Report on the doctoral thesis by Martin Francu

"Isoperimetric problem, Sobolev spaces and the Heisenberg group"

The thesis of Martin Francu deals with two major topics, namely embeddings of Sobolev-type spaces built upon rearrangement-invariant spaces in the context of Carnot-Caratheodory spaces, and the approximation of the least concave majorant of a function. The common ground of the two topics is the use of rearrangement techniques and the application of rearrangement-invariant spaces. Both topics are important and interesting.

The text is structured into four chapters. The first chapter contains an introduction and all the necessary background material. All the definitions are given here as well as motivating examples and the history of the problem. The material includes nonincreasing rearrangement, rearrangement-invariant spaces, Sobolev embeddings, and the explanation of the connection between Sobolev-type inequalities and the isoperimetric profile of the underlying domains. One of the most important ingredients is the definition of the Carnot-Caratheodory space and the Heisenberg group. The last subsection is devoted to the explanation of the basic stuff concerning the least concave majorant.

The second chapter is entitled "Higher-order Sobolev-type embeddings on Carnot-Caratheodory spaces". It contains results from the applicant's paper having the same title, which was published in Math. Nachrichten in 2017. In this paper, a sufficient condition is given for higher-order Sobolev embeddings on bounded domains for Carnot-Cartheodory spaces. The condition is expressed in terms of boundedness of a one-dimensional integral operator. Its form reflects the isoperimetric properties of the underlying domain. The results are fairly general and are applied to plenty of examples including Lebesgue, Lorentz and Orlicz spaces. It is interesting that in the case of Heisenberg group the sufficient condition is necessary as well.

The third chapter is entitled "Compact embeggings on Carnot-Caraheodory spaces". It contains results from another paper by the author, entitled "Higher-order compact embeddings of function spaces on Carnot-Caratheodory spaces", accepted for publication to Banach Journal of Functional Analysis. In this paper, a sufficient condition is given for higher-order compact embeddings on bounded domains in Carnot-Caratheodory spaces in the class of rearrangement-invariant spaces. The condition is given in terms of compactness of a one-dimensional operator. Again, the general results are illustrated with examples involving particular customary function spaces.

The last chapter of the thesis contains results from the paper by M. Francu, R. Kerman and G. Sinnamon entitled "A new algorithm for approximating the least concave majorant", published in Czechoslovak Mathematical Journal in 2017. The least concave majorant of a given continuous function on an interval is a useful mathematical concept in many areas of analysis, for instance in theory of interpolation or in the theory of function spaces. For example it found application in criteria for embeddings between classical Lorentz spaces. In the thesis an approximation of the least concave majorant is found by means of cubic splines. This technique improves the known method of finding a convex hull of finitely many points in a plane. An interesting new quantitative formula is given for functions four times continuously differentiable.

Altogether, the thesis of Martin Francu brings several new results, that are not only interesting from the point of view of pure mathematics, but also in applications in other areas.

The author has undoubtedly proved his ability to carry out creative mathematical research and obtain deep new results. The thesis evidently fulfils the requirements for doctoral dissertations. I strongly recommend that the title Ph.D. be awarded to Martin Francu.