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Isoperimetric problem, Sobolev spaces and the Heisenberg group

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Very special thanks to my supervisor, without his undying faith in convergence of this process, it wouldn't have converged.

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Isoperimetrický problém, Sobolevovy prostory a Heisenbergova grupa

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Abstrakt:

V této disertační práci studujeme vnoření prostorů funkcí definovaných na Carnotových–Carathéodoryových prostorech. Hlavními výsledky práce jsou podmínky pro sobolevovské vnoření vyššího řádu mezi prostory s normou invariantní vůči nerostoucímu přerovnání. Ve speciálním případě, kdy je v pozadí ležící prostor s mírou takzvanou X-PS doménou v Heisenbergově grupě, dostáváme úplnou charakterizaci Sobolevova vnoření. Další sada hlavních výsledků se týká kompaktnosti zmíněných vnoření (v těchto případech získáváme postačující podmínky). Z obecných výsledků vyvozujeme specifická vnoření pro důležité konkrétní případy prostorů funkcí. V závěrečné části práce uvádíme nový algoritmus pro aproximaci nejmenší konkávní majoranty funkce definované na intervalu s odhadem chyby této aproximace.

Klíčová slova:

sobolevovské vnoření, Carnotovy–Carathéodoryovy prostory, Heisenbergova grupa, kompaktní vnoření, nejmenší konkávní majoranta

Title: Isoperimetric problem, Sobolev spaces and the Heisenberg group

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Abstract:

In this thesis we study embeddings of spaces of functions defined on Carnot-Carathéodory spaces. Main results of this work consist of conditions for Sobolev-type embeddings of higher order between rearrangement-invariant spaces. In a special case when the underlying measure space is the so-called X-PS domain in the Heisenberg group we obtain full characterization of a Sobolev embedding. The next set of main results concerns compactness of the above-mentioned embeddings. In these cases we obtain sufficient conditions. We apply the general results to important particular examples of function spaces. In the final part of the thesis we present a new algorithm for approximation of the least concave majorant of a function defined on an interval complemented with the estimate of the error of such approximation.

Keywords:

Sobolev-type embedding, Carnot-Carathéodory spaces, Heisenberg group, compact embedding, least concave majorant

Contents

Chapter 1

Introduction

The content of this thesis can be divided into two parts concerned with two slightly different topics. The first part is focused on rearrangement-invariant function spaces and their embeddings in the context of Carnot-Carathéodory spaces. The second part is concerned with the approximation of the least concave majorant of a function. Both topic have in common symmetrization techniques and applications to rearrangement-invariant function spaces.

1.1 Non-increasing rearrangements

Consider a function f defined on $D = \{0, 1, \dots, N\}$ for some $N \in \mathbb{N}$. In this case the non-increasing rearrangement of f has natural meaning: a function $\bar{f}$ which attains the same values (with the same multiplicity), while it holds that $\bar{f}(k) \leq \bar{f}(l)$ when $l \leq k$. Some of interesting properties of such rearrangements are

$$\sum_{0 \leq i \leq D} f(i)g(i) \leq \sum_{0 \leq i \leq D} \bar{f}(i)\bar{g}(i)$$

and

$$\sum_{0 \leq i < D} \left|\bar{f}(i) - \bar{f}(i+1)\right| \leq \sum_{0 \leq i < D} \left|f(i) - f(i+1)\right|.$$

Such essential non-increasing rearrangements which can be easily generalized to sequences were considered for example in [26] (where the famous Hardy-Littlewood-Pólya theorem takes its origin).

If we allow D to be a more complicated set, the notion of non-increasing rearrangement becomes much more ambiguous. Numerous kinds of rearrangements possess interesting properties.

The most widely used way to introduce a non-increasing rearrangement of a function $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \in \mathbb{N}$, is through a generalized inversion of its distribution function, as it was done for instance in [26], [50] and [4]. Let us recall the basic definitions.

Definition 1 (distribution function)**.** Suppose that (R, μ) is a measure space and denote by $\mathfrak{M}_0(R, \mu)$ the set of all μ-measurable functions that are finite μ-almost everywhere.

The *distribution function* μ_f of a function f in $\mathfrak{M}_0(R, \mu)$ is given by

$$\mu_f(\lambda) = \mu\left\{x \in R : |f(x)| > \lambda\right\} \text{ for all } \lambda \geq 0.$$

Two functions are called *equimeasurable* if they have the same distribution function.

In this thesis, we will restrict our attention to cases when μ is the classical Lebesgue measure. If no measure is explicitly stated, the Lebesgue measure is considered. The measure μ is used in the previous and the following definition to emphasize the possible generality of these definitions.

Definition 2 (non-increasing rearrangement)**.** Suppose f belongs to $\mathfrak{M}_0(R, \mu)$. The *non-increasing rearrangement* of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf\left\{\lambda : \mu_f(\lambda) \leq t\right\} \text{ for } t \geq 0.$$

Since this definition of the non-increasing rearrangement of a function is the most widely used one in the context of function spaces, it is the default notion we will have in mind when we mention a non-increasing rearrangement in this thesis.

The non-increasing rearrangement of a function $f : R \to \mathbb{R}$ is $f^* : [0, \infty) \to \mathbb{R}$. Notice that the domain space of f^* can be different from that of f. The domain of f^* is always $[0, \infty)$ independently of the domain set of f. This is used in the theory of rearrangement-invariant spaces. It is shown in the celebrated Luxemburg representation theorem ([4, Chapter 2, Theorem 4.10]) that every rearrangement-invariant space has its representative function space over $[0, \infty)$. Hence many statements can be proved once for this, representative, function space and then carried over to other function spaces.

The non-increasing rearrangement possesses similar properties to its counterpart from the beginning of this section:

$$\int_R fg\, d\mu \leq \int_{[0,\infty)} f^*(x)g^*(x)\, dx \text{ for all } f, g \in \mathfrak{M}_0(R, \mu)$$

and

$$\int_{[0,\infty)} (f^*)'(x)\, dx \leq \int_{\mathbb{R}^n} |\nabla f|\,(x_n)\, dx_n \text{ for all } f \in \mathfrak{M}_0(\mathbb{R}^n).$$

As an example of an application of properties of the non-increasing rearrangement can be taken the sufficiency part of the characterization of Sobolev-like embeddings, see e.g. [12].

Another approach to the concept of the non-increasing rearrangement is the so-called spherical rearrangement, or symmetrical rearrangement. Such rearrangement is a radial function having the same distribution function as the original one. The spherical rearrangement, $f^\star$, can be expressed through the formula

$$f^\star(x) = f^*(\kappa_n |x|^n), \text{ for } x \in \mathbb{R}^n,$$

where κ_n is the volume of the unit ball.

The spherical rearrangement has similar properties as the non-increasing rearrangement but, unlike the non-increasing rearrangement from Definition 2, its domain space is $\mathbb{R}^n$. Since f^* is admissible in any measure space, the operation of the spherical rearrangement can be defined on functions over any measure space, its result always being a function from $\mathbb{R}^n$ to $\mathbb{R}$. The spherical rearrangement is used in proofs of classical isoperimetric inequalities, see for instance [10] or [6].

The key property of the spherical rearrangement is that its level sets are the solutions to the classical isoperimetric problem. This leads to another possible class of rearrangements built on a certain family of sets, following the approach of [39]. This is interesting for instance in the case of the Heisenberg group where solutions to isoperimetric problem are not spheres.

Building on the idea of a spherical rearrangement, we can define the class of functions preserving even differential properties of higher orders. A symmetral of $f \in \mathfrak{M}(0,1)$, as used in [12], is defined by

$$u_{k,f}(x) = \int_{\kappa_n |x|^n}^1 \frac{1}{s_1^{1-\frac{1}{n}}} \int_{s_1}^1 \cdots \int_{s_k} \frac{f^*(s_k)}{s_k^{1-\frac{1}{n}}} \, ds_k \ldots \, ds_2 \, ds_1.$$

The key property of the symmetral of the k-th order is the preservation of the magnitude of the k-th order derivative. In other words, the inequality

$$|\nabla_k u_{k,f}(x)| = \frac{\partial^k}{\partial x^k}(f^*)(\kappa_n |x|^n), \text{ for a.e. } x \text{ such that } |x| \leq 1,$$

holds for all f weakly differentiable up to order k. This short summary is by no means exhaustive. It was meant to illustrate that the simple idea of rearranging of functions has a surprising variety of applications.

1.2 Rearrangement-invariant function spaces

This section is dedicated to an overview of the basic theory of rearrangement-invariant function spaces - a special case of Banach function spaces. A basic general reference concerning this topic is [4].

The Banach function spaces approach builds on common features of customary examples of function spaces rather than their specific properties. Hence the following definition which collects their basic properties usually expected from a function space.

Let (R, μ) be a measure space and let $\mathfrak{M}_+(R, \mu)$ be the cone of μ-measurable functions on R whose values lie in $[0, \infty]$.

Definition 3 (Banach function space)**.** A mapping $\varrho : \mathfrak{M}_+(R, \mu) \to [0, \infty]$ is called a *Banach function norm* if, for all f, g and $\{f_n\}_{n\in\mathbb{N}}$ in $\mathfrak{M}_+(R, \mu)$, every $a \geq 0$, and for all μ-measurable $E \subset \Omega$, the following properties hold:

- A.1 $\varrho(f) = 0$ if and only if $f = 0$ μ-a.e. Moreover, $\varrho(af) = a\varrho(f)$ and $\varrho(f+g) \leq \varrho(f) + \varrho(g)$.
- A.2 If $0 \leq g \leq f$ μ-a.e. then $\varrho(g) \leq \varrho(f)$.
- A.3 If $0 \leq f_n \uparrow f$ μ-a.e. then $\varrho(f_n) \uparrow \varrho(f)$.
- A.4 If $\mu(E) < \infty$ then $\varrho(\chi_E) < \infty$, where χ_E denotes the characteristic function of E.
- A.5 If $\mu(E) < \infty$ then $\int_E f \, d\mu \leq C_E \varrho(f)$ for some constant C_E, $0 < C_E < \infty$, depending on E and ϱ but independent of f.

The collection $\mathbf{X}(R, \mu) = \mathbf{X}_\varrho(R, \mu)$ of all functions $f \in \mathfrak{M}(R, \mu)$ for which $\varrho(|f|) < \infty$ is called a *Banach function space*. For each $f \in \mathbf{X}(R, \mu)$, define

$$\|f\|_{\mathbf{X}(R,\mu)} = \varrho\left(|f|\right).$$

Let us note that different Banach function norms can define the same Banach function space. This ambiguity leads us to express, when possible, statements rather in terms of function spaces than underlying function norms.

Definition 4 (rearrangement-invariant function space)**.** Suppose $\varrho : \mathfrak{M}_+(R, \mu) \to [0, \infty]$ fulfills, in addition to properties from definition of a Banach function space, the following condition:

- A.6 $\varrho(f) = \varrho(g)$ for every pair of equimeasurable functions f and g in $\mathfrak{M}_+(R, \mu)$.

Then ϱ is called a *rearrangement-invariant Banach function norm.*

If ϱ satisfies even A.6, then the collection $\mathbf{X}(R, \mu) = \mathbf{X}_\varrho(R, \mu)$ of all functions $f \in \mathfrak{M}(R, \mu)$ for which $\varrho(|f|) < \infty$ is called a *rearrangement-invariant Banach function space.* As in the case of Banach function spaces, for each $f \in \mathbf{X}(R, \mu)$, define

$$\|f\|_{\mathbf{X}(R,\mu)} = \varrho\left(|f|\right).$$

Rearrangement-invariant function spaces reflect information about function spaces which is somehow independent of the underlying measure space.

Theorem 5 (Luxemburg representation theorem)**.** *Let ϱ be a rearrangement-invariant function norm over a σ-finite, non-atomic measure space (R, μ). Then there is a (not necessarily unique) rearrangement-invariant function norm $\bar{\varrho}$ over $\mathbb{R}^+$ such that*

$$\varrho(f) = \bar{\varrho}(f^*), \text{ for all } f \in \mathfrak{M}_0^+(R, \mu).$$

Consequently, rearrangement-invariant function spaces over such measure spaces can be fully determined by the representation function space.

Moreover, the representation norm is unique in case when the underlying measure space is infinite. In the case of finite measure spaces, the representation norm is unique if we consider rearrangement-invariant function norms on $[0, \mu(R))$.

From now on we shall solely work in the following situation. The measure space R will always be a subset of the Euclidean space $\mathbb{R}^n$ and μ will be the classical n-dimensional Lebesgue measure. In this situation, we will use the symbol Ω in place of R, where $\Omega \subset \mathbb{R}^n$.

Since we will always consider the Lebesgue measure, we will stop referring to it explicitly and we will denote the Lebesgue measure of a given set by $|K|$, for all $K \subset \mathbb{R}^n$ measurable.

Moreover, we shall for simplicity assume that $|\Omega| = 1$. All the results remain valid also in the cases when $|\Omega|$ is any finite number with trivial changes only.

The idea of abstracting from complicated measure spaces and working with a considerably simpler measure space $\mathbb{R}^+$ (equipped with the Lebesgue measure) is very useful. The application to Sobolev embeddings is of particular interest to us. In this approach, the differential structure of underlying spaces is represented by the isoperimetric function. Again, relevant information on a complex matter is extracted to setting of the well-known measure space $\mathbb{R}^+$. In the following Section 1.7 we will examine that this technique is applicable even to such complicated settings as Carnot-Carathéodory spaces.

A basic example of a function norm is the standard *Lebesgue norm* $\|\cdot\|_{L^p(0,1)}$, for $p \in [1, \infty]$, upon which the Lebesgue spaces $L^p(\Omega)$ are built.

In many situations, the class of Lebesgue spaces is not fine enough. One example of such case is the problem of optimal embeddings described in Section 1.3. Such problems call for finer scales of function spaces.

The most common extension of the Lebesgue spaces is the class of *Lorentz spaces.* Assume that $1 \leq p, q \leq \infty$. We define the functional $\|\cdot\|_{L^{p,q}(0,1)}$ as

$$\|f\|_{L^{p,q}(0,1)} = \left\| s^{\frac{1}{p} - \frac{1}{q}} f^*(s) \right\|_{L^q(0,1)} \tag{1.1}$$

for $f \in \mathfrak{M}_+(0, 1)$. Using the elementary maximal operator

$$f^{**}(s) = \frac{1}{s} \int_0^s f^*(t)\, dt, \quad f \in \mathfrak{M}_+(0, 1),\ s \in (0, 1),$$

we define $\|\cdot\|_{L^{(p,q)}(0,1)}$ as

$$\|f\|_{L^{(p,q)}(0,1)} = \left\| s^{\frac{1}{p}-\frac{1}{q}} f^{**}(s) \right\|_{L^q(0,1)}, \text{ where } f \in \mathfrak{M}_+(0,1).$$

One can show that

$$L^{p,q}(\Omega) = L^{(p,q)}(\Omega) \quad \text{if } 1 < p \leq \infty, \tag{1.2}$$

with equivalent norms. If one of the conditions

$$\begin{cases} 1 < p < \infty, \; 1 \leq q \leq \infty, \\ p = q = 1, \\ p = q = \infty, \end{cases} \tag{1.3}$$

is satisfied, then $\|\cdot\|_{L^{p,q}(0,1)}$ is equivalent to a rearrangement-invariant function norm. The corresponding rearrangement-invariant space $L^{p,q}(\Omega)$ is called a *Lorentz space.*

Let us recall that $L^{p,p}(\Omega) = L^p(\Omega)$ for every $p \in [1,\infty]$.

At a few occasions, we shall need also the so-called *Lorentz-Zygmund space* (see [3], [17]). Assume now that $1 \leq p, q \leq \infty$, and a third parameter $\alpha \in \mathbb{R}$ is called into play. We define the functionals $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ and $\|\cdot\|_{L^{(p,q;\alpha)}(0,1)}$ as

$$\begin{cases} \|f\|_{L^{p,q;\alpha}(0,1)} = \left\| s^{\frac{1}{p}-\frac{1}{q}} \log^{\alpha}\left(\frac{2}{s}\right) f^{*}(s) \right\|_{L^q(0,1)}, \\ \|f\|_{L^{(p,q;\alpha)}(0,1)} = \left\| s^{\frac{1}{p}-\frac{1}{q}} \log^{\alpha}\left(\frac{2}{s}\right) f^{**}(s) \right\|_{L^q(0,1)}, \end{cases} \tag{1.4}$$

respectively, for $f \in \mathfrak{M}_+(0,1)$. If one of the following conditions

$$\begin{cases} 1 < p < \infty, \; 1 \leq q \leq \infty, \; \alpha \in \mathbb{R}; \\ p = 1, \; q = 1, \; \alpha \geq 0; \\ p = \infty, \; q = \infty, \; \alpha \leq 0; \\ p = \infty, \; 1 \leq q < \infty, \; \alpha + \frac{1}{q} < 0, \end{cases} \tag{1.5}$$

is satisfied, then $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ is equivalent to a rearrangement-invariant function norm, called a *Lorentz-Zygmund norm* (it is a consequence of more general theorem from[42]). The corresponding rearrangement-invariant space $L^{p,q;\alpha}(\Omega)$ is a *Lorentz-Zygmund space.*

It is shown in [42, Theorem 3.16] that

$$L^{(p,q;\alpha)}(\Omega) = \begin{cases} L^{p,q;\alpha}(\Omega) & \text{if } 1 < p \leq \infty; \\ L^{1,1;\alpha+1}(\Omega) & \text{if } p = q = 1, \; \alpha > -1, \end{cases} \tag{1.6}$$

and

$$L^p(\Omega) \hookrightarrow L^{(1,q)}(\Omega) \quad \text{for every } 1 < p \leq \infty, \; 1 \leq q \leq \infty.$$

A generalization of the Lebesgue spaces in a different direction is provided by the Orlicz spaces. Let $A : [0, \infty) \to [0, \infty]$ be a Young function, namely a convex (non trivial), left-continuous function vanishing at 0. Any such function takes the form

$$A(t) = \int_0^t a(\tau)d\tau \qquad \text{for } t \geq 0, \tag{1.7}$$

for some non-decreasing, left-continuous function $a : [0, \infty) \to [0, \infty]$ which is neither identically equal to 0, nor to ∞. The Orlicz space $L^A(\Omega)$ is the rearrangement-invariant space associated with the *Luxemburg function norm* defined as

$$\|f\|_{L^A(0,1)} = \inf \left\{ \lambda > 0 : \int_0^1 A\left(\frac{f(s)}{\lambda}\right) ds \leq 1 \right\} \tag{1.8}$$

for $f \in \mathfrak{M}_+(0, 1)$. In particular, $L^A(\Omega) = L^p(\Omega)$ if $A(t) = t^p$ for some $p \in [1, \infty)$, and $L^A(\Omega) = L^\infty(\Omega)$ if $A(t) = \infty \cdot \chi_{(1,\infty)}(t)$.

We denote by $L^p \log^\alpha L(\Omega)$ the Orlicz space associated with a Young function equivalent to $t^p(\log t)^\alpha$ near infinity, where either $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \geq 0$. The notation $\exp L^\beta(\Omega)$ will be used for the Orlicz space built upon a Young function equivalent to e^{t^β} near infinity, where $\beta > 0$. It is of interest to note that $L^p \log^\alpha L(\Omega) = L^{p,p;\frac{\alpha}{p}}(\Omega)$, while $\exp L^\beta(\Omega) = L^{\infty,\infty;-\frac{1}{\beta}}(\Omega)$ Also, $\exp\exp L^\beta(\Omega)$ stands for the Orlicz space associated with a Young function equivalent to $e^{e^{t^\beta}}$ near infinity.

Having established some classes of function spaces we can turn our attention to relationships between them.

Definition 6 (embedding of function spaces)**.** Let $\mathbf{X}$ and $\mathbf{Y}$ be two Banach function spaces. We say that $\mathbf{X}$ is *continuously embedded into* $\mathbf{Y}$ if there exists $C > 0$ such that

$$\|f\|_{\mathbf{X}} \leq C \|f\|_{\mathbf{Y}},$$

for all $f \in \mathfrak{M}$. The fact that $\mathbf{X}$ is continuously embedded into $\mathbf{Y}$ is denoted by $\mathbf{X} \hookrightarrow \mathbf{Y}$.

Embeddings of function spaces have been extensively studied as can be documented by the following short overview of basic embeddings in introduced classes of function spaces. Of course, this overview is by no means exhaustive.

Embedding relations of Lebesgue spaces on domain of finite measure are quite simple. If $1 \leq q \leq r \leq \infty$ then $L^r(\Omega) \hookrightarrow L^q(\Omega)$ with equality if and only if $q = r$.

If the first parameter is fixed, then the Lorentz spaces are nested. More precisely, we have

$$L^{p,q}(\Omega) \hookrightarrow L^{p,r}(\Omega) \tag{1.9}$$

whenever $0 < p \leq \infty$ and $0 < q \leq r \leq \infty$.

Let $0 < p_i, q_i \leq \infty$, $i = 0, 1$, and let $\alpha, \beta \in \mathbb{R}$. Assume that $p_0 < p_1$. Then the embedding $L_{p_1,q_1;\alpha} \hookrightarrow L_{p_0,q_0;\beta}$ holds. When $p_0 = p_1$, a finer interplay between the remaining parameters comes into picture, but we will avoid these details since the Lorentz-Zygmund spaces are not in the focus of this thesis.

There is a lot to be said about embeddings of general rearrangement-invariant function spaces as well. It is known that for any Banach function spaces $\mathbf{X}$ and $\mathbf{Y}$, if $\mathbf{X} \subset \mathbf{Y}$ then also $\mathbf{X} \hookrightarrow \mathbf{Y}$. Moreover, if the underlying measure space is finite, it is known that any rearrangement-invariant space lies in a sense in between spaces L^1 and L^∞.

Theorem 7 ($L^\infty \hookrightarrow \mathbf{X} \hookrightarrow L^1$)**.** *Let $\mathbf{X}$ be a rearrangement-invariant space over a finite measure spaces (R, μ). Then*

$$L^\infty(R, \mu) \hookrightarrow \mathbf{X} \hookrightarrow L^1(R, \mu).$$

Another basic relation between two rearrangement-invariant function spaces is being associated.

Definition 8 (associated function norm)**.** Suppose that $\varrho : \mathfrak{M}_+(R, \mu) \to [0, \infty]$. Let us call the functional defined on $\mathfrak{M}_+(R, \mu)$ by

$$\varrho'(g) = \sup \left\{ \int_R fg : f \in \mathfrak{M}_+(R, \mu), \varrho(f) \leq 1 \right\}, \ g \in \mathfrak{M}_+(R, \mu),$$

the *associated norm* to ϱ.

The associated norm ϱ' is a Banach function norm if ϱ is a Banach function norm.

It turns out that associated spaces often align with representations dual space. That is apparent from the example of Lebesgue space as $(L^p(\Omega))' = L^{p'}(\Omega)$, where $p' = 1 - \frac{1}{p}$ is the same exponent as of the dual spaces. Very similar is the situation in the case of Lorentz spaces: if $1 < p < \infty, 1 \leq q \leq \infty$ or $p = q = 1$ or $p = q = \infty$ then $(L^{p,q}(\Omega))' = L^{p',q'}(\Omega)$.

Assume that one of the conditions in (1.5) is satisfied. Then the associate space $(L^{p,q;\alpha})'(\Omega)$ of the Lorentz–Zygmund space $L^{p,q;\alpha}(\Omega)$ satisfies (up to equivalent norms)

$$\left(L^{p,q;\alpha}\right)'(\Omega) = \begin{cases} L^{p',q';-\alpha}(\Omega) & \text{if} \quad 1 < p < \infty, \ 1 \leq q \leq \infty, \ \alpha \in \mathbb{R}; \\ L^{\infty,\infty;-\alpha}(\Omega) & \text{if} \quad p = 1, \ q = 1, \ \alpha \geq 0; \\ L^{1,1;-\alpha}(\Omega) & \text{if} \quad p = \infty, \ q = \infty, \ \alpha \leq 0; \\ L^{(1,q';-\alpha-1)}(\Omega) & \text{if} \quad p = \infty, \ 1 \leq q < \infty, \ \alpha + \frac{1}{q} < 0. \end{cases} \tag{1.10}$$

See [42, Theorems 6.11 and 6.12] for more details.

The associated norm ϱ' is a Banach function norm if ϱ is a Banach function norm. The following theorem shows that the operation of taking the associate norm creates pairs of Banach function norms, unlike representations of dual spaces where a more complicated structure can occur.

Theorem 9 (G.G. Lorentz and W.A.J. Luxemburg). *Every Banach function space* $\mathbf{X}$ *coincides with its second associate space* $\mathbf{X}''$.

Associated spaces allow for reformulation of the Hölder inequality in terms of general rearrangement-invariant function spaces.

Theorem 10 (Hölder inequality). *Let* $\mathbf{X}(\Omega)$ *be a rearrangement-invariant function space with associate space* $\mathbf{X}'(\Omega)$. *If* $f \in \mathbf{X}(\Omega)$ *and* $g \in \mathbf{X}'(\Omega)$, *then* fg *is integrable and*

$$\int_\Omega |fg| \, d\mu \leq \int_0^\infty f^*(s) g^*(s) \, ds \leq \|f\|_{\mathbf{X}(\Omega)} \|g\|_{\mathbf{X}'(\Omega)} .$$

This yields a new representations of ϱ and ϱ' given by

$$\varrho'(g) = \sup \left\{ \int_0^\infty f^*(s) g^*(s) \, ds : \varrho(f) \leq 1 \right\}, \text{ for } g \in \mathfrak{M}_0^+$$

and

$$\varrho(g) = \sup \left\{ \int_0^\infty f^*(s) g^*(s) \, ds : \varrho'(f) \leq 1 \right\}, \text{ for } g \in \mathfrak{M}_0^+.$$

Sometimes, the most convenient way to define a function norm is through its associated norm. This is the case in optimal domain and range function spaces used in theory of higher-order Sobolev embeddings which will be outlined in Section 1.3.

The fact that the non-increasing rearrangement captures information relevant to rearrangement-invariant norms is shown in the Hardy-Littlewood-Pólya principle.

Theorem 11 (Hardy-Littlewood-Pólya principle). *Suppose that* f *and* g *belong to* $\mathfrak{M}_0(\Omega)$. *Let* $\mathbf{X}$ *be any rearrangement-invariant function space on* Ω. *If*

$$\int_0^t f^*(s) \, ds \leq \int_0^t g^*(s) \, ds, \text{ for all } t > 0$$

then

$$\|f\|_{\mathbf{X}(\Omega)} \leq \|g\|_{\mathbf{X}(\Omega)} .$$

1.3 Sobolev embeddings

Sobolev embeddings, or Sobolev inequalities, constitute a very important part of modern functional analysis. This field has been the subject of intensive study for several decades and a vast amount of literature on this subject is available. We shall name only few items: [48], [49], [35], [40], [38], [1], [5], [29], [12]. Our aim is to introduce a Sobolev-type space based on the class of Banach function spaces. The straightforward definition is as follows.

Given a Banach function space $\mathbf{X}(\Omega)$ and a positive integer $m \in \mathbb{N}$, the m-th order *Sobolev type space* built upon $\mathbf{X}(\Omega)$ is the normed linear space $\mathcal{V}^m\mathbf{X}(\Omega)$ of all functions on Ω whose m-th order weak derivatives belong to $\mathbf{X}(\Omega)$, equipped with a natural norm induced by $\mathbf{X}(\Omega)$.

Unfortunately such spaces are not necessarily normed (only seminormed) and moreover they might not be complete as the norm is affected only by the highest-order derivative. Consequently, all constant functions have the same zero norm. A norm which reflects derivatives of lower orders needs to be considered in order to ensure normability and completeness of such function space.

We are interested in restricting only the m-th order derivative. Therefore, we would like to use as weak condition on lower-order derivatives as possible. In the world of rearrangement-invariant function spaces, the largest space on Ω (of finite measure) is $L^1(\Omega)$. Consequently, a natural choice is to require that $\nabla^m f \in \mathbf{X}(\Omega)$ and $\nabla^i f \in L^1(\Omega)$, for $i = 0, 1, \ldots, m-1$ where $\nabla^i f$ denotes the vector of all i-th order weak derivatives of f, in particular $\nabla^0 f = f$. Note that in general it can happen that $\nabla^1 f \in L^1(\Omega)$ but $f \notin L^1(\Omega)$. However, if we adopt certain conditions on the isoperimetric function of Ω (see Section 1.4), then $\nabla^1 f \in L^1(\Omega)$ implies $f \in L^1(\Omega)$.

Definition 12 (Sobolev space)**.** Given a Banach function space $\mathbf{X}(\Omega)$ of measurable functions on Ω, and a positive integer $m \in \mathbb{N}$, the m-th order *Sobolev space* built upon $\mathbf{X}(\Omega)$ is the normed linear space $W^m\mathbf{X}(\Omega)$ of all functions on Ω whose derivatives up to order m belong to $\mathbf{X}(\Omega)$, equipped with norm

$$\|f\|_{W^m\mathbf{X}(\Omega)} = \sum_{i=0}^{m} \left\|\nabla^i f\right\|_{\mathbf{X}(\Omega)} .$$

Assume that the isoperimetric function of Ω satisfies

$$I_\Omega(s) \geq Cs \text{ for } s \in \left[0, \tfrac{1}{2}\right],$$

for some $C > 0$. Then we define the m-th order Sobolev space $V^m\mathbf{X}(\Omega)$ as

$$V^m\mathbf{X}(\Omega) = \{u : u \text{ is } m\text{-times weakly differentiable in } \Omega, \text{ and } |\nabla^m \mathbf{X}| \in \mathbf{X}(\Omega)\}.$$

The space $V^m\mathbf{X}(\Omega)$ is equipped with the norm

$$\|f\|_{V^m\mathbf{X}(\Omega)} = \sum_{i=0}^{m-1} \left\|\nabla^i f\right\|_{L^1(\Omega)} + \|\nabla^m f\|_{\mathbf{X}(\Omega)}, \text{ for all } f \in \mathfrak{M}_+(\Omega).$$

We say that the space $V^m\mathbf{X}(\Omega)$ or $W^m\mathbf{X}(\Omega)$ is a *Sobolev space built upon* $\mathbf{X}(\Omega)$.

Let us note that the notion of Sobolev space is fundamentally connected to the differential operator of gradient. In Section 1.7, we will generalize the notion of Sobolev spaces to a custom differential operator corresponding to the intrinsic structure of Carnot-Carathéodory spaces.

It is a natural question to ask if properties of a function can be determined by its derivative. In context of function spaces, this problem is usually formulated as a so-called Sobolev embedding or Sobolev inequality.

In its most classical form, the *Sobolev inequality* asserts that for Ω with a Lipschitz boundary, given $1 < p < n$ and setting $p^* = \frac{np}{n-p}$, there exists $C > 0$ such that

$$\left(\int_\Omega |u(x)|^{p^*}\,dx\right)^{\frac{1}{p^*}} \leq C\left(\int_\Omega |(\nabla u)(x)|^p + |u(x)|^p\,dx\right)^{\frac{1}{p}} \quad \text{for all } W^1L^p(\Omega).$$

We can restate this result in the form of a *Sobolev embedding*, namely,

$$W^1L^p(\Omega) \hookrightarrow L^{p^*}(\Omega), \qquad 1 < p < n. \tag{1.11}$$

Sobolev embeddings can be easily generalized to the setting of Banach function spaces.

Definition 13 (Sobolev embedding)**.** Suppose that $\mathbf{X}(\Omega)$ and $\mathbf{Y}(\Omega)$ are two rearrangement-invariant function spaces. Fix $m \geq 1$. A Sobolev embedding amounts to the boundedness of the identity operator from the Sobolev space $V^m\mathbf{X}(\Omega)$ into another function space $\mathbf{Y}(\Omega)$, in other words

$$V^m\mathbf{X}(\Omega) \to \mathbf{Y}(\Omega). \tag{1.12}$$

When $m = 1$, we refer to (1.12) as a first-order embedding; otherwise, we call it a higher-order embedding.

We recognize $\mathbf{X}(\Omega)$ as the *domain space* of the embedding (1.12), and the space $\mathbf{Y}(\Omega)$ on the right as its *range space*.

Let us consider (1.11) in the previously established situation when Ω has Lipschitz boundary. One might be wondering if there is a function space smaller than L^{p^*}, such that (1.11) still holds with it as a range space. Or, if there is a larger domain space than L^p which can be inserted into (1.11) without compromising its validity.

The answer is dependent on an environment within which it is investigated. The embedding (1.11) can not be improved within the environment of Lebesgue spaces. If we replace the domain space $L^p(\Omega)$ in (1.11) by a larger Lebesgue space, say, $L^q(\Omega)$ with $q < p$, then the resulting embedding

$$W^1L^q(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

can no longer be true. Likewise, if we replace the range space $L^{p^*}(\Omega)$ by a smaller Lebesgue space, say $L^r(\Omega)$, $r > p^*$, then, again, the resulting embedding

$$W^1L^p(\Omega) \hookrightarrow L^r(\Omega)$$

does not hold any more.

The embedding (1.11) cannot be effectively improved in the environment of Lebesgue spaces, hence it can be considered optimal in a sense. Therefore, we need to use a finer class of function spaces to get a finer result.

Lebesgue scale is too rough to describe all the interesting details about embeddings. This is apparent when considering the example of the so-called limiting case of the embedding (1.11), corresponding to the case $p = n$. When we let p tend to n from the left, then, of course, p^* tends to ∞. However, the limiting embedding

$$W^1L^n(\Omega) \hookrightarrow L^\infty(\Omega)$$

is unfortunately not true. There are unbounded functions in $W^1L^n(\Omega)$. In the limiting case, the Lebesgue spaces environment yields only the embedding

$$W^1L^n(\Omega) \hookrightarrow L^q(\Omega) \qquad \text{for every } q < \infty. \tag{1.13}$$

Again, this information is optimal within the environment of Lebesgue spaces, where no improvement is available. This result does not provide any definite range function space. Such a space can be obtained, but not among Lebesgue spaces. We need a refinement of the Lebesgue scale.

The Lorentz spaces (see Section 1.2), allows for the following refinement of (1.11):

$$W^1L^p(\Omega) \hookrightarrow L^{p^*,p}(\Omega), \qquad 1 < p < n. \tag{1.14}$$

Note that, thanks to embedding (1.9) and the obvious inequality $p < p^*$, the range space in (1.14) is smaller than the range space in (1.11). The embedding (1.14) is introduced in [43], but it follows from results in [41] and [27] as well.

1.4 Isoperimetric inequalities

Isoperimetric problem is very old, it even plays its role in the myth of Queen Dido. When Dido fled to North Africa, she was supposedly offered by a local ruler a piece of land which she was able to encompass with hide of a bull. Dido then cut the hide into long stripes and marked a vast circle with it. The city of Carthage was then found in this area.

In its classical formulation, as outlined above, isoperimetric problem sets the task to find a set with the largest possible area having fixed perimeter. In plane, the solution to this problem is naturally a circle. In higher dimensions, the solution to isoperimetric problem is a ball.

In order to solve the isoperimetric problem in more general setting, we need a more general definition of the perimeter. We will use the definition of perimeter based on concept of variation of a function.

Definition 14 (variation)**.** Let $u \in L^1_{\text{loc}}(\Omega)$, the *variation* of u with respect to Ω is defined as

$$\operatorname{Var}(u, \Omega) = \sup_{\phi \in \mathcal{F}_\Omega} \int_\Omega u(x) \operatorname{div} \phi(x)\, dx,$$

where

$$\mathcal{F}_\Omega = \left\{ \phi = (\phi_1, \phi_2, \dots, \phi_n) \in \mathcal{C}_0^1(\Omega, \mathbb{R}^n) : \sup_{x \in \Omega} \left(\sum_{j=1}^n |\phi_j(x)|^2 \right)^{\frac{1}{2}} \le 1 \right\}.$$

Definition 15 (perimeter)**.** If $E \subset \mathbb{R}^n$ is measurable, then the *perimeter* of E relative to Ω is defined by

$$P(E, \Omega) = \operatorname{Var}(\chi_E, \Omega).$$

We return back to the mythological motivation. If we consider founding a city on the coast, then the coast itself can be used as a natural boundary, giving a different solution to the isoperimetric problem. Obviously, the "shape" of the domain affects the solution of the isoperimetric problem. The purpose of the *isoperimetric function* is to extract this information from a given domain.

Definition 16 (isoperimetric function)**.** Suppose that $\Omega \subset \mathbb{R}^n$ is open such that $|\Omega| = 1$. Then we define the *isoperimetric function* of Ω by formula

$$I_\Omega(s) = \begin{cases} \inf \left\{ P(E, \Omega) : E \subset \Omega, s \le |E| \le \frac{1}{2} \right\} & \text{for } s \in \left[0, \frac{1}{2}\right], \\ I_\Omega(1-s) \text{ for } s \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Let us note that in connection to Sobolev embeddings, the key feature of the isoperimetric function is its asymptotic behavior near zero. Therefore, we don't need to work with the isoperimetric function itself, but any function which is estimating its behavior near zero will do. More precisely, considering $I : [0, 1] \to \mathbb{R}$, we assume that there is a $c > 0$ such that

$$I_\Omega(s) \ge c I(cs) \text{ for } s \in \left[0, \tfrac{1}{2}\right]. \tag{1.15}$$

In many situations, the asymptotic behavior of the isoperimetric function is known. The assumption that Ω belongs to some classical class of domains usually implies certain behavior of the isoperimetric function. Let us recall the definition of John domains, as it is most widely used besides the class of domains having Lipschitz boundary.

Definition 17 (John domain)**.** Recall that a bounded open set Ω in $\mathbb{R}^n$ is called a *John domain* if there exist a constant $c \in (0, 1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\omega : [0, l] \to \Omega$, parameterized by arclength, such that $\omega(0) = x$, $\omega(l) = x_0$, and $\operatorname{dist}(\omega(r), \partial\Omega) \ge cr$ for $r \in [0, l]$.

Both domains with Lipschitz boundary and John domains have isoperimetric function which can be estimated in the sense of (1.15) by the function

$$I(s) = s^{1-\frac{1}{n}} \text{ for } s \in [0, 1].$$

The next natural step is to actually define specific classes of domains by their isoperimetric behavior. This idea leads us to the concept of the so-called Maz'ya classes.

Definition 18 (Maz'ya classes)**.** Given $\alpha \in \left[1-\frac{1}{n}, 1\right]$, we denote by $\mathcal{I}_\alpha$ the *Maz'ya class* of all Euclidean domains Ω satisfying (1.15) with $I(s) = s^\alpha$ for $s \in [0,1]$.

Later, we will work with a different class of domains, the so-called X-PS domains. From our point of view, the most important property of such classes is that we know their X-isoperimetric function. More details can be found in Section 1.6.

1.5 Connection between isoperimetric inequalities and Sobolev embeddings

Sobolev inequalities and isoperimetric inequalities were initially studied separately (Sobolev [48, 49], Gagliardo [21] and Nirenberg [40] on the Sobolev inequalities side and De Giorgi [16] in the realm of isoperimetric inequalities). Their intimate relation was discovered by Maz'ya in [34, 35]. Independently, Federer and Fleming [18] also exploited De Giorgi's isoperimetric inequality to exhibit the best constant in the special case of the Sobolev inequality for functions such that $\nabla f \in L^n(\Omega)$.

These results became a basis for an extensive research effort yielding a vast literature, to name just a few: [2, 5, 10, 38, 51].

The approach to Sobolev embeddings based on an isoperimetric inequality has a significant advantage consisting in the possibility to apply achieved results to a variety of rearrangement-invariant function spaces, and also in the fact that it typically yields sharp results.

On the other hand, it is not so easily applicable to higher-order embeddings. The customary techniques that are crucial in the derivation of first-order Sobolev inequalities from isoperimetric inequalities, such as symmetrization, or just truncation, cannot be adapted to the proof of higher-order Sobolev inequalities. This can be overcome by subsequent applications of an optimal first-order results as was discovered in [12].

Let us give a short example illustrating how the magnitude of derivatives can be connected to the isoperimetric function in order to demonstrate the basis of their relation.

Definition 19 (level set)**.** Let $u \in \mathfrak{M}_0(\Omega)$ have bounded variation and $t \in \mathbb{R}$. We define the *level set* of u at the value t by

$$E_{u,\Omega}(t) = \{x \in \Omega : u(x) > t\}.$$

The core of the connection between the isoperimetric inequality and the Sobolev inequalities lies in the co-area formula. The idea is to express the integral of the norm of a derivative as an integral of perimeters of level sets. Consider a $L^1(\Omega)$ norm of ∇u. Then we have

$$\int_\Omega |\nabla u| = \mathrm{Var}(u;\Omega) = \int_{-\infty}^{\infty} P(E_{u,\Omega}(t),\Omega)\, dt.$$

Now, we estimate the integrand on the right side of the inequality above by using the isoperimetric function, namely

$$P(E_u(t) \geq I_\Omega(t).$$

Consequently, we get

$$\int_\Omega |\nabla u| \geq \int_{-\infty}^{\infty} I_\Omega(|E_{u,\Omega}(t)|)\, dt.$$

The actual application of this principle is more delicate, but the rough idea remains the same. This argument is a crucial step in the proof of the following Pólya-Szegö principle, which yields a connection between the derivative of a function and its rearrangement.

Theorem 20 (Pólya-Szegö principle)**.** *Assume $u \in V^1\mathbf{X}(\Omega)$ and let $\mathbf{X}$ be a rearrangement-invariant function space. Then u^* is locally absolutely continuous and*

$$C \left\| \left(-\frac{du^*}{ds} I_{u,\Omega}(s) \right) \right\|_{\mathbf{X}(\Omega)} \leq \|\nabla u\|_{\mathbf{X}(\Omega)}$$

with $C > 0$ independent of $u \in V^1\mathbf{X}(\Omega)$.

The principal result is the *reduction principle* - a sufficient condition for a higher-order Sobolev embedding.

Theorem 21 (reduction principle)**.** *Assume that there is a function $I(s)$ satisfying both* (1.15) *and*

$$\inf_{t>0} \frac{I(t)}{t} > 0. \tag{1.16}$$

Let $m \in \mathbb{N}$ and let $\mathbf{X}(0,1)$ and $\mathbf{Y}(0,1)$ be rearrangement-invariant function spaces. If there exists a constant C such that

$$\left\| \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds \right\|_{\mathbf{Y}(0,1)} \leq C\, \|f\|_{\mathbf{X}(0,1)} \tag{1.17}$$

for every nonnegative $f \in \mathbf{X}(0,1)$, then

$$V^m\mathbf{X}(\Omega) \to \mathbf{Y}(\Omega). \tag{1.18}$$

Next topic we are interested in is the compactness of embeddings. Let us start by recalling its definition.

Definition 22 (compact embedding)**.** Let $\mathbf{X}$ and $\mathbf{Y}$ be two normed vector spaces and suppose that $\mathbf{X} \subset \mathbf{Y}$. We say that $\mathbf{X}$ is compactly embedded in $\mathbf{Y}$, and write $\mathbf{X} \hookrightarrow\hookrightarrow \mathbf{Y}$, if $\mathbf{X} \hookrightarrow \mathbf{Y}$ and the identity operator from $\mathbf{X}$ into $\mathbf{Y}$ is compact: every sequence in such a bounded set has a subsequence that is Cauchy in the norm $\|.\|_{\mathbf{Y}}$. We similarly write $T\colon \mathbf{X} \to\to \mathbf{Y}$ to denote that an operator T is compact from $\mathbf{X}$ to $\mathbf{Y}$.

The properties of Banach function spaces imply that the continuous embedding is ensured by the assumption $\mathbf{X} \subset \mathbf{Y}$.

As a reference let us recall the characterization of compact embeddings between Sobolev spaces built upon rearrangement-invariant function spaces shown in [45]. This provides a context for results in Section 1.7.

Theorem 23. *Assume that* $\Omega \subset \mathbb{R}^n$*,* $n \geq 2$*,* $I : (0,1] \to (0,\infty)$ *such that*(1.15) *and* (1.16) *holds. Let* $m \in \mathbb{N}$ *and let* $\|.\|_{\mathbf{X}(0,1)}$ *and* $\|.\|_{\mathbf{Y}(0,1)}$ *be rearrangement-invariant norms. Then*

$$H_I^m : \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1)$$

implies

$$V^m\mathbf{X}(\Omega) \hookrightarrow\hookrightarrow \mathbf{Y}(\Omega).$$

This general principle then can be easily applied to examples of rearrangement-invariant function norms to obtain characterizations of particular compact embeddings. This will be demonstrated in Section 1.7.

1.6 Carnot-Carathéodory spaces

The motivation behind the Carnot-Carathéodory spaces (constrained dynamics spaces) is probably best illustrated with the example of parallel parking. Let us point out that while it is very easy to move a car in the direction in which it is currently heading, it is impossible to move it directly in the direction sideways. Of course, the steering system allows to change the heading direction, but only while moving. This practical problem is so iconic it is even taught in the driving school.

We would like to develop a mathematical theory to describe this behavior. A good example of application of this theory is [30].

We will view a car as a system specified by the following properties: the position in the parking lot, rotation along the top-down axis and rotation of the steering wheel. In principle, there are just two basic actions to control a vehicle: turning the steering wheel and driving forward or backward.

While the first action simply changes the rotation of the steering wheel, the effect of driving depends on other factors. If the steering wheel is in the straight position, only the position (and not the rotation) of the vehicle changes according to vehicle's direction. But in general, depending on the steering system state, the rotation of the vehicle is changing while moving.

We adopt the following mathematical model to view possible vehicle configurations. We will use four parameters, namely $(x, y, r, s) \in \mathbb{R}^4$. Here the values of x and y are the coordinates in the plane representing a parking lot, r is the representation of the rotation of the vehicle along the z-axis and, finally, s represents the configuration of the vehicle's steering system.

The basic action of steering is then represented by adding a certain multiplication of the action vector $a_s = (0, 0, 0, 1)$ to the current state of the system. The second basic action has a more complicated representation as its effect depends on the current state of system. Let us represent the second action as

$$a_d(x, t, r, s) = (\cos(r), \sin(r), s, 0),\ (x, y, r, s) \in \mathbb{R}^n.$$

Adding a multiple of such a vector to a given state changes the spatial coordinates according to the current direction and, moreover, it affects the rotation of the car according to the state of the steering system (the steering wheel rotation). We represent both actions as vector fields acting on $\mathbb{R}^4$, hence they can be written in form

$$a_d(f) = \cos(r)\partial_x f + \sin(r)\partial_y f + s\partial_r f$$

and

$$a_s = \partial_s f,$$

where $f \in \mathcal{C}^\infty(\mathbb{R}^n)$.

To reflect possible movements of the car (which can happen only as a result of a combination of given two basic actions), we allow only movements along the so-called horizontal paths. A piecewise $\mathcal{C}^1$ curve $\gamma : [0, T] \to \mathbb{R}^4, T > 0$, is called a horizontal path (with respect to a_d and a_s) if $\gamma'(t) \in \text{span}\,\{a_d(\gamma(t)), a_s(\gamma(t))\}$ whenever $\gamma'(t)$ exists, $t \in [0, T]$. Please note that the linear span of the vectors $a_d(x)$ and $a_s(x)$ depends on x, in other words the possible directions of the movement are (in general) different at each point.

Since only the movement along horizontal paths is allowed in our system, it is natural to measure the distance based only on such paths. This leads to the definition of a structure based on the vector fields a_d and a_s which in the most of interesting cases is a metric space. Moreover this structure allows for redefinition of many other notions such as perimeter, dimension or gradient to reflect the custom structure defined by vector fields a_s and a_d.

We will establish the above-illustrated concept in more rigorous settings. The role of the vector fields a_d and a_s will be played by a set of vector fields $X = \{X_1, X_2, \ldots, X_m\}$ for some $m \in \mathbb{N}$. We assume that X_i, $i = 1, \ldots, m$ is of the form

$$X_i = \sum_{j=1}^{m} b_{i,j}(x)\partial_{x_j},$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $b_{i,j} \in \mathcal{C}^\infty(\mathbb{R}^n)$ (with respect to the classical Euclidean topology). Naturally the following notions will be dependent of the choice of vector fields in X.

Definition 24 (horizontal curve)**.** A piecewise $\mathcal{C}^1$-curve $\gamma : [0,T] \to \mathbb{R}^n$, $T > 0$, is called *horizontal* (more precisely X-horizontal) if whenever $\gamma'(t)$ exists then

$$\gamma'(t) = \sum_{j=1}^{m} c_j(t)\mathbf{X}_j(\gamma(t)),$$

where $c_j : (0,T) \to \mathbb{R}$ are measurable and satisfying $\sum_{j=1}^{m} c_j^2(t) \leq 1$ for $0 \leq t \leq T$. The horizontal length of γ is defined by $l_h(\gamma) = T$. The family of all horizontal curves will be denoted by $\mathcal{H}_X$.

As pointed out above, the family of all horizontal curves specifies a custom metric space, where only horizontal curves are considered to influence distance.

Definition 25 (Carnot-Carathéodory space)**.** The distance function corresponding to the family of horizontal curves with respect to X is given by

$$d_X(x,y) = \inf\left\{l_h(\gamma) : \gamma \in \mathcal{H}_X, \gamma(0) = x, \gamma(l_h(\gamma)) = y\right\},\ x, y \in \mathbb{R}^n.$$

If d_X is a metric, then the metric space $(\mathbb{R}^n, d_X)$ is called the *Carnot-Carathéodory space* generated by the system X.

We now return to the example of car parking. Although no linear combination of the actions a_d and a_s enables the car to move in the direction perpendicular to the heading direction, a consecutive application of the actions a_s, a_d leads to the desired change. Such behavior can be described through commutators of our action vector fields.

Definition 26 (commutator)**.** Let X and Y be two smooth vector fields on $\mathbb{R}^n$. Then the *commutator* (or the Lie bracket) of X and Y is defined by

$$[X,Y](f) = X \circ Y(f) - Y \circ X(f),\ f \in \mathcal{C}^\infty(\mathbb{R}^n).$$

Let us note that although the definition of $[X,Y]$ suggests that it is a differential form of the second order, the commutator is, in fact, always a first-order differential operator.

The real-life experience with cars suggests that one can get from any state to any other state just by steering and driving, but is it true for our toy-example Carnot-Carathéodory space generated by a_d and a_s? The answer is given by an application of the Chow connectivity theorem (can be found in [23]).

Theorem 27 (Chow connectivity theorem)**.** *Let $X_1, \ldots, X_m$ be $\mathcal{C}^\infty$-smooth vector fields on a connected manifold V such that successive commutators of these fields span each tangent space $T_v(V)$, $v \in V$. Then every two points in V can be joined by a piecewise smooth curve in V where each piece is a segment of an integral curve of one of the fields X_i.*

A bit of algebra shows that the commutator of a_d and a_s is

$$l_{ds} = [a_d, a_s](f) = \partial_r.$$

The vector fields a_d, a_s and l_{ds} are linearly independent, but we need a fourth vector field for a basis. While the combination of a_s and l_{ds} will not help as the commutator $[a_s, l_{ds}]$ is zero vector field, the last possible combination yields the desired vector. We have

$$[a_d, l_{ds}] = -\sin(r)\partial_x + \cos(r)\partial_y.$$

Indeed, the vectors $a_d(x), a_s(x), l_{ds}(x)$ and $[a_d, l_{ds}](x)$ form a basis of a tangent space for all $x \in \mathbb{R}^n$, therefore, by the Chow connectivity theorem, there exists a horizontal path between all pairs of points and $(\mathbb{R}^4, d_{a_d,a_s})$ is a metric space. The topology generated by the Carnot-Carathéodory metric of a system X in general might not agree with the Euclidean topology. However, in this thesis we will always assume that these topologies are the same. The condition from Theorem 27 requiring that successive commutators of vector fields span the tangent space at every point is often referred to as the *Hörmander condition.* This condition implies the equivalence of topologies.

We will now introduce counterparts to basic notions of differential theory in the context of Carnot-Carathéodory spaces. Since the intrinsic structure of Carnot-Carathéodory spaces is being generated by a set of vector fields $X = \{X_1, X_2, \ldots, X_m\}$, the definitions are analogous to the standard ones.

For a function $f \in L^1_{\text{loc}}(\Omega)$ its distributional derivative along the vector field X_j, $X_j f$, is defined by the identity

$$\langle X_j f, \phi \rangle = \int_\Omega f X_j^* \phi \, dx \quad \text{for every } \phi \in \mathcal{C}_0^\infty(\Omega),$$

where $X_j^*(\cdot) = -\sum_{k=1}^n \frac{\partial}{\partial x_k}(b_{j,k}\,\cdot)$ denotes the formal adjoint of X_j. If f is a non-smooth function then $X_j f$ will be meant in the distributional sense.

The following notion of the (first-order) gradient is widely used in the context of Carnot-Carathéodory spaces. We will generalize it to higher orders. The literature dealing with higher-order embeddings in the context of Carnot-Carathéodory spaces is scarce. Our notion of higher order gradient agrees with the one introduced in [36].

Definition 28 (X-gradient and X-variation)**.** Suppose that $f \in L^1_{loc}(\Omega)$. If derivatives $X_1 f, X_2 f, \ldots, X_m f$ exist, then the vector of X*-gradient* of a function f is defined by

$$X\nabla f = (X_1 f, X_2 f, \ldots, X_m f)\,.$$

Moreover, let us introduce the *higher-order derivatives* as

$$XD_\alpha(\cdot) = X_{\alpha_1}\left(X_{\alpha_2}\left(\ldots X_{\alpha_k}\left(\cdot\right)\ldots\right)\right),$$

where $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{1, \ldots, m\}^k$. Provided that $XD_\alpha f$ exists for all $\alpha \in \{1, \ldots, m\}^k$, the X*-gradient of order* k is defined as a vector of length m^k of the following form:

$$X\nabla^k f = \left(XD_\alpha(f)\colon \alpha \in \{1, \ldots, m\}^k\right).$$

Naturally, the norm of the X-gradient of order k reads as

$$\left|X\nabla^k f\right|^2 = \sum_{\alpha\in m^k} (XD_\alpha(f))^2.$$

Analogously to Definitions 14 and 15, we can introduce notions of X-variation and X-perimeter in the following way.

Definition 29 (X-variation)**.** If we denote

$$\mathcal{F}_\Omega = \left\{ \phi = \{\phi_1, \phi_2, \dots, \phi_m\} \in \mathcal{C}_0^1(\Omega, \mathbb{R}^m) : \sup_{x\in\Omega} \left(\sum_{j=1}^m |\phi_j(x)|^2 \right)^{\frac{1}{2}} \leq 1 \right\},$$

then, for a given $f \in L^1_{\text{loc}}(\Omega)$, the X*-variation* of f with respect to Ω is defined as

$$\text{Var}_X(f, \Omega) = \sup_{\phi\in\mathcal{F}_\Omega} \int_\Omega u(x) \sum_{j=1}^m X_j^* \phi_j(x)\, dx.$$

Let us note that the set of functions with bounded X-variation is denoted as $BV_X(\Omega)$ and forms a Banach space with respect to the norm

$$\|\cdot\|_{BV_X} = \|\cdot\|_{L_1(\Omega)} + \text{Var}_X(\cdot, \Omega).$$

If $X\nabla f \in L^1(\Omega)$, then

$$\text{Var}_X(f, \Omega) \leq \hat{C} \left\| X\nabla f \right\|_{L^1}, \tag{1.19}$$

where $\hat{C} > 0$ depends only on m.

Definition 30 (X-perimeter, X-isoperimetric function)**.** If $E \subset \mathbb{R}^n$ is measurable, then the X*-perimeter* of E relative to Ω is defined by

$$P_X(E, \Omega) = \text{Var}_X(\chi_E, \Omega),$$

where χ_E denotes the characteristic function of E. The X*-isoperimetric function* of Ω is given by the following formula

$$I_{X,\Omega}(s) = \inf \left\{ P_X(E, \Omega) \colon E \subset \Omega, s \leq |E| \leq \frac{1}{2} \right\} \quad \text{for } s \in \left[0, \frac{1}{2}\right],$$

and $I_{X,\Omega}(s) = I_{X,\Omega}(1-s)$ if $s \in \left(\frac{1}{2}, 1\right]$. Sets of finite perimeter are called X-*Caccioppoli* sets.

The isoperimetric function is often unknown. Fortunately, in [22] it is shown that if we adopt some additional assumptions we can nail it down for the so-called X-PS domains.

The first such condition is the following version of the doubling condition: for any set $U \subset \mathbb{R}^n$ with $\operatorname{diam}(U) < \infty$ there exist constants $C_1 > 0$ and $R_0 < \infty$ such that for $x_0 \in U$ and $0 < R < R_0$ one has

$$|B(x_0, 2R)| \leq C_1 \, |B(x_0, R)| \,. \tag{1.20}$$

It was shown in [39] that the Hörmander condition implies the doubling condition.

The second restriction is the following version of the Poincaré inequality: for any set $U \subset \mathbb{R}^n$ with $\operatorname{diam}(U) < \infty$, there exist constants $C_2 > 0$, $R_0 < \infty$ and $\alpha \geq 1$ such that for $x_0 \in U$, $0 < R < R_0$ and every Lipschitz function u in $\alpha B = B(x_0, \alpha R)$, we have for any $\lambda > 0$

$$\left| \left\{ x \in B \colon \left| u(x) - \int_B u(x) \, dx \right| > \lambda \right\} \right| \leq \frac{C_2}{\lambda} \int_{\alpha B} |X \nabla u(y)| \, dy. \tag{1.21}$$

The third restriction is that $(\mathbb{R}^n, d)$ is complete and it is a length-space; that is,

$$d(x, y) = \inf l(\gamma_{xy}), \tag{1.22}$$

where the infimum is extended over all continuous curves γ joining x to y, and $l(\gamma_{xy})$ denotes its metric length.

Definition 31 (homogeneous dimension)**.** Assume that (1.20) holds. Let $U \subset \mathbb{R}^n$ and denote by C the smallest constant in (1.20). Then the homogeneous dimension relative to U (and X) is defined by

$$Q = \log_2(C).$$

Let us point out that the homogeneous dimension might not (and usually does not) agree with the classical dimension, n, of the underlying set $\mathbb{R}^n$ (neither with the dimension of tangent bundle). We will illustrate this on the prominent example of the Heisenberg group.

Consider $\mathbb{R}^3$ and set $\mathbb{H} = \{H_1, H_2\}$, where

$$H_1 = \partial_{x_1} - \frac{1}{2} x_2 \partial_{x_3} \text{ and } H_2 = \partial_{x_2} + \frac{1}{2} x_1 \partial_{x_3}.$$

The Carnot-Carathéodory space generated by $\mathbb{H}$ is referred to as the Heisenberg group. Let us note that the homogeneous dimension of the Heisenberg group is 4. It can be easily verified that

$$[H_1, H_2] = \partial_{x_3},$$

consequently, the Heisenberg group satisfies the Hörmander condition. Moreover, it satisfies all three conditions (1.20), (1.21) and (1.22) ([22, Remark 1.2]).

Various classes of domains has been studied in this settings, see for example [28], [37], We will recall the definition of the X-PS domains.

Definition 32 (X-PS domain). An open set $\Omega \subset \mathbb{R}^n$ is called *X-PS domain* if there exist a covering $\{B\}_{B\in\mathcal{F}}$ of Ω by metric balls and numbers $N > 0$, $\alpha \geq 1$ and $\nu \geq 1$ such that

1. $\sum_{B\in\mathcal{F}} \chi_{(\alpha+1)B}(x) \leq N\chi_\Omega(x)$ for every $x \in \Omega$.
2. There exists a (central) ball $B_0 \in \mathcal{F}$ such that for any $B \in \mathcal{F}$ one can find a chain $B_0, B_1, \ldots, B_{s(B)} = B$, with $B_i \cap B_{i+1} \neq \emptyset$ and $|B_i \cap B_{i+1}| \geq \frac{1}{N} \max\left(|B_i|, |B_{i+1}|\right)$.
3. For any $i = 0, \ldots, s(B)$, one has $B \subset \nu B_i$.

Here we used the following notation: let $B = B(x,r), x \in \mathbb{R}^n, r > 0$, and $\alpha > 0$ the αB denotes the set $\alpha B = \{y \in \mathbb{R}^n : d(x,r) < \alpha r\}$.

Under these assumptions the isoperimetric function is known. It is analogous to the isoperimetric function of Lipschitz or John domain in the classical setting, but depending on the homegeneous dimension.

Theorem 33 (isoperimetric function of X-PS domains). *Suppose that* (1.20) *and* (1.21) *hold and for a bounded set $U \subset \mathbb{R}^n$ let $\Omega \subset \bar{\Omega} \subset U$ be an X-PS domain with* $\operatorname{diam}(\Omega) \leq \frac{R_0}{2}$. *Then for any X-Caccioppoli set $E \subset \mathbb{R}^n$ we have*

$$\min\left\{|E \cap \Omega|, |E^c \cap \Omega|\right\}^{\frac{Q-1}{Q}} \leq C \operatorname{diam}(\Omega)\, |\Omega|^{-\frac{1}{Q}}\, P_X(E, \Omega).$$

The relatively unknown notion of X-PS domains is actually in a very close relation to John domains relative to the Carnot-Carathéodory metric.

Definition 34 (X-John domain). An open set $\Omega \subset \mathbb{R}^n$ is called an *X-John domain* if there exist a constant $c \in (0,1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\omega\colon [0,l] \to \Omega$, parametrized by arc-length, such that $\omega(0) = x$, $\omega(l) = x_0$, and

$$d(\omega(r), \partial\Omega) \geq cr \text{ for } r \in [0,l],$$

where $\partial\Omega$ denotes boundary of Ω.

The class of X-PS domains contains that of X-John domains if (1.20) holds, which is always assumed in this text. On the other hand, if certain geodesic segment property is satisfied then the class of X-John domains contains the class of X-PS domains. Both inclusions are shown in [22, Theorem 1.30]. Consequently, both classes coincide if both (1.20) and the geodesic segment property are satisfied.

1.7 Sobolev-like embeddings on Carnot-Caratheodory spaces

Since the Banach function spaces are depending only on the measure structure of considered domain, they behave in the context of Carnot-Carathéodory spaces exactly the same way as in the classical setting. The Sobolev spaces built over the Carnot-Carathéodory spaces are analogous to the classical ones too. The difference is only in using the customary gradient operator defined in the previous section.

Definition 35 (Sobolev space)**.** Given a Banach function space $\mathbf{X}(\Omega)$ and a positive integer $d \in \mathbb{N}$, the d-th order *Sobolev space* built upon $\mathbf{X}(\Omega)$ is the normed linear space $W^d_X\mathbf{X}(\Omega)$ of all functions on Ω whose derivatives up to the order d (with respect to the set of vector fields X) exist and belong to $\mathbf{X}(\Omega)$, equipped with a natural norm

$$\|f\|_{W^d_X\mathbf{X}(\Omega)} = \sum_{i=0}^{d} \left\|X\nabla^i f\right\|_{\mathbf{X}(\Omega)}.$$

Assume that the X-isoperimetric function of Ω satisfies

$$I_{X,\Omega}(s) \geq Cs \text{ for } s \in \left[0, \tfrac{1}{2}\right] \tag{1.23}$$

for some $C > 0$. Then we define the m-th order Sobolev space $V^m\mathbf{X}(\Omega)$ as

$$V^m_X\mathbf{X}(\Omega) = \{u : u \text{ is } m\text{-times weakly differentiable in } \Omega, \text{ and } |X\nabla^m\mathbf{X}| \in \mathbf{X}(\Omega)\}.$$

The space $V^m_X\mathbf{X}(\Omega)$ is equipped with the norm

$$\|f\|_{V^m_X\mathbf{X}(\Omega)} = \sum_{i=0}^{m-1} \left\|X\nabla^i f\right\|_{L^1(\Omega)} + \|X\nabla^m f\|_{\mathbf{X}(\Omega)} \text{ for all } f \in \mathfrak{M}_+(\Omega).$$

The topic of Sobolev embeddings in the context of Carnot-Carathéodory spaces or Carnot groups (or sub-Riemannian geometry) has been a subject to extensive research, such as [39], [22], [13], [14], [24], [33], [44], [11], [15] or [7] to name just a few.

Some results are concerned with particular type of Carnot-Carathéodory spaces. In this brief review, this trend will be represented by results connected to the Heisenberg group. A good reference for this topic is [7], a monograph dealing with isoperimetric problem on the Heisenberg group and related questions, such as some Sobolev-type embeddings.

Let us start with the case when the domain is the whole Heisenberg group. The embeddings (between Sobolev spaces built upon Lebesgue spaces) are analogous to the classical ones, with the only difference that the exponent parameter behaves the same way as if the dimension was 4. We have

$$W^1_{\mathbb{H}}L^p(\mathbb{R}^3) \to \begin{cases} L^{\frac{4-p}{4p}}(\mathbb{R}^3), & \text{for } 1 \le p < 4, \\ \mathcal{C}^{0,1-\frac{4}{p}}(\mathbb{R}^3), & \text{for } p > 4. \end{cases} \tag{1.24}$$

One way to explain this is that the embeddings are not governed by the dimension but by the homogeneous dimension, which is in this case equal to 4. Theorem 33 indicates that the homogeneous dimension then affects the isoperimetric function.

Let us now shift our attention to results in more general settings. From our point of view, [22] is the key reference. It provides tools vital to techniques used in [12] and [45], such as knowledge of asymptotic behavior of the isoperimetric function of X-PS domains, co-area formula and prerequisites for the Maz'ya truncation technique.

To provide some context to our results, let us state the main results focused on the Sobolev-type inequalities from [22], as we are following the line of research focused on X-PS domains.

Theorem 36. *Suppose* (1.20) *and* (1.21) *hold and, for a bounded set* $U \subset \mathbb{R}^n$*, let* $\Omega \subset \bar{\Omega} \subset U$ *be an* X*-PS domain with* $\operatorname{diam}(\Omega) < \frac{R_0}{2}$*. Then there exists a constant* $C > 0$ *such that for any* $u \in W^1_X L^1(\Omega)$

$$\left(\frac{1}{|\Omega|} \int_\Omega |u - u_\Omega|^{\frac{Q}{Q-1}} \, dx \right)^{\frac{Q-1}{Q}} \le C \operatorname{diam}(\Omega) \left(\frac{1}{|\Omega|} \int_\Omega |X\nabla u| \, dx \right),$$

where $u_\Omega = \int_\Omega u(x)\, dx$.

Theorem 36 implies that the following versions of Poincaré inequality hold for domains fulfilling assumptions of Theorem 36:

1. Let $1 \le p < Q$. Then, for any $1 \le k \le \frac{Q}{Q-p}$,

$$\left(\frac{1}{|\Omega|} \int_\Omega |u - u_\Omega|^{kp} \, dx \right)^{\frac{1}{kp}} \le C \operatorname{diam}(\Omega) \left(\frac{1}{|\Omega|} \int_\Omega |X\nabla u|^p \, dx \right)^{\frac{1}{p}}$$

   for any $u \in W^1_X L^p(\Omega)$ and some C independent of u and Ω.

2. Under these assumptions Ω supports the Poincaré inequality, that is,

$$\int_\Omega |u - u_\Omega|^p \, dx \le C \operatorname{diam}(\Omega)^p \int_\Omega |X\nabla u|^p \, dx, \; 1 \le p < \infty,$$

   for any $u \in W^1_X L^p(\Omega)$.

Now we give an overview of our results. Their novelty consists in utilization of the rearrangement-invariant function spaces approach in the context of Carnot-Carathéodory spaces. The achieved knowledge can be easily applied to any rearrangement-invariant space. As an example, we apply the theory to Sobolev spaces built upon Lebesgue and Lorentz spaces. Another novel aspect of our results is that they deal with embeddings of higher order.

Now we are in the position to state the central result. It yields a sufficient condition on higher-order Sobolev-type embeddings in great generality.

Analogously to the classical case, we will be working with a function estimating the behavior of isoperimetric function near zero. We will assume that there is a $c > 0$ such that

$$I_{X,\Omega}(s) \geq cI(cs) \text{ for } s \in \left[0, \tfrac{1}{2}\right]. \tag{1.25}$$

In addition, we assume

$$\inf_{t\in(0,1)} \frac{I(t)}{t} > 0. \tag{1.26}$$

Theorem 37 (reduction theorem)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is open, $|\Omega| = 1$, such that there is some non-decreasing function I satisfying* (1.25) *and* (1.26)*. Let $m \in \mathbb{N}$ and let $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ be rearrangement-invariant function norms. If there exists a constant $C > 0$ such that*

$$\left\| \int_t^t \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds \right\|_{\mathbf{Y}(0,1)} \leq C \|f\|_{\mathbf{X}(0,1)}$$

for every nonnegative $f \in \mathfrak{M}_+(0,1)$, then

$$V_X^m \mathbf{X}(\Omega) \to \mathbf{Y}(\Omega).$$

This principle can be easily applied provided the asymptotic behavior of the isoperimetric function is known. This yields the following condition on embeddings in the context of X-PS domains.

Theorem 38 (reduction theorem for X-PS domains)**.** *Let us consider the case when X is such that the conditions (1.20), (1.21) are fulfilled. Assume that $\Omega \subset \mathbb{R}^n$ is an X-PS domain, $|\Omega| = 1$, denote by Q the homogeneous dimension relative to Ω. Let $m \in \mathbb{N}$, and let $\|.\|_{\mathbf{X}(0,1)}$ and $\|.\|_{\mathbf{Y}(0,1)}$ be rearrangement-invariant function norms. If there exists a constant C such that*

$$\left\| \int_t^1 f(s) s^{-1+\frac{m}{Q}}\, ds \right\|_{\mathbf{Y}(0,1)} \leq C \|f\|_{\mathbf{X}(0,1)} \tag{1.27}$$

for every nonnegative $f \in \mathfrak{M}_+(0,1)$, then

$$V_X^m \mathbf{X}(\Omega) \to \mathbf{Y}(\Omega). \tag{1.28}$$

An embedding between particular function spaces can be obtained by application of Theorem 38. Let us demonstrate this on examples of Lebesgue and Lorentz spaces.

Theorem 39. *Assume that the conditions (1.20), (1.21) are fulfilled, let $\Omega \subset \mathbb{R}^n$ be an X-PS domain and denote by Q the homogeneous dimension relative to Ω. Let $m \in \mathbb{N}$ and $p \in [1,\infty]$. Then the following embeddings hold:*

$$V_X^m L^p(\Omega) \to \begin{cases} L^{\frac{Qp}{Q-mp}}(\Omega) & \text{if } m < Q \text{ and } 1 \le p < \frac{Q}{m}, \\ L^r(\Omega) & \text{for } r \in [1,\infty), \text{ if } m < Q \text{ and } p = \frac{Q}{m}. \\ L^\infty(\Omega) & \text{otherwise.} \end{cases} \tag{1.29}$$

Theorem 40. *Assume that the conditions (1.20), (1.21) are fulfilled, let $\Omega \subset \mathbb{R}^n$ be an X-PS domain and denote by Q the homogeneous dimension relative to Ω and X. Let $m \in \mathbb{N}$, $1 < p < \infty$ and $1 \le q \le \infty$, then*

$$V_X^m L^{p,q}(\Omega) \to \begin{cases} L^{\frac{Qp}{Q-mp},q}(\Omega) & \text{if } m < Q \text{ and } 1 \le p < \frac{Q}{m}, \\ L^{\infty,q;-1}(\Omega) & \text{if } m < Q,\ p = \frac{Q}{m} \text{ and } q > 1. \\ L^\infty(\Omega) & \text{otherwise.} \end{cases} \tag{1.30}$$

To obtain a result analogous to (1.24) (but for bounded X-PS domains), we can apply Theorem 39 to the Heisenberg group.

Corollary 41 (embeddings of Lebesgue spaces on the Heisenberg group)**.** *Let $\Omega \subset \mathbb{R}^3$ be a $\mathbb{H}$-PS domain. Let $m \in \mathbb{N}$ and $p, q \in [1,\infty]$. Then the following embeddings hold:*

$$V_{\mathbb{H}}^m L^p(\Omega) \to \begin{cases} L^{\frac{4p}{4-mp}}(\Omega) & \text{if } m < 4 \text{ and } 1 \le p < \frac{4}{m}, \\ L^r(\Omega) & \text{for } r \in [1,\infty), \text{ if } m < 4 \text{ and } p = \frac{4}{m}, \\ L^\infty(\Omega) & \text{otherwise.} \end{cases} \tag{1.31}$$

It is a natural question whether the sufficient condition is necessary as well. The original results in classical settings are in fact dealing with this question, see [12]. Unfortunately, techniques used to achieve these results are hard to apply to the customary structure of Carnot-Carathéodory spaces in its generality. However, thanks to extensive available knowledge base, it was possible to employ the same techniques in the case of the Heisenberg group.

Theorem 42. *Let $m \in \mathbb{N}$. Let $\|.\|_{\mathbf{X}(0,1)}$ and $\|.\|_{\mathbf{Y}(0,1)}$ be rearrangement-invariant function norms such that*

$$V_{\mathbb{H}}^m \mathbf{X}(\Omega) \to \mathbf{Y}(\Omega) \tag{1.32}$$

for all $\mathbb{H}$-John domains $\Omega \subset \mathbb{R}^3$. Then there exists a constant C such that

$$\left\| \int_t^1 f(s) s^{-1+\frac{m}{Q}}\, ds \right\|_{\mathbf{Y}(0,1)} \le C \|f\|_{\mathbf{X}(0,1)} \tag{1.33}$$

for all $f \in \mathfrak{M}_+(0,1)$.

It is worth noting that a customary spherical symmetral based on the so-called Korányi gauge (see [31]) was employed in the proof of Theorem 42. Let us define the *Korányi gauge* by

$$\|(x_1,x_2,x_3)\|_{\mathbb{H}}^4 = (x_1^2 + x_2^2)^2 + 16x_3^2, \quad \text{for } (x_1,x_2,x_3) \in \mathbb{R}^3.$$

Korányi ball with centre $(0,0,0)$ and diameter s is defined as follows:

$$B_{\mathbb{H}}(s) = \{x \in \mathbb{H} : \|x\|_H \le s\}.$$

Let us note that $|B_{\mathbb{H}}(s)| = Ks^4$, for some $K > 0$ and that $B_{\mathbb{H}}$ is a $\mathbb{H}$-John domain ([8, Corollary 5]).

To a given function $f \in \mathfrak{M}_+(0,1)$, let us define the m-th *order* $\mathbb{H}$-*symmetral*

$$u_{f,m}(x) = g(\|x\|_{\mathbb{H}}) = \int_{K\|x\|_{\mathbb{H}}^4}^1 \int_{t_1}^1 \cdots \int_{t_{m-1}}^1 f(t_m) t_m^{-m+\frac{m}{4}} \, dt_m \ldots dt_2 \, dt_1,$$

where $x \in B = B_{\mathbb{H}}(1)$. The motivation behind the $\mathbb{H}$-symmetral is the following property. Let $f \in \mathfrak{M}(0,1)$, $m \in \mathbb{N}$, then

$$|\mathbb{H}\nabla_m(u_{f,m})(x)| \;\le\; \hat{C}\left(f(K\|x\|_{\mathbb{H}}^4) + \sum_{j=1}^{m-1} \|x\|_{\mathbb{H}}^{4j-m} \int_{K\|x\|_{\mathbb{H}}^4}^1 f(s) s^{m-\frac{m}{4}} \, ds\right),$$

for $x \in B$.

We will shift our attention to compact embeddings now.

It was possible to adapt techniques from [45] to achieve reduction principle analogous to Theorem 37 for compact embeddings.

Theorem 43 (reduction theorem for compact embeddings)**.** *Assume that* (1.20), (1.21) *and* (1.22) *are fulfilled. Let* $\Omega \subset \mathbb{R}^n$ *be open and* $m \in \mathbb{N}$. *Suppose that there is some non-decreasing function* $I\colon [0,1] \to \mathbb{R}$ *satisfying* (1.25) *and* (1.26), *let* $\|\cdot\|_{\mathbf{X}(0,1)}$ *and* $\|\cdot\|_{\mathbf{Y}(0,1)}$ *be rearrangement-invariant norms, then*

$$H_I^m \colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1) \tag{1.34}$$

implies

$$V_X^m \mathbf{X}(\Omega) \hookrightarrow\hookrightarrow \mathbf{Y}(\Omega). \tag{1.35}$$

Particular compact embeddings can be derived from the reduction theorem for compact embeddings in similar manner to Theorems 38 – 41.

Theorem 44 (reduction principle for compact embeddings for X-PS domains)**.** *Assume that* (1.20), (1.21) *and* (1.22) *are fulfilled. Let* $m \in \mathbb{N}$ *and* Ω *be a* X*-PS domain with homogeneous dimension* Q. *Suppose that* $\|\cdot\|_{\mathbf{X}(0,1)}$ *and* $\|\cdot\|_{\mathbf{Y}(0,1)}$ *are rearrangement-invariant function spaces.*

If $m \le Q$ *and*

$$\mathcal{Q}_Q^m \colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1), \tag{1.36}$$

here

$$\mathcal{Q}_Q^m f(t) = \int_t^1 s^{1-\frac{m}{Q}} |f|(s) \, ds, \; f \in \mathfrak{M}_+(0,1), \; t \in [0,1].$$

Then

$$V_X^m \mathbf{X}(\Omega) \hookrightarrow\hookrightarrow \mathbf{Y}(\Omega). \tag{1.37}$$

In particular, the assumption that $Q = m$ implies that (1.37) is satisfied for all $\|\cdot\|_{\mathbf{Y}(0,1)}$ if $\mathbf{X}(0,1) \neq L^1(0,1)$. Furthermore, if $m > Q$ then (1.37) is fulfilled for all choices of $\mathbf{X}(0,1)$ and $\mathbf{Y}(0,1)$.

Again, we will demonstrate an application of the general statement from Theorem 44 on examples of compact embeddings between Sobolev spaces built upon Lebesgue and Lorentz spaces.

Theorem 45. *Assume that (1.20), (1.21) and (1.22) are fulfilled. Let $m \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^n$ be an X-PS domain. Let $p_1, p_2, q_1, q_2 \in [1, \infty]$ be such that the triples $(p_1, q_1, 0)$ and $(p_2, q_2, 0)$ satisfy one of conditions in (1.5). Let Q denote the homogeneous dimension of Ω, assume $Q > 2$ and $m < Q$, then each of the following conditions*

1. $p_1 < \frac{Q}{m}$ *and* $p_2 < \frac{p_1}{1-\frac{mp_1}{Q}}$;
2. $p_1 = \frac{Q}{m}$ *and* $p_2 < \infty$;
3. $p_1 > \frac{Q}{m}$;

ensures that

$$V_X L^{p_1,q_1}(\Omega) \hookrightarrow\hookrightarrow L^{p_2,q_2}(\Omega). \tag{1.38}$$

Let us note that cases when $m = Q$ and $m \geq Q$, missing from Theorem 45, are already covered by Theorem 44. Let us explicitly state the conditions on compact embeddings of Sobolev spaces built upon Lebesgue spaces which are implied by Theorem 45.

Corollary 46. *Assume that (1.20), (1.21) and (1.22) are fulfilled. Let $m \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^n$ be an X-PS domain. Let $p_1, p_2 \in [1, \infty]$. Let Q denote the homogeneous dimension of Ω, assume $Q > 2$ and $m < Q$, then each of the following conditions*

1. $p_1 < \frac{Q}{m}$ *and* $p_2 < \frac{p_1}{1-\frac{mp_1}{Q}}$;
2. $p_1 = \frac{Q}{m}$ *and* $p_2 < \infty$;
3. $p_1 > \frac{Q}{m}$;

ensures that

$$V_X L^{p_1}(\Omega) \hookrightarrow\hookrightarrow L^{p_2}(\Omega). \tag{1.39}$$

1.8 Least concave majorant

In this section we will focus on the least concave majorant and the level function. We start with recalling definitions and basic properties. Then we give a short summary of results from [20]. We illustrate the key idea behind the algorithm to find the least concave majorant (and therefore the level function) of a cubic spline and conclude by restating the estimate of error involved.

We are dealing with a topic which is fundamentally different from the context of the previous section. We will be working with a continuous function $F\colon I \to \mathbb{R}$, where $I = [a, b]$, $a, b \in \mathbb{R}$. Although the notions of the least concave majorant and the level function can be introduced in much more general settings (more general measure space), we will restrict ourselves to the situation considered in [20].

Definition 47 (least concave majorant)**.** Suppose F is a continuous function on the interval I. Denote by $\hat{F}$ the least concave majorant of F, namely,

$$\hat{F}(x) = \inf \left\{G(x) : G \geq F, G \text{ concave}\right\}.$$

The notion of the least concave majorant has been intensively studied and a lot of information is available in the literature. It has a broad variety of applications. For example, it can be shown that

$$\hat{F}(x) = \sup \left\{ \frac{\beta - x}{\beta - \alpha} F(\alpha) + \frac{x - \alpha}{\beta - \alpha} F(\beta) : a \leq \alpha \leq x \leq \beta \leq b \right\}, x \in I.$$

The least concave majorant is in a close relation to the level function. The level function was introduced by Halperin in [25] (and further developed in [32]). It has been in a constant use ever since and many it's important applications have been found. To give a few examples, let us mention the improvement of the Hölder inequality (see e.g. Sinnamon [46, 47]) or the characterization of the dual spaces of certain classical Lorentz spaces ([9]).

Definition 48 (level function)**.** Let f be a measurable function on I. Then there exists a function f°, called the *level function* of f, and a partition of $I = E \cup F$, $E \cap K = \emptyset$, where $K = \cup_{j=1}^{k} I_j$, I_j are disjoint intervals of finite measure, such that f° is decreasing, and

$$f^\circ(x) = \begin{cases} f(x), & x \in E; \\ \frac{1}{|I_j|} \int_{I_j} w(x)\, dx, & c \in I_j, 1 \leq j \leq k. \end{cases}$$

The connection between the least concave majorant and the level function is that the level function, f°, coincides almost everywhere with the derivative of the least concave majorant of $F(x) = \int_a^x f(x)$.

The key property of the level function is that

$$\int_a^x f(x)\, dx \leq \int_a^x f^\circ(x)\, dx \leq \int_0^{x-a} f^*(x)\, dx \quad \text{for } x \in I. \tag{1.40}$$

To illustrate an application of the level function let us recall the useful concept of the so-called down norm.

Definition 49 (down norm)**.** Let $\mathbf{X}$ be a rearrangement-invariant function space. We define the *down norm* of a measurable function f to be

$$\|f\|_{\mathbf{X}\downarrow} = \sup\left\{\int_{\mathbb{R}} |f(x)|\, g(x)\, dx : g \geq 0, g \text{ non-increasing, } \|g\|_{\mathbf{X}'} \leq 1\right\}. \quad (1.41)$$

The down norm is tightly connected to the level function as the following equality holds

$$\|f\|_{\mathbf{X}\downarrow} = \|f^{\circ}\|_{\mathbf{X}}$$

for all measurable functions. One clearly has

$$\|f\|_{\mathbf{X}\downarrow} \leq \|f\|_{\mathbf{X}}\,.$$

The significance of the down norm is that the inequality

$$\int_{\mathbb{R}} fg\, d\lambda \leq \|f\|_{\mathbf{X}\downarrow} \|g\|_{\mathbf{X}}$$

holds for all f and all non-negative, non-increasing functions g. This allows us to improve, in a sense, the Hölder inequality (10) by using the (possibly smaller) down norm.

Now, we will focus on a method to find the least concave majorant of a function. If F is continuous, so is $\hat{F}$. Consequently, continuity of F implies that the set $K = \{x : \hat{F}(x) \neq F(x)\}$ is open. Obviously, K is either empty or a union of open intervals. Let us refer to these intervals as *component intervals*. The key observation is that any component interval, say (α, β), satisfies several important properties, which we shall now describe in detail. First, one has

$$\hat{F}(\alpha) = F(\alpha), \quad \hat{F}(\beta) = F(\beta). \quad (1.42)$$

Next, the least concave majorant is linear on (α, β), the linear segment joining points $[\alpha, F(\alpha)]$ and $[\beta, F(\beta)]$ is acting as a sort of bridge over some convex part contained in (α, β). We have

$$F(x) \leq \hat{F}(x) = F(\alpha) + (x_{\alpha})\frac{F(\beta) - F(\alpha)}{\beta - \alpha}, \quad \text{for } \alpha < x < \beta. \quad (1.43)$$

Moreover, the derivatives at endpoints of component interval agree with the slope of the linear segment of $\hat{F}$, more precisely,

$$(\hat{F})'(\alpha) = F'(\alpha) = \frac{F(\beta) - F(\alpha)}{\beta - \alpha} = F'(\beta) = (\hat{F})'(\beta). \quad (1.44)$$

Let us call a *bridge interval* any interval which satisfies conditions (1.42), (1.43) and (1.44). Unfortunately, not every bridge interval is a component interval. That is a consequence of locality of all three conditions. However, bridge intervals constitute a good pool of intervals to select desired component intervals from. It turns out that the largest (in sense of inclusion) bridge intervals are always the desired component intervals. As our discussion is about to steer towards splines, let us recall the definition of relevant cubic splines.

Definition 50 (clamped cubic spline)**.** Let ϱ be a partition of I given by points

$$a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b,$$

where $k \in \mathbb{N}$.

A function $s : I \to \mathbb{R}$ will be called a cubic spline (with respect to ϱ) if $s \in \mathcal{C}^2(I)$ and for each $i = 0, 1, \ldots, k-1$ there exists a polynomial p_i of degree up to 3 such that

$$s(x) = p(x) \text{ for all } x \in [x_k, x_{k+1}].$$

Given a function $f \in \mathcal{C}(I)$ which is differentiable at a and b, the clamped cubic spline approximation (interpolant) of f with respect to the partition ϱ is the unique cubic spline (with respect to partition ϱ) s such that

$$s(x_i) = f(x_i), i = 0, 1, \ldots, k$$

and

$$f'(a) = s'(a), f'(b) = s'(b).$$

The nature of conditions (1.42), (1.43) and (1.44) allows us to find all bridge intervals of a cubic spline and hence we are able to compute its least concave majorant. The least concave majorant of a clamped cubic spline approximating a function is then seen as a good approximation of the least concave majorant of the original function.

Theorem 51. *Let ϱ be a partition of the interval $I = [a, b]$ and suppose $G \in \mathcal{C}^4([a, b])$. Let F be the clamped cubic spline interpolating G on ϱ. Then*

$$\|f^\circ - g^\circ\|_{L^\infty(I)} \leq \|f - g\|_{L^\infty(I)} \leq \frac{1}{24} \left\|G^{(4)}\right\|_{L^\infty(I)} \|\varrho\|^3$$

and for each $x \in [a, b]$,

$$\left|\hat{F}(x) - \hat{G}(x)\right| \leq \frac{\min\{x - a, b - x\}}{24} \left\|G^{(4)}\right\|_{L^\infty(I)} \|\varrho\|^3 .$$

Here $\hat{F}$ and $\hat{G}$ denote the least concave majorants of F and G, respectively, $f = F'$, $g = G'$; $f^\circ = (\hat{F})'$, and $g^\circ = (\hat{G})'$. The symbol $\|\varrho\|$ denotes the length of the longest interval in a partition ϱ.

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Chapter 2

Higher-order Sobolev-type embeddings on Carnot-Carathéodory spaces

Higher-order Sobolev-type embeddings on Carnot-Carathéodory spaces

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A sufficient condition for higher-order Sobolev-type embeddings on bounded domains of Carnot-Carathéodory spaces is established for the class of rearrangement-invariant function spaces. The condition takes form of a one-dimensional inequality for suitable integral operators depending on the isoperimetric function relative to the Carnot-Carathéodory structure of the relevant sets. General results are then applied to particular Sobolev spaces built upon Lebesgue, Lorentz and Orlicz spaces on John domains in the Heisenberg group. In the case of the Heisenberg group, the condition is shown to be necessary as well.

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1 Introduction

Sobolev-type embeddings constitute a very important concept in functional analysis with wide range of applications. Their primary use concerns functions defined on domains in an Euclidean space, but they are of interest also in other situations. The investigation of Sobolev embeddings on all kinds of domains including very bad ones is considered classical nowadays, and not only in the Euclidean setting (for example their investigation on domains equipped with the product probability measures, or, in particular, the Gauss measure, is notorious - see e.g. [3, 1]).

Sobolev embeddings are of particular interest in connection with the isoperimetric inequality and its various modifications. Although these two subjects had been investigated at first along separate lines, it was observed by Maz'ya [27, 28] and also by Federer and Fleming [16] in early 1960's that there is in fact an intimate connection between them.

Since Hörmander's pioneering paper on hypoellipticity [21] the Sobolev embeddings involving general differential operators connected to non-commutative vector fields became immensely important, mainly due to their applications in the study of both linear and non-linear partial differential equations arising from a system of smooth vector fields. These considerations lead one to the study of Carnot-Carathéodory spaces, and, in particular, the Heisenberg group.

The principal goal of this paper is to establish sufficient conditions for the validity of Sobolev-type embeddings in the setting of Carnot-Carathéodory spaces. We shall consider such embeddings in a fairly general situation, namely for Sobolev spaces built upon the so-called rearrangement-invariant spaces. This enables us to formulate our results in terms of wide class of function spaces. The custom structure of given Carnot-Carathéodory space is mediated by the isoperimetric function relative to it. The topic of Sobolev embeddings in the context of Carnot-Carathéodory spaces has received wide attention (see e.g. the papers [23], [26], [13], [12], [11], [19], [9], [33], [10], [29], [34], or the monograph [4]). However, the existing literature seems to be restricted to certain particular classes of function spaces and the possibility of use of the rearrangement-invariant spaces has been neglected so far. We will restrict ourselves to bounded domains.

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Carnot-Carathéodory spaces are determined by a certain system of vector fields on $\mathbb{R}^n$, let us denote it by X. This system generates metric and differential structure with many notions, which are counterparts to their analogues from the classical differential geometry, for instance, the perimeter of sets relative to Carnot-Carathéodory structure.

A central role in our approach is played by the *isoperimetric function*. The isoperimetric function, relative to a given set $\Omega \subset \mathbb{R}^n$ and a system of vector fields X, is defined as

$$I_{X,\Omega}(s) = \min \{P_X(E) : E \subset \Omega;\ |E| = s\};\ s \in (0, |\Omega|).$$

In general, the isoperimetric function cannot be evaluated exactly. This is in fact possible only in very special situations (e.g. for balls in classical Euclidean spaces or for half-spaces in $\mathbb{R}^n$ equipped with the Gaussian measure). However, in many customary cases the asymptotic behaviour of the isoperimetric function near zero is known, and this fact can be used to formulate exact requirements on embeddings (cf. e.g. [8]).

In the present paper we establish sufficient conditions for Sobolev embeddings built on rearrangement-invariant spaces on the Carnot-Carathéodory spaces. Our approach involves certain essential tools from [17] as well as sharp rearrangement iteration techniques developed recently in [8].

Our principal result is a reduction-type theorem which provides a sufficient condition for the validity of a higher-order Sobolev embedding in the fairly general setting of the rearrangement-invariant spaces in terms of a function which estimates from below the behaviour of the isoperimetric function of the underlying domain near zero. We also give applications of the general theorem to embeddings of Sobolev spaces build upon Lebesgue, Lorentz and Orlicz spaces on the Heisenberg group.

The simple setting of the Heisenberg group allows us to show that the considered condition is necessary as well.

The paper is structured as follows. In the following section we collect some basic notions and known results. The main results are presented in the third section. In the fourth section an auxiliary lemma is proved and further facts are observed. The final section contains proofs of the main results.

2 Carnot-Carathéodory spaces and rearrangement-invariant function spaces

Throughout this paper we will be considering $\mathbb{R}^n$ equipped with the standard Lebesgue measure (denoted as $|\cdot|$) and a special differential structure. All integrals will be considered with respect to Lebesgue measure of appropriate dimension. In most cases, a general system of vector fields will be involved, while in some particular cases we will restrict ourselves to specific tangent fields of the Heisenberg group. This fact will be clearly indicated where appropriate.

By Ω we will denote an open set in $\mathbb{R}^n$ with $|\Omega| = 1$. The restriction on volume of Ω is adopted for convenience only. In the cases when Ω is having a different (finite) measure, analogous results hold with different constants. Sometimes additional properties will be required.

We denote by $\mathfrak{M}(\Omega)$ the set of all Lebesgue-measurable functions on Ω whose values belong to $[-\infty, \infty]$. Moreover, let us denote $\mathfrak{M}_+(\Omega) = \{u \in \mathfrak{M}(\Omega) : u \geq 0\}$.

Assume that $X = \{X_1, \ldots, X_m\}$ is a system of vector fields given by

$$X_j = \sum_{k=1}^{n} b_{j,k}(x) \frac{\partial}{\partial x_k},\ j = 1, \ldots, m,$$

where $b_{j,k} \in \mathcal{C}^\infty(\mathbb{R}^n)$.

The simplest choice of $\left\{X_j = \frac{\partial}{\partial x_j}, j = 1, \ldots, n\right\}$ would yield the classical Euclidean case.

A piecewise $\mathcal{C}^1$-curve $\gamma : [0, T] \to \mathbb{R}^n$ is called *horizontal* if whenever $\gamma'(t)$ exists then

$$\gamma'(t) = \sum_{j=1}^{m} c_j(t) X_j(\gamma(t)),$$

where $c_j : (0, T) \to \mathbb{R}$ are measurable, satisfying $\sum_{j=1}^m c_j^2(t) \leq 1$ for $0 \leq t \leq T$. The horizontal length of γ is defined by $l_h(\gamma) = T$.

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Let us denote by $\mathcal{H}$ the family of all horizontal curves. Distance function corresponding X is defined by

$$d(x,y) = \inf\{l_h(\gamma) : \gamma \in \mathcal{H}, \gamma(0) = x, \gamma(l_h(\gamma)) = y\},\ x, y \in \mathbb{R}^n.$$

If $d(x,y)$ is a metric, then this metric space is called the Carnot-Carathéodory space, generated by the system X.

Throughout this paper we assume that this distance function is a metric, specially that $d(x,y) < \infty$ for all $x, y \in \mathbb{R}^n$, and that the topology generated by it is the same as the classical Euclidean topology. This is ensured if the system X enjoys the so-called Hörmander finite-rank condition. For more details on Carnot-Carethéodory spaces, we refer the reader for instance to [18], [31] or [20].

For a function $f \in L^1_{\mathrm{loc}}(\Omega)$ its distributional derivative along the vector field X_j, $X_j f$, is defined by the identity

$$\langle X_j f, \phi \rangle = \int_\Omega f X_j^* \phi \, dx \ \text{ for every } \phi \in \mathcal{C}_0^\infty(\Omega),$$

where $X_j^*(.) = -\sum_{k=1}^n \frac{\partial}{\partial x_k}(b_{j,k}\,.)$ denotes the formal adjoint of X_j. Throughout the paper, if f is a non-smooth function, $X_j f$ will be meant in the distributional sense. We define the vector of *X-gradient* of a function f as

$$X\nabla f = (X_1 f, X_2 f, \ldots, X_m f)\,.$$

Moreover, let us introduce the *higher-order derivatives* operators as

$$XD_\alpha(.) = X_{\alpha_1}\left(X_{\alpha_2}\left(\ldots X_{\alpha_k}\left(.\right)\ldots\right)\right),$$

where $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{1, \ldots, m\}^k$. The *$X$-gradient of order k* is defined as a vector of length m^k of the following form:

$$X\nabla^k f = \left(XD_\alpha(f) : \alpha \in \{1, \ldots, m\}^k\right).$$

Naturally, the norm of the X-gradient of order k reads as

$$\left|X\nabla^k f\right|^2 = \sum_{\alpha \in m^k} (XD_\alpha(f))^2.$$

The X-variation and the X-perimeter can be defined as follows: If we denote

$$\mathcal{F}_\Omega = \left\{ \phi = \{\phi_1, \phi_2, \ldots \phi_m\} \in \mathcal{C}_0^1(\Omega \to \mathbb{R}^m) : \sup_{x\in\Omega} \left(\sum_{j=1}^m |\phi_j(x)|^2 \right)^{\frac{1}{2}} \le 1 \right\},$$

then, for a given $u \in L^1_{loc}(\Omega)$, the *X-variation* of u with respect to Ω is defined as

$$\mathrm{Var}_X(u,\Omega) = \sup_{\phi \in \mathcal{F}_\Omega} \int_\Omega u(x) \sum_{j=1}^m X_j^* \phi_j(x)\, dx.$$

The set of functions with bounded X-variation is denoted as $BV_X(\Omega)$ and forms a Banach space with respect to the norm

$$\|.\|_{BV_X} = \|.\|_{L_1(\Omega)} + \mathrm{Var}_X(.,\Omega).$$

If $X\nabla f \in L^1(\Omega)$, then

$$\mathrm{Var}_X(f,\Omega) \le \hat{C}\, \|X\nabla f\|_{L^1} \tag{1}$$

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where $\hat{C} > 0$ depends only on m.

If $E \subset \mathbb{R}^n$ is measurable, then the *X-perimeter* of E relative to Ω is defined by

$$P_X(E, \Omega) = \mathrm{Var}_X(\chi_E, \Omega),$$

where χ_E denotes the characteristic function of E. The *X-isoperimetric function* of Ω is given by the following formula

$$I_{X,\Omega}(s) = \inf \left\{ P_X(E, \Omega) : E \subset \Omega, s \leq |E| \leq \frac{1}{2} \right\} \text{ for } s \in \left[0, \frac{1}{2}\right],$$

and $I_{X,\Omega}(s) = I_{X,\Omega}(1-s)$ if $s \in \left(\frac{1}{2}, 1\right]$.

An open set $\Omega \subset \mathbb{R}^n$ is called an *X-John domain* if there exist a constant $c \in (0,1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\omega : [0, l] \to \Omega$, parametrised by arc-length, such that $\omega(0) = x$, $\omega(l) = x_0$, and

$$d(\omega(r), \partial_X \Omega) \geq cr \text{ for } r \in [0, l].$$

Now we turn our attention to the rearrangement-invariant function spaces. The basic reference for reader interested in more details is [2]. We recall the non-increasing rearrangement and distribution function.

Let $u \in \mathfrak{M}(\Omega)$, then

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}|, \quad t \in [0, \infty),$$

is the *distribution function* of u.

Let (R, λ) and (S, μ) be two measurable spaces. Functions $u \in \mathfrak{M}(R, \lambda)$ and $v \in \mathfrak{M}(S, \mu)$ are called equimeasurable if $\mu_u = \mu_v$ (on $\mathbb{R}^+$).

The *non-increasing rearrangement* of function $u \in \mathfrak{M}(R, \lambda)$ is then defined as

$$u^*(t) = \inf \{s \geq 0 : \mu_u(s) \leq t\}, \quad t \in [0, \infty).$$

A mapping $\rho : \mathfrak{M}_+(R, \lambda) \to [0, \infty]$ is called a *rearrangement-invariant Banach function norm* if, for all f, g and $\{f_n\}_{n \in \mathbb{N}}$ in $\mathfrak{M}_+(R, \lambda)$, every $a \geq 0$, and for all Lebesgue measurable $E \subset \Omega$, the following properties hold:

1. $\rho(f) = 0$ if and only if $f = 0$ a.e.. Moreover, $\rho(af) = a\rho(f)$ and $\rho(f+g) \leq \rho(f) + \rho(g)$.
2. If $0 \leq g \leq f$ a.e. then $\rho(g) \leq \rho(f)$.
3. If $0 \leq f_n \uparrow f$ a.e. then $\rho(f_n) \uparrow \rho(f)$.
4. If $|E| < \infty$ then $\rho(\chi_E) < \infty$.
5. If $|E| < \infty$ then $\int_E f \, d\lambda \leq C_E \rho(f)$, for some constant C_E, $0 < C_E < \infty$, depending on E and ρ but independent of f.
6. $\varrho(f) = \varrho(g)$ for every pair of equimeasurable functions f and g in $\mathfrak{M}_+(R, \lambda)$.

The collection $\mathbf{X} = \mathbf{X}_\rho$ of all functions $f \in \mathfrak{M}(R, \lambda)$ for which $\rho(|f|) < \infty$ is called a *rearrangement-invariant Banach function space* (r.i. space). For each $f \in \mathbf{X}$, define

$$\|f\|_{\mathbf{X}} = \rho(|f|).$$

With any r.i. Banach function norm ϱ, it is associated another functional ϱ' is defined on $\mathfrak{M}_+(R, \lambda)$ by

$$\varrho'(g) = \sup \left\{ \int_\Omega fg : f \in \mathfrak{M}_+(R, \lambda), \varrho(f) \leq 1 \right\}, \ g \in \mathfrak{M}_+(R, \lambda).$$

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It turns out that ϱ' is a r.i. Banach function norm, which is called *associate function norm* of ϱ. Let us note that if $|\Omega| < \infty$ then to any r.i. Banach function space $\mathbf{X}(\Omega)$ there exists a representation rearrangement-invariant Banach function norm $\varrho_{\mathbf{X}} : \mathfrak{M}_+(0,1) \to [0,\infty]$ such that

$$\|f\|_{\mathbf{X}(\Omega)} = \varrho_{\mathbf{X}}\left(f^*\left(|\Omega|\cdot\right)\right),\ f \in \mathbf{X}(\Omega).$$

This allows us to work sometimes with function spaces over simple measurable space $(0,1)$ instead of with function spaces over Ω.

Next, we recall some examples of r.i. spaces and some of their basic properties. This matter is depending only on the properties of measurable space (Ω, λ_n) which is the same as in the classical Euclidean case. More detailed treatment of this topic, together with proofs of subsequent statements can be found in [32]. The most common examples of a rearrangement-invariant Banach function norms are the Lebesgue norms, defined for $f \in \mathfrak{M}(\Omega)$ as

$$\|f\|_{L^p(\Omega)} = \left(\int_0^{|\Omega|} \left(f^*\right)^p(t)\right)^{\frac{1}{p}} dt$$

when $p \in [1,\infty)$, and

$$\|f\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_\Omega |f|$$

when $p = \infty$. The associate space to $L^p(\Omega)$ is $L^{p'}(\Omega) = L^{\frac{p-1}{p}}(\Omega)$.

Another examples are the Lorentz spaces $L^{p,q}(\Omega)$, determined by the functional

$$\|f\|_{L^{p,q}(\Omega)} = \left\|s^{\frac{1}{p}-\frac{1}{q}} f^*(s)\right\|_{L^q(0,|\Omega|)} \text{ where } 1 \le p,q \le \infty,\ f \in \mathfrak{M}_+(\Omega),$$

where the expression $\frac{1}{\infty}$ is considered to be zero. The function $\|.\|_{L^{p,q}(\Omega)}$ is equivalent to a rearrangement-invariant function norm if one of the following conditions is satisfied:

$$\begin{aligned} 1 < p < \infty, 1 \le q \le \infty,\\ p = q = 1,\\ p = q = \infty. \end{aligned}$$

It is easy to see that $L^{p,p}(\Omega) = L^p(\Omega)$ for $1 \le p \le \infty$ and $\left(L^{p,q}\right)'(\Omega) = L^{p',q'}(\Omega)$.

Let us set

$$\|f\|_{L^{p,q;\alpha}(0,1)} = \left\|s^{\frac{1}{p}-\frac{1}{q}} \log^\alpha\left(\frac{2}{s}\right) f^*(s)\right\|_{L^q(0,1)},$$

for $f \in \mathfrak{M}_+(0,1)$. If one of the following conditions

$$\begin{aligned} 1 < p < \infty, 1 \le q \le \infty, \alpha \in \mathbb{R};\\ p = 1, q = 1, \alpha \ge 0;\\ p = \infty, q = \infty, \alpha \le 0;\\ p = \infty, 1 \le q \le \infty, \alpha + \tfrac{1}{q} < 0, \end{aligned}$$

is satisfied, then $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ is equivalent to a rearrangement-invariant function norm, called a *Lorentz-Zygmund function norm*.

The last class of rearrangement-invariant function spaces that we will recall here are *Orlicz spaces*. Let $A : [0,\infty) \to [0,\infty]$ be a convex (non-trivial), left-continuous function vanishing at 0. Such function is called a *Young function*. The *Orlicz space* $L^A(\Omega)$ is the rearrangement-invariant space associated with the *Luxemburg function norm* defined by

$$\|f\|_{L^A(0,1)} = \inf\left\{\lambda > 0 : \int_0^1 A\left(\frac{f(s)}{\lambda}\right) ds \le 1\right\} \text{ for } f \in \mathfrak{M}_+(0,1).$$

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In the rest of this section we will recall some basic notions from the theory of function spaces.

Let $m \in \mathbb{N}$ and let $\mathbf{X}(\Omega)$ be a rearrangement-invariant Banach function space. We define the m-th order Sobolev space $V_X^m\mathbf{X}(\Omega)$ as the set of all functions $f \in \mathfrak{M}(\Omega)$ such that $X\nabla^m f$ exists in a distributional sense and it is represented by locally integrable functions such that $|X\nabla^m f| \in \mathbf{X}(\Omega)$.

If we adopt an additional restriction on behaviour of the isoperimetric function near zero, then functions in $V_X^m\mathbf{X}(\Omega)$ will also lie in $V_X^k L^1(\Omega)$ for $k < m$. Namely, if

$$I_{\Omega,X}(s) \geq \bar{C}s \text{ for } s \in \left[0, \frac{1}{2}\right] \tag{2}$$

with some constant $\bar{C} > 0$, then

$$V_X^m\mathbf{X}(\Omega) \to V_X^k L^1(\Omega), \text{ for } k = 0, 1, \dots, m-1.$$

The exact statement and its proof is given in the Proposition 11.

Provided that (2) holds, the $V_X^m\mathbf{X}(\Omega)$ forms a normed linear space with respect to the norm

$$\|u\|_{V_X^m\mathbf{X}(\Omega)} = \sum_{k=0}^{m-1} \left\|X\nabla^k u\right\|_{L^1(\Omega)} + \|X\nabla^m u\|_{\mathbf{X}(\Omega)}.$$

Given two function spaces $\mathbf{X}(\Omega)$ and $\mathbf{Y}(\Omega)$, the notation $\mathbf{X}(\Omega) \to \mathbf{Y}(\Omega)$ represents the fact that there exists a constant C independent of $f \in \mathbf{X}(\Omega)$ such that

$$\|f\|_{\mathbf{Y}} \leq C \|f\|_{\mathbf{X}}.$$

In such case we say that $\mathbf{X}(\Omega)$ is *embedded* into $\mathbf{Y}(\Omega)$. By saying that $\mathbf{Y}$ is the *optimal target* in $\mathbf{X}(\Omega) \to \mathbf{Y}(\Omega)$ we mean that for any function space $\mathbf{Z}$ satisfying $\mathbf{X}(\Omega) \to \mathbf{Z}(\Omega)$ one necessarily has $\mathbf{Y}(\Omega) \to \mathbf{Z}(\Omega)$.

3 Main results

In this section we state our principal results. We shall often work with a certain monotone function I on $[0,1]$ that estimates the isoperimetric function $I_{\Omega,X}$ from below, rather than with $I_{\Omega,X}$ itself. More precisely, in the statements below, we shall often assume that there exists a nondecreasing function $I : [0,1] \to [0,\infty)$ and a constant $c > 0$ such that

$$I_{X,\Omega}(s) \geq cI(cs) \quad \text{for } s \in \left[0, \frac{1}{2}\right] \tag{3}$$

and

$$\inf_{t\in(0,1)} \frac{I(t)}{t} > 0. \tag{4}$$

Theorem 1 (Reduction theorem) *Assume that $\Omega \subset \mathbb{R}^n$ is open, $|\Omega| = 1$, such that there is some non-deceasing function I satisfying (3) and (4). Let $m \in \mathbb{N}$, and let $\|.\|_{\mathbf{X}(0,1)}$ and $\|.\|_{\mathbf{Y}(0,1)}$ be rearrangement-invariant function norms. If there exists a constant $C > 0$ such that*

$$\left\| \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds \right\|_{\mathbf{Y}(0,1)} \leq C \|f\|_{\mathbf{X}(0,1)} \tag{5}$$

for every nonnegative $f \in \mathfrak{M}_+(0,1)$, then

$$V_X^m\mathbf{X}(\Omega) \to \mathbf{Y}(\Omega). \tag{6}$$

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Corollary 2 (Sobolev embeddings into L^∞) *Assume that $\Omega \subset \mathbb{R}^n$ is open, $|\Omega| = 1$, such that there is some non-deceasing function I satisfying (3) and (4). Let $m \in \mathbb{N}$, and let $\|.\|_{\mathbf{X}(0,1)}$ be rearrangement-invariant function norm. If*

$$\left\| \frac{1}{I(s)} \left(\int_0^s \frac{dr}{I(r)} \right)^{m-1} \right\|_{\mathbf{X}'(0,1)} < \infty, \tag{7}$$

then

$$V_X^m \mathbf{X}(\Omega) \to L^\infty(\Omega).$$

In this generality, the isoperimetric function is usually unknown. However, in [17] it was shown that the isoperimetric function can be evaluated if some additional conditions holds. The first such condition is the following version of the doubling condition: for any set $U \subset \mathbb{R}^n$ with $\operatorname{diam}(U) < \infty$, there exist constants $C_1 > 0$ and $R_0 < \infty$ such that for $x_0 \in U$ and $0 < R < R_0$ one has

$$|B(x_0, 2R)| \le C_1 \, |B(x_0, R)| \, . \tag{8}$$

It was shown in [31] that the finite-rank Hörmander condition implies the doubling condition.

The second restriction is the following version of the Poincaré inequality: for any set $U \subset \mathbb{R}^n$ with $\operatorname{diam}(U) < \infty$, there exist constants $C_2 > 0$, $R_0 < \infty$ and $\alpha \ge 1$ such that for $x_0 \in U$, $0 < R < R_0$ and every Lipschitz function u in $\alpha B = B(x_0, \alpha R)$, we have for any $\lambda > 0$

$$\left| \left\{ x \in B : \left| u(x) - \int_B u(x)\, dx \right| > \lambda \right\} \right| \le \frac{C_2}{\lambda} \int_{\alpha B} |X \nabla u(y)| \, dy. \tag{9}$$

Let $U \subset \mathbb{R}^n$ and denote by C the smallest constant in (8). Then the homogeneous dimension relative to U (and X) is defined by

$$Q = \log_2(C).$$

An open set $\Omega \subset \mathbb{R}^n$ is called *X-PS domain* if there exist a covering $\{B\}_{B \in \mathcal{F}}$ of Ω by metric balls and numbers $N > 0$, $\alpha \ge 1$, and $\nu \ge 1$ such that

1. $\sum_{B \in \mathcal{F}} \chi_{(\alpha+1)B}(x) \le N \chi_\Omega(x)$ for every $x \in \Omega$.
2. There exists (central) ball $B_0 \in \mathcal{F}$ such that for any $B \in \mathcal{F}$ one can find a chain $B_0, B_1, \dots, B_{s(B)} = B$, with $B_i \cap B_{i+1} \ne \emptyset$ and $|B_i \cap B_{i+1}| \ge \frac{1}{N} \max\left(|B_i|, |B_{i+1}|\right)$.
3. For any $i = 0, \dots, s(B)$, one has $B \subset \nu B_i$.

If the metric space $(\mathbb{R}^n, d)$ is in addition complete and a length-space, then it is shown in [17] that metric balls with small diameter are X-PS domains.

The class of X-PS domains is larger than the class of non-tangentially accessible domains (introduced in [22]), the class of extension domains (introduced in [24]) and X-John domain. In [30] it is shown that if X is generated by structure of a step two homogeneous group then any $\mathcal{C}^{1,1}$ domain is a X-NTA domain and consequently a X-PS domain. More examples of X-NTA and therefore X-PS domains can be found in [5]. However, the task of finding X-PS domains in a general setting is rather non-trivial.

In [17, Theorem 1.18] it is shown that if $\Omega \subset \mathbb{R}^n$ is an X-PS domain (and conditions (8) and (9) are in hold), then

$$I_{X,\Omega}(s) \le \begin{cases} \dfrac{C}{\operatorname{diam}(\Omega)|\Omega|^{-\frac{1}{Q}}} s^{\frac{Q-1}{Q}}, & \text{for } s \in \left[0, \frac{1}{2}\right]; \\ \dfrac{C}{\operatorname{diam}(\Omega)|\Omega|^{-\frac{1}{Q}}} (1-s)^{\frac{Q-1}{Q}}, & \text{for } s \in \left(\frac{1}{2}, 1\right], \end{cases} \tag{10}$$

where Q is the homogeneous dimension relative to Ω.

The known and sufficiently regular isoperimetric function of X-PS domains yields the following simplification of the condition in the reduction theorem.

Theorem 3 (Reduction theorem for X-PS domains) *Let us consider the case when X is such that the conditions (8), (9) are fulfilled. Assume that $\Omega \subset \mathbb{R}^n$ is an X-PS domain, $|\Omega| = 1$, denote by Q the homogeneous dimension relative to Ω. Let $m \in \mathbb{N}$, and let $\|.\|_{\mathbf{X}(0,1)}$ and $\|.\|_{\mathbf{Y}(0,1)}$ be rearrangement-invariant function norms. If there exists a constant C such that*

$$\left\| \int_t^1 f(s) s^{-1+\frac{m}{Q}} \, ds \right\|_{\mathbf{Y}(0,1)} \le C \|f\|_{\mathbf{X}(0,1)} \tag{11}$$

for every nonnegative $f \in \mathfrak{M}(0,1)$, then

$$V_X^m \mathbf{X}(\Omega) \to \mathbf{Y}(\Omega). \tag{12}$$

We shall now state some particular embeddings which follow from our main theorems.

Theorem 4 *Assume that the conditions (8), (9) are fulfilled, let $\Omega \subset \mathbb{R}^n$ be an X-PS domain and denote by Q the homogeneous dimension relative to Ω. Let $m \in \mathbb{N}$ and $p \in [1, \infty]$. Then the following embeddings hold:*

$$V_X^m L^p(\Omega) \to \begin{cases} L^{\frac{Qp}{Q-mp}}(\Omega) & \text{if } m < Q \text{ and } 1 \le p < \frac{Q}{m}, \\ L^r(\Omega) & \text{for } r \in [1,\infty), \text{ if } m < Q \text{ and } p = \frac{Q}{m}. \\ L^\infty(\Omega) & \text{otherwise.} \end{cases} \tag{13}$$

Theorem 5 *Assume that the conditions (8), (9) are fulfilled, let $\Omega \subset \mathbb{R}^n$ be an X-PS domain and denote Q the homogeneous dimension relative to Ω and X. Let $m \in \mathbb{N}$, $1 < p < \infty$ and $1 \le q \le \infty$, then*

$$V_X^m L^{p,q}(\Omega) \to \begin{cases} L^{\frac{Qp}{Q-mp},q}(\Omega) & \text{if } m < Q \text{ and } 1 \le p < \frac{Q}{m}, \\ L^{\infty,q;-1}(\Omega) & \text{if } m < Q, p = \frac{Q}{m} \text{ and } q > 1. \\ L^\infty(\Omega) & \text{otherwise.} \end{cases} \tag{14}$$

Theorem 4 covers the cases $L^{1,1}(\Omega) = L^1(\Omega)$ and $L^{\infty,\infty}(\Omega) = L^\infty(\Omega)$, therefore all settings of parameters p and r which leads to r.i. function norm are covered in Theorems 4 and 5.

Theorem 6 *Assume that the conditions (8), (9) are fulfilled, let $\Omega \subset \mathbb{R}^n$ be an X-PS domain and denote by Q the homogeneous dimension relative to Ω. Let $m \in \mathbb{N}$ and let A be a Young function fulfilling*

$$\int_0^s \left(\frac{t}{A(t)} \right)^{\frac{m}{Q-m}} dt < \infty, \text{ for some } s > 0. \tag{15}$$

Let us introduce the following notation of a Young function $A_{m,Q}$

$$A_{m,Q}(t) = \int_0^{\mathcal{H}^{-1}_{\frac{m}{Q}}(t)} \frac{A(\tau)}{\tau} \, d\tau,$$

where

$$\mathcal{H}_p(\tau) = \left(\int_0^\tau \left(\frac{\varrho}{A(\varrho)} \right)^{\frac{1}{p-1}} d\varrho \right)^{1-\frac{1}{p}}.$$

If $m < Q$ and the integral

$$\int_s^\infty \left(\frac{1}{A(t)} \right)^{\frac{m}{4-m}} dt, \tag{16}$$

for some $s > 0$ diverges, then

$$V_X^m L^A(\Omega) \to L^{A_{m,Q}}(\Omega). \tag{17}$$

Moreover, if one of the following conditions holds:

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- $m \geq Q$,
- $m < Q$ *and the integral (16) converges,*

then

$$V_X^m L^A(\Omega) \to L^\infty(\Omega). \tag{18}$$

In the rest of this section we will turn our attention to the setting of the Heisenberg group. It can be represented as $\mathbb{R}^3$ (with the corresponding group operation) endowed with the system of vector fields

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} \text{ and } X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3}.$$

Let us denote

$$\mathbb{H} = \{X_1, X_2\}.$$

The Carnot-Carathéodory space generated by $\mathbb{H}$ satisfies the Hörmander condition and conditions (8), (9). The homogeneous dimension of $\mathbb{H}$-PS domains is $Q = 4$. Moreover, in the case of the Heisenberg group the $\mathbb{H}$-PS domains coincide with $\mathbb{H}$-John domains as it is shown in [17].

Theorem 4, Theorem 5 and Theorem 6 yield the following embeddings on the Heisenberg group.

Corollary 7 (Imbeddings of Lebesgue spaces on the Heisenberg group) *Let* $\Omega \subset \mathbb{R}^3$ *be a* $\mathbb{H}$*-PS domain. Let* $m \in \mathbb{N}$ *and* $p, q \in [1, \infty]$. *Then the following embeddings hold:*

$$V_{\mathbb{H}}^m L^p(\Omega) \to \begin{cases} L^{\frac{4p}{4-mp}}(\Omega) & \text{if } m < 4 \text{ and } 1 \leq p < \frac{4}{m}, \\ L^r(\Omega) & \text{for } r \in [1,\infty), \text{ if } m < 4 \text{ and } p = \frac{4}{m}, \\ L^\infty(\Omega) & \text{otherwise.} \end{cases} \tag{19}$$

Corollary 8 (Imbeddings of Lorentz spaces on the Heisenberg group) *Let* $\Omega \subset \mathbb{R}^3$ *be a* $\mathbb{H}$*-PS domain. Let* $m \in \mathbb{N}$, $1 < p < \infty$ *and* $1 \leq q \leq \infty$, *then*

$$V_{\mathbb{H}}^m L^{p,q}(\Omega) \to \begin{cases} L^{\frac{4p}{4-mp},q}(\Omega) & \text{if } m < 4 \text{ and } 1 \leq p < \frac{4}{m}, \\ L^{\infty,q;-1}(\Omega) & \text{if } m < 4, p = \frac{4}{m} \text{ and } q > 1, \\ L^\infty(\Omega) & \text{otherwise.} \end{cases} \tag{20}$$

Corollary 9 (Imbeddings of Orlicz spaces on the Heisenberg group) *Let* $\Omega \subset \mathbb{R}^3$, *an* $\mathbb{H}$*-PS domain. Let* $m \in \mathbb{N}$ *and let* A *be a Young function fulfilling*

$$\int_0^s \left(\frac{t}{A(t)}\right)^{\frac{m}{4-m}} dt < \infty, \text{ for some } s > 0.$$

If $m < 4$ *and the integral*

$$\int_s^\infty \left(\frac{1}{A(t)}\right)^{\frac{m}{4-m}} dt \tag{21}$$

for some $s > 0$ *diverges, then we have*

$$V_{\mathbb{H}}^m L^A(\Omega) \to L^{A_{m,4}}(\Omega). \tag{22}$$

Moreover, if one of the following conditions holds

- $m \geq 4$,
- $m < 4$ *and the integral (21) converges,*

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then

$$V^m_{\mathbb{H}} L^A(\Omega) \to L^\infty(\Omega). \tag{23}$$

In the case of the Heisenberg group the sufficient condition can be proved to be necessary as well.

Theorem 10 *Let $m \in \mathbb{N}$. Let $\|.\|_{\mathbf{X}(0,1)}$ and $\|.\|_{\mathbf{Y}(0,1)}$ be rearrangement-invariant function norms such that*

$$V^m_{\mathbb{H}} \mathbf{X}(\Omega) \to \mathbf{Y}(\Omega) \tag{24}$$

for all $\mathbb{H}$-John domains $\Omega \subset \mathbb{R}^3$. Then there exists a constant C such that

$$\left\| \int_t^1 f(s) s^{-1+\frac{m}{Q}}\, ds \right\|_{\mathbf{Y}(0,1)} \le C \|f\|_{\mathbf{X}(0,1)} \tag{25}$$

for all $f \in \mathfrak{M}_+(0,1)$.

4 Background theorems

We will state some theorems that will be used in proofs in Section 5. First, let us recall the Carnot-Carathéodory co-area formula, [17, Theorem 5.2]:

Let $\Omega \subset \mathbb{R}^n$ be open, $u \in BV_X(\Omega)$, and for $t \in \mathbb{R}$ denote

$$E_u(t) = \{x \in \Omega : u(x) > t\}.$$

Then the following holds:

- $P_X(E_u(t), \Omega) < \infty$ for a.e. $t \in \mathbb{R}$.
- Moreover,

$$\mathrm{Var}_X(u;\Omega) = \int_{-\infty}^{\infty} P_X(E_u(t), \Omega)\, dt. \tag{26}$$

- Conversely, if $u \in L^1(\Omega)$ and $\int_{-\infty}^{\infty} P_X(E_u(t), \Omega)\, dt < \infty$, then $u \in BV_X(\Omega)$.

Proposition 11 *Let $\Omega \subset \mathbb{R}^n$, with $|\Omega| < \infty$, $m \in \mathbb{N}$. Suppose that*

$$I_{\Omega,X}(s) \ge \bar{C} s \text{ for } s \in \left[0, \frac{1}{2}\right], \tag{27}$$

with some constant $\bar{C} > 0$. Then

$$V^m_X \mathbf{X}(\Omega) \to V^k_X L^1(\Omega), \text{ for } k = 0, 1, \ldots, m-1.$$

Proof. First, we will show that

$$C \|u - \mathrm{med}(u)\|_{L^1(\Omega)} \le \|X \nabla u\|_{L^1(\Omega)} \tag{28}$$

for every $u \in V^1_X L^1(\Omega)$, where $\mathrm{med}(u)$ is the real number satisfying

$$|\{u < \mathrm{med}(u)\}| \le \frac{|\Omega|}{2} \text{ and } |\{u > \mathrm{med}(u)\}| \le \frac{|\Omega|}{2}$$

and $C > 0$ is some constant independent of u. Without loss of generality, we can assume that $\mathrm{med}(u) = 0$, because both sides of inequality will yield the same value with function u as with function $u - \mathrm{med}(u)$. Let us set $u_+ = \frac{1}{2}(|u| + u)$ and $u_- = \frac{1}{2}(|u| - u)$. We have that $u_\pm \in L^1(\Omega)$, [17, Lemma 3.5]. Note that

$$|\{u_\pm > t\}| \le \frac{1}{2}, \text{ for } t > 0. \tag{29}$$

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Of course, thanks to (27), (29) and the isoperimetric inequality, we have

$$P_X(\{u_\pm > t, \Omega\}) \geq I_{X,\Omega}(|\{u_\pm > t, \Omega\}|) \geq \bar{C}\,|\{u_\pm > t, \Omega\}|\,, \tag{30}$$

where $\bar{C}$ is a constant from (27). Consequently, the inequalities (30) and (26) yield

$$\begin{aligned} \bar{C}\,\|u_\pm\|_{L^1(\Omega)} &= \bar{C}\int_0^\infty |\{u_\pm > t\}|\;dt \leq \int_0^\infty P_X\left(\{u_\pm > t\}\,,\Omega\right)\,dt \\ &\leq \int_\Omega |X\nabla u_\pm(x)|\;dx. \end{aligned}$$

It follows that $u - \operatorname{med}(u)$ is in $L^1(\Omega)$ for all $u \in V^1_X L^1(\Omega)$. Consequently, $u \in L^1(\Omega)$.

Since we are considering Ω of finite measure, we have $\mathbf{X}(\Omega) \to L^1(\Omega)$. Iterated use of the embedding $V^1_X L^1(\Omega) \to L^1(\Omega)$ yields our claim. □

Lemma 12 (Generalised Polya-Szegö principle) *Let $\Omega \subset \mathbb{R}^n$, Ω is open, and let $\mathbf{X}$ be an r.i. space, assume $u \in V^1_X\mathbf{X}(\Omega)$. Then u^* is locally absolutely continuous and*

$$C\left\|\left(-\frac{du^*}{ds}\right)I_{X,\Omega}(s)\right\|_{\mathbf{X}(0,1)} \leq \|X\nabla u\|_{\mathbf{X}(\Omega)}\,, \tag{31}$$

with $C > 0$ independent of $u \in V^1_X\mathbf{X}(\Omega)$.

P r o o f. Let us set

$$\phi(s) = \left(-\frac{du^*}{ds}\right)I_{X,\Omega}(s),\ 0 < s < 1.$$

If we show that

$$\int_0^s \phi^*(r)\,dr \leq \int_0^s |X\nabla u|^*(r)\,dr,\ 0 < s < 1, \tag{32}$$

then (31) will follow by applying the Hardy-Littlewood-Pólya principle, see [2, Chapter 2, Theorem 4.6]. Let $0 \leq a < b \leq |\Omega|$, we denote

$$D^u_{a,b} = \{x \in \Omega : u^*(a) > |u(x)| > u^*(b)\}\,.$$

We define $v = v_{a,b}$ by

$$v(x) = \begin{cases} (u^*(a) - u^*(b))\operatorname{sign} u(x) & x \in \{y : |u(y)| \geq u^*(a)\}, \\ u(x) - u^*(b)\operatorname{sign} u(x) & x \in \{y : u^*(a) > |u(y)| > u^*(b)\}, \\ 0 & x \in \{y : u^*(b) \geq |u(y)|\}. \end{cases}$$

Function v can be obtained from u through several applications of subtraction of a constant, multiplication by (-1) and truncation operator defined

$$u^+(x) = \max[u(x), 0],$$

for $x \in \Omega$, $u \in \mathfrak{M}(\Omega)$. But [17, Lemma 3.5] yields that

$$X\nabla u^+ = \begin{cases} X\nabla u & \text{a. e. on } \{x \in \Omega : u(x) \geq 0\}\,, \\ 0 & \text{otherwise,} \end{cases}$$

for $u \in V^1_X L^1(\Omega)$. Therefore we have that $|X\nabla u| = |X\nabla v|$ almost everywhere in $D^u_{a,b}$ and $X\nabla v = 0$ elsewhere. Consequently we have that $v \in V^1_X L^1(\Omega)$.

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If we apply the equalities (1) and (26) to the function v, we get

$$\int_{D^u_{a,b}} |X\nabla u|\; dx = \int_\Omega |X\nabla v|\; dx \geq C \int_{u^*(b)}^{u^*(a)} P_X\left(E_v(t),\Omega\right)\; dt, \tag{33}$$

where C is a constant dependent only on m. Now, we use the definition of $I_{X,\Omega}$ to get

$$\begin{aligned} \int_{u^*(b)}^{u^*(a)} P_X\left(E_v(t),\Omega\right)\; dt &\geq \int_{u^*(b)}^{u^*(a)} I_{X,\Omega}\left(|E_v(t)|\right)\; dt \\ &\geq (u^*(a)-u^*(b)) \min\left[I_{X,\Omega}(a), I_{X,\Omega}(b)\right]. \end{aligned} \tag{34}$$

The latter inequality together with the inequality

$$|\{x\in\Omega : u^*(a) > |u(x)| > u^*(b)\}| \leq b-a, \tag{35}$$

and the absolute continuity of Lebesgue integral ensures that u^* is locally absolutely continuous. We apply change of variables $u^*(s)=t$ to the combination of equations (33) and (34) to get

$$\int_{D^u_{a,b}} |X\nabla u(x)|\; dx \geq \int_a^b I_{X,\Omega}\left(s\right)\left(-\frac{du^*}{dt}\right)(s)\, ds.$$

The Hardy-Littlewood inequality, together with (35) yields that for any disjoint system of intervals

$$\{(a_i,b_i), i=1,\dots,k; a_i,b_i\in[0,|\Omega|)\}$$

we have

$$\int_{\cup(a_i,b_i)} \phi(r)\, dr \leq \int_0^{\sum b_i-a_i} |X\nabla u|^*\,(r)\, dr.$$

Consequently, the outer regularity of the Lebesgue measure yields

$$\sup_{E\subset(0,|\Omega|):|E|=s} \int_E \phi(r)\, dr \leq \int_0^s |X\nabla u|^*\,(r)\, dr,\;\; 0<s<|\Omega|\,.$$

Since $\mathbb{R}^1$ is resonant, we get

$$\int_0^s \phi^*(r)\, dr \leq \int_0^s |X\nabla u|^*\,(r)\, dr,$$

for $0<s<|\Omega|$. Therefore (32) holds and an application of Hardy-Littlewood-Polya principle completes the proof. □

In the remaining part of this section we survey some auxiliary tools that will be useful in the proof of our main results. We follow the notation from [8]. Since these facts are connected only to certain operators defined on functions on real line, the Carnot-Carathéodory structure won't interfere.

Let $I:[0,1]\to[0,\infty)$ be a measurable function satisfying (4). We define the following two operators from $\mathfrak{M}_+(0,1)$ into $\mathfrak{M}_+(0,1)$ by

$$H_I(f)(t) = \int_t^1 \frac{f(s)}{I(s)}\, ds,$$

and

$$R_I(f)(t) = \frac{1}{I(t)}\int_0^t f(s)\, ds \text{ for } t\in(0,1].$$

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Moreover, given $j \in \mathbb{N}$, we set

$$H_I^j(f) = \underbrace{H_I \circ H_I \circ \cdots \circ H_I}_{j-\text{times}}(f).$$

We also set $H_I^0 = \mathrm{Id}$.

Given a rearrangement-invariant function space $\mathbf{X}(0,1)$, we set

$$\|f\|_{\mathbf{X}'_{m,I}(0,1)} = \left\| \frac{1}{I(t)} \int_0^t \left(\int_s^t \frac{dr}{I(r)} \right)^{m-1} f^*(s)\, ds \right\|_{\mathbf{X}'(0,1)}.$$

Next, define $\|.\|_{\mathbf{X}_j(0,1)}$ as the rearrangement-invariant function norm whose associate norm $\|.\|_{\mathbf{X}'_j(0,1)}$ is given, via iteration, by $\|.\|_{\mathbf{X}'_0(0,1)} = \|.\|_{\mathbf{X}'(0,1)}$, and, for $j \geq 1$, by

$$\|f\|_{\mathbf{X}'_j(0,1)} = \|R_I(f^*)\|_{\mathbf{X}'_{j-1}(0,1)}, \text{ for } f \in \mathfrak{M}_+(0,1).$$

We recall that it was shown in [8] that for every $m \in \mathbb{N}$ and $f \in \mathfrak{M}_+(0,1)$, one has

$$H_I^m f(t) = \frac{1}{(m-1)!} \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)}\, dr \right)^{m-1} ds. \tag{36}$$

Moreover, if we define the norm $\|.\|_{\mathbf{X}_{m,I}}$ as the associated norm to $\|.\|_{\mathbf{X}'_{m,I}(0,1)}$, then

$$H_I^j : \mathbf{X}(0,1) \to \mathbf{X}_{j,I}(0,1),\ j \in \mathbb{N}, \tag{37}$$

and $\mathbf{X}_{j,I}$ is the optimal target among all r.i. spaces.

Furthermore,

$$\mathbf{X}_1(0,1) = \mathbf{X}_{1,I}(0,1) \text{ and } \mathbf{X}_j(0,1) = \underbrace{\Big(\ldots(\mathbf{X}_{1,I})_{1,I}\ldots\Big)_{1,I}}_{j-\text{times}}(0,1),$$

for $j \in \mathbb{N}$. In particular,

$$(\mathbf{X}_k)_h(0,1) = \mathbf{X}_{k+h}(0,1). \tag{38}$$

5 Proofs of the main results

The proof of the reduction theorem is split into two steps. First we prove our claim in the case $m = 1$, using an approach similar to the one from [15]. The second step is then based on an iteration of the first-order result.

Proof of Theorem 1. *Step one.* Assume that $m = 1$. Fix $u \in V_X^1\mathbf{X}(\Omega)$ and let $t \in [0, \frac{1}{2}]$, set $\alpha = \lim_{t\to\frac{1}{2}-} u^*(t)$, then we have

$$\begin{aligned}
u^*(t) &= \int_t^{\frac{1}{2}} -\left(I_{X,\Omega}\frac{du^*}{ds}\right)(s)\, \frac{1}{I_{X,\Omega}(s)}\, ds + \alpha \\
&\leq \int_t^{\frac{1}{2}} \left|-\left(I_{X,\Omega}\frac{du^*}{ds}\right)(s)\right| \frac{1}{I_{X,\Omega}(s)}\, ds + \alpha \\
&\leq \int_t^{\frac{1}{2}} \left|-\left(I_{X,\Omega}\frac{du^*}{ds}\right)(s)\right| \frac{1}{c}\frac{1}{I(cs)}\, ds + \alpha,
\end{aligned}$$

where c is the constant from (3). Moreover, the property 4. from definition of r.i. Banach function norm yields

$$\left\|\chi_{(0,\frac{1}{2})}(t)\alpha\right\|_{\mathbf{Y}} \leq \alpha C_{\mathbf{Y}} \leq 2C_{\mathbf{Y}}\, \|u^*(t)\|_{L^1(0,1)},$$

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where $C_{\mathbf{Y}} > 0$ is a constant independent of u. Consequently, we have

$$\begin{aligned} \|u\|_{\mathbf{Y}(\Omega)} &= \|u^*(t)\|_{\mathbf{Y}(0,1)} \leq 2 \left\| u^*(t)\, \chi_{[0,\frac{1}{2}]}(t) \right\|_{\mathbf{Y}(0,1)} \\ &\leq \frac{2}{c} \left\| \chi_{[0,\frac{1}{2}]}(t) \int_t^{\frac{1}{2}} \left| \left(I_{X,\Omega} \frac{du^*}{ds} \right)(s) \right| \frac{1}{I(cs)}\, ds \right\|_{\mathbf{Y}(0,1)} + 2C_{\mathbf{Y}} \|u^*\|_{L_1(0,1)} \\ &\leq \frac{2}{c} \left\| \int_t^1 \left| \left(I_{X,\Omega} \frac{du^*}{ds} \right)(s) \right| \frac{1}{I(cs)}\, ds \right\|_{\mathbf{Y}(0,1)} + 2C_{\mathbf{Y}} \|u^*\|_{L_1(0,1)} . \end{aligned}$$

Using the assumption of this theorem and boundedness of dilations on rearrangement-invariant spaces, we get

$$\begin{aligned} \|u\|_{\mathbf{Y}(\Omega)} &\leq \frac{2}{c} \left\| \left(I_{X,\Omega} \frac{du^*}{ds} \right)(cs)\; ds \right\|_{\mathbf{X}(0,1)} + 4C_{\mathbf{Y}} \|u^*\|_{L_1(0,1)} \\ &\leq \frac{2C'}{c} \left\| \left(I_{X,\Omega} \frac{du^*}{ds} \right)(s)\; ds \right\|_{\mathbf{X}(0,1)} + 4C_{\mathbf{Y}} \|u^*\|_{L_1(0,1)} , \end{aligned}$$

where C' is the constant of boundedness of dilation operator on $\mathbf{Y}(0,1)$ ([2, Chapter 3, Proposition 5.11]).

An application of Lemma 12 then yields our claim.

Step two. Now suppose that $m > 1$. From (37) with $j = 1$, we get

$$\left\| H_I^1 f \right\|_{\mathbf{X}_{1,I}(0,1)} = \left\| \int_t^1 \frac{f(s)}{I(s)}\, ds \right\|_{\mathbf{X}_{1,I}(0,1)} \leq C \|f\|_{\mathbf{X}(0,1)} , \text{ for } f \in \mathfrak{M}_+(0,1),$$

thus (5) holds with $m = 1$ and $\mathbf{Y}(0,1) = \mathbf{X}_{1,I}(0,1)$. Hence we get (from the case $m = 1$)

$$V_X^1 \mathbf{X}(\Omega) \to \mathbf{X}_1(\Omega).$$

Consequently, if we apply this embedding to all spaces $\mathbf{X}_j(\Omega)$, for $j = 0, \ldots, m-1$, we get

$$V_X^1 \mathbf{X}_j(\Omega) \to \mathbf{X}_{j+1}(\Omega),$$

in other words, there are constants $C_j > 0$, $j = 0, \ldots, m-1$, such that

$$\|u\|_{\mathbf{X}_{j+1}(\Omega)} \leq C_j \left(\|u\|_{L_1(\Omega)} + \|X\nabla u\|_{\mathbf{X}_j(\Omega)} \right),$$

for all $u \in V_X^1 \mathbf{X}_j(\Omega)$, $j = 0, \ldots, m-1$. Therefore, there is a constant $C' > 0$ such that

$$\begin{aligned} \|u\|_{\mathbf{X}_m(\Omega)} &\leq C_m \|u\|_{V_X^1 \mathbf{X}_{m-1}(\Omega)} = C_m \left(\|u\|_{L_1(\Omega)} + \|X\nabla u\|_{\mathbf{X}_{m-1}(\Omega)} \right) \\ &\leq C_m \left(\|u\|_{L_1(\Omega)} + C_{m-1} \left(\|X\nabla u\|_{L_1(\Omega)} + \left\|X\nabla^2 u\right\|_{\mathbf{X}_{m-2}(\Omega)} \right) \right) \\ &\;\;\vdots \\ &\leq \left(\Pi_{i=1}^m \max\{1, C_i\} \right) \left(\sum_{j=1}^{m-1} \left\| X\nabla^j u \right\|_{L^1(\Omega)} + \|X\nabla^m u\|_{\mathbf{X}(\Omega)} \right) \\ &\leq C' \|u\|_{V_X^m \mathbf{X}(\Omega)} \end{aligned}$$

holds for all $u \in V_X^1 \mathbf{X}_j(\Omega)$. Hence,

$$V_X^m \mathbf{X}(\Omega) \to \mathbf{X}_m(\Omega).$$

The assumption of this theorem implies that

$$H_I^m : \mathbf{X}(0,1) \to \mathbf{Y}(0,1).$$

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Optimality of the space $\mathbf{X}_{m,I}(0,1)$ as a target in (37) yields

$$\mathbf{X}_{m,I}(0,1) \to \mathbf{Y}(0,1).$$

Altogether, then there are constants $D, D', D'' > 0$ such that for all $u \in V_X^m\mathbf{X}(\Omega)$ one has

$$\|u\|_{\mathbf{Y}(\Omega)} = \|u^*\|_{\mathbf{Y}(0,1)} \le D\,\|u^*\|_{\mathbf{X}_{m,I}(0,1)} \le D'\,\|u^*\|_{\mathbf{X}_m(0,1)} \le D''\,\|u\|_{V_X^m\mathbf{X}(\Omega)}.$$

In other words

$$V_X^m\mathbf{X}(\Omega) \to \mathbf{Y}(\Omega),$$

as desired. □

Proof of Corollary 2. We have for $f \in \mathfrak{M}_+(0,1)$

$$\left\|\int_t^1 \frac{f(s)}{I(s)}\left(\int_t^s \frac{dr}{I(r)}\right)^{m-1} ds\right\|_{L^\infty(0,1)} = \int_0^1 \frac{f(s)}{I(s)}\left(\int_0^s \frac{dr}{I(r)}\right)^{m-1} ds.$$

Therefore,

$$\sup_{f\ge 0, \|f\|_{\mathbf{X}(0,1)}\le 1}\left\|\int_t^1 \frac{f(s)}{I(s)}\left(\int_t^s \frac{dr}{I(r)}\right)^{m-1} ds\right\|_{L^\infty(0,1)} = \\ = \left\|\frac{1}{I(s)}\left(\int_0^s \frac{dr}{I(r)}\right)^{m-1}\right\|_{\mathbf{X}'(0,1)}.$$

This shows that (5) with $\mathbf{Y}(0,1) = L^\infty(0,1)$ is equivalent to (7), and the application of Theorem 1 yields the claim. □

Proof of Theorem 3. Assumptions of this Theorem combined with (10) yield that the function $I(s) = s^{\frac{Q-1}{Q}}$ satisfies conditions (3) and (4). For $f \in \mathfrak{M}_+(0,1)$ it holds

$$\begin{aligned}\int_t^1 \frac{f(s)}{I(s)}\left(\int_t^s \frac{dr}{I(r)}\right)^{m-1} ds &= \int_t^1 f(s)s^{-\frac{Q-1}{Q}}\left(\int_t^s \frac{dr}{r^{\frac{Q-1}{Q}}}\right)^{m-1} ds \\ &\le \int_t^1 f(s)s^{-\frac{Q-1}{Q}}\left(\int_0^s \frac{dr}{r^{\frac{Q-1}{Q}}}\right)^{m-1} ds \\ &\le \int_t^1 f(s)s^{-1+\frac{m}{Q}}\,ds.\end{aligned}$$

Therefore, the inequality (11) yields

$$\left\|\int_t^1 \frac{f(s)}{I(s)}\left(\int_t^s \frac{dr}{I(r)}\right)^{m-1} ds\right\|_{\mathbf{Y}(0,1)} \le \left\|\int_t^1 f(s)s^{-1+\frac{m}{Q}}\,ds\right\|_{L^1(\Omega)} \le C\,\|f\|_{\mathbf{X}(0,1)},$$

where C is a constant from (11). This means that the assumptions of Theorem 1 are verified and its application completes the proof. □

Proof of Theorem 4. Function $I(s) = s^{\frac{Q-1}{Q}}$ is estimating the isoperimetric function $I_{X,\Omega}$ near zero in sense of conditions (3) and (4). In [8, proof of Theorem 6.8] it is shown that

$$\left(L^p\right)_{m,I}(0,1) \to L^{\frac{Qp}{Q-mp}}(0,1), \text{ if } m < Q \text{ and } 1 \le p < \frac{Q}{m},$$

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and

$$(L^p)_{m,I}(0,1) \to L^r(0,1) \text{ for } r \in [1,\infty), \text{ if } m < Q \text{ and } p = \frac{Q}{m}.$$

This, together with (37) and (36) ensures that there exist a constants $C_1, C_2 > 0$ independent of $f \in \mathfrak{M}(0,1)$ such that

$$\left\| \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds \right\|_{L^{\frac{Qp}{Q-mp}}(0,1)} \le C_1 \|f\|_{L^p(0,1)}$$

if $m < Q$, $1 \le p < \frac{Q}{m}$, and

$$\left\| \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds \right\|_{L^r(0,1)} \le C_2 \|f\|_{L^p(0,1)},$$

for $r \in [1,\infty)$, provided that $m < Q$ and $p = \frac{Q}{m}$. First two cases in (19) thus follow by application of Theorem 1.

The proof of the last case relies on Corollary 2. There is a constant $C_3 > 0$ depending on m and Q such that

$$\begin{aligned} \Phi &= \left\| \frac{1}{I(s)} \left(\int_0^s \frac{dr}{I(r)} \right)^{m-1} \right\|_{L^{p'}(0,1)}^{p'} = C \left\| s^{\frac{m}{Q}-1} \right\|_{L^{p'}(0,1)}^{p'} \\ &= C \int_0^1 \left(s^{\frac{m}{Q}-1} \right)^{p'} ds = C \int_0^1 s^{\left(\frac{m}{Q}-1\right)\frac{p}{p-1}} ds. \end{aligned}$$

If $m \ge Q$, then $\Phi < \infty$ because the exponent of s inside the integral on the last line is not negative. If $m < Q$ and $p > \frac{Q}{m}$, then

$$\left(\frac{m}{Q} - 1 \right) \frac{p}{p-1} > \left(\frac{m}{Q} - 1 \right) \frac{\frac{Q}{m}}{\frac{Q}{m} - 1} = -1.$$

Therefore $\Phi < \infty$ if $m \ge Q$ or $m < Q$ and $p > \frac{Q}{m}$. Assumptions of Corollary 2 are verified and it's application will provide the rest of (13). □

Proof of Theorem 5. It is shown in [8, proof of Theorem 6.9] that if $m < Q$ and $1 \le p < \frac{Q}{m}$ then

$$\|f\|_{(L^{p,q})'_{m,I}(0,1)} \approx_c \|f\|_{\left(L^{\frac{Qp}{Q-mp},q}\right)'(0,1)},$$

where the equivalence holds up to constants depending on p and q. Therefore, (37) yields that

$$H_I^j : (L^{p,q})_{m,I}(0,1) \to L^{\frac{Qp}{Q-mp},q}(0,1)$$

and the claim follows from Theorem 1.

Additionally, it is shown in [8] that if $m < Q$, $p = \frac{Q}{m}$ and $q > 1$ then

$$\|f\|_{(L^{p,q})'_{m,I}(0,1)} \approx_c \|f\|_{(L^{\infty,q;-1})'(0,1)}.$$

Application of Theorem 1 then yields the claim.

To achieve embeddings into $L^\infty(\Omega)$, we will use Corollary 2. The inequality (7) in our setting reads as

$$\left\| s^{\frac{m}{Q}-1} \right\|_{L^{p',q'}(0,1)} < \infty. \tag{39}$$

This can be verified by simple computation. □

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Proof of Theorem 6. It is shown in [6, Theorem 3.5, with choice of $p = \frac{m}{Q}$ and $q = \frac{p}{p-1}$] that for A fulfilling (15) it holds with some constant $C_1 > 0$ that

$$\left\| \int_t^1 f(s) s^{-1+\frac{m}{Q}} \, ds \right\|_{L^{A_{m,Q}}(0,1)} \leq C_1 \|f\|_{L^A(0,1)} , \; f \in \mathfrak{M}(0,1),$$

if integral (16) diverges for some $s > 0$ and $m < Q$. Therefore, Theorem 3 implies our claim in that case. It is shown in [6] as well, that if $m < Q$ and (16) converges for some $s > 0$, then there is a constant $C_2 > 0$ such that

$$\left\| \int_t^1 f(s) s^{-1+\frac{m}{Q}} \, ds \right\|_{L^\infty(0,1)} \leq C_2 \|f\|_{L^A(0,1)} , \; f \in \mathfrak{M}(0,1),$$

which yields the second case. The remaining case when $m \geq Q$ follows from the application of Theorem 3. We have that if $\varepsilon = -1 + \frac{m}{Q} \geq 0$, then for some $C_3 > 0$

$$\left\| \int_0^1 f(s) s^{\varepsilon} \, ds \right\|_{L^\infty(0,1)} \leq \int_0^1 f(s) s^{\varepsilon} \, ds \leq C_3 \|f\|_{L^A(0,1)} , \; f \in \mathfrak{M}(0,1),$$

since $L^A(0,1) \subset L^1(0,1)$. □

Corollaries 7, 8 and 9 are just particular cases of appropriate more general results above.

Proof of Theorem 10 uses the notion of *Korányi gauge* (see [25]) on the Heisenberg group:

$$\|(x_1, x_2, x_3)\|_{\mathbb{H}}^4 = (x_1^2 + x_2^2)^2 + 16 x_3^2, \quad \text{for } (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Korányi ball with centre $(0,0,0)$ and diameter s is defined as follows:

$$B_{\mathbb{H}}(s) = \{ x \in \mathbb{H} : \|x\|_H \leq s \} .$$

Let us note that $|B_{\mathbb{H}}(s)| = K s^4$, for some $K > 0$ and that $B_{\mathbb{H}}$ is a $\mathbb{H}$-John domain [5, Corollary 5].

To a given function $f \in \mathfrak{M}_+(0,1)$, let us define the m-th *order symmetral*

$$u_{f,m}(x) = g(\|x\|_{\mathbb{H}}) = \int_{K\|x\|_{\mathbb{H}}^4}^1 \int_{t_1}^1 \ldots \int_{t_{m-1}}^1 f(t_m) t_m^{-m+\frac{m}{4}} \, dt_m \ldots \, dt_2 \, dt_1 .$$

where $x \in B = B_{\mathbb{H}}(1)$. Functions $u_{f,m}$ are fabricated in such way that norm of their m-th order gradient can be controlled by properties of function f, as it is shown in the following lemma.

Lemma 13 *Let $f \in \mathfrak{M}(0,1)$, $m \in \mathbb{N}$, then*

$$|\mathbb{H}\nabla_m(u_{f,m})(x)| \quad \leq \quad \hat{C} \left(f(K\|x\|_{\mathbb{H}}^4) + \sum_{j=1}^{m-1} \|x\|_{\mathbb{H}}^{4j-m} \int_{K\|x\|_{\mathbb{H}}^4}^1 f(s) s^{m-\frac{m}{4}} \, ds \right),$$

for $x \in B$.

P r o o f. Let us begin by stating few technical observations. Consider $k \geq 1$, $\alpha \in \{1,2\}^k$ and $x \in B$, then

1. $$\mathbb{H}D_\alpha \|x\|_{\mathbb{H}} = \sum_{i=1}^{h} C_i \frac{x_1^{e_{i,1}} x_2^{e_{i,2}} x_3^{e_{i,3}}}{\|x\|_{\mathbb{H}}^{p_i}}$$

   for some constants $e_{i,1}, e_{i,2}, e_{i,3}, p_i$ and C_i, where h depends only on k. This can be shown by induction with respect to k through simple computation.

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2.

$$|\mathbb{H}D_\alpha \|x\|_{\mathbb{H}}| \leq \frac{C}{\|x\|_{\mathbb{H}}^{k-1}},$$

where C depends only on k. This inequality is easily verified for $k = 1$. The case when $k > 1$ follows by induction. According to the previous observation, $(k-1)$-th derivatives must be written as a linear combination of expressions

$$P_{k-1,i} = \frac{x_1^{e_{k-1,i,1}} x_2^{e_{k-1,i,2}} x_3^{e_{k-1,i,3}}}{\|x\|_{\mathbb{H}}^{p_{k-1,i}}}, i = 1, \dots, h.$$

Thanks to the induction assumption it must hold for all exponents $p_{k-1,i}$, $e_{k-1,i,1}$, $e_{k-1,i,2}$ and $e_{k-1,i,3}$ that

$$p_{k-1,i} - (e_{k-1,i,1} + e_{k-1,i,2} + 2e_{k-1,i,3}) < k - 2.$$

Computation of $X_{\alpha_1}(P_{k-1,i})$ then yields the desired inequality for k.

3. $\mathbb{H}D_\alpha u_{f,m}(x)$ can be expressed in the following way:

$$\mathbb{H}D_\alpha u_{f,m}(x) = \sum_{i=1}^{n} C_i g^{(l_i)}(\|x\|_{\mathbb{H}})\, \mathbb{H}D_{\alpha_{i,1}} \|x\|_{\mathbb{H}} \dots \mathbb{H}D_{\alpha_{i,h}} \|x\|_{\mathbb{H}},$$

where $l_i \in \mathbb{N}$, $\alpha_{i,j}$ are multi-indices which satisfy

$$\sum_{j=1}^{h} (|\alpha_{i,j}| - 1) = |\alpha| - l_i, \text{ for all } i = 1, \dots, n, \tag{40}$$

and n is a constant depending only on k. Particularly, if $l_i = k$ then $|\alpha_{i,j}| = 1$ for all j. This can be shown by induction as well. The induction step consists of verification that the application of vector field X_{α_1} preserve condition (40).

4. There is $C' > 0$ such that

$$\left| g^{(l)}(\|x\|_{\mathbb{H}}) \right| \leq C' \sum_{j=1}^{l} \|x\|_{\mathbb{H}}^{4j-l} \int_{K\|x\|_{\mathbb{H}}}^{1} f(s) s^{m-\frac{m}{4}}\, ds,\ x \in B.$$

This inequality follows from the fact that $g^{(l)}(\|x\|_{\mathbb{H}})$ is a linear combination of the expressions

$$R_{l,j} = \|x\|_{\mathbb{H}}^{4j-l} \int_{K\|x\|_{\mathbb{H}}^4}^{1} \dots \int_{t_{m-1}}^{1} f(t_m) t_m^{m-\frac{m}{4}}\, dt_m \dots dt_{j+1}.$$

Simple estimates obtained by extending domains of integration of integrals yield that there is $\hat{C} > 0$ such that

$$R_{l,j} \leq \hat{C} \|x\|_{\mathbb{H}}^{4j-l} \int_{K\|x\|_{\mathbb{H}}^4}^{1} f(s) s^{m-\frac{m}{4}}\, ds,\ f \in \mathfrak{M}(0,1).$$

Combining given observations we get

$$\begin{aligned}
|\mathbb{H}\nabla_m(u_f)(x)| &\leq C \sum_{l=1}^{m} \frac{g^{(l)}(\|x\|_{\mathbb{H}})}{\|x\|_{\mathbb{H}}^{m-l}} \\
&\leq \hat{C} \sum_{j=1}^{l} \|x\|_{\mathbb{H}}^{4j-m} \int_{K\|x\|_{\mathbb{H}}^4}^{1} f(s) s^{m-\frac{m}{4}}\, ds \\
&\leq \hat{C} \left(f(K\|x\|_{\mathbb{H}}^4) + \sum_{j=1}^{m-1} \|x\|_{\mathbb{H}}^{4j-m} \int_{K\|x\|_{\mathbb{H}}^4}^{1} f(s) s^{m-\frac{m}{4}}\, ds \right).
\end{aligned}$$

□

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Proof of Theorem 10. Fix $f \in \mathfrak{M}_+(0,1)$. Constants $C_1, C_2, \dots, C_6$ used below are independent of choice of f. Using the previous lemma we get with some $C_1 > 0$

$$\||\mathbb{H}\nabla_m(u_f)|\|_{\mathbf{X}(B)} \leq C_1 \left(\left\| f(K\|x\|_{\mathbb{H}}^4) \right\|_{\mathbf{X}(B)} + \sum_{j=1}^{m-1} \left\| \|x\|_{\mathbb{H}}^{4j-m} \int_{K\|x\|_{\mathbb{H}}}^1 f(s) s^{m-\frac{m}{4}}\, ds \right\|_{\mathbf{X}(B)} \right)$$

and therefore

$$\left\| |\mathbb{H}\nabla_m(u_f)|^* \right\|_{\mathbf{X}(0,1)} \leq C_1 \left(\|f\|_{\mathbf{X}(0,1)} + \sum_{j=1}^{m-1} \left\| j^{4j-m} \int_j^1 f(s) s^{m-\frac{m}{4}}\, ds \right\|_{\mathbf{X}(0,1)} \right).$$

Since, for $j = 1, \dots, m_1$,

$$\left\| t^{4j-m} \int_t^1 f(s) s^{m-\frac{m}{4}}\, ds \right\|_{L^1(0,1)} \leq \|f\|_{L^1(0,1)}$$

and

$$\left\| t^{4j-m} \int_t^1 f(s) s^{m-\frac{m}{4}}\, ds \right\|_{L^\infty(0,1)} \leq \|f\|_{L^\infty(0,1)},$$

the standard interpolation theory ([2, Chapter 3, Theorem 2.12]) yields that

$$\left\| t^{4j-m} \int_t^1 f(s) s^{m-\frac{m}{4}}\, ds \right\|_{\mathbf{X}(0,1)} \leq C_2 \|f\|_{\mathbf{X}(0,1)},$$

Hence we have established following estimate

$$\left\| |\mathbb{H}\nabla_m(u_f)|^* \right\|_{\mathbf{X}(0,1)} \leq C_1(C_2+1)m \|f\|_{\mathbf{X}(0,1)}. \tag{41}$$

The assumption of this Theorem yields that there exists a constant $C_3 > 0$ such that

$$\begin{aligned} \||\mathbb{H}\nabla_m(u_f)|\|_{\mathbf{X}(B)} &\geq C_3 \left\| u_f^* \right\|_{\mathbf{Y}(0,1)} \\ &= C_3 \left\| \int_t^1 \int_{t_1}^1 \dots \int_{t_{m-1}}^1 f(t_m) t_m^{-m+\frac{m}{4}}\, dt_m \dots dt_2\, dt_1 \right\|_{\mathbf{Y}(0,1)}. \end{aligned}$$

Sequential applications of the Fubini Theorem then yield

$$\begin{aligned} &\int_{t_{m-1}}^1 f(t_m) t_m^{-m+\frac{m}{4}}\, dt_m \dots dt_2\, dt_1 \\ &\quad = \int_t^1 f(t_m) t_m^{-m+\frac{m}{4}} \int_t^{t_1} \dots \int_t^{t_{m-1}} dt_{m-1} \dots dt_1\, dt_m \\ &\quad = \int_t^1 f(t_m) t_m^{-m+\frac{m}{4}} \frac{(t_m - t)^{m-1}}{(m-1)!}\, dt_m \end{aligned}$$

If $s \in (2t, 1)$ then there exists a constant $C_4 > 0$ such that $\frac{(s-t)^{m-1}}{(m-1)!} \geq s^{m-1}$. This observation yields

$$\begin{aligned} \int_t^1 f(s) s^{-m+\frac{m}{4}} \frac{(s-t)^{m-1}}{(m-1)!}\, ds &\geq \chi_{(0,1)}(2t) \int_{2t}^1 f(s) s^{-m+\frac{m}{4}} \frac{(s-t)^{m-1}}{(m-1)!}\, ds \\ &\geq C_4 \chi_{(0,1)}(2t) \left(\int_{2t}^1 f(s) s^{-1+\frac{m}{4}}\, ds \right). \end{aligned}$$

Let us set

$$\phi(t) = \int_t^1 f(s) s^{\frac{m}{4}-1}\, ds \chi_{(0,1)}(t).$$

Boundedness of the dilation operator on r.i. function spaces then yields that there is a constant $C_5 > 0$ such that

$$\|\phi(t)\|_{\mathbf{Y}(0,1)} \leq C_5 \|\phi(2t)\|_{\mathbf{Y}(0,1)}.$$

Consequently, there is a constant $C_6 > 0$ such that

$$\| |\mathbb{H}\nabla_m(u_f)| \|_{\mathbf{X}(B)} \geq C_6 \left\| \int_t^1 f(s) s^{\frac{m}{4}-1}\, ds \right\|_{\mathbf{Y}(0,1)}. \tag{42}$$

Altogether, by combining inequalities (41) and (42) we get

$$\left\| \int_t^1 f(s) s^{\frac{m}{4}-1}\, ds \right\|_{\mathbf{Y}(0,1)} \leq \frac{1}{C_6} \| |\mathbb{H}\nabla_m(u_f)| \|_{\mathbf{X}(B)} \leq \frac{C_1(C_2+1)m}{C_6} \|f\|_{\mathbf{X}(0,1)},$$

which concludes the proof. □

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Chapter 3

Compact embeddings on Carnot-Carathéodory spaces

HIGHER-ORDER COMPACT EMBEDDINGS OF FUNCTION SPACES ON CARNOT-CARATHÉODORY SPACES

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Abstract. A sufficient condition for higher-order compact embeddings on bounded domains in Carnot-Carathéodory spaces is established for the class of rearrangement-invariant function spaces. The condition is expressed in terms of compactness of a suitable one-dimensional integral operator depending on the isoperimetric function relative to the Carnot-Carathéodory structure of the relevant sets. The general result is then applied to particular Sobolev spaces built upon Lebesgue and Lorentz spaces.

1. Introduction

One of the most important characteristics of Sobolev spaces is how they relate to other spaces. This sort of information is usually expressed in terms of (continuous) embeddings, and compact embeddings. Compact embeddings are of particular interest from the point of view of applications of Sobolev spaces in mathematical physics, calculus of variations, economical sciences and the probability theory. A compact embedding can be used to pave a path pointing towards a solution to a given partial differential equation or to show the discreteness of the spectra of linear elliptic partial differential operators defined over bounded domains.

One of the first classical compactness results originated in a lemma by Rellich [33] and was later proved specifically for Sobolev spaces by Kondrachov [24]. These results of course found their way to classical textbooks such as [1]. Compact embeddings of Sobolev spaces have been ever since the subject of an extensive research as a very important topic of functional analysis. Obtained results extend far beyond the original context of underlining measurable space $\mathbb{R}^n$ to various classes or domains in different measurable spaces.

It has been realized that the quality of an embedding of a Sobolev space into another appropriate space is closely connected to the isoperimetric profile of the underlying domain, and even that Sobolev embeddings can be derived from isoperimetric inequalities. The problem of Sobolev inequalities and function spaces embeddings can thus be approached through isoperimetric inequality. Their deep connection was observed by Maz'ya [26] and [27] and also by Federer and Fleming [13] in early 1960's.

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The approach to Sobolev spaces via isoperimetric inequalities allows one to consider Sobolev embeddings from much wider perspective than that of the classical Euclidean setting. Examples of important non-Euclidean embeddings include, for instance, the Gaussian-Sobolev embeddings studied in the connection with the so-called logarithmic Sobolev inequalities (see e.g. [16, 2]), a central subject in the investigation of hypercontractive semigroups. On the other hand, investigation of Sobolev embeddings has been carried out on Carnot-Carathéodory spaces, where Sobolev spaces are build with respect to a different differential operator, and whose pivotal example is the Heisenberg chain.

While the Sobolev embeddings on Carnot-Carathéodory spaces have been studied to some extent ([8, 11, 10, 9, 12, 15, 17, 18, 20, 28, 25, 31, 6]), very little is known about compactness of such embeddings. In this paper we concentrate on this problem.

The isoperimetric approach was successfully applied in the context of Carnot-Carathéodory spaces to the problem of establishing higher-order Sobolev-type embeddings in [14]. Our main aim here is to determine when a Sobolev embedding on a Carnot-Carathéodory space is compact. We intend to work under quite general setting of rearrangement-invariant spaces.

The Carnot-Carathéodory spaces (also known as Sub-Riemannian spaces) possess an exciting range of applications ranging from quantum mechanics (where we can also find the origin of the most famous example, the Heisenberg group), through the control theory to exotic applications such as automatic animation of physically plausible trajectories of vehicles in computer graphics [23]. This paper continues in this trend by applying the readily prepared tools in [15], [7] and [14] to adapting state-of-art proofs from [35] to Carnot-Carathéodory spaces settings.

One of the main advantages of the isoperimetric approach to embeddings of Sobolev spaces is the possibility to extend the embeddings to the classes of rearrangement-invariant function spaces and higher-order embeddings. Moreover, it allows us to reduce sufficient condition on embeddings over Carnot-Carathéodory spaces to condition on certain one-dimensional operator over $\mathbb{R}$.

This paper is structured as follows. In Section 2 we collect a necessary background material. In particular we fix all the indispensable basic notions concerning Carnot-Carathéodory spaces, the rearrangement spaces, and the isoperimetric inequalities (as our approach is built on the combination of these three topics). In Section 3 we state the main theorems. In Section 4, for the readers convenience, the authors collect known results that will be used in the proof. In the final section we present the proofs of the main results.

2. Settings

Let X be a system of vector fields $X_1, \dots, X_m$ such that

$$X_i = \sum_{j=1}^{n} b_{i,j}(x) \frac{\partial}{\partial x_j},$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $b_{i,j} \colon \mathbb{R}^n \to \mathbb{R}$, $b_ij \in \mathcal{C}^\infty(\mathbb{R}^n)$ (with respect to the classical Euclidean topology).

The simplest choice of $\left\{X_j = \frac{\partial}{\partial x_j}, j = 1, \ldots, n\right\}$ would yield the classical Euclidean case.

A piecewise $\mathcal{C}^1$-curve $\gamma\colon [0,T] \to \mathbb{R}^n$, $T > 0$, is called *horizontal* if whenever $\gamma'(t)$ exists then

$$\gamma'(t) = \sum_{j=1}^{m} c_j(t) X_j(\gamma(t)),$$

where $c_j\colon (0,T) \to \mathbb{R}$ are measurable and satisfying $\sum_{j=1}^m c_j^2(t) \le 1$ for $0 \le t \le T$. The horizontal length of γ is defined by $l_h(\gamma) = T$.

Let us denote by $\mathcal{H}$ the family of all horizontal curves. The distance function corresponding to X is defined by

$$d(x,y) = \inf\{l_h(\gamma)\colon \gamma \in \mathcal{H}, \gamma(0) = x, \gamma(l_h(\gamma)) = y\},\ x, y \in \mathbb{R}^n.$$

If $d(x,y)$ is a metric, then $\mathbb{R}^n$ equipped with $d(x,y)$ as metric is called the Carnot-Carathéodory space, generated by the system X.

Throughout this paper we assume that the distance function is a metric, especially that $d(x,y) < \infty$ for all $x, y \in \mathbb{R}^n$, and that the topology generated by it is the same as the classical Euclidean topology. It is known that this is ensured if the system X enjoys the so-called Hörmander finite-rank condition.

In this paper we assume that $\Omega \subset \mathbb{R}^n$ is open with $|\Omega| < \infty$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure.

For a function $f \in L^1_{\text{loc}}(\Omega)$ its distributional derivative along the vector field X_j, $X_j f$, is defined by the identity

$$\langle X_j f, \phi\rangle = \int_\Omega f X_j^* \phi\, dx \quad \text{for every } \phi \in \mathcal{C}_0^\infty(\Omega),$$

where $X_j^*(\cdot) = -\sum_{k=1}^n \frac{\partial}{\partial x_k}(b_{j,k}\,\cdot)$ denotes the formal adjoint of X_j. Throughout the paper, if f is a non-smooth function, $X_j f$ will be meant in the distributional sense. If derivatives $X_1 f, X_2 f, \ldots, X_m f$ exist, then the vector of X-*gradient* of a function f is defined by

$$X\nabla f = (X_1 f, X_2 f, \ldots, X_m f)\,.$$

Moreover, let us introduce the *higher-order derivatives* as

$$XD_\alpha(.) = X_{\alpha_1}\left(X_{\alpha_2}\left(\ldots X_{\alpha_k}\left(\cdot\right)\ldots\right)\right),$$

where $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{1, \ldots, m\}^k$. Provided that $XD_\alpha f$ exists for all $\alpha \in \{1, \ldots, m\}^k$, the X-*gradient of order* k is defined as a vector of length m^k of the following form:

$$X\nabla^k f = \left(XD_\alpha(f)\colon \alpha \in \{1, \ldots, m\}^k\right).$$

Naturally, the norm of the X-gradient of order k reads as

$$\left|X\nabla^k f\right|^2 = \sum_{\alpha\in m^k} (XD_\alpha(f))^2.$$

The X-variation and the X-perimeter can be defined as follows: if we denote

$$\mathcal{F}_\Omega = \left\{ \phi = \{\phi_1, \phi_2, \dots \phi_m\} \in \mathcal{C}^1_0(\Omega \to \mathbb{R}^m)\colon \sup_{x\in\Omega} \left(\sum_{j=1}^m |\phi_j(x)|^2 \right)^{\frac{1}{2}} \le 1 \right\},$$

then, for a given $u \in L^1_{loc}(\Omega)$, the *X-variation* of u with respect to Ω is defined as

$$\mathrm{Var}_X(u,\Omega) = \sup_{\phi\in\mathcal{F}_\Omega} \int_\Omega u(x) \sum_{j=1}^m X_j^* \phi_j(x)\, dx.$$

The set of functions with bounded X-variation is denoted as $BV_X(\Omega)$ and forms a Banach space with respect to the norm

$$\|\cdot\|_{BV_X} = \|\cdot\|_{L_1(\Omega)} + \mathrm{Var}_X(\cdot,\Omega).$$

If $X\nabla f \in L^1(\Omega)$, then

$$\mathrm{Var}_X(f,\Omega) \le \hat{C}\, \|X\nabla f\|_{L^1} \tag{2.1}$$

where $\hat{C} > 0$ depends only on m.

If $E \subset \mathbb{R}^n$ is measurable, then the *X-perimeter* of E relative to Ω is defined by

$$P_X(E,\Omega) = \mathrm{Var}_X(\chi_E,\Omega),$$

where χ_E denotes the characteristic function of E. The *X-isoperimetric function* of Ω is given by the following formula

$$I_{X,\Omega}(s) = \inf \left\{ P_X(E,\Omega)\colon E \subset \Omega, s \le |E| \le \frac{1}{2} \right\} \quad \text{for } s \in \left[0, \frac{1}{2}\right],$$

and $I_{X,\Omega}(s) = I_{X,\Omega}(1-s)$ if $s \in \left(\frac{1}{2}, 1\right]$.

Through this paper we will assume certain regularity of $I_{X,\Omega}(s)$, namely: suppose that there is some non-decreasing function $I\colon [0,1] \to \mathbb{R}$ satisfying

$$I_{X,\Omega}(s) \ge cI(cs) \quad \text{for } s \in \left[0, \frac{1}{2}\right] \tag{2.2}$$

with some constant $c > 0$, and

$$\inf_{t\in(0,1)} \frac{I(t)}{t} > 0. \tag{2.3}$$

In this generality, the isoperimetric function is usually unknown. However, in [15] it was shown that the isoperimetric function can be evaluated if some additional conditions hold.

The first such condition is the following version of the doubling condition: for any set $U \subset \mathbb{R}^n$ with $\mathrm{diam}(U) < \infty$, there exist constants $C_1 > 0$ and $R_0 < \infty$ such that for $x_0 \in U$ and $0 < R < R_0$ one has

$$|B(x_0, 2R)| \le C_1\, |B(x_0, R)|\,. \tag{2.4}$$

It was shown in [30] that the finite-rank Hörmander condition implies the doubling condition.

The second restriction is the following version of the Poincaré inequality: for any set $U \subset \mathbb{R}^n$ with $\mathrm{diam}(U) < \infty$, there exist constants $C_2 > 0$, $R_0 < \infty$

and $\alpha \geq 1$ such that for $x_0 \in U$, $0 < R < R_0$ and every Lipschitz function u in $\alpha B = B(x_0, \alpha R)$, we have for any $\lambda > 0$

$$\left| \left\{ x \in B \colon \left| u(x) - \int_B u(x)\, dx \right| > \lambda \right\} \right| \leq \frac{C_2}{\lambda} \int_{\alpha B} |X \nabla u(y)|\, dy. \tag{2.5}$$

The third restriction is that $(\mathbb{R}^n, d)$ is complete and it is a length-space; that is

$$d(x, y) = \inf l(\gamma_{xy}), \tag{2.6}$$

where γ is a continuous curve joining x to y, and $l(\gamma_{xy})$ denotes its metric length.

Let $U \subset \mathbb{R}^n$ and denote by C the smallest constant in (2.4). Then the homogeneous dimension relative to U (and X) is defined by

$$Q = \log_2(C).$$

Let us recall definitions of John domains and X-PS domains in the following two paragraphs.

An open set $\Omega \subset \mathbb{R}^n$ is called an *X-John domain* if there exist a constant $c \in (0, 1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\omega \colon [0, l] \to \Omega$, parametrized by arc-length, such that $\omega(0) = x$, $\omega(l) = x_0$, and

$$d(\omega(r), \partial\Omega) \geq cr \quad \text{for } r \in [0, l],$$

where $\partial\Omega$ denotes boundary of Ω.

An open set $\Omega \subset \mathbb{R}^n$ is called *X-PS domain* if there exist a covering $\{B\}_{B \in \mathcal{F}}$ of Ω by metric balls and numbers $N > 0$, $\alpha \geq 1$, and $\nu \geq 1$ such that

(1) $\sum_{B \in \mathcal{F}} \chi_{(\alpha+1)B}(x) \leq N \chi_\Omega(x)$ for every $x \in \Omega$.
(2) There exists a (central) ball $B_0 \in \mathcal{F}$ such that for any $B \in \mathcal{F}$ one can find a chain $B_0, B_1, \ldots, B_{s(B)} = B$, with $B_i \cap B_{i+1} \neq \emptyset$ and $|B_i \cap B_{i+1}| \geq \frac{1}{N} \max(|B_i|, |B_{i+1}|)$.
(3) For any $i = 0, \ldots, s(B)$, one has $B \subset \nu B_i$.

Though the class of John domains is better known in the context of Sobolev-type embeddings, we will state our results by means of the notion of X-PS domains. The class of X-PS domains contains that of X-John domains if (2.4) holds, which is always assumed in this paper. On the other hand, if certain geodesic segment property is satisfied then the class of X-John domains contains the class of X-PS domains. Both inclusions are shown in [15, Theorem 1.30]. Consequently, both classes coincide if both (2.4) and the geodesic segment property are satisfied.

If the metric space $(\mathbb{R}^n, d)$ is in addition complete and a length-space, then it is shown in [15] that metric balls with small diameter are X-PS domains.

The class of X-PS domains is larger than the class of non-tangentially accessible domains (introduced in [19]), the class of extension domains (introduced in [22]). In [29] it is shown that if X is generated by structure of a step two homogeneous group then any $\mathcal{C}^{1,1}$ domain is an X-NTA domain and consequently an X-PS domain. More examples of X-NTA and therefore X-PS domains can be found in [4]. However, the task of finding X-PS domains in a general setting is rather non-trivial.

In [15, Theorem 1.18] it is shown that if $\Omega \subset \mathbb{R}^n$ is an X-PS domain (and conditions (2.4) and (2.5) are in hold), then

$$I_{X,\Omega}(s) \leq \begin{cases} \frac{C}{\operatorname{diam}(\Omega)|\Omega|^{-\frac{1}{Q}}} s^{\frac{Q-1}{Q}}, & \text{for } s \in \left[0, \frac{1}{2}\right]; \\ \frac{C}{\operatorname{diam}(\Omega)|\Omega|^{-\frac{1}{Q}}} (1-s)^{\frac{Q-1}{Q}}, & \text{for } s \in \left]\frac{1}{2}, 1\right], \end{cases} \tag{2.7}$$

where Q is the homogeneous dimension relative to Ω.

Now we turn our attention to the rearrangement-invariant function spaces. The basic references and new ones for reader interested in more details are [3, 5, 21, 32, 36]. We recall the non-increasing rearrangement and distribution function.

Let $u \in \mathfrak{M}(\Omega)$, then

$$\mu_u(t) = |\{x \in \Omega \colon |u(x)| > t\}|, \quad t \in [0, \infty),$$

is the *distribution function* of u.

Let (R, λ) and (S, μ) be two measurable spaces. Functions $u \in \mathfrak{M}(R, \lambda)$ and $v \in \mathfrak{M}(S, \mu)$ are called equimeasurable if $\mu_u = \mu_v$ (on $\mathbb{R}^+$). In such case we write $u \sim v$.

The *non-increasing rearrangement* of function $u \in \mathfrak{M}(R, \lambda)$ is then defined as

$$u^*(t) = \inf\{s \geq 0 \colon \mu_u(s) \leq t\}, \quad t \in [0, \infty).$$

A mapping $\varrho \colon \mathfrak{M}_+(R, \lambda) \to [0, \infty]$ is called a *Banach function norm* if, for all f, g and $\{f_n\}_{n\in\mathbb{N}}$ in $\mathfrak{M}_+(R, \lambda)$, every $a \geq 0$, and for all Lebesgue measurable $E \subset \Omega$, the following properties hold:

(1) $\varrho(f) = 0$ if and only if $f = 0$ a.e.. Moreover, $\varrho(af) = a\varrho(f)$ and $\varrho(f+g) \leq \varrho(f) + \varrho(g)$.
(2) If $0 \leq g \leq f$ a.e. then $\varrho(g) \leq \varrho(f)$.
(3) If $0 \leq f_n \uparrow f$ a.e. then $\varrho(f_n) \uparrow \varrho(f)$.
(4) If $|E| < \infty$ then $\varrho(\chi_E) < \infty$.
(5) If $|E| < \infty$ then $\int_E f \, d\lambda \leq C_E \varrho(f)$, for some constant C_E, $0 < C_E < \infty$, depending on E and ϱ but independent of f.

If, in addition, ϱ satisfies $\varrho(f) = \varrho(g)$ for every pair of equimeasurable functions f and g in $\mathfrak{M}_+(R, \lambda)$, then ϱ is called a *rearrangement-invariant Banach function norm.*

The collection $\mathbf{X}(R, \mu) = \mathbf{X}_\varrho(R, \mu)$ of all functions $f \in \mathfrak{M}(R, \lambda)$ for which $\varrho(|f|) < \infty$ is called a *rearrangement-invariant Banach function space* (r.i. space). For each $f \in \mathbf{X}(R, \mu)$, define

$$\|f\|_{\mathbf{X}}(R, \mu) = \varrho(|f|).$$

Let us recall that there is functional ϱ' defined on $\mathfrak{M}_+(R, \lambda)$ by

$$\varrho'(g) = \sup\left\{\int_R fg \colon f \in \mathfrak{M}_+(R, \lambda), \varrho(f) \leq 1\right\}, \; g \in \mathfrak{M}_+(R, \lambda),$$

associated with r.i. Banach function norm ϱ. It turns out that ϱ' is an r.i. Banach function norm, which is called *associate function norm* of ϱ.

Let us note that if $|\Omega| < \infty$ then to any r.i. Banach function space $\mathbf{X}(\Omega)$ there exists a representation rearrangement-invariant Banach function norm

$$\varrho_{\mathbf{X}(\Omega)} \colon \mathfrak{M}_+(0,1) \to [0,\infty]$$

such that

$$\|f\|_{\mathbf{X}(\Omega)} = \varrho_{\mathbf{X}} \left(f^* \left(|\Omega| \cdot\right)\right), \ f \in \mathbf{X}(\Omega).$$

This allows us to work sometimes with function spaces over simple measurable space $(0,1)$ instead of with function spaces over Ω.

Let us now give some examples of r.i. norms. A basic example are the Lebesgue norms $L^p(0,1)$, $p \in [1,\infty]$, defined for all $f \in \mathfrak{M}(0,1)$ by

$$\|f\|_{L^p(0,1)} = \begin{cases} \left(\int_0^1 |f(x)|^p \, dx\right)^{\frac{1}{p}}, \ p < \infty; \\ \operatorname{esssup}_{x\in(0,1)} |f(x)|, \ p = \infty. \end{cases}$$

The corresponding r.i. spaces $l^p(R,\mu)$ are then called the Lebesgue spaces.

One can consider also more general functionals $\|\cdot\|_{L^{p,q}(0,1)}$ and $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$. They are given for any $f \in \mathfrak{M}(0,1)$ by

$$\|f\|_{L^{p,q}(0,1)} = \left\| f^*(s) s^{\frac{1}{p}-\frac{1}{q}} \right\|_{L^q(0,1)}$$

and

$$\|f\|_{L^{p,q,\alpha}(0,1)} = \left\| f^*(s) s^{\frac{1}{p}-\frac{1}{q}} \left(\log \frac{2}{s}\right)^{\alpha} \right\|_{L^q(0,1)},$$

respectively. Here, we assume that $p \in [1,\infty]$, $\alpha \in \mathbb{R}$, and use the convention that $\frac{1}{\infty} = 0$. Note that $\|\cdot\|_{L^p(0,1)} = \|\cdot\|_{L^{p,p}(0,1)}$ and $\|\cdot\|_{L^{p,q}(0,1)} = \|\cdot\|_{L^{p,q;0}(0,1)}$ for every such p and q. However, it turns out that under these assumptions on p, q and α, $\|\cdot\|_{L^{p,q}(0,1)}$ and $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ do not have to be r.i. norms. To ensure that $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ is equivalent to a r.i. norm, we need to assume that one of the following conditions is satisfied:

$$p = q = 1, \alpha \geq 0; \tag{2.8}$$

$$1 < p < \infty; \tag{2.9}$$

$$p = \infty, q < \infty, \alpha + \tfrac{1}{q} < 0; \tag{2.10}$$

$$p = q = \infty, \alpha \leq 0. \tag{2.11}$$

In this case, $\|\cdot\|_{L^{p,q}(0,1)}$ is called Lorentz norm, $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ is called Lorentz-Zygmund norm and the corresponding r.i. spaces $L^{p,q}(0,1)$ and $L^{p,q;\alpha}(0,1)$ are called Lorentz spaces and Lorentz-Zygmund spaces, respectively.

Let $m \in \mathbb{N}$ and let $\mathbf{X}(\Omega)$ be a rearrangement-invariant Banach function space. We define the m-th order Sobolev space $V^m_X\mathbf{X}(\Omega)$ as the set of all functions $f \in \mathfrak{M}(\Omega)$ such that $X\nabla^m f$ exists in a distributional sense and it is represented by locally integrable functions such that $|X\nabla^m f| \in \mathbf{X}(\Omega)$.

In [14] it is proved that if

$$I_{\Omega,X}(s) \geq \bar{C}s \text{ for } s \in \left[0, \frac{1}{2}\right] \tag{2.12}$$

with some constant $\bar{C} > 0$ then

$$V_X^m \mathbf{X}(\Omega) \subset V_X^k L^1(\Omega) \text{ for } k < m. \tag{2.13}$$

Provided that (2.12) holds, the $V_X^m \mathbf{X}(\Omega)$ forms a normed linear space with respect to the norm

$$\|u\|_{V_X^m \mathbf{X}(\Omega)} = \sum_{k=0}^{m-1} \left\|X\nabla^k u\right\|_{L^1(\Omega)} + \|X\nabla^m u\|_{\mathbf{X}(\Omega)}\,.$$

Moreover, by $W_X^m \mathbf{X}(\Omega)$ we denote the set of all functions $f \in (\Omega)$ such that for all $k = 0, 1, \dots m$, $X\nabla^k f$ exists in a distributional sense and it is represented by locally integrable functions such that $\left|X\nabla^k f\right| \in \mathbf{X}(\Omega)$. $W_X^m(\Omega)$ forms a normed linear space with respect to norm

$$\|u\|_{W_X^m \mathbf{X}(\Omega)} = \sum_{k=0}^{m} \left\|X\nabla^k u\right\|_{\mathbf{X}(\Omega)}\,.$$

Let (R, μ) be a measurable space. Given two function spaces $\mathbf{X}(R, \mu)$ and $\mathbf{Y}(R, \mu)$ (not necessarily rearrangement invariant), the notation

$$\mathbf{X}(R, \mu) \to \mathbf{Y}(R, \mu)$$

represents the fact that there exists a constant C independent of $f \in \mathbf{X}(R, \mu)$ such that

$$\|f\|_{\mathbf{Y}(R,\mu)} \le C \|f\|_{\mathbf{X}(R,\mu)}\,.$$

In such case we say that $\mathbf{X}(R, \mu)$ is *embedded* into $\mathbf{Y}(R, \mu)$. By saying that $\mathbf{Y}(R, \mu)$ is the *optimal target* in $\mathbf{X}(R, \mu) \to \mathbf{Y}(R, \mu)$ we mean that for any function space $\mathbf{Z}(R, \mu)$ satisfying $\mathbf{X}(R, \mu) \to \mathbf{Z}(R, \mu)$ one necessarily has $\mathbf{Y}(R, \mu) \to \mathbf{Z}(R, \mu)$.

If $\mathbf{X}(R, \mu) \to \mathbf{Y}(R, \mu)$ then the identity operator Id is continuous from $\mathbf{X}(R, \mu)$ to $\mathbf{Y}(R, \mu)$. If it is, in addition, compact we write

$$\mathbf{X}(R, \mu) \hookrightarrow\hookrightarrow \mathbf{Y}(R, \mu).$$

In such case we say that function space $\mathbf{X}(R, \mu)$ is *compactly embedded* into $\mathbf{Y}(R, \mu)$. The fact that operator T is compact from function space $\mathbf{X}(R, \mu)$ into $\mathbf{Y}(R, \mu)$ is denoted as

$$T\colon \mathbf{X}(R, \mu) \to\to \mathbf{Y}(R, \mu).$$

Suppose that $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ are rearrangement-invariant norms. We say that $\mathbf{X}(R, \mu)$ is *almost-compactly embedded* int $\mathbf{Y}(R, \mu)$ and write

$$\mathbf{X}(R, \mu) \overset{*}{\hookrightarrow} \mathbf{Y}(R, \mu)$$

if

$$\lim_{k\to\infty} \sup_{\|f\|_{\mathbf{X}(R,\mu)} \le 1} \|\chi_{E_k} f\|_{\mathbf{Y}(R,\mu)} = 0$$

is satisfied for every sequence $(E_k)_{k=1}^{\infty}$ of μ-measurable subset of R fulfilling $\chi_{E_k} \to 0$ μ-a.e.

3. The main theorems

The connection between Sobolev embeddings and certain Hardy-type operators in the setting of Carnot-Carathéodory spaces is established in [14]. Here, we are going to extend this connection to compactness of Sobolev embeddings.

Let $J\colon (0,1] \to (0,\infty)$ be a measurable function satisfying (2.3), we shall consider the operator $H_J\colon \mathfrak{M}(0,1) \to \mathfrak{M}(0,1)$ defined by

$$H_J f(t) = \int_t^1 \frac{|f(s)|}{J(s)}\, ds, \ f \in \mathfrak{M}(0,1) \text{ and } t \in (0,1). \tag{3.1}$$

Furthermore, given $j \in \mathbb{N}$, we define the operator H_J^j by

$$\underbrace{H_J \circ H_J \circ \cdots \circ H_J}_{j-\text{times}}(f). \tag{3.2}$$

Theorem 3.1. *Assume that (2.4), (2.5) and (2.6) are fulfilled. Let $\Omega \subset \mathbb{R}^n$ be open, let $m \in \mathbb{N}$. Suppose that there is some non-decreasing function $I\colon [0,1] \to \mathbb{R}$ satisfying (2.2) and (2.3), let $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ be rearrangement-invariant norms, then*

$$H_I^m\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1) \tag{3.3}$$

implies

$$V_X^m \mathbf{X}(\Omega) \hookrightarrow\hookrightarrow \mathbf{Y}(\Omega). \tag{3.4}$$

Remark 3.2. According to [35] the following two conditions are equivalent under the assumptions of Theorem 3.1:

- $H_I^m\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1)$,
- $\lim_{a\to 0+} \sup_{\|f\|_{\mathbf{X}(0,1)}\le 1} \left\| H_I^m \left(\chi_{(0,a)} f\right)\right\|_{\mathbf{Y}(0,1)} = 0.$

Adopting some additional conditions allows us to reformulate the condition from Theorem 3.1 in terms of a simpler operator which we shall denote K_I^m.

Suppose that $I\colon (0,1] \to (0,\infty)$ is a nondecreasing function satisfying (2.3) and let $m \in \mathbb{N}$. Set

$$J(t) = \frac{(I(t))^m}{t^{m-1}}, \ t \in (0,1]. \tag{3.5}$$

Let us observe that J is measurable on $(0,1]$ and fulfills (2.3). Consider the operator K_I^m defined by

$$K_I^m f(t) = \int_t^1 |f(s)| \frac{s^{m-1}}{(I(s))^m}\, ds, \ f \in \mathfrak{M}(0,1), \ t \in (0,1).$$

If, up to multiplicative constants depending on I,

$$\int_0^s \frac{dr}{I(r)} \approx \frac{s}{I(s)}, \ s \in (0,1), \tag{3.6}$$

then the sufficient condition for (3.4) can be reformulated with operator K_I^m.

Theorem 3.3. *Assume that (2.4), (2.5) and (2.6) are fulfilled. Suppose that there is some non-decreasing function $I\colon [0,1] \to \mathbb{R}$ satisfying (2.2) and (3.6).*

Let $m \in \mathbb{N}$ and let $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ be rearrangement-invariant norms.

(1) *Suppose that*

$$\lim_{t\to 0_+} \frac{t^{m-1}}{(I(t))^m} \neq 0. \tag{3.7}$$

Then

$$K_I^m\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1) \tag{3.8}$$

implies

$$V_X^m \mathbf{X}(\Omega) \hookrightarrow\hookrightarrow \mathbf{Y}(\Omega). \tag{3.9}$$

(2) *Suppose that*

$$\lim_{t\to 0_+} \frac{t^{m-1}}{(I(t))^m} = 0. \tag{3.10}$$

Then (3.9) *is satisfied for all pairs of rearrangement-invariant norms* $\|\cdot\|_{\mathbf{X}(0,1)}$ *and* $\|\cdot\|_{\mathbf{Y}(0,1)}$.

If we restrict our consideration to X-PS domains, where the isoperimetric function is known, we can use a yet simpler operator. Given $Q > 0$ and $m \in \mathbb{N}$, we define

$$\mathcal{Q}_Q^m f(t) = \int_t^1 |f(s)|\, s^{\frac{m}{Q}-1}\, ds, \ \ f \in \mathfrak{M}(0,1) \text{ and } t \in (0,1).$$

Theorem 3.4 (Reduction principle for X-PS domains)**.** *Assume that* (2.4), (2.5) *and* (2.6) *are fulfilled. Let* $m \in \mathbb{N}$ *and* Ω *be a* X*-PS domain with homogeneous dimension* Q*. Suppose that* $\|\cdot\|_{\mathbf{X}(0,1)}$ *and* $\|\cdot\|_{\mathbf{Y}(0,1)}$ *are rearrangement-invariant function spaces.*

If $m \leq Q$ *and*

$$\mathcal{Q}_Q^m\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1), \tag{3.11}$$

then

$$V_X^m \mathbf{X}(\Omega) \hookrightarrow\hookrightarrow \mathbf{Y}(\Omega). \tag{3.12}$$

In particular, the assumption that $Q = m$ *implies that* (3.12) *is satisfied for all* $\|\cdot\|_{\mathbf{Y}(0,1)}$ *if* $\mathbf{X}(0,1) \neq L^1(0,1)$.

Furthermore, if $m > Q$ *then* (3.12) *is fulfilled for all choices of* $\mathbf{X}(0,1)$ *and* $\mathbf{Y}(0,1)$.

The principle introduced in the Theorem 3.1 and further developed in Theorems 3.3 and Theorem 3.4 will be applied to the class of Lorentz spaces.

Theorem 3.5. *Assume that* (2.4), (2.5) *and* (2.6) *are fulfilled. Let* $m \in \mathbb{N}$ *and let* $\Omega \subset \mathbb{R}^n$ *be an* X*-PS domain. Let* $p_1, p_2, q_1, q_2 \in [1,\infty]$ *be such that the triples* $(p_1, q_1, 0)$ *and* $(p_2, q_2, 0)$ *satisfy one of conditions* (2.8)-(2.11)*. Let* Q *denote the homogeneous dimension of* Ω*, assume* $Q > 2$ *and* $m < Q$*, then each of the following conditions*

(1) $p_1 < \frac{Q}{m}$ *and* $p_2 < \frac{p_1}{1-\frac{mp_1}{Q}}$;

(2) $p_1 = \frac{Q}{m}$ *and* $p_2 < \infty$;

(3) $p_1 > \frac{Q}{m}$;

ensures that

$$V_X L^{p_1,q_1}(\Omega) \hookrightarrow\hookrightarrow L^{p_2,q_2}(\Omega). \tag{3.13}$$

Let us note that cases when $m = Q$ and $m \geq Q$, missing from Theorem 3.5 are already covered in Theorem 3.4. Let us explicitly state the conditions on compact embeddings of Sobolev spaces build upon Lebesgue spaces which are implied by Theorem 3.5.

Corollary 3.6. *Assume that (2.4), (2.5) and (2.6) are fulfilled. Let $m \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^n$ be an X-PS domain. Let $p_1, p_2 \in [1, \infty]$. Let Q denote the homogeneous dimension of Ω, assume $Q > 2$ and $m < Q$, then each of the following conditions*

(1) $p_1 < \frac{Q}{m}$ and $p_2 < \frac{p_1}{1-\frac{mp_1}{Q}}$;
(2) $p_1 = \frac{Q}{m}$ and $p_2 < \infty$;
(3) $p_1 > \frac{Q}{m}$;

ensures that

$$V_X L^{p_1}(\Omega) \hookrightarrow\hookrightarrow L^{p_2}(\Omega). \tag{3.14}$$

4. Support theorems

In this section we collect known theorems which will be used in proofs of theorems from the section 3. First we recall some facts about the operator H_J^j and its connection to compact and almost compact embeddings of r.i. spaces. Then we shift our attention to compact and almost compact embeddings of Lebesgue spaces on Carnot-Carathéodory spaces, adapting known results to this settings if necessary. We conclude this section by brief summary of some properties of functions from Sobolev-like spaces over Carnot-Carathéodory spaces.

Assume that $J\colon (0,1] \to (0,\infty)$ is a measurable function satisfying

$$\inf \frac{J(t)}{t} > 0. \tag{4.1}$$

It is proved in [7, Remark 8.2] that

$$H_J^j f(t) = \frac{1}{(j-1)!} \int_t^1 \frac{|f|}{J(s)} \left(\int_t^s \frac{dr}{J(r)} \right)^{j-1} ds,$$

for $f \in \mathfrak{M}(0,1), t \in (0,1)$.

Let $j \in \mathbb{N}$ and $\|\cdot\|_{\mathbf{X}(0,1)}$ be an r.i. norm. For every $f \in \mathfrak{M}(0,1)$ we define the functional $\|\cdot\|_{(X_{j,J}^r)'(0,1)}$ by

$$\|f\|_{(X_{j,J}^r)'(0,1)} = \frac{1}{(j-1)!} \left\| \frac{1}{J(s)} \int_0^s \left(\int_t^s \frac{dr}{J(r)} \right)^{j-1} f^*(t)\, dt \right\|_{\mathbf{X}'(0,1)}.$$

It is shown in [7, Proposition 8.3] that $\|\cdot\|_{(\mathbf{X}_{j,J}^r)'(0,1)}$ is an r. i. norm and its associate norm $\|\cdot\|_{(\mathbf{X}_{j,J}^r)(0,1)}$ fulfills

$$H_J^j\colon \mathbf{X}(0,1) \to \mathbf{X}_{j,J}^r(0,1). \tag{4.2}$$

Lemma 4.1. *Let $J\colon (0,1] \to (0,\infty)$ be a measurable function then H_J^1 is not compact from $L^1(0,1)$ into $L^\infty(0,1)$.*

Proof. We will follow the argument from the end of the proof of Lemma 4.1 in [35].

Since $\frac{1}{J(t)} > 0$, $t \in (0,1)$ there exists $\varepsilon > 0$ and set $M \subset (0,1)$ with $|M| = \frac{1}{2}$ such that $\frac{1}{J(s)} \geq \varepsilon$ for $s \in M$. Let $(x_n)_{n=1}^{\infty}$ be sequence of points in $[0,1)$ such that $|(x_n,1) \cap M| = \frac{1}{2^n}$, $n \in \mathbb{N}$ and set

$$f_n(t) = 2^n \chi_{(x_n,x_{n+1})\cap M}(t), \quad \text{for } t \in (0,1), n \in \mathbb{N}.$$

We have

$$\|f\|_{L^1(0,1)} = 2^n \left|(x_n,x_{n+1}) \cap M\right| = 2^n(|(x_n,1)\cap M| - |(x_{n+1},1)\cap M|) = \frac{1}{2},$$

hence $(f_n)_{n=1}^{\infty}$ is bounded in $L^1(0,1)$.

Fix $m,n \in \mathbb{N}$, $m < n$.

$$\left\|H_J^1(f_n) - H_J^1(f_m)\right\|_{L^\infty(0,1)} \geq |H_J f_n(x_n) - H_J f_m(x_n)|$$

$$= \left| \int_{x_n}^1 \frac{2^n \chi_{(x_n,x_{n+1})\cap M}(s)}{J(s)}\,ds - \underbrace{\int_{x_n}^1 \frac{2^m \chi_{(x_m,x_{m+1})\cap M}(s)}{J(s)}\,ds}_{=0 \text{ since } m<n} \right|$$

$$= 2^n \int_{x_n}^{x_{n+1}} \frac{\chi_M(s)}{J(s)}\,ds \geq 2^n \varepsilon \frac{1}{2^{n+1}} = \frac{\varepsilon}{2}.$$

Consequently, the sequence $(H_J^1(f_n))_{n=1}^{\infty}$ is not a Cauchy sequence in $L^\infty(0,1)$ and H_J^1 is not compact from $L^1(0,1)$ into $L^\infty(0,1)$. □

Let us restate the following two characterizations of compactness of operator H_J^j ([35, Theorem 4.1 and Theorem 4.2]).

Theorem 4.2. *Let $J\colon (0,1] \to (0,\infty)$ be a measurable function satisfying (2.3) and let $j \in \mathbb{N}$. Suppose that $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ are rearrangement-invariant norms. Consider the following two conditions:*

(a) $H_J^j\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1)$,

(b) $\lim_{a\to 0+} \sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1} \left\|H_J^j(\chi_{(0,a)}f)\right\|_{\mathbf{Y}(0,1)} = 0$.

If $\mathbf{X}(0,1) = L^1(0,1)$, $\mathbf{Y}(0,1) = L^\infty(0,1)$, $j = 1$ and

$$\lim_{a\to 0+} \operatorname{ess\,sup}_{t\in(0,a)} \frac{1}{J(t)} = 0,$$

then (b) is satisfied but (a) is not. In all other cases, (a) holds if and only if (b) holds.

Theorem 4.3. *Let $J\colon (0,1] \to (0,\infty)$ be a measurable function satisfying (2.3), and let $j \in \mathbb{N}$. Suppose that $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ are rearrangement-invariant norms. If*

$$\mathbf{Y}(0,1) \neq L^\infty(0,1) \text{ or } \int_0^1 \frac{dr}{J(r)} = \infty,$$

then the following conditions are equivalent

(1) $H_J^j\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1)$;

(2) $\mathbf{X}^r_{j,J}(0,1) \overset{*}{\hookrightarrow} \mathbf{Y}(0,1)$.

Assume that $\mathbf{X}(0,1)$ and J are such that, in addition to (4.1), it holds that

$$\left\| \left(\int_t^1 \frac{dr}{J(r)} \right)^j \right\|_{\mathbf{X}(0,1)} < \infty. \tag{4.3}$$

For every $f \in \mathfrak{M}(0,1)$ define the function $\|\cdot\|_{\mathbf{Y}^d_{j,J}(0,1)}$ by

$$\|f\|_{\mathbf{Y}^d_{j,J}(0,1)} = \sup_{h\sim f} \left\|H^j_J h\right\|_{\mathbf{Y}(0,1)} + \|f\|_{L^1(0,1)}.$$

Important properties of $\mathbf{Y}^d_{j,J}(0,1)$ are summarised in the following proposition and theorem from [35, Proposition 4.5, Theorem 4.6].

Proposition 4.4. *Let $J\colon (0,1] \to (0,\infty)$ be a measurable function satisfying (2.3) and let $j \in \mathbb{N}$. Suppose that $\|\cdot\|_{\mathbf{Y}(0,1)}$ is a rearrangement-invariant norm fulfilling (4.3). Then $\|\cdot\|_{(\mathbf{Y})^d_{j,J}(0,1)}$ is a rearrangement-invariant norm and*

$$H^j_J\colon \mathbf{Y}^d_{j,J}(0,1) \to \mathbf{Y}(0,1).$$

Theorem 4.5. *Let $J\colon (0,1] \to (0,\infty)$ be a measurable function satisfying (2.3) and let $j \in \mathbb{N}$. Suppose that $\mathbf{X}(0,1)$ is a r.i. function space such that $\mathbf{X}(0,1) \neq L^1(0,1)$. Assume that $\mathbf{Y}(0,1)$ is a r.i. function space fulfilling (4.3). Then the following conditions are equivalent:*

(1) $H^j_J\colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1)$*;*
(2) $\mathbf{X}(0,1) \overset{*}{\hookrightarrow} \mathbf{Y}^d_{j,J}(0,1)$.

When certain conditions are satisfied the norm $\|\cdot\|_{(L^\infty)^d_{1,I}(0,1)}$ can be approximated by a simpler one as is shown in the next lemma from [35, Lemma 5.6].

Lemma 4.6. *Let $I\colon (0,1] \to (0,\infty)$ be a non-decreasing function satisfying (2.3) and*

$$\int_0^1 \frac{ds}{I(s)} < \infty.$$

Then

$$\|f\|_{(L^\infty)^d_{1,I}(0,1)} \approx \int_0^1 \frac{f^*(s)}{I(s)}\, ds, f \in \mathfrak{M}(0,1).$$

up to multiplicative constants depending on I.

Useful conditions on compactness of operator H^j_J formulated through its boundedness stated in Lemma 4.7 are proved in [35, Remark 4.8].

Lemma 4.7. *Let $\mathbf{X}(0,1) \neq L^1(0,1)$ and $\mathbf{Y}(0,1)$ are rearrangement-invariant function spaces. Suppose that $j \in \mathbb{N}$ and $J\colon (0,1] \to (0,\infty)$ is a measurable function satisfying*

$$\inf_{t\in(0,1]} \frac{J(t)}{t} > 0$$

then

$$H^j_J\colon L^1(0,1) \to \mathbf{Y}(0,1),$$

implies

$$H_J^j \colon \mathbf{X}(0,1) \hookrightarrow\hookrightarrow \mathbf{Y}(0,1).$$

We continue by stating the characterization of almost compact embeddings of r.i. spaces from [34, Theorem 3.1].

Theorem 4.8. *Let $\mathbf{X}(R,\mu)$ and $\mathbf{Y}(R,\mu)$ be Banach function spaces over a totally σ-finite measure space (R,μ). Then $\mathbf{X}(R,\mu) \overset{*}{\hookrightarrow} \mathbf{Y}(R,\mu)$ if and only if for every sequence $(f_n)_{n=1}^{\infty}$ of μ-measurable functions on R satisfying $\|f_n\|_{\mathbf{X}(R,\mu)} \le 1$ and $f_n \to 0$ μ-a.e., one has $\|f_n\|_{\mathbf{Y}(R,\mu)} \to 0$.*

The following lemma is an adaptation of Lemma 5.5 from [35].

Lemma 4.9. *Assume that (2.4), (2.5) and (2.6) are fulfilled, Ω is an open domain and $m \in \mathbb{N}$. Let $\|\cdot\|_{\mathbf{X}}$ be an rearrangement-invariant function space. Then every sequence $(u_k)_{k=1}^{\infty}$ bounded in $V_X^m\mathbf{X}(\Omega)$ contains a subsequence $(u_{k_l})_{l=1}^{\infty}$ converging a.e. in Ω.*

To prove Lemma 4.9 we need compact embedding $V_X^1L^1(B_{x_l}) \hookrightarrow\hookrightarrow L^1(B_{x_l})$. which was conveniently already proved in [15, Theorem 1.28].

Theorem 4.10. *Assume that (2.4), (2.5) and (2.6), and let $\Omega \subset \mathbb{R}^n$ be an X-PS domain with $\operatorname{diam}(\Omega) < \frac{R_0}{2}$, where R_0 is the constant from (2.4). Then, one has the following:*

(1) The embedding $BV_X(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ holds for any $1 \le q < \frac{Q}{Q-1}$.
(2) For any $1 \le p < Q$ the embedding $V_X^1L^p(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ holds provided that $1 \le q < \frac{Qp}{Q-p}$.
(3) For any $Q \le p < \infty$ and any $1 \le q < \infty$, the embedding $V_X^1L^p(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ holds.

Proof of Lemma 4.9. Since Ω is open, for all $x \in \Omega$ there exists a ball (with respect to metric d) B_x such that $x \in B_x$ and $B_x \subset \Omega$. There is a sequence $(x_l)_l^{\infty}$ of points in Ω such that $\{B_{x_l}, x_l = 1,2,\dots\}$ is a covering of Ω because the topology generated by the metric d is equivalent to the Euclidean topology.

Let $(u_k)_{k=1}^{\infty}$ be bounded in $V_X^m\mathbf{X}(\Omega)$. Balls with respect to metric d are an X-PS domain. X-PS domains fulfills (2.12) with a specific constant (consequence of Theorem 1.18 in [15]). Proposition 11 in [14] yields that $V_X^1\mathbf{X}(B_{x_l}) \to L^1(Bx_l)$ for all all $l = 1,2,\dots\infty$. Moreover, the proof of Proposition 11 yields that the embedding constant is dependent only on constant from (2.12), therefore there exists a constant which holds for all embeddings $V_X^1\mathbf{X}(B_{x_l}) \to L^1(B_{x_l})$, $l \in \mathbb{N}$. Consequently, there exists $C > 0$ such that

$$\|u_k\|_{L^1(B_{x_l})} \le C\,\|u_k\|_{V_X^m\mathbf{X}(B_{x_l})} \le C\,\|u_k\|_{V_X^m\mathbf{X}(\Omega)}$$

for all $l \in \mathbb{N}$. Sequence $(u_k)_{k=1}^{\infty}$ is therefore bounded in $V_X^1L^1(B_{x_l})$ with the same bounding constant for all $l \in \mathbb{N}$.

Now, for every $l \in \mathbb{N}$ Theorem 4.10 yields that

$$V_X^1L^1(B_{x_l}) \hookrightarrow\hookrightarrow L^1(B_{x_l}).$$

Consequently, a bounded sequence $(v_k)_{k=1}^{\infty}$ in $V^1_X L^1(B_{x_l})$ contains a subsequence $(v_{k_r})_{r=1}^{\infty}$ such that v_{k_r} converges in $L^1(B_{x_l})$.

We start with selecting a subsequence of $(u_k)_{k=1}^{\infty}$, let us name it $(u_{k_{1,r}})_{r=1}^{\infty}$, such that it converges in $L^1(B_{x_1})$. Then there exists a subsequence of $(u_{k_{1,r}})_{r=1}^{\infty}$, $(u_{k_{2,r}})_{r=1}^{\infty}$, such that it converges in $L^1(B_{x_2})$ and $L^1(B_{x_1})$.

By repeating this step, we get a sequence $(u_{k_{l,r}})_{r=1}^{\infty}$ for each $l \in \mathbb{N}$ which converges (with respect to r) in $L^1(B_{x_s})$ for all $s \in \mathbb{N}$, $s \leq l$.

The diagonal sequence, $(u_{k_{l,l}})_{l=1}^{\infty}$ is then the desired subsequence of $(u_k)_{k=1}^{\infty}$ which converges a.e. on Ω. □

Theorem 4.11 (Reduction theorem for non-compact embeddings)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is open. Suppose that there is some non-deceasing function $I\colon [0,1] \to \mathbb{R}$ satisfying (2.2) and (2.3). Let $m \in \mathbb{N}$, and let $\|\cdot\|_{\mathbf{X}(0,1)}$ and $\|\cdot\|_{\mathbf{Y}(0,1)}$ be rearrangement-invariant function norms. If there exists a constant $C > 0$ such that*

$$\left\| \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds \right\|_{\mathbf{Y}(0,1)} \leq C \|f\|_{\mathbf{X}(0,1)} \tag{4.4}$$

for every nonnegative $f \in \mathfrak{M}_+(0,1)$, then

$$V^m_X \mathbf{X}(\Omega) \to \mathbf{Y}(\Omega). \tag{4.5}$$

Remark 4.12. The condition (4.4) can be restated as $H^m_I\colon \mathbf{X}(0,1) \to \mathbf{Y}(0,1)$.

Lemma 4.13 ($W^1_X \mathbf{X}(\Omega) = V^1_X \mathbf{X}$)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is open. Suppose that there is some non-deceasing function $I\colon [0,1] \to \mathbb{R}$ satisfying (2.2) and*

$$\int_0^s \frac{dt}{I(t)} < \infty, \text{ for some } s > 0. \tag{4.6}$$

Let $\Omega \subset \mathbb{R}^n$ is open. Then there exists a constant $C > 0$ such that

$$\|f\|_{\mathbf{X}(\Omega)} \leq C \|X\nabla f\|_{\mathbf{X}(\Omega)}, \; f \in \mathfrak{M}(0,1).$$

Consequently, up to a multiplicative constants, we get

$$\|f\|_{V^1_X \mathbf{X}(\Omega)} \approx \|f\|_{\mathbf{X}(\Omega)} + \|X\nabla f\|_{\mathbf{X}(\Omega)}, f \in V^1_X \mathbf{X}(\Omega).$$

Proof. We will follow the approach of the Proposition 4.5 in [7]. We define

$$J(t) = \begin{cases} I(s), & s \in [0, \frac{1}{3}] \\ I(\frac{1}{3}), & s \in \,]\frac{1}{3}, 1] \end{cases},$$

Then $J(t) > cs$ for some constant $c > 0$ thanks to (4.6) and J fulfills

$$I_{\Omega,X}(s) \geq c' J(c's)$$

for some $c' > 0$ and s near zero. A simple computation shows that $H^1_J\colon L^1(0,1) \to L^1(0,1)$ and $H^1_J L^\infty(0,1) \to L^\infty(0,1)$. The interpolation theorem of Calderón ([3, Chapter 3, Theorem 2.12]) then yields that $H^1_J\colon \mathbf{X}(0,1) \to \mathbf{X}(0,1)$. Application of Theorem 4.11 then implies the desired embedding

$$V^1_X \mathbf{X}(\Omega) \to \mathbf{X}(\Omega).$$

□

Let us restate here Lemma 3.5 from [15] which allows us to use the well known Maz'ya truncation technique.

Lemma 4.14. *Let $1 \leq p < \infty$, and Ω be a bounded open set in $\mathbb{R}^n$. If $u \in W^1_X L^p(\Omega)$ and $F \in \mathcal{C}^1(\mathbb{R})$, $F' \in L^\infty(\mathbb{R})$, then we have the following:*

(1) $F \circ u \in W^1_X L^p(\Omega)$ and

$$X_j(F \circ u) = (F' \circ u) X_j u \text{ in } \mathcal{D}'(\Omega) \text{ for } 1 \leq j \leq m.$$

(2) Also, one has $u^+, u^-, |u| \in V^1_X L^1(\Omega)$ and

$$X\nabla u^+ = \begin{cases} X\nabla u \text{ a. e. on } \{x \in \Omega \colon u(x) \geq 0\} \\ 0 \quad \text{otherwise} \end{cases}$$

$$X\nabla u^- = \begin{cases} -X\nabla u \text{ a. e. on } \{x \in \Omega \colon u(x) < 0\} \\ 0 \quad \text{otherwise} \end{cases}$$

$$X\nabla |u| = \begin{cases} X\nabla u \text{ a. e. on } \{x \in \Omega \colon u(x)0\} \\ 0 \quad \text{a. e. on } \{x \in \Omega \colon u(x) = 0\} \\ -X\nabla u \text{ a.e. on } \{x \in \Omega \colon u(x) < 0\}. \end{cases}$$

5. Proof of the main theorem

Proof of Theorem 3.1. Assume first that $\mathbf{Y}(\Omega) \neq L^\infty(\Omega)$ or $\int_0^1 \frac{ds}{I(s)} = \infty$. In this case Theorem 4.3 yields $\mathbf{X}^r_{m,I}(0,1) \overset{*}{\hookrightarrow} \mathbf{Y}(0,1)$ and consequently $\mathbf{X}^r_{m,I}(\Omega) \overset{*}{\hookrightarrow} \mathbf{Y}(\Omega)$.

Assume that $(u_k)_{k=1}^\infty$ is a sequence bounded in $V^m_X \mathbf{X}(\Omega)$. Lemma 4.9 ensures that there is subsequence $(u_{k_l})_{l=1}^\infty$ which converges to some function u a. e. on Ω. The embedding (4.2) implies that $H^m_I \mathbf{X}(0,1) \to \mathbf{X}^r_{m,I}(0,1)$. Theorem 4.11 then yields that $V^m_X \mathbf{X}(\Omega) \to \mathbf{X}^r_{m,I}(\Omega)$. Hence, $(u_{k_l})_{l=1}^\infty$ is bounded in $\mathbf{X}^r_{m,I}(\Omega)$. Therefore, the almost compact embedding $\mathbf{X}^r_{m,I}(\Omega) \overset{*}{\hookrightarrow} \mathbf{Y}(\Omega)$ and Theorem 4.8 yield that $u_{k_l} \to u \in \mathbf{Y}(\Omega)$. Thus, $V^m_X \mathbf{X}(\Omega) \hookrightarrow\hookrightarrow \mathbf{Y}(\Omega)$.

In following we will focus on the remaining case $\mathbf{Y}(\Omega) = L^\infty(\Omega)$ and $\int_0^1 \frac{ds}{I(s)} < \infty$. Assume first that $m = 1$.

Lemma 4.1 ensures that $\mathbf{X}(0,1) \neq L^1(0,1)$ since we are assuming

$$H^1_J \colon \mathbf{X}(0,1) \to\to L^\infty(0,1).$$

This, together with observation that

$$\left\| \int_t^1 \frac{dr}{I(r)} \right\|_{L^\infty(0,1)} = \int_0^1 \frac{dr}{I(r)} < \infty, \tag{5.1}$$

and Theorem 4.5 yields that

$$\mathbf{X}(0,1) \overset{*}{\hookrightarrow} (L^\infty)^d_{1,I}(0,1). \tag{5.2}$$

Moreover, the inequality (5.1) and Lemma 4.4 yields that assumptions of Theorem 4.11 are met with $\mathbf{X}(0,1) = (L^\infty)^d_{1,I}(0,1)$ and $\mathbf{Y}(0,1) = L^\infty(0,1)$. Consequently, we get

$$V^1_X (L^\infty)^d_{1,I}(\Omega) \to L^\infty(\Omega). \tag{5.3}$$

Assume that $(u_k)_{k=1}^{\infty}$ in $V_X^1\mathbf{X}(\Omega)$ is a bounded sequence. Since $\int_0^1 \frac{ds}{I(s)} < \infty$, the Lemma 4.13 ensures that $(u_k)_{k=1}^{\infty}$ is bounded in $W_X^1\mathbf{X}(\Omega)$ as well. Without loss of generality we may assume that

$$\|u_k\|_{W_X^1\mathbf{X}(\Omega)} \leq 1,\ k \in \mathbb{N}. \tag{5.4}$$

Lemma 4.9 then assure that there is a subsequence $(v_k)_{k=1}^{\infty}$ which converges in measure to some function v. Indeed, our goal is to show that $(v_k)_{k=1}^{\infty}$ is a Cauchy sequence in $L^\infty(\Omega)$ and thus it converges to v in $L^\infty(\Omega)$, which will imply that $V_X^1\mathbf{X}(\Omega)$ compactly embedded into $L^\infty(\Omega)$.

Fix $\varepsilon > 0$ and $k, l \in \mathbb{N}$. Let us introduce the following notation:

$$d(x) \ = \ |v_k(x) - v_l(x)| = \min\left\{d(x), \frac{\varepsilon}{2}\right\} + \max\left\{d(x) - \frac{\varepsilon}{2}, 0\right\}$$

for $x \in \Omega$. Moreover, let us write $e(x) = \max\left\{d(x) - \frac{\varepsilon}{2}, 0\right\}$, $x \in \Omega$.

Differentiability of v_k and v_l combined with Lemma 4.14 ensures that $d - \frac{\varepsilon}{2}$ and e are both differentiable almost everywhere in Ω. Because e is being derived from v_k and v_l by substraction, absolute value operator and truncation by constant, standard argumentation accompanied by Lemma 4.14 yields that

$$|X\nabla e(x)| = \chi_{\left\{d \geq \frac{\varepsilon}{2}\right\}}(x)\, |X\nabla v_k(x) - X\nabla v_l(x)|\,, \tag{5.5}$$

for almost every $x \in \Omega$.

Consequently, we have

$$\begin{aligned}
\|d\|_{L^\infty(\Omega)} &\leq \left\|\min\left\{d(x), \frac{\varepsilon}{2}\right\}\right\|_{L^\infty(\Omega)} + \|e\|_{L^\infty(\Omega)} \\
&\leq \frac{\varepsilon}{2} + C\,\|e\|_{W_X^1 (L^\infty)^d_{1,I}(\Omega)} \\
&\leq \frac{\varepsilon}{2} + C\left\|\chi_{\left\{d > \frac{\varepsilon}{2}\right\}}\, |X\nabla(v_k - v_l)|\right\|_{(L^\infty)^d_{1,I}(\Omega)} + C\left\|\chi_{\left\{d > \frac{\varepsilon}{2}\right\}}\, |e|\right\|_{(L^\infty)^d_{1,I}(\Omega)}
\end{aligned}$$

where we have used (5.3), Lemma 4.13 and (5.5). Therefore, we get

$$\begin{aligned}
\|d\|_{L^\infty(\Omega)} &\leq \frac{\varepsilon}{2} + C\Bigg(\left\|\chi_{\left\{d > \frac{\varepsilon}{2}\right\}}\, |X\nabla v_k|\right\|_{(L^\infty)^d_{1,I}(\Omega)} + \left\|\chi_{\left\{d > \frac{\varepsilon}{2}\right\}}\, |X\nabla v_l|\right\|_{(L^\infty)^d_{1,I}(\Omega)} + \\
&\qquad \left\|\chi_{\left\{d > \frac{\varepsilon}{2}\right\}}\, |v_k|\right\|_{(L^\infty)^d_{1,I}(\Omega)} + \left\|\chi_{\left\{d > \frac{\varepsilon}{2}\right\}}\, |v_l|\right\|_{(L^\infty)^d_{1,I}(\Omega)}\Bigg) \\
&= \frac{\varepsilon}{2} + C\Bigg(\left\|\left(\chi_{\left\{d > \frac{\varepsilon}{2}\right\}}\, |X\nabla v_k|\right)^*\right\|_{(L^\infty)^d_{1,I}(0,1)} \\
&\qquad + \left\|\left(\chi_{\left\{d > \frac{\varepsilon}{2}\right\}}\, |X\nabla v_l|\right)^*\right\|_{(L^\infty)^d_{1,I}(0,1)} \\
&\qquad + \left\|\chi_{\left\{d > \frac{\varepsilon}{2}\right\}} v_k^*\right\|_{(L^\infty)^d_{1,I}(0,1)} + \left\|\chi_{\left\{d > \frac{\varepsilon}{2}\right\}} v_l^*\right\|_{(L^\infty)^d_{1,I}(0,1)}\Bigg)
\end{aligned}$$

Since (5.4) is in hold, we have that

$$\left\|\left(\chi_{\left\{d > \frac{\varepsilon}{2}\right\}}\, |X\nabla v_k|\right)^*\right\|_{(L^\infty)^d_{1,I}(0,1)} \leq \sup_{\|f\|_{\mathbf{X}(0,1)} \leq 1} \left\|\chi_{\left(0, \left|\left\{d > \frac{\varepsilon}{2}\right\}\right|\right)} f^*\right\|_{(L^\infty)^d_{1,I}(0,1)}$$

and

$$\left\|\chi_{\{d>\frac{\varepsilon}{2}\}}v_k^*\right\|_{(L^\infty)^d_{1,I}(0,1)} \leq \sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1}\left\|\chi_{(0,|\{d>\frac{\varepsilon}{2}\}|)}f^*\right\|_{(L^\infty)^d_{1,I}(0,1)},$$

for $k\in\mathbb{N}$. But (5.2) yields that there is $\delta>0$ such that

$$\sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1}\left\|\chi_{(0,\delta)}f^*\right\|_{(L^\infty)^d_{1,I}(0,1)} < \frac{\varepsilon}{8C}.$$

Since $(v_k)_{k=1}^\infty$ converges in measure to v, we can find $k_0\in\mathbb{N}$ such that for every $k\geq k_0$

$$\left|\left\{x\in\Omega\colon\ |v_k(x)-v(x)|>\frac{\varepsilon}{4}\right\}\right|<\frac{\delta}{2}.$$

Moreover, for all $k,l\geq k_0$, it holds

$$\left\{x\in\Omega\colon\ |d|\geq\frac{\varepsilon}{2}\right\}\subset\left\{x\in\Omega\colon\ |v_k(x)-v(x)|\geq\frac{\varepsilon}{4}\right\}\cup\left\{x\in\Omega\colon\ |v_l(x)-v(x)|\geq\frac{\varepsilon}{4}\right\}$$

and

$$\left|\left\{x\in\Omega\colon\ |d|\geq\frac{\varepsilon}{2}\right\}\right|\leq\delta.$$

Consequently, for $k,l>n_0$, we have

$$\begin{aligned}\|d\|_{L^\infty(\Omega)} &\leq \frac{\varepsilon}{2}+4C\sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1}\left\|\chi_{(0,|\{d>\frac{\varepsilon}{2}\}|)}f^*\right\|_{(L^\infty)^d_{1,I}(0,1)}\\ &\leq \frac{\varepsilon}{2}+4C\sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1}\left\|\chi_{(0,\delta)}f^*\right\|_{(L^\infty)^d_{1,I}(0,1)}\leq\varepsilon.\end{aligned}$$

Therefore $(v_k)_{k=1}^\infty$ is a Cauchy sequence in $L^\infty(\Omega)$ and $V^1_X\mathbf{X}(\Omega)$ is compactly embedded into $L^\infty(\Omega)$.

Next, we will deal with the case $m>1$. (We still assume that $\mathbf{Y}(\Omega)=L^\infty(\Omega)$ and $\int_0^1\frac{ds}{I(s)}<\infty$.) According to Lemma 4.6, for every $f\in\mathfrak{M}(0,1)$, then

$$\|g\|_{(L^\infty)^d_{1,I}(0,1)}\approx\int_0^1\frac{g^*(s)}{I(s)}\,ds=\|H_Ig^*\|_{L^\infty(0,1)}$$

up to multiplicative constants depending on I. Thus, whenever $f\in\mathcal{M}(0,1)$ and $a\in(0,1)$, then

$$\begin{aligned}\left\|H_I^m(\chi_{(0,a)}f)\right\|_{L^\infty(0,1)} &= \left\|H_I\left(H_I^{m-1}(\chi_{(0,a)}f)\right)\right\|_{L^\infty(0,1)}\\ &\approx \left\|H_I^{m-1}\left(\chi_{(0,a)}f\right)\right\|_{(L^\infty)^d_{1,I}(0,1)},\end{aligned}$$

up to multiplicative constants depending on I. As it is stated in Remark 3.2 the assumption (3.3) is equivalent to

$$\lim_{a\to 0+}\sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1}\left\|H_I^m\left(\chi_{(0,a)}f\right)\right\|_{L^\infty(0,1)}=0,$$

hence it is also equivalent to

$$\lim_{a\to 0+}\sup_{\|f\|_{\mathbf{X}(0,1)}\leq 1}\left\|H_I^{m-1}\left(\chi_{(0,a)}f\right)\right\|_{(L^\infty)^d_{1,I}(0,1)}=0.$$

In order to use the previously proved case of this proof, we will show that $(L^\infty)^d_{1,I}(0,1)\neq L^\infty(0,1)$. Consider functions $\chi_{(0,a)}$ for $a\in(0,1)$. We have

$\left\|\chi_{(0,a)}\right\|_{L^\infty(0,1)} = 1$. On the other hand, up to multiplicative constants depending on I, we have

$$\lim_{a\to 0+} \left\|\chi_{(0,a)}\right\|_{(L^\infty)^d_{1,I}(0,1)} \approx \lim_{a\to 0+} \int_0^a \frac{ds}{I(s)} = 0.$$

If $(L^\infty)^d_{1,I}(0,1) = L^\infty(0,1)$, then $\|\cdot\|_{(L^\infty)^d_{1,I}(0,1)}$ must be equivalent to $\|\cdot\|_{L^\infty(0,1)}$ up to multiplicative constants. Consequently, $(L^\infty)^d_{1,I}(0,1) \neq L^\infty(0,1)$, because it is impossible to have constant $c > 0$ such that $\|f\|_{L^\infty(0,1)} \leq c\,\|f\|_{(L^\infty)^d_{1,I}(0,1)}$, for all $f \in L^\infty(0,1)$.

Since $(L^\infty)^d_{1,I}(0,1) \neq L^\infty(0,1)$, the previous part of proof implies that

$$V_X^{m-1}\mathbf{X}(\Omega) \hookrightarrow\hookrightarrow (L^\infty)^d_{1,I}(\Omega). \tag{5.6}$$

Let $(u_k)_{k=1}^\infty$ be a bounded sequence in $V_X^m\mathbf{X}(\Omega)$. Then $(u_k)_{k=1}^\infty$ is bounded in $L^1(\Omega)$, so $(\int_\Omega u_k(x)\,dx)_{k=1}^\infty$ is a bounded sequence of real numbers and we can find a subsequence $(u_k^0)_{k=1}^\infty$ of $(u_k)_{k=1}^\infty$ such that the sequence $(\int_\Omega u_k^0(x)\,dx)_{k=1}^\infty$ is convergent.

Consider sequences $(X_i u_k^0)_{k=1}^\infty$, $X_i \in X$, $i = 1,\dots,m$. Owing to boundedness of $(u_k^0)_{k=1}^\infty$ in $V_X^m\mathbf{X}(\Omega)$, $(X_i u_k^0)_{k=1}^\infty$ is bounded in $V_X^{m-1}\mathbf{X}(\Omega)$. Compact embedding (5.6) then yields that, we can inductively find $(u_k^i)_{k=1}^\infty$, subsequence of $(u_k^{i-1})_{k=1}^\infty$, $i = 1,2,\dots,m$ such that $(X_i u_k^i)_{k=1}^\infty$ is convergent in $(L^\infty)^d_{1,I}(\Omega)$. Consequently, $(X_i u_k^m)_{k=1}^\infty$ is a Cauchy sequence in $(L^\infty)^d_{1,I}(\Omega)$ for every $j = 1,2,\dots,m$.

To conclude the proof, we need to show that $(u_k^m)_{k=1}^\infty$ is a Cauchy sequence in $L^\infty(\Omega)$.

Let $\varepsilon > 0$. Assumptions (2.2) and (2.3) ensures (2.12). Therefore the inequality (2.13) with $\mathbf{X}(\Omega) = (L^\infty)^d_{1,I}(\Omega)$ yields that there exists a constant $C > 0$ such that

$$\left\|u - \int_\Omega u(x)\,dx\right\|_{L^\infty(\Omega)} \leq C\,\|X\nabla u\|_{(L^\infty)^d_{1,I}(\Omega)} \leq C\sum_{j=1}^m \|X_j u\|_{(L^\infty)^d_{1,I}(\Omega)}.$$

Because $(X_j u_k^m)_{k=1}^\infty$, $j = 1,\dots,m$, is a Cauchy sequence in $(L^\infty)^d_{1,I}(\Omega)$, there exists $k_0 \in \mathbb{N}$ such that

$$\|X_j u_l^m - X_j u_l^m\|_{(L^\infty)^d_{1,I}(\Omega)} \leq \frac{\varepsilon}{Cm},$$

for all $j = 1,\dots,m$, whenever $k, l > k_0$. Sequence $(u_k^m - \int_\Omega u_k^m(x)\,dx)_{k=1}^\infty$ is Cauchy sequence in $L^\infty(\Omega)$ since

$$\left\|u_l^m - \int_\Omega u_l^m(x)\,dx - u_k^m - \int_\Omega u_k^m(x)\,dx\right\| \leq C\sum_{j=1}^m \|X_j u_l^m - X_j u_l^m\|_{(L^\infty)^d_{1,I}(\Omega)} < \varepsilon,$$

for $k, l > k_0$. $L^\infty(\Omega)$ is complete therefore $(u_k^m - \int_\Omega u_k^m(x)\,dx)_{k=1}^\infty$ is a convergent sequence. Sequence $(\int_\Omega u_k^m(x)\,dx)_{k=1}^\infty$ is a subsequence of $(\int_\Omega u_k^0(x)\,dx)_{k=1}^\infty$ which is convergent in $L^\infty(\Omega)$. Therefore $(u_k^m)_{k=1}^\infty$ is convergent sequence in $L^\infty(\Omega)$ as well. This concludes the proof. ∎

Proof of Theorem 3.3. It is proved in [7, Proposition 8.6] that if (3.6) holds than for any r.i. space $\mathbf{Y}$ and $f \in \mathfrak{M}(0,1)$ we have

$$\|H_I^m f\|_{\mathbf{Y}(0,1)} \approx \|K_I^m f\|_{\mathbf{Y}(0,1)},$$

up to multiplicative constants depending on m and I.

Moreover, it is shown in [35, Proof of Theorem 5.3] that, under the same assumptions, it holds

$$\big\|H_I^m(\chi_{(0,a)} f)\big\|_{\mathbf{Y}(0,1)} \approx \big\|K_I^m(\chi_{(0,a)} f)\big\|_{\mathbf{Y}(0,1)}, \tag{5.7}$$

for a given $a \in (0,1)$, up to multiplicative constants depending on m and I.

At the same place, it is shown that in this situation

$$\lim_{t\to 0+} \operatorname{ess\,sup}_{s\in(0,t)} \frac{1}{J(s)} = 0$$

holds if and only if

$$\lim_{t\to 0+} \frac{t^{m-1}}{(I(t))^m} = 0.$$

Therefore, if (3.7) holds, Theorem 4.2 and fact that $K_I^m = H_J$ yields that

$$\lim_{a\to 0+} \sup_{\|f\|_{\mathbf{X}(0,1)}\le 1} \big\|K_I^m(\chi_{(0,a)} f)\big\|_{\mathbf{Y}(0,1)} = 0$$

is equivalent to

$$K_I^m \colon \mathbf{X}(0,1) \to\to \mathbf{Y}(0,1).$$

Remark 3.2 together with (5.7) yields that (3.8) implies (3.3). Therefore the assumptions of Theorem 3.1 are satisfied and it ensures that (3.9) holds.

Assume now, that (3.10) is in force. In such case, Theorem 4.2 yields that

$$\lim_{a\to 0+} \sup_{\|f\|_{L^1(0,1)}\le 1} \big\|H_J^j(\chi_{(0,a)} f)\big\|_{L^\infty(0,1)} = 0.$$

Remark 3.2, together with Theorem 3.1 yields that

$$V_X^m L^1(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega).$$

Standard embeddings $\mathbf{X}(\Omega) \hookrightarrow L^1(\Omega)$ and $L^\infty(\Omega) \hookrightarrow \mathbf{Y}(\Omega)$ (which are valid for all rearrangement-invariant spaces $\mathbf{X}(\Omega)$ and $\mathbf{Y}(\Omega)$) then concludes the proof. □

Proof of Theorem 3.4. Consider function $I(t) = t^{1-\frac{1}{Q}}$. It follows from the fact that Ω is X-PS domain and (2.7) that (2.2) holds with such $I(t)$. Simple computation yields that (3.6) holds as well.

An application of the Theorem 3.3 will yield the claim. We have

$$\lim_{t\to 0+} \frac{t^{m-1}}{(I(t))^m} = \lim_{t\to 0+} \frac{t^{m-1}}{t^{m-\frac{m}{Q}}} = \lim_{t\to 0+} t^{\frac{m}{Q}-1}. \tag{5.8}$$

Therefore, if $m > Q$, then $\lim_{t\to 0+} \frac{t^{m-1}}{(I(t))^m} = 0$ and Theorem 3.3 yields the claim.

On the other hand, if $m \le Q$ then $\lim_{t\to 0+} \frac{t^{m-1}}{(I(t))^m} \neq 0$. We have

$$\mathcal{Q}_Q^m f(t) = \int_t^1 |f(x)|\, s^{\frac{m}{Q}-1}\, ds = \int_t^1 |f(s)| \frac{s^{m-1}}{\left(s^{1-\frac{1}{Q}}\right)^m}\, ds = K_I^m f(t),$$

for $f \in \mathfrak{M}(0,1)$ and $t \in (0,1)$. Consequently, if (3.11) holds (and $Q \le m$), so does (3.8) and the Theorem 3.3 yields the required embedding.

If $Q = m$,then we have

$$\mathcal{Q}_m^m f(t) = \int_t^1 |f(s)|\, ds = H_1^1 f(t),\ f \in \mathfrak{M}(0,1).$$

Simple computation now yields that $\mathcal{Q}_m^m \colon L^1(0,1) \to L^\infty(0,1)$. Consequently, since $L^\infty(0,1) \hookrightarrow \mathbf{Y}(0,1)$, we get $\mathcal{Q}_m^m \colon L^1(0,1) \to \mathbf{Y}(0,1)$. Application of Lemma 4.7 then yields the result. □

The proof of Theorem 3.5 rests on the following characterization of almost compact embeddings between Lorentz-Zygmund spaces from [35, Proposition 7.12].

Theorem 5.1. *Let $p_1, p_2, q_1, q_2 \in [1,\infty]$, $\alpha_1, \alpha_2 \in \mathbb{R}$ be such that both triples $(p,q,\alpha) = (p_1,q_1,\alpha_1)$ and $(p,q,\alpha) = (p_2,q_2,\alpha_2)$ satisfy one of the conditions (2.8)-(2.11). Then*

$$L^{p_1,q_1:\alpha_1}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2:\alpha_2}(0,1)$$

holds if and only of $p_1 > p_2$, or $p_1 = p_2$ and the following conditions are satisfied:

$$\textit{if } p_1 = p_2 < \infty \textit{ and } q_1 \le q_2 \textit{ then } \alpha_1 > \alpha_2;$$

$$\textit{if } p_1 = p_2 = \infty \textit{ or } q_1 > q_2 \textit{ then } \alpha_1 + \frac{1}{q_1} > \alpha_2 + \frac{1}{q_2}.$$

In particular, if $p_1, p_2, q_1, q_2 \in [1,\infty]$ are such that both triplets $(p,q,\alpha) = (p_1,q_1,0)$ and $(p,q,\alpha) = (p_2,q_2,0)$ satisfy one of the conditions (2.8)-(2.11) then

$$L^{p_1,q_1}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2}(0,1)$$

holds if and only if $p_1 > p_2$.

Proof of Theorem 3.5. Using the Theorem 3.4 we can reduce (3.13) to proving that

$$\mathcal{Q}_Q^m \colon L^{p_1,q_1}(0,1) \to\to L^{p_1,q_1}(0,1). \tag{5.9}$$

We are going to derive the embedding (5.9) from assumption of the Theorem 3.5.

Assume that $p_1 > \frac{Q}{m}$, then Theorem 5.1 yields that

$$L^{p_1,q_1}(0,1) \overset{*}{\hookrightarrow} L^{\frac{Q}{m},1}(0,1).$$

Since $\int_0^1 \frac{1}{s^{1-\frac{m}{Q}}}\, ds < \infty$, application of Lemma 4.6 yields that

$$\|f\|_{L^{\frac{Q}{m},1}(0,1)} = \int_0^1 \frac{f^*(s)}{s^{1-\frac{m}{Q}}}\, ds \approx \|f\|_{(L^\infty)^d_{1,s^{1-\frac{m}{Q}}}(0,1)} \tag{5.10}$$

Hence we get

$$L^{p_1,q_1}(0,1) \overset{*}{\hookrightarrow} (L^\infty)^d_{1,s^{1-\frac{m}{Q}}}(0,1). \tag{5.11}$$

Now, we want to use the Theorem 4.5 to get required compactness of the operator $\mathcal{Q}_Q^m$. To this end, we need to verify assumptions of the corresponding theorems.

Since $p_1 > \frac{Q}{m}$ and $m < Q$, $L^{p_1,q_1}(0,1) \neq L^1(0,1)$. The role of function J in the claim of the Theorem 4.5 plays function $J = s^{1-\frac{m}{Q}}(0,1)$, which satisfy condition (2.3). Finally,

$$\left\| \left(\int_t^1 \frac{ds}{J(s)} \right)^1 \right\|_{L^\infty(0,1)} = \int_0^1 \frac{1}{s^{1-\frac{m}{Q}}}\, ds < \infty,$$

which is the condition (4.3) with $J = s^{1-\frac{m}{Q}}(0,1)$, $j = 1$ and $\mathbf{Y}(0,1) = L^\infty(0,1)$. Theorem 4.5 combined with (5.11) now yields

$$\mathcal{Q}_Q^m \colon L^{p_1,q_1}(0,1) \to\to L^\infty(0,1).$$

The previous result together with the embedding $L^\infty(0,1) \hookrightarrow L^{p_2,q_2}(0,1)$, now yields the claim.

Suppose now that $p_1 \leq \frac{Q}{m}$. Assumptions of this theorem excludes cases when $L^{p_2,q_2}(0,1) = L^\infty(0,1)$ from consideration. This allows us to use the Theorem 4.3 to reduce (5.9) to

$$(L^{p_1,q_1})^r_{1,s^{1-\frac{m}{Q}}}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2}(0,1). \tag{5.12}$$

It is shown in [7, Proposition 8.3] that $\mathbf{X}^r_{j,J}(0,1)$ is the smallest rearrangement-invariant space such that

$$H^j_J \colon \mathbf{X}(0,1) \to\to \mathbf{X}^r_{j,J}(0,1)$$

holds. In [7, Theorem 6.8.] the following characterization of such rearrangement-invariant space is given.

$$(L^{p_1,q_1})^r_{1,s^{1-\frac{m}{Q}}}(0,1) = \begin{cases} L^{\frac{p_1}{1-\frac{mp_1}{Q}},q_1}(0,1), & \text{if } \frac{m}{Q} < 1 \text{ and } 1 \leq p_1 < \frac{Q}{m}, \\ L^{\infty,q_1;-1}(0,1), & \text{if } \frac{m}{Q} < 1 \text{ and } q_1 > 1, \\ L^\infty(0,1), & \text{otherwise.} \end{cases}$$

Consequently, if $p_1 < \frac{Q}{m}$ and

$$p_2 < \frac{p_1}{1 - \frac{mp_1}{Q}}$$

then the Theorem 5.1 ensures that (5.12) holds.

If $p_1 = \frac{Q}{m}$ then, again, Theorem 5.1 yields that (5.12) holds if $p_2 < \infty$. □

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Chapter 4

A new algorithm for approximating the least concave majorant

A NEW ALGORITHM FOR APPROXIMATING THE LEAST CONCAVE MAJORANT

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Abstract. The least concave majorant, $\hat{F}$, of a continuous function F on a closed interval, I, is defined by

$$\hat{F}(x) = \inf\{G(x)\colon\ G \geqslant F,\ G \text{ concave}\}, \quad x \in I.$$

We present an algorithm, in the spirit of the Jarvis March, to approximate the least concave majorant of a differentiable piecewise polynomial function of degree at most three on I. Given any function $F \in \mathcal{C}^4(I)$, it can be well-approximated on I by a clamped cubic spline S. We show that $\hat{S}$ is then a good approximation to $\hat{F}$.

We give two examples, one to illustrate, the other to apply our algorithm.

Keywords: least concave majorant; level function; spline approximation

MSC 2010: 26A51, 52A41, 46N10

1. Introduction

Suppose F is a continuous function on the interval $I = [a, b]$. Denote by $\hat{F}$ the least concave majorant of F, namely,

$$\hat{F}(x) = \inf\{G(x)\colon\ G \geqslant F,\ G \text{ concave}\},$$

which can be shown to be given by

$$\hat{F}(x) = \sup\Big\{\frac{\beta - x}{\beta - \alpha}F(\alpha) + \frac{x - \alpha}{\beta - \alpha}F(\beta)\colon\ a \leqslant \alpha \leqslant x \leqslant \beta \leqslant b\Big\}, \quad x \in I.$$

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This concave function has application in such diverse areas as mathematical economics, statistics, and abstract interpolation theory. See, for example, [3], [2], [11], [1], [10] and [8]. We observe that $\hat{F}$ is continuous on I, and it is differentiable there when F is.

Our aim in this paper is to give a new algorithm to approximate $\hat{F}$, together with an estimate of the error entailed. If F is a continuous or, stronger yet, a differentiable piecewise polynomial of degree at most three, then so is $\hat{F}$. If not, then F may be approximated by a clamped cubic spline and the least concave majorant of the approximating function is seen to be a good approximation to $\hat{F}$. To estimate the error in Theorem 16 below we use a known result for the approximation error involving such cubic splines from [4], together with a new result on $(\hat{F})'$, which in [9], page 70, and [5] is denoted by $(F')^\circ$ and is referred to as the level function of F' in the unweighted case. See the aforementioned Theorem 16.

The simple structure of $\hat{F}$ will be the basis of our algorithm. Since F and $\hat{F}$ are continuous, the zero set, Z_F, of $\hat{F} - F$ is closed; of course, $\hat{F} = F$ on Z_F. The connected components of Z_F^c are intervals open in the relative topology of I on which $\hat{F}$ is a strict linear majorant of F; indeed, if, for definiteness, the component interval with endpoints α and β is a subset of the interior of I, then

$$\hat{F}(\alpha) = F(\alpha), \quad \hat{F}(\beta) = F(\beta), \tag{1}$$

$$F(x) < \hat{F}(x) = F(\alpha) + (x - \alpha)\frac{F(\beta) - F(\alpha)}{\beta - \alpha}, \quad \alpha < x < \beta, \tag{2}$$

and, if F is differentiable on I,

$$(\hat{F})'(\alpha) = F'(\alpha) = \frac{F(\beta) - F(\alpha)}{\beta - \alpha} = F'(\beta) = (\hat{F})'(\beta). \tag{3}$$

Our task is thus to find the component intervals of Z_F^c. This will be done using a refinement of the Jarvis March algorithm; see [7]. To begin, we determine the set of points, D, at which F attains its maximum value, M, and then take $C = [c_1, c_2]$ to be the smallest closed interval containing D. Of course, in many cases D consists of one point and $c_1 = c_2$.

It turns out that $\hat{F}$ increases to M on $[a, c_1]$, is identically equal to M on C, then decreases on $[c_2, b]$.

To describe in general terms how the algorithm works we focus on $[a, c_1]$, $a < c_1$, and take F to be a differentiable function which is piecewise cubic. As such, there is a partition, P, of $[a, c_1]$ on each subinterval of which F is a cubic polynomial. By refining the partition, if necessary, to include critical points and points of inflection of F, we may assume that this polynomial is either strictly concave, linear or strictly convex, and is either increasing or decreasing on its subinterval. It is the subintervals where the associated cubic polynomial is increasing and strictly concave that are of interest. It is important to point out that for a piecewise cubic function, Z_F^c has only finitely many components.

Now, $\hat{F}$ on a component of Z_F^c may be thought of as a kind of linear bridge over a convex part of F. With this in mind, we call an interval, say $J = (\alpha, \beta)$, a bridge interval if, on it, F satisfies

$$F(x) < F(\alpha) + (x - \alpha)\frac{F(\beta) - F(\alpha)}{\beta - \alpha}, \quad \alpha < x < \beta, \tag{4}$$

and

$$F'(\alpha) = \frac{F(\beta) - F(\alpha)}{\beta - \alpha} = F'(\beta). \tag{5}$$

We include endpoints of I as possible endpoints of bridge intervals. In such case, the corresponding part of (5) is omitted. An illustrating example of bridge intervals and least concave majorant of a function can be found in Figure 7. It might be helpful to reader to check the demonstrative Example 1 in Section 7 while reading the formal description of algorithm. The algorithm is there applied to a particular spline.

Proceeding systematically from c_1 to a (the procedure from c_2 to b is similar) our algorithm determines, in a finite number of steps, a finite number of pairwise disjoint bridge intervals with endpoints in the intervals of increasing strict concavity referred to in the above paragraph. The desired components are among these bridge intervals.

The technical details of all this are elaborated in Section 2. Proofs of results stated in that section are given in the next one and the algorithm itself is justified in the one following that. Remarks on the implementation of the procedure are made in Section 5. Section 6 has estimates of the error incurred when approximating an absolutely continuous function by a clamped cubic spline, while in the final section two examples are given.

2. The algorithm

In this section we describe our algorithm in more detail. This will require us to first state some lemmas whose proof will be given in the next section.

Suppose that F is a continuous function on some interval $I = [a, b]$ and let $\hat{F}$, Z_F^c, M, D and $C = [c_1, c_2]$ be as in the introduction.

Lemma 1. *If F is a continuous function on I, then the least concave majorant, $\hat{F}$, of F on $I = [a, b]$ is continuous on I, with $\hat{F}(a) = F(a)$ and $\hat{F}(b) = F(b)$. Moreover, on each component interval, J, of Z_F^c, with endpoints α and β, $\hat{F}$ is the linear function, l, interpolating F at the points α and β.*

Lemma 2. *Suppose F is differentiable on (a, b) and (α, β) is a component of Z_F^c. Then $\hat{F}$ is differentiable on (a, b), $(\hat{F})'(x) = F'(x)$ for $x \in (a, b) \cap Z_F$, and $(\hat{F})'(x) = (F(\beta) - F(\alpha))/(\beta - \alpha)$ for $x \in [\alpha, \beta]$. In particular, $F'(x) = (\hat{F})'(x) = (F(\beta) - F(\alpha))/(\beta - \alpha)$ if $x = \alpha \in (a, b)$ or $x = \beta \in (a, b)$. Moreover, if F' is continuous on (a, b), then so is $(\hat{F})'$.*

Lemma 3. *Let F be a continuous function on I, then $\hat{F} \equiv M$ on C. Moreover, $\hat{F}$ is strictly increasing on (a, c_1) and strictly decreasing on (c_2, b).*

Lemma 4. *Let F be a continuous function, suppose $C = [c_1, c_2]$ is as in the introduction, and suppose $x, y, z \in (a, b)$ are such that F is strictly convex on (x, z) and $y \in (x, z)$. Then $F(y) \neq \hat{F}(y)$.*

Suppose F is differentiable as well. If $y \in (a, c_1)$ and $F'(y) \leqslant 0$ then $F(y) \neq \hat{F}(y)$. Analogously, if $y \in (c_2, b)$ and $F'(y) \geqslant 0$ then $F(y) \neq \hat{F}(y)$.

Lemma 5. *Let F be a continuous function. If $J = (\alpha, \beta)$ is a component interval of Z_F^c then either $J \subset (a, c_1)$, $J \subset (c_1, c_2)$ or $J \subset (c_2, b)$.*

Suppose that F is piecewise cubic and differentiable on I, and suppose $J \subset [a, c_1]$. Denote by P the closed intervals determined by the partition of $[a, c_1]$ inherited from the piecewise cubic structure of F, together with any critical points and points of inflection of F in $[a, c_1]$.

Lemma 6. *Suppose that F is piecewise cubic and differentiable on I. Let $J = (\alpha, \beta) \subset [a, c_1)$ be a component interval of Z_F^c. Then either $\alpha = a$ or there is an interval $K = [k_1, k_2]$ in P containing α on which F is strictly concave and increasing. Similarly, either $\beta = c_1$ or there is an interval $L = [l_1, l_2]$ in P containing β on which F is strictly concave and increasing. Moreover, $K \neq L$.*

Leaving aside the case $c_1 = c_2 = b$ our goal is to select the components of Z_F^c from among the bridge intervals of the form $[a, b_1)$ or (a_1, b_1), $a_1 > a$, such that a_1 and b_1 lie in distinct intervals in $\mathcal{P}$ with disjoint interiors on which intervals F is strictly concave and increasing.

Let $\mathcal{P}$ be the collection of intervals in P where F is strictly concave and increasing.

Given a pair of intervals in $\mathcal{P}$ that can have the endpoints of a bridge interval in them, one determines those endpoints, if they exist, by the study of a certain sextic polynomial equation. The details of the most complicated case are described in the following lemma.

Lemma 7. *Let $L = [l_1, l_2]$ and $R = [r_1, r_2]$ belong to $\mathcal{P}$ with $l_2 \leqslant r_1$. Suppose*

$$F(x) = \begin{cases} P_L(x) = Ax^3 + Bx^2 + Cx + D & \text{on } L, \\ P_R(x) = Wx^3 + Xx^2 + Yx + Z & \text{on } R, \end{cases}$$

with $AW \neq 0$. Assume

$$J = P'_L(L) \cap P'_R(R) \neq \emptyset.$$

Then, if there is a bridge interval $I_1 = (a_1, b_1)$ with $a_1 \in L$ and $b_1 \in R$, this bridge interval is such that

$$a_1 = (P'_L)^{-1}(y_0) \quad \text{and} \quad b_1 = (P'_R)^{-1}(y_0), \tag{6}$$

where y_0 is a point in J satisfying the sextic equation

$$(\mu_1^2 - \mu_2^2 - \mu_3^2\delta)^2 - 4\mu_2^2\mu_3^2\gamma\delta = 0,$$

in which

$$\gamma = 3Ay + B^2 - 3AC, \quad \delta = 3Wy + X^2 - 3WY, \quad \mu_2 = \frac{-2\gamma}{27A^2}, \quad \mu_3 = \frac{2\delta}{27W^2}$$

and

$$\mu_1 = \frac{1}{3}\Big(\frac{X}{W} - \frac{B}{A}\Big)y + \Big(Z + \frac{2X^3}{27W^2} - \frac{YX}{3W}\Big) - \Big(D + \frac{2B^3}{27A^2} - \frac{BC}{3A}\Big).$$

The verification that a given interval $J = (\alpha, \beta) \subset (a, c_1)$ satisfies condition (4) can be achieved using the following criterion: Assume that $\alpha \in L = [l_1, l_2] \in \mathcal{P}$, $\beta \in R = [r_1, r_2] \in \mathcal{P}$, $l_2 < r_1$, and that l is a linear function interpolating F on J. Then J satisfies (4) if for every $K = [k_1, k_2]$ in P with $K \subset [l_2, r_1]$,

$$l(k_1) - F(k_1) > 0 \quad \text{and} \quad l(k_2) - F(k_2) > 0,$$

and, in addition, if $K \in \mathcal{P}$, then

$$l(\varrho) - F(\varrho) > 0$$

for any root, ϱ, in K of the quadratic

$$F'(x) = \frac{F(\beta) - F(\alpha)}{\beta - \alpha}.$$

Obvious modifications of the above must also hold for $[a_1, l_2]$ and $[r_1, b_2]$. This criterion can be proved using elementary calculus.

We are now able to describe an iterative procedure that selects the component intervals of Z_F^c from a class of bridge intervals. We will focus our description on the case of finding all component intervals contained in (a, c_1) as the case in which the component intervals are contained in (c_2, b) is analogous while the component intervals in (c_1, c_2) are determined trivially by Lemma 3.

If $a = c_1$, then there is no such component interval. In the following, we exclude, at first, the case $c_1 = c_2 = b$, so that $c_1 < b$. Set $\mathcal{P}_0 = \mathcal{P}$.

We claim that $\mathcal{P}_0$ cannot be empty. As a consequence of Lemma 5 we have that $\hat{F}(c_1) = F(c_1)$, since c_1 cannot be in the interior of any component interval. The point c_1 is a local maximum of F. The choice of P ensures that there is an interval (x, c_1) such that F is increasing and concave on it, hence $\mathcal{P}_0$ must contain at least one interval.

Assume $\mathcal{P}_0$ has exactly one interval. The fact that c_1 is a local maximum of F ensures that this interval is of a form $[x, c_1)$. Suppose now that $x = a$, then $F = \hat{F}$ on $[a, c_1]$, since the function

$$m(t) = \begin{cases} F(t), & t \in [a, c_1], \\ M, & t \in (c_1, b], \end{cases}$$

is a concave majorant of F. (It is a concave function extended linearly with slope equel to that of the tangent line at the endpoint.)

Suppose now that $x \neq a$. We have $F \neq \hat{F}$ on (a, x) — if there were $y \in (a, x)$ such that $F(y) = \hat{F}(y)$, then F would have to be increasing and strictly concave on some neighbourhood by Lemma 4 and Lemma 6. This is a contradiction to the assumption that $[x, c_1]$ is the only interval in $\mathcal{P}_0$. Since $F \neq \hat{F}$ on (a, x) there must be a component interval containing (a, x). On the other hand, Lemma 3 implies that $F(c_1) = \hat{F}(c_1)$, hence this component interval must be a subset of (a, c_1).

The desired component interval is of a form (a, β), $\beta \in [x, c_1)$. If we choose β to be the unique solution to the equation

$$F'(\beta) = \frac{F(\beta) - F(a)}{\beta - a},$$

then the interval (a, β) will be the component interval, since it is the only interval which satisfies the necessary conditions (3).

Suppose next that $\mathcal{P}_0$ has at least two intervals and take $R = [r_1, r_2]$ to be that interval in $\mathcal{P}_0$ closest to c_1.

We seek first a component interval of the form (a, r), $r \in R$, as if R were the only interval in $\mathcal{P}_0$. If no such interval exists, let $L = [l_1, l_2]$ be the interval in $\mathcal{P}_0$ closest to a, then use Lemma 7 to test for a bridge interval $W = (w_1, w_2)$ with $w_1 \in L$ and $w_2 \in R$.

It is important to point out that Lemma 7 only places a restriction on bridge intervals, it does not guarantee them. Once the sextic is solved, condition (2) must still be verified for the proposed bridge interval. This means iterating through each partition subinterval contained in the proposed bridge interval and solving a maximum problem to verify that F lies underneath the proposed linear $\hat{F}$.

In a true Jarvis March points, rather than intervals, are ordered according to the angle of a tangent line. In the case of intervals associated to piecewise *cubic* functions such an ordering is computationally expensive.

Should there be no such W carry out the same test on the interval in $\mathcal{P}_0$ closest to the right of L, if one exists.

If, in moving systematically to the right in this way, we find no W, we discard R from $\mathcal{P}_0$ to get $\mathcal{P}_1$ and repeat the above procedure.

If, on the contrary, we find such a W, it will be a component interval. Say $w_1 \in N = [n_1, n_2]$, $N \in \mathcal{P}_0$.

We next form $\mathcal{P}_1$ by discarding from $\mathcal{P}_0$ all intervals to the right of the point w_1, for example R, and, in addition, replace N by the interval $[n_1, w_1]$ (if $n_1 < w_1$, otherwise just discard N). We then carry out the above-described procedure with $\mathcal{P}_1$, if $\mathcal{P}_1 \neq \emptyset$.

Continuing in this way we see that $\mathcal{P}_{n+1}$ has at least one less interval than $\mathcal{P}_n$, so the algorithm terminates after a finite number of steps.

Finally, in the case $c_1 = c_2 = b$ there may be a component interval of Z_F^c of the form (r, b), $r \in [a, b)$. This may be found in a way similar to those of the form (a, r).

Remark 8. We now comment briefly on how one can modify our algorithm to deal with piecewise cubic functions that are only continuous. In this case the notion of a bridge interval has to be changed since the function F might not be differentiable at the endpoint of a component intervals of Z_F^c and hence that end point needn't belong to an interval of strict concavity. Accordingly, we say that (α, β) is a bridge interval if conditions (4) and (5) hold and, in addition,

$$F'(\alpha-) \geqslant \frac{F(\beta) - F(\alpha)}{\beta - \alpha} \geqslant F'(\alpha+) \quad \text{and} \quad F'(\beta-) \geqslant \frac{F(\beta) - F(\alpha)}{\beta - \alpha} \geqslant F'(\beta+).$$

Again, Lemma 6 must be modified to compensate for the F need not be differentiable. To do this we allow for *three* possibilities, namely, $\alpha = a$, α is contained in an interval of strict concavity of F *or* α is one of the points at which $F'(\alpha-) > F'(\alpha+)$; a similar change must be made at the β. These changes necessitate our including all points of discontinuity of F' as degenerate intervals in $\mathcal{P}$.

The iterations of our algorithm proceed much as in the differentiable case, with the difference that when some point, say x, is selected from $\mathcal{P}_i$ we must check if (α, x) (or (β, x)) is a bridge interval in the new sense. This can be done in a manner similar to the one we described for determining if (α, β) is a bridge interval in the old sense.

3. Proof of Lemmas 1–7

Proof of Lemma 1. Since $\hat{F}$ is concave it is continuous on the interior of I. This continuity ensures that for all $\varepsilon > 0$, there exists a slope m such that the graph of F lies under the line

$$l_a(x) = F(a) + m(x - a) + \varepsilon.$$

But then l_a would be a concave majorant of F, so

$$F(x) \leqslant \hat{F}(x) \leqslant l_a(x), \quad x \in I.$$

As $\varepsilon > 0$ is arbitrary, $\hat{F}$ is continuous at a, with $\hat{F}(a) = F(a)$. A similar argument shows $\hat{F}$ is continuous at b, with $\hat{F}(b) = F(b)$.

Let J and l be as in the statement of Lemma 1 and suppose y is a point at which $F - l$ achieves its maximum value on I. Since F lies below the line $l + F(y) - l(y)$, so does $\hat{F}$. In particular, $\hat{F}(y) \leqslant F(y)$, so $\hat{F}(y) = F(y)$ and hence $y \notin J^\circ$. But, $\hat{F}(\alpha) = F(\alpha)$ and $\hat{F}(\beta) = F(\beta)$, so, by concavity, $\hat{F}$ lies above l on J and below l off J°. Thus,

$$F(y) - l(y) \leqslant \hat{F}(y) - l(y) \leqslant 0,$$

whence

$$F \leqslant l + F(y) - l(y) \leqslant l.$$

This means $\hat{F}$ lies below l on J. It follows that $\hat{F} = l$ on J. □

Proof of Lemma 2. If $x \in (a, b) \cap Z_F$ then $\hat{F}(x) = F(x)$. Since $\hat{F}$ is a concave majorant of F for any w and y satisfying $a < w < x < y < b$ we have

$$\frac{F(y) - F(x)}{y - x} \leqslant \frac{\hat{F}(y) - \hat{F}(x)}{y - x} \leqslant \frac{\hat{F}(x) - \hat{F}(w)}{x - w} \leqslant \frac{F(x) - F(w)}{x - w} = \frac{F(w) - F(x)}{w - x}.$$

Since F is differentiable at x, the squeeze theorem shows that $(\hat{F})'(x)$ exists and equals $F'(x)$.

Lemma 1 shows that, on (α, β), $\hat{F}$ is a line with slope $(F(\beta) - F(\alpha))/(\beta - \alpha)$. So it is differentiable on (α, β) and has one-sided derivatives at the points α and β.

If α or β is in $(a,b)\cap Z_F$ the derivative of $\hat{F}$ exists there and, of course, coincides with its one-sided derivative. If $\alpha = a$ or $\beta = b$, the endpoints of the domain of $\hat{F}$, then $(\hat{F})'$ is a necessarily just a one-sided derivative. We conclude that $(\hat{F})' = (F(\beta) - F(\alpha))/(\beta - \alpha)$ on the closed interval $[\alpha, \beta]$.

Evidently, $(\hat{F})'$ is continuous at each $x \in X_F^c$. Suppose F' is continuous at $x \in (a,b)\cap Z_F$. If $a < w < x < y < b$ then any component of Z_F^c that intersects (w,y) has at least one endpoint in (w,y). It follows that $(\hat{F})'(w,y) \subset F'(w,y)$. Since F' is continuous at x, so is $(\hat{F})'$. □

P r o o f of Lemma 3. To verify the first statement, one needs only to observe that between two points in D (at which $F = M$) $\hat{F} = M$.

The second statement follows from a simple contradiction argument: Assume that there are $x_1, x_2 \in (a, c_1)$, $x_1 < x_2$, such that $\hat{F}(x_1) \geqslant \hat{F}(x_2)$. Then $\hat{F}(x_1) < \hat{F}(c_1)$ implies that

$$\hat{F}(x_2) < \hat{F}(x_1)\frac{c_1 - x_2}{c_1 - x_1} + \hat{F}(c_1)\Big(1 - \frac{c_1 - x_2}{c_1 - x_1}\Big).$$

But this contradicts the concavity of $\hat{F}$. Consequently, we have $\hat{F}(x_1) < \hat{F}(x_2)$. An analogous argument shows that $\hat{F}$ is strictly decreasing on (c_2, b). □

P r o o f of Lemma 4. The second part follows from Lemma 3, as $\hat{F}$ is strictly increasing on (a, c_1) and strictly decreasing on (c_2, b). This leads to contradiction as if $F(y) = \hat{F}(y)$ then $F'(y) = (\hat{F})'(y)$ by Lemma 2 if y is an isolated point of Z_F and trivially otherwise.

To prove the first part: suppose for contradiction that $\hat{F}(y) = F(y)$. Then

$$\hat{F}(y) = F(y) \leqslant F(x)\frac{y-x}{z-x} + F(z)\frac{z-y}{z-x} \leqslant \hat{F}(x)\frac{y-x}{z-x} + \hat{F}(z)\frac{z-y}{z-x},$$

which is in contradiction with the strict concavity of $\hat{F}$. □

P r o o f of Lemma 5. For x in a bridge interval $J = (\alpha, \beta)$, condition (4) yields

$$F(x) \leqslant F(\alpha)\frac{\beta - x}{\beta - \alpha} + F(\beta)\frac{x - \alpha}{\beta - \alpha} \leqslant M,$$

with equality only with $F(\alpha) = F(\beta) = M$. Thus, J intersects C only if both endpoints are contained in C. The conclusion follows. □

P r o o f of Lemma 6. When $a = \alpha$ or $b = c_1 = \beta$ there is nothing to prove. Assume first, then, that $\alpha > a$ and choose $K = [k_1, k_2] \in P$ such that $\alpha \in [k_1, k_2)$. For any $x \in J \cap (\alpha, k_2)$, Lemmas 2 and 3 combine to give

$$F(x) < \hat{F}(x) = F(\alpha) + (x - \alpha)F'(\alpha).$$

Since F lies below its tangent line, it is neither linear nor strictly convex on $[k_1, k_2]$.

Thus, F must be strictly concave on K. Lemma 3 implies that $\hat{F}$ is strictly increasing on (a, c_1), hence $(\hat{F})'(\alpha) > 0$. Lemma 2 yields that $(\hat{F})'$ exists and $(\hat{F})'(\alpha) = F'(\alpha)$. The choice of P ensures that F is monotone on K. Hence F is increasing on K.

A similar argument yields F strictly concave and increasing on $L = [l_1, l_2]$ when $\beta < c_1$. □

P r o o f of Lemma 7. Since F' is decreasing on L and R, $J = [c, d]$, with $c = \max[F'(l_2), F'(r_2)]$ and $d = \min[F'(l_1), F'(r_1)]$.

Now,

$$F'(x) = \begin{cases} P_L'(x) = 3Ax^2 + 2Bx + C & \text{on } L, \\ P_R'(x) = 3Wx^2 + 2Xx + Y & \text{on } R, \end{cases}$$

with F' decreasing on both intervals. So, the unique roots, $a(y)$ and $b(y)$, of

$$P_L'(a(y)) = y \quad \text{and} \quad P_R'(b(y)) = y, \quad y \in J,$$

can be obtained from the formulas

$$a(y) = -\frac{1}{3A}\left[B \pm \sqrt{3Ay + B^2 - 3AC}\,\right]$$

and

$$b(y) = -\frac{1}{3W}\left[X \pm \sqrt{3Wy + X^2 - 3WY}\,\right].$$

We now seek $y \in J$ such that

$$\frac{F(b(y)) - F(a(y))}{b(y) - a(y)} = y$$

or

$$F(b(y)) - yb(y) - (F(a(y)) - ya(y)) = 0. \tag{7}$$

Figures 1 and Figures 2 below illustrate the geometric meaning of equation (7). Letting

$$\gamma(y) = 3Ay + B^2 - 3AC \quad \text{and} \quad \delta(y) = 3Wy + X^2 - 3WY,$$

equation (7) is equivalent to

$$\mu_1 + \mu_2\sqrt{\gamma} + \mu_3\sqrt{\delta} = 0, \tag{8}$$

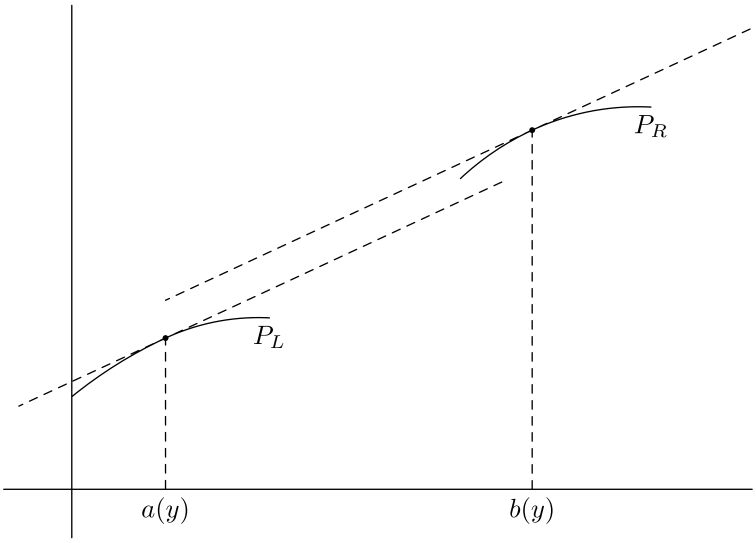

Figure 1. For each $y \in (P_L)'(L) \cap (P_R)'(R)$ there exist exactly one $a(y) \in L$ and $b(y) \in R$ such that $P_L'(a(y)) = R_L'(b(y)) = y$.

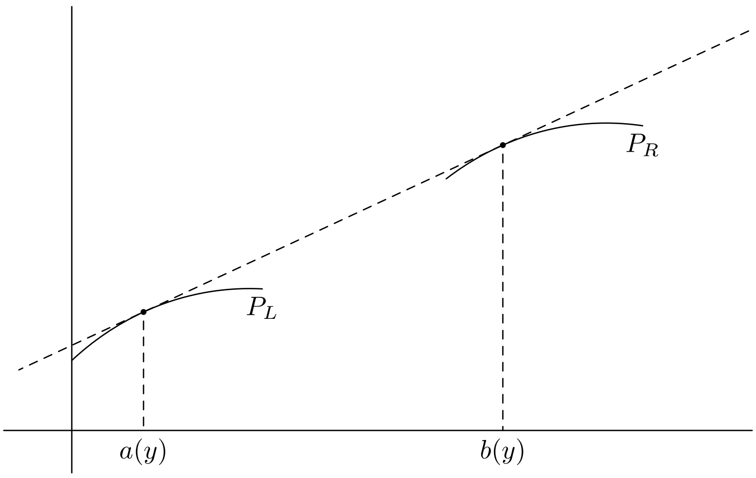

Figure 2. There is a $y_0 \in (P_L)'(L) \cap (P_R)'(R)$ such that the corresponding $a(y_0)$ and $b(y_0)$ referred to in the caption of Figure 1 satisfy $y_0 = (F(b(y_0)) - F(a(y_0)))/(b(y_0) - a(y_0)) = (P_R(b(y_0)) - P_L(a(y_0)))/(b(y_0) - a(y_0))$, whence $P_L'(a(y_0)) = P_R'(b(y_0)) = y_0 = (F(b(y_0)) - F(a(y_0)))/(b(y_0) - a(y_0))$.

with μ_1, μ_2, μ_3 linear functions of y, namely,

$$\mu_2(y) = -\frac{2\gamma}{27A^2}, \quad \mu_3 = \frac{2\delta}{27W^2}$$

and

$$\mu_1(y) = \frac{1}{3}\Big(\frac{X}{W} - \frac{B}{A}\Big)y + \Big(Z + \frac{2X^2}{27W^2} - \frac{YX}{3W}\Big) - \Big(D + \frac{2B^2}{27A^2} - \frac{CB}{3A}\Big).$$

We claim the solution of (7) is a root of the sextic polynomial equation

$$(\mu_1^2 - \mu_2^2\gamma - \mu_3^2\delta)^2 - 4\mu_2^2\mu_3^2\gamma\delta = 0. \tag{9}$$

Indeed, isolating μ_1 in (8), then squaring both sides gives

$$\mu_1^2 = \mu_2^2\gamma + \mu_3^2\delta + 2\mu_2\mu_3\sqrt{\gamma}\sqrt{\delta}. \tag{10}$$

Isolating the term in (10) with the square roots and squaring both sides yields (9). □

The following remark is given to make the appearance of the sextic equation seem more natural.

Remark 9. Suppose, for definiteness, the $a(y)$ and $b(y)$ referred to in the proof of Lemma 7 are given by

$$a(y) = \frac{-B}{3A} + \frac{1}{3A}\sqrt{3Ay + B^2 - 3AC} \quad \text{and} \quad b(y) = \frac{-X}{3W} + \frac{1}{3W}\sqrt{3WY + X^2 - 3WY}.$$

Then, equation (7) can be written as

$$\begin{aligned} P_R\Big(\frac{-X}{3W} + \frac{1}{3W}\sqrt{3WY + X^2 - 3WY}\Big) - P_L\Big(\frac{-B}{3A} + \frac{1}{3A}\sqrt{3Ay + B^2 - 3AC}\Big) \\ = y\Big(\frac{-X}{3W} + \frac{B}{3A} + \frac{1}{3W}\sqrt{3WY + X^2 - 3WY} - \frac{1}{3A}\sqrt{3Ay + B^2 - 3AC}\Big). \end{aligned}$$

In our original proof of Lemma 7 we rearranged the terms in this version of (7), then squared both sides. We repeated this procedure a few times to get rid of the square roots and so arrive at the sextic equation (9).

4. Justification of the algorithm

The purpose of this section is to prove

Theorem 10. *Let F be a differentiable piecewise cubic function. Then the bridge intervals coming out of the algorithm are precisely the component intervals of Z_f^c.*

For simplicity, we consider only the components in $[a, c_1)$. We begin with the preparatory

Lemma 11. *Suppose that F is absolutely continuous on $I_0 = [a, b]$. Let $I = (a_1, b_1)$ be a bridge interval with the right hand endpoint in an interval R on which F is strictly concave and increasing. If $J = (a_2, b_2)$ is another bridge interval such that $I \cap J \neq \emptyset$, $b_2 \in R$ and $b_1 < b_2$, then $a_2 < a_1$.*

P r o o f. Let

$$l_I(x) = F(a_1) + (x - a_1)\frac{F(b_1) - F(a_1)}{b_1 - a_1}$$

and, similarly,

$$l_J(x) = F(a_2) + (x - a_2)\frac{F(b_2) - F(a_2)}{b_2 - a_2}.$$

Assume, if possible, $a_1 < a_2$. Then, $a_2 < b_1$, otherwise $I \cap J = \emptyset$. So,

$$l_J(a_2) = F(a_2) < l_I(a_2), \tag{11}$$

since I is a bridge interval. The latter also implies

$$F(b_2) = l_I(a_2) + (b_1 - a_2)F'(b_1) + \int_{b_1}^{b_2} F'(t)\,\mathrm{d}t;$$

further, J being a bridge interval, we have

$$F(b_2) = l_J(a_2) + (b_2 - a_2)F'(b_2).$$

Therefore,

$$0 = l_J(a_2) - l_I(a_2) + (b_2 - a_2)F'(b_2) - (b_1 - a_2)F'(b_1) - \int_{b_1}^{b_2} F'(t)\,\mathrm{d}t.$$

The strict concavity of F on R ensures that $F'(t) > F'(b_2)$ for $t \in R$, $t < b_2$. Thus

$$\begin{aligned} l_I(a_2) - l_J(a_2) &= (b_2 - a_2)F'(b_2) - (b_1 - a_2)F'(b_1) - \int_{b_1}^{b_2} F'(t)\,\mathrm{d}t \\ &< (b_2 - a_2)F'(b_2) - (b_1 - a_2)F'(b_2) - \int_{b_1}^{b_2} F'(t)\,\mathrm{d}t \\ &= (b_2 - b_1)F'(b_2) - \int_{b_1}^{b_2} F'(t)\,\mathrm{d}t < 0. \end{aligned}$$

Consequently,

$$l_I(a_2) - l_J(a_2) < 0,$$

thereby contradicting (11). □

P r o o f of Theorem 10. As a consequence of Lemma 5 one gets that the component intervals are split into three groups: component intervals contained in $[a, c_1]$, $[c_1, c_2]$ and component intervals which are subsets of $[c_2, b]$. We begin by observing that component intervals of Z_F^c in $[a, c_1]$ are the maximal bridge intervals there.

To the end of showing every bridge interval coming out of the algorithm is a component interval of Z_F^c, fix an iteration, say the nth, of the procedure. Let $R = [r_1, r_2]$ be that interval in $\mathcal{P}_n$ closest to c_1. According to Lemma 11, if there are bridge intervals with righthand endpoint in R, the one closest to c_1 will be the bridge interval chosen by the algorithm and, moreover, will be a maximal bridge interval.

We next prove *all* component intervals of Z_F^c (in $[a, c_1)$) come out of the algorithm. Assume, if possible, $M = (m_1, m_2)$ is a component not obtained by the algorithm. Let $S = [s_1, s_2]$ be that member of $\mathcal{P}$ such that $m_2 \in S$.

Now, either S was chosen as an R in some iteration or it was not. If it was chosen and M is not the bridge interval with righthand endpoint in S closest to c_1, then another bridge interval, $N = (n_1, n_2)$, is; in particular, M and N satisfy the hypotheses of Lemma 11, with $m_2 < n_2$. We conclude $M \subset N$, which contradicts the maximality of M.

Finally, suppose S was not chosen. Then there is a last iteration, say the nth, such that $S \in \mathcal{P}_n$. Let $T \in \mathcal{P}_n$ be the interval in $\mathcal{P}_n$ closest to c_2.

If T does not contain the righthand endpoint of a bridge interval, S, it will be chosen in the next iteration, which cannot be. So, let $N = (n_1, n_2)$ be a bridge interval, indeed a component interval of Z_F^c, having $n_2 \in T$. Now, n_1 cannot be to the right of S as that would entail $S \in \mathcal{P}_{n+1}$. Again, n_1 cannot lie to the left of S nor can we have $n_1 < m_2$, since either would contradict the maximality of M. The only possibility left is $n_1 \in S$, $n_1 \geqslant m_2$.

Should we have $n_1 > s_1$, M would arise from $[s_1, n_1]$ in the next iteration. This leaves the case $s_1 = m_2 = n_1$. All intervals in $\mathcal{P}_n$ contained in $[n_1, c_1] = [m_2, c_2]$ will be discarded at the end of the nth step. But, according to Lemma 6, there exists an interval in $\mathcal{P}_{n+1}$ with m_2 as its right hand endpoint, which interval will be the one in $\mathcal{P}_{n+1}$ closest to c_1. As m_2 belongs to that interval M would come out of the $(n+1)$-st step of the algorithm contrary to our assumption. □

5. Implementation of the algorithm

In this section we discuss ways to make the algorithm more efficient. Suppose, then, that F is a differentiable piecewise polynomial and that we are searching for component intervals contained in $[a, c_1]$. In a given iteration we have chosen the interval $R = [r_1, r_2]$ furthest to the right in the current version of $\mathcal{P}$ and we are about to seek in it and, in an appropriate interval L to the left, endpoints of a bridge interval. It turns out we needn't do this for all L.

We have developed a few simple criteria to determine those L which cannot contain the left endpoint of a bridge interval with right endpoint in R.

One natural test is to require of L that $F'(L) \cap F'(R) \neq \emptyset$.

Lemma 12 below implies that there must be an intervening interval in P between L and R on which F is convex (or linear). We split the intervals in $\mathcal{P}$ into groups such that intervals in the same group are not separated by any intervening convex or linear interval. Then, bridge intervals cannot have endpoints in intervals from the same group. Consequently L is a viable candidate only if it belongs to a group other then R.

Moreover, for L to be a viable candidate it must lie to the left of the set of points at which F equals its maximum value on $[a, r_1]$. This is a consequence of Lemma 13 as it ensures that otherwise no bridge interval has endpoints in L and R.

Of course, there are more such criteria. We now state and proof the two Lemmas referred to above.

Lemma 12. *Let F be a differentiable piecewise polynomial function. Every bridge interval has to contain an interval from P on which F is not strictly concave.*

Proof. Suppose for contradiction that there is a bridge interval $B = (b_1, b_2)$ such that F is strictly concave on (b_1, b_2). Condition 5 then yields that $F'(b_1) = F'(b_2)$. At the same time, strict concavity of F yields that F' is decreasing on B, which leads to contradiction. □

Lemma 13. *Assume F is a cubic spline, suppose $R = [r_1, r_2] \subset [a, c_1]$ is an interval on which F is strictly concave and increasing, with $m_2 \in R$ such that $\hat{F}(m_2) = F(m_2)$. Given $s < r_1$ satisfying $F(s) = \max\{F(x)\colon x \in [a, r_1]\}$ and an $m_1 < r_1$ for which $M = (m_1, m_2)$ is a component interval of Z_F^c, one has $m_1 \in [a, s]$.*

Proof. Assume, if possible, $m_1 \in (s, r_1]$. Then $\hat{F}(s) \geqslant F(s) \geqslant F(m_1) = \hat{F}(m_1)$ by hypothesis, and $\hat{F}(m_2) > \hat{F}(m_1)$, since $\hat{F}$ is increasing on $[a, c_1]$ according to Lemma 3. Hence

$$\begin{aligned}\hat{F}(m_1) &\leqslant \frac{m_2 - m_1}{m_2 - s}\hat{F}(s) + \frac{m_1 - s}{m_2 - s}\hat{F}(m_2) \\ &= \hat{F}(s) + (m_1 - s)\frac{\hat{F}(m_2) - \hat{F}(s)}{m_2 - s},\end{aligned}$$

which contradicts the concavity of $\hat{F}$. □

6. Error estimates

Given an absolutely continuous function G on a closed interval I of finite length, we choose F to be the clamped cubic spline interpolating G at the points of a partition ϱ of I. This permits us to take advantage of the following special case of optimal error bounds for cubic spline interpolation obtained by Hall and Meyer in [4].

Proposition 14. *Suppose $G \in \mathcal{C}^4(I)$ and let $\varrho := [x_0, \ldots, x_{n+1}]$ be a partition of I. Denote by F the clamped cubic spline interpolating G at the nodes of ϱ. Then*

$$|G'(x) - F'(x)| \leqslant \frac{1}{24} \|G^{(4)}\|_\infty \, \|\varrho\|^3, \quad x \in I,$$

where $\|\cdot\|_\infty$ denotes the usual supremum norm and

$$\|\varrho\| := \sup\{|x_k - x_{k-1}|\colon\ k = 1, \ldots, n\}.$$

To estimate the error involved in approximating the least concave majorant, we first consider the sensitivity of the level function to changes in the original function. We recall that the level function, f°, of f is given by $f^\circ = (\hat{F})'$, where $F' = f$.

Theorem 15. *Suppose F and G are absolutely continuous functions defined on a finite interval I. Then $f = F'$, $g = G'$ and we denote by f° and g° the level functions of f and g, respectively. Then $\hat{F}$ and $\hat{G}$ are also absolutely continuous on I, and*

$$\|f^\circ - g^\circ\|_\infty = \|(\hat{F})' - (\hat{G})'\|_\infty \leqslant \|f - g\|_\infty.$$

Here $\hat{F}$ and $\hat{G}$ denote the least concave majorants of F and G, respectively, and $f = F'$, $g = G'$, while $f^\circ = (\hat{F})'$, and $g^\circ = (\hat{G})'$.

P r o o f. Set

$$Z_F = \{x \in I\colon\ F(x) = \hat{F}(x)\}, \quad Z_G = \{x \in I\colon\ G(x) = \hat{G}(x)\}$$

and observe that $f^\circ = f$ almost everywhere on Z_F and $g^\circ = g$ almost everywhere on Z_G. By Lemma 1, $\hat{F}$ is continuous and is of constant slope on each component of the complement of Z_F. It follows that $\hat{F}$ is absolutely continuous on I. Since $\hat{G}$ is continuous and is of constant slope on each component of the complement of Z_G, $\hat{G}$ is absolutely continuous on I as well. We consider several cases to establish that $|f^\circ(x) - g^\circ(x)| \leqslant \|f - g\|_\infty$ for almost every $x \in I$.

Case 1: $x \in Z_F$ and $x \in Z_G$. For almost every such x,

$$|f^\circ(x) - g^\circ(x)| = |f(x) - g(x)| \leqslant \|f - g\|_\infty.$$

Case 2: $x \in Z_G$ but $x \notin Z_F$. Then x is in the interior of some component interval $[a, b]$ of F. By Lemma 1, $\hat{F}(a) = F(a)$ and $\hat{F}(b) = F(b)$. Since $\hat{F}$ has constant slope on $[a, b]$,

$$\int_a^x f = F(x) - F(a) \leqslant \hat{F}(x) - \hat{F}(a) = (x - a)f^\circ(x)$$

and

$$\int_x^b f = F(b) - F(x) \geqslant \hat{F}(b) - \hat{F}(x) = (b - x)f^\circ(x).$$

Also, since $\hat{G}(x) = G(x)$ and g° is non-increasing,

$$\int_a^x g = G(x) - G(a) \geqslant \hat{G}(x) - \hat{G}(a) = \int_a^x g^\circ \geqslant (x - a)g^\circ(x)$$

and

$$\int_x^b g = G(b) - G(x) \leqslant \hat{G}(b) - \hat{G}(x) = \int_x^b g^\circ \leqslant (b - x)g^\circ(x).$$

Combining these four inequalities, we obtain

$$\begin{aligned} -\|f - g\|_\infty \leqslant \frac{1}{x - a}\int_a^x (f - g) &\leqslant f^\circ(x) - g^\circ(x) \\ &\leqslant \frac{1}{b - x}\int_x^b (f - g) \leqslant \|f - g\|_\infty. \end{aligned}$$

Thus, $|f^\circ(x) - g^\circ(x)| \leqslant |f - g\|_\infty$.

Case 3: $x \in Z_F$ but $x \notin Z_G$. Just reverse the roles of F and G in Case 2.

Case 4: $x \notin Z_F$ and $x \notin Z_G$. Suppose without loss of generality that $g^\circ(x) \leqslant f^\circ(x)$. Let a be the left-hand endpoint of the component interval of G containing x, and let b be the right-hand endpoint of the component interval of F containing x. By Lemma 1, $\hat{G}(a) = G(a)$ and $\hat{F}(b) = F(b)$. Since g° is constant on (a, x) and non-increasing on (x, b) we have

$$(b - a)g^\circ(x) \geqslant \int_a^b g^\circ = \hat{G}(b) - \hat{G}(a) \geqslant G(b) - G(a) = \int_a^b g.$$

Since f° is non-increasing on (a, x) and constant on (x, b), we have

$$(b - a)f^\circ(x) \leqslant \int_a^b f^\circ = \hat{F}(b) - \hat{F}(a) \leqslant F(b) - F(a) = \int_a^b f.$$

Combining these, we have

$$f^\circ(x) - g^\circ(x) \leqslant \frac{1}{b-a}\int_a^b (f-g) \leqslant \|f-g\|_\infty.$$

This completes the proof. □

The last result can be combined with Proposition 14 to give the desired error estimates.

Theorem 16. *Let ϱ be a partition of the interval $[a,b]$ and suppose $G \in \mathcal{C}^4([a,b])$. Let F be the clamped cubic spline interpolating G on ϱ. Then*

$$\|f^\circ - g^\circ\|_\infty \leqslant \|f-g\|_\infty \leqslant \frac{1}{24}\|G^{(4)}\|_\infty\,\|\varrho\|^3$$

and for each $x \in [a,b]$,

$$|\hat{F}(x) - \hat{G}(x)| \leqslant \frac{\min\{x-a, b-x\}}{24}\,\|G^{(4)}\|_\infty\,\|\varrho\|^3.$$

Here $\hat{F}$ and $\hat{G}$ denote the least concave majorants of F and G, respectively, and $f = F'$, $g = G'$; $f^\circ = (\hat{F})'$, and $g^\circ = (\hat{G})'$.

P r o o f. The first inequality is just Theorem 15 together with the result from [4]. For the second, observe that by Lemma 1, $\hat{F}(a) = F(a)$ and $\hat{G}(a) = G(a)$, and since a is in the partition ϱ, $G(a) = F(a)$. Thus, $\hat{F}(a) = G(a)$. Since both $\hat{F}$ and $\hat{G}$ are concave and hence absolutely continuous,

$$|\hat{F}(x) - \hat{G}(x)| = \left|\int_a^x f^\circ(x) - g^\circ(x)\right| \leqslant \int_a^x \|f^\circ - g^\circ\|_\infty \leqslant \frac{x-a}{24}\,\|G^{(4)}\|_\infty\,\|\varrho\|^3.$$

A similar argument, using integration on $[x,B]$, shows that

$$|\hat{F}(x) - \hat{G}(x)| \leqslant \frac{b-x}{24}\,\|G^{(4)}\|_\infty\,\|\varrho\|^3$$

and completes the proof. □

We present here two examples involving our algorithm.

Example 1. With our first example we illustrate the flow of the algorithm. Let s be the continuously differentiable, piecewise cubic function defined on $[0, 10]$ by

$$s(x) = s_n(x) \quad \text{on } [n-1, n],\ n = 1, 2, \ldots, 10,$$

where

$$\begin{aligned}
&s_1(x) = -1.1x^3 + 1.1x^2 + x + 1, && s_2(x) = 1.3x^3 - 5.3x^2 + 6.6x - 0.6,\\
&s_3(x) = -0.9x^3 + 6x^2 - 12.2x + 9.4, && s_4(x) = -1.5x^3 + 16x^2 - 56x + 67,\\
&s_5(x) = 3, && s_6(x) = 0.5x^3 - 8.75x^2 + 50x - 90.75,\\
&s_7(x) = 2 + (x - 6.5)^2, && s_8(x) = -0.5x^3 + 10.75x^2 - 76x + 179,\\
&s_9(x) = x^3 - 25.5x^2 + 216x - 605, && s_{10}(x) = 0.6x^3 - 16.6x^2 + 153x - 467.3.
\end{aligned}$$

The graph of s is given in Figure 3.

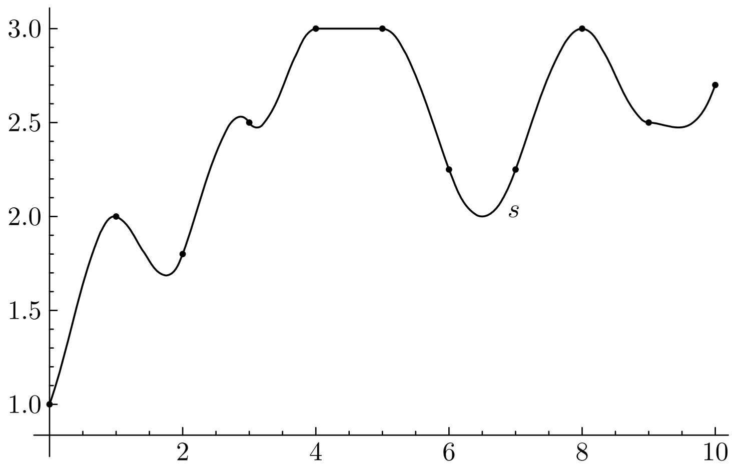

Figure 3. Graph of s with marked points where prescribed polynomials change.

To begin, s attains its maximum value of 3 on $D = [4, 5] \cup \{8\}$. So, $\hat{s}(x) = 3$ on $C = [4, 8]$.

Since $s < 3$ on $(5, 8)$ it will be a component interval. We next seek the component intervals in $[0, 4]$. By adding to the partition those points in $[0, 4]$ for which s' or s'' changes sign we get a refined partition where, on each subinterval, s is monotone and either strictly convex or strictly concave. The first derivative of s changes sign at 0.97687, 1.75204, 2.8701 and $3.\bar{1}$. The second derivative changes sign at $0.\bar{3}$, 1.35897, $2.\bar{2}$ and $3.\bar{5}$. We are interested in subintervals of $[0, 4]$ where s is strictly concave and increasing. These are $I_1 = [0.\bar{3}, 0.97687]$, $I_2 = [2.\bar{2}, 2.87011]$ and $I_3 = [3.\bar{5}, 4]$. Thus, $\mathcal{P}_0 = \{I_1, I_2, I_3\}$. Clearly, I_3 is the interval in $\mathcal{P}_0$ furthest to the right.

There are no bridge intervals with left endpoint 0 and right endpoint in I_3.

Indeed, there *are* two candidate intervals of the form $[a, r]$, $r \in I_3$, such that

$$s'(r) = \frac{s(r) - s(0)}{r} = \frac{s(r) - 1}{r}, \tag{12}$$

but, for neither candidate does one have (3), that is,

$$s(x) < x\left[\frac{s(r) - 1}{r}\right], \quad x \in (0, r).$$

This can be seen in Figure 4.

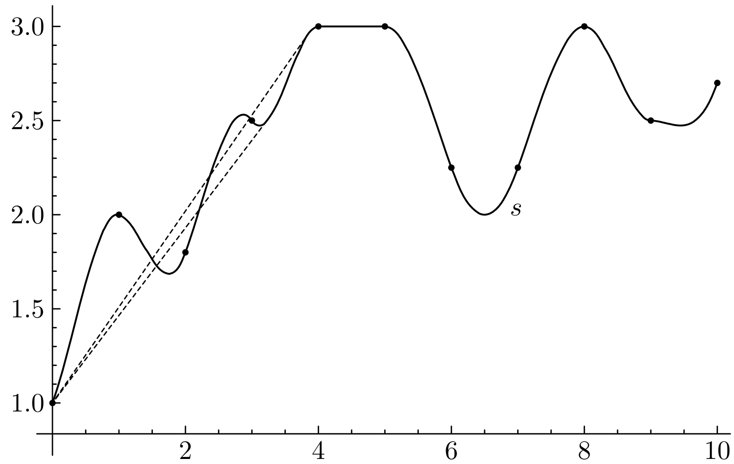

Figure 4. The two intervals with left-hand endpoint being 0 which satisfy the first condition (12) are $[0, 3.24826]$ and $[0, 3.84606]$. But neither of them can meet the second condition from the definition of a bridge interval.

Again, there are two intervals with right endpoint in I_3 and left endpoint in I_1 for which (1) and (2) holds. These are

$$I_{1,1} = (0.89359, 3.90772) \quad \text{and} \quad I_{1,2} = (0.92390, 3.16878).$$

However, only on $I_{1,1}$ is (3) satisfied. The situation is depicted in Figure 5.

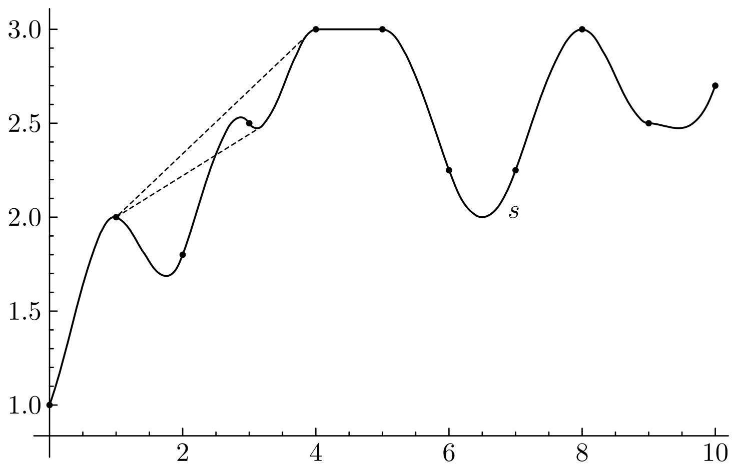

Figure 5. This figure pictures the bridge interval joining intervals I_1 and I_3 and the other candidate.

Since no interval with left endpoint in I_2 can have the left endpoint smaller than the left endpoint of $I_{1,1}$, the interval $I_{1,1}$ is the desired component interval. This completes the first iteration of our algorithm.

To form $\mathcal{P}_1$ for the second iteration we, of course, discard I_3. We also discard I_2, since it is contained int $I_{1,1}$. This leaves in $\mathcal{P}_1$ only interval I_1', as $(0.\bar{3}, 0.89359) = I_1 \setminus I_{1,1}$.

There is one bridge interval with right endpoint in I_1' and left endpoint 0. It is $(0, 0.5)$, therefore $(0, 0.5)$ is a component interval. We have thus found all component intervals in $[0, 4]$.

We now seek component intervals contained in $[8, 10]$. To begin we must ad to the partition points 8,9,10 the critical point 8.5 and the inflection points $9.\bar{2}$ and $9.\bar{4}$. It is then found that the intervals on which s is strictly concave and increasing are $J_1 = [8, 8.5]$ and $J_2 = [9, 9.\bar{2}]$.

The interval $[8.05353, 10]$ is a bridge interval with left endpoint in J_1 and right endpoint 10.

The unique component interval is $[8, 10]$. See Figure 6.

The graph of $\hat{s}$ appears in Figure 7.

Example 2. Consider the trimodal density function discussed in [6], namely,

$$f(x) = 0.5\varphi(x-3) + 3\varphi(10(x-3.8)) + 2\varphi(10(x-4.2)),$$

in which

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}.$$

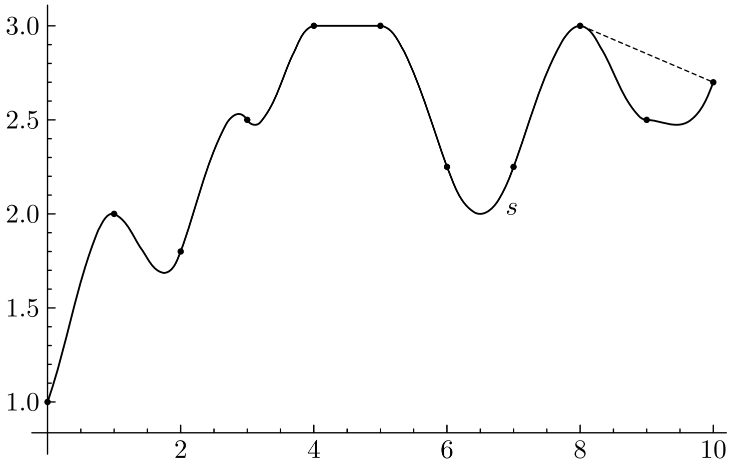

Figure 6. This figure shows the component interval $(8.05353, 10)$.

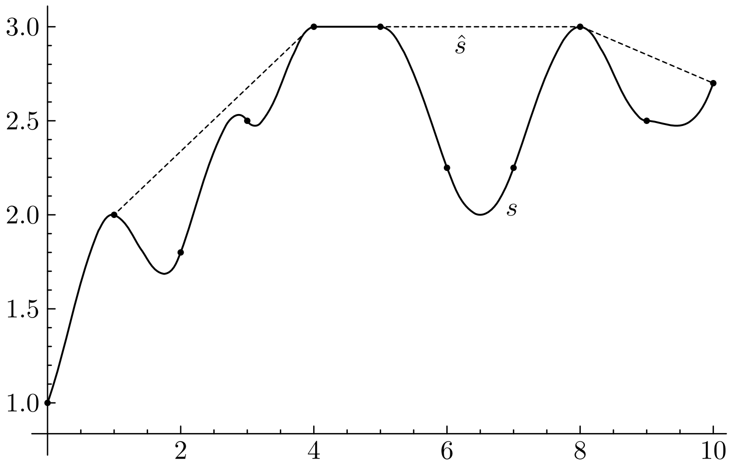

Figure 7. The least concave majorant of s is linear interpolation of s from the end-points of a component interval and agrees with s elsewhere.

We wish to approximate the least concave majorant of $F(x) = \int_0^x f(y)\,\mathrm{d}y$ on $[0,6]$. Now, $\|F^{(4)}\|_\infty \leqslant 700$, so to ensure that the clamped cubic spline S_F approximating F on $[0,6]$ satisfies $|f^\circ(x) - (S_F')^\circ(x)| \leqslant 0.001$ on $[0,6]$, we solve the equation $\frac{700}{24}\|\varrho\|^3 = 0.001$ to obtain $\|\varrho\| = 0.03249$. Dividing $[0,6]$ into $85 > \frac{6}{0.03249}$ equal subintervals, we apply the algorithm to identify the component intervals of $Z_{S_f}^C$. The approximation $\int_0^x (\hat{S}_f')^\circ$ to $\hat{F}(x)$ is accurate to within 0.003.

Figure 8 shows the graph of $F(y)$ and the approximation to its least concave majorant, $\hat{S}_F$.

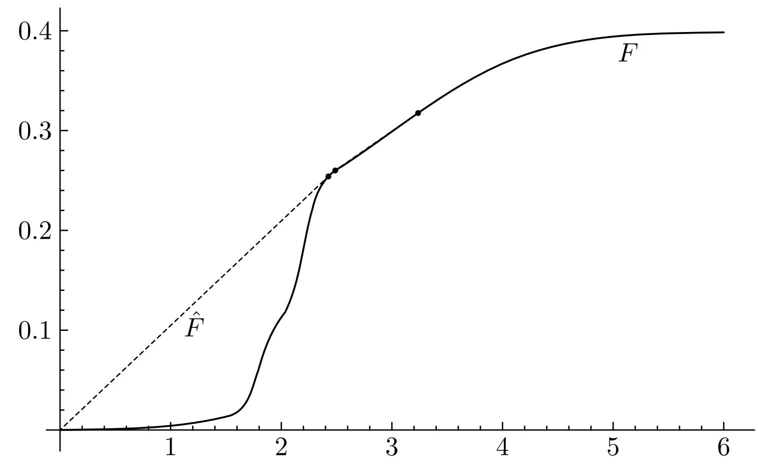

Figure 8. The trimodal density function F with its least concave majorant $\hat{F}$. The bridge intervals are $(0, 2.42575)$ and $(2.48781, 3.23693)$.

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