}$ is given by (3.3.5). Moreover, if $R < 0$, then C_2 depends on C_1, A, α, β and R and, if $R = \infty$, then the constants C_1 and C_3 only depend on each other and on α and β.

Corollary 3.3.3. *Let $0 < \alpha < 1$, $\beta > 0$ and $\alpha + 1/\beta \geq 1$ and let A be a Young function. Then*

$$\left\| \int_{t^\beta}^{R} f(s)\, s^{\alpha-1}\, \mathrm{d}s \right\|_{L^\infty(0,R^{1/\beta})} \leq C \|f\|_{L^A(0,R)}$$

if and only if either $R < \infty$ and

$$\int^\infty \left(\frac{t}{A(t)} \right)^{\frac{\alpha}{1-\alpha}} \mathrm{d}t < \infty.$$

or $R = \infty$ and

$$\int_0^\infty \left(\frac{t}{A(t)} \right)^{\frac{\alpha}{1-\alpha}} \mathrm{d}t < \infty.$$

Proposition 3.3.4. *Let α, β be as in Theorem 3.5.2 and let A and B be Young functions. Then there exists a constant $C_1 > 0$ such that*

$$\left\| \int_{s^\beta}^{\infty} f(r)\, r^{\alpha-1}\, \mathrm{d}r \right\|_{L^B(0,\infty)} \leq C_1 \|f\|_{L^A(0,\infty)} \tag{3.3.12}$$

for every $f \in L^A(0,\infty)$ if and only if there is a constant $C_2 > 0$ such that

$$\int_0^\infty B\left(\frac{H^\infty_{\alpha,\beta} f(t)}{C_2 \left(\int_0^\infty A(|f(s)|)\,\mathrm{d}s \right)^\alpha} \right) \mathrm{d}t \leq \left(\int_0^\infty A(|f(t)|)\,\mathrm{d}t \right)^{\frac{1}{\beta}} \tag{3.3.13}$$

for each $f \in L^A(0,\infty)$.

In order to prove the sufficiency in Theorem 3.3.2 there is no need to have full information about our operator; it is enough to know the endpoint estimates only. This enables us to introduce a more general result, an example of interpolation theorem in Orlicz spaces. Several such results appear in literature, mostly in the work of Andrea Cianchi. Let us mention for instance [18, 19, 23]. Our proof is based on [24, Theorem 3.6]

Theorem 3.3.5. *Let α and β be as in Theorem 3.3.2, let A be a Young function such that* (3.3.3) *and* (3.3.1) *holds and let $A^{\infty}_{\alpha,\beta}$ be as in* (3.3.2)*. Suppose that $(\mathcal{R},\mu)$ and $(\mathcal{S},\nu)$ are σ-finite non-atomic measure spaces and Assume that T is a linear operator satisfying*

$$T\colon L^1(\mathcal{R},\mu) \to L^{\frac{1}{\beta(1-\alpha)}}(\mathcal{S},\nu) \tag{3.3.14}$$

$$T\colon L^{\frac{1}{\alpha},1}(\mathcal{R},\mu) \to L^{\infty}(\mathcal{S},\nu) \tag{3.3.15}$$

with operator norms C_1 and C_2, respectively. Then

$$\int_{\mathcal{S}} A^{\infty}_{\alpha,\beta}\left(\frac{Tf(t)}{C\Big(\int_{\mathcal{R}} A(|f|)\,\mathrm{d}\mu\Big)^{\alpha}}\right)\mathrm{d}\nu(t) \le \Big(\int_{\mathcal{R}} A(|f|)\,\mathrm{d}\mu\Big)^{\frac{1}{\beta}} \tag{3.3.16}$$

for every $f \in L^A(\mathcal{R})$, where C is a constant depending on C_1, C_2 and α, β.

Proof. Our first task is to prove that the operator T is well defined for every f such that $\int_{\mathcal{R}} A(|f|)\,\mathrm{d}\nu < \infty$. Recall that we will use the numbers t_∞ and H_∞ associated to A as in the definition of $A^{\infty}_{\alpha,\beta}$.

Given any such f and any $0 < t < t_\infty$, decompose f as $f = f^t + f_t$, where

$$f^t = \max\{|f|-t, 0\}\operatorname{sign} f \quad \text{and} \quad f_t = \min\{|f|, t\}\operatorname{sign} f.$$

It suffices to show that $f^t \in L^1(\mathcal{R})$ and $f_t \in L^{\frac{1}{\alpha},1}(\mathcal{R})$. Since A is a Young function, $A(t)/t$ is non-decreasing and also $A(t)/t \le a(t)$, by (2.1.2). Thus

$$\begin{aligned}\|f^t\|_{L^1(\mathcal{R})} &= \int_t^{t_\infty} \mu(\{|f|>s\})\,\mathrm{d}s \le \frac{t}{A(t)}\int_t^{t_\infty} a(s)\mu(\{|f|>s\})\,\mathrm{d}s\\ &\le \frac{t}{A(t)}\int_{\mathcal{R}} A(|f|)\,\mathrm{d}\mu\end{aligned}$$

and hence $f^t \in L^1(\mathcal{R})$. As for the f_t, we have, by Hölder inequality,

$$\begin{aligned}\|f_t\|_{L^{\frac{1}{\alpha},1}(\mathcal{R})} &= \frac{1}{\alpha}\int_0^t \mu(\{|f|>s\})^{\alpha}\,\mathrm{d}s\\ &\le \frac{1}{\alpha}\Big(\int_0^t \Big(\frac{s}{A(s)}\Big)^{\frac{\alpha}{1-\alpha}}\mathrm{d}s\Big)^{1-\alpha}\Big(\int_0^t \frac{A(s)}{s}\mu(\{|f|>s\})\,\mathrm{d}s\Big)^{\alpha}\\ &\le \frac{1}{\alpha}H^{\infty}_{\alpha}(t)\Big(\int_{\mathcal{R}} A(|f|)\,\mathrm{d}\mu\Big)^{\alpha},\end{aligned} \tag{3.3.17}$$

where H^{∞}_{α} is given by (3.3.4). Thus, $f_t \in L^{\frac{1}{\alpha},1}(\mathcal{R})$.

In order to establish the inequality (3.3.16), recall that

$$A^{\infty}_{\alpha,\beta}(t) = \int_0^t \frac{D^{\infty}_{\alpha,\beta}(s)}{s}\,\mathrm{d}s, \quad 0 \le t < H_\infty. \tag{3.3.18}$$

To make the notation lighter, let us write just D in place of $D^{\infty}_{\alpha,\beta}$. Further, denote

$$K = \frac{C_2}{\alpha}\Big(\int_{\mathcal{R}} A(|f|)\,\mathrm{d}\mu\Big)^{\alpha}. \tag{3.3.19}$$

Our first step is to show that if $H_\infty < \infty$, then

$$\nu(\{|Tf| > 2Ks\}) = 0 \quad \text{for } s \ge H_\infty. \tag{3.3.20}$$

Indeed, taking the limit $t \to \infty$ in (3.3.17), we obtain

$$\|f\|_{L^{\frac{1}{\alpha},1}(\mathcal{R})} \le \frac{1}{\alpha} H_\infty \Big(\int_{\mathcal{R}} A(|f|)\,\mathrm{d}\mu \Big)^{\alpha},$$

which, in combination with (3.3.15) and (3.3.19), gives

$$\|Tf\|_{L^\infty(\mathcal{S})} \le H_\infty K$$

and (3.3.20) follows. Next, by (3.3.18) and (3.3.20),

$$\int_{\mathcal{S}} A^{\infty}_{\alpha,\beta}\Big(\frac{Tf}{2K}\Big)\,\mathrm{d}\nu = \int_0^{H_\infty} \frac{D(s)}{s} \nu(\{|Tf| > 2Ks\})\,\mathrm{d}s \tag{3.3.21}$$

Let us now work with the latter integral in (3.3.21). In what follows, we will use the abbreviated notation $q = \frac{1}{\beta(1-\alpha)}$. Choose any $\varepsilon > 0$ and, by integration by parts, we get

$$\begin{aligned}\int_\varepsilon^{H_\infty} \frac{D(s)}{s} \nu(\{|Tf| > 2Ks\})\,\mathrm{d}s = &\int_\varepsilon^{H_\infty} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{D(t)}{t^q}\Big) \int_t^{H_\infty} \nu(\{|Tf| > 2Ks\}) s^{q-1}\,\mathrm{d}s \\ &+ \frac{D(\varepsilon)}{\varepsilon^q} \int_\varepsilon^{H_\infty} \nu(\{|Tf| > 2Ks\}) s^{q-1}\,\mathrm{d}s.\end{aligned} \tag{3.3.22}$$

Now, denote $\sigma(t) = H_\alpha^{\infty -1}(t)$ for $0 \le t < H_\infty$ and observe that

$$\nu(\{|Tf| > 2Ks\}) \le \nu(\{|Tf_{\sigma(t)}| > Ks\}) + \nu(\{|Tf^{\sigma(t)}| > Ks\}). \tag{3.3.23}$$

Also, thanks to the assumption (3.3.15) and by (3.3.17), we have

$$\|Tf_{\sigma(t)}\|_{L^\infty(\mathcal{S})} \le C_2 \|f_{\sigma(t)}\|_{L^{\frac{1}{\alpha},1}(\mathcal{R})} \le \frac{C_2}{\alpha} H_\alpha(\sigma(t)) \Big(\int_{\mathcal{R}} A(|f|)\,\mathrm{d}\mu \Big)^{\alpha} \le Kt,$$

which implies that $\nu(\{|Tf_{\sigma(t)}| > Ks\}) = 0$ for every $t \le s < H_\infty$, whence the first term on the right hand side of (3.3.23) vanishes. The equation (3.3.22) now continues by

$$\begin{aligned}\int_\varepsilon^{H_\infty} \frac{D(s)}{s} \nu(\{|Tf| > 2Ks\})\,\mathrm{d}s = &\int_\varepsilon^{H_\infty} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{D(t)}{t^q}\Big) \int_t^{H_\infty} \nu(\{|Tf^{\sigma(t)}| > Ks\}) s^{q-1}\,\mathrm{d}s \\ &+ \frac{D(\varepsilon)}{\varepsilon^q} \int_\varepsilon^{H_\infty} \nu(\{|Tf^{\sigma(\varepsilon)}| > Ks\}) s^{q-1}\,\mathrm{d}s.\end{aligned} \tag{3.3.24}$$

It also follows from (3.3.14) that

$$\int_t^{H_\infty} \nu(\{|Tf^{\sigma(t)}| > Ks\})\, s^{q-1}\,\mathrm{d}s \le \frac{1}{q} \Big(\frac{C_1}{K}\Big)^q \Big(\int_{\sigma(t)}^{t_\infty} \mu(\{|f| > s\})\,\mathrm{d}s \Big)^q \tag{3.3.25}$$

for $0 \le t < H_\infty$. Denote the right hand side of (3.3.25) by $J(t)$ and let us plug the estimate (3.3.25) into (3.3.24). We obtain

$$
\begin{aligned}
&\int_{\varepsilon}^{H_\infty} \frac{D(s)}{s} \nu(\{|Tf| > 2Ks\}) \,\mathrm{d}s \\
&\quad = -\int_{\varepsilon}^{H_\infty} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{D(t)}{t^q}\Big) \int_t^{\infty} J'(y) \,\mathrm{d}y + \frac{D(\varepsilon)}{\varepsilon^q} J(\varepsilon) \\
&\quad = -\int_{\varepsilon}^{H_\infty} J'(y) \int_{\varepsilon}^{y} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{D(t)}{t}\Big) \,\mathrm{d}t \,\mathrm{d}y + \frac{D(\varepsilon)}{\varepsilon^q} J(\varepsilon) \\
&\quad = -\int_{\varepsilon}^{H_\infty} J'(y) \Big(\frac{D(y)}{y^q} - \frac{D(\varepsilon)}{\varepsilon^q}\Big) + \frac{D(\varepsilon)}{\varepsilon^q} J(\varepsilon) \\
&\quad = -\int_{\varepsilon}^{H_\infty} J'(y) \frac{D(y)}{y^q} \,\mathrm{d}y,
\end{aligned}
$$

hence, letting $\varepsilon \to 0^+$, we have

$$
\int_0^{H_\infty} \frac{D(s)}{s} \nu(\{|Tf| > 2Ks\}) \,\mathrm{d}s = -\int_0^{H_\infty} \frac{D(y)}{y^q} J'(y) \,\mathrm{d}y.
$$

Owing to the definition of J and thanks to the relation

$$
\frac{D(H_\alpha^\infty(t))}{H_\alpha^\infty(t)^q} = \Big(\frac{A(t)}{t}\Big)^q, \quad 0 < t < t_\infty,
$$

we may continue by

$$
\begin{aligned}
&\int_0^{H_\infty} \frac{D(s)}{s} \nu(\{|Tf| > 2Ks\}) \,\mathrm{d}s \\
&= \Big(\frac{C_1}{K}\Big)^q \int_0^{H_\infty} \frac{D(y)}{y^q} \Big(\int_{\sigma(y)}^{t_\infty} \nu(\{|f| > s\}) \,\mathrm{d}s\Big)^{q-1} \mu(\{|f| > \sigma(y)\}) \sigma'(y) \,\mathrm{d}y \\
&= \Big(\frac{C_1}{K}\Big)^q \int_0^{t_\infty} \frac{D(H_\alpha^\infty(t))}{H_\alpha^\infty(t)^q} \Big(\int_t^{t_\infty} \nu(\{|f| > s\}) \,\mathrm{d}s\Big)^{q-1} \mu(\{|f| > t\}) \,\mathrm{d}t \\
&\le \Big(\frac{C_1}{K}\Big)^q \int_0^{t_\infty} \frac{D(H_\alpha^\infty(t))}{H_\alpha^\infty(t)^q} \Big(\frac{t}{A(t)}\Big)^{q-1} \\
&\qquad \Big(\int_t^{t_\infty} \frac{A(s)}{s} \mu(\{|f| > s\}) \,\mathrm{d}s\Big)^{q-1} \mu(\{|f| > t\}) \,\mathrm{d}t \\
&\le \Big(\frac{C_1}{K}\Big)^q \int_0^{t_\infty} \frac{A(t)}{t} \Big(\int_t^{t_\infty} \frac{A(s)}{s} \mu(\{|f| > s\}) \,\mathrm{d}s\Big)^{q-1} \mu(\{|f| > t\}) \,\mathrm{d}t \\
&= \frac{1}{q} \Big(\frac{C_1}{K}\Big)^q \Big(\int_0^{t_\infty} \frac{A(s)}{s} \mu(\{|f| > s\}) \,\mathrm{d}s\Big)^q \\
&= \frac{1}{q} \Big(\frac{C_1}{K}\Big)^q \Big(\int_{\mathcal{R}} A(|f|) \,\mathrm{d}\mu\Big)^q.
\end{aligned}
$$

Altogether, by the last inequality with (3.3.21), we conclude that

$$
\int_{\mathcal{S}} A_{\alpha,\beta}^\infty\Big(\frac{Tf}{2K}\Big) \,\mathrm{d}\nu \le \frac{1}{q} \Big(\frac{C_1}{K}\Big)^q \Big(\int_{\mathcal{R}} A(|f|) \,\mathrm{d}\mu\Big)^q
$$

which, thanks to the choice of K in (3.3.19) and by the meaning of q, rewrites as

$$
\int_{\mathcal{S}} A_{\alpha,\beta}^\infty\left(\frac{Tf(t)}{\frac{2C_2}{\alpha}\Big(\int_{\mathcal{R}} A(|f|) \,\mathrm{d}\mu\Big)^\alpha}\right) \le \beta(1-\alpha) \Big(\frac{\alpha C_1}{C_2}\Big)^{\frac{1}{\beta(1-\alpha)}} \Big(\int_{\mathcal{R}} A(|f|) \,\mathrm{d}\mu\Big)^{\frac{1}{\beta}}
$$

whence (3.3.16) follows. □

Note that the definition of the Young function $A_{\alpha,\beta}$, (3.3.2), uses the Young function A itself. In what follows we will work with the associate Young function $\widetilde{A}$ instead. First observe that, by Proposition 3.2.3, (3.3.1) is equivalent to

$$\int^{\infty} \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s = \infty. \tag{3.3.26}$$

We now define $\Phi_\alpha\colon [0,\infty) \to [0,\infty]$ by

$$\Phi_\alpha(t) = \int_1^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \quad \text{for } t > 1, \tag{3.3.27}$$

and also $F_\alpha\colon [0,\infty) \to [0,\infty)$ by

$$F_\alpha(t) = t^{1/(1-\alpha)} \int_1^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \quad \text{for } t > 1, \tag{3.3.28}$$

or, shortly, $F_\alpha(t) = t^{1/(1-\alpha)}\Phi_\alpha(t)$, $t \geq 0$. Clearly, Φ_α and F_α are non-decreasing, whence we may denote by Φ_α^{-1} the generalised left-continuous inverse of Φ_α and by F_α^{-1} the generalised right-continuous inverse function to F_α. Further, if (3.3.26), then both Φ_α^{-1} and F_α^{-1} are finite valued. The next proposition summarises the relation between Φ_α, F_α and $A_{\alpha,\beta}$ and hence gives another equivalent expressions of the Young function $A_{\alpha,\beta}$.

Proposition 3.3.6. *Let A be a Young function. Then*

$$A_{\alpha,\beta}(t) \approx \Big(t\,\Phi_\alpha^{-1}\big(t^{\frac{1}{1-\alpha}}\big)\Big)^{\frac{1}{\beta(1-\alpha)}} \quad \textit{near infinity} \tag{3.3.29}$$

and

$$A_{\alpha,\beta}^{-1}(t) \simeq \frac{t^{\beta(1-\alpha)}}{F_\alpha^{-1}(t^\beta)} \quad \textit{near infinity,} \tag{3.3.30}$$

where Φ_α and F_α are the functions given by (3.3.27) *and* (3.3.28)*, respectively.*

Furthermore, if $I_A < 1/\alpha$, then

$$A_{\alpha,\beta}^{-1}(t) \simeq A^{-1}(t^\beta)\,t^{-\alpha\beta} \quad \textit{near infinity.} \tag{3.3.31}$$

Similar formulas also hold for the global variant, $A_{\alpha,\beta}^{\infty}$, provided that the relevant integrals converge at zero. Observe that, by Proposition 3.2.3, (3.3.3) is equivalent to

$$\int_0 \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s < \infty. \tag{3.3.32}$$

Now, if (3.3.32) holds, we may define $\Phi_\alpha^\infty\colon [0,\infty) \to [0,\infty]$ by

$$\Phi_\alpha^\infty(t) = \int_0^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \quad \text{for } t \geq 0 \tag{3.3.33}$$

and also $F_\alpha^\infty\colon [0,\infty) \to [0,\infty)$ by

$$F_\alpha^\infty(t) = t^{1/(1-\alpha)} \int_0^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \quad \text{for } t \geq 0. \tag{3.3.34}$$

Similarly as above, Φ_α^∞ and F_α^∞ are non-decreasing and hence the generalised left-continuous inverse functions $\Phi_\alpha^{\infty-1}$ and $F_\alpha^{\infty-1}$ are well-defined. The alternative expressions of $A_{\alpha,\beta}^\infty$ then read as follows.

Proposition 3.3.7. *Let A be a Young function satisfying* (3.3.32) *and* (3.3.26). *Then*

$$A^{\infty}_{\alpha,\beta}(t) \approx \Big(t\,(\Phi^{\infty}_{\alpha})^{-1}(t^{\frac{1}{1-\alpha}})\Big)^{\frac{1}{\beta(1-\alpha)}} \quad \text{for } t \geq 0 \tag{3.3.35}$$

and

$$(A^{\infty}_{\alpha,\beta})^{-1}(t) \simeq \frac{t^{\beta(1-\alpha)}}{(F^{\infty}_{\alpha})^{-1}(t^{\beta})} \quad \text{for } t \geq 0, \tag{3.3.36}$$

where Φ^{∞}_{α} and F^{∞}_{α} are the functions given by (3.3.33) *and* (3.3.34), *respectively. Furthermore, if $I^{\infty}_{A} < 1/\alpha$, then*

$$(A^{\infty}_{\alpha,\beta})^{-1}(t) \simeq A^{-1}(t^{\beta})\,t^{-\alpha\beta} \quad \text{for } t \geq 0. \tag{3.3.37}$$

Part of this result appeared in [21, Lemma 2], although we shall introduce more comprehensive approach. Also, we will prove only the global variant; the near infinity one needs only trivial modifications.

Proof of Proposition 3.3.7. First observe that Φ^{∞}_{α} and F^{∞}_{α} are well defined if and only if $A^{\infty}_{\alpha,\beta}$ is well defined thanks to Proposition 3.2.3. Throughout the proof, we will denote the functions Φ^{∞}_{α} and F^{∞}_{α} by Φ and F, for brevity. Also, write E in the place of H^{∞}_{α}.

Let us show (3.3.29). Assume first that (3.3.1), or, equivalently (3.3.26), holds, whence $A^{\infty}_{\alpha,\beta}$ and Φ^{-1} are finite-valued and $H_{\infty} = \infty$. Owing to (3.3.6), it suffices to show that

$$\Big(t\,\Phi^{-1}(t^{\frac{1}{1-\alpha}})\Big)^{\frac{1}{\beta(1-\alpha)}} \approx \left(t\,\frac{A(E^{-1}(t))}{E^{-1}(t)}\right)^{\frac{1}{\beta(1-\alpha)}} \quad \text{for } t > 0,$$

or, equivalently,

$$\Phi^{-1}(t^{\frac{1}{1-\alpha}}) \approx \frac{A(E^{-1}(t))}{E^{-1}(t)} \quad \text{for } t > 0. \tag{3.3.38}$$

Let us temporarily denote $\varrho(t) = A(t)/t$. Since A is a Young function ϱ is non-decreasing, its generalised left-continuous inverse ϱ^{-1} is well defined and, due to 2.1.2, possess

$$\widetilde{A}(t) \leq t\varrho^{-1}(t) \leq \widetilde{A}(4t). \tag{3.3.39}$$

We have

$$\begin{aligned}
\Phi(\varrho(t)) &= \int_0^{\varrho(t)} \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \leq \int_0^{\varrho(t)} \frac{\varrho^{-1}(s)}{s^{1/(1-\alpha)}}\,\mathrm{d}s \\
&= \int_0^{\varrho(t)} \frac{1}{s^{1/(1-\alpha)}} \int_0^{\varrho^{-1}(s)} \mathrm{d}y\,\mathrm{d}s = \int_0^t \int_{\varrho(y)}^{\varrho(t)} \frac{\mathrm{d}s}{s^{1/(1-\alpha)}}\,\mathrm{d}y \\
&\leq \int_0^t \int_{\varrho(y)}^{\infty} \frac{\mathrm{d}s}{s^{1/(1-\alpha)}}\,\mathrm{d}y = c\int_0^t \varrho(y)^{-\frac{\alpha}{1-\alpha}}\mathrm{d}y \\
&= c\int_0^t \left(\frac{y}{A(y)}\right)^{\frac{\alpha}{1-\alpha}} \mathrm{d}y = cE(t)^{\frac{1}{1-\alpha}} \quad \text{for } t > 0,
\end{aligned} \tag{3.3.40}$$

where we used first inequality in (3.3.39), Fubini's theorem with trivial estimates. Here, c represents a constant depending on α. Therefore, passing to inverses, (3.3.40) implies that

$$\varrho(E^{-1}(t)) \leq \Phi^{-1}(Ct^{\frac{1}{1-\alpha}}) \quad \text{for } t > 0$$

with some constant C. This proves one of the inequalities in (3.3.38). Conversely, using similar computation as in (3.3.40),

$$
\begin{aligned}
\Phi(4\varrho(t)) &= \int_0^{4\varrho(t)} \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s = c\int_0^{\varrho(t)} \frac{\widetilde{A}(4s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \\
&\geq c\int_0^{\varrho(t)} \frac{\varrho^{-1}(s)}{s^{1/(1-\alpha)}}\,\mathrm{d}s = \int_0^t \int_{\varrho(y)}^{\varrho(t)} \frac{\mathrm{d}s}{s^{1/(1-\alpha)}}\,\mathrm{d}y \\
&= c\int_0^t \varrho(y)^{-\frac{\alpha}{1-\alpha}}\mathrm{d}y - ct\varrho(t)^{-\frac{\alpha}{1-\alpha}} \\
&= cE(t)^{\frac{1}{1-\alpha}} - ct\varrho(t)^{-\frac{\alpha}{1-\alpha}} \quad \text{for } t>0.
\end{aligned}
\tag{3.3.41}
$$

Also here and in what follows, c denotes some constant depending on α, which may differ in each step. Next, we show that

$$
t\varrho(t)^{-\frac{\alpha}{1-\alpha}} \leq c\Phi(4\varrho(t)). \tag{3.3.42}
$$

Let us denote $J=\widetilde{A}$ and J^{-1} its left-continuous inverse. Then

$$
\begin{aligned}
\Phi(2y) &= \int_0^{2y} \frac{J(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \geq \int_y^{2y} \frac{J(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \\
&\geq J(y)\int_y^{2y} \frac{\mathrm{d}s}{s^{1/(1-\alpha)+1}} = cJ(y)y^{-\frac{1}{1-\alpha}} \quad \text{for } y>0.
\end{aligned}
\tag{3.3.43}
$$

Further, we have that

$$
J(2\varrho(t)) \geq A(t). \tag{3.3.44}
$$

Indeed, by (2.1.5),

$$
2\varrho(t) = \frac{2A(t)}{t} \geq \frac{2A(t)}{A^{-1}(A(t)} \geq \widetilde{A}^{-1}(A(t))
$$

whence

$$
J(2\varrho(t)) \geq J\Big(\widetilde{A}^{-1}(A(t))\Big) \geq A(t).
$$

Plugging $\varrho(t)$ into (3.3.43) and using (3.3.44), we obtain

$$
\Phi(4\varrho(t)) \geq cJ(2\varrho(t))\,\varrho(t)^{-\frac{1}{1-\alpha}} \geq cA(t)\frac{t}{A(t)}\,\varrho^{1-\frac{1}{1-\alpha}} = ct\varrho(t)^{-\frac{\alpha}{1-\alpha}}
$$

which is (3.3.42). Coupling (3.3.41) with (3.3.42), we get

$$
c\Phi(c\varrho(t)) \geq E(t)^{\frac{1}{\alpha-1}} \quad \text{for } t>0,
$$

which, passing to inverses, gives

$$
\varrho(E^{-1}(t)) \geq c\Phi^{-1}(ct^{\frac{1}{\alpha-1}}) \quad \text{for } t>0,
$$

the remaining inequality in (3.3.38). Conversely, if (3.3.1) fails, than (3.3.29) holds near zero as above and $A^{\infty}_{\alpha,\beta}=\infty$ near infinity. Also, Φ is bounded in this case and hence $\Phi^{-1}=\infty$ near infinity.

Let us now focus on (3.3.36). If we denote

$$
J(t) = \Big(t\,\Phi^{-1}(t^{\frac{1}{1-\alpha}})\Big)^{\frac{1}{\beta(1-\alpha)}} \quad \text{for } t\geq 0,
$$

then (3.3.35) implies that $J \approx A^{\infty}_{\alpha,\beta}$ globally and thus $J^{-1} \simeq {A^{\infty}_{\alpha,\beta}}^{-1}(t)$ for $t > 0$. If (3.3.1) holds, then J^{-1} and F^{-1} are finite valued and one can verify that

$$J^{-1}(t)F^{-1}(t^{\beta}) = t^{\beta(1-\alpha)} \quad \text{for } t \geq 0 \tag{3.3.45}$$

Therefore

$$(A^{\infty}_{\alpha,\beta})^{-1}(t) \simeq J^{-1}(t) = \frac{t^{\beta(1-\alpha)}}{F^{-1}(t^{\beta})} \quad \text{for } t > 0. \tag{3.3.46}$$

which gives (3.3.36). On the other hand, if (3.3.1) fails, then (3.3.45) and hence (3.3.46) holds only near zero. Further $F(t) \simeq t^{1/(1-\alpha)}$ near infinity and ${A^{\infty}_{\alpha,\beta}}^{-1}$ is constant near infinity whence (3.3.36) holds also in this case.

As for the "furthermore" part, the assumption $I^{\infty}_{A} < 1/\alpha$ implies that $F^{\infty}_{\alpha} \approx \widetilde{A}$ globally and thus

$$(A^{\infty}_{\alpha,\beta})^{-1}(t) \simeq \frac{t^{\beta(1-\alpha)}}{\widetilde{A}^{-1}(t^{\beta})} \simeq A^{-1}(t^{\beta})\, t^{-\alpha\beta} \quad \text{for } t > 0$$

thanks to (3.3.36) and (2.1.5). □

Proof of Theorem 3.3.2. Let us denote the operator (3.1.1) by H, i.e.

$$Hf(t) = \int_{t^{\beta}}^{R} f(s)\, s^{\alpha-1}\, \mathrm{d}s \quad \text{for } 0 < t < R^{\frac{1}{\beta}}$$

whenever the integral converges. We have,

$$\begin{aligned}
\|Hf\|_{L^{\frac{1}{\beta(1-\alpha)}}(0,R^{\frac{1}{\beta}})} &\leq \|Hf\|_{L^{\frac{1}{\beta(1-\alpha)},1}(0,R^{\frac{1}{\beta}})} \\
&\leq \int_0^{R^{\frac{1}{\beta}}} \int_{t^{\beta}}^{R} t^{\frac{1}{\beta(1-\alpha)}-1} |f(s)|\, s^{\alpha-1}\, \mathrm{d}s \\
&= \int_0^{R} |f(s)|\, s^{\alpha-1} \int_0^{s^{\frac{1}{\beta}}} t^{\frac{1}{\beta(1-\alpha)}-1}\, \mathrm{d}t\, \mathrm{d}s \\
&= \tfrac{1}{\beta(1-\alpha)} \int_0^{R} |f(s)|\, \mathrm{d}s = \tfrac{1}{\beta(1-\alpha)} \|f\|_{L^1(0,R)}
\end{aligned}$$

whence (3.3.14) is satisfied with $C_1 = \frac{1}{\beta(1-\alpha)}$ and, by Hardy-Littlewood inequality, cf. [9, Theorem 2.2, Chapter 2]

$$\|Hf\|_{L^{\infty}(0,R^{\frac{1}{\beta}})} \leq \int_0^{R} |f(s)|\, s^{\alpha-1}\, \mathrm{d}s \leq \int_0^{R} f^*(s)\, s^{\alpha-1}\, \mathrm{d}s = \|f\|_{L^{\frac{1}{\alpha},1}(0,R)}$$

and (3.3.15) follows with $C_2 = 1$. Now, assume that (3.3.3) holds. If $R < \infty$, (3.3.3) may be considered as satisfied without loss of generality, since the change of the behaviour of a Young function near zero does not affect the space. By Theorem 3.3.5, the modular inequality (3.3.16) holds with C depending only on α and β. Now, (3.3.11) holds with B replaced by $A^{\infty}_{\alpha,\beta}$. This proves the necessity for $R = \infty$ and the necessity for $R < \infty$ follows once we use (3.3.7).

Conversely, assume that (3.3.11) holds for some Young functions A and B with some constant C. By the double use of Hölder inequality in Orlicz spaces

(2.3.3) together with Fubini's theorem, we get

$$
\begin{aligned}
2C &\geq 2 \sup_{f\in L^A(0,R)} \frac{\left\| \int_{t^\beta}^R f(s)\, s^{\alpha-1}\, \mathrm{d}s \right\|_{L^B(0,R^{\frac{1}{\beta}})}}{\|f\|_{L^A(0,R)}} \\
&\geq \sup_{f\in L^A(0,R)} \sup_{g\in L^{\widetilde{B}}(0,R^{1/\beta})} \frac{\int_0^{R^{\frac{1}{\beta}}} g(t) \int_{t^\beta}^R f(s)\, s^{\alpha-1}\, \mathrm{d}s\, \mathrm{d}t}{\|f\|_{L^A(0,R)} \|g\|_{L^{\widetilde{B}}(0,R^{1/\beta})}} \\
&\geq \sup_{g\in L^{\widetilde{B}}(0,R^{1/\beta})} \sup_{f\in L^A(0,R)} \frac{\int_0^R f(s)\, s^{\alpha-1} \int_0^{s^{\frac{1}{\beta}}} g(t)\, \mathrm{d}t\, \mathrm{d}t}{\|f\|_{L^A(0,R)} \|g\|_{L^{\widetilde{B}}(0,R^{1/\beta})}} \\
&\geq \sup_{g\in L^{\widetilde{B}}(0,R^{1/\beta})} \frac{\left\| s^{\alpha-1} \int_0^{s^{\frac{1}{\beta}}} g(t)\, \mathrm{d}t \right\|_{L^{\widetilde{A}}(0,R)}}{\|g\|_{L^{\widetilde{B}}(0,R^{1/\beta})}}.
\end{aligned}
\tag{3.3.47}
$$

Let $0 < r < R^{\frac{1}{\beta}}$ be fixed and suppose that g has support in the interval $[0, r]$. Then

$$
\left\| s^{\alpha-1} \int_0^{s^{\frac{1}{\beta}}} g(t)\, \mathrm{d}t \right\|_{L^{\widetilde{A}}(0,R)} \geq \left\| s^{\alpha-1} \int_0^{s^{\frac{1}{\beta}}} g(t)\, \mathrm{d}t \right\|_{L^{\widetilde{A}}(r^\beta,R)} \geq \int_0^r g(t)\, \mathrm{d}t\, \|s^{\alpha-1}\|_{L^{\widetilde{A}}(r^\beta,R)}
$$

and (3.3.47) continues as

$$
\begin{aligned}
2C &\geq \sup_{g\in L^{\widetilde{B}}(0,r)} \frac{\left\| s^{\alpha-1} \int_0^{s^{\frac{1}{\beta}}} g(t)\, \mathrm{d}t \right\|_{L^{\widetilde{A}}(0,R)}}{\|g\|_{L^{\widetilde{B}}(0,r)}} \\
&\geq \sup_{g\in L^{\widetilde{B}}(0,r)} \frac{\int_0^r g(t)\, \mathrm{d}t\, \|s^{\alpha-1}\|_{L^{\widetilde{A}}(r^\beta,R)}}{\|g\|_{L^{\widetilde{B}}(0,r)}} \\
&\geq \|1\|_{L^B(0,r)}\, \|s^{\alpha-1}\|_{L^{\widetilde{A}}(r^\beta,R)}.
\end{aligned}
\tag{3.3.48}
$$

By (2.3.1),

$$
\|1\|_{L^B(0,r)} = \frac{1}{B^{-1}(\frac{1}{r})} \tag{3.3.49}
$$

and, thanks to Lemma 3.2.1, for $R < \infty$ $[R = \infty]$

$$
\|s^{\alpha-1}\|_{L^{\widetilde{A}}(r^\beta,R)} = \frac{r^{\beta(1-\alpha)}}{(F_\alpha^\infty)^{-1}(\frac{\alpha-1}{r^\beta})} \quad \text{near infinity} \quad [\text{for } t > 0]. \tag{3.3.50}
$$

Here F_α^∞ is given by (3.3.34). Observe that (3.3.48) in particular implies that $\|s^{\alpha-1}\|_{L^{\widetilde{A}}(r^\beta,R)}$ is finite. Therefore, by Lemma 3.2.1, the condition

$$
\int_0 \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\, \mathrm{d}s < \infty
$$

holds. This implies (3.3.3), thanks to Lemma 3.2.3.

Combining (3.3.48), (3.3.49) and (3.3.50) and using the change of variables, we obtain that there is a constant C_1, depending on C, α and β, such that

$$C_1 B^{-1}(t) \geq \frac{t^{\beta(\alpha-1)}}{(F_\alpha^\infty)^{-1}(t^\beta)} \quad \text{near infinity} \quad [\text{for } t > 0]. \tag{3.3.51}$$

Using (3.3.30) together with (2.1.5), the inequality (3.3.51) rewrites as

$$B^{-1}(t) \gtrsim (A_{\alpha,\beta}^\infty)^{-1}(t) \quad \text{near infinity} \quad [\text{for } t > 0].$$

The claim now follows by taking the inverses and eventually by (3.3.7). □

Proof of Proposition 3.3.4. Let A and B be Young functions and assume (3.3.12). We will make use of the following scaling argument. Suppose that $N > 0$ is given and let us define A_N and B_N by

$$A_N(t) = \frac{A(t)}{N} \quad \text{and} \quad B_N(t) = \frac{1}{N^{1/\beta}} B(tN^{-\alpha}) \quad \text{for } t \geq 0.$$

We claim that

$$H_{\alpha,\beta}^\infty \colon L^{A_N}(0,\infty) \to L^{B_N}(0,\infty) \tag{3.3.52}$$

in which the operator norm does not depend on N. Indeed, thanks to the characterisation in Theorem 3.3.2, the inequality (3.3.12) is equivalent to

$$B(t) \leq A_{\alpha,\beta}^\infty(Ct)$$

where C is some positive constant. It is easy to observe that $(A_N)_{\alpha,\beta}^\infty$, the Young function associated to A_N as in (3.3.5), satisfies

$$(A_N)_{\alpha,\beta}^\infty = \frac{1}{N^{1/\beta}} A_{\alpha,\beta}^\infty(tN^{-\alpha}) \quad \text{for } t \geq 0,$$

hence, we infer, by the change of variables, that

$$B_N(t) \leq (A_N)_{\alpha,\beta}^\infty(Ct) \quad \text{for } t \geq 0 \tag{3.3.53}$$

with the same constant C. Owing to Theorem 3.3.2 again, (3.3.53) implies (3.3.52).

Next, let $f \in L^A(0,\infty)$. Assume that $\int_0^\infty A(|f|) < \infty$ otherwise there is nothing to prove, and set $N = \int_0^\infty A(|f|)$. By the definition of the Luxemburg norm, $\|f\|_{L^{A_N}(0,\infty)} \leq 1$ and thus, by (3.3.12), $\|H_{\alpha,\beta}^\infty f\|_{L^{B_N}(0,\infty)} \leq C_1$. Therefore

$$\int_0^\infty B_N\Big(\frac{H_{\alpha,\beta}^\infty f(t)}{C_1}\Big)\,\mathrm{d}t \leq 1$$

and (3.3.13) follows by the definition of B_N. □

3.4 Optimal Orlicz target spaces

Let us turn our attention back to Theorem 3.3.2. Let L^A be a given Orlicz space. The description of the optimal Orlicz space L^B for which (3.1.2) holds is straightforward.

Theorem 3.4.1. *Let $0 < \alpha < 1$, $\beta > 0$, and $\alpha + 1/\beta \geq 1$. Suppose that A is a Young function satisfying* (3.3.3) *and let $A^\infty_{\alpha,\beta}$ be the Young function defined by* (3.3.5)*. Then*

$$H^\infty_{\alpha,\beta}\colon L^A(0,\infty) \to L^{A^\infty_{\alpha,\beta}}(0,\infty), \tag{3.4.1}$$

and $L^{A^\infty_{\alpha,\beta}}(0,\infty)$ is the optimal (i.e. smallest) Orlicz target space that renders (3.4.1) *true. Moreover, the embedding norm depends only on α and β*

In particular, if $I^\infty_A < 1/\alpha$, then

$${A^\infty_{\alpha,\beta}}^{-1}(t) \simeq A^{-1}(t^\beta)\, t^{-\alpha\beta} \quad \text{for } t > 0. \tag{3.4.2}$$

Conversely, if (3.3.3) *is not satisfied, then there does not exist an Orlicz target space $L^B(0,\infty)$ for which* (3.1.3) *holds.*

Proof. Let A an $A^\infty_{\alpha,\beta}$ be the Young functions from the assumptions. The boundedness of $H^\infty_{\alpha,\beta}$ follows directly by Theorem 3.3.2. Next, assume that

$$H^\infty_{\alpha,\beta}\colon L^A(0,\infty) \to L^B(0,\infty). \tag{3.4.3}$$

Then, again, by Theorem 3.3.2, A satisfies (3.3.3), $A^\infty_{\alpha,\beta}$ is well defined and B is dominated by $A^\infty_{\alpha,\beta}$ globally, which ensures that $L^{A^\infty_{\alpha,\beta}}(0,\infty) \to L^B(0,\infty)$. That proves the optimality. By the same argument, we infer that (3.3.3) is a necessary condition for existence of any Young function B in (3.4.3).

The relation (3.4.2) follows by Proposition 3.3.7, namely by (3.3.37). □

The integral inequality in the optimal case is also available. The next corollary is an immediate consequence of Theorem 3.4.1 and Proposition 3.3.4.

Corollary 3.4.2. *Let α, β, A and $A^\infty_{\alpha,\beta}$ be as in Theorem 3.4.1. Then there exists a constant $C > 0$ such that*

$$\int_0^\infty A^\infty_{\alpha,\beta}\left(\frac{H^\infty_{\alpha,\beta} f(t)}{C\left(\int_0^\infty A(|f(s)|)\,\mathrm{d}s\right)^\alpha}\right)\mathrm{d}t \leq \left(\int_0^\infty A(|f(t)|)\,\mathrm{d}t\right)^{\frac{1}{\beta}}$$

for every $f \in L^A(0,\infty)$. Moreover, the Young function $A^\infty_{\alpha,\beta}$ is optimal.

A parallel result to that of Theorem 3.4.1 on bounded sets reads similarly. Let us stress that the behaviour near zero of Young functions involved is now immaterial. Observe also that the existence of the optimal target space is guaranteed without any constrains.

Theorem 3.4.3. *Let $0 < \alpha < 1$, $\beta > 0$, and $\alpha + 1/\beta \geq 1$. Suppose that A is a Young function and let $A_{\alpha,\beta}$ be the Young function defined by* (3.3.2)*. Then*

$$H_{\alpha,\beta}\colon L^A(0,1) \to L^{A_{\alpha,\beta}}(0,1), \tag{3.4.4}$$

and $L^{A_{\alpha,\beta}}(0,1)$ is the optimal (i.e. smallest) Orlicz target space that renders (3.4.4) *true.*

In particular, if $I_A < 1/\alpha$, then

$$A^{-1}_{\alpha,\beta}(t) \simeq A^{-1}(t^\beta)\, t^{-\alpha\beta} \quad \text{near infinity.} \tag{3.4.5}$$

Proof. The validity of (3.4.4) and the optimality is a consequence of Theorem 3.3.2; the simplified equation (3.4.5) holds by (3.3.31) of Proposition 3.3.6. □

3.5 Second reduction principle in Orlicz spaces

We shall first introduce an auxiliary Young function which importance will reveal in the subsequent section. Let B be a given Young function and let $0 < \alpha < 1$, $\beta > 0$ and $\alpha + 1/\beta \geq 1$. Define $B_{\alpha,\beta} \colon [0,\infty) \to [0,\infty]$ by

$$B_{\alpha,\beta}(t) = \int_0^t \frac{G_{\alpha,\beta}^{-1}(s)}{s} \mathrm{d}s \quad \text{for } t \geq 0, \tag{3.5.1}$$

where $G_{\alpha,\beta} \colon [0,\infty) \to [0,\infty)$ is given by

$$G_{\alpha,\beta}(t) = \begin{cases} tB^{-1}(1) & \text{if } 0 \leq t \leq 1, \\ t \inf\limits_{1 \leq s \leq t} B^{-1}(s^{1/\beta}) s^{\alpha-1} & \text{if } t > 1. \end{cases} \tag{3.5.2}$$

Remark 3.5.1. Observe that the function $G_{\alpha,\beta}$ is increasing, as shown via the alternate formula

$$G_{\alpha,\beta}(t) = t^\alpha \inf_{1 \leq s < \infty} B^{-1}(s^{1/\beta}) \max\Big\{1, \tfrac{t}{s}\Big\}^{1-\alpha} \quad \text{for } t \geq 1, \tag{3.5.3}$$

and hence its inverse $G_{\alpha,\beta}^{-1}$ is well-defined.

Also, $B_{\alpha,\beta}$ is actually a Young function. Indeed, since $G_{\alpha,\beta}$ is increasing, $G_{\alpha,\beta}^{-1}$ is increasing as well. Thus, since the function $G_{\alpha,\beta}(t)/t$ is non-increasing, the function $G_{\alpha,\beta}^{-1}(t)/t$ is non-decreasing. These facts also ensure that

$$B_{\alpha,\beta}^{-1}(t) \simeq G_{\alpha,\beta}(t) \quad \text{for } t > 0. \tag{3.5.4}$$

Under the additional assumption that

$$\sup_{0<t<1} \frac{B(t)}{t^{\frac{1}{\beta(1-\alpha)}}} < \infty, \tag{3.5.5}$$

we also define

$$B_{\alpha,\beta}^{\infty}(t) = \int_0^t \frac{{G_{\alpha,\beta}^{\infty}}^{-1}(s)}{s} \mathrm{d}s \quad \text{for } t \geq 0, \tag{3.5.6}$$

where $G_{\alpha,\beta}^{\infty} \colon [0,\infty) \to [0,\infty)$ is given by

$$G_{\alpha,\beta}^{\infty}(t) = t \inf_{0<s\leq t} B^{-1}(s^{1/\beta}) s^{\alpha-1} \quad \text{for } t > 0. \tag{3.5.7}$$

Note that (3.5.5) guarantees that $G_{\alpha,\beta}^{\infty}$ is positive on $(0,\infty)$. Furthermore, by an argument similar to that of Remark 3.5.1, $B_{\alpha,\beta}^{\infty}$ is a Young function, and

$$ {B_{\alpha,\beta}^{\infty}}^{-1}(t) \simeq G_{\alpha,\beta}^{\infty}(t) \quad \text{for } t > 0. \tag{3.5.8}$$

Our next version of Reduction principle for Hardy operator in Orlicz spaces now reads as follows.

Theorem 3.5.2. *Let $0 < \alpha < 1$, $\beta > 0$ and $\alpha + 1/\beta \geq 1$, and $R \in (0,\infty]$. Suppose that A and B are Young functions and let $B_{\alpha,\beta}$ and $B_{\alpha,\beta}^{\infty}$ be the Young functions defined in* (3.5.1) *and* (3.5.6) *respectively. Then there exists a constant $C_1 > 0$ such that*

$$\Big\| \int_{s^\beta}^R f(r)\, r^{\alpha-1}\, \mathrm{d}r \Big\|_{L^B(0,R^{1/\beta})} \leq C_1 \|f\|_{L^A(0,R)} \tag{3.5.9}$$

for every $f \in L^A(0,R)$ *if and only if either* $R < \infty$ *and there exists a constant* $C_2 > 0$ *such that*

$$\int_1^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \le \frac{\widetilde{B_{\alpha,\beta}}(C_2 t)}{t^{1/(1-\alpha)}} \quad \text{for } t > 1$$

or $R = \infty$*,* B *obeys* (3.5.5) *and there exists a constant* $C_3 > 0$ *such that*

$$\int_0^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \le \frac{\widetilde{B^{\infty}_{\alpha,\beta}}(C_3 t)}{t^{1/(1-\alpha)}} \quad \text{for } t > 0.$$

Moreover, if $R < 0$*, then* C_2 *depends on* C_1*,* A*,* α*,* β *and* R *and, if* $R = \infty$*, then the constants* C_1 *and* C_3 *only depend on each other and on* α *and* β*.*

In the case when $R = \infty$, there is also an integral version of the inequality (3.5.9) on hand. Despite the statement is exactly the same as in Proposition 3.3.4 and repeated here just for the sake of self-contained nature of the whole section, the proof is derived differently – see the end of this section.

Proposition 3.5.3. *Let* α*,* β *be as in Theorem 3.5.2 and let* A *and* B *be Young functions. Then there exists a constant* $C_1 > 0$ *such that*

$$\left\| \int_{s^\beta}^\infty f(r)\, r^{\alpha-1}\,\mathrm{d}r \right\|_{L^B(0,\infty)} \le C_1 \|f\|_{L^A(0,\infty)} \tag{3.5.10}$$

for every $f \in L^A(0,\infty)$ *if and only if there is a constant* $C_2 > 0$ *such that*

$$\int_0^\infty B\left(\frac{H^{\infty}_{\alpha,\beta} f(t)}{C_2 \left(\int_0^\infty A(|f(s)|)\,\mathrm{d}s \right)^\alpha} \right) \mathrm{d}t \le \left(\int_0^\infty A(|f(t)|)\,\mathrm{d}t \right)^{\frac{1}{\beta}} \tag{3.5.11}$$

for each $f \in L^A(0,\infty)$*.*

The proof of Theorem 3.5.2 is significantly different from that of Theorem 3.3.2 and it consists of two steps. In the first one, we characterise the boundedness of our operator between Orlicz and weak Orlicz space, and in the second one, we show that any such boundedness to a weak Orlicz space can be strengthen to the strong one. Let us first focus on those partial results; the proof of Theorem 3.5.2 will be then their straightforward consequence.

The next two propositions deals with the reduction principle to a weak Orlicz space. The underlying idea of the proof follows the observation that the boundedness of the operator is fully dependent just on characteristic functions. Such result was first published in [57, Theorem B] for bounded domains and later in [27, Proposition 5.2] for unbounded one.

We moreover treat the cases on a bounded and on an unbounded domains separately since their proofs are not entirely the same. Let us start with the reduction on the whole of $(0,\infty)$.

Proposition 3.5.4. *Let* $0 < \alpha < 1$*,* $\beta > 0$ *and* $\alpha + 1/\beta \ge 1$*, and let* A *and* B *be Young functions. Then the following two assertions are equivalent.*

(i) *There exists a constant* $C_1 > 0$ *such that*

$$\|H^{\infty}_{\alpha,\beta} f\|_{M^B(0,\infty)} \le C_1 \|f\|_{L^A(0,\infty)} \tag{3.5.12}$$

for every $f \in L^A(0,\infty)$*.*

(ii) *B fulfills condition* (3.5.5) *and there exists a constant* $C_2 > 0$ *such that*

$$\int_0^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \le \frac{\widetilde{B^\infty_{\alpha,\beta}}(C_2 t)}{t^{1/(1-\alpha)}} \quad \textit{for } t > 0, \tag{3.5.13}$$

where $B^\infty_{\alpha,\beta}$ *is the Young function defined in* (3.5.6).

Moreover the constants C_1 *and* C_2 *only depend on each other and on* α *and* β.

Proof. A duality argument (2.8.2) combined with equation (2.3.3), tells us that inequality (3.5.12) is equivalent to

$$\left\| t^{\alpha-1}\int_0^{t^{\frac{1}{\beta}}} f(s)\,\mathrm{d}s \right\|_{L^{\widetilde{A}}(0,\infty)} \le C_1 \|f\|_{\Lambda^{\widetilde{A}}(0,\infty)} \tag{3.5.14}$$

for every $f \in \Lambda^{\widetilde{A}}(0,\infty)$, where $\Lambda^{\widetilde{A}}(0,\infty)$ is defined as in (2.4.3). We claim that inequality (3.5.14) is in turn equivalent to

$$\left\| t^{\alpha-1}\int_0^{t^{\frac{1}{\beta}}} f^*(s)\,\mathrm{d}s \right\|_{L^{\widetilde{A}}(0,\infty)} \le C_1 \|f^*\|_{\Lambda^{\widetilde{A}}(0,\infty)} \tag{3.5.15}$$

for every $f \in \Lambda^{\widetilde{A}}(0,\infty)$. The fact that (3.5.14) implies (3.5.15) is trivial. The reverse implication follows from a basic property of rearrangements, which implies that

$$\int_0^{t^{\frac{1}{\beta}}} f(s)\,\mathrm{d}s \le \int_0^{t^{\frac{1}{\beta}}} f^*(s)\,\mathrm{d}s \quad \text{for } t > 0, \tag{3.5.16}$$

see [9, Lemma 2.1, Chapter 2], and from the equality

$$\|f\|_{\Lambda^{\widetilde{A}}(0,\infty)} = \|f^*\|_{\Lambda^{\widetilde{A}}(0,\infty)},$$

which is a consequence of (2.6.2).

Next, by Proposition 2.8.1, inequality (3.5.15) is equivalent to the same inequality restricted just to characteristic functions of the sets of finite measure, namely to the inequality

$$\left\| t^{\alpha-1}\int_0^{t^{\frac{1}{\beta}}} \chi_{(0,r)}(s)\,\mathrm{d}s \right\|_{L^{\widetilde{A}}(0,\infty)} \le C_1 \|\chi_{(0,r)}\|_{\Lambda^{\widetilde{A}}(0,\infty)} \quad \text{for } r > 0. \tag{3.5.17}$$

Owing to the equality

$$\|\chi_{(0,r)}\|_{M^B(0,\infty)}\, \|\chi_{(0,r)}\|_{\Lambda^{\widetilde{A}}(0,\infty)} = r \quad \text{for } r > 0$$

(see [9, Theorem 5.2, Chapter 2]), and to equation (2.4.5) with A replaced by B,

$$\|\chi_{(0,r)}\|_{\Lambda^{\widetilde{A}}(0,\infty)} \simeq r\, B^{-1}(1/r) \quad \text{for } r > 0, \tag{3.5.18}$$

up to absolute equivalence constants. On the other hand, computations show that

$$\left\| t^{\alpha-1}\int_0^{t^{\frac{1}{\beta}}} \chi_{(0,r)}(s)\,\mathrm{d}s \right\|_{L^{\widetilde{A}}(0,\infty)} \simeq r\, \|t^{\alpha-1}\chi_{(r^\beta,\infty)}(t)\|_{L^{\widetilde{A}}(0,\infty)} \quad \text{for } r > 0, \tag{3.5.19}$$

up to equivalence constants depending on α and β. The right-hand side of (3.5.19) is finite if and only if the integral on the left-hand side of (3.5.13) converges. Moreover, if this is the case, then, by Lemma 3.2.1,

$$\|t^{\alpha-1}\chi_{(r^\beta,\infty)}(t)\|_{L^{\widetilde{A}}(0,\infty)} = \frac{r^{\beta(\alpha-1)}}{F^{-1}(r^{-\beta})} \quad \text{for } r > 0, \tag{3.5.20}$$

where $F\colon (0,\infty) \to [0,\infty)$ is the (increasing) function defined by

$$F(t) = \frac{1}{1-\alpha}\, t^{\frac{1}{1-\alpha}} \int_0^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\, \mathrm{d}s \quad \text{for } t > 0. \tag{3.5.21}$$

Combining (3.5.18), (3.5.20) and (3.5.17) tells us that (3.5.15), and hence (3.5.12), is equivalent to the existence of a constant C_3, depending on α and β, such that

$$\frac{1}{F^{-1}(s)} \le C_3 B^{-1}(s^{1/\beta})\, s^{\alpha-1} \quad \text{for } s > 0. \tag{3.5.22}$$

Taking the infimum, as $s \in (0,t]$, of both sides of (3.5.22), and making use of the fact that the function F is increasing, yield

$$\frac{1}{F^{-1}(t)} \le C_3 \inf_{0<s\le t} B^{-1}(s^{1/\beta})\, s^{\alpha-1} \quad \text{for } t > 0.$$

On the other hand, the latter inequality trivially implies (3.5.22). Hence (3.5.22) is equivalent to

$$\frac{1}{F^{-1}(t)} \le C_3 \frac{G^{\infty}_{\alpha,\beta}(t)}{t} \quad \text{for } t > 0.$$

It follows from (3.5.22) that (3.5.5) holds if and only if F and hence $\widetilde{A}$ is not identically zero. Finally, by equations (3.5.8) and (2.1.5), inequality (3.5.12) is equivalent to

$$\widetilde{B^{\infty}_{\alpha,\beta}}^{-1}(t) \le C_4 F^{-1}(t) \quad \text{for } t > 0, \tag{3.5.23}$$

for some constant C_4 depending on α and β. Hence, the conclusion follows, on taking the inverses of both sides of (3.5.23). □

The next proposition now deals with the operator on the interval $(0,1)$. The case for an general bounded interval $(0,R)$ can be obtained simply by the composition with the dilation operator. Note also that in the previous proposition the constants C_1 and C_2 were independent of the Young functions involved. This is no longer true for the result on a bounded domain where a dependence on a Young function A appears.

Proposition 3.5.5. *Let $0 < \alpha < 1$, $\beta > 0$ and $\alpha + 1/\beta \ge 1$, and let A and B be Young functions. Then the following two assertions are equivalent.*

(i) *There exists a constant $C_1 > 0$ such that*

$$\|H_{\alpha,\beta} f\|_{M^B(0,1)} \le C_1 \|f\|_{L^A(0,1)} \tag{3.5.24}$$

for every $f \in L^A(0,1)$.

(ii) *There exists a constant* $C_2 > 0$ *such that*

$$\int_1^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \le \frac{\widetilde{B_{\alpha,\beta}}(C_2 t)}{t^{1/(1-\alpha)}} \quad \textit{near infinity,} \tag{3.5.25}$$

where $B_{\alpha,\beta}$ *is the Young function defined in* (3.5.1).

Moreover the constants C_1 *and* C_2 *depend on each other on* A *and on* α *and* β.

Proof. I order to prove this statement, we will follow the same scheme as in the proof of Proposition 3.5.4 and we shall comment the principal differences only. First, as above, we infer that (3.5.24) is equivalent to the existence of a constant C_2 such that

$$\left\| t^{\alpha-1}\int_0^{t^{\frac{1}{\beta}}} \chi_{(0,r)}(s)\,\mathrm{d}s \right\|_{L^{\widetilde{A}}(0,1)} \le C_2\, r B^{-1}\left(\frac{1}{r}\right) \quad \text{for } r\in(0,1). \tag{3.5.26}$$

Here C_2 depend only on C_1. Next, we show that (3.5.26) is equivalent to the same inequality for r restricted only to some neighbourhood of zero. Indeed, assume that (3.5.26) hold on $(0,r_0]$ for some r_0 fixed. Let $r\in(r_0,1)$ and denote by c the largest integer such that $cr_0<r$. Then, by (3.5.16),

$$\begin{aligned}\int_0^{t^{\frac{1}{\beta}}} \chi_{(0,r)}(s)\,\mathrm{d}s &= \int_0^{t^{\frac{1}{\beta}}} \chi_{(0,r_0)}(s)\,\mathrm{d}s + \int_0^{t^{\frac{1}{\beta}}} \chi_{(r_0,2r_0)}(s)\,\mathrm{d}s + \cdots + \int_0^{t^{\frac{1}{\beta}}} \chi_{(cr_0,r)}(s)\,\mathrm{d}s\\ &\le c\int_0^{t^{\frac{1}{\beta}}} \chi_{(0,r_0)}(s)\,\mathrm{d}s \le \frac{1}{r_0}\int_0^{t^{\frac{1}{\beta}}} \chi_{(0,r_0)}(s)\,\mathrm{d}s\end{aligned}$$

and (3.5.26) follows with C_2 replaced by C_2/r_0.

We observe that

$$\left\| t^{\alpha-1}\int_0^{t^{\frac{1}{\beta}}} \chi_{(0,r)}(s)\,\mathrm{d}s \right\|_{L^{\widetilde{A}}(0,1)} \simeq r\|t^{\alpha-1}\chi_{(r^\beta,1)}(t)\|_{L^{\widetilde{A}}(0,1)} \quad \text{for } r\in(0,2^{-1/\beta}). \tag{3.5.27}$$

where the constants depend only on α and β. Clearly, it is

$$\left\| t^{\alpha-1}\int_0^{t^{\frac{1}{\beta}}} \chi_{(0,r)}(s)\,\mathrm{d}s \right\|_{L^{\widetilde{A}}(0,1)} \le \|t^{\alpha+\frac{1}{\beta}-1}\chi_{(0,r^\beta)}(t)\|_{L^{\widetilde{A}}(0,1)} + \|t^{\alpha-1}\chi_{(r^\beta,1)}(t)\|_{L^{\widetilde{A}}(0,1)}$$

and for $r\in(0,2^{-1/\beta})$ we have

$$\begin{aligned}\|t^{\alpha+\frac{1}{\beta}-1}\chi_{(0,r^\beta)}(t)\|_{L^{\widetilde{A}}(0,1)} &\le r^{\beta(\alpha-1)+1}\|\chi_{(0,r^\beta)}\|_{L^{\widetilde{A}}(0,1)}\\ &= r^{\beta(\alpha-1)+1}\|\chi_{(r^\beta,2r^\beta)}\|_{L^{\widetilde{A}}(0,1)} \le 2^{\beta(1-\alpha)}r\|t^{\alpha-1}\chi_{(r^\beta,1)}\|_{L^{\widetilde{A}}(0,1)}.\end{aligned}$$

Also, by Lemma 3.2.1,

$$\|t^{\alpha-1}\chi_{(r^\beta,1)}(t)\|_{L^{\widetilde{A}}(0,1)} \simeq \frac{r^{\beta(\alpha-1)}}{F^{-1}(r^{-\beta})} \quad \text{near zero,} \tag{3.5.28}$$

where F is defined as in (3.5.21). The assumption (3.2.1) can be rendered as satisfied since otherwise we can modify the Young function A on $(0,1)$ without affecting the space $L^A(0,1)$. The constant in (3.5.28) now also depends on A.

Combining (3.5.26), (3.5.27) and (3.5.28) gives us that (3.5.24) is equivalent to the existence of a constant C_3, depending on C_1, α, β and A, such that

$$\frac{1}{F^{-1}(s)} \leq C_3 B^{-1}(s^{1/\beta})\, s^{\alpha-1} \quad \text{near infinity.} \tag{3.5.29}$$

The rest of the proof is now very similar. We take the infimum, as $s \in (1,t]$, of both sides of (3.5.29) to obtain that (3.5.29) is equivalent to the inequality

$$\frac{1}{F^{-1}(t)} \leq C_3 \frac{G_{\alpha,\beta}(t)}{t} \quad \text{near infinity.}$$

Therefore, by (3.5.4) and (2.1.5), inequality (3.5.24) is equivalent to

$$\widetilde{B_{\alpha,\beta}}^{-1}(t) \leq C_4 F^{-1}(t) \quad \text{near infinity} \tag{3.5.30}$$

for some constant C_4 depending on C_1, α, β and A. On taking inverses of both sides of (3.5.30), we get

$$\int_0^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \leq \frac{\widetilde{B_{\alpha,\beta}}(C_4 t)}{t^{1/(1-\alpha)}} \quad \text{near infinity,} \tag{3.5.31}$$

which is equivalent to (3.5.25) since the integrals on the left hand side of (3.5.31) and (3.5.25) are comparable near infinity. The resulting inequality therefore does not depend on the values of $\widetilde{A}$ on $(0,1)$. □

The next key result reveals that any Orlicz the boundedness to a weak space can actually be lifted to the strong one. This idea already appeared in similar form in [29, Lemmas 5.3 and 5.4]; we will follow the proof from [27, Lemma 5.3].

Proposition 3.5.6. *Let α, β, A and B be as in Proposition 3.5.4. If*

$$H^{\infty}_{\alpha,\beta}\colon L^A(0,\infty) \to M^B(0,\infty), \tag{3.5.32}$$

then

$$H^{\infty}_{\alpha,\beta}\colon L^A(0,\infty) \to L^B(0,\infty). \tag{3.5.33}$$

Moreover, the norms of the operator $H^{\infty}_{\alpha,\beta}$ in (3.5.32) *and* (3.5.33) *are equivalent, up to multiplicative constants independent of A and B.*

Proof. Throughout this proof, we adopt the abridged notation H for $H^{\infty}_{\alpha,\beta}$. Given $N>0$, define the Young functions A_N and B_N as

$$A_N(t) = \frac{A(t)}{N} \quad \text{and} \quad B_N(t) = \frac{1}{N^{1/\beta}} B\big(tN^{-\alpha}\big) \quad \text{for } t \geq 0. \tag{3.5.34}$$

We claim that equation (3.5.32) implies that

$$H\colon L^{A_N}(0,\infty) \to M^{B_N}(0,\infty)\,, \tag{3.5.35}$$

with operator norm independent of N. To prove this claim, we make use of Proposition 3.5.4, which tells us that (3.5.32) is equivalent to the existence of a positive constant C such that

$$\int_0^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \leq \frac{\widetilde{B^{\infty}_{\alpha,\beta}}(Ct)}{t^{1/(1-\alpha)}} \quad \text{for } t>0, \tag{3.5.36}$$

where $B^\infty_{\alpha,\beta}$ is the Young function defined by (3.5.6). One can verify that the function $(B_N)^\infty_{\alpha,\beta}$, associated with B_N as in (3.5.6), satisfies

$$(B_N)^\infty_{\alpha,\beta} = \frac{B^\infty_{\alpha,\beta}}{N}\,,$$

and that inequality (3.5.36) holds with A and $B^\infty_{\alpha,\beta}$ replaced by A_N and $(B_N)^\infty_{\alpha,\beta}$, respectively, with the same constant C. Proposition 3.5.4 again tells us that (3.5.35) holds, with operator norm independent of N.

Now, given any function $f \in \mathcal{M}_+(0,\infty)$ such that

$$0 < \int_0^\infty A\big(f(r)\big)\,\mathrm{d}r \le 1\,, \tag{3.5.37}$$

set

$$N = \int_0^\infty A\big(f(r)\big)\,\mathrm{d}r.$$

Thanks to (3.5.35), we have that

$$\|Hf\|_{M^{B_N}(0,\infty)} \le C\|f\|_{L^{A_N}(0,\infty)} \le C\,, \tag{3.5.38}$$

for some constant C independent of N and f, since, by the very definition of Luxemburg norm in Orlicz spaces,

$$\|f\|_{L^{A_N}(0,\infty)} \le 1. \tag{3.5.39}$$

Equations (3.5.38)–(3.5.39), inequality (2.4.2) and equation (3.5.34) tell us

$$C \ge \|Hf\|_{M^{B_N}(0,\infty)} \ge \sup_{0<t<\infty} \frac{t}{B_N^{-1}\Big(\frac{1}{|\{Hf>t\}|}\Big)} = \sup_{0<t<\infty} \frac{t}{N^\alpha B^{-1}\Big(\frac{N^{1/\beta}}{|\{Hf>t\}|}\Big)}$$

for $t > 0$, namely

$$|\{Hf > t\}|\, B\left(\frac{t}{C\Big(\int_0^\infty A\big(f(r)\big)\,\mathrm{d}r\Big)^\alpha}\right) \le \left(\int_0^\infty A\big(f(r)\big)\,\mathrm{d}r\right)^{\frac{1}{\beta}} \quad \text{for } t > 0. \tag{3.5.40}$$

From inequality (3.5.40), via assumption (3.5.37) and property (2.1.3) applied to B, one can deduce that

$$|\{Hf > t\}|\, B\Big(\frac{t}{C}\Big) \le \left(\int_0^\infty A\big(f(r)\big)\,\mathrm{d}r\right)^{\alpha+\frac{1}{\beta}} \quad \text{for } t > 0. \tag{3.5.41}$$

Clearly, inequality (3.5.41) continues to hold even if the integral on the right-hand side vanishes.

Our next task is to derive a strong type inequality from the weak type inequality (3.5.41). This will be accomplished via a discretization argument. If the (non-negative) function Hf is unbounded, denote by $\{s_k\}_{k\in\mathbb{Z}}$ a sequence in $(0,\infty)$ such that

$$Hf(s_k) = 2^k \quad \text{for } k \in \mathbb{Z}. \tag{3.5.42}$$

In the case when Hf is bounded, we define the sequence $\{s_k\}$ similarly, save that now the index k ranges from $-\infty$ to the smallest $K \in \mathbb{Z}$ such that $Hf(0) \le 2^K$.

We then set $s_K = 0$, and define s_k again by (3.5.42) for $k \leq K-1$. In what follows, we shall treat these two cases simultaneously, and K will denote either ∞, or an integer, according to whether Hf is unbounded or bounded, respectively.

Notice that s_k is non-increasing, since Hf is non-increasing. Define

$$f_k = f\chi_{[s_k^\beta, s_{k-1}^\beta)} \quad \text{for } k < K.$$

If $k < K$, then

$$Hf(s) \leq Hf(s_k) = 2^k \quad \text{for } s \in (s_k, s_{k-1}).$$

Hence,

$$\begin{aligned}\int_0^\infty B\left(\frac{Hf(s)}{4C}\right) \mathrm{d}s &= \sum_{k<K} \int_{s_{k+1}}^{s_k} B\left(\frac{Hf(s)}{4C}\right) \mathrm{d}s \\ &\leq \sum_{k<K} \int_{s_{k+1}}^{s_k} B\left(\frac{2^{k+1}}{4C}\right) \mathrm{d}s = \sum_{k<K} (s_k - s_{k+1}) B\left(\frac{2^{k-1}}{C}\right).\end{aligned} \tag{3.5.43}$$

Given any $k < K$,

$$\begin{aligned}Hf_k(s) &\geq \int_{s_k^\beta}^\infty f_k(r)\, r^{\alpha-1}\, \mathrm{d}r = \int_{s_k^\beta}^\infty f(r)\chi_{[s_k^\beta, s_{k-1}^\beta)}(r)\, r^{\alpha-1}\, \mathrm{d}r = \int_{s_k^\beta}^{s_{k-1}^\beta} f(r)\, r^{\alpha-1}\, \mathrm{d}r \\ &= Hf(s_k) - Hf(s_{k-1}) = 2^{k-1} \quad \text{for } s \in [s_{k+1}, s_k).\end{aligned}$$

Consequently,

$$[s_{k+1}, s_k) \subset \left\{Hf_k \geq 2^{k-1}\right\} \quad \text{for } k < K. \tag{3.5.44}$$

From inclusion (3.5.44) and inequality (3.5.41) we obtain that

$$(s_k - s_{k+1}) B\left(\frac{2^{k-1}}{C}\right) \leq \left|\left\{Hf_k \geq 2^{k-1}\right\}\right| B\left(\frac{2^{k-1}}{C}\right) \leq \left(\int_0^\infty A\big(f_k(r)\big)\, \mathrm{d}r\right)^{\alpha+\frac{1}{\beta}} \tag{3.5.45}$$

for $k < K$. Coupling (3.5.43) with (3.5.45), and exploiting the fact that $\alpha + 1/\beta \geq 1$ yield

$$\begin{aligned}\int_0^\infty B\left(\frac{Hf(s)}{4C}\right) \mathrm{d}s &\leq \sum_{k<K} \left(\int_0^\infty A\big(f_k(r)\big)\, \mathrm{d}r\right)^{\alpha+\frac{1}{\beta}} \\ &\leq \left(\sum_{k<K} \int_0^\infty A\big(f_k(r)\big)\, \mathrm{d}r\right)^{\alpha+\frac{1}{\beta}} \leq \left(\int_0^\infty A\big(f(r)\big)\, \mathrm{d}r\right)^{\alpha+\frac{1}{\beta}},\end{aligned} \tag{3.5.46}$$

for every function $f \in \mathcal{M}_+(0,\infty)$ satisfying the second inequality in (3.5.37). Inequality (3.5.46) implies equation (3.5.33). □

Corollary 3.5.7. *Let α, β, A and B be as in Proposition 3.5.4. If*

$$H_{\alpha,\beta} \colon L^A(0,1) \to M^B(0,1), \tag{3.5.47}$$

then

$$H_{\alpha,\beta} \colon L^A(0,1) \to L^B(0,1). \tag{3.5.48}$$

In particular, the space $L^A(0,1)$ is the optimal Orlicz domain in (3.5.47) *if and only if it is the optimal Orlicz domain in* (3.5.48).

Proof. Suppose that A and B are Young functions such that (3.5.47) holds. Proposition 3.5.5 tells us that (3.5.47) implies that

$$\int_1^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}} \mathrm{d}s \leq \frac{\widetilde{B_{\alpha,\beta}}(Ct)}{t^{1/(1-\alpha)}} \quad \text{for } t \geq t_0,$$

for some constants $C > 0$ and $t_0 > 1$. Let us denote by $\widehat{A}$ and $\widehat{B}$ two Young functions which agree with A and B near infinity, and are modified near zero in such a way that $\widehat{B}$ satisfies condition (3.5.5), and condition (3.5.13) holds with A and $B^{\infty}_{\alpha,\beta}$ replaced by $\widehat{A}$ and $\widehat{B}^{\infty}_{\alpha,\beta}$. Hence, by Proposition (3.5.4),

$$H^{\infty}_{\alpha,\beta} \colon L^{\widehat{A}}(0,\infty) \to M^{\widehat{B}}(0,\infty),$$

and therefore, by Proposition 3.5.6,

$$H^{\infty}_{\alpha,\beta} \colon L^{\widehat{A}}(0,\infty) \to L^{\widehat{B}}(0,\infty). \tag{3.5.49}$$

Equation (3.5.49) implies (3.5.48) since $L^{\widehat{A}}(0,1) = L^{A}(0,1)$, and $L^{\widehat{B}}(0,1) = L^{B}(0,1)$, up to equivalent norms. □

Proof of Theorem 3.5.2. In the case $R = \infty$, the assertion is a direct consequence of Propositions 3.5.4 and 3.5.6.

If $R < \infty$, then, thanks to (2.8.3), the dilation operator is bounded on any Orlicz space and hence (3.5.9) holds if and only if the same inequality holds with R replaced by 1. Thus, the result follows due to Proposition 3.5.5 together with Corollary (3.5.7). □

Proof of Proposition 3.5.3. Let A and B be Young functions and assume (3.3.12). We use of the same scaling argument that already appeared in the proof of Proposition 3.5.6. Let $N > 0$ be given and define A_N and B_N as in (3.5.34). We claim that

$$H^{\infty}_{\alpha,\beta} \colon L^{A_N}(0,\infty) \to L^{B_N}(0,\infty) \tag{3.5.50}$$

in which the operator norm does not depend on N. Indeed, thanks to the characterisation in Theorem 3.5.2, the inequality (3.5.10) is equivalent to

$$\int_0^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}} \, \mathrm{d}s \leq \frac{\widetilde{B^{\infty}_{\alpha,\beta}}(Ct)}{t^{1/(1-\alpha)}} \quad \text{for } t > 0, \tag{3.5.51}$$

where C is a positive constant. One might observe that $(B_N)^{\infty}_{\alpha,\beta}$, the Young function associated to B_N as in (3.5.6), satisfies $(B_N)^{\infty}_{\alpha,\beta} = B^{\infty}_{\alpha,\beta}/N$. Hence, by the substitution in (3.5.51), we infer that

$$\int_0^t \frac{\widetilde{A_N}(s)}{s^{1/(1-\alpha)+1}} \, \mathrm{d}s \leq \frac{\widetilde{(B_N)^{\infty}_{\alpha,\beta}}(Ct)}{t^{1/(1-\alpha)}} \quad \text{for } t > 0 \tag{3.5.52}$$

with the same constant C. Thanks to Theorem 3.5.2, (3.5.52) implies (3.5.50).

Next, let $f \in L^A(0,\infty)$. Assume that $\int_0^\infty A(|f|) < \infty$ otherwise there is nothing to prove, and set $N = \int_0^\infty A(|f|)$. By the definition of the Luxemburg norm, $\|f\|_{L^{A_N}(0,\infty)} \leq 1$ and thus, by (3.5.10), $\|H^{\infty}_{\alpha,\beta} f\|_{L^{B_N}(0,\infty)} \leq C_1$. Therefore

$$\int_0^\infty B_N\Big(\frac{H^{\infty}_{\alpha,\beta} f(t)}{C_1}\Big) \, \mathrm{d}t \leq 1$$

and (3.5.11) follows by the definition of B_N. □

3.6 Optimal Orlicz domain spaces

The aim of this section is to present the solution of the optimal Orlicz domain space L^A for the boundedness of the operators $H^{\infty}_{\alpha,\beta}$ and $H_{\alpha,\beta}$ in (3.1.3) and (3.1.4) respectively for a given space L^B. Let us start with the variant on $(0,\infty)$.

Theorem 3.6.1. *Let $0<\alpha<1$, $\beta>0$, and $\alpha+1/\beta\geq 1$. Suppose that B is a Young function satisfying* (3.5.5) *and let $B^{\infty}_{\alpha,\beta}$ be the Young function defined by* (3.5.6)*. If*

$$I^{\infty}_{B^{\infty}_{\alpha,\beta}} < \frac{1}{\alpha}, \tag{3.6.1}$$

then

$$H^{\infty}_{\alpha,\beta}\colon L^{B^{\infty}_{\alpha,\beta}}(0,\infty)\to L^B(0,\infty), \tag{3.6.2}$$

and $L^{B^{\infty}_{\alpha,\beta}}(0,\infty)$ is the optimal (i.e. largest) Orlicz domain space that renders (3.6.2) *true. Moreover, the embedding norm depends only on α and β*

Conversely, if (3.6.1) *is not satisfied, then no optimal Orlicz domain space exists in* (3.6.2) *in the sense that any Orlicz space $L^A(0,\infty)$ which makes* (3.1.3) *true can be replaced with a strictly larger Orlicz space from which the operator $H^{\infty}_{\alpha,\beta}$ is still bounded into $L^B(0,\infty)$.*

In particular, if $i^{\infty}_B > \frac{1}{\beta(1-\alpha)}$, then (3.6.1) *is equivalent to $I^{\infty}_B<\infty$ and*

$${B^{\infty}_{\alpha,\beta}}^{-1}(t) \simeq B^{-1}(t^{1/\beta})\, t^{\alpha} \quad \text{for } t>0. \tag{3.6.3}$$

In addition, if (3.5.5) *is not satisfied, then there does not exist an Orlicz domain space $L^A(0,\infty)$ for which* (3.1.3) *holds.*

The proof of Theorem 3.6.1 continues at the end of this section. The sharp embedding (3.5.12) is also equivalent to the corresponding inequality in its integral form. The next corollary is a direct consequence of Proposition 3.5.3.

Corollary 3.6.2. *Let α, β, B and $B^{\infty}_{\alpha,\beta}$ be as in Theorem 3.6.1 and suppose that*

$$I^{\infty}_{B^{\infty}_{\alpha,\beta}} < \frac{1}{\alpha}. \tag{3.6.4}$$

Then there exists a constant $C>0$ such that

$$\int_0^{\infty} B\left(\frac{H^{\infty}_{\alpha,\beta}f(t)}{C\left(\int_0^{\infty} B^{\infty}_{\alpha,\beta}(|f(s)|)\,\mathrm{d}s\right)^{\alpha}}\right)\mathrm{d}t \leq \left(\int_0^{\infty} B^{\infty}_{\alpha,\beta}(|f(t)|)\,\mathrm{d}t\right)^{\frac{1}{\beta}}$$

for every $f\in L^{B^{\infty}_{\alpha,\beta}}(0,\infty)$.

In particular, if $i^{\infty}_B > \frac{1}{\beta(1-\alpha)}$, then (3.6.4) *is equivalent to $I^{\infty}_B<\infty$ and*

$${B^{\infty}_{\alpha,\beta}}^{-1}(t) \simeq B^{-1}(t^{1/\beta})\, t^{\alpha} \quad \text{for } t>0.$$

The local version of Theorem 3.6.1 is also available. Naturally, the behaviour of Young functions does not play any role and hence all necessary relations which now have to be taken "near infinity" only. Note that in this case it is not possible to have an integral integral version of the statement in general. The proof of this result is similar (and even simpler) to that of Theorem 3.6.1 and therefore omitted.

Theorem 3.6.3. *Let $0 < \alpha < 1$, $\beta > 0$, and $\alpha + 1/\beta \geq 1$. Suppose that B is a Young function and let $B_{\alpha,\beta}$ be the Young function defined by* (3.5.1). *If*

$$I_{B_{\alpha,\beta}} < \frac{1}{\alpha}, \tag{3.6.5}$$

then

$$H_{\alpha,\beta} \colon L^{B_{\alpha,\beta}}(0,1) \to L^B(0,1)\,, \tag{3.6.6}$$

and $L^{B_{\alpha,\beta}}(0,1)$ is the optimal (i.e. largest) Orlicz domain space that renders (3.6.6) *true.*

In particular, if $i_B > \frac{1}{\beta(1-\alpha)}$, then (3.6.5) *is equivalent to $I_B < \infty$ an*

$$B_{\alpha,\beta}^{\infty\;-1}(t) \simeq B^{-1}(t^{1/\beta})\, t^{\alpha} \quad \textit{near infinity.} \tag{3.6.7}$$

Conversely, if (3.6.5) *is not satisfied, then no optimal Orlicz domain space exists in* (3.6.6)*, in the sense that any Orlicz space $L^A(0,1)$ which makes* (3.1.4) *true can be replaced with a strictly larger Orlicz space from which the operator $H_{\alpha,\beta}$ is still bounded into $L^B(0,1)$.*

In order to prove the result of this section, i.e., to prove Theorems 3.6.1 and 3.6.3, we need to introduce several auxiliary lemmas focused on a construction of specific Young functions. They all share the same background idea. They enable us to essentially enlarge Young functions appearing on left hand sides of some specific integral inequalities under several conditions imposed on right hand sides.

Lemma 3.6.4. *Let $n \in \mathbb{N}$, $0 < \alpha < 1$ and let D be a Young function such that $D(t)/t^{1/(1-\alpha)}$ is non-decreasing near infinity,*

$$\lim_{t\to\infty} \frac{D(t)}{t^{1/(1-\alpha)}} = \infty \tag{3.6.8}$$

and

$$\sup_{1<t<\infty} \frac{t^{1/(1-\alpha)}}{D(Kt)} \int_1^t \frac{D(s)}{s^{1/(1-\alpha)+1}}\, \mathrm{d}s = \infty \tag{3.6.9}$$

for every $K \geq 1$. Suppose that E is a Young function such that

$$\int_1^t \frac{E(s)}{s^{1/(1-\alpha)+1}}\, \mathrm{d}s \leq \frac{D(Ct)}{t^{1/(1-\alpha)}}, \quad t > 1, \tag{3.6.10}$$

for some $C \leq 1$. Then there exists a Young function E_1 essentially dominating E near infinity and satisfying

$$\int_1^t \frac{E_1(s)}{s^{1/(1-\alpha)+1}}\, \mathrm{d}s = \frac{D(5Ct)}{t^{1/(1-\alpha)}}, \quad t > 1. \tag{3.6.11}$$

Proof. Let D and E be the Young functions from the statement. Fix $t > 1$ and define the set G_t by

$$G_t = \left\{ s \in (1,\infty) : \tfrac{E(s)}{s} \geq \tfrac{D(t)}{t} \right\}.$$

We may assume that $E(s)/s$ is a non-decreasing mapping from $(1,\infty)$ onto some neighbourhood of infinity, and hence the sets G_t are nonempty for every $t > 0$. If not, i.e. $\lim_{s\to\infty} E(s)/s < \infty$, then $E(s) \leq cs$ on $(1,\infty)$ for some $c > 0$ and thus

$$\int_1^\infty \frac{E(s)}{s^{1/(1-\alpha)+1}}\, \mathrm{d}s \leq \int_1^\infty \frac{\mathrm{d}s}{s^{1/(1-\alpha)}} < \infty$$

and we may take $E_1(t) = t^\xi$ on $(1,\infty)$ for some $1 < \xi < 1/(1-\alpha)$ which satisfies the requirements trivially.

Let us define $\tau = \tau_t = \inf G_t$. Observe that, by the continuity of Young function E,

$$\frac{E(\tau)}{\tau} = \frac{D(t)}{t}, \quad t > 1. \tag{3.6.12}$$

Also,

$$\limsup_{t\to\infty} \frac{E(\tau_t)}{\tau_t} \cdot \frac{t}{E(Kt)} = \infty \tag{3.6.13}$$

for every $K \geq 1$. Indeed, suppose that there is some $K \geq 1$ for which (3.6.13) is violated. We then have some $L > 0$ such that

$$\frac{E(Kt)}{t} \geq L\frac{E(\tau)}{\tau}, \quad t > 1,$$

which in connection with (3.6.12) gives $E(Ks) \geq L\,D(s)$ on $(1,\infty)$. Thus

$$\begin{aligned}\frac{D(CKt)}{t^{1/(1-\alpha)}} \geq \int_1^{Kt} \frac{E(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \geq K^{\frac{1}{\alpha-1}} \int_1^t \frac{E(Ks)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \\ \geq LK^{\frac{1}{\alpha-1}} \int_1^t \frac{D(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s, \quad t > 1,\end{aligned}$$

which contradicts (3.6.9) and therefore (3.6.13) holds true.

Next, by(3.6.13), we can take an increasing sequence $t_k \in (1,\infty)$, $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty} \frac{E(\tau_k)}{\tau_k} \cdot \frac{t_k}{E(kt_k)} = \infty, \tag{3.6.14}$$

where we set $\tau_k = \tau_{t_k}$. Without loss of generality we may assume that $2t_k < \tau_k$ for every $k \in \mathbb{N}$. For contradiction, suppose that there exists a subsequence $\{k_j\}$ such that $\tau_{k_j} \leq 2t_{k_j}$. Then, since $E(s)/s$ does not increase and thanks to (2.1.3),

$$\frac{E(\tau_{k_j})}{\tau_{k_j}} \cdot \frac{t_{k_j}}{E(k_j t_{k_j})} \leq \frac{E(2t_{k_j})}{2t_{k_j}} \cdot \frac{t_{k_j}}{E(k_j t_{k_j})} \leq \frac{E(2t_{k_j})}{E(2t_{k_j})} \cdot \frac{1}{k_j} = \frac{1}{k_j} \to 0 \quad \text{as} \quad k \to \infty,$$

which is impossible due to (3.6.14). We may also require that t_{k+1} is chosen in a way that $2t_k \leq \tau_k < t_{k+1}$. Furthermore, from (3.6.8) take t_{k+1} big enough so that

$$2\frac{D(t_k)}{t_k^{1/(1-\alpha)}} \leq \frac{D(t_{k+1})}{t_{k+1}^{1/(1-\alpha)}}. \tag{3.6.15}$$

We now define a function E_1 by the formula

$$E_1(t) = \begin{cases} E(t_k) + \dfrac{E(\tau_k) - E(t_k)}{\tau_k - t_k}\,(t - t_k), & t \in (t_k, \tau_k),\ k \in \mathbb{N}, \\ E(t), & \text{otherwise.} \end{cases} \tag{3.6.16}$$

Obviously, E_1 is a well-defined Young function and $E_1 \geq E$. Moreover, for $k \in \mathbb{N}$, $2E(t_k) \leq E(\tau_k)$ by (2.1.3) and therefore

$$\frac{E_1(2t_k)}{E(kt_k)} = \frac{E(t_k) + \frac{E(\tau_k)-E(t_k)}{\tau_k - t_k}\,t_k}{E(kt_k)} \geq \frac{E(\tau_k) - E(t_k)}{E(kt_k)} \cdot \frac{t_k}{\tau_k} \geq \frac{1}{2} \cdot \frac{E(\tau_k)}{\tau_k} \cdot \frac{t_k}{E(kt_k)}$$

and the latter tends to infinity as $k \to \infty$ by (3.6.14). Consequently

$$\limsup_{t\to\infty} \frac{E_1(t)}{E(\lambda t)} = \infty$$

for every $\lambda \geq 1$ and E_1 essentially dominates E. It remains to show that E_1 fulfills (3.6.11). Let $t > 1$ be fixed and let $j \in \mathbb{N}$ be such that $t \in [t_j, t_{j+1})$. Then we have

$$\int_1^t \frac{E_1(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \leq \int_1^t \frac{E(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s + \sum_{k=1}^{j} \frac{E(\tau_k)-E(t_k)}{\tau_k - t_k} \int_{t_k}^{\tau_k} \frac{s-t_k}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s. \tag{3.6.17}$$

By the assumption, the former integral is dominated by the right hand side of (3.6.10). Let us follow with estimates of the latter sum. Thanks to $2t_k < \tau_k$ and (3.6.12),

$$\frac{E(\tau_k)-E(t_k)}{\tau_k - t_k} \leq \frac{2E(\tau_k)}{\tau_k} = \frac{2D(t_k)}{t_k} \tag{3.6.18}$$

and since $1/(1-\alpha) > 1$, we have

$$\int_{t_k}^{\tau_k} \frac{(s-t_k)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \leq \int_{t_k}^{\infty} \frac{1}{s^{1/(1-\alpha)}}\,\mathrm{d}s \leq \frac{1}{t_k^{1/(1-\alpha)-1}}. \tag{3.6.19}$$

Combination of (3.6.17) and (3.6.18) with (3.6.19) then gives

$$\int_1^t \frac{E_1(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \leq \frac{D(Ct)}{t^{1/(1-\alpha)}} + 2\sum_{k=1}^{j} \frac{D(t_k)}{t_k^{1/(1-\alpha)}}. \tag{3.6.20}$$

It follows from (3.6.15) that

$$\frac{D(t_k)}{t_k^{1/(1-\alpha)}} \leq 2^{k-j} \frac{D(t_j)}{t_j^{1/(1-\alpha)}} \leq 2^{k-j} \frac{D(t)}{t^{1/(1-\alpha)}}, \quad j \leq k < \infty,$$

whence

$$\sum_{k=1}^{j} \frac{D(t_k)}{t_k^{1/(1-\alpha)}} \leq \frac{D(t)}{t^{1/(1-\alpha)}} \sum_{k=1}^{j} 2^{k-j} \leq \frac{2D(t)}{t^{1/(1-\alpha)}}. \tag{3.6.21}$$

The inequality (3.6.27) thus follows by (3.6.20) and (3.6.21). □

Lemma 3.6.5. *Let E be a Young function satisfying*

$$\int_1^\infty \frac{E(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s < \infty. \tag{3.6.22}$$

Then there is a Young function E_1 such that E_1 essentially dominates E near infinity and also

$$\int_1^\infty \frac{E_1(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s < \infty. \tag{3.6.23}$$

Proof. Assume (3.6.22). The procedure is almost the same as the proof of Lemma 3.6.4 with only several modifications. Let us write $D(t) = t^{1/(1-\alpha)}\eta(t)$ where η is some function decreasing to zero at infinity and satisfying

$$\int_1^\infty \eta(s)\frac{\mathrm{d}s}{s} = \infty.$$

Define τ_t exactly the same to obtain (3.6.12) with our D. We also prove (3.6.13) for every $K \geq 1$ since, by contradiction, we have $E(Ks) \geq LD(s)$ on $(1, \infty)$ and hence

$$\int_1^\infty \frac{E(s)}{s^{1/(1-\alpha)+1}} \, \mathrm{d}s \geq K^{\frac{1}{\alpha-1}} \int_1^\infty \frac{E(Ks)}{s^{1/(1-\alpha)+1}} \, \mathrm{d}s \geq LK^{\frac{1}{\alpha-1}} \int_1^\infty \eta(s) \frac{\mathrm{d}s}{s} = \infty$$

which is impossible by (3.6.22). The sequence t_k is then chosen as in (3.6.14) with the difference that instead of (3.6.15) we require

$$\sum_{k=1}^\infty \eta(t_k) < \infty.$$

Now, we set the Young function E_1 by the same formula as in (3.6.16). We then get that E_1 dominates essentially E near infinity and finally

$$\int_1^\infty \frac{E_1(s)}{s^{1/(1-\alpha)+1}} \, \mathrm{d}s \leq \int_1^\infty \frac{E(s)}{s^{1/(1-\alpha)+1}} \, \mathrm{d}s + \sum_{k=1}^\infty \eta(t_k) < \infty.$$

by the combination of the estimates (3.6.17), (3.6.18) and (3.6.19). This proves (3.6.23). □

Lemma 3.6.6. *Let $n \in \mathbb{N}$, $0 < \alpha < 1$ and let D be a Young function such that $D(t)/t^{1/(1-\alpha)}$ is non-decreasing near zero,*

$$\lim_{t \to 0} \frac{D(t)}{t^{1/(1-\alpha)}} = 0 \tag{3.6.24}$$

and

$$\sup_{0<t<1} \frac{t^{1/(1-\alpha)}}{D(Kt)} \int_0^t \frac{D(s)}{s^{1/(1-\alpha)+1}} \, \mathrm{d}s = \infty \tag{3.6.25}$$

for every $K \geq 1$. Suppose that E is a Young function such that

$$\int_0^t \frac{E(s)}{s^{1/(1-\alpha)+1}} \, \mathrm{d}s \leq \frac{D(Ct)}{t^{1/(1-\alpha)}}, \quad 0 < t < 1, \tag{3.6.26}$$

for some $C \leq 1$. Then there exists a Young function E_1 essentially dominating E near zero and satisfying

$$\int_0^t \frac{E_1(s)}{s^{1/(1-\alpha)+1}} \, \mathrm{d}s \leq \frac{D(5Ct)}{t^{1/(1-\alpha)}}, \quad 0 < t < 1. \tag{3.6.27}$$

Proof. Let D and E be the Young functions from the statement. Fix $t \in (0,1)$ and define the set G_t by

$$G_t = \left\{ s \in (0,1) : \tfrac{E(s)}{s} \leq \tfrac{D(t)}{t} \right\}.$$

We claim that $E(s)/s$ is a non-decreasing mapping from $(0,1)$ onto some neighbourhood of zero, and hence the sets G_t are nonempty for every $t \in (0,1)$. Indeed, if $\lim_{s \to 0+} E(s)/s > 0$, then $E(s) \geq cs$ on $(0,1)$ for some $c > 0$ and thus

$$\int_0 \frac{E(s)}{s^{1/(1-\alpha)+1}} \, \mathrm{d}s \geq c \int_0 \frac{\mathrm{d}s}{s^{1/(1-\alpha)}} = \infty$$

which contradicts (3.6.26).

Let us define $\tau = \tau_t = \sup G_t$. Observe that, by the continuity of Young functions,

$$\frac{E(\tau)}{\tau} = \frac{D(t)}{t}, \quad 0 < t < 1. \tag{3.6.28}$$

Also,

$$\limsup_{t\to 0+} \frac{E(\tau_t)}{\tau_t} \cdot \frac{t}{E(Kt)} = \infty \tag{3.6.29}$$

for every $K \geq 1$. Indeed, suppose that there is some $K \geq 1$ for which (3.6.29) is violated. We then have some $L > 0$ such that

$$\frac{E(Kt)}{t} \geq L\frac{E(\tau)}{\tau}, \quad 0 < t < 1,$$

which in connection with (3.6.28) gives $E(Ks) \geq L\,D(s)$ on $(0,1)$. Thus

$$\begin{aligned}\frac{D(CKt)}{t^{1/(1-\alpha)}} \geq \int_0^{Kt} \frac{E(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s = K^{\frac{1}{\alpha-1}} \int_0^t \frac{E(Ks)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \\ \geq LK^{\frac{1}{\alpha-1}} \int_0^t \frac{D(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s, \quad 0 < t < 1,\end{aligned}$$

which contradicts (3.6.25) and therefore (3.6.29) holds true.

Next, by(3.6.29), we can take a decreasing sequence $t_k \in (0,1)$, $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty} \frac{E(\tau_k)}{\tau_k} \cdot \frac{t_k}{E(kt_k)} = \infty, \tag{3.6.30}$$

where we set $\tau_k = \tau_{t_k}$. Without loss of generality we may assume that $2t_k < \tau_k$ for every $k \in \mathbb{N}$. For contradiction, suppose that there exists a subsequence $\{k_j\}$ such that $\tau_{k_j} \leq 2t_{k_j}$. Then, since $E(s)/s$ does not increase and thanks to (2.1.3),

$$\frac{E(\tau_{k_j})}{\tau_{k_j}} \cdot \frac{t_{k_j}}{E(k_j t_{k_j})} \leq \frac{E(2t_{k_j})}{2t_{k_j}} \cdot \frac{t_{k_j}}{E(k_j t_{k_j})} \leq \frac{E(2t_{k_j})}{E(2t_{k_j})} \cdot \frac{1}{k_j} = \frac{1}{k_j} \to 0 \quad \text{as} \quad k \to \infty,$$

which is impossible due to (3.6.30). We may also require that t_{k+1} is chosen in a way that $2t_{k+1} \leq \tau_{k+1} < t_k$, which is ensured if $\tau_t \to 0$ as $t \to 0$. To observe that, by (3.6.28), we need to have $\lim_{t\to 0+} D(t)/t = 0$ which is however guaranteed by the stronger condition (3.6.24). Furthermore, from (3.6.24) take t_{k+1} small enough so that

$$\frac{D(t_{k+1})}{t_{k+1}^{1/(1-\alpha)}} \leq \frac{1}{2} \cdot \frac{D(t_k)}{t_k^{1/(1-\alpha)}}. \tag{3.6.31}$$

We now define a function E_1 by the formula

$$E_1(t) = \begin{cases} E(t_k) + \dfrac{E(\tau_k) - E(t_k)}{\tau_k - t_k}\,(t - t_k), & t \in (t_k, \tau_k),\ k \in \mathbb{N}, \\ E(t), & \text{otherwise.} \end{cases}$$

Obviously, E_1 is a well-defined Young function and $E_1 \geq E$. Moreover, for $k \in \mathbb{N}$, $2E(t_k) \leq E(\tau_k)$ by (2.1.3) and therefore

$$\frac{E_1(2t_k)}{E(kt_k)} = \frac{E(t_k) + \frac{E(\tau_k)-E(t_k)}{\tau_k - t_k}\,t_k}{E(kt_k)} \geq \frac{E(\tau_k) - E(t_k)}{E(kt_k)} \cdot \frac{t_k}{\tau_k} \geq \frac{1}{2} \cdot \frac{E(\tau_k)}{\tau_k} \cdot \frac{t_k}{E(kt_k)}$$

and the latter tends to infinity as $k \to \infty$ by (3.6.30). Consequently

$$\limsup_{t\to 0+} \frac{E_1(t)}{E(\lambda t)} = \infty$$

for every $\lambda \geq 1$ and E_1 essentially dominates E. It remains to show that E_1 fulfills (3.6.27). Let $t \in (0,1)$ be fixed and let $j \in \mathbb{N}$ be such that $t \in [t_j, t_{j+1})$. Then we have

$$\int_0^t \frac{E_1(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \leq \int_0^t \frac{E(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s + \sum_{k=j}^{\infty} \frac{E(\tau_k)-E(t_k)}{\tau_k - t_k} \int_{t_k}^{\tau_k} \frac{s-t_k}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s. \tag{3.6.32}$$

By the assumption, the former integral is dominated by the right hand side of (3.6.26). Let us follow with estimates of the latter sum. Thanks to $2t_k < \tau_k$ and (3.6.28),

$$\frac{E(\tau_k)-E(t_k)}{\tau_k - t_k} \leq \frac{2E(\tau_k)}{\tau_k} = \frac{2D(t_k)}{t_k} \tag{3.6.33}$$

and since $1/(1-\alpha) > 1$, we have

$$\int_{t_k}^{\tau_k} \frac{(s-t_k)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \leq \int_{t_k}^{\infty} \frac{1}{s^{1/(1-\alpha)}}\,\mathrm{d}s \leq \frac{1}{t_k^{1/(1-\alpha)-1}}. \tag{3.6.34}$$

Combination of (3.6.32) and (3.6.33) with (3.6.34) then gives

$$\int_0^t \frac{E_1(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \leq \frac{D(Ct)}{t^{1/(1-\alpha)}} + 2\sum_{k=j}^{\infty} \frac{D(t_k)}{t_k^{1/(1-\alpha)}}. \tag{3.6.35}$$

It follows from (3.6.31) that

$$\frac{D(t_k)}{t_k^{1/(1-\alpha)}} \leq 2^{j-k} \frac{D(t_j)}{t_j^{1/(1-\alpha)}} \leq 2^{j-k} \frac{D(t)}{t^{1/(1-\alpha)}}, \quad j \leq k < \infty,$$

whence

$$\sum_{k=j}^{\infty} \frac{D(t_k)}{t_k^{1/(1-\alpha)}} \leq \frac{D(t)}{t^{1/(1-\alpha)}} \sum_{k=j}^{\infty} 2^{j-k} \leq \frac{2D(t)}{t^{1/(1-\alpha)}}. \tag{3.6.36}$$

The inequality (3.6.27) thus follows by (3.6.35) and (3.6.36). □

Lemma 3.6.7. *Let E be a Young function satisfying*

$$\int_0^1 \frac{E(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s < \infty. \tag{3.6.37}$$

Then there is a Young function E_1 such that E_1 essentially dominates E near zero and also

$$\int_0^1 \frac{E_1(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s < \infty. \tag{3.6.38}$$

Proof. Assume that E is given and fulfills (3.6.37). Let us define $d_k = 1/\log(k+1)$, $k \in \mathbb{N}$, and let t_k, $k \in \mathbb{N}$, satisfy $t_k \leq t_{k-1}/k$ and

$$\int_0^{t_k} \frac{E(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \leq d_k \quad \text{for } k \in \mathbb{N}.$$

Then define D_1 by

$$D_1(t) = \sum_{k=1}^{\infty} d_k t^{\frac{1}{1-\alpha}} \chi_{[t_{k+1},t_k)}(t) + C t^{\frac{1}{1-\alpha}} \chi_{[t_1,\infty)}(t), \quad t \geq 0,$$

where C is a constant which dominates the integral in (3.6.37), and set D by

$$D(t) = \int_0^{2t} \frac{D_1(s)}{s} \, ds, \quad t \geq 0.$$

Since D_1 is non-decreasing, D is a Young function and $D_1(t) \leq D(t)$.

We shall show that E and D satisfy the assumptions of Lemma 3.6.6. Clearly $D(t)/t^{1/(1-\alpha)}$ is non-decreasing and $\lim_{t\to 0+} D(t)/t^{1/(1-\alpha)} = \lim_{k\to\infty} d_k = 0$. Next, let $t \in (0,1)$ be arbitrary and let $j \in \mathbb{N}$ be such that $t \in [t_{j+1}, t_j)$. Then

$$\begin{aligned} \int_0^t \frac{D(s)}{s^{1/(1-\alpha)+1}} \, ds &\geq \int_0^t \frac{D_1(s)}{s^{1/(1-\alpha)+1}} \, ds \geq \sum_{k=j+1}^{\infty} \int_{t_{k+1}}^{t_k} \frac{D_1(s)}{s^{1/(1-\alpha)+1}} \, ds \\ &\geq \sum_{k=j+1}^{\infty} d_k \int_{t_{k+1}}^{t_k} \frac{ds}{s} \geq \sum_{k=j+1}^{\infty} d_k \log(k+1) = \infty \end{aligned}$$

which gives (3.6.25) and also

$$\int_0^t \frac{E(s)}{s^{1/(1-\alpha)+1}} \, ds \leq \int_0^{t_j} \frac{E(s)}{s^{1/(1-\alpha)+1}} \, ds = d_j = \frac{D_1(t)}{t^{1/(1-\alpha)}} \leq \frac{D(t)}{t^{1/(1-\alpha)}}$$

which is (3.6.26). Lemma 3.6.6 gives us a Young function E_1 essentially dominating E such that

$$\int_0^t \frac{E_1(s)}{s^{1/(1-\alpha)+1}} \, ds \leq \frac{D(5t)}{t^{1/(1-\alpha)}}, \quad 0 < t < 1.$$

Then we have (3.6.38) as a special case. □

We next analyze connections between the Boyd indices of a Young function B, and those of the Young functions $B_{\alpha,\beta}$ and $B_{\alpha,\beta}^{\infty}$ defined in (3.5.1) and (3.5.6), respectively.

Let us preliminarily observe that

$$1 \leq I_{B_{\alpha,\beta}} \leq \frac{1}{\alpha} \qquad \left[1 \leq I_{B_{\alpha,\beta}^{\infty}}^{\infty} \leq \frac{1}{\alpha} \right]$$

for every B. Indeed, as one gets from (3.5.3) and (3.5.4),

$$B_{\alpha,\beta}^{-1}(t)\, t^{-\alpha} \simeq \inf_{1\leq s<\infty} B^{-1}(s^{1/\beta}) \max\left\{1, \tfrac{t}{s}\right\}^{1-\alpha} \quad \text{for } t \geq 1, \tag{3.6.39}$$

and that the right-hand side of (3.6.39) is a non-decreasing function. As for the global version, we get

$$(B_{\alpha,\beta}^{\infty})^{-1}(t)\, t^{-\alpha} \simeq \inf_{0\leq s<\infty} B^{-1}(s^{1/\beta}) \max\left\{1, \tfrac{t}{s}\right\}^{1-\alpha} \quad \text{for } t > 0, \tag{3.6.40}$$

and the conclusion follows as above.

The next lemma tells us that, under a suitable lower bound for the lower Boyd index of B, the infimum on the right-hand side of equations (3.5.2) and (3.5.7) can be disregarded.

Lemma 3.6.8. *Let $0 < \alpha < 1$, $\beta > 0$ and $\alpha + 1/\beta \geq 1$ and let B be a Young function. Assume that*

$$i_B > \frac{1}{\beta(1-\alpha)}. \tag{3.6.41}$$

Then

$$\inf_{1\leq s\leq t} B^{-1}(s^{1/\beta})\, s^{\alpha-1} \simeq B^{-1}(t^{1/\beta})\, t^{\alpha-1} \quad \textit{near infinity.} \tag{3.6.42}$$

Hence,

$$B^{-1}_{\alpha,\beta}(t) \simeq B^{-1}(t^{1/\beta})\, t^{\alpha} \quad \textit{near infinity.}$$

Conversely if (3.6.42) *holds, then* $i_B \geq \frac{1}{\beta(1-\alpha)}$.

Lemma 3.6.9. *Let $0 < \alpha < 1$, $\beta > 0$ and $\alpha + 1/\beta \geq 1$. Let B be a Young function, and assume in addition that* (3.5.5) *holds. If*

$$i_B^\infty > \frac{1}{\beta(1-\alpha)}, \tag{3.6.43}$$

then

$$\inf_{0<s\leq t} B^{-1}(s^{1/\beta})\, s^{\alpha-1} \simeq B^{-1}(t^{1/\beta})\, t^{\alpha-1} \quad \textit{for } t > 0. \tag{3.6.44}$$

Hence,

$$B^{\infty}_{\alpha,\beta}{}^{-1}(t) \simeq B^{-1}(t^{1/\beta})\, t^{\alpha} \quad \textit{for } t > 0. \tag{3.6.45}$$

Conversely if (3.6.44) *holds, then* $i_B^\infty \geq \frac{1}{\beta(1-\alpha)}$.

We limit ourselves to proving Lemma 3.6.9; the proof of Lemma 3.6.8 requires minor modifications.

Proof of Lemma 3.6.9. If B is infinite for large values of its argument, then the its generalized inverse B^{-1} is constant near infinity, and equation (3.6.44) holds trivially.

In the remaining part of this proof, we may thus assume that the function B is finite-valued. Equation (3.6.44) is equivalent to

$$\inf_{0<s\leq t} \widetilde{B}(s)\, s^{\frac{1}{\beta(1-\alpha)-1}} \simeq \widetilde{B}(t)\, t^{\frac{1}{\beta(1-\alpha)-1}} \quad \text{for } t > 0. \tag{3.6.46}$$

Indeed, owing to (2.1.5), condition (3.6.44) is equivalent to

$$\inf_{0<s\leq t} \frac{s^{\beta(\alpha-1)+1}}{\widetilde{B}^{-1}(s)} \simeq \frac{t^{\beta(\alpha-1)+1}}{\widetilde{B}^{-1}(t)} \quad \text{for } t > 0, \tag{3.6.47}$$

and equation (3.6.47) is in turn equivalent to (3.6.46). On the other hand, by Proposition 3.2.2, condition (3.6.43) is equivalent to

$$I_B^\infty < \eta\,, \tag{3.6.48}$$

where we have set $\eta = \frac{1}{\beta(\alpha-1)+1}$. The same proposition ensures that condition (3.6.48) is equivalent to the inequality

$$\int_t^\infty \widetilde{B}(s)\, s^{-\eta-1}\, \mathrm{d}s \leq \widetilde{B}(kt)\, t^{-\eta} \quad \text{for } t > 0, \tag{3.6.49}$$

for some constant $k > 1$. Hence it suffices to show that (3.6.49) implies (3.6.46). To this purpose, denote by $\rho \in [0, t]$ a number satisfying

$$\inf_{0<s\le t} \widetilde{B}(s)\, s^{-\eta} = \widetilde{B}(\rho t)\, (\rho t)^{-\eta} \quad \text{for } t > 0.$$

By the same argument as in the derivation (iii) from (i) in Proposition 3.2.2, one has that

$$\widetilde{B}(\rho t)\, (\rho t)^{-\eta} k^{\eta} \ge \int_{\rho t/k}^{\infty} \widetilde{B}(s)\, s^{-\eta-1} \mathrm{d}s \ge \int_{\rho t}^{t} \widetilde{B}(s)\, s^{-\eta-1} \mathrm{d}s \ge \widetilde{B}(\rho t)\, (\rho t)^{-\eta} \log \frac{1}{\rho}$$

for $t > 0$, whence $k^{\eta} \ge \log \frac{1}{\rho}$, and

$$\rho \ge e^{-k^{\eta}} > 0 \quad \text{for } t > 0.$$

In the proof of Proposition 3.2.2 it is also shown that $\widetilde{B}$ satisfies the Δ_2-condition. Hence, there exists a positive constant c such that

$$\widetilde{B}(\rho t) \ge \widetilde{B}(t e^{-k\eta}) \ge c\widetilde{B}(t) \quad \text{for } t > 0.$$

Consequently,

$$\inf_{0<s\le t} \widetilde{B}(s)\, s^{-\eta} = \widetilde{B}(\rho t)\, (\rho t)^{-\eta} \ge c\rho^{-\eta} \widetilde{B}(t)\, t^{-\eta} \quad \text{for } t > 0,$$

whence (3.6.46) follows.

Finally, if (3.6.44) is in force, then $B^{-1}(t)\, t^{\beta(\alpha-1)}$ is equivalent to a non-increasing function, and therefore $i_B^{\infty} \ge \frac{1}{\beta(1-\alpha)}$. □

We conclude the preliminary result section by showing that, under assumption (3.6.41) or (3.6.43), the upper Boyd indices of B and $B_{\alpha,\beta}$, or of B and $B^{\infty}_{\alpha,\beta}$ are determined by each other. In what follows, we adopt the convention that $\frac{1}{\infty} = 0$.

Lemma 3.6.10. *Let α and β be as in Lemma 3.6.8 and let B be a Young function. Assume that condition* (3.6.41) *holds. Then*

$$\frac{1}{I_{B_{\alpha,\beta}}} = \alpha + \frac{1}{\beta I_B}.$$

In particular, $I_{B_{\alpha,\beta}} < 1/\alpha$ if and only if $I_B < \infty$.

Lemma 3.6.11. *Let α and β be as in Lemma 3.6.9 and let B be a Young function that satisfies condition* (3.5.5)*. If* (3.6.43) *holds, then*

$$\frac{1}{I^{\infty}_{B^{\infty}_{\alpha,\beta}}} = \alpha + \frac{1}{\beta I^{\infty}_B}. \tag{3.6.50}$$

In particular, $I^{\infty}_{B^{\infty}_{\alpha,\beta}} < 1/\alpha$ if and only if $I^{\infty}_B < \infty$.

As before, we only prove Lemma 3.6.11; the proof if Lemma 3.6.10 is similar.

Proof of Lemma 3.6.11. By Lemma 3.6.9, assumption (3.6.43) implies equation (3.6.45). Thereby,

$$h^{\infty}_{B^{\infty}_{\alpha,\beta}}(t) \simeq \sup_{s>0} \frac{{B^{\infty}_{\alpha,\beta}}^{-1}(st)}{{B^{\infty}_{\alpha,\beta}}^{-1}(s)} \simeq t \sup_{s>0} \frac{B^{-1}\Big((st)^{1/\beta}\Big)\,(st)^{\alpha-1}}{B^{-1}\Big(s^{1/\beta}\Big)\,s^{\alpha-1}} \simeq t^{\alpha}\, h^{\infty}_{B}\Big(t^{1/\beta}\Big)$$

for $t > 0$. Equation (3.6.50) is therefore a consequence of the definition of global upper Boyd index. □

Now, we are ready prove the Theorem 3.6.1.

Proof of Theorem 3.6.1. Let B be a Young function satisfying (3.5.5). Note that since (3.5.8) and $G^{\infty}_{\alpha,\beta}$ increases to infinity, ${B^{\infty}_{\alpha,\beta}}^{-1}$ is not constant near infinity and hence $B^{\infty}_{\alpha,\beta}$ is finite-valued.

Let us assume (3.6.1). By Proposition 3.2.2, (3.6.1) is equivalent to the inequality

$$\int_0^t \frac{\widetilde{B^{\infty}_{\alpha,\beta}}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \le \frac{\widetilde{B^{\infty}_{\alpha,\beta}}(Ct)}{t^{1/(1-\alpha)}} \quad \text{for } t > 0 \tag{3.6.51}$$

with some constant $C > 0$. Theorem 3.5.2 then guarantees (3.6.2).

We now prove the optimality. Suppose that $L^A(0,\infty)$ satisfies (3.1.3). Then, by Theorem 3.5.2, there is a constant $C > 0$ such that

$$\int_0^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \le \frac{\widetilde{B^{\infty}_{\alpha,\beta}}(Ct)}{t^{1/(1-\alpha)}} \quad \text{for } t > 0. \tag{3.6.52}$$

Thus,

$$\begin{aligned}\widetilde{B^{\infty}_{\alpha,\beta}}(2Ct) &\ge t^{1/(1-\alpha)} \int_0^{2t} \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \ge t^{1/(1-\alpha)} \int_t^{2t} \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \\ &\ge \widetilde{A}(t)\, t^{1/(1-\alpha)} \int_t^{2t} \frac{\mathrm{d}s}{s^{1/(1-\alpha)+1}} \ge \widetilde{A}(t) C_\alpha\end{aligned}$$

for $t > 0$ and hence $\widetilde{B^{\infty}_{\alpha,\beta}}$ dominates $\widetilde{A}$ globally or, equivalently, A dominates $B^{\infty}_{\alpha,\beta}$ globally, whence, by (2.3.4), $L^A(0,\infty) \to L^{B^{\infty}_{\alpha,\beta}}(0,\infty)$ and $L^{B^{\infty}_{\alpha,\beta}}(0,\infty)$ is optimal.

Conversely, suppose that (3.6.1) fails. Then, by Proposition 3.2.2, the inequality (3.6.51) is violated for every $C > 0$. The failure of (3.6.51) occurs under one of these two conditions, namely

$$\sup_{1<t<\infty} \frac{t^{1/(1-\alpha)}}{\widetilde{B^{\infty}_{\alpha,\beta}}(Ct)} \int_1^t \frac{\widetilde{B^{\infty}_{\alpha,\beta}}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s = \infty \quad \text{for every } C > 0 \tag{3.6.53}$$

or

$$\sup_{0<t<1} \frac{t^{1/(1-\alpha)}}{\widetilde{B^{\infty}_{\alpha,\beta}}(Ct)} \int_0^t \frac{\widetilde{B^{\infty}_{\alpha,\beta}}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s = \infty \quad \text{for every } C > 0. \tag{3.6.54}$$

Now, let A be a Young function such that (3.1.3), i.e., thanks to Theorem 3.5.2, the inequality (3.6.52) holds for some $C > 0$. In both cases, we will show that there is a Young function A_1 such that $L^A(0,\infty) \subsetneq L^{A_1}(0,\infty)$ and also

$$H^{\infty}_{\alpha,\beta}\colon L^{A_1}(0,\infty) \to L^B(0,\infty). \tag{3.6.55}$$

or equivalently, by Theorem 3.5.2,

$$\int_0^t \frac{\widetilde{A_1}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \le \frac{\widetilde{B^{\infty}_{\alpha,\beta}}(Ct)}{t^{1/(1-\alpha)}} \quad \text{for } t>0, \tag{3.6.56}$$

where C is a possibly different constant. If (3.6.53) holds, we modify the Young function A only near infinity and, similarly, if (3.6.54) holds, we do that near zero.

Let us start with the "near infinity" case. First observe that

$$\widetilde{B^{\infty}_{\alpha,\beta}}(t)/t^{1/(1-\alpha)} \quad \text{is non-decreasing on } (0,\infty). \tag{3.6.57}$$

Indeed, by (3.6.40), ${B^{\infty}_{\alpha,\beta}}^{-1}(t)/t^{\alpha}$ is non-decreasing which, in combination with (2.1.5), tells us that ${\widetilde{B^{\infty}_{\alpha,\beta}}}^{-1}(t)/t^{1-\alpha}$ is non-increasing whence (3.6.57). Therefore, the fraction $\widetilde{B^{\infty}_{\alpha,\beta}}(t)/t^{1/(1-\alpha)}$ is either bounded near infinity or

$$\lim_{t\to\infty} \frac{\widetilde{B^{\infty}_{\alpha,\beta}}(t)}{t^{1/(1-\alpha)}} = \infty. \tag{3.6.58}$$

In the former case, (3.6.52) reads as

$$\int_0^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \le \frac{\widetilde{B^{\infty}_{\alpha,\beta}}(Ct)}{t^{1/(1-\alpha)}}, \quad 0<t<1, \tag{3.6.59}$$

and

$$\int_1^{\infty} \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s < \infty.$$

Denote $E=\widetilde{A}$. By Lemma 3.6.5, there is a modification of E on $(1,\infty)$, say E_1, such that E_1 essentially dominates E near infinity and also

$$\int_1^{\infty} \frac{E_1(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s < \infty.$$

Let now A_1 be a Young function which coincides with A near zero and $A_1=\widetilde{E_1}$ near infinity. Then A essentially dominates A_1 near infinity and (3.6.59) holds with A replaced by A_1. We therefore have (3.6.56) which yields (3.6.55).

Now, assume (3.6.53) and (3.6.58). The condition (3.6.52) now splits into (3.6.59) and

$$\int_1^t \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s \le \frac{\widetilde{B^{\infty}_{\alpha,\beta}}(Ct)}{t^{1/(1-\alpha)}}, \quad 1<t<\infty. \tag{3.6.60}$$

By Lemma 3.6.4, there exists a modified Young function A_1 such that A is essentially larger than A_1 near infinity and also satisfies (3.6.60) with A_1 in place of A. If we keep $A_1=A$ near zero, then (3.6.59) remains valid for A_1. That gives us (3.6.56) also in this case.

Let us work "near zero". We again distinguish two cases, when $\widetilde{B^{\infty}_{\alpha,\beta}}(t)/t^{1/(1-\alpha)}$ is equivalent to a constant function near zero and when

$$\lim_{t\to 0} \frac{\widetilde{B^{\infty}_{\alpha,\beta}}(t)}{t^{1/(1-\alpha)}} = 0. \tag{3.6.61}$$

In the constant case, (3.6.52) boils down to (3.6.60) and

$$\int_0^1 \frac{\widetilde{A}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s < \infty.$$

Lemma 3.6.7 ensures that there is an essentially smaller Young function than A near zero, A_1 say, such that

$$\int_0^1 \frac{\widetilde{A_1}(s)}{s^{1/(1-\alpha)+1}}\,\mathrm{d}s < \infty.$$

Let $A_1 = A$ near infinity. We thus have that (3.6.60) holds for A replaced by A_1 and therefore (3.6.56) is true also in this case.

Finally, assume (3.6.54) and (3.6.61). The inequality (3.6.52) splits into (3.6.59) and (3.6.60). On using Lemma 3.6.6, one gets a modified Young function A_1 such that A essentially dominates A_1 near zero and still satisfies (3.6.59). Again, if we set $A_1 = A$ near infinity, we obtain (3.6.56).

As for the proof of the "in particular" statement, assume that the condition $i_B^\infty > \frac{1}{\beta(1-\alpha)}$ is satisfied. Then the simplified relation (3.6.3) holds by Lemma 3.6.9 and (3.6.1) is equivalent to $I_B^\infty < \infty$ due to Lemma 3.6.11.

The necessity of the condition (3.5.5) for the existence of any Orlicz space $L^A(0,\infty)$ satisfying (3.1.3) follows by Theorem 3.5.2. □

4. Sobolev type embeddings

4.1 Introduction

The present chapter deals with Orlicz-Sobolev embeddings, namely embeddings of Sobolev type, involving norms in Orlicz spaces. The family of Orlicz spaces includes that of the usual Lebesgue spaces, and provides a flexible, well suited framework for a unified description of Sobolev embeddings. Orlicz-Sobolev spaces are an appropriate functional setting for the analysis of nonlinear partial differential equations and variational problems governed by nonlinearities of non-necessarily polynomial type. The study of these problems has received an increasing attention over the years – see e.g. [1, 6, 7, 12, 13, 14, 15, 39, 50, 52, 54, 68, 69, 72] – and is motivated, among other reasons, by applications to mathematical models for physical phenomena, such as nonlinear elasticity and non-Newtonian fluid-mechanics.

A basic version of the Orlicz-Sobolev embeddings to be considered amounts to

$$W_0^{m,A}(\Omega) \to L^B(\Omega)\,, \tag{4.1.1}$$

where Ω is an open subset of Euclidean space $\mathbb{R}^n$, $n \geq 2$, having Lebesgue measure $|\Omega|$, A and B are Young functions, $L^B(\Omega)$ is the Orlicz space on Ω built upon B, and $W_0^{m,A}(\Omega)$ is the m-th order Orlicz-Sobolev space built upon A. The subscript 0 denotes that functions vanishing (in a suitable sense) on the boundary $\partial\Omega$, together with their derivatives up to the order $m-1$, are taken into account. The arrow " $\to$ " stands for continuous inclusion. Precise definitions on these topics are recalled in Section 2.

We are concerned with the optimal form of the relevant embeddings. Given A, we say that $L^B(\Omega)$ is the optimal Orlicz target space in (4.1.1) if it is the smallest Orlicz space on Ω that renders (4.1.1) true. The expression "smallest" means that if (4.1.1) holds with $L^B(\Omega)$ replaced with another Orlicz space $L^{\widehat{B}}(\Omega)$, then $L^B(\Omega) \to L^{\widehat{B}}(\Omega)$. Analogously, given B, the space $W_0^{m,A}(\Omega)$ is said to be the optimal Orlicz-Sobolev domain in (4.1.1) if it is the largest Orlicz-Sobolev space on Ω for which (4.1.1) holds. Namely, if, whenever (4.1.1) holds with $W_0^{m,A}(\Omega)$ replaced by another Orlicz-Sobolev space $W_0^{m,\widehat{A}}(\Omega)$, then $W_0^{m,\widehat{A}}(\Omega) \to W_0^{m,A}(\Omega)$.

The question of best possible Orlicz target spaces in Sobolev type embeddings has attracted the attention of various authors over the years. In particular, embeddings for the critical Sobolev space $W_0^{m,\frac{n}{m}}(\Omega)$, and for special Orlicz-Sobolev spaces "close" to it, have been investigated in several contributions, including [73, 61, 66, 70, 42, 55, 68, 35]. Results for arbitrary Orlicz-Sobolev spaces, which however need not provide the optimal Orlicz target, can be found in [34, 2].

The optimal Orlicz target problem has been solved in general in [24] for $m = 1$ (see also [22] for an alternate formulation of the solution), and in [23] for arbitrary $m \in \mathbb{N}$. We present this result in Theorem 4.3.1. As it is shown, given any Orlicz-Sobolev space $W_0^{m,A}(\Omega)$, there always exists an optimal target Orlicz space $L^B(\Omega)$ in (4.1.1), and the function B admits an explicit expression in terms of A, n and m. Thus, the class of Orlicz spaces is closed under the operation of associating an optimal target in Sobolev embeddings. By contrast, this property is not enjoyed by the smaller family of Lebesgue spaces, namely in the context of classical Sobolev embeddings. Actually, if $A(t) = t^p$ for some $p \geq 1$, so that $W_0^{m,A}(\Omega)$ agrees with the usual Sobolev space $W_0^{m,p}(\Omega)$, and $|\Omega| < \infty$, one has that

$$W_0^{m,p}(\Omega) \to \begin{cases} L^{\frac{mp}{n-mp}}(\Omega) & \text{if } 1 \leq m < n \text{ and } 1 \leq p < \frac{n}{m}, \\ \exp L^{\frac{n}{n-m}}(\Omega) & \text{if } 1 \leq m < n \text{ and } p = \frac{n}{m}, \\ L^\infty(\Omega) & \text{if either } 1 \leq m < n \text{ and } p > \frac{n}{m}, \text{ or } m \geq n, \end{cases} \tag{4.1.2}$$

all targets being optimal in the class of Orlicz spaces. Here, $\exp L^{\frac{n}{n-m}}(\Omega)$ denotes the Orlicz space associated with the Young function $e^{t^{\frac{n}{n-m}}} - 1$. The first and the third embedding in (4.1.2) are nothing but the classical Sobolev embedding. The second one was independently obtained by Yudovich [73], Pokhozhaev [61], Strichartz [66], and, for $m = 1$, by Trudinger [70]. Note that, in the first and third embedding, the target is a Lebesgue space, and it is hence optimal also in this subclass, but no optimal Lebesgue target space exists in the second embedding.

The situation is different, and subtler in a sense, when the optimal Orlicz-Sobolev domain space $W_0^{m,A}(\Omega)$ in (4.1.1), for a given Young function B, is in question. Actually, the existence of such an optimal domain is not guaranteed for every B. Testing the problem on the spaces appearing in (4.1.2) may help have an idea of the possibilities that may occur. Assume that

$$L^B(\Omega) = L^q(\Omega)$$

for some $q \in [1, \infty]$. It is well known that, if $m \geq n$, then $W_0^{m,1}(\Omega) \to L^\infty(\Omega)$, and hence, in particular,

$$W_0^{m,1}(\Omega) \to L^q(\Omega), \tag{4.1.3}$$

for every $q \in [1, \infty]$. Embedding (4.1.3) continues to hold even if $1 \leq m < n$, provided that $q \leq \frac{n}{n-m}$. On the other hand, if $1 \leq m < n$ and $\frac{n}{n-m} < q < \infty$, then

$$W_0^{m,\frac{nq}{n+mq}}(\Omega) \to L^q(\Omega). \tag{4.1.4}$$

Both domain spaces in (4.1.3) and (4.1.4) are optimal among all Orlicz-Sobolev spaces [57, Example 5.2]. Instead, if $1 \leq m < n$,

$$\text{“no optimal Orlicz-Sobolev space”} \to \exp L^{\frac{n}{n-m}}(\Omega) \tag{4.1.5}$$

and

$$\text{“no optimal Orlicz-Sobolev space”} \to L^\infty(\Omega), \tag{4.1.6}$$

see [51, Theorem 4.3] and [30, Theorem 6.4 (ii)], respectively, for the case when $m = 1$, and [57, Example 5.1 (b)] for arbitrary $m \in \mathbb{N}$. Equation (4.1.5) means that any Orlicz-Sobolev space that is continuously embedded into the Orlicz space $\exp L^{\frac{n}{n-m}}(\Omega)$ can be replaced with a strictly larger Orlicz-Sobolev space which is still continuously embedded into $\exp L^{\frac{n}{n-m}}(\Omega)$. Equation (4.1.6), as well as similar statements about non-existence of optimal Orlicz-Sobolev domain spaces in what follows, has to be interpreted in an analogous sense. In particular, interestingly enough, the space $W_0^{m,\frac{n}{m}}(\Omega)$, appearing on the left-hand side of (4.1.2) when $p = \frac{n}{m}$, turns out not to be optimal for Orlicz-Sobolev embeddings into $\exp L^{\frac{n}{n-m}}(\Omega)$.

As far as we know, these were the only instances for which the answer to the optimal Orlicz-Sobolev domain problem is available in the literature. The recent contribution [57] provides a solution to an analogous problem for Orlicz-Sobolev embeddings of weak type, namely into Marcinkiewicz spaces, and the paper [27] fills in this gap, and address this question in full generality. A necessary and sufficient condition is established on the Young function B for an optimal Orlicz-Sobolev domain $W_0^{m,A}(\Omega)$ to exist in (4.1.1). Moreover, the optimal Young function A when such an optimal domain does exist is exhibited. This is the content of Theorem 4.4.2.

The aim of this chapter is to collect most of the available results from the literature and present them in a unified and comprehensive form.

As mentioned above, we are not confined just to (4.1.1), but we also include other related embedding problems. A natural variant amounts to

$$W^{m,A}(\Omega) \to L^B(\Omega), \tag{4.1.7}$$

where $W^{m,A}(\Omega)$ is an Orlicz-Sobolev space of functions that are not subject to any boundary condition. Under suitable regularity assumptions on Ω, which are indispensable even in the classical Sobolev embedding, we show that the conclusions are exactly the same as for (4.1.1) – see Theorems 4.3.3 and 4.4.4.

Embeddings of the form (4.1.7), with $\Omega = \mathbb{R}^n$, namely

$$W^{m,A}(\mathbb{R}^n) \to L^B(\mathbb{R}^n), \tag{4.1.8}$$

are the subject of Theorems 4.3.7 and 4.4.9. The point here is that, unlike the case of sets Ω of finite measure, the behavior of the Young functions A and B near 0 plays a role as well.

Finally, in Theorems 4.3.8 and 4.4.10 the more general issue is faced of optimal Orlicz-Sobolev domains for embeddings into Orlicz spaces with respect to a Frostman measure μ on $\overline{\Omega}$. These read

$$W^{m,A}(\Omega) \to L^B(\overline{\Omega}, \mu), \tag{4.1.9}$$

where Ω is a bounded Lipschitz domain, $\overline{\Omega}$ denotes the closure of Ω, and μ is a Borel measure on $\overline{\Omega}$ such that

$$\mu\Big(B_r(x) \cap \overline{\Omega}\Big) \le C r^\gamma \quad \text{for every } x \in \mathbb{R}^n \text{ and } r > 0, \tag{4.1.10}$$

for some constants $C > 0$ and $\gamma \in [n-m, n]$. Here, $B_r(x)$ denotes the ball centered at x, with radius r. The restriction $\gamma \ge n-m$ is imposed to guarantee that a trace operator on $\overline{\Omega}$, endowed with the measure μ, be well defined on the space $W^{m,A}(\Omega)$, whatever the Young function A is.

Of course, measures μ supported in Ω, and hence embeddings into Orlicz spaces $L^B(\Omega, \mu)$, are included as special cases. On the other hand, measures μ supported in $\partial\Omega$ correspond to trace inequalities in a classical sense. In particular, on denoting by $\mathcal{H}^\gamma$ the γ-dimensional Hausdorff measure, the choice $\mu = \mathcal{H}^{n-1}|_{\partial\Omega}$ turns (4.1.9) into the boundary trace embedding

$$\mathrm{Tr}\colon W^{m,A}(\Omega) \to L^B(\partial\Omega) \tag{4.1.11}$$

enucleated in Corollaries 4.3.12 and 4.4.13. Here, we thus recover the original result of [24]. Another customary specialization of μ amounts to the case when

$\mu = \mathcal{H}^d|_{\Omega \cap \mathcal{N}_d}$, where $d \in \mathbb{N}$, and $\mathcal{N}_d$ denotes a d-dimensional compact submanifold of $\mathbb{R}^n$. Embedding (4.1.9) takes the form

$$\mathrm{Tr}\colon W^{m,A}(\Omega) \to L^B(\Omega \cap \mathcal{N}_d) \tag{4.1.12}$$

in this case, with $d \in [n-m, n]$, see Corollaries 4.3.13 and 4.4.14. Clearly, $\Omega \cap \mathcal{N}_d$ can, in particular, equal the intersection of Ω with a d-dimensional affine subspace of $\mathbb{R}^n$.

4.2 Reduction principles

A key ingredient in our approach is the use of so-called reduction principles for Sobolev type embeddings. They assert that a wide class of Sobolev and trace inequalities, including those considered here, are in fact equivalent to considerably simpler one-dimensional inequalities for suitable Hardy type operators. The relevant operators are defined as

$$H_{\alpha,\beta} f(s) = \int_{s^\beta}^1 f(r)\, r^{\alpha-1}\, \mathrm{d}r \quad \text{for } s > 0 \tag{4.2.1}$$

for any function $f \in \mathcal{M}(0,1)$ making the integral in (4.2.1) converge. The exponents α and β satisfy the constraints $0 < \alpha < 1$, $0 < \beta < \infty$ and $\alpha + 1/\beta \geq 1$, and depend on the Sobolev inequality in question.

Given any open set $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$, embedding (4.1.1) is equivalent to the inequality

$$\|u\|_{L^B(\Omega)} \leq C_1 \|\nabla^m u\|_{L^A(\Omega)} \tag{4.2.2}$$

for some constant C_1 and for every $u \in W_0^{m,A}(\Omega)$. The pertinent reduction principle asserts that inequality (4.2.2) holds if and only if

$$\|H_{\frac{m}{n},1} f\|_{L^B(0,1)} \leq C_2 \|f\|_{L^A(0,1)} \tag{4.2.3}$$

for some constant C_2, and for every non-negative $f \in L^A(0,1)$. See [24, Proof of Theorem 1] for $m = 1$, and [45, Theorem A] and [31, Theorem 6.1] for arbitrary m. Moreover, the constants C_1 and C_2 depend on each other, and on n, m and $|\Omega|$.

Embedding (4.1.7) in a John domain Ω amounts to the inequality

$$\|u\|_{L^B(\Omega)} \leq C_1 \|u\|_{W^{m,A}(\Omega)} \tag{4.2.4}$$

for every $u \in W^{m,A}(\Omega)$. Inequality (4.2.4) is again equivalent to (4.2.3) ([24, Proof of Theorem 2] for $m = 1$, and [31, Theorem 6.1] for any m). However, in this case the mutual dependence of the constants C_1 and C_2 involves full information on Ω, and not just on $|\Omega|$.

A characterization of embeddings on the whole $\mathbb{R}^n$ requires a combination of the Hardy inequality (4.2.3), which only depends on the behavior of the functions A and B near infinity, with a condition on their decay near zero. Specifically, the inequality

$$\|u\|_{L^B(\mathbb{R}^n)} \leq C \|u\|_{W^{m,A}(\mathbb{R}^n)} \tag{4.2.5}$$

holds for some constant C, and for every $u \in W^{m,A}(\mathbb{R}^n)$ if and only if inequality (4.2.3) holds, and

$$A \text{ dominates } B \text{ near zero,} \tag{4.2.6}$$

see [3].

The reduction principle for embedding (4.1.9) into Orlicz spaces, with respect to Frostman measures, applies to bounded Lipschitz domains Ω in $\mathbb{R}^n$. It provides us with a sufficient condition for the validity of (4.1.9) in terms of an appropriate Hardy type inequality, and it is also necessary if the decay in (4.1.10) is sharp, in the sense that there exist $x_0 \in \overline{\Omega}$ and positive constants c and $R > 0$ such that

$$\mu(B_r(x_0)) \cap \overline{\Omega}) \geq cr^\gamma \quad \text{if } 0 < r < R. \tag{4.2.7}$$

The relevant principle asserts that, if (4.1.10) and (4.2.7) are in force for some $\gamma \in [n-m, n]$, then the inequality

$$\|u\|_{L^B(\overline{\Omega},\mu)} \leq C_1 \|u\|_{W^{m,A}(\Omega)} \tag{4.2.8}$$

holds for some constant C_1 and for every $u \in W^{m,A}(\Omega)$ if and only if

$$\|H_{\frac{m}{n},\frac{n}{\gamma}} f\|_{L^B(0,1)} \leq C_2 \|f\|_{L^A(0,1)} \tag{4.2.9}$$

for some constant C_2, and for every non-negative $f \in L^A(0,1)$. The constants C_1 and C_2 depend on each other, and on n, m, γ, Ω and on the constants appearing in (4.1.10) and (4.2.7). The equivalence of inequalities (4.2.8) and (4.2.9) is established in [32]. Let us mention that the special case when μ is the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$ is treated in [26]. The case when $\gamma \in \mathbb{N}$, and μ is the γ-dimensional Hausdorff measure restricted to a γ-dimensional affine subspace of $\mathbb{R}^n$ is dealt with in [28].

4.3 Optimal Orlicz target spaces

Let us begin by considering embedding (4.1.1). As a preliminary observation, note that, when

$$m \geq n, \tag{4.3.1}$$

the optimal Orlicz target $L^B(\Omega)$ in (4.1.1) corresponds to the choice

$$B(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ \infty, & t > 1, \end{cases}$$

namely

$$L^B(\Omega) = L^\infty(\Omega).$$

Indeed, under assumption (4.3.1), one classically has

$$W_0^{m,1}(\Omega) \to L^\infty(\Omega),$$

whence the optimality of L^∞ follows, since

$$W_0^{m,A}(\Omega) \to W_0^{m,1}(\Omega) \to L^\infty(\Omega) \to L^B(\Omega) \tag{4.3.2}$$

for any Young functions A and B. We may thus restrict our attention to the case when

$$1 \le m < n.$$

Let us set $H_n\colon [t_0, \infty) \to [0, \infty)$ by

$$H_n(t) = \left(\int_{t_0}^{t} \left(\frac{s}{A(s)}\right)^{\frac{m}{n-m}} \mathrm{d}s\right)^{1-\frac{m}{n}} \quad \text{for } t \ge t_0, \tag{4.3.3}$$

where the constant t_0 is chosen in a way that the integral converges in the right neighbourhood of t_0. Under the assumption

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{m}{n-m}} \mathrm{d}t = \infty, \tag{4.3.4}$$

The function H_n is increasing onto $[0, \infty)$ and hence the inverse H_n^{-1} is well defined on $[0, \infty)$.

We then define $A_n\colon [0, \infty) \to [0, \infty)$ by

$$A_n(t) = \int_0^t \frac{D_n(s)}{s} \, \mathrm{d}s \quad \text{for } t \ge 0, \tag{4.3.5}$$

where $D_n\colon [0, \infty) \to [0, \infty]$ is given by

$$D_n(s) = \begin{cases} \left(s \, \dfrac{A(H_n^{-1}(s))}{H_n^{-1}(s)}\right)^{\frac{n}{n-m}} & \text{if (4.3.4) holds} \\ \infty & \text{otherwise} \end{cases}$$

for $s > 1$ and $D_n(s) = 0$ for $s \in [0, 1]$. Observe that since $A(s)/s$ is nondecreasing, H_n^{-1} is increasing and $s^{n/(n-m)-1}$ increases, $D_n(s)/s$ is nondecreasing and hence A_n is a Young function.

Theorem 4.3.1 [Optimal Orlicz target under vanishing boundary conditions]. *Let $n \ge 2$ and $1 \le m < n$ and let A be a Young function. Suppose that A_n is the Young function defined by* (4.3.5). *Then*

$$W_0^{m,A}(\Omega) \to L^{A_n}(\Omega) \tag{4.3.6}$$

and the target space $L^{A_n}(\Omega)$ is the optimal Orlicz target space in (4.3.6) *Moreover, if $I_A < \frac{n}{m}$, then A_n possess the simplified relation*

$$A_n^{-1}(t) \simeq A^{-1}(t) \, t^{-\frac{m}{n}} \quad \textit{near infinity.} \tag{4.3.7}$$

Under an additional assumption on the decay of A near 0, which reads

$$\int_0 \left(\frac{s}{A(s)}\right)^{\frac{m}{n-m}} \mathrm{d}s < \infty. \tag{4.3.8}$$

embedding (4.3.6) is equivalent to a Sobolev inequality in integral form. Inequalities in this form are usually better suited for applications to the theory of partial differential equations. The relevant integral inequality requires a slight variant in the definition near zero of the Young function in the optimal Orlicz target space. Let us first define $H_n^\infty\colon [0,\infty) \to [0,\infty)$ by

$$H_n^\infty(t) = \left(\int_0^t \left(\frac{s}{A(s)}\right)^{\frac{m}{n-m}} \mathrm{d}s\right)^{1-\frac{m}{n}} \quad \text{for } t > 0 \tag{4.3.9}$$

and set $H^\infty = \lim_{t\to\infty} H_n^\infty(t)$. Note that it is possible to have $H^\infty = \infty$. The function A_n^∞ is then defined by

$$A_n^\infty(t) = \int_0^t \frac{D_n^\infty(s)}{s} \mathrm{d}s \quad \text{for } t > 0, \tag{4.3.10}$$

in which D_n^∞ is given by

$$D_n^\infty(s) = \begin{cases} \left(s\,\dfrac{A(H_n^{\infty -1}(s))}{H_n^{\infty -1}(s)}\right)^{\frac{n}{n-m}}, & 0 \le t < H^\infty, \\ \infty, & t \ge H^\infty. \end{cases}$$

Repeating the argument as for A_n, we infer that A_n is a Young function. The integral inequality then reads as follows. Note that the relevant condition on upper Boyd index of A_n has to be now replaced by the corresponding global variant.

Corollary 4.3.2. *Let n, m and Ω be as in Theorem 4.3.1. Let A be a Young function satisfying* (4.3.8)*, and let A_n^∞ be the Young function defined by* (4.3.10)*. Then there exists a constant C such that*

$$\int_\Omega A_n^\infty\left(\frac{|u(x)|}{C\Big(\int_\Omega A(|\nabla^m u|)\,\mathrm{d}y\Big)^{m/n}}\right)\mathrm{d}x \le \int_\Omega A(|\nabla^m u|)\,\mathrm{d}x \tag{4.3.11}$$

for every $u \in W_0^{m,A}(\Omega)$. In particular, if $I_A^\infty < \frac{n}{m}$, then A_n^∞ fulfills

$$A_n^{\infty -1}(t) \simeq A^{-1}(t)\, t^{-\frac{m}{n}} \quad \textit{for } t > 0.$$

Companion results to Theorem 4.3.1 and Corollary 4.3.2 hold for embedding (4.1.7) between spaces of functions with unrestricted boundary values, provided that Ω is a John domain. Like for embedding (4.1.1), the only non-trivial case is when $1 \le m < n$. Indeed, if $m \ge n$, the same chain as in (4.3.2) holds with $W_0^{m,A}(\Omega)$ and $W_0^{m,1}(\Omega)$ replaced by $W^{m,A}(\Omega)$ and $W^{m,1}(\Omega)$, respectively, and hence $L^\infty(\Omega)$ is the optimal Orlicz target space in (4.1.7).

Theorem 4.3.3 [Optimal Orlicz target without boundary conditions]. *Let $n \ge 2$ and $1 \le m < n$, and let A be a Young function. Assume that Ω is a John domain in $\mathbb{R}^n$. Let A_n be the Young function defined by* (4.3.5)*. Then*

$$W^{m,A}(\Omega) \to L^{A_n}(\Omega) \tag{4.3.12}$$

and the space $L^{A_n}(\Omega)$ is the optimal Orlicz target space in (4.3.6)*. Further, if $I_A < \frac{n}{m}$, then A_n obeys*

$$A_n^{-1}(t) \simeq A^{-1}(t)\, t^{-\frac{m}{n}} \quad \textit{near infinity.}$$

Remark 4.3.4. An integral inequality analogous to (4.3.11), corresponding to embedding (4.3.12), holds under the assumption (3.3.3) and with A_n replaced by A_n^∞.

Example 4.3.5. Let $L^A(\Omega) = L^p(\log L)^\alpha(\Omega)$ be the Zygmund class, where either $p > 1$ and $\alpha \in \mathbb{R}$ or $p = 1$ and $\alpha \geq 0$. Assume that $1 \leq m < n$. The computations yield to

$$A_n(t) \text{ is equivalent to } \begin{cases} t^{\frac{np}{n-mp}}(\log t)^{\frac{n\alpha}{n-mp}} & \text{if } 1 \leq p < \frac{n}{m}, \\ \exp t^{\frac{n}{n-m(1+\alpha)}} & \text{if } p = \frac{n}{m} \text{ and } \alpha < \frac{m-n}{m}, \\ \exp\exp t^{\frac{n}{n-m}} & \text{if } p = \frac{n}{m} \text{ and } \alpha = \frac{m-n}{m}, \\ \infty & \text{otherwise}, \end{cases}$$

near infinity, whence an application of Theorem 4.3.1 tells us that

$$W_0^m L^p(\log L)^\alpha(\Omega) \to \begin{cases} L^{\frac{np}{n-mp}}(\log L)^{\frac{n\alpha}{n-mp}}(\Omega) & \text{if } 1 \leq p < \frac{n}{m}, \\ \exp L^{\frac{n}{n-m(1+\alpha)}}(\Omega) & \text{if } p = \frac{n}{m} \text{ and } \alpha < \frac{m-n}{m}, \\ \exp\exp L^{\frac{n}{n-m}}(\Omega) & \text{if } p = \frac{n}{m} \text{ and } \alpha = \frac{m-n}{m}, \\ L^\infty(\Omega) & \text{otherwise}, \end{cases} \tag{4.3.13}$$

for any open set Ω with $|\Omega| < \infty$, and the target spaces are optimal among all Orlicz spaces. By Theorem 4.3.3, the embeddings (4.3.13) also hold with the optimal targets for any John domain Ω if we replace W_0^m by W^m.

Example 4.3.6. Assume that $L^A(\Omega) = L^p(\log\log L)^\alpha$, namely the Orlicz space built upon the Young function obeying $A(t) = t^p(\log\log t)^\alpha$ near infinity, in which p and α are as in Example 4.3.5. Let also $1 \leq m < n$. The calculations show that

$$A_n(t) \text{ is equivalent to } \begin{cases} t^{\frac{np}{n-mp}}(\log\log t)^{\frac{n\alpha}{n-mp}} & \text{if } 1 \leq p < \frac{n}{m}, \\ e^{t^{\frac{n}{n-m}}(\log t)^{\frac{\alpha m}{n-m}}} & \text{if } p = \frac{n}{m}, \\ \infty & \text{if } p > \frac{n}{m}, \end{cases}$$

near infinity, and thus, owing to Theorem 4.3.1,

$$W_0^m L^p(\log\log L)^\alpha(\Omega) \to \begin{cases} L^{\frac{np}{n-mp}}(\log\log L)^{\frac{n\alpha}{n-mp}}(\Omega) & \text{if } 1 \leq p < \frac{n}{m}, \\ \exp(L^{\frac{n}{n-m}}(\log L)^{\frac{\alpha m}{n-m}})(\Omega) & \text{if } p = \frac{n}{m}, \\ L^\infty(\Omega) & \text{if } p > \frac{n}{m} \end{cases}$$

for any open set Ω with $|\Omega| < \infty$, and the target spaces are the optimal in the class of Orlicz spaces. A parallel result holds for any John domain Ω, provided that W_0^m is replaced by W^m, thanks to Theorem 4.3.3.

The next result is a counterpart of Theorem 4.3.3 in the case when $\Omega = \mathbb{R}^n$. The decay near zero of the involved Young functions is also relevant now. A Young function $\overline{A}$ obeying

$$\overline{A}(t) = \begin{cases} \infty & \text{near infinity}, \\ A(t) & \text{near zero}, \end{cases} \tag{4.3.14}$$

and a Young function $\overline{A}_n$ obeying

$$\overline{A}_n(t) = \begin{cases} A_n(t) & \text{near infinity,} \\ A(t) & \text{near zero} \end{cases} \tag{4.3.15}$$

come into play in the present situation.

Let us stress that, if $m \geq n$, then the answer to the optimal domain problem is still easier than in the case when $1 \leq m < n$, but not as trivial as when Ω is a John domain, since the optimal domain space is not just $L^\infty(\mathbb{R}^n)$ in general.

Theorem 4.3.7 [Optimal Orlicz target on $\mathbb{R}^n$]. *Let $n \geq 2$ and $m \in \mathbb{N}$, and let A be a Young function.*
(i) *Assume that $m \geq n$. Let $\overline{A}$ be a Young function satisfying* (4.3.14)*. Then*

$$W^{m,A}(\mathbb{R}^n) \to L^{\overline{A}}(\mathbb{R}^n)\,, \tag{4.3.16}$$

and $L^{\overline{A}}(\mathbb{R}^n)$ is the optimal Orlicz target space in (4.3.16)*.*
(ii) *Assume that $1 \leq m < n$. Let B_n be the Young function defined by* (4.4.1)*, and let $\overline{A}_n$ be a Young function satisfying* (4.3.15)*. Then*

$$W^{m,A}(\mathbb{R}^n) \to L^{\overline{A}_n}(\mathbb{R}^n)\,, \tag{4.3.17}$$

and $L^{\overline{A}_n}(\mathbb{R}^n)$ is the optimal Orlicz target space in (4.3.17)*.*

In particular, if $I_B < \frac{n}{m}$, then

$$\overline{A}_n^{-1}(t) \simeq \begin{cases} A^{-1}(t)\, t^{-\frac{m}{n}} & \textit{near infinity,} \\ A^{-1}(t) & \textit{near zero.} \end{cases}$$

The last results concern the Orlicz-Sobolev embedding (4.1.9) with a measure μ satisfying (4.1.10) and (4.2.7). By the same reason as for (4.1.7), the optimal Orlicz target space in these embeddings is $L^\infty(\Omega)$, provided that $m \geq n$.

If, instead, $1 \leq m < n$, the optimal Orlicz target in (4.1.9) is built upon the Young function A_γ defined, for $\gamma \in [n-m, n]$, by

$$A_\gamma(t) = \int_0^t \frac{D_\gamma(s)}{s} \mathrm{d}s \quad \text{for } t \geq 0, \tag{4.3.18}$$

where $D_\gamma \colon [0,\infty) \to [0,\infty]$ is given by

$$D_\gamma(s) = \begin{cases} \left(s \, \dfrac{A(H_n^{-1}(s))}{H_n^{-1}(s)} \right)^{\frac{\gamma}{n-m}} & \text{if (3.3.1) holds} \\ \infty & \text{otherwise} \end{cases}$$

for $s > 1$ and $D_n(s) = 0$ for $s \in [0,1]$. Here, H_n is as in (4.3.3).

In particular, if μ is the Lebesgue measure, then conditions (4.1.10) and (4.2.7) hold with $\gamma = n$, and A_γ coincides with the function A_n given by (4.3.5).

Theorem 4.3.8 [Optimal Orlicz target for embeddings with measure]. *Let $n \geq 2$, and let $1 \leq m < n$. Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, and let μ be a Borel measure satisfying conditions* (4.1.10) *and* (4.2.7) *for* *some $\gamma \in [n-m, n]$. Let A be a Young function, and let A_γ be the Young function defined by* (4.3.18). *Then*

$$W^{m,A}(\Omega) \to L^{A_\gamma}(\overline{\Omega}, \mu), \tag{4.3.19}$$

and $L^{A_\gamma}(\Omega)$ is the optimal Orlicz target space in (4.3.19).

In particular, if $I_A < \frac{n}{m}$, then

$$A_\gamma^{-1}(t) \simeq A^{-1}(t^{\frac{n}{\gamma}})\, t^{-\frac{m}{\gamma}} \quad \textit{near infinity.}$$

An integral version of embedding (4.3.19) is also available under the assumption (4.3.8) It involves a modified version of the function A_γ, say A_γ^∞, given by

$$A_\gamma^\infty(t) = \int_0^t \frac{D_\gamma^\infty(s)}{s} \mathrm{d}s \quad \text{for } t > 0, \tag{4.3.20}$$

where D_γ^∞ is defined by

$$D_\gamma^\infty(s) = \begin{cases} \left(s\, \dfrac{A(H_n^{\infty-1}(s))}{H_n^{\infty-1}(s)}\right)^{\frac{\gamma}{n-m}}, & 0 \le t < H^\infty, \\ \infty, & t \ge H^\infty. \end{cases}$$

Here H_n^∞ is the function from (4.3.9) and $H^\infty = \lim_{t\to\infty} H_n^\infty(t)$.

Corollary 4.3.9. *Let n, m, γ, Ω and μ be as in Theorem 4.3.8. Let A be a Young function satisfying* (4.3.8), *and let A_γ^∞ be the Young function defined by* (4.3.20). *Then there exists a constant C such that*

$$\int_{\overline{\Omega}} A_n^\infty\left(\frac{|u(x)|}{C\left(\sum_{k=0}^m \int_\Omega A(|\nabla^k u|)\,\mathrm{d}y\right)^{m/n}}\right) \mathrm{d}\mu(x) \le \left(\sum_{k=0}^m \int_\Omega A(|\nabla^k u|)\,\mathrm{d}x\right)^{\frac{\gamma}{n}} \tag{4.3.21}$$

for every $u \in W^{m,A}(\Omega)$. In particular, if $I_A^\infty < \frac{n}{m}$ then

$$A_\gamma^{\infty-1}(t) \simeq A^{-1}(t^{\frac{n}{\gamma}})\, t^{-\frac{m}{\gamma}} \quad \textit{for } t > 0.$$

Example 4.3.10. Suppose that $L^A(\Omega) = L^p(\log L)^\alpha(\Omega)$, the same Zygmund class as in Example 4.3.5, and suppose that n, m, γ, Ω and μ are as in Theorem 4.3.8. An application of Theorem 4.3.8 tells us that

$$W^m L^p(\log L)^\alpha(\Omega) \to \begin{cases} L^{\frac{\gamma p}{n-mp}}(\log L)^{\frac{\gamma\alpha}{n-mp}}(\overline{\Omega}, \mu) & \text{if } 1 \le p < \frac{n}{m}, \\ \exp L^{\frac{n}{n-m(1+\alpha)}}(\overline{\Omega}, \mu) & \text{if } p = \frac{n}{m} \text{ and } \alpha < \frac{m-n}{m}, \\ \exp\exp L^{\frac{n}{n-m}}(\overline{\Omega}, \mu) & \text{if } p = \frac{n}{m} \text{ and } \alpha = \frac{m-n}{m}, \\ L^\infty(\overline{\Omega}, \mu) & \text{otherwise,} \end{cases}$$

in which the target spaces are optimal within the class of Orlicz spaces.

Example 4.3.11. Assume that $L^A(\Omega) = L^p(\log\log L)^\alpha$, the Orlicz space from Example 4.3.6. Let further n, m, γ, Ω and μ be as in Theorem 4.3.8. The calculations show that

$$A_\gamma(t) \text{ is equivalent to } \begin{cases} t^{\frac{\gamma p}{n-mp}}(\log\log t)^{\frac{\gamma\alpha}{n-mp}} & \text{if } 1 \le p < \frac{n}{m}, \\ e^{t^{\frac{n}{n-m}}(\log t)^{\frac{\alpha m}{n-m}}} & \text{if } p = \frac{n}{m}, \\ \infty & \text{if } p > \frac{n}{m}, \end{cases}$$

near infinity, hence, by Theorem 4.3.8,

$$W^m L^p(\log\log L)^\alpha(\Omega) \to \begin{cases} L^{\frac{\gamma p}{n-mp}}(\log\log L)^{\frac{\gamma\alpha}{n-mp}}(\overline{\Omega},\mu) & \text{if } 1 \le p < \frac{n}{m}, \\ \exp(L^{\frac{n}{n-m}}(\log L)^{\frac{\alpha m}{n-m}})(\overline{\Omega},\mu) & \text{if } p = \frac{n}{m}, \\ L^\infty(\overline{\Omega},\mu) & \text{if } p > \frac{n}{m}, \end{cases}$$

and the target spaces are the optimal Orlicz ones.

The optimal Orlicz target space in (4.1.11) agrees with that in (4.1.9), with $\gamma = n-1$. Namely, it is built upon the Young function A_{n-1} defined as in (4.3.18), with $\gamma = n-1$. This is the content of Corollary 4.3.12 below, and follows from Theorem 4.3.8, and from the fact that, if Ω is a bounded Lipschitz domain, then the measure $\mu = \mathcal{H}^{n-1}|_{\partial\Omega}$ fulfills conditions (4.1.10) and (4.2.7) with $\gamma = n-1$.

Corollary 4.3.12 [Optimal Orlicz target for boundary traces]. *Let $n \ge 2$ and $1 \le m < n$. Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Let A be a Young function, and let A_{n-1} be the Young function defined by* (4.3.18)*, with $\gamma = n-1$. Then*

$$\mathrm{Tr}\colon W^{m,A}(\Omega) \to L^{A_{n-1}}(\partial\Omega), \tag{4.3.22}$$

and $W^{m,B_{n-1}}(\Omega)$ is the optimal Orlicz target space in (4.3.22)*. In particular, if $I_A < \frac{n}{m}$, then*

$$A_{n-1}^{-1}(t) \simeq A^{-1}(t^{\frac{n}{n-1}})\, t^{-\frac{m}{n-1}} \quad \textit{near infinity.}$$

We conclude this section by specializing Theorem 4.3.8 to embeddings of the form (4.1.12) into Orlicz spaces defined on the intersection of Ω with d-dimensional compact submanifolds $\mathcal{N}_d$ of $\mathbb{R}^n$. Since the measure $\mu = \mathcal{H}^d|_{\Omega\cap\mathcal{N}_d}$ satisfies conditions (4.1.10) and (4.2.7), with $\gamma = d$, from Theorem 4.3.8 we infer the following corollary.

Corollary 4.3.13 [Optimal Orlicz target for traces on submanifolds]. *Let $n \ge 2$, $1 \le m < n$, and let $d \in \mathbb{N}$, with $n - m \le d \le n$. Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, and let $\mathcal{N}_d$ be a d-dimensional compact submanifold of $\mathbb{R}^n$ such that $\Omega \cap \mathcal{N}_d \ne \emptyset$. Let A be a Young function, and let A_d be the Young function defined by* (4.3.18)*, with $\gamma = d$. Then*

$$\mathrm{Tr}\colon W^{m,A}(\Omega) \to L^{A_d}(\Omega \cap \mathcal{N}_d),$$

and $L^{A_d}(\Omega)$ is the optimal Orlicz-Sobolev domain space in (4.4.26)*. In particular, if $I_A < \frac{n}{m}$, then*

$$A_{n-1}^{-1}(t) \simeq A^{-1}(t^{\frac{n}{d}})\, t^{-\frac{m}{d}} \quad \textit{near infinity.}$$

Proof of Theorem 4.3.1. The validity of the inequality (4.2.2) is equivalent to the Hardy type inequality (4.2.3). The optimal Orlicz target space thus always exists and coincides with $L^{A_n}(\Omega)$ thanks to Theorem 3.4.3. The relation (4.3.7) follows from (3.4.2). □

Proof of Corollary 4.3.2. Let us fix $u \in W_0^{m,A}(\Omega)$ and let, without loss of generality, $\int_\Omega A(|\nabla^m u|)\,\mathrm{d}y < \infty$, otherwise (4.3.11) is trivially satisfied. Since A is assumed to fulfill (4.3.8), we have, by Theorem 3.4.1 that

$$H^\infty_{\frac{m}{n},1}\colon L^A(0,\infty) \to L^{A_{\frac{m}{n}}^\infty}(0,\infty)$$

with the operator norm independent of A. Let $N > 0$ be given and set $A_N = A/N$. Then A_N satisfies (4.3.8) and the Young function $(A_N)_n^\infty$, associated to A_N as in (4.3.10), obeys

$$(A_N)_n^\infty(t) = \frac{A_n^\infty(tN^{-\frac{m}{n}})}{N} \quad \text{for } t \geq 0. \tag{4.3.23}$$

Thus, we infer that

$$H^\infty_{\frac{m}{n},1}\colon L^{A_N}(0,\infty) \to L^{(A_N)_n^\infty}(0,\infty)$$

with the operator norm independent of N. In particular,

$$H_{\frac{m}{n},1}\colon L^{A_N}(0,1) \to L^{(A_N)_n^\infty}(0,1)$$

and consequently, owing to the equivalence of inequalities (4.2.2) and (4.2.3),

$$\|u\|_{L^{(A_N)_n^\infty}(\Omega)} \leq C\|\nabla^m u\|_{L^{A_N}(\Omega)} \tag{4.3.24}$$

for every $u \in W_0^{m,(A_N)}(\Omega)$, with the constant C independent of N. On choosing

$$N = \int_\Omega A(|\nabla^m u|)\,\mathrm{d}y,$$

and observing that $\|\nabla^m u\|_{L^{A_N}(\Omega)} \leq 1$ with such choice of N, inequality (4.3.24) gives $\|u\|_{L^{(A_N)_n^\infty}(\Omega)} \leq C$. Therefore

$$\int_\Omega (A_N)_n^\infty\left(\frac{|u(x)|}{C}\right)\mathrm{d}x \leq 1,$$

whence, by (4.3.23),

$$\int_\Omega A_n^\infty\left(\frac{|u(x)|}{CN^{m/n}}\right)\mathrm{d}x \leq N,$$

which is nothing but (4.3.11). □

Proof of Theorem 4.3.3. The appropriate reduction principle asserts that the inequalities (4.2.4) and (4.2.3) are equivalent. The embedding (4.3.12) thus holds true and the space $L^{A_n}(\Omega)$ is the optimal Orlicz target by Theorem 3.4.3. □

Proof of Theorem 4.3.7. The reduction principle for the inequality (4.2.5) is the core ingredient here. It proposes that such inequality holds if and only if the inequality (4.2.3) and the property (4.2.6) hold simultaneously. If $m \geq n$, then inequality (4.2.3) holds with $L^B(0,1) = L^\infty(0,1)$ and (4.2.6) is trivially satisfied for $B = \bar{A}$. The optimality is also straightforward since $L^\infty(0,1)$ is the smallest possible Orlicz space and the inequality (4.2.6) is saturated. On the other hand, if $1 \leq m < n$, then, by Theorem 3.4.3, the optimal Orlicz target space in (4.2.3) agrees with $L^{A_n}(0,1)$ and, (4.2.6) holds for $B = \bar{A}_n$. Clearly, such choice is sharp. □

Proof of Theorem 4.3.8. Here we use the reduction principle for Sobolev embeddings with measure, which guarantees the equivalence of inequalities (4.2.8) and (4.2.9). The rest is due to Theorem 3.4.3. □

Proof of Corollary 4.3.9. We use the same scaling argument as in the proof of Corollary 4.3.2. We again set $A_N = A/N$ instead of A to obtain

$$(A_N)^\infty_\gamma(t) = N^{-\frac{\gamma}{n}} A^\infty_\gamma(tN^{-\frac{m}{n}}).$$

The inequality (4.3.21) then follows by the choice

$$N = \sum_{k=0}^{m} \int_\Omega A(|\nabla^k u|)\, \mathrm{d}y. \qquad \square$$

4.4 Optimal Orlicz domain spaces

Let us begin by the embedding (4.1.1). Similarly as at the beginning of the previous section, observe that if

$$m \geq n,$$

then the optimal Orlicz-Sobolev domain $W_0^{m,A}(\Omega)$ in (4.1.1) reads as

$$W_0^{m,A}(\Omega) = W_0^{m,1}(\Omega)$$

as one infers from (4.3.2).

In the sequel we thus assume

$$1 \leq m < n.$$

Under this circumstance, the existence of an optimal Orlicz-Sobolev space $W^{m,A}(\Omega)$ in (4.1.1) is not guaranteed anymore. Our first main result asserts that the existence of such an optimal space depends on the local upper Boyd index of the Young function B_n given by

$$B_n(t) = \int_0^t \frac{G_n^{-1}(s)}{s} \mathrm{d}s \quad \text{for } t \geq 0, \tag{4.4.1}$$

where $G_n\colon [0,\infty) \to [0,\infty)$ is defined as

$$G_n(t) = t \inf_{1 \leq s \leq t} B^{-1}(s)\, s^{\frac{m}{n}-1} \quad \text{for } t \geq 1.$$

and $G_n(t) = tB^{-1}(1)$ for $t \in [0,1)$. Moreover, whenever it exists, the function A in the optimal domain space in (4.1.1) equals B_n.

Remark 4.4.1. Observe that the function G_n is increasing, as shown via the alternate formula

$$G_n(t) = t^{\frac{m}{n}} \inf_{1 \leq s < \infty} B^{-1}(s) \max\Big\{1, \tfrac{t}{s}\Big\}^{1-\frac{m}{n}} \quad \text{for } t \geq 1,$$

and hence its inverse G_n^{-1} is well-defined.

Also, the function B_n is actually a Young function. Indeed, since G_n is increasing, G_n^{-1} is increasing as well. Thus, since the function $G_n(t)/t$ is non-increasing, the function $G_n^{-1}(t)/t$ is non-decreasing. These facts also ensure that B_n is equivalent to G_n^{-1} globally.

Theorem 4.4.2 [Optimal Orlicz-Sobolev domain under vanishing boundary conditions]. *Let $n \geq 2$ and $1 \leq m < n$, and let B be a Young function. Let B_n be the Young function defined by* (4.4.1). *Assume that Ω is an open set in $\mathbb{R}^n$ with $|\Omega| < \infty$. If*

$$I_{B_n} < \frac{n}{m}, \tag{4.4.2}$$

then

$$W_0^{m,B_n}(\Omega) \to L^B(\Omega), \tag{4.4.3}$$

and $W_0^{m,B_n}(\Omega)$ is the optimal Orlicz-Sobolev domain space in (4.4.3).

Conversely, if (4.4.2) *fails, then no optimal Orlicz-Sobolev domain space exists in* (4.1.1), *in the sense that any Orlicz-Sobolev space $W_0^{m,A}(\Omega)$ for which embedding* (4.1.1) *holds can be replaced with a strictly larger Orlicz-Sobolev space for which* (4.1.1) *is still true.*

In particular, if $i_B > \frac{n}{n-m}$, then condition (4.4.2) *is equivalent to $I_B < \infty$, and*

$$B_n^{-1}(t) \simeq B^{-1}(t)\, t^{\frac{m}{n}} \quad \textit{near infinity.} \tag{4.4.4}$$

Under a mild additional assumption on the decay of B near 0, which reads

$$\sup_{0<t<1} \frac{B(t)}{t^{\frac{n}{n-m}}} < \infty\,, \tag{4.4.5}$$

embedding (4.4.3) is also equivalent to a Sobolev inequality in integral form. A slight variant in the definition near zero of the Young function i the optimal Orlicz-Sobolev domain. This function will be denoted by B_n^∞, and is defined as

$$B_n^\infty(t) = \int_0^t \frac{G_n^{\infty\,-1}(s)}{s} \mathrm{d}s \quad \text{for } t \geq 0, \tag{4.4.6}$$

where

$$G_n^\infty(t) = t \inf_{0<s\leq t} B^{-1}(s)\, s^{\frac{m}{n}-1} \quad \text{for } t \geq 0.$$

Note that condition (4.4.2) on the local upper Boyd index of B_n has now to be replaced with a parallel condition on the global upper Boyd index of B_n^∞.

Corollary 4.4.3. *Let n, m and Ω be as in Theorem 4.4.2. Let B be a Young function satisfying* (4.4.5), *and let B_n^∞ be the Young function defined by* (4.4.6). *If*

$$I^\infty_{B_n^\infty} < \frac{n}{m}, \tag{4.4.7}$$

then there exists a constant C such that

$$\int_\Omega B\left(\frac{|u(x)|}{C\Big(\int_\Omega B_n^\infty(|\nabla^m u|)\,\mathrm{d}y\Big)^{m/n}}\right)\mathrm{d}x \leq \int_\Omega B_n^\infty(|\nabla^m u|)\,\mathrm{d}x \tag{4.4.8}$$

for every $u \in W_0^{m,B_n^\infty}(\Omega)$.

In particular, if $i_B^\infty > \frac{n}{n-m}$, then condition (4.4.7) *is equivalent to $I_B^\infty < \infty$, and*

$$B_n^{\infty\,-1}(t) \simeq B^{-1}(t)\, t^{\frac{m}{n}} \quad \textit{for } t \geq 0.$$

As we have seen for the Orlicz targets, there are also the corresponding variants of Theorem 4.4.2 and Corollary 4.4.3 for the embedding (4.1.7) between spaces of functions with unrestricted boundary values, provided that Ω is a John domain. Also here, the only non-trivial case is when $1 \le m < n$ otherwise, by the same argument as in (4.3.2), $W^{m,1}(\Omega)$ is the optimal Orlicz-Sobolev domain space in (4.1.7).

Theorem 4.4.4 [Optimal Orlicz-Sobolev domain without boundary conditions]. *Let $n \ge 2$ and $1 \le m < n$, and let B be a Young function. Assume that Ω is a John domain in $\mathbb{R}^n$. Let B_n be the Young function defined by* (4.4.1). *If* (4.4.2) *holds, then*

$$W^{m,B_n}(\Omega) \to L^B(\Omega), \tag{4.4.9}$$

and $W^{m,B_n}(\Omega)$ is the optimal Orlicz-Sobolev domain space in (4.4.9).

Conversely, if (4.4.2) *fails, then no optimal Orlicz-Sobolev domain space exists in* (4.1.7)*, in the sense that any Orlicz-Sobolev space $W^{m,A}(\Omega)$ for which embedding* (4.1.7) *holds can be replaced with a strictly larger Orlicz-Sobolev space for which* (4.1.7) *is still true.*

In particular, if $i_B > \frac{n}{n-m}$, then condition (4.4.2) *is equivalent to $I_B < \infty$, and*

$$B_n^{-1}(t) \simeq B^{-1}(t)\, t^{\frac{m}{n}} \quad \textit{near infinity.}$$

Remark 4.4.5. An integral inequality analogous to (4.4.8), corresponding to embedding (4.4.9), holds under assumption (4.4.5), and with B_n replaced by B_n^∞.

Example 4.4.6. Consider the case when $L^B(\Omega)$ is a Zygmund space of the form $L^q(\log L)^\alpha(\Omega)$, where either $q \in (1,\infty)$ and $\alpha \in \mathbb{R}$, or $q = 1$ and $\alpha \ge 0$. Assume that $1 \le m < n$, the only non-trivial case in view of the discussion above. Computations show that

$$B_n(t) \text{ is equivalent to } \begin{cases} t^{\frac{nq}{n+mq}} (\log t)^{\frac{n\alpha}{n+mq}} & \text{if } q > \frac{n}{n-m},\ \alpha \in \mathbb{R}, \\ t\,(\log t)^{\alpha(1-\frac{m}{n})} & \text{if } q = \frac{n}{n-m},\ \alpha > 0, \\ t & \text{otherwise,} \end{cases}$$

near infinity. Moreover,

$$I_{B_n} = \begin{cases} \frac{nq}{n+mq} & \text{if } q > \frac{n}{n-m},\ \alpha \in \mathbb{R}, \\ 1 & \text{otherwise,} \end{cases}$$

whence $I_{B_n} < n/m$. Therefore, by Theorem 4.4.2,

$$\left.\begin{array}{lll} \text{if } q > \frac{n}{n-m},\ \alpha \in \mathbb{R}, & & W_0^m L^{\frac{nq}{n+mq}} (\log L)^{\frac{n\alpha}{n+mq}}(\Omega) \\ \text{if } q = \frac{n}{n-m},\ \alpha > 0, & & W_0^m L(\log L)^{\alpha(1-\frac{m}{n})}(\Omega) \\ \text{otherwise,} & & W_0^{m,1}(\Omega) \end{array}\right\} \to L^q(\log L)^\alpha(\Omega) \tag{4.4.10}$$

for any open set Ω with $|\Omega| < \infty$, and the domain spaces are optimal among all Orlicz-Sobolev spaces. By Theorem 4.4.4, the same embeddings continue to hold, with optimal domain spaces, for any John domain Ω, provided that W_0^m is replaced by W^m.

Let us point out that, by Example 4.3.5, the space $L^q(\log L)^\alpha(\Omega)$ is in turn the optimal Orlicz target space in (4.4.10). Thus, the domain and target spaces are mutually optimal in (4.4.10).

Example 4.4.7. We deal here with the target space $L^q \exp\sqrt{\log L}(\Omega)$, with $q \in [1,\infty)$, namely the Orlicz space built upon a Young function $B(t) = t^q e^{\sqrt{\log t}}$ near infinity. Assume as above that $1 \le m < n$. If $q < \frac{n}{n-m}$, then $B(t)\, t^{\frac{n}{m-n}} = t^{q+\frac{n}{m-n}} e^{\sqrt{\log t}}$ near infinity, a decreasing function. Thus, $B^{-1}(s)\, s^{\frac{m}{n}-1}$ is increasing near infinity, and $B_n(t)$ is equivalent to t near infinity.

Suppose next that $q \ge \frac{n}{n-m}$. Then the function $B(t)\, t^{\frac{n}{m-n}}$ is increasing near infinity, so that $B^{-1}(s)\, s^{\frac{m}{n}-1}$ is decreasing, and

$$B_n^{-1}(t) \simeq B^{-1}(t)\, t^{\frac{m}{n}}$$

near infinity. One can verify that

$$B^{-1}(s) \simeq s^{\frac{1}{q}} e^{-q^{-\frac{3}{2}}\sqrt{\log s}}$$

near infinity. Hence,

$$B_n(t) \quad \text{is equivalent to} \quad t^{\frac{nq}{n+mq}} e^{\left(\frac{n}{n+mq}\right)^{\frac{3}{2}}\sqrt{\log t}}$$

near infinity. In particular, $I_{B_n} = \frac{nq}{n+mq} < \frac{n}{m}$. Altogether, by Theorem 4.4.2, one has that

$$\left.\begin{array}{ll} \text{if } q \ge \frac{n}{n-m}, & W_0^m L^{\frac{nq}{n+mq}} \exp\Big(\Big(\frac{n}{n+mq}\Big)^{\frac{3}{2}}\sqrt{\log L}\Big)(\Omega) \\ \text{otherwise,} & W_0^{m,1}(\Omega) \end{array}\right\} \to L^q \exp\sqrt{\log L}(\Omega)$$

for any open set Ω with $|\Omega| < \infty$, and the domain spaces are optimal among all Orlicz-Sobolev spaces. A parallel result holds in any John domain Ω, with W_0^m replaced by W^m, owing to Theorem 4.4.4.

Example 4.4.8. If the Young function B grows so fast near infinity that $i_B = \infty$, then it immediately follows from Theorems 4.4.2 and 4.4.4 that no optimal Orlicz-Sobolev domain space exists in embeddings (4.1.1) and (4.1.7). This is the case, for instance, when $L^B(\Omega)$ agrees with one of the following spaces:

$$\exp\Big((\log L)^\alpha\Big)(\Omega) \quad \exp\Big(L^q(\log L)^\beta\Big)(\Omega),$$

or

$$\exp L^\beta(\Omega),\ \exp\Big(\exp L^\beta\Big)(\Omega),\ \dots,\ \exp\Big(\cdots(\exp L^\beta)\Big)(\Omega),$$

or

$$L^\infty(\Omega)\,,$$

where $\alpha > 1$, $\beta > 0$ and $q \in [1,\infty)$.

The next result is a counterpart of Theorem 4.4.4 in the situation when $\Omega = \mathbb{R}^n$. As we have seen in the optimal range case, the decay near zero of the involved Young functions also plays a role here. Namely, the Young function $\overline{B}$ obeying

$$\overline{B}(t) = \begin{cases} t & \text{near infinity,} \\ B(t) & \text{near zero,} \end{cases} \tag{4.4.11}$$

and a Young function $\overline{B}_n$ satisfying

$$\overline{B}_n(t) = \begin{cases} B_n(t) & \text{near infinity,} \\ B(t) & \text{near zero} \end{cases} \tag{4.4.12}$$

are relevant her.

Let us point out that, if $m \geq n$, then the solution to the optimal domain problem is not as trivial as when Ω is a John domain, however, it is still easier.

Theorem 4.4.9 [Optimal Orlicz-Sobolev domain on $\mathbb{R}^n$]. *Let $n \geq 2$ and $m \in \mathbb{N}$, and let B be a Young function.*
(i) *Assume that $m \geq n$. Let $\overline{B}$ be a Young function satisfying* (4.4.11)*. Then*

$$W^{m,\overline{B}}(\mathbb{R}^n) \to L^B(\mathbb{R}^n), \tag{4.4.13}$$

and $W^{m,\overline{B}}(\mathbb{R}^n)$ is the optimal Orlicz-Sobolev domain space in (4.4.13)*.*
(ii) *Assume that $1 \leq m < n$. Let B_n be the Young function defined by* (4.4.1)*, and let $\overline{B}_n$ be a Young function satisfying* (4.4.12)*. If* (4.4.2) *holds, then*

$$W^{m,\overline{B}_n}(\mathbb{R}^n) \to L^B(\mathbb{R}^n), \tag{4.4.14}$$

and $W^{m,\overline{B}_n}(\mathbb{R}^n)$ is the optimal Orlicz-Sobolev domain space in (4.4.14)*.*

Conversely, if (4.4.2) *fails, then no optimal Orlicz-Sobolev domain space exists in* (4.1.8)*, in the sense that any Orlicz-Sobolev space $W^{m,A}(\mathbb{R}^n)$ for which embedding* (4.1.8) *holds can be replaced with a strictly larger Orlicz-Sobolev space for which* (4.1.8) *is still true.*

In particular, if $i_B > \frac{n}{n-m}$, then condition (4.4.2) *is equivalent to $I_B < \infty$, and*

$$\overline{B}_n^{-1}(t) \simeq \begin{cases} B^{-1}(t)\, t^{\frac{m}{n}} & \textit{near infinity,} \\ B^{-1}(t) & \textit{near zero.} \end{cases}$$

The last results of this section concern the Orlicz-Sobolev embedding (4.1.9) with a measure μ obeying (4.1.10) and (4.2.7). By the same reason as for (4.1.7), the optimal Orlicz-Sobolev domain space in these embeddings is $W^{m,1}(\Omega)$, provided that $m \geq n$.

If, instead, $1 \leq m < n$, the optimal Orlicz-Sobolev domain in (4.1.9), when it exists, is built upon the Young function B_γ defined, for $\gamma \in [n-m, n]$, as

$$B_\gamma(t) = \int_0^t \frac{G_\gamma^{-1}(s)}{s} \mathrm{d}s \quad \text{for } t \geq 0, \tag{4.4.15}$$

where $G_\gamma \colon [0, \infty) \to [0, \infty)$ is given by

$$G_\gamma(t) = t \inf_{1 \leq s \leq t} B^{-1}\left(s^{\frac{\gamma}{n}}\right) s^{\frac{m}{n}-1} \quad \text{for } t \geq 1,$$

and $G_\gamma(t) = tB^{-1}(1)$ for $t \in [0, 1)$. In particular, if μ is the Lebesgue measure, then conditions (4.1.10) and (4.2.7) hold with $\gamma = n$, and B_γ agrees with the function B_n given by (4.4.1).

Theorem 4.4.10 [Optimal Orlicz-Sobolev domain for embeddings with measure]. *Let $n \geq 2$, and let $1 \leq m < n$. Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, and let μ be a Borel measure satisfying conditions* (4.1.10) *and* (4.2.7) *for some $\gamma \in [n-m, n]$. Let B be a Young function, and let B_γ be the Young function defined by* (4.4.15)*. If*

$$I_{B_\gamma} < \frac{n}{m}, \tag{4.4.16}$$

then

$$W^{m,B_\gamma}(\Omega) \to L^B(\overline{\Omega}, \mu)\,, \tag{4.4.17}$$

and $W^{m,B_\gamma}(\Omega)$ is the optimal Orlicz-Sobolev domain space in (4.4.17)*.*

Conversely, if (4.4.16) *fails, then no optimal Orlicz-Sobolev domain space exists in* (4.1.9)*, in the sense that any Orlicz-Sobolev space $W^{m,A}(\Omega)$ for which embedding* (4.1.9) *holds can be replaced with a strictly larger Orlicz-Sobolev space for which* (4.1.9) *is still true.*

In particular, if $i_B > \frac{\gamma}{n-m}$, then condition (4.4.16) *is equivalent to $I_B < \infty$, and*

$$B_\gamma^{-1}(t) \simeq B^{-1}\Big(t^{\frac{\gamma}{n}}\Big)\, t^{\frac{m}{n}} \quad \textit{near infinity.}$$

An integral version of embedding (4.4.17) holds under the assumption that

$$\sup_{0<t<1} \frac{B(t)}{t^{\frac{\gamma}{n-m}}} < \infty\,. \tag{4.4.18}$$

It involves a modified version of the function B_γ given by

$$B_\gamma^\infty(t) = \int_0^t \frac{G_\gamma^{\infty\,-1}(s)}{s}\mathrm{d}s \quad \text{for } t \geq 0, \tag{4.4.19}$$

where $G_\gamma^\infty\colon [0,\infty) \to [0,\infty)$ is defined by

$$G_\gamma^\infty(t) = t \inf_{0<s\leq t} B^{-1}\Big(s^{\frac{\gamma}{n}}\Big)\, s^{\frac{m}{n}-1} \quad \text{for } t \geq 0.$$

Corollary 4.4.11. *Let n, m, γ, Ω and μ be as in Theorem 4.4.10. Let B be a Young function satisfying* (4.4.18)*, and let B_γ^∞ be the Young function defined by* (4.4.19)*. If*

$$I^\infty_{B_\gamma^\infty} < \frac{n}{m}, \tag{4.4.20}$$

then there exists a constant C such that

$$\int_{\overline{\Omega}} B\left(\frac{|u(x)|}{C\Big(\sum_{k=0}^m \int_\Omega B_\gamma^\infty(|\nabla^k u|)\,\mathrm{d}y\Big)^{m/n}}\right)\mathrm{d}\mu(x) \leq \left(\sum_{k=0}^m \int_\Omega B_\gamma^\infty(|\nabla^k u|)\,\mathrm{d}x\right)^{\frac{\gamma}{n}} \tag{4.4.21}$$

for every $u \in W^{m,B_\gamma^\infty}(\Omega)$.

In particular, if $i_B^\infty > \frac{\gamma}{n-m}$, then condition (4.4.20) *is equivalent to $I_B^\infty < \infty$, and*

$$B_\gamma^{\infty\,-1}(t) \simeq B^{-1}\Big(t^{\frac{\gamma}{n}}\Big)\, t^{\frac{m}{n}} \quad \textit{for } t \geq 0.$$

Example 4.4.12. Assume that $L^B(\Omega) = L^q(\log L)^\alpha(\Omega)$, the same Zygmund space as in Example 4.4.6, where either $q \in (1,\infty)$ and $\alpha \in \mathbb{R}$, or $q = 1$ and $\alpha \geq 0$. Assume that $1 \leq m < n$, the case when $m \geq n$ being trivial. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, and let μ be a Borel measure fulfilling conditions (4.1.10) and (4.2.7). Then

$$B_\gamma(t) \text{ is equivalent to } \begin{cases} t^{\frac{nq}{\gamma+mq}} (\log t)^{\frac{n\alpha}{\gamma+mq}} & \text{if } q > \frac{\gamma}{n-m},\ \alpha \in \mathbb{R}, \\ t\,(\log t)^{\frac{\alpha(n-m)}{\gamma}} & \text{if } q = \frac{\gamma}{n-m},\ \alpha > 0, \\ t & \text{otherwise,} \end{cases}$$

near infinity. Hence,

$$I_{B_\gamma} = \begin{cases} \frac{nq}{\gamma+mq} & \text{if } q > \frac{\gamma}{n-m},\ \alpha \in \mathbb{R}, \\ 1 & \text{otherwise.} \end{cases}$$

Since $I_{B_\gamma} < n/m$, Theorem 4.4.10 tells us that

$$\left.\begin{array}{lll} \text{if } q > \frac{\gamma}{n-m},\ \alpha \in \mathbb{R}, & & W^m L^{\frac{nq}{\gamma+mq}} (\log L)^{\frac{n\alpha}{\gamma+mq}}(\Omega) \\ \text{if } q = \frac{\gamma}{n-m},\ \alpha > 0, & & W^m L(\log L)^{\frac{\alpha(n-m)}{\gamma}}(\Omega) \\ \text{otherwise,} & & W^{m,1}(\Omega) \end{array}\right\} \to L^q(\log L)^\alpha(\overline{\Omega},\mu)\,, \tag{4.4.22}$$

the domain spaces being optimal among all Orlicz-Sobolev spaces. Observe, that, by Example 4.3.10, the space $L^q(\log L)^\alpha(\overline{\Omega},\mu)$ is the optimal Orlicz target space in (4.4.22).

The optimal Orlicz-Sobolev domain space in (4.1.11) agrees with that in (4.1.9), with $\gamma = n-1$. Namely, it is built upon the Young function B_{n-1} defined as in (4.4.15), with $\gamma = n-1$. This is the content of Corollary 4.4.13 below, and follows from Theorem 4.4.10, and from the fact that, if Ω is a bounded Lipschitz domain, then the measure $\mu = \mathcal{H}^{n-1}|_{\partial\Omega}$ fulfills conditions (4.1.10) and (4.2.7) with $\gamma = n-1$.

Corollary 4.4.13 [Optimal Orlicz-Sobolev domain for boundary traces]. *Let $n \geq 2$ and $1 \leq m < n$. Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Let B be a Young function, and let B_{n-1} be the Young function defined by* (4.4.15), *with $\gamma = n-1$. If*

$$I_{B_{n-1}} < \frac{n}{m}, \tag{4.4.23}$$

then

$$W^{m,B_{n-1}}(\Omega) \to L^B(\partial\Omega), \tag{4.4.24}$$

and $W^{m,B_{n-1}}(\Omega)$ is the optimal Orlicz-Sobolev domain space in (4.4.24).

Conversely, if (4.4.23) *fails, then no optimal Orlicz-Sobolev domain space exists in* (4.1.11), *in the sense that any Orlicz-Sobolev space $W^{m,A}(\Omega)$ for which embedding* (4.1.11) *holds can be replaced with a strictly larger Orlicz-Sobolev space for which* (4.1.11) *is still true.*

In particular, if $i_B > \frac{n-1}{n-m}$, then condition (4.4.23) *is equivalent to $I_B < \infty$, and*

$$B_{n-1}^{-1}(t) \simeq B^{-1}\Big(t^{\frac{n-1}{n}}\Big)\, t^{\frac{m}{n}} \quad \textit{near infinity.}$$

We conclude this section by specializing Theorem 4.4.10 to embeddings of the form (4.1.12) into Orlicz spaces defined on the intersection of Ω with d-dimensional compact submanifolds $\mathcal{N}_d$ of $\mathbb{R}^n$. Since the measure $\mu = \mathcal{H}^d|_{\Omega\cap\mathcal{N}_d}$ satisfies conditions (4.1.10) and (4.2.7), with $\gamma = d$, from Theorem 4.4.10 we infer the following corollary.

Corollary 4.4.14 [Optimal Orlicz-Sobolev domain for traces on submanifolds]. *Let $n \geq 2$, $1 \leq m < n$, and let $d \in \mathbb{N}$, with $n-m \leq d \leq n$. Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, and let $\mathcal{N}_d$ be a d-dimensional compact submanifold of $\mathbb{R}^n$ such that $\Omega \cap \mathcal{N}_d \neq \emptyset$. Let B be a Young function, and let B_d be the Young function defined by* (4.4.15)*, with $\gamma = d$. If*

$$I_{B_d} < \frac{n}{m}, \tag{4.4.25}$$

then

$$W^{m,B_d}(\Omega) \to L^B(\Omega \cap \mathcal{N}_d)\,, \tag{4.4.26}$$

and $W^{m,B_d}(\Omega)$ is the optimal Orlicz-Sobolev domain space in (4.4.26)*.*

Conversely, if (4.4.25) *fails, then no optimal Orlicz-Sobolev domain space exists in* (4.1.12)*, in the sense that any Orlicz-Sobolev space $W^{m,A}(\Omega)$ for which embedding* (4.1.12) *holds can be replaced with a strictly larger Orlicz-Sobolev space for which* (4.1.12) *is still true.*

In particular, if $i_B > \frac{d}{n-m}$, then condition (4.4.25) *is equivalent to $I_B < \infty$, and*

$$B_d^{-1}(t) \simeq B^{-1}\Big(t^{\frac{d}{n}}\Big)\, t^{\frac{m}{n}} \quad \textit{near infinity.}$$

Proof of Theorem 4.4.2. The fact that an optimal Orlicz domain space in (4.2.2) exists if and only if (4.4.2) holds, and that, in the affirmative case, it agrees with $W_0^{m,B_n}(\Omega)$, follows from Theorem 3.6.1, via the equivalence of the Sobolev inequality (4.2.2) and of the Hardy type inequality (4.2.3). The property (2.7.1) also plays a role here. The assertion about the validity of equation (4.4.4) is a consequence of 3.6.7. □

Proof of Corollary 4.4.3. Fix $u \in W_0^{m,B_n^\infty}(\Omega)$, and assume, without loss of generality, that $\int_\Omega B_n^\infty(|\nabla^m u|)\,\mathrm{d}y < \infty$, otherwise (4.4.8) is trivially satisfied. By Proposition 3.2.2, assumption (4.4.7) is equivalent to the existence of a constant $C_1 > 0$, such that

$$\int_0^t \frac{\widetilde{B_n^\infty}(s)}{s^{n/(n-m)+1}}\,\mathrm{d}s \leq \frac{\widetilde{B_n^\infty}(C_1 t)}{t^{n/(n-m)}} \quad \text{for } t > 0. \tag{4.4.27}$$

Given $N > 0$, let B_N be the Young function defined as

$$B_N(t) = \frac{B\Big(tN^{-\frac{m}{n}}\Big)}{N} \quad \text{for } t \geq 0.$$

Then, the Young function $(B_N)_n^\infty$ associated with B_N as in (4.4.6) satisfies

$$(B_N)_n^\infty = \frac{B_n^\infty}{N}.$$

One can thus verify that inequality (4.4.27) continues to hold with B_n^∞ replaced by $(B_N)_n^\infty$, and with the same constant C_1, whatever N is. Hence, by Theorem 3.5.2

$$\|H^\infty_{\frac{m}{n},1}f\|_{L^{B_N}(0,\infty)} \le C_2\|f\|_{L^{(B_N)^\infty_n}(0,\infty)}$$

for every $f \in L^{(B_N)^\infty_n}(0,\infty)$ and for some constant for some C_2 independent of N. In particular,

$$\|H_{\frac{m}{n},1}f\|_{L^{B_N}(0,1)} \le C_2\|f\|_{L^{(B_N)^\infty_n}(0,1)}$$

for every $f \in L^{(B_N)^\infty_n}(0,1)$. Therefore, owing to the equivalence of inequalities (4.2.2) and (4.2.3),

$$\|u\|_{L^{B_N}(\Omega)} \le C\|\nabla^m u\|_{L^{(B_N)^\infty_n}(\Omega)} \tag{4.4.28}$$

for every $u \in W_0^{m,(B_N)^\infty_n}(\Omega)$, where the constant C is independent of N. On choosing

$$N = \int_\Omega B_n^\infty(|\nabla^m u|)\,dy\,,$$

and observing that $\|\nabla^m u\|_{L^{(B_N)^\infty_n}(\Omega)} \le 1$ with this choice of N, inequality (4.4.28) yields $\|u\|_{L^{B_N}(\Omega)} \le C$. Therefore

$$\int_\Omega B_N\left(\frac{|u(x)|}{C}\right)dx \le 1\,,$$

whence, by the definition of B_N,

$$\int_\Omega B\left(\frac{|u(x)|}{CN^{m/n}}\right)dx \le N\,,$$

namely (4.4.8). □

Proof of Theorem 4.4.4. The proof follows along the same lines as that of Theorem 4.4.2. Here, the equivalence of inequalities (4.2.4) and (4.2.3) comes into play. □

Proof of Theorem 4.4.9. The reduction principle for inequality (4.2.5) is relevant in this proof. Recall that such a principle asserts that this inequality is equivalent to the simultaneous validity of inequality (4.2.3) and of property (4.2.6). Now, assume that condition (4.4.2) holds. Then, by Theorem 3.6.3, inequality (4.2.3) holds with either $L^A(0,1) = L^1(0,1)$, or $L^A(0,1) = L^{B_n}(0,1)$, according to whether $m \ge n$ or $1 \le m < n$. On the other hand, (4.2.6) trivially holds by the very definition of $\overline{B}$ and $\overline{B}_n$. Thus, inequality (4.2.5), with $A = \overline{B}$ or $A = \overline{B}_n$, holds, and embedding (4.4.13) or (4.4.14), respectively, follows. Moreover, $W^{m,\overline{B}}(\mathbb{R}^n)$, or $W^{m,\overline{B}_n}(\mathbb{R}^n)$ is optimal in (4.4.13) or (4.4.14). Indeed, if inequality (4.2.5) holds for some Young function A, then, by the reduction principle, A has to dominate B near 0, and inequality (4.2.3) must hold. By the optimality of the domain space $L^1(0,1)$ or $L^{B_n}(0,1)$ in (4.2.3), the function $A(t)$ has to dominate t or $B_n(t)$ near infinity. Thus, A dominates $\overline{B}$ or $\overline{B}_n$ globally, whence $W^{m,A}(\mathbb{R}^n) \to W^{m,\overline{B}}(\mathbb{R}^n)$, or $W^{m,A}(\mathbb{R}^n) \to W^{m,\overline{B}_n}(\mathbb{R}^n)$, thus proving the optimality of $W^{m,\overline{B}}(\mathbb{R}^n)$, or $W^{m,\overline{B}_n}(\mathbb{R}^n)$.

Conversely, suppose that condition (4.4.2) fails. Then, by Theorem 3.6.3, there does not exist an optimal Orlicz space $L^A(0,1)$ in (4.2.3). As a consequence of the reduction principle, and of (2.7.1) and (2.7.2), there does not exist an optimal domain Orlicz-Sobolev space in embedding (4.1.8). □

Proof of Theorem 4.4.10. This is a consequence of Theorem 3.6.3 and of the reduction principle for Sobolev embeddings with measure, which asserts the equivalence of inequalities (4.2.8) and (4.2.9). □

Proof of Corollary 4.4.11. The proof of inequality (4.4.21) relies upon a scaling argument as in the proof of Corollary 4.4.3. Here, $B(t)$ has to be replaced by $B_N(t) = N^{-\frac{\gamma}{n}} B(tN^{-\frac{m}{n}})$, where

$$N = \sum_{k=0}^{m} \int_{\Omega} B_{\gamma}^{\infty}(|\nabla^k u|) \, dy \,. \qquad \square$$

5. Fractional maximal operator

5.1 Introduction

Let $n \in \mathbb{N}$ and $0 < \gamma < n$ be fixed. The fractional maximal operator M_γ is defined for any locally integrable function f in Ω by

$$M_\gamma f(x) = \sup_{Q \ni x} |Q|^{\frac{\gamma}{n}-1} \int_Q |f(y)| \,\mathrm{d}y, \quad x \in \Omega,$$

where the supremum is taken over all cubes $Q \subseteq \Omega$ containing x and having the sides parallel to the coordinate axes.

Our first aim is to analyse the boundedness of M_γ acting between Orlicz spaces. More specifically, given Orlicz spaces $L^A(\Omega)$ and $L^B(\Omega)$, we want to decide whether

$$M_\gamma \colon L^A(\Omega) \to L^B(\Omega). \tag{5.1.1}$$

We show that (5.1.1) is equivalent to one-dimensional inequalities involving only the Young functions A and B. Resulting inequality is then much more easier to verify. We will follow the result from [56] which partly overlaps with [20, Section 2]. Note that such question was also studied in [41], where (5.1.1) is studied under some restrictive assumptions.

The principal innovation of paper [56] however lays in the description of optimal Orlicz spaces in (5.1.1). More specifically, given $L^A(\Omega)$, we seek for the smallest Orlicz space $L^B(\Omega)$ such that (5.1.1) holds. By "smallest" we mean that if (5.1.1) holds with $L^B(\Omega)$ replaced with another Orlicz space $L^{\widehat{B}}(\Omega)$, then $L^B(\Omega) \to L^{\widehat{B}}(\Omega)$. Instead of smallest we also often say optimal.

Let us briefly look at the situation in the class of the Lebesgue spaces. It is well known that

$$M_\gamma \colon L^p(\mathbb{R}^n) \to \begin{cases} L^{\frac{np}{n-\gamma p}}(\mathbb{R}^n), & 1 < p < \frac{n}{\gamma}, \\ L^\infty(\mathbb{R}^n), & p = \frac{n}{\gamma}, \end{cases}$$

and this result is sharp within Lebesgue spaces. However, there is no Lebesgue space $L^q(\mathbb{R}^n)$ for which

$$M_\gamma \colon L^1(\mathbb{R}^n) \to L^q(\mathbb{R}^n).$$

The situation in the class of Orlicz spaces is much more subtler and not many results are available in the literature. The authors of [37] characterised the boundedness of M_γ (they work with more general operator, in fact) on classical Lorentz spaces. In the cases when such spaces coincide with Orlicz spaces, we may recover the following result (see Section 2 for the definitions of the spaces involved).

$$M_\gamma \colon L^p(\log L)^{\mathbb{A}}(\mathbb{R}^n) \to \begin{cases} L^{\frac{n}{n-\gamma}}(\log L)^{\frac{n\mathbb{A}}{n-\gamma}-1}(\mathbb{R}^n), & p = 1,\ \alpha_0 < 0,\ \alpha_\infty > 0, \\ L^{\frac{np}{n-\gamma p}}(\log L)^{\frac{n\mathbb{A}}{n-\gamma p}}(\mathbb{R}^n), & 1 < p < \frac{n}{\gamma}, \\ \exp L^{-\frac{n}{\gamma\mathbb{A}}}(\mathbb{R}^n), & p = \frac{n}{\gamma},\ \alpha_0 > 0,\ \alpha_\infty < 0, \end{cases} \tag{5.1.2}$$

however, it does not say anything about its sharpness. In Theorem 5.3.2, we give the complete characterization of the existence of the optimal Orlicz target and, in the affirmative case, we give its full description. It turns out that the spaces obtained in (5.1.2) are optimal in the cases when $p > 1$. If $p = 1$, then the target space $L^{\frac{n}{n-\gamma}}(\log L)^{\frac{n\mathbb{A}}{n-\gamma}-1}(\mathbb{R}^n)$ is not the best possible Orlicz space and even more, the optimal Orlicz target does not exist in this case. This means that any Orlicz space $L^B(\mathbb{R}^n)$ for which $M_\gamma \colon L^p(\log L)^{\mathbb{A}}(\mathbb{R}^n) \to L^B(\mathbb{R}^n)$ may be replaced by essentially smaller space, whence there is an "open" set of all the eligible Orlicz spaces. The details on this particular case are discussed in Example 5.3.3.

One can also ask the converse problem, i.e. when the target space $L^B(\Omega)$ is given and we seek for the largest possible $L^A(\Omega)$ rendering (5.1.1) true. Analogously, by "largest" we mean that if (5.1.1) holds with $L^A(\Omega)$ replaced by $L^{\widehat{A}}(\Omega)$, then $L^{\widehat{A}}(\Omega) \to L^A(\Omega)$. Again, we shorten this notion to the word "optimal" since no confusion with the above situation is likely to happen. The solution of this task is the subject of Theorem 5.3.6, where we give the complete description of optimal domains. If we return back to the example in (5.1.2), one gets that the Orlicz domain $L^A(\log L)^{\mathbb{A}}(\mathbb{R}^n)$ is the optimal one in all three cases. See Example 5.3.7 for further details.

5.2 Reduction principle

We start by introducing a crucial tool often named as a reduction principle. Such a principle translates the boundedness of the operator M_γ acting between Orlicz spaces on $\mathbb{R}^n$ or on open sets Ω of finite measure to a much simpler one-dimensional inequality containing only the Young functions and the parameters n and γ. That enables us to simplify our analysis and helps us to understand the behaviour of the operator and the spaces involved.

At first, we need to introduce several constructions of Young functions. Their importance will then be resembled in the following theorem. It is no doubt that the two above-mentioned cases have to be distinguished

Clearly, when $|\Omega| < \infty$, the behaviour of Young functions near zero is irrelevant, since, all Young functions which coincide near infinity result in the same Orlicz space with equivalent norms.

Let us start with the local variant. Let A be a given Young function and define A_γ by

$$A_\gamma(t) = \int_0^t \frac{G_\gamma^{-1}(s)}{s} \mathrm{d}s \quad \text{for } t \geq 0, \tag{5.2.1}$$

where $G_\gamma \colon [0,\infty) \to [0,\infty)$ is a non-decreasing function defined by

$$G_\gamma(t) = \begin{cases} tA^{-1}(1) & \text{for } 0 \leq t < 1, \\ \sup\limits_{1 \leq s \leq t} A^{-1}(s)\, s^{-\frac{\gamma}{n}} & \text{for } t \geq 1 \end{cases}$$

Here G_γ^{-1} represents its generalized right-continuous inverse.

Remark 5.2.1. Note that A_γ is a Young function. Indeed, as might be observed by the formula, we have

$$\frac{G_\gamma(t)}{t} = \frac{1}{t} \sup_{1 \leq s < \infty} \frac{A^{-1}(s)}{s} \min\{t, s\}^{1-\frac{\gamma}{n}}$$

and hence $G_\gamma(t)/t$ is nondecreasing. Therefore A_γ is a Young function and moreover, by (2.1.2), A_γ is equivalent to G_γ^{-1} near infinity.

Let B be a given Young function and define (with the little abuse of notation)

$$B_\gamma(t) = \int_0^t \frac{E_\gamma^{-1}(s)}{s}\,\mathrm{d}s \quad \text{for } t \geq 0. \tag{5.2.2}$$

Here, $E_\gamma\colon [0,\infty) \to [0,\infty)$ is given by

$$E_\gamma(t) = \begin{cases} 0, & \text{for } 0 \leq t < 1, \\ t^{\frac{\gamma}{n}} F_\gamma^{\infty-1}(t) & \text{for } t \geq 1, \end{cases}$$

where $F_\gamma\colon [1,\infty) \to [0,\infty)$ is defined by

$$F_\gamma(t) = t^{n/(n-\gamma)} \int_1^t \frac{B(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s \quad \text{for } t \geq 1, \tag{5.2.3}$$

and where F_γ^{-1} stands for the generalised left-continuous inverse of F_γ.

Remark 5.2.2. Observe that B_γ is a well-defined Young function. The assumption (5.2.6) guarantees that F_γ is nondecreasing. Thus F_γ^{-1} is nondecreasing and so is E_γ. Next, since $F_\gamma(t)/t^{n/(n-\gamma)}$ is nondecreasing, $F_\gamma^{-1}(t)/t^{1-\frac{\gamma}{n}}$ is nonincreasing, hence $E_\gamma(t)/t$ is nonincreasing and $E_\gamma^{-1}(s)/s$ is nondecreasing. Therefore, B_γ is convex. In addition, by (2.1.2), B_γ is equivalent to E_γ^{-1} near infinity.

The first principal result then reads as follows.

Theorem 5.2.3 [Reduction principle in Orlicz spaces on finite-measure sets]. *Let $n \in \mathbb{N}$ and $0 < \gamma < n$. Let $\Omega \subset \mathbb{R}^n$ be a open set such that $|\Omega| < \infty$. Suppose that A and B are Young functions and let A_γ and B_γ be the Young functions defined by* (5.2.1) *and* (5.2.2)*, respectively. The following assertions are equivalent:*

(i) *There exists a constant $C_1 > 0$ such that*

$$\|M_\gamma f\|_{L^B(\Omega)} \leq C_1 \|f\|_{L^A(\Omega)}$$

for every $f \in L^A(\Omega)$;

(ii) *There exists a constant $C_2 > 0$ such that*

$$\int_1^t \frac{B(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s \leq \frac{A_\gamma(C_2 t)}{t^{n/(n-\gamma)}} \quad \textit{for } t > 1;$$

(iii) *There exists a constant $C_3 > 0$ such that*

$$B_\gamma(t) \leq A(C_3 t) \quad \textit{near infinity.}$$

If $\Omega = \mathbb{R}^n$, then also the values of Young functions near zero come into play. Let again A be given and assume that

$$\inf_{0<t<1} A(t)\, t^{-\frac{n}{\gamma}} > 0. \tag{5.2.4}$$

We then define A_γ^∞ by

$$A_\gamma^\infty(t) = \int_0^t \frac{G_\gamma^{\infty-1}(s)}{s} \mathrm{d}s \quad \text{for } t \geq 0, \tag{5.2.5}$$

where $G_\gamma^\infty \colon [0, \infty) \to [0, \infty)$ is a non-decreasing function defined by

$$G_\gamma^\infty(t) = \sup_{0<s\leq t} A^{-1}(s)\, s^{-\frac{\gamma}{n}} \quad \text{for } t \geq 0$$

and $G_\gamma^{\infty-1}$ represents its generalized right-continuous inverse. By a similar argument as in Remark 5.2.1, we infer that A_γ^∞ is a Young function such that A_γ^∞ is equivalent to $G_\gamma^{\infty-1}$ globally.

Conversely, let B be a given Young function satisfying

$$\int_0 \frac{B(s)}{s^{n/(n-\gamma)+1}} \, \mathrm{d}s < \infty. \tag{5.2.6}$$

We define

$$B_\gamma^\infty(t) = \int_0^t \frac{E_\gamma^{\infty-1}(s)}{s} \, \mathrm{d}s \quad \text{for } t \geq 0, \tag{5.2.7}$$

in which $E_\gamma^\infty \colon [0, \infty) \to [0, \infty)$ is defined by

$$E_\gamma^\infty(t) = t^{\frac{\gamma}{n}} F_\gamma^{\infty-1}(t) \quad \text{for } t \geq 0,$$

where $F_\gamma^\infty \colon [0, \infty) \to [0, \infty)$ is given by

$$F_\gamma^\infty(t) = t^{n/(n-\gamma)} \int_0^t \frac{B(s)}{s^{n/(n-\gamma)+1}} \, \mathrm{d}s \quad \text{for } t \geq 0. \tag{5.2.8}$$

and where $F_\gamma^{\infty-1}$ stands for the generalised left-continuous inverse of F_γ^∞. Similarly as in Remark 5.2.2, B_γ^∞ is a Young function such that B_γ^∞ is equivalent to $E_\gamma^{\infty-1}$ globally.

The corresponding reduction principle then reads as follows.

Theorem 5.2.4 [Reduction principle in Orlicz spaces on $\mathbb{R}^n$]. *Let $n \in \mathbb{N}$ and $0 < \gamma < n$. Suppose that A and B are Young functions and let A_γ^∞ and B_γ^∞ be the Young functions defined by* (5.2.5) *and* (5.2.7)*, respectively. The following assertions are equivalent:*

(i) *There exists a constant $C_1 > 0$ such that*

$$\|M_\gamma f\|_{L^B(\mathbb{R}^n)} \leq C_1 \|f\|_{L^A(\mathbb{R}^n)}$$

for every $f \in L^A(\mathbb{R}^n)$;

(ii) *A satisfies* (5.2.4) *and there exists a constant $C_2 > 0$ such that*

$$\int_0^t \frac{B(s)}{s^{n/(n-\gamma)+1}} \, \mathrm{d}s \leq \frac{A_\gamma^\infty(C_2 t)}{t^{n/(n-\gamma)}} \quad \textit{for } t > 0;$$

(iii) *B satisfies* (5.2.6) *and there exists a constant $C_3 > 0$ such that*

$$B_\gamma^\infty(t) \leq A(C_3 t) \quad \textit{for } t \geq 0;$$

(iv) *There is a constant $C_4 >$ such that*

$$\int_{\mathbb{R}^n} B\Biggl(\frac{M_\gamma f(x)}{C_4\Bigl(\int_{\mathbb{R}^n} A(|f(y)|)\,\mathrm{d}y\Bigr)^{\gamma/n}}\Biggr)\,\mathrm{d}x \le \int_{\mathbb{R}^n} A\Bigl(|f(x)|\Bigr)\,\mathrm{d}x$$

for every $f \in L^A(\mathbb{R}^n)$.

Moreover, the constants C_1, C_2, C_3 and C_4 depend on each other and on n and γ.

We would like to point out the philosophy behind the criteria (ii) and (iii) in our reduction principle. They look completely different at a first glance and they rely on auxiliary Young functions A_γ (or A_γ^∞) and B_γ (or B_γ^∞), respectively. In situations when both A and B are given, there is no significantly better choice of a condition to check. However, imagine that we have one A and a bunch of candidates B to choose from. Then the condition (ii) comes handy as we compute A_γ once and we check the inequality against every choice of B. The condition (iii) is then welcome in the reciprocal case.

5.3 Optimal Orlicz spaces

This section is devoted to description of optimal spaces in (5.1.1). It is no surprise that the Young functions A_γ and B_γ (and A_γ^∞ with B_γ^∞ in the global case) play the major role in the problem of establishing the corresponding optimal Orlicz spaces. Let us begin with the targets.

Theorem 5.3.1 [Optimal Orlicz target on finite-measure sets]. *Let $n \in \mathbb{N}$, $0 < \gamma < n$ and let $\Omega \subseteq \mathbb{R}^n$ be an open set of finite measure. Suppose that A is a Young function and let A_γ as in* (5.2.1). *If*

$$i_{A_\gamma} > \frac{n}{n-\gamma}, \tag{5.3.1}$$

then

$$M_\gamma\colon L^A(\Omega) \to L^{A_\gamma}(\Omega) \tag{5.3.2}$$

and $L^{A_\gamma}(\Omega)$ is the smallest among all Orlicz spaces in (5.3.2).

Conversely, if (5.3.1) *fails, then there is no optimal Orlicz space in* (5.1.1) *in a sense that any Orlicz space $L^B(\Omega)$ for which* (5.1.1) *holds true can be replaced by a strictly smaller Orlicz space for which* (5.1.1) *is still valid.*

In particular, if $I_A < \frac{n}{\gamma}$, then (5.3.1) *is equivalent to $i_A > 1$ and*

$$A_\gamma^{-1}(t) \simeq A^{-1}(t)\, t^{-\frac{\gamma}{n}} \quad \textit{near infinity.} \tag{5.3.3}$$

The result for infinite-measure sets then reads as follows.

Theorem 5.3.2 [Optimal Orlicz target on $\mathbb{R}^n$]. *Let $n \in \mathbb{N}$, $0 < \gamma < n$ and suppose that A is a Young function satisfying* (5.2.4) *and set A_γ^∞ as in* (5.2.1). *If*

$$i^\infty_{A_\gamma^\infty} > \frac{n}{n-\gamma}, \tag{5.3.4}$$

then

$$M_\gamma\colon L^A(\mathbb{R}^n) \to L^{A_\gamma^\infty}(\mathbb{R}^n) \tag{5.3.5}$$

and $L^{A_\gamma^\infty}(\mathbb{R}^n)$ *is the smallest among all Orlicz spaces in* (5.3.5).

Conversely, if (5.3.4) *fails, then there is no optimal Orlicz space in* (5.1.1) *in a sense that any Orlicz space* $L^B(\mathbb{R}^n)$ *for which* (5.1.1) *holds true can be replaced by a strictly smaller Orlicz space for which* (5.1.1) *is still valid.*

In particular, if $I_A^\infty < \frac{n}{\gamma}$, *then* (5.3.4) *is equivalent to* $i_A^\infty > 1$ *and*

$$A_\gamma^{\infty\,-1}(t) \simeq A^{-1}(t)\, t^{-\frac{\gamma}{n}} \quad \textit{for } t > 0. \tag{5.3.6}$$

In addition, if (5.2.4) *is not satisfied, then there does not exist an Orlicz target space* $L^B(\mathbb{R}^n)$ *for which* (5.1.1) *holds.*

Example 5.3.3. Let $\Omega = \mathbb{R}^n$. Assume that $1 \le p_0, p_\infty \le \infty$ and $\alpha_0, \alpha_\infty \in \mathbb{R}$. If $p_0 = 1$ then let $\alpha_0 \le 0$ and if $p_\infty = 1$ then let $\alpha_\infty \ge 0$. Suppose that

$$A(t) \text{ is equivalent to } \begin{cases} t^{p_0}\,\ell(t)^{\alpha_0} & \text{near zero,} \\ t^{p_\infty}\,\ell(t)^{\alpha_\infty} & \text{near infinity.} \end{cases}$$

Let us consider the nontrivial cases only, i.e. let us assume that (5.2.4) is satisfied. This implies that either $1 \le p_0 < \frac{n}{\gamma}$ or $p_0 = \frac{n}{\gamma}$ and $\alpha_0 \ge 0$. Computations show that

$$A_\gamma^\infty(t) \text{ is equivalent to } \begin{cases} t^{\frac{np_0}{n-\gamma p_0}}\,\ell(t)^{\frac{n\alpha_0}{n-\gamma p_0}}, & \begin{array}{l} 1 \le p_0 < \frac{n}{\gamma},\ \alpha_0 \in \mathbb{R} \text{ or} \\ p_0 = 1,\ \alpha_0 \le 0, \end{array} \\ \exp(-t^{-\frac{n}{\gamma\alpha_0}}), & p_0 = \frac{n}{\gamma},\ \alpha_0 > 0, \\ 0, & p_0 = \frac{n}{\gamma},\ \alpha_0 = 0, \end{cases}$$

near zero and

$$A_\gamma^\infty(t) \text{ is equivalent to } \begin{cases} t^{\frac{np_\infty}{n-\gamma p_\infty}}\,\ell(t)^{\frac{n\alpha_\infty}{n-\gamma p_\infty}}, & \begin{array}{l} 1 \le p_\infty < \frac{n}{\gamma},\ \alpha_\infty \in \mathbb{R} \text{ or} \\ p_\infty = 1,\ \alpha_\infty \ge 0, \end{array} \\ \exp(t^{-\frac{n}{\gamma\alpha_\infty}}), & \text{if } p_\infty = \frac{n}{\gamma},\ \alpha_\infty < 0, \\ \infty, & \begin{array}{l} p_\infty = \frac{n}{\gamma},\ \alpha_\infty \ge 0 \text{ or} \\ p_\infty > \frac{n}{\gamma},\ \alpha_\infty \in \mathbb{R}, \end{array} \end{cases}$$

near infinity. Moreover,

$$i_{A_\gamma^\infty}^\infty = \begin{cases} \min\left\{\frac{np_0}{n-\gamma p_0}, \frac{np_\infty}{n-\gamma p_\infty}\right\}, & 1 \le \min\{p_0, p_\infty\} < \frac{n}{\gamma}, \\ \infty, & p_0 = p_\infty = \frac{n}{\gamma}, \end{cases}$$

whence $i_{A_\gamma^\infty}^\infty > \frac{n}{n-\gamma}$ if and only if both $p_0 > 1$ and $p_\infty > 1$. Therefore, by Theorem 5.3.2,

$$M_\gamma\colon L^A(\mathbb{R}^n) \to L^{A_\gamma^\infty}(\mathbb{R}^n)$$

and the range spaces are optimal among all Orlicz spaces.

If $p_0 = 1$ or $p_\infty = 1$, then every Young function B satisfying (5.1.1) can be essentially enlarged near zero or near infinity, respectively.

Example 5.3.4. Set $\Omega = \mathbb{R}^n$ and let $1 \le p_0, p_\infty \le \infty$. Suppose that

$$A(t) \text{ is equivalent to } \begin{cases} t^{p_0}\, e^{-\sqrt{\log 1/t}} & \text{near zero,} \\ t^{p_\infty}\, e^{\sqrt{\log t}} & \text{near infinity.} \end{cases}$$

In order to ensure (5.2.4), assume that $1 \le p_0 < \frac{n}{\gamma}$. We have that $A(t)t^{-\frac{n}{\gamma}}$ is decreasing near zero hence $A^{-1}(s)s^{-\frac{\gamma}{n}}$ is increasing near zero. Thus the supremum in the definition of A_γ^∞ may be disregarded and A_γ^∞ obeys (5.3.3) near zero. Calculation shows that

$$A^{-1}(s) \simeq s^{\frac{1}{p_0}}\, e^{-p_0^{-\frac{3}{2}}\sqrt{\log 1/s}} \quad \text{near zero}$$

and then

$$A_\gamma^\infty(t) \quad \text{is equivalent to} \quad t^{\frac{np_0}{n-\gamma p_0}}\, e^{-(\frac{n}{n-\gamma p_0})^{\frac{3}{2}}\sqrt{\log 1/t}} \quad \text{near zero}$$

If $p_\infty \ge \frac{n}{\gamma}$, then $A(t)t^{-\frac{n}{\gamma}}$ is increasing, $A^{-1}(s)s^{-\frac{\gamma}{n}}$ is decreasing, hence G_γ^∞ is a constant function and $A_\gamma^\infty = \infty$ near infinity. If $1 \le p_\infty < \frac{n}{\gamma}$, then, similarly as before, A_γ^∞ satisfies (5.3.3) near infinity. To sum it up,

$$A_\gamma^\infty(t) \text{ is equivalent to } \begin{cases} t^{\frac{np_\infty}{n-\gamma p_\infty}}\, e^{(\frac{n}{n-\gamma p_\infty})^{\frac{3}{2}}\sqrt{\log t}}, & 1 \le p_\infty < \frac{n}{\gamma}, \\ \infty, & p_\infty \ge \frac{n}{\gamma}, \end{cases}$$

near infinity. In conclusion

$$i_{A_\gamma^\infty}^\infty = \begin{cases} \min\left\{\frac{np_0}{n-\gamma p_0}, \frac{np_\infty}{n-\gamma p_\infty}\right\}, & 1 \le p_\infty < \frac{n}{\gamma}, \\ \frac{np_0}{n-\gamma p_0}, & p_\infty \ge \frac{n}{\gamma}, \end{cases}$$

and $i_{A_\gamma^\infty}^\infty > \frac{n}{n-\gamma}$ if and only if both $p_0 > 1$ and $p_\infty > 1$. If this is the case, then, by Theorem 5.3.2,

$$M_\gamma \colon L^A(\mathbb{R}^n) \to L^{A_\gamma^\infty}(\mathbb{R}^n)$$

and the range space is optimal among all Orlicz spaces, otherwise any Young function B satisfying (5.1.1) can be essentially enlarged near zero or near infinity, respectively.

The characterization of optimal Orlicz domain spaces follows. Again, we distinguish two cases, whether the measure of Ω is finite or infinite.

Theorem 5.3.5 [Optimal Orlicz domain on finite-measure sets]. *Let $n \in \mathbb{N}$, $0 < \gamma < n$ and let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying $|\Omega| < \infty$. Suppose that B is a Young function and let B_γ be given by* (5.2.2). *Then*

$$M_\gamma \colon L^{B_\gamma}(\Omega) \to L^B(\Omega) \tag{5.3.7}$$

and $L^{B_\gamma}(\Omega)$ is the largest possible Orlicz space satisfying (5.3.7).

In addition, if $i_B > \frac{n}{n-\gamma}$, then B_γ obeys the simpler relation

$$B_\gamma^{-1}(t) \simeq t^{\frac{\gamma}{n}}\, B^{-1}(t), \quad \textit{near infinity.} \tag{5.3.8}$$

Theorem 5.3.6 [Optimal Orlicz domain on $\mathbb{R}^n$]. *Let $n \in \mathbb{N}$, $0 < \gamma < n$. Suppose that B is a Young function fulfilling* (5.2.6) *and let B_γ be the Young function defined by* (5.2.2). *Then*

$$M_\gamma \colon L^{B_\gamma^\infty}(\mathbb{R}^n) \to L^B(\mathbb{R}^n) \tag{5.3.9}$$

and $L^{B_\gamma^\infty}(\mathbb{R}^n)$ is the largest possible Orlicz space satisfying (5.3.9).

In addition, if $i_B^\infty > \frac{n}{n-\gamma}$, then B satisfies (5.2.6) *and B_γ^∞ obeys the simpler relation*

$$B_\gamma^{\infty-1}(t) \simeq t^{\frac{\gamma}{n}}\, B^{-1}(t), \quad \textit{for } t > 0. \tag{5.3.10}$$

Conversely, if (5.2.6) *fails, then there is no Orlicz space $L^A(\mathbb{R}^n)$ for which* (5.1.1) *holds.*

Example 5.3.7. Let $\Omega = \mathbb{R}^n$ and let $1 < q_0, q_\infty \le \infty$ and $\alpha_0, \alpha_\infty \in \mathbb{R}$. Suppose that B is a Young function such that

$$B(t) \text{ is equivalent to } \begin{cases} t^{q_0}\,\ell(t)^{\alpha_0} & \text{near zero,} \\ t^{q_\infty}\,\ell(t)^{\alpha_\infty} & \text{near infinity.} \end{cases}$$

The condition (5.2.6) requires that either $q_0 > n/(n-\gamma)$ or $q_0 = n/(n-\gamma)$ and $\alpha_0 < -1$. One can compute that

$$B_\gamma^\infty(t) \text{ is equivalent to } \begin{cases} t^{\frac{nq_0}{n+\gamma q_0}}\,\ell(t)^{\frac{n\alpha_0}{n+\gamma q_0}}, & q_0 > \frac{n}{n-\gamma}, \\ t\,\ell(t)^{(1-\frac{\gamma}{n})(\alpha_0+1)}, & q_0 = \frac{n}{n-\gamma},\ \alpha_0 < -1, \end{cases}$$

near zero and

$$B_\gamma^\infty(t) \text{ is equivalent to } \begin{cases} t^{\frac{nq_\infty}{n+\gamma q_\infty}}\,\ell(t)^{\frac{n\alpha_\infty}{n+\gamma q_\infty}}, & q_\infty > \frac{n}{n-\gamma}, \\ t\,\ell(t)^{(1-\frac{\gamma}{n})(\alpha_\infty+1)}, & q_\infty = \frac{n}{n-\gamma},\ \alpha_\infty > -1, \\ t\,\ell(\ell(t))^{(1-\frac{\gamma}{n})}, & q_\infty = \frac{n}{n-\gamma},\ \alpha_\infty = -1, \\ t, & q_\infty = \frac{n}{n-\gamma},\ \alpha_\infty < -1 \text{ or } \\ & q_\infty < \frac{n}{\gamma}, \end{cases}$$

near infinity. By Theorem 5.3.6,

$$M_\gamma \colon L^{B_\gamma^\infty}(\mathbb{R}^n) \to L^B(\mathbb{R}^n)$$

and $L^{B_\gamma^\infty}(\mathbb{R}^n)$ is the optimal Orlicz space.

Example 5.3.8. Let $\Omega = \mathbb{R}^n$ and $1 \le q_0, q_\infty \le \infty$. We deal with the Young function B such that

$$B(t) \text{ is equivalent to } \begin{cases} t^{q_0}\, e^{-\sqrt{\log 1/t}} & \text{near zero,} \\ t^{q_\infty}\, e^{\sqrt{\log t}} & \text{near infinity.} \end{cases}$$

The condition (5.2.6) forces that $q_0 \ge n/(n-\gamma)$. If $q_0 > n/(n-\gamma)$ then $i_B^\infty > n/(n-\gamma)$ and B_γ^∞ satisfies the simplified relation (5.3.8) near zero. In the case when $q_0 = n/(n-\gamma)$ then

$$F_\gamma^\infty(t) \simeq t^{\frac{n}{n-\gamma}} e^{-\sqrt{\log 1/t}} \sqrt{\log 1/t} \quad \text{near zero}$$

and

$$B_\gamma^{\infty -1}(s) \simeq F_\gamma^{\infty -1}(s)\, s^{\frac{\gamma}{n}} \simeq s e^{-(1-\frac{\gamma}{n})^{\frac{3}{2}}\sqrt{\log 1/s}} (\log 1/s)^{-\frac{n-\gamma}{2n}} \quad \text{near zero.}$$

Calculating the inverses, we get that

$$B_\gamma^\infty(t) \text{ is equivalent to } \begin{cases} t^{\frac{nq_0}{n+\gamma q_0}} e^{-(\frac{n}{n+\gamma q_0})^{\frac{3}{2}}\sqrt{\log 1/t}}, & q_0 > \frac{n}{n-\gamma}, \\ t e^{-(1-\frac{n}{\gamma})^{\frac{3}{2}}\sqrt{\log 1/t}} (\log 1/t)^{\frac{2n}{n-\gamma}}, & q_0 = \frac{n}{n-\gamma}, \end{cases}$$

near zero.

Let us also sketch the calculations near infinity. If $q_\infty < n/(n-\gamma)$, then

$$\int^\infty \frac{B(s)}{s^{n/(n-\gamma)+1}}\, \mathrm{d}s < \infty$$

whence $F_\gamma^\infty(t) \simeq t^{\frac{n}{n-\gamma}}$ and $B_\gamma^\infty(t)$ is equivalent to t near infinity. If $q_\infty > n/(n-\gamma)$ then B_γ^∞ obeys the relation (5.3.8) near infinity, and finally, when $q_\infty = n/(n-\gamma)$, then

$$F_\gamma^\infty(t) \simeq t^{\frac{n}{n-\gamma}} e^{\sqrt{\log t}} \sqrt{\log t} \quad \text{near zero.}$$

In conclusion, we have

$$B_\gamma^\infty(t) \text{ is equivalent to } \begin{cases} t^{\frac{nq_\infty}{n+\gamma q_\infty}} e^{(\frac{n}{n+\gamma q_\infty})^{\frac{3}{2}}\sqrt{\log t}}, & q_\infty > \frac{n}{n-\gamma}, \\ t e^{(1-\frac{n}{\gamma})^{\frac{3}{2}}\sqrt{\log t}} (\log t)^{\frac{2n}{n-\gamma}}, & q_\infty = \frac{n}{n-\gamma}, \\ t, & q_\infty < \frac{n}{n-\gamma}, \end{cases}$$

near infinity and by Theorem 5.3.6,

$$M_\gamma \colon L^{B_\gamma^\infty}(\mathbb{R}^n) \to L^B(\mathbb{R}^n),$$

in which $L^{B_\gamma^\infty}(\mathbb{R}^n)$ is optimal within Orlicz spaces.

Concerning optimality, one may naturally ask the question if the relation "be optimal Orlicz space for someone" is symmetric. We will look closely what is meant by this now.

Let us start on the target side, so let us have some Young function B fixed. Assume for the sake of this example that $\Omega \subseteq \mathbb{R}^n$ is open and $|\Omega| < \infty$. By Theorem 5.3.5, the optimal Orlicz domain always exists and is described by the Young function B_γ. At this stage we may set $A = B_\gamma$ and try to use Theorem 5.3.1 to investigate the corresponding best possible Orlicz target.

We illustrate what is happening on a basic example. Assume that $B(t) = t^q$ near infinity and $q > \frac{n}{n-\gamma}$ (we will be ignoring the behaviour near zero for the sake of this paragraph, the careful reader may adapt the Young functions also near zero). Then $B_\gamma(t)$ is equivalent to $t^{\frac{nq}{n+\gamma q}}$ near infinity. Set $A = B_\gamma$ and observe that $A_\gamma(t)$ is equivalent to t^q, whence $i_{A_\gamma} = q > \frac{n}{n-\gamma}$. Thus A_γ is equivalent to B and both domain L^A and range L^B are optimal Orlicz spaces.

Now, if $B(t) = t^{\frac{n}{n-\gamma}}$ near infinity, then B_γ is equivalent to $t\log(t)^{(1-\frac{n}{\gamma})}$ near infinity. Denoting $A = B_\gamma$, $A_\gamma(t)$ is equivalent to $t^{n/(n-\gamma)}\log(t)$ which exceeds $B(t)$.

However, $i_{A_\gamma} = \frac{n}{n-\gamma}$ thus B is not the optimal Orlicz space to L^A and moreover, no such Orlicz space exists.

From this example we may guess that the borderline lays somewhere around the space $L^{\frac{n}{n-\gamma}}$. Indeed, the proper classification of this phenomenon relies on the Boyd index of B as the following theorem shows.

Theorem 5.3.9 [Orlicz range reiteration on finite-measure sets]. *Let B be a Young function and let $\Omega \subseteq \mathbb{R}^n$ be such that $|\Omega| < \infty$. If B satisfies*

$$i_B > \frac{n}{n-\gamma}, \tag{5.3.11}$$

then the Young function B_γ from (5.2.2) *obeys* (5.3.8) *and*

$$M_\gamma\colon L^{B_\gamma}(\Omega) \to L^B(\Omega) \tag{5.3.12}$$

where both domain and target spaces are optimal within all Orlicz spaces.

Conversely, if (5.3.11) *fails, then $L^{B_\gamma}(\Omega)$ is optimal Orlicz domain and no optimal Orlicz target space exists in* (5.3.12)*.*

Theorem 5.3.10 [Orlicz range reiteration on $\mathbb{R}^n$]. *Let B be a Young function obeying*

$$i_B^\infty > \frac{n}{n-\gamma}. \tag{5.3.13}$$

Then the Young function B_γ^∞ from (5.2.7) *satisfies* (5.3.10) *and*

$$M_\gamma\colon L^{B_\gamma^\infty}(\mathbb{R}^n) \to L^B(\mathbb{R}^n) \tag{5.3.14}$$

where both domain and target spaces are optimal among all Orlicz spaces.

Conversely, if (5.3.13) *fails, then $L^{B_\gamma^\infty}(\mathbb{R}^n)$ is optimal Orlicz domain and no optimal Orlicz target space exists in* (5.3.14)*.*

In this iteration scheme, we may also assume that A is given and we try to make one step further and then one step back, or more precisely, we can compute A_γ, then set $B = A_\gamma$ and then analyze the relation of B_γ and A. The main difference between this case and the previous one is that even the success after first step is not guaranteed any more. So one has to restrict his attention to the positive cases only. Let us look at similar trivial example.

Assume that $A(t) = t^{p_0}$ near zero and $A(t) = t^{p_\infty}$ near infinity, where $1 < p_0 < \frac{n}{\gamma} < p_\infty < \infty$. Calculations gives that $A_\gamma(t)$ is equivalent to $t^{\frac{np_0}{n-\gamma p_0}}$ and to ∞ near infinity. Since $i_{A_\gamma} > \frac{n}{n-\gamma}$, M_γ acts from L^A to L^{A_γ} and the range is optimal within Orlicz spaces. If we set $B = A_\gamma$, then $B_\gamma(t)$ is equivalent to t^{p_0} near zero and to $t^{\frac{n}{\gamma}}$ near infinity. We see that B_γ coincides with A near zero while there is a significant improvement near infinity.

To state the result in its full generality, we need to introduce a way how to define the improved Young function to A.

Let A be a Young function satisfying (5.2.4) and let $A_{\sup}$ be given by

$$A_{\sup}(t) = \int_0^t \frac{G_{\sup}^{-1}(s)}{s}\,\mathrm{d}s, \quad t > 0. \tag{5.3.15}$$

where $G_{\sup}$ is defined by

$$G_{\sup}(t) = \begin{cases} A^{-1}(t), & 0 \le t < 1, \\ t^{\frac{n}{\gamma}} \sup\limits_{1<s\le t} A^{-1}(s)\, t^{-\frac{n}{\gamma}}, & t \ge 1. \end{cases}$$

Using similar arguments as in Remark 5.2.1, one can easily observe that $G_{\sup}$ is increasing and $A_{\sup}$ is well-defined Young function equivalent to $G_{\sup}^{-1}$.

The spirit of this improvement lays in the observation that any domain A can be always replaced by $A_{\sup}$. This is the essence of the next theorem.

Theorem 5.3.11 [Orlicz domain reiteration on finite-measure sets]. *Let $n \in \mathbb{N}$, $0 < \gamma < n$, $\Omega \subseteq \mathbb{R}^n$ satisfy $|\Omega| < \infty$. Suppose that A is a Young function and let $A_{\sup}$ be the Young function from* (5.3.15)*. Then* (5.1.1) *holds if and only if*

$$M_\gamma \colon L^{A_{\sup}}(\Omega) \to L^B(\Omega). \tag{5.3.16}$$

Furthermore, if

$$i_{A_{\sup}} > 1, \tag{5.3.17}$$

then

$$M_\gamma \colon L^{A_{\sup}}(\Omega) \to L^{A_\gamma}(\Omega)$$

and both domain and target spaces are optimal in the class of Orlicz spaces.

Conversely, if (5.3.17) *fails, then $L^A(\Omega)$ can be replaced by $L^{A_{\sup}}(\Omega)$ and no optimal Orlicz target exists.*

A parallel result continues to hold on $\mathbb{R}^n$ if we modify properly the definition of $A_{\sup}$ near zero. Let A be a Young function satisfying (5.2.4) and let $A^\infty_{\sup}$ be given by

$$A^\infty_{\sup}(t) = \int_0^t \frac{{G^\infty_{\sup}}^{-1}(s)}{s}\,\mathrm{d}s, \quad t > 0. \tag{5.3.18}$$

where $G^\infty_{\sup}$ is defined by

$$G^\infty_{\sup}(t) = t^{\frac{n}{\gamma}} \sup_{0<s\le t} A^{-1}(s)\, t^{-\frac{n}{\gamma}}, \quad t > 0.$$

Similarly as above, $G^\infty_{\sup}$ is increasing and $A^\infty_{\sup}$ is Young function.

Theorem 5.3.12 [Orlicz domain reiteration on $\mathbb{R}^n$]. *Let $n \in \mathbb{N}$ and $0 < \gamma < n$. Assume that A and B are Young functions and suppose that A satisfies* (5.2.4)*. Let $A_{\sup}$ be the Young function from* (5.3.18)*. Then* (5.1.1) *holds if and only if*

$$M_\gamma \colon L^{A_{\sup}}(\mathbb{R}^n) \to L^B(\mathbb{R}^n).$$

Moreover, if

$$i^\infty_{A^\infty_{\sup}} > 1, \tag{5.3.19}$$

then

$$M_\gamma \colon L^{A^\infty_{\sup}}(\mathbb{R}^n) \to L^{A^\infty_\gamma}(\mathbb{R}^n)$$

and both domain and target spaces are optimal in the class of Orlicz spaces.

Conversely, if (5.3.19) *fails, then $L^A(\mathbb{R}^n)$ can be replaced by $L^{A^\infty_{\sup}}(\mathbb{R}^n)$ and no optimal Orlicz target exists.*

At the end of this section, we present special cases of the reduction principle for the spaces L^1 and L^∞.

Corollary 5.3.13 [Endpoint embeddings]. *Let $n \in \mathbb{N}$, $0 < \gamma < n$, $\Omega \subseteq \mathbb{R}^n$ with $|\Omega| < \infty$ and suppose that A and B are Young functions. Then the following statements hold true:*
(i)

$$M_\gamma \colon L^A(\Omega) \to L^\infty(\Omega) \quad \left[M_\gamma \colon L^A(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)\right]$$

if and only if there is a constant $C > 0$ such that

$$A(t) \geq C t^{\frac{n}{\gamma}} \quad \text{near infinity} \quad [\text{for } t \geq 0];$$

(ii)

$$M_\gamma \colon L^1(\Omega) \to L^B(\Omega) \quad \left[M_\gamma \colon L^1(\mathbb{R}^n) \to L^B(\mathbb{R}^n)\right]$$

if and only if

$$\int^\infty \frac{B(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s < \infty \quad \left[\int_0^\infty \frac{B(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s < \infty\right].$$

5.4 Proofs

We start with an auxiliary reduction principle for fractional maximal operator. Note that the result remains valid also if we replace Orlicz spaces by any rearrangement-invariant function spaces. For further details on this general setting we refer to [36, Section 4].

Proposition 5.4.1. *Let $n \in \mathbb{N}$, $0 < \gamma < n$ and let A and B be Young functions. The following statements are equivalent:*

(i) *There is a constant $C_1 > 0$ such that*

$$\|M_\gamma f\|_{L^B(\mathbb{R}^n)} \leq C_1 \|f\|_{L^A(\mathbb{R}^n)}, \quad f \in L^A(\mathbb{R}^n);$$

(ii) *There is a constant $C_2 > 0$ such that*

$$\left\| t^{\frac{\gamma}{n}-1} \int_0^t g(s)\,\mathrm{d}s \right\|_{L^B(0,\infty)} \leq C_2 \|g\|_{L^A(0,\infty)}, \quad g \in L^A(0,\infty).$$

Moreover, the constants C_1 and C_2 depend on each other, on n and γ.

Proof. Assume (ii). By [25, Theorem 1.1], there is a constant $c_1 > 0$, depending only on n and γ, such that

$$(M_\gamma f)^*(t) \leq c_1 \sup_{t \leq s < \infty} s^{\frac{\gamma}{n}} f^{**}(s) \quad \text{for } t > 0$$

and for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Here, f^{**} is the function defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\,\mathrm{d}s \quad \text{for } t > 0.$$

Then

$$\|M_\gamma f\|_{L^B(\mathbb{R}^n)} = \|(M_\gamma f)^*\|_{L^B(0,\infty)} \le c_1 \Big\| \sup_{t\le s<\infty} s^{\frac{\gamma}{n}} f^{**}(s) \Big\|_{L^B(0,\infty)}.$$

The supremum may be dropped paying another constant $c_2 = c_2(n,\gamma)$, thanks to [45, Theorem 3.9]. Hence

$$\|M_\gamma f\|_{L^B(\mathbb{R}^n)} \le c_1 c_2 \|t^{\frac{\gamma}{n}} f^{**}(t)\|_{L^B(0,\infty)} \le c_1 c_2 C_2 \|f^*\|_{L^A(0,\infty)} = C_1 \|f\|_{L^A(\mathbb{R}^n)},$$

due to the assumption (ii) where we take $g = f^*$.

Conversely, suppose (i). Let $\varphi\colon (0,\infty) \to [0,\infty)$ be a nonincreasing function. By [25, Theorem 1.1], there is a function f on $\mathbb{R}^n$ such that $f^* = \varphi$ and

$$(M_\gamma f)^*(t) \ge c_3 \sup_{t\le s<\infty} s^{\frac{\gamma}{n}} f^{**}(s) \quad \text{for } t \in (0,\infty)$$

where $c_3 = c_3(n,\gamma)$ is a positive constant independent of f. We have

$$\begin{aligned} C_1\|\varphi\|_{L^A(0,\infty)} &= C_1\|f\|_{L^A(\mathbb{R}^n)} \ge \|M_\gamma f\|_{L^B(\mathbb{R}^n)} = \|(M_\gamma f)^*\|_{L^B(0,\infty)} \\ &\ge c_3 \Big\| \sup_{t\le s<\infty} s^{\frac{\gamma}{n}} f^{**}(s) \Big\|_{L^A(0,\infty)} \ge c_3 \Big\| t^{\frac{\gamma}{n}-1} \int_0^t \varphi(s)\,\mathrm{d}s \Big\|_{L^A(0,\infty)}, \end{aligned}$$

hence (ii) holds for every nonincreasing function φ with $C_2 = C_1/c_3$. The inequality (ii) for any function then follows by Hardy-Littlewood inequality (2.4.1). □

As for the local results, we have the parallel statement. Note, that the information about the embedding norm is lost in this case. The proof of is similar to that of Proposition 5.4.1 and is omitted.

Proposition 5.4.2. *Let $n \in \mathbb{N}$, $0 < \gamma < n$, let $\Omega \subseteq \mathbb{R}^n$ be an open set of finite measure and let A and B be Young functions. Then* (5.1.1) *holds true if and only if*

$$\Big\| t^{\frac{\gamma}{n}-1} \int_0^t g(s)\,\mathrm{d}s \Big\|_{L^B(0,1)} \le C_2 \|g\|_{L^A(0,1)}, \quad g \in L^A(0,1).$$

Proof of Theorem 5.2.4. We begin with the preliminary statement, equivalent to (i). Recall the Hardy operator $H_{\frac{\gamma}{n}}$, defined by

$$H_{\frac{\gamma}{n}} f(t) = \int_t^\infty f(s) s^{\frac{\gamma}{n}-1}\,\mathrm{d}s, \quad t \in (0,\infty),$$

for $f \in \mathcal{M}(0,\infty)$ and its associate operator, $H'_{\frac{\gamma}{n}}$ say, given by

$$H'_{\frac{\gamma}{n}} g(t) = t^{\frac{\gamma}{n}-1} \int_0^t g(s)\,\mathrm{d}s, \quad t \in (0,\infty),$$

for $g \in \mathcal{M}(0,\infty)$. See (2.8.1) for details. By Proposition 5.4.1, (i) holds if and only if $H'_{\frac{\gamma}{n}}$ is bounded from $L^A(0,\infty)$ to $L^B(0,\infty)$ with the operator norm comparable to C_1. Hence, by the duality argument as in (3.3.47), (i) is equivalent to the following statement.

(i') There exist a constant $C_1' > 0$ such that

$$\|H_{\frac{\gamma}{n}} f\|_{L^{\widetilde{A}}(0,\infty)} \le C_1' \|f\|_{L^{\widetilde{B}}(0,\infty)}.$$

Moreover the constants C_1 and C_1' are comparable.

Now, we can prove the theorem with the help of results from Chapter 3. We show (i')⇔(ii) first. We infer from Theorem 3.5.2 that (i') holds true if and only if

$$\sup_{0<t<1} \widetilde{A}(t)\, t^{\frac{n}{n-\gamma}} < \infty \tag{5.4.1}$$

and there is a constant $C_2' > 0$, comparable to C_1', such that the inequality

$$\int_0^t \frac{\widetilde{\widetilde{B}}(s)}{s^{n/(n-\gamma)+1}}\, \mathrm{d}s \le \frac{\widetilde{(\widetilde{A})^{\infty}_{\gamma/n,1}}(C_2' t)}{t^{n/(n-\gamma)}} \quad \text{for } t > 0, \tag{5.4.2}$$

holds true, in which $(\widetilde{A})^{\infty}_{\gamma/n,1}$ is a Young function associated to $\widetilde{A}$ as in (3.5.6) for $\alpha = \gamma/n$ and $\beta = 1$. By (3.5.8), we have

$$(\widetilde{A})^{\infty\;-1}_{\gamma/n,1}(t) \simeq t \inf_{0<s\le t} \widetilde{A}^{-1}(s)\, s^{\frac{\gamma}{n}-1} \quad \text{for } t > 0$$

which, passing to conjugates due to (2.1.5), gives

$$\widetilde{(\widetilde{A})^{\infty}_{\gamma/n,1}}^{\,-1}(t) \simeq \sup_{0<s\le t} A^{-1}(s)\, s^{-\frac{\gamma}{n}} \quad \text{for } t > 0,$$

whence, by the definition (5.2.5),

$$\widetilde{(\widetilde{A})^{\infty}_{\gamma/n,1}}^{\,-1}(t) \simeq A^{\infty\,-1}_{\gamma} \quad \text{for } t > 0$$

and $\widetilde{(\widetilde{A})^{\infty}_{\gamma/n,1}}$ is globally equivalent to A^{∞}_{γ}. Since also $\widetilde{\widetilde{B}} = B$, (5.4.2) is equivalent to

$$\int_0^t \frac{B(s)}{s^{n/(n-\gamma)+1}}\, \mathrm{d}s \le \frac{A^{\infty}_{\gamma}(C_2 t)}{t^{n/(n-\gamma)}} \quad \text{for } t > 0,$$

where C_2 a constant comparable to C_2'. Observe also that (5.4.1) holds if and only if (5.2.4) holds. This proves (ii).

The equivalence of (i') and (iii) is a consequence of Theorem 3.3.2 which asserts that

$$\int_0 \left(\frac{s}{\widetilde{B}(s)}\right)^{\frac{\gamma}{n-\gamma}} \mathrm{d}s < \infty \tag{5.4.3}$$

and that (i') is equivalent to the existence of a constant $C_3' > 0$ such that

$$\widetilde{A}(t) \le (\widetilde{B})^{\infty}_{\gamma/n,1}(C_3' t) \quad \text{for } t > 0, \tag{5.4.4}$$

where $(\widetilde{B})^{\infty}_{\gamma/n,1}$ is the young function associated to $\widetilde{B}$ as in (3.3.5) in which we set $\alpha = \gamma/n$ and $\beta = 1$. Proposition 3.2.3 gives that (5.4.3) is equivalent to (5.2.6). From Proposition 3.3.7, we infer that

$$(\widetilde{B})^{\infty\;-1}_{\gamma/n,1}(t) \simeq \frac{t^{1-\frac{n}{\gamma}}}{F^{\infty\,-1}_{\gamma}(t)} \quad \text{for } t > 0.$$

This, together with the definition of B_γ^∞ in (5.2.7) and by (2.1.5) yields to

$$(\widetilde{B})_{\gamma/n,1}^{\infty}{}^{-1}(t) \simeq \widetilde{B_\gamma^\infty}^{-1}(t) \quad \text{for } t \geq 0,$$

whence $(\widetilde{B})_{\gamma/n,1}^{\infty}$ is globally equivalent to $\widetilde{B_\gamma^\infty}$. Using this and passing to the conjugate functions in (5.4.4) equivalently gives that there is a constant $C_3 > 0$ comparable to C_3' such that

$$B_\gamma^\infty(t) \leq A(C_3 t) \quad \text{for } t \geq 0$$

which is (iii).

Let us establish (iv) from (i). Suppose that $N > 0$ is given and set $A_N = A/N$, a scaled Young function. Then

$$(A_N)_\gamma(t) = \frac{1}{N} A_\gamma\Big(tN^{-\frac{\gamma}{n}}\Big), \quad t \geq 0,$$

where $(A_N)_\gamma$ is a Young function associated to A_N as in (5.2.1). Define also B_N by

$$B_N(t) = \frac{1}{N} B\Big(tN^{-\frac{\gamma}{n}}\Big), \quad t \geq 0.$$

We claim that

$$\|M_\gamma f\|_{L^{B_N}(\mathbb{R}^n)} \leq C_2 \|f\|_{L^{A_N}(\mathbb{R}^n)} \tag{5.4.5}$$

for all $f \in L^{A_N}(\mathbb{R}^n)$ with the constant C_2 independent of N. Indeed, as one can readily check by the change of variables, (iii) holds with A and B replaced by A_N and B_N, respectively with the same constant C_3. The claim follows by the already proven equivalence of (i) and (iii). Now, let $f \in L^A(\mathbb{R}^n)$. If $\int_{\mathbb{R}^n} A(|f|) = \infty$, then there is nothing to prove. Otherwise, set $N = \int_{\mathbb{R}^n} A(|f|)$. It is $\|f\|_{L^{A_N}(\mathbb{R}^n)} \leq 1$ and, by (5.4.5), $\|f\|_{L^{B_N}(\mathbb{R}^n)} \leq C_2$. Therefore

$$\int_{\mathbb{R}^n} B_N\left(\frac{M_\gamma f(x)}{C_2}\right) \mathrm{d}x \leq 1$$

and (ii) follows by the definition of B_N. The converse implication (ii)⇒(i) is trivial. □

Proof of Theorem 5.2.3. The proof is very similar to that of Theorem 5.2.4; one just hast to use the local results from Chapter 3 instead. The details are omitted, for brevity. □

Proof of Theorem 5.3.2. We have already observed that the boundedness of M_γ between $L^A(\mathbb{R}^n)$ and $L^B(\Omega)$ is equivalent to the boundedness of a Hardy-type operator $H_{\frac{n}{\gamma}}$ between associated Orlicz spaces, i.e., (5.1.1) is equivalent to

$$H_{\frac{n}{\gamma}} \colon L^{\widetilde{B}}(0,\infty) \to L^{\widetilde{A}}(0,\infty) \tag{5.4.6}$$

and, therefore, the problem of optimal Orlicz target space in (5.1.1) reduces to the problem of optimal Orlicz domain space in (5.4.6). At this stage, Theorem 3.6.1 comes into play. It asserts that if $\widetilde{A}$ obeys (5.4.1) and if the Young function $(\widetilde{A})_{\gamma/n,1}^{\infty}$, associated to $\widetilde{A}$ as in (3.5.6), satisfies

$$I^{\infty}_{(\widetilde{A})_{\gamma/n,1}^{\infty}} < \frac{n}{\gamma} \tag{5.4.7}$$

then

$$H_{\frac{n}{\gamma}}\colon L^{(\widetilde{A})^{\infty}_{\gamma/n,1}}(0,\infty)\to L^{\widetilde{A}}(0,\infty) \tag{5.4.8}$$

and the domain space in (5.4.8) is optimal within Orlicz spaces. Conversely, if (5.4.7) fails, then no optimal Orlicz domain space exists (in the sense that any Orlicz domain might be improved). Also, if (5.4.1) is not fulfilled, then no domain Orlicz space in (5.4.8) exists.

As we observed in the proof of Theorem 5.2.4, $(\widetilde{A})^{\infty}_{\gamma/n,1}$ is equivalent to $\widetilde{A^{\infty}_{\gamma}}$ and (5.4.1) holds if and only if (5.2.4) holds. Further, (5.4.7) is equivalent to (5.3.4) and the claim is thus recovered. The special relation (5.3.6) then also follows from (3.6.3) by analogous duality argument. □

Proof of Theorem 5.3.1. The proof is similar to that of Theorem 5.3.2. Here, the Theorem 3.6.3 from Chapter 3 plays the major together with passing to conjugate Young functions. □

Before we continue, we look closer to the relation between the Young function B and the auxiliary function F_γ. It is immediate that F_γ dominates B near infinity, since

$$\begin{aligned} F_\gamma(2t) &\geq t^{n/(n-\gamma)}\int_1^{2t}\frac{B(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s \geq t^{n/(n-\gamma)}\int_t^{2t}\frac{B(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s \\ &\geq B(t)\,t^{n/(n-\gamma)}\int_t^{2t}\frac{\mathrm{d}s}{s^{n/(n-\gamma)+1}} \geq B(t)C_{n,\gamma} \quad \text{for } t>1 \end{aligned} \tag{5.4.9}$$

and, due to Proposition 3.2.2, if $i_B > \frac{n}{n-\gamma}$, then F_γ and B are equivalent. The next lemma shows that if $i_B \leq \frac{n}{n-\gamma}$ then also i_{F_γ} has to be small.

Lemma 5.4.3. *Let B be a Young function and let F_γ be defined as in* (5.2.3). *Then F_γ is a Young function and*

$$i_{F_\gamma} > \frac{n}{n-\gamma} \quad \textit{if and only if} \quad i_B > \frac{n}{n-\gamma}.$$

Furthermore, if this is the case, then F_γ is equivalent to B near infinity.

Proof. Suppose that $i_B > \frac{n}{n-\gamma}$. Then, by Proposition 3.2.2 and by (5.4.9) B is equivalent t F_γ and therefore $i_{F_\gamma} > \frac{n}{n-\gamma}$.

Conversely, suppose that $i_{F_\gamma} > \frac{n}{n-\gamma}$. Then, thanks to Proposition 3.2.2, there exist $\sigma > 1$ and $c > 1$ such that

$$F_\gamma(\sigma t) \geq c\sigma^{\frac{n}{n-\gamma}} F_\gamma(t) \quad \text{near infinity.} \tag{5.4.10}$$

Let us write $c = 1+\varepsilon$ for some $\varepsilon > 0$. Then (5.4.10) becomes

$$\int_1^t \frac{B(s)}{s^{n/(n-\gamma)+1}}\mathrm{d}s \leq \frac{1}{\varepsilon}\int_t^{\sigma t}\frac{B(s)}{s^{n/(n-\gamma)+1}}\mathrm{d}s \leq B(\sigma t)\frac{1}{\varepsilon}\int_t^{\sigma t}\frac{1}{s^{n/(n-\gamma)+1}}\mathrm{d}s \leq \frac{B(kt)}{t^{n/(n-\gamma)}}$$

for every $t > 0$ and for a suitable constant k. Another use of Proposition 3.2.2 gives that $i_B > \frac{n}{n-\gamma}$. □

A parallel observation as in (5.4.9) holds for the global variant F_γ^∞. A variant of Lemma 5.4.3 is also on hand. Its proof is omitted, for brevity.

Lemma 5.4.4. *Let B be a Young function satisfying* (5.2.6) *and let F_γ^∞ be defined as in* (5.2.8). *Then F_γ^∞ is a Young function and*

$$i_{F_\gamma^\infty}^\infty > \frac{n}{n-\gamma} \quad \textit{if and only if} \quad i_B^\infty > \frac{n}{n-\gamma}.$$

Furthermore, if this is the case, then F_γ^∞ is equivalent to B.

Proof of Theorem 5.3.6. Assume that B satisfies (5.2.6). Then B_γ^∞ is well-defined and (5.3.9) holds by Theorem 5.2.4. To observe the optimality, assume that $L^A(\mathbb{R}^n)$ satisfies (5.1.1). Then, again, due to Theorem 5.2.4, A dominates B_γ^∞, whence $L^A(\mathbb{R}^n) \to L^{B_\gamma^\infty}(\mathbb{R}^n)$ and $L^{B_\gamma^\infty}(\mathbb{R}^n)$ is optimal.

If (5.2.6) is violated, then no Orlicz space $L^A(\mathbb{R}^n)$ might satisfy (5.1.1) since otherwise it would contradict Theorem 5.2.4.

Furthermore, if $i_B^\infty > \frac{n}{n-\gamma}$, then Proposition 3.2.2 gives that F_γ^∞ is equivalent to B and the formula (5.3.10) is immediate. □

Proof of Theorem 5.3.5. Here, the argument relies on Theorem 5.2.3 which asserts that the choice $A = B_\gamma$ guarantees (5.3.7). As for the optimality, if $L^A(\Omega)$ possess (5.1.1), then, by Theorem 5.2.3, A dominates B_γ near infinity and $L^A(\Omega) \to L^{B_\gamma}(\Omega)$ proving the optimality.

Next, if $i_B > \frac{n}{n-\gamma}$, then F_γ is equivalent to B thanks to Proposition 3.2.2 and (5.3.8) follows. □

Proof of Theorem 5.3.9. The boundedness of M_γ in (5.3.12) and the optimality of the domain space is a consequence of Theorem 5.3.5.

As for the range, set $A = B_\gamma$ and let us compute A_γ by (5.2.1). We have

$$A_\gamma^{-1}(t) \simeq G_\gamma(t) = \sup_{1<s\le t} B_\gamma^{-1}(s)\, s^{-\frac{\gamma}{n}} \simeq \sup_{1<s\le t} F_\gamma^{-1}(s) = F_\gamma^{-1}(t) \quad \text{near infinity},$$

whence A_γ is equivalent to F_γ near infinity. Therefore $i_{A_\gamma} > \frac{n}{n-\gamma}$ if and only if $i_{F_\gamma} > \frac{n}{n-\gamma}$ which is due to Lemma 5.4.3 the same as $i_B > \frac{n}{n-\gamma}$. The optimality of the range in (5.3.12) is then driven by the condition (5.3.11), thanks to Theorem 5.3.1. □

Proof of Theorem 5.3.10. Note, that the condition (5.3.13) ensures that B_γ^∞ is well defined. Then the rest of the proof follows the same scheme as that of Theorem 5.3.9. □

Proof of Theorem 5.3.11. Let A be given and let A_{sup} be the Young function from (5.3.15). Then the Young function $(A_{\text{sup}})_\gamma$ associated to A_{sup} as in (5.2.1) satisfies

$$\begin{aligned}(A_{\text{sup}})_\gamma^{-1}(t) &\simeq \sup_{1<s\le t} s^{-\frac{n}{\gamma}} A_{\text{sup}}^{-1}(s)\\ &\simeq \sup_{1<s\le t}\sup_{1<y\le s} y^{-\frac{n}{\gamma}} A^{-1}(y)\\ &= \sup_{1<y\le t} y^{-\frac{n}{\gamma}} A^{-1}(y) \simeq A_\gamma^{-1}(t) \quad \text{for } t>1,\end{aligned}$$

in other words, $(A_{\text{sup}})_\gamma$ is equivalent to A_γ. Therefore, by Theorem 5.2.4, criterion (ii), (5.1.1) holds if and only if (5.3.16) holds. Also,

$$(A_{\text{sup}})_\gamma^{-1}(t) \simeq t^{-\frac{n}{\gamma}} A_{\text{sup}}^{-1}(t) \quad \text{for } t>1$$

and thus

$$h_{(A_{\sup})_\gamma}(t) = \limsup_{s\to\infty} \frac{(A_{\sup})_\gamma^{-1}(st)}{(A_{\sup})_\gamma^{-1}(s)} \simeq t^{-\frac{n}{\gamma}} \limsup_{s\to\infty} \frac{A_{\sup}^{-1}(st)}{A_{\sup}^{-1}(s)} = t^{-\frac{n}{\gamma}} h_{A_{\sup}}(t)$$

for $t > 1$, whence, by the definition of the local lower Boyd index,

$$\frac{1}{i_{(A_{\sup})_\gamma}} + \frac{\gamma}{n} = \frac{1}{i_{A_{\sup}}}$$

and therefore $i_{(A_{\sup})_\gamma} > \frac{n}{n-\gamma}$ if and only if $i_{A_{\sup}} > 1$. The claim now follows by Theorem 5.3.1. □

Proof of Theorem 5.3.12. Let A be a Young function such that (5.2.4) holds true. Then observe that $A_{\sup}$ satisfies (5.2.4) as well. The remaining part of the proof is analogous to that for the local version. □

Proof of Corollary 5.3.13. Assume that $|\Omega| < \infty$. (i) Suppose that $B(t) = 0$ on $[0, 1]$ and $B(t) = \infty$ for $t > 1$. Then $L^B(\Omega) = L^\infty(\Omega)$ and, by (5.2.2), B_γ is equivalent to $t^{\frac{n}{\gamma}}$ near infinity. By Theorem 5.2.3 (the equivalence of (i) and (iii)), we have (i).

(ii) Let us set $A(t) = t$, $t \geq 0$, so $L^A(\Omega) = L^1(\Omega)$. Clearly, by (5.2.1), $A_\gamma(t)$ is equivalent to $t^{\frac{n}{n-\gamma}}$ near infinity and thus (ii) follows by Theorem 5.2.3, the equivalence of (i) and (ii).

The proof for $\Omega = \mathbb{R}^n$ is similar. □

6. Hardy-Littlewood maximal operator

6.1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. The Hardy–Littlewood maximal operator, M, is defined for every locally integrable function f on Ω by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, \mathrm{d}y \quad \text{for } x \in \Omega,$$

where the supremum is extended over all cubes Q contained in Ω, whose edges are parallel to the coordinate axes of $\mathbb{R}^n$, that contain x.

The operator M is merely sublinear, rather than linear, and it is clearly a contraction on $L^\infty(\Omega)$. On the other hand, if $|\Omega| = \infty$, Mf is never integrable unless $f \equiv 0$. For every locally-integrable function f on Ω, one has $|f| \leq Mf$ almost everywhere. The most important information (for our purpose) concerning the operator M, now classical, states that there exist positive constants c, c', depending only on n, such that

$$c(Mf)^*(t) \leq f^{**}(t) \leq c'(Mf)^*(t) \quad \text{for } t \geq 0, \tag{6.1.1}$$

for every locally-integrable function on Ω. The first inequality in (6.1.1) was established in works of R. M. Gabriel [40], F. Riesz [62] and N. Wiener [71], while the second was added later through the efforts of C. Herz [43] (for one dimension) and C. Bennett and R. Sharpley [8] (for higher dimensions). The result is summarized and proved in [9, Chapter 3, Theorem 3.8].

6.2 Reduction principle

In this scope, we would like to give the necessary and sufficient conditions of the boundedness

$$M\colon L^A(\Omega) \to L^B(\Omega) \tag{6.2.1}$$

in terms of the Young functions A and B itself. Such a result was first obtained by H. Kita in [48].

Since properties of function spaces defined on a set Ω may significantly differ in the cases when the measure $|\Omega|$ is finite or infinite, it is no surprise that we will treat such cases separately. Let us start with the global variant.

Theorem 6.2.1 [Reduction principle in Orlicz spaces, case $|\Omega| = \infty$]. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set such that $|\Omega| = \infty$ let A and B be Young functions. The following statements are equivalent.*

(i) *There is a constant $C_1 > 0$ such that*

$$\|Mf\|_{L^B(\Omega)} \leq C_1 \|f\|_{L^A(\Omega)}$$

for every $f \in L^A(\Omega)$;

(ii) *There is a constant* $C_2 > 0$ *such that*

$$\int_0^t \frac{B(s)}{s^2}\,\mathrm{d}s \le \frac{A(C_2 t)}{t} \quad \text{for } t > 0;$$

(iii) *There is a constant* $C_3 > 0$ *such that*

$$\int_\Omega B\left(\frac{Mf(x)}{C_3}\right)\mathrm{d}x \le \int_\Omega A(|f(x)|)\,\mathrm{d}x$$

for every $f \in L^A(\Omega)$.

Moreover the constants C_1, C_2 *and* C_2 *are comparable.*

The local version then follows. Since only the values of Young functions near infinity are relevant in this case, it is no longer possible to have an integral inequality in general. Let us also point out that we are able to obtain a qualitative result only; the information about the norm of the operator is now lost.

Theorem 6.2.2 [Reduction principle in Orlicz spaces, case $|\Omega| < \infty$]. *Let* $\Omega \subseteq \mathbb{R}^n$ *be an open set of finite Lebesgue measure and suppose that* A *and* B *are Young functions. Then* (6.2.1) *holds if and only if there is a constant* $C > 0$ *such that*

$$\int_1^t \frac{B(s)}{s^2}\,\mathrm{d}s \le \frac{A(Ct)}{t} \quad \text{for } t > 1. \tag{6.2.2}$$

In the case when A and B coincide, we may simplify the condition (6.2.2).

Corollary 6.2.3. *Let* A *be a Young function and suppose that* $\Omega \subseteq \mathbb{R}^n$. *Then*

$$M\colon L^A(\Omega) \to L^A(\Omega) \tag{6.2.3}$$

if and only if either $|\Omega| < \infty$ *and* $\widetilde{A}$ *satisfies* Δ_2 *condition near infinity or* $|\Omega| = \infty$ *and* $\widetilde{A}$ *obeys* Δ_2 *condition globally.*

Before we prove the reduction principles, we need an auxiliary lemma.

Lemma 6.2.4. *Let* f *be a locally integrable function on* $\mathbb{R}^n$. *Then*

$$\frac{1}{t}\int_{\{f^* \ge t\}} f^*(s)\,\mathrm{d}s \le |\{f^{**} \ge t\}| \quad \text{for } t > 0 \tag{6.2.4}$$

and

$$\frac{1}{t}\int_{\{f^* \ge t\}} f^*(s)\,\mathrm{d}s \ge |\{f^{**} \ge 2t\}| \quad \text{for } t > 0. \tag{6.2.5}$$

Proof. Let $t \in (0, \infty)$ be given and denote $s_0 = \sup\{s > 0 : f^{**}(s) = t\}$. We have $f^{**}(s_0) = t$, by the continuity of f^{**}, thereby

$$|\{f^{**} \ge t\}| = s_0 = \frac{1}{t}\int_0^{s_0} f^* \ge \frac{1}{t}\int_{\{f^* \ge t\}} f^*,$$

since $\{f^* \ge t\} \subseteq (0, s_0)$. This gives (6.2.4). To prove (6.2.5), define $f_t = \min\{f^*, t\}$ and $f^t = f^* - f_t$. Then

$$\{f^{**} \ge 2t\} \subseteq \{(f_t)^{**} > t\} \cup \{(f^t)^{**} \ge t\} = \{(f^t)^{**} \ge t\}.$$

Now, for $s_0 = \sup\{s : (f^t)^{**} \ge t\}$, we infer

$$|\{f^{**} \ge 2t\}| \le |\{(f^t)^{**} \ge t\}| = s_0 = \frac{1}{t}\int_0^{s_0} f^t \le \frac{1}{t}\int_{\{f^* \ge t\}} f^*. \qquad \square$$

Proof of Theorem 6.2.1. Assume (i) and suppose that $E \subseteq \Omega$ is any measurable set such that $|E| = t$ for fixed $t > 0$. Testing the inequality in (i) on the function $f = \chi_E$, we obtain

$$\|M\chi_E\|_{L^B(\Omega)} \leq C_1\|\chi_E\|_{L^A(\Omega)}. \tag{6.2.6}$$

By (2.3.1), the right hand side of (6.2.6) equals to $1/A^{-1}(1/t)$. Next, on applying the rearrangement, we infer, thanks to (2.6.2) and the second inequality in (6.1.1), that

$$\|M\chi_E\|_{L^B(\Omega)} = \|(M\chi_E)^*\|_{L^B(0,\infty)} \geq \frac{1}{c'}\|(\chi_E)^{**}\|_{L^B(0,\infty)}. \tag{6.2.7}$$

We can now combine a trivial estimate with Lemma 3.2.1 to get

$$\|(\chi_E)^{**}\|_{L^B(0,\infty)} = \left\|\frac{1}{s}\int_0^s \chi_{(0,t)}\right\|_{L^B(0,\infty)} \geq t\left\|\frac{1}{s}\chi_{(t,\infty)}\right\|_{L^B(0,\infty)} = \frac{1}{F^{-1}(\frac{1}{t})}, \tag{6.2.8}$$

in which $F\colon [0,\infty) \to [0,\infty]$ is given by

$$F(t) = t\int_0^t \frac{B(s)}{s^2}\,\mathrm{d}s \quad \text{for } t \geq 0 \tag{6.2.9}$$

and F^{-1} represents its generalised right-continuous inverse. Coupling (6.2.6), (6.2.7) and (6.2.8), we infer that there is a constant C_2, depending on C_1 and c' such that

$$\frac{1}{F^{-1}(\frac{1}{t})} \leq C_2\frac{1}{A^{-1}(1/t)} \quad \text{for } t > 0,$$

which rewrites as

$$F(t) \leq A(C_2t) \quad \text{for } t \geq 0$$

and hence (ii) is fulfilled by the definition of F.

Assume (ii) and suppose that B possesses the representation

$$B(t) = \int_0^t b(s)\,\mathrm{d}s.$$

We have, by Fubini's theorem, that

$$\int_\Omega B\left(\frac{cMf(x)}{4}\right)\mathrm{d}x = \int_0^\infty b(s)|\{cMf \geq 4s\}|\,\mathrm{d}s. \tag{6.2.10}$$

Next, by the basic property of rearrangement and due to the first inequality in (6.1.1), we get

$$\{cMf > 4s\} = \{c(Mf)^* > 4s\} \subseteq \{f^{**} > 4s\} \quad \text{for } s > 0. \tag{6.2.11}$$

Using (6.2.11) in (6.2.10), we continue by

$$\begin{aligned}\int_\Omega B\left(\frac{cMf(x)}{4}\right)\mathrm{d}x &\leq \int_0^\infty b(s)|\{f^{**} \geq 4s\}|\,\mathrm{d}s = \frac{1}{2}\int_0^\infty b\left(\frac{s}{2}\right)|\{f^{**} \geq 2s\}|\,\mathrm{d}s\\ &\leq \int_0^\infty \frac{B(s)}{s}|\{f^{**} \geq 2s\}|\,\mathrm{d}s \leq \int_0^\infty \frac{B(s)}{s^2}\int_{\{f^* \geq s\}} f^*(t)\,\mathrm{d}t\,\mathrm{d}s\end{aligned}$$

where we used the property (2.1.2) of the Young function B and the inequality (6.2.4) of Lemma 6.2.4. Next, using Fubini's theorem and the assumed inequality in (ii) yields

$$\begin{aligned}\int_\Omega B\left(\frac{cMf(x)}{4}\right)\mathrm{d}x &\le \int_0^\infty \frac{B(s)}{s^2}\int_s^\infty |\{f^*\ge t\}|\,\mathrm{d}t\,\mathrm{d}s\\ &= \int_0^\infty |\{f^*\ge t\}|\int_0^t \frac{B(s)}{s^2}\,\mathrm{d}s\,\mathrm{d}t\\ &\le \int_0^\infty |\{f^*\ge t\}|\frac{A(C_3t)}{t}\,\mathrm{d}t \le \int_\Omega A\Big(C_3|f(x)|\Big)\,\mathrm{d}x.\end{aligned}$$

Then (iii) follows by considering f/C_3 instead of f.

Finally, it is easy to see that (iii) implies (i). □

Proof of Theorem 6.2.2. Assume that (6.2.1) holds on finite-measure open set $\Omega \subseteq \mathbb{R}^n$. We may test such embedding on functions $f = \chi_E$, in which E is any measurable set with $|E| = t$. Here only values of t near zero are relevant. We can also modify the Young function B near zero such that the integral

$$\int_0 \frac{B(s)}{s^2}\,\mathrm{d}s$$

converges leaving the space $L^B(\Omega)$ unchanged. Then, by the same calculations as in the proof of Theorem 6.2.1, we get that

$$F(t) \le A(Ct) \quad \text{near infinity,} \tag{6.2.12}$$

where C is some positive constant and F is given by (6.2.9). Ignoring the values of B near zero, (6.2.12) implies (6.2.2).

Assume, conversely, that (6.2.2) holds. Let $\widehat{A}$ and $\widehat{B}$ be Young functions such that $\widehat{A} = A$ and $\widehat{B} = B$ near infinity and such that

$$\int_0^t \frac{\widehat{B}(s)}{s^2}\,\mathrm{d}s \le \frac{\widehat{A}(Ct)}{t} \quad \text{for } t > 0;$$

holds for some constant $C > 0$. Then, by Theorem 6.2.1, we obtain that

$$M\colon L^{\widehat{A}}(\mathbb{R}^n) \to L^{\widehat{B}}(\mathbb{R}^n)$$

and, in particular,

$$M\colon L^{\widehat{A}}(\Omega) \to L^{\widehat{B}}(\Omega)$$

and (6.2.1) follows, since $L^{\widehat{A}}(\Omega) = L^A(\Omega)$ and $L^{\widehat{B}}(\Omega) = L^B(\Omega)$ up to equivalent norms. □

Before we prove Corollary 6.2.3, we need an auxiliary result dealing with several equivalent conditions on Young function. The Δ_2 is one of them.

Proposition 6.2.5. *Let E be a finite-valued Young function and denote $t_0 = \sup\{t \ge 0,\ E(t) = 0\}$. Then the following conditions are equivalent.*

(i) *There exists a constant $K > 1$ such that*

$$\int_0^t \frac{E(s)}{s^2}\,\mathrm{d}s \le \frac{E(Kt)}{t} \quad \textit{for } t > 0 \quad \left[\int_1^t \frac{E(s)}{s^2}\,\mathrm{d}s \le \frac{E(Kt)}{t} \quad \textit{near infinity}\right];$$

(ii) *There exist constants* $\varrho > 1$ *and* $c > 1$ *such that*

$$E(\varrho t) \geq c\varrho E(t) \quad \textit{for } t > 0 \quad [\textit{near infinity}];$$

(iii) *There exist constants* σ *and* $c > 1$ *such that*

$$\sigma E^{-1}(t) \geq cE^{-1}(\sigma t) \quad \textit{for } t > 0 \quad [\textit{near infinity}];$$

(iv) *There exist constants* $\lambda > 1$ *and* $c > 1$ *such that*

$$\widetilde{E}(ct) \leq \lambda \widetilde{E}(t) \quad \textit{for } t > 0 \quad [\textit{near infinity}];$$

(v) *The Young function* $\widetilde{E}$ *satisfies the* Δ_2 *condition globally [near infinity];*

(vi) *There exists a constant* $C > 1$ *such that*

$$\int_0^t \frac{\mathrm{d}s}{E^{-1}(s)} \leq \frac{Ct}{E^{-1}(t)} \quad \textit{for } t > 0 \quad \left[\int_1^t \frac{\mathrm{d}s}{E^{-1}(s)} \leq \frac{Ct}{E^{-1}(t)} \quad \textit{near infinity}\right];$$

(vii) *There exists a constant* $C > 1$ *such that*

$$\int_t^\infty \frac{E^{-1}(s)}{s^2} \mathrm{d}s \leq C\frac{E^{-1}(t)}{t} \quad \textit{for } t > 0 \quad [\textit{near infinity}];$$

(viii) *There exists a constant* $C > 1$ *such that*

$$\int_t^\infty \frac{\mathrm{d}s}{E(s)} \leq \frac{Ct}{E(t)} \quad \textit{for } t \in (t_0, \infty) \quad [\textit{near infinity}];$$

Proof. We prove the statement in its global form only. The proof of the version "near infinity" is analogous.

"(i) ⇒ (ii)". Assume (i) and assume for contradiction that to fixed $c > 1$ and any $\varrho > K$ there exists a $t > 0$ such that $E(\varrho t) < c\varrho E(t)$. By the inequality (i) with $\varrho t/K$ in place of t we get

$$K\frac{E(\varrho t)}{\varrho t} \geq \int_0^{\varrho t/K} \frac{E(s)}{s^2} \,\mathrm{d}s \geq \int_t^{\varrho t/K} \frac{E(s)}{s^2} \,\mathrm{d}s \geq \frac{E(t)}{t} \int_t^{\varrho t/K} \frac{\mathrm{d}s}{s} > \frac{E(\varrho t)}{c\varrho t} \log \frac{\varrho}{K}$$

Hence

$$cK > \log \frac{\varrho}{K}$$

for every $\varrho > K$ which is impossible.

"(ii) ⇒ (iii)". Let us replace t by $E^{-1}(t)$ in (ii). By the right continuity of E^{-1}, we have

$$E(\varrho E^{-1}(t)) \geq c\varrho E(E^{-1}(t)) \geq c\varrho t. \tag{6.2.13}$$

Now, on applying the nondecreasing function E^{-1} on both sides of (6.2.13) and denoting $\sigma = c\varrho > 1$, we get (iii).

"(iii) ⇒ (vi)". By the repeated use of (iii) we have

$$E^{-1}(\sigma^{-k}t) \geq \left(\frac{c}{\sigma}\right)^k E^{-1}(t) \quad \text{for } t \in (0, \infty). \tag{6.2.14}$$

Since $E(s)/s$ is nondecreasing, $s/E^{-1}(s)$ is nondecreasing as well and

$$\begin{aligned}\int_0^t \frac{\mathrm{d}s}{E^{-1}(s)} &= \sum_{k=0}^{\infty} \int_{t\sigma^{-k-1}}^{t\sigma^{-k}} \frac{\mathrm{d}s}{E^{-1}(s)} \le \log\sigma \sum_{k=0}^{\infty} \frac{t\sigma^{-k}}{E^{-1}(t\sigma^{-k})} \\ &\le \frac{t}{E^{-1}(t)} \log\sigma \sum_{k=0}^{\infty} \sigma^{-k}\Big(\frac{\sigma}{c}\Big)^k = \frac{t}{E^{-1}(t)} \frac{c\log\sigma}{c-1}\end{aligned}$$

for any $t \in (0,\infty)$. That gives (vi).

"(vi) $\Rightarrow$ (i)". If $t \in (0, t_0)$, then

$$\int_0^t \frac{E(s)}{s^2}\,\mathrm{d}s = 0$$

and (i) holds trivially. Assume that $t \ge t_0$. Since E is finite-valued, it is strictly increasing and hence $E^{-1}(E(t)) = t$. Let us now plug $E(t)$ in the inequality (vi) in place of t. We have

$$\begin{aligned}C\frac{E(t)}{t} = C\frac{E(t)}{E^{-1}(E(t))} &\ge \int_0^{E(t)} \frac{\mathrm{d}s}{E^{-1}(s)} = \int_{E(t_0)}^{E(t)} \frac{\mathrm{d}s}{E^{-1}(s)} \\ &= \int_{t_0}^t \frac{a(s)}{E^{-1}(E(s))}\,\mathrm{d}s \ge \int_{t_0}^t \frac{E(s)}{s^2}\,\mathrm{d}s = \int_0^t \frac{E(s)}{s^2}\,\mathrm{d}s.\end{aligned}$$

"(iii) $\Rightarrow$ (vii)". By the iterative use of (ii), we again obtain (6.2.14). Then

$$\begin{aligned}\int_t^\infty \frac{E^{-1}(s)}{s^2}\,\mathrm{d}s &= \sum_{k=0}^{\infty} \int_{t\sigma^k}^{t\sigma^{k+1}} \frac{E^{-1}(s)}{s^2}\,\mathrm{d}s \le \log\sigma \sum_{k=0}^{\infty} \frac{E^{-1}(t\sigma^k)}{t\sigma^k} \\ &\le \frac{E^{-1}(t)}{t} \log\sigma \sum_{k=0}^{\infty} \sigma^{-k}\Big(\frac{\sigma}{c}\Big)^k = \frac{E^{-1}(t)}{t} \log\sigma \frac{c\log\sigma}{c-1}\end{aligned}$$

for any $t \in (0,\infty)$, which proves (vii).

"(vii) $\Rightarrow$ (viii)". By the substitution $s \mapsto 1/s$ and $t \mapsto 1/t$, (vii) is equivalent to

$$\int_0^t E^{-1}\big(\tfrac{1}{s}\big)\,\mathrm{d}s \le CtE^{-1}\big(\tfrac{1}{t}\big) \quad \text{for } t \in (0,\infty). \tag{6.2.15}$$

Denote $F(t) = 1/E(t)$, $t \in (t_0,\infty)$. Then F is strictly monotone, $F'(t) = -a(t)/E^2(t)$ and $F^{-1}(s) = E^{-1}(1/s)$. Then, by (6.2.15),

$$\begin{aligned}CtE^{-1}\big(\tfrac{1}{t}\big) \ge \int_0^t F^{-1}(s)\,\mathrm{d}s &= \int_{F^{-1}(0)}^{F^{-1}(t)} F^{-1}(F(s))F'(s)\,\mathrm{d}s \\ &= \int_{E^{-1}(1/t)}^{\infty} \frac{sa(s)}{E^2(s)}\,\mathrm{d}s = \int_{E^{-1}(1/t)}^{\infty} \frac{\mathrm{d}s}{E(s)}\end{aligned}$$

for any $t \in (0,\infty)$ and (viii) follows by interchanging t and $1/E(t)$.

"(viii) $\Rightarrow$ (ii)". Assume by contradiction that to fixed $c > 1$ and any $\varrho > 1$ there is a $t > 0$ such that $E(\varrho t) < c\varrho E(t)$. Then, E is positive on (t,∞) and, by (viii),

$$\frac{Ct}{E(t)} \ge \int_t^\infty \frac{\mathrm{d}s}{E(s)} \ge \int_t^{\varrho t} \frac{\mathrm{d}s}{E(s)} \ge \frac{\varrho t}{E(\varrho t)} \log\varrho > \frac{t}{E(t)} \cdot \frac{\log\varrho}{c},$$

whence $cC > \log\varrho$ for any $\varrho > 1$. A contradiction.

Finally, "(ii) $\Leftrightarrow$ (iv)" follows easily by (2.1.6) and "(iv) $\Leftrightarrow$ (v)" is obvious. $\square$

Proof of Corollary 6.2.3. If $|\Omega| < \infty$, then Theorem 6.2.2 asserts that (6.2.3) is equivalent to (6.2.2)with $A = B$. This is however equivalent to the validity of Δ_2 for $\widetilde{A}$ near infinity, due to Proposition 6.2.5, provided that A is finite valued. If this is not the case, then $L^A(\Omega) = L^\infty(\Omega)$ and the claim holds trivially. The case $|\Omega| = \infty$ is similar. □

6.3 Optimal domain spaces

The objective of the present scope is to describe the optimal Orlicz domain space in (6.2.1). It seems that our result is not covered by literature, although the tool, used in the proof is well known. It relies upon the reverse integral inequality which is available in the work of H. Kita, see [46, 47, 49] for the details. We would like to also mention the paper [64], where the local boundedness to L^1 is characterised.

Let B be a given Young function. We define $A_B\colon [0,\infty) \to [0,\infty]$ by

$$A_B(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1, \\ \displaystyle\int_1^t \int_1^y \frac{B(s)}{s^2}\, \mathrm{d}s\, \mathrm{d}y & \text{if } t > 1. \end{cases} \tag{6.3.1}$$

Then, clearly A_B is a Young function, as it possesses the representation (2.1.1). It turns out that the optimal Orlicz domain problem is quite straightforward.

Theorem 6.3.1 [Optimal Orlicz domain, case $|\Omega| < \infty$]. *Assume that $\Omega \subseteq \mathbb{R}^n$ is open set such that $|\Omega| < \infty$. Let B be a Young function and let A_B be the Young function from* (6.3.1). *Then*

$$M\colon L^{A_B}(\Omega) \to L^B(\Omega) \tag{6.3.2}$$

and $L^{A_B}(\Omega)$ is the optimal Orlicz domain space in (6.3.2).

Concerning the global variant, we have to take the values near zero into account. Let again B be given and assume that

$$\int_0 \frac{B(s)}{s^2}\, \mathrm{d}s < \infty. \tag{6.3.3}$$

We define $A_B^\infty\colon [0,\infty) \to [0,\infty]$ by

$$A_B^\infty(t) = \int_0^t \int_0^y \frac{B(s)}{s^2}\, \mathrm{d}s\, \mathrm{d}y, \quad \text{for } t \ge 0. \tag{6.3.4}$$

Again, A_B^∞ is a Young function. The characterization of optimal Orlicz domain space now reads as follows. Let us stress that its existence is not guaranteed any more in general.

Theorem 6.3.2 [Optimal Orlicz domain, case $|\Omega| = \infty$]. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying $|\Omega| = \infty$. Let B be a Young function. If B satisfies* (6.3.3), *then*

$$M\colon L^{A_B^\infty}(\Omega) \to L^B(\Omega). \tag{6.3.5}$$

where A_B^∞ is the Young function given by (6.3.4). *Furthermore, the Orlicz space $L^{A_B^\infty}(\Omega)$ is the optimal Orlicz domain space in* (6.3.5).

Conversely, if (6.3.3) *fails, then there is no Orlicz domain space with respect to M.*

Proof of Theorem 6.3.2. Let B be a Young function obeying (6.3.3) and let A_B^∞ be as in (6.3.4). We have

$$t\int_0^t \frac{B(s)}{s^2}\,\mathrm{d}s \le \int_t^{2t}\int_0^y \frac{B(s)}{s^2}\,\mathrm{d}s\,\mathrm{d}y \le A_B^\infty(2t) \quad \text{for } t \ge 0$$

and hence (6.3.5) follows by Theorem 6.2.1.

To prove the optimality, assume that A is a Young function satisfying (6.2.1). Then, by Theorem 6.2.1 together with (2.1.2), we infer that

$$A_B^\infty(t) \le t\int_0^t \frac{B(s)}{s^2}\,\mathrm{d}s \le A(Ct) \quad \text{for } t > 0$$

for some $C > 0$ whence $L^A(\Omega) \to L^{A_B^\infty}(\Omega)$, proving the optimality. The necessity of the condition (6.3.3) also follows by Theorem 6.2.1. □

The proof of Theorem 6.3.1 is then very same and hence omitted.

Even though the problem of optimal Orlicz domain space was easy to solve, the space $L^{A_B}(\Omega)$ (or $L^{A_B^\infty}(\Omega)$) possesses more interesting property. It can be shown that it is also the optimal partner to $L^B(\Omega)$ in a much wider class of function spaces, namely in rearrangement-invariant ones.

Theorem 6.3.3 [Optimal r.i. domain, case $|\Omega| < \infty$]. *Let Ω, B and A_B be as in Theorem 6.3.1. Then* (6.3.2) *holds and the Orlicz space $L^{A_B}(\Omega)$ is the optimal rearrangement-invariant space in* (6.3.2).

Theorem 6.3.4 [Optimal r.i. domain, case $|\Omega| = \infty$]. *Let Ω, B and A_B^∞ be as in Theorem 6.3.2. If B obeys* (6.3.3) *then* (6.3.5) *holds and the Orlicz space $L^{A_B^\infty}(\Omega)$ is the optimal rearrangement-invariant space in* (6.3.5).

Conversely, if (6.3.3) *fails, no rearrangement-invariant space with respect to M exists.*

We restrict ourselves only to the case of infinite-measure set, the local version being similar.

Proof of Theorem 6.3.4. The boundedness of M between $L^{A_B^\infty}(\Omega)$ and $L^B(\Omega)$ follows by Theorem 6.3.2. Let us focus on the optimality. We claim that

$$\int_\Omega A_B^\infty(|f(x)|)\,\mathrm{d}x \le \int_\Omega B(c'Mf(x))\,\mathrm{d}x, \quad f \in \mathcal{M}(\Omega). \tag{6.3.6}$$

Indeed, thanks to the basic property of rearrangements and by the second inequality in (6.1.1), we have

$$\{f^{**} > s\} \subseteq \{c'(Mf)^* > s\} = \{c'Mf > s\} \quad \text{for } s > 0,$$

whence, by Fubini's theorem and due to the inequality (6.2.4) of Lemma (6.2.4),

$$\begin{aligned}\int_\Omega A_B^\infty(|f(x)|)\,\mathrm{d}x &= \int_0^\infty \int_0^y \frac{B(s)}{s^2}\,|\{f^* \geq y\}|\,\mathrm{d}s\,\mathrm{d}y\\ &= \int_0^\infty \frac{B(s)}{s^2} \int_s^\infty |\{f^* \geq y\}|\,\mathrm{d}y\,\mathrm{d}s\\ &= \int_0^\infty \frac{B(s)}{s^2} \int_{\{f^*\geq s\}} f^*(t)\,\mathrm{d}t\,\mathrm{d}s\\ &\leq \int_0^\infty \frac{B(s)}{s}\,|\{f^{**} \geq s\}|\,\mathrm{d}s\\ &\leq \int_0^\infty \frac{B(s)}{s}\,|\{c'Mf \geq s\}|\,\mathrm{d}s\\ &\leq \int_\Omega B(c'Mf(x))\,\mathrm{d}x.\end{aligned}$$

Applying (6.3.6) to the function $f/c'\|Mf\|_{L^B(\Omega)}$, we get

$$\int_\Omega A_B^\infty\left(\frac{|f(x)|}{c'\|Mf\|_{L^B(\Omega)}}\right)\mathrm{d}x \leq \int_\Omega B\left(\frac{Mf(x)}{\|Mf\|_{L^B(\Omega)}}\right)\mathrm{d}x \leq 1$$

and therefore

$$\|f\|_{L^{A_B^\infty}(\Omega)} \leq c'\|Mf\|_{L^B(\Omega)} \tag{6.3.7}$$

for every measurable f on Ω.

We are now about to use the characterization of the optimal rearrangement-invariant space available in [36, Theorem 3.2]. It proposes that the functional

$$\varrho(f) = \|g^{**}\|_{L^B(0,\infty)} \quad \text{for } g \in \mathcal{M}(\Omega). \tag{6.3.8}$$

is a well-defined rearrangement-invariant norm such that the corresponding space $X_\varrho(\Omega)$ obeys

$$M\colon X_\varrho(\Omega) \to L^B(\Omega) \tag{6.3.9}$$

provided that

$$\tfrac{1}{t} \in L^B(1,\infty). \tag{6.3.10}$$

Moreover, $X_\varrho(\Omega)$ is the optimal r.i. space to $L^B(\Omega)$ with respect to M. Conversely, if (6.3.10) fails, then no r.i. domain space exists in (6.3.9).

Observe that, by Lemma 3.2.1, the condition (6.3.10) is equivalent to the convergence condition (6.3.3). Next, by (6.3.10), the first inequality in (6.1.1) and by the definition (6.3.8), we have

$$c\|Mf\|_{L^B(\Omega)} \leq \|f^{**}\|_{L^B(0,\infty)} = \|f\|_{X_\varrho(\Omega)} \tag{6.3.11}$$

Coupling (6.3.7) with (6.3.11), we infer that

$$c\|f\|_{L^{A_B^\infty}(\Omega)} \leq cc'\|Mf\|_{L^B(\Omega)} \leq c'\|f\|_{X_\varrho(\Omega)}$$

for any measurable f on Ω. Thus $X_\varrho(\Omega) \to L^{A_B^\infty}(\Omega)$ and therefore $X_\varrho(\Omega) = L^{A_B^\infty}(\Omega)$, by the optimality of $X_\varrho(\Omega)$. $\square$

6.4 Optimal target spaces

Despite the fact that the problem of optimal Orlicz domain space for Maximal operator is easily solved, the question of optimal Orlicz target space remains open. In this section we would like to collect some ideas which might be helpful to answer the question of existence and possible construction of optimal Orlicz target space.

Throughout this section we assume that $\Omega = \mathbb{R}^n$, for simplicity. Since the local problem is similar (and sometimes even simpler), we believe that the reader will be able to adapt our observations to this case, if necessary.

Let us start with preliminary discussion. We have already observed in Section 6.2 that the Maximal operator is bounded between Orlicz spaces $L^A(\mathbb{R}^n)$ and $L^B(\mathbb{R}^n)$ if and only if

$$\int_0^t \frac{B(s)}{s^2}\,\mathrm{d}s \le \frac{A(Kt)}{t} \quad \text{for } t > 0 \tag{6.4.1}$$

for some positive constant K. To find the optimal Orlicz space means to define a Young functions which make the inequality (6.4.1) sharp. If A and B are any Young functions fulfilling (6.4.1), then B is dominated by A, as it can be easily observed by

$$A(2Kt) \ge 2t \int_t^{2t} \frac{B(s)}{s^2}\,\mathrm{d}s \ge B(t)2t \int_t^{2t} \frac{\mathrm{d}s}{s^2} = B(t) \quad \text{for } t \ge 0.$$

Therefore when such A and B are equivalent, the optimality is clearly attained. By Corollary 5.3.13 this happens if and only if $\widetilde{A}$ (and hence $\widetilde{B}$) enjoys the Δ_2 condition. Hence, in the sequel, we sort such cases out of our attention.

A possible explanation, why finding optimal A for given B is much simpler than the reciprocal case, is the fact that the function A in the relation (6.4.1) sits alone on the right hand side, while B is locked “inside” the integral on the left.

The experience with such type of inequality for Hardy-type operator from Chapter 3 may cause some concern, since we proved that there is no best possible Young function to fit in the integral unless the integration is meaningless. However, there is still a hope, as the following example suggests.

Example 6.4.1. Let $\mathbb{A} = [\alpha_0, \alpha_\infty]$ where $\alpha_0 < -1$ and $\alpha_\infty \ge 0$. Then

$$M\colon L(\log L)^{\mathbb{A}+1}(\mathbb{R}^n) \to L(\log L)^{\mathbb{A}}(\mathbb{R}^n)$$

and both domain and target spaces are optimal among all Orlicz spaces. Furthermore, both domain and target spaces are even the optimal spaces in the class of rearrangement-invariant spaces.

The proof of this example is given below. This example then strengthens the faith that the optimal Orlicz domains may exist in general. In the sequel we will mention several approaches which can be tried to arrive at the optimal Young function B provided A is given. Some of them are more naive than others, though we believe that any sort of positive thoughts might guide us to the eventual satisfying solution.

The following simple lemma is needed in the proof of Example 6.4.1.

Lemma 6.4.2. *Let E be a Young function. Then*

$$\sup_{0<t<\infty} \frac{f^{**}(t)}{E^{-1}\left(\frac{1}{t}\right)} \simeq \sup_{0<t<\infty} \frac{f^{*}(t)}{E^{-1}\left(\frac{1}{t}\right)} \tag{6.4.2}$$

for any $f \in \mathcal{M}(0,\infty)$ if and only if $\widetilde{E} \in \Delta_2$.

Proof. Let us assume (6.4.2) and set $f = E^{-1}(1/t)$. Then f is nonincreasing and we obtain that there is a constant $C > 0$ such that

$$\int_0^t E^{-1}\left(\tfrac{1}{s}\right) \mathrm{d}s \le CtE^{-1}\left(\tfrac{1}{t}\right) \quad \text{for } t > 0 \tag{6.4.3}$$

must be fulfilled. By the change of variables $s \mapsto 1/s$ and $t \mapsto 1/t$, (6.4.3) is equivalent to

$$\int_t^\infty \frac{E^{-1}(s)}{s^2} \mathrm{d}s \le C\frac{E^{-1}(t)}{t} \quad \text{for } t > 0$$

which is, by Proposition 6.2.5, equivalent to $\widetilde{E} \in \Delta_2$.

Conversely, assume that $\widetilde{E} \in \Delta_2$ and hence that (6.4.3) holds. If we denote the right hand side of (6.4.2) by K, we have

$$f^*(s) \le KE^{-1}\left(\tfrac{1}{s}\right), \quad \text{for } s > 0. \tag{6.4.4}$$

Integrating (6.4.4) over the interval $(0,t)$ we arrive at

$$f^{**}(t) \le \frac{K}{t}\int_0^t E^{-1}\left(\frac{1}{s}\right) \mathrm{d}s \le CKE^{-1}\left(\frac{1}{t}\right), \quad \text{for } t > 0,$$

whence

$$\sup_{0<t<\infty} \frac{f^{**}(t)}{E^{-1}\left(\frac{1}{t}\right)} \le CK.$$

That gives (6.4.2) since the opposite inequality is obvious. □

Proof of Example 6.4.1. The optimality of the domain follows from Theorem 6.3.2; The condition $\alpha_0 < -1$ comes from the constrain (6.3.3).

We shall use the characterization of the optimal r.i. target space with respect to M from [36]. It asserts that if

$$\log \tfrac{1}{t} \in L^{\widetilde{A}}(0,1), \tag{6.4.5}$$

then the functional

$$\sigma(g) = \left\| \int_t^\infty g^*(s) \frac{\mathrm{d}s}{s} \right\|_{L^{\widetilde{A}}(0,\infty)}$$

is an r.i. norm for which the corresponding space $Y'_\sigma(\mathbb{R}^n)$, associated to $Y_\sigma(\mathbb{R}^n)$, satisfies

$$M\colon L^A(\mathbb{R}^n) \to Y'_\sigma(\mathbb{R}^n) \tag{6.4.6}$$

and the space $Y'_\sigma(\mathbb{R}^n)$ is the optimal r.i. target space in (6.4.6). Moreover, if (6.4.5) fails, no r.i. target space exists in (6.4.6).

Let us apply this theorem to our space $L^A(\mathbb{R}^n)$. The condition (6.4.5) has to be satisfied, since $L^A(\mathbb{R}^n)$ came as an optimal r.i. domain for $L^B(\mathbb{R}^n)$. We compute σ now. Recall that A is a Young function equivalent to $t\ell(t)^{\mathbb{A}+1}$. The Orlicz space $L^A(\mathbb{R}^n)$, namely, the space $L(\log L)^{\mathbb{A}+1}(\mathbb{R}^n)$, then coincides with the Lorentz space $\Lambda^A(\mathbb{R}^n)$. Thus, by (2.4.7), $L^{\widetilde{A}}(\mathbb{R}^n) = M^{\widetilde{A}}(\mathbb{R}^n)$ and σ obeys

$$\sigma(g) = \sup_{0<t<\infty} \frac{1}{\widetilde{A}^{-1}(\frac{1}{t})} \int_0^t \int_y^\infty g^*(s) \frac{\mathrm{d}s}{s}\, \mathrm{d}y.$$

Since A satisfies the Δ_2, we have, by Lemma 6.4.2, that

$$\sigma(g) \simeq \sup_{0<t<\infty} tA^{-1}(\tfrac{1}{t}) \int_t^\infty g^*(s) \frac{\mathrm{d}s}{s},$$

where we used (2.1.5). Calculating the inverse, we have $A^{-1}(s) \simeq s\ell(s)^{-\mathbb{A}-1}$. Denote also $\overline{\mathbb{A}} = [\alpha_\infty, \alpha_0]$. Then

$$\begin{aligned}
\sigma(g) &\simeq \sup_{0<t<\infty} \ell^{-\overline{\mathbb{A}}-1}(t) \int_t^\infty g^*(s) \frac{\mathrm{d}s}{s} \\
&= \sup_{0<t<\infty} \ell^{-\overline{\mathbb{A}}-1}(t) \int_t^\infty g^*(s) \ell^{-\overline{\mathbb{A}}}(s) \ell^{\overline{\mathbb{A}}}(s) \frac{\mathrm{d}s}{s} \\
&\leq \Big(\sup_{0<s<\infty} g^*(s)\ell^{-\overline{\mathbb{A}}}(s) \Big) \Big(\sup_{0<t<\infty} \ell^{-\overline{\mathbb{A}}-1}(t) \int_t^\infty \ell^{\overline{\mathbb{A}}}(s) \frac{\mathrm{d}s}{s} \Big) \\
&\simeq \sup_{0<s<\infty} g^*(s)\ell^{-\overline{\mathbb{A}}}(s) \\
&\leq \sup_{0<s<\infty} g^{**}(s)\ell^{-\overline{\mathbb{A}}}(s)
\end{aligned}$$

whence $M^{\widetilde{B}}(\mathbb{R}^n) \to Y_\sigma(\mathbb{R}^n)$, where $B(t)$ is equivalent to $t\ell(t)^{\mathbb{A}}$. On the other side,

$$\begin{aligned}
\sigma(g) &\geq \max\Big\{ \sup_{0<t<1} \ell^{-\alpha_\infty}(t) \int_t^{\sqrt{t}} g^*(s) \frac{\mathrm{d}s}{s}, \sup_{1<t<\infty} \ell^{-\alpha_0}(t) \int_t^{t^2} g^*(s) \frac{\mathrm{d}s}{s} \Big\} \\
&\geq \max\Big\{ \sup_{0<t<1} \ell^{-\alpha_\infty}(t) g^*(\sqrt{t}) \log(t^{-\frac{1}{2}}), \sup_{1<t<\infty} \ell^{-\alpha_0}(t) g^*(t^2) \log t \Big\} \\
&\simeq \max\Big\{ \sup_{0<t<1} \ell^{1-\alpha_\infty}(\sqrt{t}) g^*(\sqrt{t}), \sup_{1<t<\infty} \ell^{1-\alpha_0}(t^2) g^*(t^2) \Big\} \\
&\simeq \max\Big\{ \sup_{0<t<1} \ell^{1-\alpha_\infty}(t) g^*(t), \sup_{1<t<\infty} \ell^{1-\alpha_0}(t) g^*(t) \Big\} \\
&\simeq \sup_{0<s<\infty} g^*(s)\ell^{-\overline{\mathbb{A}}}(s) \\
&\simeq \sup_{0<s<\infty} g^{**}(s)\ell^{-\overline{\mathbb{A}}}(s).
\end{aligned}$$

The last equivalence is due to Lemma 6.4.2 applied to $\widetilde{B}$. Thus $Y_\sigma(\mathbb{R}^n) \to M^{\widetilde{B}}(\mathbb{R}^n)$, whence, by (2.4.6), the space $\Lambda^B(\mathbb{R}^n)$ is the optimal r.i. target space for $L^B(\mathbb{R}^n)$. Since $\Lambda^B(\mathbb{R}^n)$ coincides with $L^B(\mathbb{R}^n)$, it is also optimal within Orlicz spaces. □

Embedding to weak Orlicz space

First idea we would like to demonstrate relies upon the version of reduction principle, where the target space $L^B(\mathbb{R}^n)$ is replaced by the weak Orlicz space

$M^B(\mathbb{R}^n)$. There is one advantage and one disadvantage of this approach. Let us look at the bright side first. Such principle gives us a necessary and sufficient condition on A and B in which B is standing outside any integrals or any norms. From this, we may guess the optimal form of B. The next step in this case, would be to prove the statement in the following form.

Let A and B be Young functions, such that

$$M\colon L^A(\mathbb{R}^n) \to M^B(\mathbb{R}^n) \tag{6.4.7}$$

then

$$M\colon L^A(\mathbb{R}^n) \to L^B(\mathbb{R}^n) \tag{6.4.8}$$

holds as well.

Such statement seems reasonable and we have met it before in similar form for Hardy-type operator – see Proposition 3.5.6. However, getting to the disadvantages, the proof cannot be reproduced and it is not clear whether such a claim holds. Nevertheless, let us present the reduction principle for weak Orlicz spaces.

Proposition 6.4.3. *Let A and B be Young functions. The following statements are equivalent.*

(i) *There exists a constant $C_1 > 0$ such that*

$$\|Mf\|_{M^B(\mathbb{R}^n)} \le C_1 \|f\|_{L^A(\mathbb{R}^n)}$$

for every $f \in L^A(\mathbb{R}^n)$;

(ii) *There exists a constant $C_2 > 0$ such that*

$$t \left\| \log \frac{1}{st} \right\|_{L^{\widetilde{A}}(0,1/t)} \le C_2 B^{-1}(t) \quad \textit{for } t > 0.$$

Moreover the all the constants C_1 and C_2 are comparable.

We begin with more general preliminary reduction in r.i. spaces.

Lemma 6.4.4. *Let X and Y be rearrangement-invariant spaces with their associate spaces X' and Y', respectively. It is equivalent:*

(i) *There exists a constant $C_1 > 0$ such that*

$$\|Mf\|_{Y(\mathbb{R}^n)} \le C_1 \|f\|_{X(\mathbb{R}^n)}$$

for every $f \in X(\mathbb{R}^n)$;

(ii) *There exists a constant $C_2 > 0$ such that*

$$\left\| \frac{1}{t} \int_0^t f(s)\, \mathrm{d}s \right\|_{Y(0,\infty)} \le C_2 \|f\|_{X(0,\infty)}.$$

for every $f \in X(0,\infty)$.

(iii) *There exists a constant $C_3 > 0$ such that*

$$\left\| \int_t^\infty g(s) \frac{\mathrm{d}s}{s} \right\|_{X'(0,\infty)} \le C_3 \|g\|_{Y'(0,\infty)}$$

(iii)* *There exists a constant* $C_3^* > 0$ *such that*

$$\Big\| \int_t^\infty g^*(s) \frac{\mathrm{d}s}{s} \Big\|_{X'(0,\infty)} \leq C_3^* \|g^*\|_{Y'(0,\infty)}$$

Moreover the all the constants C_1, C_2, C_3 *and* C_3^* *are comparable.*

Proof. Let us start with the observation that (i) is equivalent to

$$\|(Mf)^*\|_{Y(0,\infty)} \leq C_1 \|f^*\|_{X(0,\infty)}$$

for every $f \in X(0,\infty)$ due to (2.6.2). Then, by the inequalities (6.1.1), we infer that (i) is equivalent to the existence of a constant $C_2 > 0$ such that

$$\Big\| \frac{1}{t} \int_0^t f^*(s)\,\mathrm{d}s \Big\|_{Y(0,\infty)} \leq C_2 \|f^*\|_{X(0,\infty)}. \tag{6.4.9}$$

for every $f \in X(0,\infty)$. We claim that (6.4.9) is equivalent to (ii). Indeed, one implication follows just by restriction to nonincreasing functions and the converse one is due to Hardy-Littlewood inequality (2.4.1). Next, by duality (2.8.2), (ii) is equivalent to (iii) for some $C_3 > 0$ and for every $g \in Y'(0,\infty)$. The final equivalence of (iii) and (iii)* is a consequence of [32, Corollary 9.8] of a more general principle established in [32, Theorem 9.5] in connection with sharp higher-order Sobolev-type embeddings and whose extension to unbounded intervals was given in [60, Theorem 1.10]. □

Proof of Proposition 6.4.3. Let us apply Lemma 6.4.4 to $X = L^A$ and $Y = M^B$. By the equivalence of (i) and (iii)*, we have

$$\Big\| \int_s^\infty g^*(y) \frac{\mathrm{d}y}{y} \Big\|_{L^{\widetilde{A}}(0,\infty)} \leq C_3^* \|g^*\|_{\Lambda^{\widetilde{B}}(0,\infty)} \tag{6.4.10}$$

for all $g \in \Lambda^{\widetilde{B}}(0,\infty)$ and some $C_3^* > 0$. We also used (2.4.6) here. Now, by Proposition 2.8.1, (6.4.10) is equivalent to the same inequality restricted to functions of the form χ_E, in which E stands for any measurable subset of $(0,\infty)$. Thus, using the rearrangement, (6.4.10) rewrites as

$$\Big\| \int_s^\infty \chi_{(0,t)}(y) \frac{\mathrm{d}y}{y} \Big\|_{L^{\widetilde{A}}(0,\infty)} \leq C_3^* \|\chi_{(0,t)}\|_{\Lambda^{\widetilde{B}}(0,\infty)} \tag{6.4.11}$$

for $t > 0$. Computing the left hand side of (6.4.11), we have

$$\Big\| \int_s^\infty \chi_{(0,t)}(y) \frac{\mathrm{d}y}{y} \Big\|_{L^{\widetilde{A}}(0,\infty)} = \Big\| \int_s^t \frac{\mathrm{d}y}{y} \Big\|_{L^{\widetilde{A}}(0,t)} = \Big\| \log \frac{t}{s} \Big\|_{L^{\widetilde{A}}(0,t)} \tag{6.4.12}$$

and, using (2.4.5) and (2.1.5), the right hand side of (6.4.11) gives

$$\|\chi_{(0,t)}\|_{\Lambda^{\widetilde{B}}(0,\infty)} = \frac{1}{\widetilde{B}^{-1}(\frac{1}{t})} \simeq tB^{-1}(\tfrac{1}{t}). \tag{6.4.13}$$

Finally plugging (6.4.12) with (6.4.13) into (6.4.11), (ii) follows after the change of variables $t \mapsto 1/t$. □

Now, with the help of Proposition 6.4.3, we may proceed to the definition of optimal Young function B. So, let A be a given Young function such that

$$\log \frac{1}{t} \in L^{\widetilde{A}}(0,1). \tag{6.4.14}$$

We define $B_A^\infty \colon [0,\infty) \to [0,\infty]$ by

$$B_A^\infty(t) = \int_0^t \frac{G_A^{\infty\,-1}(s)}{s}\,\mathrm{d}s \quad \text{for } t \geq 0, \tag{6.4.15}$$

where $G_A^\infty \colon (0,\infty) \to [0,\infty)$ is the generalised right-continuous inverse of

$$G_A^\infty(t) = t\left\|\log\frac{1}{st}\right\|_{L^{\widetilde{A}}(0,1/t)} \qquad \text{for } t > 0.$$

Observe that B_A^∞ is a Young function. By (6.4.14), G_A^∞ is finite-valued. Next, $G_A^\infty(t)/t$ is nonincreasing, as might be seen from the alternate formula

$$\begin{aligned} \frac{G_A^\infty(t)}{t} &= \left\|\log\frac{1}{st}\right\|_{L^{\widetilde{A}}(0,1/t)} \\ &= \inf\left\{\lambda > 0 : \int_0^{1/t} \widetilde{A}\Big(\frac{1}{\lambda}\log\frac{1}{st}\Big)\,\mathrm{d}s \leq 1\right\} \\ &= \inf\left\{\lambda > 0 : \int_0^{1} \widetilde{A}\Big(\frac{1}{\lambda}\log\frac{1}{s}\Big)\,\mathrm{d}s \leq t\right\}. \end{aligned} \tag{6.4.16}$$

Using this, we infer that $G_A^{\infty\,-1}(t)/t$ is nondecreasing and hence B_A^∞ is a Young function.

Let us also mention, that unlike in the case of Hardy operator in Chapter 3, it is not possible to easily simplify the expression for the function G_A^∞ (and hence in the corresponding reduction principle). The reason is that we cannot make similar change of variables in the integral in (6.4.16) to get the λ outside.

The following characterization of optimal weak Orlicz target space is a direct consequence of Proposition 6.4.3.

Proposition 6.4.5. *Let A be a Young function satisfying* (6.4.14) *and let B_A^∞ be as in* (6.4.15). *Then*

$$M \colon L^A(\mathbb{R}^n) \to M^{B_A^\infty}(\mathbb{R}^n) \tag{6.4.17}$$

and the space $M^{B_A^\infty}(\mathbb{R}^n)$ is the optimal target space in (6.4.17) *within all Marcinkiewicz spaces.*

Conversely, if (6.4.14) *fails, then no Marcinkiewicz target space exists.*

Here, the inequality for Young functions is also available, in its week form.

Proposition 6.4.6. *Let A be a Young function satisfying* (6.4.14) *and let B_A^∞ be defined by* (6.4.15). *Then there is a constant $C > 0$, such that*

$$\sup_{t\geq 0} B_A^\infty(t)\left|\left\{(Mf)^{**} \geq Ct\right\}\right| \leq \int_{\mathbb{R}^n} A(|f(x)|)\,\mathrm{d}x \quad \textit{for } t \in L^A(\mathbb{R}^n). \tag{6.4.18}$$

Proof. Let A be a Young function satisfying (6.4.14). We will use the scaling argument. For a positive constant N, define the Young function $A_N = A/N$. Then the Young function B_N^∞ connected with A_N by the relation (6.4.15) satisfies

$$B_N^\infty = \frac{B_A^\infty}{N},$$

as one may observe by the change of variables in the definition of the Luxemburg norm. We therefore infer, by Proposition 6.4.3, that

$$\|Mf\|_{M^{B_N^\infty}(\mathbb{R}^n)} \leq C\|f\|_{L^{A_N}(\mathbb{R}^n)} \tag{6.4.19}$$

holds with an universal constant $C > 0$ independent of N. On setting

$$N = \int_{\mathbb{R}^n} A(|f(x)|)\,\mathrm{d}x$$

we infer from (6.4.19) that

$$\|Mf\|_{M^{B_N^\infty}(\mathbb{R}^n)} \leq C\|f\|_{L^{A_N}(\mathbb{R}^n)} \leq C, \tag{6.4.20}$$

provided that N is finite an positive. Here, C is independent of N and f. Equation (6.4.20) tells us that

$$C \geq \|Mf\|_{M^{B_N^\infty}(\mathbb{R}^n)} = \sup_{0<t<\infty} \frac{t}{B_N^{\infty-1}\left(\frac{1}{|\{(Mf)^{**}>t\}|}\right)} \quad \text{for } t > 0,$$

which rewrites as

$$B_N^\infty\left(\tfrac{t}{C}\right)|\{(Mf)^{**} > t\}| \leq 1 \quad \text{for } t > 0$$

and (6.4.18) follows by the choice of N. □

Lifting to an Orlicz space

Let us follow up the idea presented at the beginning of this section. The natural continuation in obtaining the optimal Orlicz target would be a lifting argument which claims that (6.4.7) and (6.4.8) are equivalent. However, the discretization argument, used in Proposition 3.5.6, does not work here.

On the other hand, it is possible to prove a variant of such lifting statement, where we play on the domain side.

Lemma 6.4.7. *Let A and B be Young functions. Then the following are equivalent.*

(i) $M\colon L^A(\mathbb{R}^n) \to L^B(\mathbb{R}^n)$;

(ii) $M\colon \Lambda^A(\mathbb{R}^n) \to L^B(\mathbb{R}^n)$.

Proof. The implication (i)→(ii) is trivial since, by (2.4.4), $\Lambda^A(\mathbb{R}^n) \to L^A(\mathbb{R}^n)$ for every A. Conversely, assume (ii). By Lemma 6.4.4, it is equivalent to

$$\left\|\frac{1}{t}\int_0^t f(s)\,\mathrm{d}s\right\|_{L^B(0,\infty)} \leq C\|f\|_{\Lambda^A(0,\infty)}. \tag{6.4.21}$$

for some $C > 0$ and every $f \in L^A(0,\infty)$. On testing the inequality (6.4.21) by the functions of the form $f = \chi_{(0,t)}$ we recover the inequality

$$\int_0^t \frac{B(s)}{s^2}\,\mathrm{d}s \le \frac{A(C't)}{t} \quad \text{for } t > 0$$

which implies (i), thanks to Theorem 6.2.1. □

If we believe that $L^{B_A^\infty}(\mathbb{R}^n)$ is our optimal Orlicz target, it is thus sufficient to verify that

$$M\colon \Lambda^A(\mathbb{R}^n) \to L^{B_A^\infty}(\mathbb{R}^n),$$

thanks to Lemma 6.4.7.

Endpoint space embeddings

At the end of this chapter, we would like to collect few available reduction principles on M acting between various types of endpoint spaces. The proofs are easy and in all cases depend on Proposition 2.8.1, passing to the characteristic functions of intervals and computation of corresponding quantities involved. The details are skipped.

Lemma 6.4.8. *Let A and B be Young functions. Then the following statements are equivalent.*

(i) *There exists a constant $C_1 > 0$ such that*

$$\|Mf\|_{M^B(\mathbb{R}^n)} \le C_1 \|f\|_{\Lambda^A(\mathbb{R}^n)}$$

for every $f \in \Lambda^A(\mathbb{R}^n)$;

(ii) *There exists a constant $C_2 > 0$ such that*

$$t \sup_{t \le s < \infty} \frac{A^{-1}(s)}{s}\left(\log\frac{s}{t} + 1\right) \le C_2 B^{-1}(t) \quad \textit{for } t > 0.$$

In addition, the constants C_1 and C_2 are comparable.

Lemma 6.4.9. *Let A and B be Young functions. Then the following statements are equivalent.*

(i) *There exists a constant $C_1 > 0$ such that*

$$\|Mf\|_{\Lambda^B(\mathbb{R}^n)} \le C_1 \|f\|_{\Lambda^A(\mathbb{R}^n)}$$

for every $f \in \Lambda^A(\mathbb{R}^n)$;

(ii) *There exists a constant $C_2 > 0$ such that*

$$\int_0^t \frac{\mathrm{d}s}{B^{-1}(s)} \le C_2 \frac{t}{A^{-1}(t)} \quad \textit{for } t > 0;$$

(iii) *There exists a constant $C_3 > 0$ such that*

$$\int_0^{B^{-1}(A(t))} \frac{B(s)}{s^2}\,\mathrm{d}s \le \frac{A(C_3 t)}{t} \quad \textit{for } t > 0.$$

Moreover, the all the constants C_1, C_2 and C_3 are comparable.

Lemma 6.4.10. *Let A and B be Young functions. Then the following statements are equivalent.*

(i) *There exists a constant $C_1 > 0$ such that*

$$\|Mf\|_{M^B(\mathbb{R}^n)} \leq C_1 \|f\|_{M^A(\mathbb{R}^n)}$$

for every $f \in M^A(\mathbb{R}^n)$;

(ii) *There exists a constant $C_2 > 0$ such that*

$$t \int_0^t A^{-1}\left(\frac{1}{s}\right) \log \frac{t}{s} \, \mathrm{d}s \leq C_2 B^{-1}(t) \quad \text{for } t > 0.$$

Furthermore, C_1 and C_2 are comparable.

7. Riesz potential

7.1 Introduction

Let $n \in \mathbb{N}$. Consider the Riesz potential operator I_γ of order $\gamma \in (0, n)$ given by

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}} \mathrm{d}y \quad \text{for } x \in \mathbb{R}^n$$

and for any measurable function f on $\mathbb{R}^n$.

We are going to make use of a special case of the O'Neil inequality. In its general form ([58, Lemma 1.5]), it states that, for the convolution of two measurable functions f, g on $\mathbb{R}^n$, defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\,\mathrm{d}y \quad \text{for } x \in \mathbb{R}^n,$$

we have

$$(f * g)^{**}(t) \le t f^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s \quad \text{for } t > 0.$$

With the particular choice

$$g(x) = |x|^{\gamma - n}, \ x \in \mathbb{R}^n,$$

we obtain that

$$(I_\gamma f)^*(t) \le \int_t^\infty s^{\frac{\gamma}{n}-1} f^{**}(s)\,\mathrm{d}s \quad \text{for } t > 0. \tag{7.1.1}$$

This inequality is known to be sharp, but merely in a broader sense than, for example, the corresponding estimate for the Hardy–Littlewood maximal operator. This was firstly observed by O'Neil in the final remark of the paper [58], where it is pointed out that the inequality can be reversed when f, g are radially decreasing positive functions. Furthermore, by an appropriately modified argument from [38, Theorem 10.2(iii)]), we get that, for every $f \in \mathcal{M}(\mathbb{R}^n)$ there exists a function $g \in \mathcal{M}(0,\infty)$ equimeasurable with f such that

$$(I_\gamma g)^*(t) \ge c \int_t^\infty s^{\frac{\gamma}{n}-1} f^{**}(s)\,\mathrm{d}s \quad \text{for } t > 0, \tag{7.1.2}$$

with some constant c, $0 < c < \infty$, depending on γ and n, but independent of f and t.

It is no surprise that we are interested in sharp results in

$$I_\gamma \colon L^A(\mathbb{R}^n) \to L^B(\mathbb{R}^n). \tag{7.1.3}$$

In the recent work [36] this task is solved in much broader class of rearrangements-invariant spaces. However, as we are now familiar with, such results do not give us the answers to our questions in Orlicz spaces. Concerning this, some work has been done already; let us mention the paper [20], in which a variant of necessary and sufficient condition on (7.1.3) is given. In this chapter, we present a modified version of such result which relies on our new knowledge from Chapter 3. We then turn our attention to obtaining the optimal Orlicz spaces in (7.1.3).

7.2 Reduction principle

Let A be a Young function satisfying

$$\inf_{0<t<1} A(t)\, t^{-\frac{n}{\gamma}} > 0. \tag{7.2.1}$$

We define A_D by

$$A_D(t) = \int_0^t \frac{G_D^{-1}(s)}{s} \mathrm{d}s \quad \text{for } t \geq 0, \tag{7.2.2}$$

where $G_D \colon [0,\infty) \to [0,\infty)$ is a non-decreasing function defined by

$$G_D(t) = \sup_{0<s\leq t} A^{-1}(s)\, s^{-\frac{\gamma}{n}} \quad \text{for } t \geq 0,$$

and G_D^{-1} represents its generalized right-continuous inverse. By a similar argument as in Remark 5.2.1, we have that A_D is a Young function. Moreover

$$A_D^{-1}(t) \simeq G_D(t) \quad \text{for } t \geq 0.$$

Let further A obey

$$\int_0 \left(\frac{s}{A(s)}\right)^{\frac{\gamma}{n-\gamma}} \mathrm{d}s < \infty. \tag{7.2.3}$$

We define $H_R^\infty \colon [0,\infty) \to [0,\infty)$ by

$$H_R^\infty(t) = \left(\int_0^t \left(\frac{s}{A(s)}\right)^{\frac{\gamma}{n-\gamma}} \mathrm{d}s\right)^{1-\frac{\gamma}{n}} \quad \text{for } t > 0$$

and set $H^\infty = \lim_{t\to\infty} H_R^\infty(t)$. Let A_R be then defined by

$$A_R(t) = \int_0^t \frac{D_R^\infty(s)}{s} \mathrm{d}s \quad \text{for } t > 0, \tag{7.2.4}$$

in which D_n^∞ is given by

$$D_R(s) = \begin{cases} \left(s\, \dfrac{A(H_R^{-1}(s))}{H_R^{-1}(s)}\right)^{\frac{n}{n-\gamma}}, & 0 \leq t < H^\infty, \\ \infty, & t \geq H^\infty. \end{cases}$$

Recall that A_R is a Young function as might be observed by Remark 3.3.1.

Let also B be given and satisfying

$$\sup_{0<t<1} B(t)\, t^{-\frac{n}{n-\gamma}} < \infty. \tag{7.2.5}$$

We define

$$B_D(t) = \int_0^t \frac{J_D^{-1}(s)}{s} \mathrm{d}s \quad \text{for } t \geq 0, \tag{7.2.6}$$

where $J_D \colon [0,\infty) \to [0,\infty)$ is given by

$$J_D(t) = t \inf_{0<s\leq t} B^{-1}(s) s^{\frac{\gamma}{n}-1} \quad \text{for } t > 0.$$

By an argument similar to that of Remark 3.5.1, we infer that B_D is a Young function, and

$$B_D^{-1}(t) \simeq J_D(t) \quad \text{for } t > 0. \tag{7.2.7}$$

Let also B obey the condition

$$\int_0 \frac{B(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s < \infty. \tag{7.2.8}$$

We thus define

$$B_R(t) = \int_0^t \frac{E_R^{-1}(s)}{s}\,\mathrm{d}s \quad \text{for } t \geq 0, \tag{7.2.9}$$

in which $E_R\colon [0,\infty) \to [0,\infty)$ is defined by

$$E_R(t) = t^{\frac{\gamma}{n}} F_R^{-1}(t) \quad \text{for } t \geq 0,$$

where $F_R\colon [0,\infty) \to [0,\infty)$ is given by

$$F_R(t) = t^{n/(n-\gamma)} \int_0^t \frac{B(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s \quad \text{for } t \geq 0,$$

and where F_R^{-1} stands for the generalised left-continuous inverse of F_R. Similarly as in Remark 5.2.2, B_R is a Young function and

$$B_R^{-1}(t) \simeq t^{\frac{\gamma}{n}} F_R^{-1}(t) \quad \text{for } t \geq 0. \tag{7.2.10}$$

The reduction principle now reads as follows.

Theorem 7.2.1. *Let A and B be Young functions. The following assertions are equivalent.*

(i) *There is a constant $C_1 > 0$ such that*

$$\|I_\gamma f\|_{L^B(\mathbb{R}^n)} \leq C_1 \|f\|_{L^A(\mathbb{R}^n)}$$

for all $f \in L^A(\mathbb{R}^n)$;

(ii) *A satisfies* (7.2.1) *and* (7.2.3) *and there is a constant C_2 such that*

$$\int_0^t \frac{B(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s \leq \frac{A_D(C_2 t)}{t^{n/(n-\gamma)}} \quad \textit{for } t \geq 0 \tag{7.2.11}$$

and

$$B(t) \leq A_R(C_2 t) \quad \textit{for } t \geq 0, \tag{7.2.12}$$

where A_D an A_R are the Young functions given by (7.2.2) *and* (7.2.4)*, respectively;*

(iii) *B satisfies* (7.2.5) *and* (7.2.8) *and there is a constant C_3 such that*

$$\int_0^t \frac{\widetilde{A}(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s \leq \frac{\widetilde{B_D}(C_3 t)}{t^{n/(n-\gamma)}} \quad \textit{for } t \geq 0 \tag{7.2.13}$$

and

$$B_R(t) \leq A(C_3 t) \quad \textit{for } t \geq 0, \tag{7.2.14}$$

where B_D an B_R are the Young functions given by (7.2.6) *and* (7.2.9)*, respectively.*

Furthermore, the constants C_1, C_2 and C_3 depend on each other and on n and γ.

7.3 Optimal Orlicz spaces

Proposition 7.3.1 [Optimal Orlicz domain space – sufficiency]. *Let B be a Young function satisfying* (7.2.8) *and assume that*

$$I_{B_D} < \frac{n}{\gamma}. \tag{7.3.1}$$

Then

$$I_\gamma \colon L^{B_R}(\mathbb{R}^n) \to L^B(\mathbb{R}^n) \tag{7.3.2}$$

and $L^{B_R}(\mathbb{R}^n)$ is the optimal Orlicz domain space in (7.3.2).

In particular, if $i_B > \frac{n}{n-\gamma}$, then (7.3.1) *is equivalent to $I_B < \infty$ and*

$$B_R^{-1}(t) \simeq B^{-1}(t)\, t^{\frac{\gamma}{n}} \quad \text{for } t \geq 0.$$

Proposition 7.3.2 [Optimal Orlicz domain space – necessity]. *Let B be a Young function satisfying* (7.2.8). *If*

$$\frac{n}{n-\gamma} < i_B \leq I_B = \infty, \tag{7.3.3}$$

then there is no optimal Orlicz domain space to B with respect to I_γ in a sense that to every $L^A(\mathbb{R}^n)$ for which (7.1.3) *holds there exists an essentially larger Orlicz space $L^{\widehat{A}}(\mathbb{R}^n)$ still satisfying* (7.1.3) *with $\widehat{A}$ in place of A.*

Furthermore, if (7.2.8) *is not satisfied then no Orlicz space $L^A(\mathbb{R}^n)$ for which* (7.1.3) *holds exists.*

Proposition 7.3.3 [Optimal Orlicz target space – sufficiency]. *Let A be a Young function satisfying* (7.2.3) *and assume that*

$$i_{A_D} > \frac{n}{n-\gamma}. \tag{7.3.4}$$

Then

$$I_\gamma \colon L^A(\mathbb{R}^n) \to L^{A_R}(\mathbb{R}^n) \tag{7.3.5}$$

and $L^{A_R}(\mathbb{R}^n)$ is the optimal Orlicz domain space in (7.3.5).

In particular, if $I_A < \frac{n}{\gamma}$, then (7.3.4) *is equivalent to $i_A > 1$ and*

$$A_R^{-1}(t) \simeq A^{-1}(t)\, t^{-\frac{\gamma}{n}} \quad \text{for } t \geq 0.$$

Proposition 7.3.4 [Optimal Orlicz target space – necessity]. *Let A be a Young function satisfying* (7.2.3). *If*

$$1 = i_A \leq I_A < \frac{n}{\gamma},$$

then there is no optimal Orlicz domain space to A with respect to I_γ in a sense that to every $L^B(\mathbb{R}^n)$ for which (7.1.3) *holds there exists an essentially smaller Orlicz space $L^{\widehat{B}}(\mathbb{R}^n)$ still satisfying* (7.1.3) *with $\widehat{B}$ in place of B.*

Furthermore, if (7.2.3) *is not satisfied then no Orlicz space $L^B(\mathbb{R}^n)$ for which* (7.1.3) *holds exists.*

7.4 Proofs

We start with an auxiliary reduction which simplifies (7.1.3) to the boundedness of one-dimensional operator, namely the operator H_γ given by

$$H_\gamma f(t) = \int_t^\infty f(s) s^{\frac{\gamma}{n}-1} \, \mathrm{d}s \quad \text{for } t \geq 0. \tag{7.4.1}$$

Proposition 7.4.1. *Let A and B be Young functions. Then*

$$I_\gamma \colon L^A(\mathbb{R}^n) \to L^B(\mathbb{R}^n) \tag{7.4.2}$$

if and only if

$$H_\gamma \colon L^A(0,\infty) \to L^B(0,\infty) \tag{7.4.3}$$

and

$$H_\gamma \colon L^{\widetilde{B}}(0,\infty) \to L^{\widetilde{A}}(0,\infty) \tag{7.4.4}$$

hold simultaneously. Moreover the norm of the embedding (7.4.2) *is comparable to those of* (7.4.3) *and* (7.4.4)

Proof. Assume (7.4.2), i.e., there is a constant $C > 0$, such that

$$\|I_\gamma g\|_{L^B(\mathbb{R}^n)} \leq C \|g\|_{L^A(\mathbb{R}^n)}$$

for every $g \in L^A(\mathbb{R}^n)$. Using the rearrangements, we infer that

$$\|(I_\gamma g)^*\|_{L^B(0,\infty)} \leq C \|g^*\|_{L^A(0,\infty)} \tag{7.4.5}$$

owing to (2.6.2). Let now $f \in L^A(0,\infty)$ be given. By (7.1.2), there is a function g equimeasurable with f such that

$$(I_\gamma g)^*(t) \geq c \int_t^\infty s^{\frac{\gamma}{n}-1} f^{**}(s) \, \mathrm{d}s \quad \text{for } t > 0,$$

whence, (7.4.5) implies that

$$\left\| \int_t^\infty s^{\frac{\gamma}{n}-1} f^{**}(s) \, \mathrm{d}s \right\|_{L^B(0,\infty)} \leq C \|f\|_{L^A(0,\infty)} \tag{7.4.6}$$

for every $f \in L^A(0,\infty)$ and possibly different $C > 0$, depending only on n and γ. By Fubini's theorem, we obtain

$$\int_t^\infty s^{\frac{\gamma}{n}-1} f^{**}(s) \, \mathrm{d}s \simeq t^{\frac{n}{\gamma}-1} \int_0^t f^*(s) \, \mathrm{d}s + \int_t^\infty f^*(s) s^{\frac{\gamma}{n}-1} \, \mathrm{d}s \quad \text{for } t > 0 \tag{7.4.7}$$

and (7.4.6) gives that

$$\left\| t^{\frac{n}{\gamma}-1} \int_0^t f^*(s) \, \mathrm{d}s \right\|_{L^B(0,\infty)} \leq C \|f\|_{L^A(0,\infty)} \tag{7.4.8}$$

and

$$\left\| \int_t^\infty s^{\frac{\gamma}{n}-1} f^*(s) \, \mathrm{d}s \right\|_{L^B(0,\infty)} \leq C \|f\|_{L^A(0,\infty)}. \tag{7.4.9}$$

Now, (7.4.8) is implies

$$\left\| t^{\frac{n}{\gamma}-1} \int_0^t f(s)\, \mathrm{d}s \right\|_{L^B(0,\infty)} \le C \|f\|_{L^A(0,\infty)}.$$

as one infers from (2.4.1), which is, by duality, equivalent to (7.4.4). Also (7.4.9) is equivalent to the same inequality for every function f instead of f^*, as follows by [60, Theorem 1.10] (See also [32, Corollary 9.8]). This gives (7.4.3).

Conversely, assume that (7.4.3) and (7.4.4) hold, i.e., there are constants C_1 and C_2 such that

$$\left\| \int_t^\infty s^{\frac{\gamma}{n}-1} f(s)\, \mathrm{d}s \right\|_{L^B(0,\infty)} \le C_1 \|f\|_{L^A(0,\infty)} \tag{7.4.10}$$

for every $f \in L^A(0,\infty)$ and

$$\left\| \int_t^\infty s^{\frac{\gamma}{n}-1} g(s)\, \mathrm{d}s \right\|_{L^{\widetilde{A}}(0,\infty)} \le C_2 \|g\|_{L^{\widetilde{B}}(0,\infty)}. \tag{7.4.11}$$

for every $g \in L^{\widetilde{B}}(0,\infty)$. Passing to the dual inequality in (7.4.11), we obtain that

$$\left\| t^{\frac{n}{\gamma}-1} \int_0^t f(s)\, \mathrm{d}s \right\|_{L^B(0,\infty)} \le C_2' \|f\|_{L^A(0,\infty)} \tag{7.4.12}$$

for each $f \in L^A(0,\infty)$. Next, we restrict the inequalities (7.4.10) and (7.4.12) to non-increasing functions f^* only. This together with (7.4.7) results in

$$\left\| \int_t^\infty s^{\frac{\gamma}{n}-1} f^{**}(s)\, \mathrm{d}s \right\|_{L^B(0,\infty)} \le C_3 \|f^*\|_{L^A(0,\infty)}$$

for every $f \in L^A(0,\infty)$, in which C_3 depends on C_1, C_2, n and γ. By the rearrangement formula for I_γ, namely (7.1.1), we infer that

$$\|(I_\gamma f)^*\|_{L^B(0,\infty)} \le C_3 \|f^*\|_{L^A(0,\infty)},$$

which is equivalent to (7.1.3), thanks to (2.6.2). □

Remark 7.4.2. It is immediate from Proposition 7.4.1 that the Riesz potential is a self-adjoint operator. Namely (7.1.3) holds if and only if

$$I_\gamma \colon L^{\widetilde{B}}(\mathbb{R}^n) \to L^{\widetilde{A}}(\mathbb{R}^n)$$

is satisfied.

Proof of Theorem 7.2.1. By Proposition 7.4.1, (i) is equivalent to (7.4.3) and (7.4.4). Now, (ii) follows by the corresponding characterizations of the operator H_γ in Orlicz spaces, namely, (7.4.3) is equivalent to (7.2.11) and (7.2.1) by Theorem 3.5.2 and (7.4.4) is characterised by (7.2.12) with (7.2.3), thanks to Theorem 3.3.2 and passing to conjugate Young functions when necessary.

Similarly, (iii) may be obtained using characterisation of (7.4.3) by Theorem 3.3.2 resulting in (7.2.14) and (7.2.8), while (7.4.4) is treated by Theorem 3.5.2 which yields to (7.2.13) and (7.2.5) after deriving the conjugate formulas. □

Lemma 7.4.3. *Let B be a Young function satisfying* (7.2.8). *Then B obeys* (7.2.5) *and B_D is dominated by B_R.*

Furthermore, if $i_B > \frac{n}{n-\gamma}$, then B_D and B_R are equivalent.

Proof. Assume that B fulfills (7.2.8). Then F_R is well defined and we have the trivial estimate

$$\begin{aligned} F_R(2t) &\geq t^{n/(n-\gamma)} \int_0^{2t} \frac{B(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s \geq t^{n/(n-\gamma)} \int_t^{2t} \frac{B(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s \\ &\geq B(t)\, t^{n/(n-\gamma)} \int_t^{2t} \frac{\mathrm{d}s}{s^{n/(n-\gamma)+1}} \geq B(t) C_{n,\gamma} \quad \text{for } t > 0. \end{aligned}$$

Using (2.1.3) and passing to inverses, we infer that

$$F_R^{-1}(s)\, s^{\frac{\gamma}{n}-1} \leq cB^{-1}(s)\, s^{\frac{\gamma}{n}-1} \quad \text{for } s > 0 \tag{7.4.13}$$

and for some $c > 0$, depending on n and γ. Next, since $F_R(t)\, t^{n/(n-\gamma)}$ is a non-decreasing function, $F_R^{-1}(s)\, s^{\frac{\gamma}{n}-1}$ is non-increasing whence (7.4.13) is equivalent to

$$F_R^{-1}(t)\, t^{\frac{\gamma}{n}-1} \leq c \inf_{0<s\leq t} B^{-1}(s)\, s^{\frac{\gamma}{n}-1} \quad \text{for } t > 0. \tag{7.4.14}$$

It follows from (7.4.14) that (7.2.5) holds. Also, (7.4.14) implies that

$$t^{\frac{\gamma}{n}}\, F_R^{-1}(t) \leq cJ_D(t) \quad \text{for } t > 0.$$

and, by (7.2.7) and by (7.2.10), we have that

$$B_R^{-1}(t) \leq CB_D^{-1}(t) \quad \text{for } t > 0$$

and for some $C > 0$, which means that B_R dominates B_D globally.

Now, let $i_B > \frac{n}{n-\gamma}$. Then, by Proposition 3.2.2, F_R is dominated by B globally. This implies that the inequality in (7.4.13) may be reversed in this case and consequently

$$B_D^{-1}(t) \leq CB_R^{-1}(t) \quad \text{for } t > 0,$$

which gives the claim. □

Proof of Proposition 7.3.1. Assume (7.2.8) and (7.3.1). From Lemma 7.4.3 we infer that (7.2.5) holds and hence both B_R and B_D are well defined. Let us show (7.3.2). This is equivalent to the simultaneous validity of conditions (7.2.13) and (7.2.14), as we infer from Theorem 7.2.1. The latter is clearly satisfied for $A = B_R$. As for the former, we have to show that

$$\int_0^t \frac{\widetilde{B_R}(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s \leq \frac{\widetilde{B_D}(Ct)}{t^{n/(n-\gamma)}} \quad \text{for } t \geq 0 \tag{7.4.15}$$

and for some $C > 0$. Observe that Proposition 3.2.2 tells us that (7.3.1) is equivalent to the existence of $C_1 > 0$ such that

$$\int_0^t \frac{\widetilde{B_D}(s)}{s^{n/(n-\gamma)+1}}\,\mathrm{d}s \leq \frac{\widetilde{B_D}(C_1 t)}{t^{n/(n-\gamma)}} \quad \text{for } t \geq 0. \tag{7.4.16}$$

Next, by Lemma 7.4.3, B_R dominates B_D and thus

$$\widetilde{B_R}(t) \leq \widetilde{B_D}(C_2 t) \quad \text{for } t \geq 0 \tag{7.4.17}$$

for some $C_2 > 0$. At this point, (7.4.15) follows easily from (7.4.16) and (7.4.17) by the change of variables.

Let further A be a Young function satisfying (7.1.3). Theorem 7.2.1, namely (7.2.14) then asserts that $L^A(\mathbb{R}^n) \to L^{B_R}(\mathbb{R}^n)$ and $L^{B_R}(\mathbb{R}^n)$ is optimal.

The rest of the claim, namely the simplified relation for Young function B_R follows by Proposition 3.2.2 and the indices property is due to Lemma 3.6.11. □

Proof of Proposition 7.3.2. Let B satisfy (7.2.8) and (7.3.3). Lemma 7.4.3 tells us that B satisfies (7.2.5) as well, B_D is correctly defined and it is equivalent to B_R, since $i_B > \frac{n}{n-\gamma}$. Concurrently, by Lemma 3.6.11, we have that $I_{B_D} = \frac{n}{\gamma}$, since, by assumption $I_B = \infty$.

Now, let A be a given Young function obeying (7.1.3). We make use of the characterization given in Proposition 7.4.1 which says that (7.1.3) holds if and only if

$$H_\gamma \colon L^A(0,\infty) \to L^B(0,\infty) \tag{7.4.18}$$

and

$$H_\gamma \colon L^{\widetilde{B}}(0,\infty) \to L^{\widetilde{A}}(0,\infty) \tag{7.4.19}$$

hold, in which H_γ is the operator given by (7.4.1). Since $I_{B_D} = \frac{n}{\gamma}$, then, by Theorem 3.6.1, there is a Young function A_1 such that $L^{A_1}(0,\infty)$ is strictly larger space than $L^A(0,\infty)$ and

$$H_\gamma \colon L^{A_1}(0,\infty) \to L^B(0,\infty). \tag{7.4.20}$$

It follows from Theorem 3.5.2 together with Proposition 3.2.2 that (7.4.18) does not hold with A replaced by B_D. This in particular implies that $L^{A_1}(0,\infty) \to L^{B_D}(0,\infty)$ or, by duality,

$$L^{\widetilde{B_D}}(0,\infty) = L^{\widetilde{B_R}}(0,\infty) \to L^{\widetilde{A_1}}(0,\infty). \tag{7.4.21}$$

Further, Theorem 3.4.1 tells us that (7.4.19) holds with $\widetilde{A}$ replaced by $\widetilde{A_R}$. Combining this with (7.4.21), we obtain

$$H_\gamma \colon L^{\widetilde{B}}(0,\infty) \to L^{\widetilde{A_1}}(0,\infty). \tag{7.4.22}$$

Using Proposition 7.4.1 again, equations (7.4.20) with (7.4.22) gives

$$I_\gamma \colon L^{A_1}(\mathbb{R}^n) \to L^B(\mathbb{R}^n)$$

and no optimal Orlicz domain space exists for B with respect to I_γ.

Conversely, if (7.2.5) fails, then no Orlicz domain space exists by Theorem 7.2.1. □

Since, by Remark 7.4.2, I_γ is self-adjoint operator, the proofs of Propositions 7.3.3 and 7.3.4 can be derived using the duality and by calculation of the conjugate Young functions. A variant of Lemma 7.4.3 is needed here. Its proof is also analogous and we omit it.

Lemma 7.4.4. *Let A be a Young function satisfying* (7.2.3). *Then B obeys* (7.2.1) *and A_D is dominated by A_R.*

Furthermore, if $I_A < \frac{n}{\gamma}$, then A_D and A_R are equivalent.

8. Laplace transform

8.1 Introduction

The Laplace transform $\mathcal{L}$ is a classical linear integral operator defined for every f on $(0,\infty)$ by

$$\mathcal{L}f(t) = \int_0^\infty f(s)e^{-ts}\,\mathrm{d}s \quad \text{for } t > 0$$

whenever the integral converges.

In our analysis, it is crucial to have the pointwise estimate of a rearrangement of $\mathcal{L}$ of measurable function f. We shall use the result of the recent paper [16], which asserts that

$$(\mathcal{L}f)^*(t) \le \int_0^{1/t} f^*(s)\,\mathrm{d}s \quad \text{for } t > 0 \tag{8.1.1}$$

and for any $f \in (L^1+L^\infty)(0,\infty)$. It will be also useful to notice that the inequality can be reversed. Indeed, assume that f is a non-negative measurable function over $(0,\infty)$. Then

$$\mathcal{L}f(t) = \int_0^\infty f(s)e^{-st}\,\mathrm{d}s \ge \int_0^{1/t} f(s)e^{-st}\,\mathrm{d}s \ge \tfrac{1}{e}\int_0^{1/t} f(s)\,\mathrm{d}s \quad \text{for } t > 0. \tag{8.1.2}$$

We are concerned with the analysis of

$$\mathcal{L}\colon L^A(0,\infty) \to L^B(0,\infty). \tag{8.1.3}$$

with the special attention to the sharpness of the results. The question of boundedness has attracted the attention of many authors, see [5, 59, 63], for instance, the sharpness stayed aside, cf. the late exceptional work [11]. However, none of these papers covers the problem addressed in Orlicz spaces. We would like to expose some of the difficulties concerning the action of Laplace transform between Orlicz spaces in this scope.

8.2 Reduction principle

Let us start with a general reduction theorem based on the rearrangement inequality (8.1.1).

Proposition 8.2.1. *Let $X(0,\infty)$ and $Y(0,\infty)$ be rearrangement-invariant function spaces with their associate spaces $X'(0,\infty)$ and $Y'(0,\infty)$, respectively. The following statements are equivalent.*

(i) *There is a constant $C_1 > 0$ such that*

$$\|\mathcal{L}f\|_{Y(0,\infty)} \le C_1\|f\|_{X(0,\infty)}$$

for every $f \in X(0,\infty)$.

(ii) *There is a constant* $C_2 > 0$ *such that*

$$\left\| \int_0^{1/t} f(s)\,\mathrm{d}s \right\|_{Y(0,\infty)} \le C_2 \|f\|_{X(0,\infty)}$$

for each $f \in X(0,\infty)$.

Moreover, the constants C_1 *and* C_2 *are comparable.*

Proof. Assume (i) and let $f \in X(0,\infty)$ be given. By (8.1.2), we have

$$\begin{aligned}\left\| \int_0^{1/t} f(s)\,\mathrm{d}s \right\|_{Y(0,\infty)} &\le \left\| \int_0^{1/t} |f(s)|\,\mathrm{d}s \right\|_{Y(0,\infty)} \\ &\le e \left\| \mathcal{L}|f| \right\|_{Y(0,\infty)} \le eC_1 \|f\|_{X(0,\infty)},\end{aligned}$$

which gives (ii).

Conversely, if (ii) holds then, by (8.1.1) together with (2.6.2), we infer

$$\begin{aligned}\|\mathcal{L}f\|_{Y(0,\infty)} = \|(Lf)^*\|_{Y(0,\infty)} &\le \left\| \int_0^{1/t} f^*(s)\,\mathrm{d}s \right\|_{Y(0,\infty)} \\ &\le \|f^*\|_{X(0,\infty)} = \|f\|_{X(0,\infty)}\end{aligned}$$

and (i) follows. □

Remark 8.2.2. Observe that $\mathcal{L}$ is self-adjoint operator. Indeed, by the sharp Hölder inequality in r.i. spaces (2.6.1),

$$\begin{aligned}&\sup_{f \in X(0,\infty)} \frac{\left\| \int_0^{1/t} f(s)\,\mathrm{d}s \right\|_{Y(0,\infty)}}{\|f\|_{X(0,\infty)}} \\ &= \sup_{f \in X(0,\infty)} \sup_{g \in Y'(0,\infty)} \frac{\int_0^\infty g(t) \int_0^{1/t} f(s)\,\mathrm{d}s\,\mathrm{d}t}{\|f\|_{X(0,\infty)} \|g\|_{Y'(0,\infty)}} \\ &= \sup_{g \in Y'(0,\infty)} \sup_{f \in X(0,\infty)} \frac{\int_0^\infty f(s) \int_0^{1/s} g(t)\,\mathrm{d}t\,\mathrm{d}t}{\|f\|_{X(0,\infty)} \|g\|_{Y'(0,\infty)}} \\ &= \sup_{g \in Y'(0,\infty)} \frac{\left\| \int_0^{1/s} g(t)\,\mathrm{d}t \right\|_{X'(0,\infty)}}{\|g\|_{Y'(0,\infty)}}\end{aligned}$$

and hence

$$\mathcal{L} \colon X(0,\infty) \to Y(0,\infty)$$

holds if and only if

$$\mathcal{L} \colon Y'(0,\infty) \to X'(0,\infty),$$

due to Proposition 8.2.1.

Now, we restrict our attention to Orlicz spaces. Unlike for the operators mentioned earlier in this thesis, the necessary and sufficient condition on A and B to ensure (8.1.3) is not known. We are however able to characterise the weak-type inequality instead.

Proposition 8.2.3. *Let A and B be Young functions. Then*

$$\mathcal{L}\colon L^A(0,\infty) \to M^B(0,\infty) \tag{8.2.1}$$

if and only if there is a constant $C > 0$ such that

$$A^{-1}(t) \le CtB^{-1}\Big(\tfrac{1}{t}\Big) \quad \text{for } t > 0. \tag{8.2.2}$$

Furthermore, C_1 is comparable to the norm of $\mathcal{L}$ in (8.2.1).

Proof. We will prove both implications at once using only equivalent steps. Proposition 8.2.1 together with the self-adjointness of the Laplace transform tells us that (8.2.1) is equivalent to

$$\bigg\|\int_0^{1/y} f(s)\,\mathrm{d}s\bigg\|_{L^{\widetilde{A}}(0,\infty)} \le C\|f\|_{\Lambda^{\widetilde{B}}(0,\infty)} \tag{8.2.3}$$

for every $f \in \Lambda^{\widetilde{B}}(0,\infty)$, where $C > 0$ is some constant depending on the norm of (8.2.1) only. Note, that we used the Orlicz and Marcinkiewicz space in Proposition 8.2.1 together with (2.3.3) and (2.4.6). Next, (8.2.3) is equivalent to

$$\bigg\|\int_0^{1/y} f^*(s)\,\mathrm{d}s\bigg\|_{L^{\widetilde{A}}(0,\infty)} \le C\|f\|_{\Lambda^{\widetilde{B}}(0,\infty)}. \tag{8.2.4}$$

Indeed, one implication is trivial, since we restrict the inequality to monotone functions only, and the reversed one is due to (2.4.1). From Proposition 2.8.1 we infer that (8.2.4) is equivalent to

$$\bigg\|\int_0^{1/y} \chi_{(0,t)}(s)\,\mathrm{d}s\bigg\|_{L^{\widetilde{A}}(0,\infty)} \le C\|\chi_{(0,t)}\|_{\Lambda^{\widetilde{B}}(0,\infty)} \quad \text{for } t > 0. \tag{8.2.5}$$

The right hand side of (8.2.5) simplifies to

$$\|\chi_{(0,t)}\|_{\Lambda^{\widetilde{B}}(0,\infty)} = \frac{1}{\widetilde{B}^{-1}(\frac{1}{t})} \simeq tB^{-1}\Big(\tfrac{1}{t}\Big) \tag{8.2.6}$$

due to (2.4.5) and (2.1.5). Calculation shows that

$$\bigg\|\int_0^{1/y} \chi_{(0,t)}(s)\,\mathrm{d}s\bigg\|_{L^{\widetilde{A}}(0,\infty)} \simeq t\|\chi_{(0,1/t)}\|_{L^{\widetilde{A}}(0,\infty)} = \frac{t}{\widetilde{A}^{-1}(t)} \simeq A^{-1}(t) \tag{8.2.7}$$

thanks to (2.4.5) and (2.1.5) again. Plugging (8.2.6) and (8.2.7) into (8.2.5) we obtain that (8.2.5) is equivalent to (8.2.2). □

A variant of parallel integral inequality is also available.

Proposition 8.2.4. *Let A and B be Young functions such that* (8.1.3). *Then there is a constant $C > 0$ such that*

$$B\bigg(\frac{t}{C\int_0^\infty A(|f(s)|)\,\mathrm{d}s}\bigg)|\{\mathcal{L}f > t\}| \le \frac{1}{\int_0^\infty A(|f(s)|)\,\mathrm{d}s} \quad \text{for } t \ge 0 \tag{8.2.8}$$

for every nonzero $f \in L^A(0,\infty)$.

Proof. Assume that A and B are satisfying (8.1.3), or equivalently, due to Proposition 8.2.3, A and B obey (8.2.2) for some $C > 0$. Let $N > 0$ be fixed and define

$$A_N(t) = \frac{A(t)}{N} \quad \text{and} \quad B_N(t) = NB\left(\frac{t}{N}\right) \quad \text{for } t \geq 0.$$

Then A_N and B_N satisfy

$$A_N^{-1}(t) \leq CtB_N^{-1}\left(\tfrac{1}{t}\right) \quad \text{for } t > 0$$

with the same $C > 0$ independent of N. Thus, by Proposition 8.2.3 again,

$$\mathcal{L}\colon L^{A_N}(0,\infty) \to M^{B_N}(0,\infty) \tag{8.2.9}$$

with the embedding norm independent of N. Now, assume that $f \in L^A(0,\infty)$ is a nonzero function such that $\int_0^\infty A(|f(s)|)\,\mathrm{d}s < \infty$. Setting $N = \int_0^\infty A(|f(s)|)\,\mathrm{d}s$ and observing that

$$\|f\|_{L^{A_N}(0,\infty)} \leq 1,$$

we infer that

$$\|\mathcal{L}f\|_{M^{B_N}(0,\infty)} \leq C, \tag{8.2.10}$$

in which $C > 0$ is (possibly different) constant of the norm of (8.2.9). The equation (8.2.10) with (2.4.2) gives us that

$$C \geq \|\mathcal{L}f\|_{M^{B_N}(0,\infty)} \geq \sup_{0<t<\infty} \frac{t}{B_N^{-1}\left(\frac{1}{|\{\mathcal{L}f>t\}|}\right)} \quad \text{for } t > 0,$$

which rewrites as

$$B_N\left(\tfrac{t}{C}\right)|\{\mathcal{L}f > t\}| \leq 1 \quad \text{for } t > 0 \tag{8.2.11}$$

and (8.2.8) follows by (8.2.11) and the choice of N. □

We would like to mention that this week type embedding (8.2.1) cannot be lifted to the strong type (8.1.3). We will demonstrate it on a simple counterexample. Let us start with the result from [16, Theorem 3.8].

Example 8.2.5. Assume that $p \in (1,\infty)$ and $q \in [1,\infty]$. Then

$$\mathcal{L}\colon L^{p,q}(0,\infty) \to L^{p',q}(0,\infty) \tag{8.2.12}$$

where $p' = p/(p-1)$. Moreover both domain and target spaces are the optimal rearrangement-invariant spaces with respect to $\mathcal{L}$.

We now look closer to (8.2.12). Assume that $1 < p \leq 2$. Then, $p \leq p'$ and, since the Lorentz spaces are nested, $L^{p',p}(0,\infty) \to L^{p'}(0,\infty)$. Thus, we obtain

$$\mathcal{L}\colon L^{p}(0,\infty) \to L^{p'}(0,\infty). \tag{8.2.13}$$

On the contrary, if $2 < p < \infty$, then $p' < p$ and we have only

$$\mathcal{L}\colon L^{p}(0,\infty) \to L^{p',p}(0,\infty)$$

or

$$\mathcal{L}\colon L^{p,p'}(0,\infty) \to L^{p'}(0,\infty).$$

Example 8.2.6. If we set $A(t) = t^p$ and $B(t) = t^{p'}$, then $L^A(0,\infty) = L^p(0,\infty)$ and $L^B(0,\infty) = L^{p'}(0,\infty)$. Then, (8.1.3) holds if and only if $1 < p \le 2$, by (8.2.13), and (8.2.1) holds for any $1 < p < \infty$.

In the sequel, we focus on a sufficient condition on A and B to ensure (8.1.3).

Theorem 8.2.7. *Let A and B be Young functions satisfying*

$$\int_0^\infty B\Big(\frac{1}{t}\Big)\widetilde{A}(t)\frac{\mathrm{d}t}{t} < \infty. \tag{8.2.14}$$

Then Laplace transform is bounded from $L^A(0,\infty)$ to $L^B(0,\infty)$.

The proof is based on an auxiliary interpolation result for Laplace transform in Orlicz spaces. It reads as follows.

Proposition 8.2.8. *Let A and B be Young functions satisfying* (8.2.14)*. Suppose that $(\mathcal{R},\mu)$ and $(\mathcal{S},\nu)$ are σ-finite non-atomic measure spaces and Assume that T is a linear operator satisfying*

$$T\colon L^1(\mathcal{R},\mu) \to L^\infty(\mathcal{S},\nu) \tag{8.2.15}$$

$$T\colon L^\infty(\mathcal{R},\mu) \to L^{1,\infty}(\mathcal{S},\nu) \tag{8.2.16}$$

with operator norms C_1 and C_2, respectively. Then

$$\int_{\mathcal{S}} B\Big(\frac{Tf(x)}{2C_1\int_{\mathcal{R}} A(|f|)\,\mathrm{d}\mu}\Big)\,\mathrm{d}\nu(x) \le \frac{C_2\cdot C}{C_1\int_{\mathcal{R}} A(|f|)\,\mathrm{d}\mu} \tag{8.2.17}$$

for every $f \in L^A(\mathcal{R})$, where the constant C depends on the value of (8.2.14)*.*

Proof. Let f satisfying $\int_{\mathcal{R}} A(|f|)\,\mathrm{d}\mu < \infty$ be fixed. We need to show that T is defined on such f. Denote $t_0 = \inf\{t > 0 : A(t) > 0\}$ and $t_\infty = \sup\{t > 0 : A(t) < \infty\}$. Let $t_0 < t < t_\infty$ and decompose f as $f = f^t + f_t$, where

$$f^t = \max\{|f| - t, 0\}\operatorname{sign} f \quad \text{and} \quad f_t = \min\{|f|, t\}\operatorname{sign} f.$$

It suffices to show that $f^t \in L^1(\mathcal{R})$ and $f_t \in L^\infty(\mathcal{R})$. The latter is obvious, since $\sup f \le t$. As for the former, $A(s)/s$ is non-decreasing and, by (2.1.2), $A(s)/s \le a(s)$, whence

$$\begin{aligned}\|f^t\|_{L^1(\mathcal{R})} &= \int_t^{t_\infty} \mu(\{|f| > s\})\,\mathrm{d}s \le \frac{t}{A(t)}\int_t^{t_\infty} a(s)\mu(\{|f| > s\})\,\mathrm{d}s \\ &\le \frac{t}{A(t)}\int_{\mathcal{R}} A(|f|)\,\mathrm{d}\mu\end{aligned} \tag{8.2.18}$$

and thus $f^t \in L^1(\mathcal{R})$. Further, let us denote by σ the generalised right-continuous inverse of $s/A(s)$. Observe that, by (8.2.14), $\widetilde{A}$ is finite-valued Young function and hence $s/A(s)$ decreases to zero as $s \to \infty$. Thus $\sigma\colon (0,\infty) \to (t_0, t_\infty)$.

Then, by (8.2.18) and (8.2.15),

$$\|Tf^{\sigma(t)}\|_{L^\infty(\mathcal{S})} \le C_1\|f^{\sigma(t)}\|_{L^1(\mathcal{R})} \le C_1 t\int_{\mathcal{R}} A(|f|)\,\mathrm{d}\mu. \tag{8.2.19}$$

Denoting $K = C_1 \int_{\mathcal{R}} A(|f|)\,\mathrm{d}\mu$, we infer that

$$\nu(\{|Tf| > 2Kt\}) \le \nu(\{|Tf_{\sigma(t)}| > Kt\}) + \nu(\{|Tf^{\sigma(t)}| > Kt\}) \tag{8.2.20}$$

in which the second term is zero, thanks to (8.2.19). Further, by (8.2.16),

$$\nu(\{Tf_{\sigma(t)} > Kt\})Kt \le \|f_{\sigma(t)}\|_{L^\infty(\mathcal{S})} \le C_2\sigma(t). \tag{8.2.21}$$

Coupling (8.2.20) with (8.2.21) we get

$$\nu(\{|Tf| > 2Kt\}) \le \frac{C_2}{K} \cdot \frac{\sigma(t)}{t},$$

whence

$$\begin{aligned}\int_{\mathcal{S}} B\Big(\frac{Tf(x)}{2K}\Big)\,\mathrm{d}\nu &= \int_0^\infty b(t)\nu(\{|Tf| > 2Kt\})\,\mathrm{d}t \\ &= \frac{C_2}{K}\int_0^\infty \frac{\sigma(t)}{t} b(t)\,\mathrm{d}t\end{aligned} \tag{8.2.22}$$

Simple analysis of σ shows that

$$\widetilde{A}(t) \le t\sigma\left(\frac{1}{t}\right) \le \widetilde{A}(4t)$$

and since $tb(t) \le B(2t)$, (8.2.22) continues by

$$\int_{\mathcal{S}} B\Big(\frac{Tf(x)}{2K}\Big)\,\mathrm{d}\nu = \frac{C_2}{K}\int_0^\infty \widetilde{A}\left(\frac{4}{t}\right) B(2t)\frac{\mathrm{d}t}{t}. \tag{8.2.23}$$

The integral on the right hand side of (8.2.23) is finite due to (8.2.14) and (8.2.17) follows by the choice of K. $\square$

The proof of the sufficient condition for the boundedness of Laplace transform now follows easily.

Proof of Theorem 8.2.7. For every measurable f on has

$$|\mathcal{L}f(t)| \le \int_0^\infty |f(s)|\, e^{-ts}\,\mathrm{d}s \le \int_0^\infty |f(s)|\,\mathrm{d}s$$

and therefore

$$\|\mathcal{L}\|_{L^\infty(0,\infty)} \le \|f\|_{L^1(0,\infty)}.$$

Next, if f is bounded and non-negative function, we infer

$$(\mathcal{L}f)^*(t) = \mathcal{L}f(t) \le \|f\|_{L^\infty(0,\infty)} \int_0^\infty e^{-st}\,\mathrm{d}s = \frac{\|f\|_{L^\infty(0,\infty)}}{t},$$

whence

$$\|\mathcal{L}f\|_{L^{1,\infty}(0,\infty)} \le \|f\|_{L^\infty(0,\infty)}.$$

For functions f which are not non-negative, we use that $\mathcal{L}f \le \mathcal{L}|f|$ and the outcome is the same. The claim now follows by Proposition 8.2.8. $\square$

We shall demonstrate on the example (8.2.6) that the condition in Theorem (8.2.7) is not necessary. Indeed, by (8.2.13), The Laplace transform is bounded from L^A to L^B for the choice $A(t) = t^p$, $B(t) = t^{p'}$ for $1 < p \le 2$. However, the condition (8.2.14) fails in this case, since $\widetilde{A}$ is equivalent to B and thus

$$\int_0^\infty B\Big(\frac{1}{t}\Big)\widetilde{A}(t)\frac{\mathrm{d}t}{t} \simeq \int_0^\infty \frac{\mathrm{d}t}{t} = \infty.$$

8.3 Orlicz optimality

The rest of the chapter is devoted to the particular case if $L^A(0,\infty) = L^p(0,\infty)$ or $L^B(0,\infty) = L^{p'}(0,\infty)$. We show that such pair of Orlicz spaces forms an optimal couple with respect to $\mathcal{L}$ if $1 < p \leq 2$. We further emphasize to show that if $p > 2$ then there is no optimal Orlicz target space for $L^A(0,\infty)$ and conversely, no optimal Orlicz domain space exists for $L^B(0,\infty)$ in this case.

Theorem 8.3.1. *Let $1 < p \leq 2$. Then*

$$\mathcal{L}\colon L^p(0,\infty) \to L^{p'}(0,\infty) \tag{8.3.1}$$

and the space $L^{p'}(0,\infty)$ is the optimal target space in (8.3.1) *among all Orlicz spaces.*

Conversely, if $p > 2$ then there is no optimal Orlicz target space for $L^p(0,\infty)$ with respect to $\mathcal{L}$.

Theorem 8.3.2. *Let $1 < p \leq 2$. Then*

$$\mathcal{L}\colon L^p(0,\infty) \to L^{p'}(0,\infty) \tag{8.3.2}$$

and the space $L^p(0,\infty)$ is the optimal domain space in (8.3.2) *among all Orlicz spaces.*

Conversely, if $p > 2$ then there is no optimal Orlicz domain space for $L^{p'}(0,\infty)$ with respect to $\mathcal{L}$.

The proofs are given at the end of the section. The method we shall use significantly differs from all of those developed in the previous chapters. Although this tool does not help us to answer the question of Orlicz optimality in its full generality, it might be used in application to much broader class of operators.

Our approach is based on the analysis of a relationship between Orlicz and Lorentz $L^{p,q}$ spaces. The idea relies on the observation that all the spaces on the scale $L^{p,q}$, for p fixed and q varying between 1 and ∞, share the same fundamental function while there is one to one correspondence between Young functions and fundamental functions, cf. (2.3.2).

Theorem 8.3.3. *Let $(\mathcal{R},\nu)$ be σ-finite non-atomic measure space and let $1 \leq p, q < \infty$.*
(i) *If $p > q$, then there is no optimal Orlicz space $L^A(\mathcal{R})$ satisfying*

$$L^A(\mathcal{R}) \to L^{p,q}(\mathcal{R}). \tag{8.3.3}$$

(ii) *If $p \leq q$, then $L^p(\mathcal{R})$ is the largest Orlicz space contained in $L^{p,q}(\mathcal{R})$.*

Theorem 8.3.4. *Let $(\mathcal{R},\nu)$ be σ-finite non-atomic measure space and let $1 \leq p, q < \infty$.*
(i) *If $p < q$, then there is no optimal Orlicz space $L^B(\mathcal{R})$ satisfying*

$$L^{p,q}(\mathcal{R}) \to L^B(\mathcal{R}).$$

(ii) *If $p \geq q$, then $L^p(\mathcal{R})$ is the smallest Orlicz space containing $L^{p,q}(\mathcal{R})$.*

Before we prove this results, we need some preparation. The next lemma exhibits a sufficient condition for the embedding of Orlicz space $L^A(\mathcal{R})$ to a Lorentz-type space $\Lambda^q_w(\mathcal{R})$ in which the norm is given by

$$\|f\|_{\Lambda^q_w(\mathcal{R})} = \Big(\int_0^{\nu(\mathcal{R})} [f^*(t)]^q \, w(t)\, \mathrm{d}t\Big)^{\frac{1}{q}}$$

or, equivalently,

$$\|f\|_{\Lambda^q_w(\mathcal{R})} \simeq \Big(\int_0^{\infty} W(\mu_f(t))\, t^{q-1}\, \mathrm{d}t\Big)^{\frac{1}{q}}.$$

Here $W(t) = \int_0^t w(s)\,\mathrm{d}s$ for $0 < t < \infty$. These spaces generalize the Lorentz $L^{p,q}(\mathcal{R})$ spaces, since, by the choice $w(t) = t^{q/p-1}$, we recover

$$\Lambda^q_w(\mathcal{R}) = L^{p,q}(\mathcal{R}).$$

Although we will need only a special instance of this result, we prove it in this more general form. Note, that our results also include those of [53], [44] and [28].

Lemma 8.3.5. *Let $0 < q < \infty$, $A(t) = \int_0^t a(s)\,\mathrm{d}s$ be a Young function and w be a nonincreasing weight. Denote*

(i)

$$\int_0^{\infty} W\Big(w^{-1}\Big(a(t)\, t^{1-q}\Big)\Big) t^{q-1}\, \mathrm{d}t < \infty, \tag{8.3.4}$$

(ii)

$$L^A(\mathcal{R}) \to \Lambda^q_w(\mathcal{R}).$$

Then (i) implies (ii).

Proof. Let w be a nonincreasing weight, A be a Young function satisfying (i) and let $f \in L^A(\mathcal{R})$ be nonzero functions. Without loss of generality we can assume that $\|f\|_{L^A(\mathcal{R})} = 1$ otherwise we will work with the function $f/\|f\|_{L^A(\mathcal{R})}$. Denote $G = W^{-1}$. Then since w is nonincreasing, G is convex. Let $t \in (0,\infty)$. Applying Young inequality to the terms $W(\mu_f(t))$ and $t^{q-1}/a(t)$ we get

$$W(\mu_f(t))\, t^{q-1} \le a(t)\, \mu_f(t) + a(t)\, \widetilde{G}\bigg(\frac{t^{q-1}}{a(t)}\bigg) \quad \text{for } t > 0$$

and integrating over $(0,\infty)$, we have

$$\|f\|^q_{\Lambda^q_w(\mathcal{R})} \simeq \int_0^{\infty} W(\mu_f(t))\, t^{q-1}\, \mathrm{d}t \le \int_{\mathcal{R}} A(|f|)\, \mathrm{d}\nu + \int_0^{\infty} a(t)\, \widetilde{G}\bigg(\frac{t^{q-1}}{a(t)}\bigg)\, \mathrm{d}t.$$

The first integral is less or equal to 1. For the second term, we have, by concavity of $\widetilde{G}$,

$$\widetilde{G}\bigg(\frac{t^{q-1}}{a(t)}\bigg) \le [\widetilde{G}]'\bigg(\frac{t^{q-1}}{a(t)}\bigg)\, t^{q-1} \quad \text{for } t > 0$$

and together with

$$[\widetilde{G}]'(s) = [G']^{-1}(s) = \Big[[W^{-1}]'\Big]^{-1}(s) = \bigg[\frac{1}{w(W^{-1}(s))}\bigg]^{-1} = W\Big(w^{-1}\Big(\tfrac{1}{s}\Big)\Big)$$

for $s > 0$ we can finally write

$$\int_0^\infty a(t)\,\widetilde{G}\Big(\frac{t^{q-1}}{a(t)}\Big)\,\mathrm{d}t \le \int_0^\infty W\Big(w^{-1}\Big(a(t)\,t^{1-q}\Big)\Big)t^{q-1}\,\mathrm{d}t,$$

which is finite by (i). We have shown that f belongs to $\Lambda^q_w(\mathcal{R})$. Since f was taken arbitrarily, we obtain the inclusion $L^A(\mathcal{R}) \subseteq \Lambda^q_w(\mathcal{R})$ which implies also the continuous embedding, due to (2.6.3). □

In general, the conditions (i) and (ii) in the previous lemma are not equivalent as the next example shows.

Example 8.3.6. Let us take $A(t) = t^q$ with $q \ge 1$. Then the condition (8.3.4) becomes

$$W\Big(w^{-1}(q)\Big)\int_0^\infty t^{q-1}\,\mathrm{d}t = \infty$$

for every weight w. Now, one can choose a decreasing w in a way that

$$L^q(\mathcal{R}) \to \Lambda^q_w(\mathcal{R}),$$

which is equivalent to

$$W(t) \le ct, \quad \text{for } t \ge 0$$

for some positive c. Try for instance

$$w(t) = \begin{cases} 2-t, & 0 \le t < 1, \\ 1/t, & 1 \le t < \infty. \end{cases}$$

However, in the case when the space $\Lambda^q_w(\mathcal{R})$ coincides with the Lorentz space $L^{p,q}(\mathcal{R})$, we have the following result. The sufficiency can be also found in [4, Lemma 4.2].

Proposition 8.3.7. *Let $1 \le q < p < \infty$ and A be a Young function. Then*

$$L^A(\mathcal{R}) \to L^{p,q}(\mathcal{R}) \tag{8.3.5}$$

if and only if either $\nu(\mathcal{R}) < \infty$ and

$$\int^\infty \Big(\frac{t^p}{A(t)}\Big)^{\frac{q}{p-q}}\frac{\mathrm{d}t}{t} < \infty.$$

or $\nu(\mathcal{R}) = \infty$ and

$$\int_0^\infty \Big(\frac{t^p}{A(t)}\Big)^{\frac{q}{p-q}}\frac{\mathrm{d}t}{t} < \infty. \tag{8.3.6}$$

Proof. We prove only the global variant, the other one is analogous. The sufficiency is the corollary of Lemma 8.3.5 in which we set $w(t) = t^{q/p-1}$.

Conversely suppose (8.3.5). This is equivalent to the existence of a positive constant C such that

$$\|u\|_{L^{p,q}(\mathcal{R})} \le C \quad \text{whenever} \quad \|u\|_{L^A(\mathcal{R})} \le 1.$$

Since modular and norm unit balls in $L^A(\mathcal{R})$ coincide, this is equivalent to

$$\|u\|_{L^{p,q}(\mathcal{R})} \leq C \quad \text{whenever} \quad \int_{\mathcal{R}} A(|u(x)|)\,\mathrm{d}x \leq 1$$

namely, by the Fubini's theorem, it is the same as the existence of a positive C' such that

$$\int_0^\infty [\mu_f(t)]^{\frac{p}{q}}\, t^{q-1}\,\mathrm{d}t \leq C' \quad \text{whenever} \quad \int_0^\infty a(t)\mu_f(t)\,\mathrm{d}t \leq 1,$$

where $A(t) = \int_0^t a(s)\,\mathrm{d}s$. This is equivalent to

$$\int_0^\infty [\phi(t)]^{\frac{p}{q}}\, t^{q-1}\,\mathrm{d}t \leq C'$$

for every nonincreasing ϕ such that

$$\int_0^\infty a(t)\phi(t)\,\mathrm{d}t \leq 1,$$

which is nothing but the embedding

$$\Lambda^1_a(0,\infty) \to \Lambda^{q/p}_v(0,\infty) \tag{8.3.7}$$

with $v(t) = t^{q-1}$. Finally, (8.3.7) is equivalent to

$$\int_0^\infty t^{\frac{p(q-1)}{p-q}}\, a(t)^{\frac{q}{q-p}}\,\mathrm{d}t < \infty, \tag{8.3.8}$$

thanks to [65, Section 2, Proposition 1]. Consequently, (8.3.8) holds if and only if (8.3.6) holds, due to (2.1.2). □

Proposition 8.3.8. *Let $1 \leq p < q < \infty$ and B be a Young function. Then*

$$L^{p,q}(\mathcal{R}) \to L^B(\mathcal{R}) \tag{8.3.9}$$

if and only if either $\nu(\mathcal{R}) < \infty$

$$\int^\infty \left(\frac{B(t)}{t^p}\right)^{\frac{q}{q-p}} \frac{\mathrm{d}t}{t} < \infty.$$

or $\nu(\mathcal{R}) = \infty$ and

$$\int_0^\infty \left(\frac{B(t)}{t^p}\right)^{\frac{q}{q-p}} \frac{\mathrm{d}t}{t} < \infty. \tag{8.3.10}$$

Proof. Again, we show the statement in its global variant only. Using duality, (8.3.9) is equivalent to

$$L^{\widetilde{B}}(\mathcal{R}) \to L^{p',q'}(\mathcal{R})$$

which is, due to Proposition 8.3.7 characterised by

$$\int_0^\infty \left(\frac{t^{p'}}{\widetilde{B}(t)}\right)^{\frac{q'}{p'-q'}} \frac{\mathrm{d}t}{t} < \infty. \tag{8.3.11}$$

Using (2.1.4), (8.3.11) is equivalent to

$$\int_0^\infty [b^{-1}(t)]^{\frac{q(p-1)}{p-q}}\, t^{\frac{p}{q-p}}\,\mathrm{d}t < \infty.$$

Now, Fubini's theorem tells us that

$$\begin{aligned}\int_0^\infty t^{\frac{p}{q-p}}\left[b^{-1}(t)\right]^{\frac{q(p-1)}{p-q}}\,\mathrm{d}t &\simeq \int_0^\infty t^{\frac{p}{q-p}}\int_{b^{-1}(t)}^\infty r^{\frac{p(q-1)}{p-q}}\,\mathrm{d}r\,\mathrm{d}t\\ &= \int_0^\infty r^{\frac{p(q-1)}{p-q}}\int_0^{b(r)} t^{\frac{q}{p-q}}\,\mathrm{d}t\,\mathrm{d}r\\ &= \int_0^\infty r^{\frac{p(q-1)}{p-q}}\, b(r)^{\frac{q}{q-p}},\end{aligned}$$

whence (8.3.11) is equivalent to (8.3.10), due to (2.1.2). □

Proposition 8.3.9. *Let $1 \le q < \infty$ and w be a nonincreasing weight. Then*

$$\begin{aligned}\Lambda^q_w(\mathcal{R}) &= \bigcup\Big\{L^A(\mathcal{R}) : A \textit{ satisfies } (8.3.4)\Big\}\\ &= \bigcup\Big\{L^A(\mathcal{R}) : L^A(\mathcal{R}) \subseteq \Lambda^q_w(\mathcal{R})\Big\}.\end{aligned}$$

Note that the equalities in Proposition 8.3.9 are considered as set equalities only, since we have not defined any norm or other structure on the unions on the right hand side.

Proof of Proposition 8.3.9. One inclusion is straightforward since every Orlicz space $L^A(\mathcal{R})$ with a Young function A satisfying (8.3.4) is contained in $\Lambda^q_w(\mathcal{R})$ thanks to Lemma 8.3.5.

Conversely suppose that $f \in \Lambda^q_w(\mathcal{R})$ and define A by

$$A(t) = \int_0^t w\Big(\mu_{f/\lambda}(s)\Big)s^{q-1}\,\mathrm{d}s$$

with

$$\lambda = \|f\|_{\Lambda^q_w(\mathcal{R})}.$$

Then A is a Young function obeying the condition (8.3.4). Indeed

$$\int_0^\infty W\Big(w^{-1}\Big(a(t)\,t^{1-q}\Big)\Big)t^{q-1}\,\mathrm{d}t = \int_0^\infty W\Big(\mu_{f/\lambda}(t)\Big)\,t^{q-1}\,\mathrm{d}t \simeq \|f\|^q_{\Lambda^q_w(\mathcal{R})} < \infty.$$

It remains to show that $f \in L^A(\mathcal{R})$. Since W is concave, we have $w(s)s \le W(s)$ and hence

$$\begin{aligned}\int_{\mathcal{R}} A\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}x &= \int_0^\infty a(t)\mu_{f/\lambda}(t)\,\mathrm{d}t = \int_0^\infty w\Big(\mu_{f/\lambda}(t)\Big)t^{q-1}\mu_{f/\lambda}(t)\,\mathrm{d}t\\ &\le \int_0^\infty W\Big(\mu_{f/\lambda}(t)\Big)t^{q-1}\,\mathrm{d}t = \frac{\|f\|^q_{\Lambda^q_w(\mathcal{R})}}{q\lambda^q} = \frac{1}{q} \le 1.\end{aligned}$$

Therefore

$$\|f\|_{L^A(\mathcal{R})} \le \|f\|_{\Lambda^q_w(\mathcal{R})} < \infty$$

and f belongs to $L^A(\mathcal{R})$. □

Proof of Theorem 8.3.3. (i). Let us set $w(t) = t^{q/p-1}$. Since $q < p$, the weight w is decreasing and, by Proposition 8.3.9,

$$L^{p,q}(\mathcal{R}) = \bigcup\Big\{L^A(\mathcal{R}) : L^A(\mathcal{R}) \subseteq L^{p,q}(\mathcal{R})\Big\}. \tag{8.3.12}$$

Let A be a Young function obeying (8.3.3). Then, since $L^{p,q}(\mathcal{R})$ itself is not an Orlicz space, there is a function $f \in L^{p,q}(\mathcal{R}) \setminus L^{p,q}(\mathcal{R})$. From (8.3.12), we obtain that there is an Orlicz space, say $L^E(\mathcal{R})$ such that $L^E(\mathcal{R}) \subseteq L^{p,q}(\mathcal{R})$ and $f \in L^E(\mathcal{R})$. Let us define $\widehat{A}$ as the largest convex minorant of the function $\min\{A, E\}$. Then $\widehat{A}$ is a Young function such that

$$L^{\widehat{A}}(\mathcal{R}) = (L^A + L^E)(\mathcal{R}) \to L^{p,q}(\mathcal{R})$$

and $L^A(\mathcal{R}) \subsetneq L^{\widehat{A}}(\mathcal{R})$. Since A was arbitrary, the assertion of (i) follows.

(ii). Since $p \leq q$, then, by (2.5.1), $L^p(\mathcal{R}) \to L^{p,q}(\mathcal{R})$. Assume that (8.3.3) for some Young function A. Then there is a constant $C > 0$ such that

$$\|f\|_{L^{p,q}(\mathcal{R})} \leq C\|f\|_{L^A(\mathcal{R})}$$

for all $f \in L^A(\mathcal{R})$. Now, plugging $f = \chi_E$ for measurable $E \subseteq \mathcal{R}$ such that $\nu(E) = t$, $0 < t < \nu(\mathcal{R})$, we infer that

$$t^{\frac{1}{p}} \leq \frac{C'}{A^{-1}\left(\frac{1}{t}\right)} \quad \text{for } 0 < t < \nu(\mathcal{R}) \tag{8.3.13}$$

and for some $C' > 0$, due to (2.3.1) and (2.5.2). Rewriting (8.3.13), we obtain that A dominates t^p near infinity provided $\nu(\mathcal{R}) < 0$ or globally otherwise, whence, by (2.3.4), $L^A(\mathcal{R}) \to L^p(\mathcal{R})$, proving the optimality. □

Since the statement of Theorem 8.3.4 is just the dual version of Theorem 8.3.3 the proof is obvious and we omit it.

Proof of Theorem 8.3.1. The use of [16, Theorem 3.8] tells us that the optimal r.i. target for $L^p(\mathcal{R})$ is the Lorentz space $L^{p',p}(\mathcal{R})$. The assertion is then consequence of Theorem 8.3.4. □

The proof of Theorem 8.3.2 then follows analogously by Theorem 8.3.3 and is skipped.

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That's all Folks!