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Twistor equation on isolated horizons

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Abstract: In the present work we investigate the solution of the univalent twistor equation on an isolated horizon that serves for the definition of the so-called Penrose mass. We start our discussion with the construction of adapted coordinates to the isolated horizon and summarizing the main results in this field that are needed for our work. We include a chapter devoted to the extremal isolated horizons and prove an important result concerning uniqueness of geometry therein. It is a generalization of the paper by Lewandowski and Pawlowski (*Class. Quantum Grav.* **31** (17), 2014), which states that the extremal isolated horizons are necessarily isometric to the intrinsic geometry of the Kerr-Newmann black hole. Further we proceed to investigation of the twistor equation on the isolated horizon. We analyze conditions of integrability and derive the time dependent solution. Consequently we solve the 2-surface twistor equation and briefly discuss the general approach to the problem of defining the Penrose charge.

Keywords: isolated horizons; uniqueness of extremal horizons; twistor equation; Penrose mass

Contents

1 Introduction

Black holes are, on the one hand, the simplest objects we can find in nature: according to the no-hair theorem, they are described by a small number of parameters which completely determine all their properties. At the same time, black holes are very mysterious objects in which the space and time display their strangest and most incomprehensible properties. They are interesting also from astrophysical point of view, since their strong fields significantly contribute to gravitational lensing, thus allowing to investigate the large scale structure of the Universe. Black holes power the most violent processes like the emission of Blandford-Znajek jets from active galactic nuclei, black hole merger was the source of historically first detected burst of gravitational waves.

From theoretical point of view, black holes are also important laboratories for testing basic physical principles. The laws of black hole thermodynamics and related Hawking effect point to the possibility that gravitational field might be only effective description of more fundamental degrees of freedom that are, presumably, described by not-yet-discovered theory of quantum gravity. Attempts to apply principles of general relativity and quantum theory simultaneously have lead not only to Hawking's satisfactory discovery of black hole evaporation but, even more importantly, also to several paradoxes showing that principles of both theories might be contradictory, see e.g. information paradox, black hole complementarity or recent firewall paradox. The existence of black hole evaporation plays a crucial role even in cosmological scenarios like Penrose's conformal cyclic cosmology.

All these considerations show that black holes are both important astrophysical objects and gates to our deeper understanding of fundamental laws of nature.

From mathematical point of view, basic properties of black holes (like laws of thermodynamics, role of the horizon in the causal structure of spacetimes, existence of singularities) were discovered during the so-called *golden age of general relativity* (roughly 1960–1975) in works by Penrose, Hawking, Bekenstein, Carter, Bardeen and others. Although these works were absolutely fundamental for the progress of black hole physics (and gravitational physics in general), they were based on somewhat restrictive notion of the *event horizon*. In realistic situations, a more general concept of black hole is needed and there indeed exist several modifications of the notion of the horizon: apparent horizon, dynamical horizon, absolute horizon, Cauchy horizon, Killing horizon…

In this thesis we are particularly interested in the so-called *isolated horizons* introduced by Ashtekar and other in the context of loop quantum gravity. Mathematical description of isolated horizons is presented in Chapter 2. First we spell out in detail the limitations of the standard concept of the event horizon and explain how these limitations are overcame by isolated horizons. Then we employ the Newman-Penrose formalism to provide geometrical characterization and derive basic properties of isolated horizons. Then we specialize to the horizons possessing axial symmetry and introduce appropriate coordinate system on the spherical cuts of the horizons and discuss the multipole moments (in the sense of Ashtekar) which can be associated with the axi-symmetric horizon.

The important point here is that the geometry of the spacetime can be conveniently split into the intrinsic geometry of the horizon and the exterior region. Assumptions imposed on the isolated horizons fix the intrinsic part of the geometry (including part of the connection) but, up to some restrictions, leave a significant freedom for the geometry outside the black hole. For example, the presence of the external matter or radiation deforming the black hole is allowed.

With this formalism at hand, we prove the main result of this thesis in chapter 3. As we explain there in detail, despite the fact that isolated horizons are typically much more general than standard black holes admitting the event horizon, there is an important exception. Extremal horizons possessing the axial symmetry have a unique intrinsic geometry, regardless of the spacetime outside the black hole. We rederive existing results on the uniqueness of extremal horizons in different coordinate system and generalize them by allowing the deficit angles around the poles of the spherical cuts of the horizon. Our uniqueness result shows that in the absence of deficit angles, extremal horizon is isometric to that of extremal Kerr-Newman solution, but our family of solutions covers also the rotating C-metric with possibly non-vanishing NUT parameter.

The next goal of the thesis which was not fully achieved is related to the problem of quasi-local energy in general relativity. In Chapter 4 we explain the origin of the difficulties with the definition of energy in GR and describe a specific suggestion by Penrose how to define the energy on the quasi-local level. This suggestion is based on the solution of the twistor equation. Here, we do not go into the details of the twistor theory but we briefly review the main points of the Penrose construction.

The motivation for this part of the thesis is to compare the aforementioned Ashtekar multipole moments with other approaches to characterize the properties of gravitational field on a quasi-local level. Here we focused on the Penrose mass but other definitions exist (Hawking mass, Bartnik mass, Ludvigsen-Vickers mass, Brown-York mass, Dougan-Mason mass . . .). The construction is quite sophisticated and beside standard relativistic apparatus it requires the tools of advanced functional analysis, topology, cohomology theory etc. For this reason we restricted our attention to the study of the properties of the twistor equation (and related 2-surface twistor equation) which is the key ingredient in the construction. Namely, in Chapter 5 we analyze in detail the integrability conditions for the twistor equation on axially symmetric isolated horizons. Then we find an explicit solution of this equation in terms of free data on the horizon. We find the expression for the Penrose charge integral but we leave aside other subtleties connected with the construction of the Penrose mass. These will be discussed in the context of isolated horizons in future work.

Finally, in the appendix A we provide the equations of Newman-Penrose formalism which is the essential mathematical tool employed in the thesis, including the description of electromagnetic and gravitational field and definitions related to quantities with spin weight.

Conventions

Throughout the thesis we use the following conventions. Small Latin indices $a, b, c \dots$ denote the tensorial *abstract* indices as they were introduced by R. Penrose [1]. On the other hand, concrete tensor components with respect to a coordinate basis are labeled by small Greek letters, e.g. $\mu, \nu, \rho, \dots$. Here and there we use $i, j, k \dots$ to denote spatial indices running through $1, 2, 3$ and I, J with values $1, 2$. We also use the standard summation convention – every pair of (non-abstract) indices, where one is upper and the other one is lower, is implicitly summed. Capital Latin indices $A, B, C, \dots$ or $A', B', C', \dots$ denotes spinorial abstract indices. In this work we deal only with two-spinors and their basis components are always explicitly labeled by $0, 1$.

The tensor quantities are usually denoted by corresponding abstract indices, but occasionally a compact, index-free, notation by the same but bold symbol is used, e.g. $F_{ab} \equiv \mathbf{F}$.

The metric tensor g_{ab} is assumed to have signature $(+---)$. The metric compatible covariant derivative is denoted by ∇_a. For the Riemann tensor we use the convention $[\nabla_a, \nabla_b]X^c = -R^c{}_{dab}X^d$, the Ricci tensor is obtained via contraction $R_{ab} = R^c{}_{acd}$. Then the Einstein field equations of general relativity read

$$R_{ab} - \frac{1}{2}R g_{ab} + \tilde{\Lambda} g_{ab} = -8\pi G\, T_{ab}.$$

2 | Isolated horizons

The motivating ideas behind introducing the concept of (weakly) isolated horizons come mainly from loop quantum gravity. One of the most challenging task of all candidates for quantum theory of gravity is to provide a micropscopic explanation for the Bekenstein-Hawking entropy of black holes which is related to the area of the event horizon. However, because of its teleological nature, the event horizon is a very rigid concept that requires the knowledge of the causal structure of the full spacetime and, hence, it cannot be characterized locally. Similar concept of the apparent horizon is defined quasi-locally but is observer-dependent and therefore not suitable for general considerations in the covariant theory. Isolated horizons have been introduced in order to obtain am invariant, quasi-locally defined horizon accessible for quantization methods of loop quantum gravity. On the other hand, isolated horizons have numerous applications in purely classical general relativity.

In GR, the prototypical solution representing physically realistic black hole is the so-called Kerr-Newman black hole. This solution describes a black hole of mass M, angular momentum $J = a\,M$ (parameter a, which enters the standard form of the metric, is usually referred to as the *spin* parameter) and possibly non-zero charge Q, although astrophysical black holes are typically uncharged. This solution is an excellent approximation of astrophysically relevant black holes and successfully models important physical phenomena, ranging from the description of accreting matter, through the highly energetic jets from active galactic nuclei, to the final states of binary black hole mergers [MHD, Blandford-Znajek,mergers]. Yet, this solution has been derived under several restricting simplifying assumptions which are not fulfilled in all relevant scenarios.

First, the solution is *stationary*, meaning that the spacetime admits a timelike Killing vector. This assumption might be a good approximation for a black hole in equilibrium with its neighborhood for a significant period of the lifetime of a black hole but it is too restrictive for the rest of the spacetime. For example, black holes arise as products of stellar collapse and their creation is usually accompanied by the emission of gravitational radiation which remains present also after the black hole has formed. Hence, the spacetime outside the black hole is not stationary. Second, Kerr solution assumes axial symmetry of the spacetime. Again, such symmetry describes the equilibrium stage of a black hole (otherwise the black hole would emit radiation) itself, but is unreasonably restrictive for the rest of spacetime.

Third, Kerr spacetime is asymptotically flat, i.e. it possesses both timelike and null infinity where the symmetries resembling the Poincaré group of the flat spacetime are recovered (more precisely, the relevant group of symmetries is the BMS group which is a direct sum of Lorentz group and infinite dimensional group of supertranslations).

Quantities like mass or angular momentum are defined as global Geroch-Hansen[] multipoles evaluated at infinity, since in general no quasi-local definition is available. Real black holes, however, are immersed in expanding universe with de Sitter asymptotics and, thus, such definitions are more questionable. Isolated horizons circumvent this problem by introducing Ashtekar's multipole moments that are evaluated on the horizon; such multipoles differ from the Kerr parameters. For the discussion on multipole moments, see section 2.8.

Finally, realistic black holes are usually surrounded by the accretion disks or strong electromagnetic fields. All kinds of matter deform the geometry around the black hole. Usually, it is possible to neglect the backreaction of the matter on the geometry but there are good reasons to study geometries and fields beyond the test field approximation. For example, some recently proposed experiments, e.g. the Event Horizon Telescope [], aim to test the no-hair theorem in astrophysical settings and might be sensitive even to slight deformations of the geometry. Full understanding of such corrections is necessary in order to correctly attribute possible deviations either to purely GR effects or to possible alternative theories of gravity.

2.1 NP tetrad and optical scalars

The starting point of our work is the so-called Newman-Penrose (NP) formalism in general relativity. Since it is not our primary interest, we just briefly sum up all relevant results important for our future work. More technical parts are included in the appendix A. Introduction to NP formalism is provided by several books and articles, see e.g. [2, 1].

Definition 1 *A set of null vectors* $\{l^a, n^a, m^a, \overline{m}^a\}$ *is said to constitute a* Newman-Penrose (NP) tetrad *if*

$$\begin{aligned} &l_a l^a = n_a n^a = m_a m^a = \overline{m}_a \overline{m}^a = 0, \\ &l_a n^a = 1,\ m_a \overline{m}^a = -1,\ \textit{otherwise zero.} \end{aligned} \tag{2.1}$$

Covariant derivatives in the direction of these vectors are conventionally denoted as

$$l^a \nabla_a \equiv D, \quad n^a \nabla_a \equiv \Delta, \quad m^a \nabla_a \equiv \delta, \quad l^a \nabla_a \equiv \overline{\delta}. \tag{2.2}$$

The metric tensor in the NP tetrad has the form

$$g_{ab} = l_a n_b + l_b n_a - m_a \overline{m}_b - \overline{m}_a m_b. \tag{2.3}$$

The NP tetrad allows us to define the orthogonal *Minkowski tetrad* via

$$\begin{aligned} T^a &= \frac{1}{\sqrt{2}}(l^a + n^a), \qquad Z^a = \frac{1}{\sqrt{2}}(l^a - n^a), \\ X^a &= \frac{1}{\sqrt{2}}(m^a + \overline{m}^a), \quad Y^a = \frac{-\mathrm{i}}{\sqrt{2}}(m^a - \overline{m}^a). \end{aligned} \tag{2.4}$$

Straightforward computation using the definition 2.1 yields

$$T_a T^a = 1, \quad Z_a Z^a = X_a X^a = Y_a Y^a = -1. \tag{2.5}$$

The metric has in this basis diagonal form $g = \mathrm{diag}(1, -1, -1, -1)$, which coincides with the metric of the flat space. This is the reason, why (2.4) is called the Minkowski tetrad, even when the spacetime is not flat. Conversely, any orthonormal tetrad which fulfills (2.5) can be used to construct an NP tetrad via

$$\begin{aligned} l^a &= \frac{1}{\sqrt{2}}(T^a + Z^a), \qquad & n^a &= \frac{1}{\sqrt{2}}(T^a - Z^a), \\ m^a &= \frac{1}{\sqrt{2}}(X^a + \mathrm{i}Y^a), \qquad & \overline{m}^a &= \frac{1}{\sqrt{2}}(X^a - \mathrm{i}Y^a). \end{aligned} \tag{2.6}$$

Definition 2 *A projector to the orthogonal complement to vectors l^a, n^a of the NP tetrad is defined by*

$$q_{ab} = -m_a\overline{m}_b - \overline{m}_a m_b. \tag{2.7}$$

An alternating or Levi-Civita tensor on this complement is

$$\varepsilon_{ab} = \mathrm{i}\,(m_a\overline{m}_b - \overline{m}_a m_b) \equiv \mathrm{i}\, m_c \wedge \overline{m}_d. \tag{2.8}$$

Definition 3 *For the congruence of curves with the tangent vector field l^a, we define [3]*

1. *expansion tensor and expansion*

$$\Theta_{\mathbf{l}\,ab} \equiv q_a^c q_b^d \nabla_{(c} l_{d)}, \quad \Theta_{\mathbf{l}} \equiv q^{ab}\nabla_a l_b,$$

2. *shear tensor*

$$\sigma_{ab} \equiv \Theta_{\mathbf{l}\,ab} - \frac{1}{2} q_{ab}\Theta_{\mathbf{l}},$$

3. *twist tensor*

$$\omega_{ab} \equiv q_a^c q_b^d \nabla_{[c} l_{d]},$$

and corresponding scalar quantities usually referred to as optical scalars

$$\Theta_{\mathbf{l}}, \quad \mathfrak{S}^2 \equiv \sigma_{ab}\sigma^{ab}, \quad \omega^2 \equiv \omega_{ab}\omega^{ab}. \tag{2.9}$$

Note: Notice that the optical tensors σ_{ab} and ω_{ab} are by definition trace-free with respect to q^{ab}.

Theorem 1 *The optical scalars* (2.9) *have the following form in terms of spin coefficients (see also [4])*

$$\begin{aligned} \Theta_{\mathbf{l}} &= -(\rho + \overline{\rho}) = 2\mathrm{Re}\,\{\rho\}, \\ \omega &= \frac{1}{\sqrt{2}}(\rho - \overline{\rho}) = \sqrt{2}\mathrm{Im}\,\{\rho\}, \\ \mathfrak{S} &= \sqrt{2}|\sigma|. \end{aligned} \tag{2.10}$$

2.2 Non-expanding horizon

In this section we outline the notion of non-expanding horizon, which provides a starting point to a local generalization of event horizons. The basic idea is to extract the minimal information of a Killing horizon, which can describe black hole in an equilibrium. Here, we refer mainly to articles by Asthekar et al. [5, 6, 7, 8, 9, 4] and very readable summary [10]. In what follows, we rather state precisely formulated definitions supplemented by necessary theorems, which proofs are omitted to keep this section reasonably short and transparent. Here and there a sketch of a proof appears. An interested reader can find details in the cited literature or do all the calculations on his/her own (many of them are just straightforward computations).

Definition 4 *A submanifold $\mathcal{H} \subset \mathcal{M}$ of a spacetime $(\mathcal{M}, g_{ab})$ is said to be a* non-expanding horizon *(NH) if the following conditions are satisfied:*

1. *$\mathcal{H}$ is null hypersurface with topology $\mathbb{R} \times \mathcal{S}^2$,*
2. *every (null) normal l^a of $\mathcal{H}$ has vanishing expansion $\Theta_{\mathbf{l}} \stackrel{\mathcal{H}}{=} 0$,*
3. *Einstein equations are satisfied on $\mathcal{H}$ and energy-momentum tensor T_{ab} is such that for every future normal vector l^a, the vector $T_a^b l^a$ is also future pointing.*

By the symbol $\stackrel{\mathcal{H}}{=}$ we mean the equality of two quantities on the horizon $\mathcal{H}$.

Note: For many calculations, some of these assumptions can be weakened without affecting the main results. For example, the topology can be assumed $\mathbb{R} \times \mathcal{K}$ where $\mathcal{K}$ is a compact 2-dimensional manifold. It is known that the topology of a horizon distorted by the external matter can be, in fact, toroidal [11]. Another possibility is to introduce deficit angles around the poles of otherwise spherical horizon possessing axial symmetry, like in the case of the C-metric [12]. See [13] for more detailed discussion.

According to the definition 4 $\mathcal{H} \sim \mathbb{R} \times \mathcal{S}^2$, so it can be covered by a continuous set of (topological) spheres $\mathcal{S}^2$ parametrized by a real number. Let us choose an arbitrary sphere of this foliation $\mathcal{S}^2_0$. We define a coordinate v on $\mathcal{H}$ by

$$v = 0 \text{ on } \mathcal{S}^2_0, \quad l^a \nabla_a v \stackrel{\mathcal{H}}{=} 1, \tag{2.11}$$

for a particular choice of smooth normal l^a. The solution exists and is unique by the method of characteristics, see e.g. [14]. Or simply, we have defined v in a such way, that the null normal l^a is a coordinate vector on $\mathcal{H}$. On the spherical cut $\mathcal{S}^2_0$ coordinates x^1, x^2 can be chosen arbitrarily (typically spherical coordinates). We propagate them along the generators of the horizon by the condition

$$Dx^1 \stackrel{\mathcal{H}}{=} Dx^2 \stackrel{\mathcal{H}}{=} 0.$$

Since $\mathcal{S}^2_0$ is a topological 2-sphere, we can find smooth orthonormal space-like vector fields[1] $X^a, Y^a \in T\mathcal{S}^2_0 \subset T\mathcal{H}$, which define vectors $m^a, \overline{m}^a$ of an NP tetrad via relations

[1]For instance normalized coordinate vectors in spherical coordinates in a local map on $\mathcal{S}^2_0$.

(2.6). Notice that $l_a X^a = l_a Y^a = 0$, because l_a is a covariant normal. In principle, at every point of $\mathcal{H}$ we can find vectors $l^a, m^a, \overline{m}^a$, but we are not guaranteed smoothness of $m^a, \overline{m}^a$. Therefore, we rather define the vector m^a (and also $\overline{m}^a$) on $\mathcal{H}$ by setting

$$Dm^a \stackrel{\mathcal{H}}{=} 0. \tag{2.12}$$

Since X^a, Y^a are some linear combinations of basis vectors $\boldsymbol{\partial}_1, \boldsymbol{\partial}_2$ corresponding to x^1, x^2 respectively, they are parallel transported to $\mathcal{H}$ as $\boldsymbol{\partial}_1, \boldsymbol{\partial}_2$ are. This means, that m^a is given by (2.6) everywhere on the horizon. The vectors $l^a, m^a, \overline{m}^a$ define by (2.1) uniquely the fourth vector of an NP tetrad n^a at every point of $\mathcal{H}$, which is also smooth. A projector to a tangent space of a two-sphere $\mathcal{S}_v^2$ of the foliation $v = $ const. is given by (2.7).

The framework of the constructed NP tetrad allows us to determine non-trivial conditions on the corresponding spin coefficients, which follow from the fact that l^a is hypersurface orthogonal and null. The essential point in our discussion also will be a restriction that the definition 4 imposes on tetrad components of the Ricci tensor (A.10).

Theorem 2 *On every non-expanding horizon $\mathcal{H}$, the following tetrad components of the Ricci tensor vanish*

$$\Phi_{00} \stackrel{\mathcal{H}}{=} \Phi_{01} \stackrel{\mathcal{H}}{=} 0.$$

In the case of electrovacuum spacetimes in addition

$$\Phi_{02} \stackrel{\mathcal{H}}{=} 0, \quad \Lambda = 0.$$

Note: The proof of this theorem is based on the energy condition 3 in the definition 4. The basic idea might be sketched as

$$\Phi_{00} \equiv -\frac{1}{2} R_{ab} l^a l^b = -\frac{1}{2}(R_{ab} - \frac{1}{2} R g_{ab} + \tilde{\Lambda} g_{ab}) l^a l^b = 4\pi T_{ab} l^a l^b,$$

where $\tilde{\Lambda}$ is the cosmological constant. The energy condition now implies that on the horizon $T_a^b l^a$ is proportional to l^b, because the only future pointing vector there is just l^a. But contraction of l_a with any vector on $\mathcal{H}$ yields automatically zero. The same argument also holds for Φ_{01}. In electrovacuum spacetimes, Einstein equations read $\Phi_{mn} = \phi_m \bar{\phi}_n$, $m, n \in \{1, 2, 3\}$, where ϕ_n are projections of an electromagnetic field tensor (see appendix A.2 and equation (A.20)). Since $0 \stackrel{\mathcal{H}}{=} \Phi_{00} = |\phi_0|^2$ the component ϕ_0 is zero on the horizon and consequently $\Phi_{0n} = 0$ for every n.

Theorem 3 *Let l^a be a null normal of non-expanding horizon $\mathcal{H}$. Integral curves of the vector field l^a are null geodesics and the spin coefficient κ vanishes on the horizon, i.e.*

$$\kappa \stackrel{\mathcal{H}}{=} 0. \tag{2.13}$$

Theorem 4 *Non-expanding horizon $\mathcal{H}$ has zero twist and zero shear. Moreover, the spin coefficients ρ, σ vanish on the horizon, or*

$$\rho \stackrel{\mathcal{H}}{=} \sigma \stackrel{\mathcal{H}}{=} 0. \tag{2.14}$$

As a consequence, the tetrad components of the Weyl tensor Ψ_0, Ψ_1 *are zero on the horizon, i.e.*

$$\Psi_0 \overset{\mathcal{H}}{=} \Psi_1 \overset{\mathcal{H}}{=} 0. \tag{2.15}$$

Moreover, in source free electrovacuum spacetimes

$$D\phi_1 \overset{\mathcal{H}}{=} 0, \quad \Delta\Phi_{00} \overset{\mathcal{H}}{=} 0 \quad \Rightarrow \quad D\Psi_2 \overset{\mathcal{H}}{=} 0. \tag{2.16}$$

Note: These properties of l^a are in fact independent of the rest of the tetrad. Let us keep l^a fixed and transform $m^a, \overline{m}^a$ by a null rotation. From (A.25) we see, that the spin coefficient κ is invariant under this transformation and $\rho, \sigma, \varepsilon$ change only by adding a term proportional to κ. But $\kappa \overset{\mathcal{H}}{=} 0$, so $\rho, \sigma, \varepsilon$ are left unchanged. To show the second statement of the theorem 4 one has to use the Ricci identities (A.13b), (A.13p) and the theorem 2. The last proposition follows from the Maxwell equation (A.19a), the Einstein equations for Φ_{mn}, theorem 2 and the Bianchi identity (A.15b).

Let us now return to the triad $l^a, m^a, \overline{m}^a$ defined only on $T\mathcal{M}|_{\mathcal{S}_0^2}$. In order to define it on the whole $T\mathcal{H}$, we propagated m^a by (2.12). However, as it will turn out soon, more suitable definition of m^a on entire $\mathcal{H}$ is given by Lie dragging. To make it work, theorems 3 and 4 are necessary.

Theorem 5 *Let* $\{l^a, n^a, m^a, \overline{m}^a\}$ *be an NP tetrad defined on a tangent space* $T\mathcal{M}|_{\mathcal{S}_0^2}$ *at points of a spherical section* $\mathcal{S}_0^2$ *of non-expanding horizon* $\mathcal{H}$*. Then the vector field* m^a *propagated to* $T\mathcal{H}$ *by a condition*

$$\pounds_l m^a \overset{\mathcal{H}}{=} 0 \tag{2.17}$$

defines together with the vector field l^a *an NP tetrad at every point of* $\mathcal{H}$*. Namely, the Lie dragging of* m^a *preserves conditions* (2.1).

The relation (2.17) *puts the following restrictions on spin coefficients*

$$\varepsilon - \overline{\varepsilon} \overset{\mathcal{H}}{=} 0, \quad \alpha + \overline{\beta} \overset{\mathcal{H}}{=} \pi. \tag{2.18}$$

From the construction we have

$$l^a \overset{\mathcal{H}}{=} \boldsymbol{\partial}_v^a, \quad , m^a \overset{\mathcal{H}}{=} A\boldsymbol{\partial}_v^a + \xi^I \boldsymbol{\partial}_I^a,$$

where A, ξ^I are some functions of (v, x^I) and $A = 0$ on $\mathcal{S}_0^2$. The directional derivatives are

$$D \equiv l^a \nabla_a \overset{\mathcal{H}}{=} \partial_v, \quad \delta \equiv m^a \nabla_a \overset{\mathcal{H}}{=} A\partial_v + \xi^I \partial_I.$$

Commutator $[D, \delta]$ acting on a scalar function is identically zero on the horizon, because of vanishing of the corresponding spin coefficients (see (A.4a)). On the other hand, its action on v, x^I yields

$$DA \overset{\mathcal{H}}{=} D\xi^I \overset{\mathcal{H}}{=} 0,$$

so A, ξ^I are fully determined by their value on $\mathcal{S}_0^2$, implying $A \overset{\mathcal{H}}{=} 0$ everywhere on the horizon. Hence, we conclude

$$l^a \overset{\mathcal{H}}{=} \boldsymbol{\partial}_v^a, \quad m^a \overset{\mathcal{H}}{=} \xi^I(x^J)\boldsymbol{\partial}_I^a. \tag{2.19}$$

The only non-tangential vector to $\mathcal{H}$ is n^a, which can be used to extend the coordinates into the whole spacetime. From every point of $\mathcal{H}$ we send an affinelly parametrized geodesic in the direction of n^a, or

$$\Delta n^a = 0 \quad \Rightarrow \quad n^a = N\boldsymbol{\partial}_r^a,$$

where r denotes the affine parameter and N is some constant. By a suitable rescaling of r, we may achieve $N = 1, r|_{\mathcal{H}} = r_0$. The parameter r defined in this way will constitute the new coordinate. The remaining coordinates and vectors are propagated analogously:

$$\begin{aligned} \Delta v &= \Delta x^{1,2} = 0, \\ \Delta l^a &= \Delta m^a = 0. \end{aligned} \tag{2.20}$$

The transport equations (A.5) together with (2.20) imply

$$\gamma = \tau = \nu = 0.$$

Vectors l^a, m^a within the horizon are given simply by (2.19), but outside it we have to consider general expansion into the coordinate basis

$$\begin{aligned} l^a &= B\boldsymbol{\partial}_v^a + U\boldsymbol{\partial}_r^a + X^I\boldsymbol{\partial}_I^a, \quad n^a = \boldsymbol{\partial}_r^a, \\ m^a &= C\boldsymbol{\partial}_v^a + \Omega\boldsymbol{\partial}_r^a + \xi^I\boldsymbol{\partial}_I^a, \quad \overline{m}^a = \overline{C}\boldsymbol{\partial}_v^a + \overline{\Omega}\boldsymbol{\partial}_r^a + \overline{\xi}^I\boldsymbol{\partial}_I^a. \end{aligned}$$

At first, it is good to realize that $n_a\boldsymbol{\partial}_I^a = 0$. On the horizon the inverse relation to the second one of (2.19) can be written in a form $\boldsymbol{\partial}_I \overset{\mathcal{H}}{=} \eta_I m^a + \varphi_I \overline{m}^a$ for some functions η_I, φ_I, so $n_a\boldsymbol{\partial}_I^a \overset{\mathcal{H}}{=} 0$ holds trivially. It is also valid everywhere, because the coordinate vectors are parallel transported along n^a. By the same argument we get $n_a\boldsymbol{\partial}_v^a = 1$. Using this results and explicit form of products $n_a l^a, n_a m^a$, we immediately obtain $B = 1, C = 0$. To sum up

$$l^a = \boldsymbol{\partial}_v^a + U\boldsymbol{\partial}_r^a + X^I\boldsymbol{\partial}_I^a, \quad n^a = \boldsymbol{\partial}_r^a, \quad m^a = \Omega\boldsymbol{\partial}_r^a + \xi^I\boldsymbol{\partial}_I^a, \quad \overline{m}^a = \overline{\Omega}\boldsymbol{\partial}_r^a + \overline{\xi}^I\boldsymbol{\partial}_I^a. \tag{2.21}$$

Directional derivatives acting on a scalar function:

$$D = \partial_v + U\partial_r + X^I\partial_I, \quad \Delta = \partial_r, \quad \delta = \Omega\partial_r + \xi^I\partial_I, \quad \overline{\delta} = \overline{\Omega}\partial_r + \overline{\xi}^I\partial_I, \tag{2.22}$$

The tetrad components U, X^I, Ω, ξ^I remain to be determined. The functions ξ^I can be chosen arbitrarily on $\mathcal{H}$ (or more precisely on $\mathcal{S}_0^2$), but $U, X^I, \Omega \overset{\mathcal{H}}{=} 0$. Appropriate equations might be derived from the general commutation relations (A.4) acting subsequently on the coordinates (v, x^1, x^2, r). The non-trivial equations are

$$D\Omega - \delta U = -\kappa + (\bar{\rho} - \bar{\varepsilon} + \varepsilon)\,\Omega + \sigma\,\bar{\Omega}, \tag{2.23a}$$

$$D\xi^I - \delta X^I = (\bar{\rho} - \bar{\varepsilon} + \varepsilon)\,\xi^I + \bar{\sigma}\,\bar{\xi}^I, \tag{2.23b}$$

$$\bar{\delta}\Omega - \delta\bar{\Omega} = \bar{\rho} - \rho + \left(\alpha - \bar{\beta}\right)\Omega - (\bar{\alpha} - \beta)\,\bar{\Omega}, \tag{2.23c}$$

$$\delta\bar{\xi}^I - \bar{\delta}\xi^I = (\bar{\alpha} - \beta)\,\bar{\xi}^I - \left(\alpha - \bar{\beta}\right)\xi^I, \tag{2.23d}$$

$$\Delta U = \varepsilon + \bar{\varepsilon} - \pi\,\Omega - \bar{\pi}\,\bar{\Omega}, \tag{2.23e}$$

$$\Delta X^I = -\pi\,\xi^I - \bar{\pi}\,\bar{\xi}^I, \tag{2.23f}$$

$$\Delta\Omega = \bar{\pi} - \mu\,\Omega - \bar{\lambda}\,\bar{\Omega}, \tag{2.23g}$$

$$\Delta\xi^I = -\mu\,\xi^I - \bar{\lambda}\,\bar{\xi}^I, \tag{2.23h}$$

and will be referred to as the *frame equations.* In addition to these we get

$$\mu = \overline{\mu}, \quad \alpha + \overline{\beta} = \pi. \tag{2.24}$$

The metric can be reconstructed according to (2.3) and it is fully determined by the functions U, X^I, Ω, ξ^I. Therefore these are also called the *metric functions.*

The spin coefficient π is related by (2.24) to α and β, which are independent. However, it seems more natural to work with quantities π and a, where

$$a \equiv m^a \overline{\delta}\,\overline{m}_a = \alpha - \overline{\beta}, \tag{2.25}$$

as these quantities have more direct geometrical meaning: a defines the connection on the sphere and π is related to the angular momentum of the horizon, see (2.36) and section 2.8. For further reference, we write explicitly

$$\alpha = \frac{1}{2}(\pi + a), \quad \beta = \frac{1}{2}(\overline{\pi} - \overline{a}) \tag{2.26}$$

Throughout the text we will use both π, a or α, β keeping these relations in mind.

2.3 Covariant derivative on the non-expanding horizon

We have defined a non-expanding horizon as a null hypersurface, i.e. its normal l^a has vanishing norm. This implies that the metric q_{ab} of the horizon is degenerate, simply because $q_{ab}l^b$ annihilates any tangent vector. So q_{ab} does not have an inverse in the standard sense, but it can be given an inverse q^{ab} in a weaker way $q_{ma}q_{nb}q^{ab} = q_{mn}$. The inverse q^{ab} is not unique, there are infinitely many of them. Consequently, there are infinitely many covariant derivatives[2] compatible with the metric q_{ab}. However, the framework of isolated horizons and vanishing of some spin coefficients allow us to choose a preferred one.

[2] If one wants to define Christoffel symbols (of the 1. kind), one has to employ also the inverse of the metric, which is not unique in our case.

Theorem 6 *Let $\mathcal{H}$ be a non-expanding horizon embedded in a spacetime $(\mathcal{M}, g_{ab})$ equipped with a metric compatible covariant derivative ∇_a. Then a derivative $\mathcal{D}_a$ defined by*

$$X^a \mathcal{D}_a Y^b \stackrel{\mathcal{H}}{=} X^a \nabla_a Y^b, \quad \forall X^a, Y^a \in T\mathcal{H} \tag{2.27}$$

is a covariant derivative on $\mathcal{H}$. Moreover, pull-back of this derivative on $\mathcal{H}$ is compatible with the induced metric q_{ab}.

Theorem 7 *Let $\mathcal{D}_a$ be the induced covariant derivative on $\mathcal{H}$ as in the previous theorem. Then there exists a 1-form $\omega_a \in T\mathcal{M}$ satisfying*

$$\mathcal{D}_a l^b \stackrel{\mathcal{H}}{=} \omega_a l^b, \tag{2.28}$$

where l^a is a null normal to $\mathcal{H}$. The 1-form ω_a is called rotation 1-form *and might be expressed as*

$$\omega_a \stackrel{\mathcal{H}}{=} (\varepsilon + \overline{\varepsilon}) n_a - \pi m_a - \overline{\pi}\, \overline{m_a}$$

Note: We use a conventional notation for the rotation 1-form ω_a, which has nothing to do neither with the twist tensor nor the twist scalar.

Definition 5 *A pair $(q_{ab}, \mathcal{D}_a)$, where q_{ab} and $\mathcal{D}_a$ are induced metric and covariant derivative on a non-expanding horizon $\mathcal{H}$, is called* intrinsic geometry *of $\mathcal{H}$.*

Theorem 8 *Consider a spherical section $\mathcal{S}_0^2$ of a non-expanding horizon $\mathcal{H}$ equipped with a rotation 1-form ω_a. Then its curl on $\mathcal{S}_0^2$ is*

$$\mathcal{D}_{[a} \tilde{\omega}_{b]} = \mathrm{Im}\,\{\Psi_2\} \varepsilon_{ab}, \tag{2.29}$$

where $\tilde{\omega}_a$ denotes pull-back of ω_a onto $\mathcal{S}_0^2$ and ε_{ab} is Levi-Civita tensor on $\mathcal{S}_0^2$.

2.4 Weakly-isolated horizons and the zeroth law of black hole thermodynamics

Definition 6 *An equivalence class $[\cdot]$ of vector fields is defined*

$$[v^a] = \{X^a, \exists \lambda \in \mathbb{R}^+ : X^a = \lambda v^a\}.$$

The weakly isolated horizon (WIH) is a pair $(\mathcal{H}, [l^a])$, where $\mathcal{H}$ is non-expanding horizon and $[l^a]$ is an equivalence class of a chosen null normal l^a of $\mathcal{H}$ such that

$$[\pounds_l, \mathcal{D}_a] l^b \stackrel{\mathcal{H}}{=} 0. \tag{2.30}$$

Note: The parameter λ in the definition 6 is supposed to be a constant, not a space time function. Since λ is defined to be positive, for a future pointing vector v^a every $X^a \in [v^a]$ is also future pointing.

Definition 7 A surface gravity $\varkappa_{(\ell)}$ *associated with the normal* l^a *of a weakly isolated horizon* $(\mathcal{H}, [l^a])$ *is defined by*

$$\varkappa_{(\ell)} \stackrel{\mathcal{H}}{=} n^a D l_a \stackrel{\mathcal{H}}{=} \varepsilon + \bar{\varepsilon} \tag{2.31}$$

We say that the WIH is extremal, *if* $\varkappa_{(\ell)} = 0$, *otherwise it is called* non-extremal.

Note: The surface gravity $\varkappa_{(\ell)}$ is not a particular number for a given WIH. Rescaling l^a in the chosen equivalence class $l'^a = cl^a$, $c \in \mathbb{R}$ yields $\varkappa'_{(\ell)} = c\varkappa_{(\ell)}$. However, the extremal horizon $\varkappa_{(\ell)} = 0$ *is* defined uniquely.

Theorem 9 (The zeroth law of black hole thermodynamics) *The surface gravity* $\varkappa_{(\ell)}$ *associated with a weakly isolated horizon* $(\mathcal{H}, [l^a])$ *is constant on* $\mathcal{H}$. *Moreover, rotational 1-form* ω_a *is time independent in the sense that* $\pounds_l \omega_a \stackrel{\mathcal{H}}{=} 0$ *and following relations for spin coefficients are satisfied:*

$$D\varepsilon \stackrel{\mathcal{H}}{=} D\pi \stackrel{\mathcal{H}}{=} D\alpha \stackrel{\mathcal{H}}{=} D\beta \stackrel{\mathcal{H}}{=} 0, \quad \delta\varepsilon \stackrel{\mathcal{H}}{=} 0. \tag{2.32}$$

2.5 Isolated horizons and free data

A stronger notion of isolation is in many physical applications more natural and plausible [15]. Here we also touch problematics what the free data are that have to be given in order to reconstruct horizon geometry completely.

Definition 8 *An isolated horizon (IH) is a weakly isolated horizon* $(\mathcal{H}, [l^a])$*, such that*

$$[\pounds_l, \mathcal{D}_a] X^b \stackrel{\mathcal{H}}{=} 0 \tag{2.33}$$

for any tangential vector field X^a.

Notes: • Every non-expanding horizon $\mathcal{H}$ can be given a (WIH) structure with a suitable choice of an equivalence class of normal vectors $[l^a]$ and the condition (2.30) is therefore only a gauge fixing. However, time independence of the horizon connection $\mathcal{D}_a$, as the requirement (2.33) might be interpreted, is non-trivial restriction on spacetime geometry. It translates to time independence of spin coefficients λ, μ.

• Imposing merely time independence of μ leads to the notion of an *almost isolated horizon* as introduced in [13].

Theorem 10 *Spin coefficients* λ, μ *are time independent in the sense*

$$D\lambda \stackrel{\mathcal{H}}{=} D\mu \stackrel{\mathcal{H}}{=} 0$$

on every isolated horizon.

Let us pay some attention to free data on an isolated horizon. We have shown that actually all the spin coefficients are independent of v on an IH. Hence, they are determined on $\mathcal{H}$ by their values on a particular slice $\mathcal{S}_v$, say $\mathcal{S}_0$. The same has been already concluded about functions ξ^I that can be chosen suitably. However, not all the remaining non-zero spin coefficients can be specified freely (see [9] or [5] for discussion).

The function π on $\mathcal{H}$ is governed by the Ricci identity (A.13g)

$$\overline{\delta}\pi \overset{\mathcal{H}}{=} 2\varepsilon\lambda - \pi^2 - a\pi \tag{2.34}$$

The Ricci identity (A.13q) determines Ψ_2. In terms of π and a it is recast as

$$\delta\pi - \overline{\delta}\overline{\pi} + \delta a + \overline{\delta}\overline{a} \overset{\mathcal{H}}{=} 2a\overline{a} + \pi\overline{a} - \overline{\pi}a - 2\Psi_2 + \Phi_{11}, \tag{2.35}$$

which, taking the real and imaginary part separately, yields

$$\begin{aligned} \mathrm{Re}\,\{\Psi_2\} &\overset{\mathcal{H}}{=} |a|^2 - \frac{1}{2}(\delta a + \overline{\delta}\overline{a}) + \Phi_{11} \\ \mathrm{Im}\,\{\Psi_2\} &\overset{\mathcal{H}}{=} -\mathrm{Im}\,\{\delta\pi - \overline{a}\pi\} \equiv -\mathrm{Im}\,\{\eth\pi\}, \end{aligned} \tag{2.36}$$

where $\eth$ is defined by (A.22). On the other hand, π and Ψ_2 enter the equation (A.13h) which, for isolated horizon, reads

$$\delta\pi = 2\varepsilon\mu - \pi\overline{\pi} + a\pi - \Psi_2. \tag{2.37}$$

Algebraically, this can be regarded as an equation for Ψ_2 but, at the same time, Ψ_2 is constrained by relations (2.36). Hence, (2.37) is an equation where the only unknown quantity is the metric on the sphere $\mathcal{S}_0^2$, provided that the spin coefficient μ is freely specified. This is reasonable, since μ is a part of the extrinsic curvature of the horizon. In the special case of extremal horizon, μ drops out of the equation (2.37) which is at the core of our uniqueness theorem for axisymmetric extremal horizons to be discussed in the chapter 3.

Free data entering the game are summarized in the following theorem.

Theorem 11 (Free data on an IH) *Let $\mathcal{H}$ be an isolated horizon, $\mathcal{N}$ a null hypersurface such that intersection of the two is a spherical slice $\mathcal{S}_v^2$ of constant coordinate v of a foliation of $\mathcal{H}$. Then functions*

$$\varepsilon,\ \lambda,\ \mu,$$

may be specified freely on $\mathcal{S}_v$ and this specification makes them uniquely determined on the entire horizon $\mathcal{H}$. The Weyl scalar Ψ_4 has to be prescribed on the hypersurface $\mathcal{N}$.

The electromagnetic field in electrovacuum spacetimes is determined by values of ϕ_1 on $\mathcal{S}_v^2$ and ϕ_2 on $\mathcal{N}$.

Note: Here we omit detailed analysis of the characteristic initial value problem, e.g. the (in)dependence of various constraints. These details, including the existence results, can be found in [16].

In practical applications, an isolated horizon is often assumed to possess some symmetries. Before going into details, we state a definition of this notion.

Definition 9 *A map $\mathfrak{F} : \mathcal{H} \mapsto \mathcal{H}$, which is diffeomorphism of an isolated horizon $\mathcal{H}$ onto itself and preserves its intrinsic geometry $(q_{ab}, \mathcal{D}_a)$ is called a* symmetry of $\mathcal{H}$.

Time independence of quantities on an IH implies that diffeomorphisms generated by the field l^a are symmetries. So a symmetry group of the IH is at least one dimensional. No other symmetries are present in general, therefore we recognize several classes of IHs [8, 17]:

Type I: geometry of the IH is spherical, symmetry group is four-dimensional,

Type II: geometry of the IH is axially symmetric, symmetry group is two-dimensional,

Type III: symmetries generated by l^a are the only symmetries, symmetry group is one-dimensional.

Later in the thesis, we restrict ourselves to the type II isolated horizons.

Note: Notice that the symmetry as it was presented here relates only to $\mathcal{H}$. The outer geometry can be affected by matter fields, radiation, etc. and is not required to share the same symmetry properties.

It has already been mentioned that an isolated horizon is a generalization of the notion of a Killing horizon. Now, we make this statement more clear.

Theorem 12 *Every Killing horizon $\mathcal{K} \equiv \{P \in \mathcal{M},\ \mathbf{g}(\mathbf{k},\mathbf{k})|_P = 0\}$ of a Killing vector $\mathbf{k}$, which has topology $\mathbb{R} \times \mathcal{S}^2$ is also an isolated horizon, provided the weak energy condition is satisfied.*

Proof. *To prove that the Killing horizon $\mathcal{K}$ is a non-expanding horizon it suffices to show that it has zero expansion, since the other two requirements of the definition 4 are assumed to be satisfied. A straightforward calculation yields*

$$\Theta_{\mathbf{k}\,ab} = q_a^c q_b^d \nabla_{(c} k_{b)} = 0,$$

because of the Killing equation $\nabla_{(c} k_{b)} = 0$. The expansion and the shear tensor are therefore automatically zero.

The defining requirement of a weakly isolated horizon (2.30) *is in fact equivalent to the zeroth law of black hole termodynamics. Thus, we want to show that the surface gravity associated with k^a is constant on $\mathcal{K}$. The vector k^a is null on $\mathcal{K}$, therefore geodesic and can serve as a preferred normal. The surface gravity is defined by*

$$k^a \nabla_a k^b = \varkappa_{\mathbf{k}} k^b. \tag{2.38}$$

Differentiation of this equation with respect to $k^c \nabla_c$ gives

$$\begin{aligned} k^c \nabla_c k^a\, \nabla_a k^b + k^c k^a\, \nabla_c \nabla_a k^b &= k^c k^b\, \nabla_c \varkappa_k + \varkappa_k\, k^c \nabla_c k^b, \\ k^c k^a \nabla_c \nabla_a k_b &= k_b k^c \nabla_c \varkappa_{\mathbf{k}}, \end{aligned} \tag{2.39}$$

where we used (2.38)*. Furthermore, the Killing field is a non-trivial restriction on the spacetime geometry, hence it has to satisfy a certain integrability conditions. These can be derived easily. The second exterior derivative of k_a has to vanish identically*

$$\nabla_{[c} \nabla_a k_{b]} \equiv 0. \tag{2.40}$$

Using the Killing equation and the definition of the Riemann tensor in (2.40) *one derives*

$$\nabla_{[c}\nabla_a k_{b]} = R_{abcd}k^d.$$

Substitution in (2.39) *immediately yields*

$$k^c\nabla_c \varkappa_{\mathbf{k}} = 0,$$

or in other words, the surface gravity $\varkappa_{\mathbf{k}}$ *is constant along* k^a*.*

To show that $\mathcal{K}$ *is an isolated horizon, one has to project the Killing equation and the integrability conditions* (2.40) *onto the NP tetrad. The resulting restrictions on the spin coefficients are precisely conditions stated in the theorem 10, which are equivalent to the definition of the isolated horizon.* ■

2.6 Axi-symmetric structures

In this section we deal with metric manifolds of two sphere topology that obey axial symmetry. We offer a definition of this notion and construct an appropriate coordinate system following mainly [17]. For an application and further reference see also [13]. In what follows, we assume a metric to have signature (++) just to keep things throughout the construction without unnecessary minuses. The opposite sign convention will be discussed at the end of the section.

Let us consider a manifold $\mathcal{S}$ of topology of a two sphere equipped with a metric q_{ab}. Then there exists the unique metric area form

$$\varepsilon_{\mu\nu} = \sqrt{|\det(q_{\rho\sigma})|}\epsilon_{\mu\nu}, \tag{2.41}$$

where $\epsilon_{\mu\nu}$ denotes a permutation symbol[3]. It allows us to define the area of $\mathcal{S}$ as

$$A = \oint_{\mathcal{S}} \varepsilon \tag{2.42}$$

and its "radius" R by the relation $A = 4\pi R^2$.

Definition 10 *A manifold* $(\mathcal{S}, q_{ab})$ *is said to be axi-symmetric if there exists a Killing field* ϕ^a *with closed orbits such that* $\phi^a = 0$ *exactly at two points of* $\mathcal{S}$*. These points will be referred to as* poles*.*

Note: Since ϕ^a is a Killing field, its orbits $\langle\phi^a\rangle$ can not intersect each other (otherwise it would violate the requirement that the orbits are closed) and they foliate the manifold $\mathcal{S}$.

The field ϕ^a is by definition a Killing field, so we immediately have $\pounds_\phi q_{ab} = 0$ and also $\pounds_\phi \varepsilon_{ab} = 0$, which allows us to introduce a coordinate in a specific way.

Theorem 13 *On* $\mathcal{S}$ *there exists a function* ζ *satisfying*

$$\mathcal{D}_a\zeta = \frac{1}{R^2}\varepsilon_{ba}\phi^b, \quad \oint_{\mathcal{S}} \zeta\varepsilon = 0. \tag{2.43}$$

[3] We use convention, where $\epsilon_{12} = \epsilon^{12} = 1$, cf. also (2.8).

Moreover this function is unique and $\zeta = \text{const.}$ *on integral curves of* ϕ^a*.*

Proof. *Let us contract the first equation by* ϕ^a*. The right hand side is zero, because* ε_{ab} *is contracted with a symmetric expression. Since the covariant derivative* $\mathcal{D}_a$ *acts as a partial one on every scalar function, we are left with*

$$\phi^a \partial_a \zeta = 0$$

in a local coordinate map. This equation has a unique solution up to a constant by the method of characteristics (see [14] and discussion after the equation (2.11)*). Characteristics coincide with integral curves of* ϕ^a*.*

The second condition of (2.43) *fixes the freedom contained in the integration constant and makes* ζ *unique.* ■

From the proof we can infer the meaning of the two requirements (2.43). The first one tells us that there is a function which is constant on integral curves of the field ϕ^a, the second one just fixes the integration constant.

The fact $\zeta = \text{const.}$ on $\langle\phi\rangle$ can also be expressed as $\pounds_\phi \zeta = 0$. Since $\mathcal{D}_a\zeta = 0$ only at the poles, the function ζ has to be monotonic on the foliation of $\mathcal{S}$. The pole where ζ has the maximum will be called the *north* pole P_N, the other one the *south* pole P_S.

Further we define a vector field on $\mathcal{S} - \{P_N, P_S\}$

$$\zeta^a = \frac{R^4}{\Phi^2} q^{ab} \mathcal{D}_b \zeta, \quad \Phi^2 \equiv q_{ab}\phi^a\phi^b. \tag{2.44}$$

Theorem 14 *The vector field* ζ^a *defined above satisfies*

$$\zeta^a \mathcal{D}_a \zeta = 1, \quad \zeta_a \phi^a = 0. \tag{2.45}$$

Proof. *A straightforward calculation where one substitutes the defining condition* (2.44) *into each of the two formulae* (2.45) *while using* (2.43) *leads to desired results.* ■

Conversely, the properties (2.45) can be used to define the vector field ζ^a itself. From (2.43) we see that the vector $\mathcal{D}^a\zeta \equiv q^{ab}\mathcal{D}_b\zeta$ is not proportional to ϕ^a. Therefore these vectors can be taken as a basis that ζ^a can be expanded into:

$$\zeta^a = X\phi^a + Y\mathcal{D}^a\zeta.$$

Functions X, Y are then determined from (2.45). Direct substitution yields

$$X = 0, \quad Y = \frac{R^4}{\Phi^2}.$$

Since ζ^a is perpendicular to the field ϕ^a and $\zeta^a\mathcal{D}_a\zeta = 1$, integral curves of ζ^a run from the south pole to the north pole.

We can always find a parameter λ of the vector field ϕ^a such that $\phi^a = (\boldsymbol{\partial}/\boldsymbol{\partial}\lambda)^a$, however the vector field ζ^a allows us to define a preferred parameter ϕ of ϕ^a in the

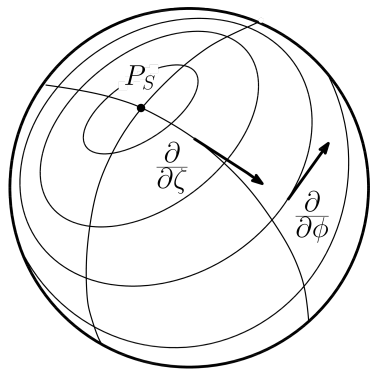

Figure 2.1: Coordinate vector fields on a topological two-sphere $\mathcal{S}$.

following way[4]. Let us choose an integral curve $\langle\zeta^a\rangle_0$ and set $\phi = 0$ on it. So $\pounds_\zeta\phi = 0$ on $\langle\zeta^a\rangle_0$. Now, we define ϕ on $\mathcal{S}$ by condition

$$\pounds_\phi\phi = 1. \tag{2.46}$$

The field ζ^a is constructed only of q_{ab} and ϕ^a, so $[\phi,\zeta]^a = \pounds_\phi\zeta^a = 0$ or the commutator $[\pounds_\phi, \pounds_\zeta]h = \pounds_{[\phi,\zeta]}h = 0$ for all test functions h. Taking ϕ as the test function, we find

$$\pounds_\phi(\pounds_\zeta\phi) = \pounds_\zeta(\pounds_\phi\phi) = \pounds_\zeta 1 \equiv 0 \quad \Rightarrow \quad \pounds_\zeta\phi = g|_{\langle\phi\rangle} = g(\zeta),$$

where the function g can depend on a particular orbit of ϕ^a, but $\zeta = \text{const.}$ on $\langle\phi^a\rangle$ so it is equivalent to dependence (only) on ζ. However, we have chosen $g(\zeta)|_{\langle\zeta^a\rangle_0} = 0$ what implies $\pounds_\zeta\phi = 0$ at every point of $\mathcal{S}$, in other words ϕ is constant on the orbits of ζ^a. It is not difficult to realize that due to this equality the range of ϕ is the same on every orbit of ϕ^a, so $\phi \in [0, \phi_{\max})$.

Functions ζ, ϕ can be regarded as independent coordinates, which are, at least in some sense, preferred ones on $\mathcal{S}$. Equations $\pounds_\zeta\phi = 0, \pounds_\phi\zeta = 0$ with defining conditions (2.46) and (2.43) tell us that ϕ^a, ζ^a are coordinate vector fields, i.e.

$$\phi^a = \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\phi}\right)^a, \quad \zeta^a = \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\zeta}\right)^a. \tag{2.47}$$

Moreover, the coordinates are orthogonal, because the vectors are orthogonal by construction. The property that ϕ^a is a Killing vector is independent of rescaling ϕ^a by an arbitrary constant. We can choose a new coordinate $\phi' = (2\pi/\phi_{\max})\phi$, which lies in the interval $\phi' \in [0, 2\pi)$. Then $\phi^a = \boldsymbol{\partial}^a_\phi = (2\pi/\phi_{\max})\boldsymbol{\partial}^a_{\phi'} \equiv (2\pi/\phi_{\max})\phi'^a$. Furthermore, we can choose ϕ'^a as the Killing field and repeat the whole construction, which will fix the range of its parameter to $[0, 2\pi)$. Hence, we can assume $\phi \in [0, 2\pi)$ without loss of generality.

A covector basis can consist of ϕ_a, ζ_a, but we choose rather $\mathcal{D}_a\zeta, \mathcal{D}_a\phi$, because these constitute a dual coordinate basis. The first relation of (2.43) defines $\mathcal{D}_a\zeta$, while $\mathcal{D}_a\phi$ can be expanded as

$$\mathcal{D}_a\phi = U\phi_a + V\zeta_a, \quad U, V \in \mathbb{R}.$$

[4]It is good to have the picture 2.1 in mind.

Transvecting with ζ^a and using $\mathcal{L}_\zeta\phi = 0$ we get $V = 0$, while contracting with ϕ^a yields $U = 1/\Phi^2$. Hence

$$\mathcal{D}_a\phi = \frac{1}{\Phi^2}\phi_a.$$

Now, we have all ingredients to express the metric in this basis. Since coordinates are orthogonal, the metric is diagonal and only two its components have to be calculated. By the same procedure, one derives

$$q_{\phi\phi} = q_{ab}\phi^a\phi^b = \Phi^2, \quad q_{\zeta\zeta} = \frac{R^4}{\Phi^2}.$$

Further, let us denote $f = \Phi^2/R^2$. Finally

$$q_{ab} = R^2\left(f\mathcal{D}_a\phi\,\mathcal{D}_b\phi + \frac{1}{f}\mathcal{D}_a\zeta\,\mathcal{D}_b\zeta\right), \quad q^{ab} = \frac{1}{R^2}\left(\frac{1}{f}\phi^a\phi^b + f\zeta^a\zeta^a\right). \tag{2.48}$$

The metrics depends only on a single function f. Notice that $\partial_\phi f = 0$, as can be inferred from

$$\frac{\partial}{\partial\phi}\Phi^2 = \phi^a\mathcal{D}_a\Phi^2 = 2\phi^a\phi^b\mathcal{D}_a\phi_b \equiv 0.$$

The last equality is valid, because ϕ^a satisfies the Killing equation $\mathcal{D}_{(a}\phi_{b)} = 0$.

Our definition of area and radius of $\mathcal{S}$ imposes a non-trivial condition on the coordinate ζ.

Lemma 1 *The definition of radius R fixes the range of the coordinate ζ to $[-1, 1]$.*

Proof. *Direct computation of area using* $\det(q_{ab}) = R^2$ *in* (2.41) *gives*

$$A = \oint_\mathcal{S}\varepsilon = R^2\int_0^{2\pi}\int_{\zeta(P_S)}^{\zeta(P_N)} \mathrm{d}\phi\,\mathrm{d}\zeta = 2\pi R^2(\zeta(P_N) - \zeta(P_S)) \stackrel{!}{=} 4\pi R^2 \quad \Rightarrow \quad \zeta(P_N) - \zeta(P_S) = 2.$$

On the other hand, the defining relation (2.43) *yields*

$$\begin{aligned} 0 &= \oint_\mathcal{S}\zeta\varepsilon = \pi R^2(\zeta^2(P_N) - \zeta^2(P_S)) = 2\pi R^2(\zeta(P_N) + \zeta(P_S)), \\ 0 &= \zeta(P_N) + \zeta(P_S) \end{aligned}$$

or $\zeta(P_N) = 1$ *and* $\zeta(P_S) = -1$ *as was to be shown.* ■

The range of coordinates is yet definitely established on $(\zeta, \phi) \in [-1, 1] \times [0, 2\pi)$. Similarly to the case of standard spherical coordinates, ϕ has the meaning of azimuthal angle and it has a discontinuity of 2π at $\langle\zeta^a\rangle_0$. This discontinuity causes no problems as the vector field ϕ^a is smooth and thus $\lim_{\phi\to0+} f = \lim_{\phi\to0-} f$. However at the poles $f = 0$, so a divergence occurs in the metric coefficients and we have to impose appropriate regularity conditions on the behavior of the metric there. Following [17], we assume that the spheres are elementary flat, i.e. no conical singularities are present. By definition it means

$$\lim_{\zeta_c\to+1/-1}\frac{O(\zeta_c)}{r_{N/S}(\zeta_c)} = 2\pi,$$

where O stands for the circumference of a circle at $\zeta = \zeta_c$, $r_{N/S}$ for its radius[5] at north and south pole, respectively:

$$O(\zeta_c) \equiv \int_0^{2\pi} \sqrt{q_{\phi\phi}}|_{\zeta_c} \mathrm{d}\phi = 2\pi R\sqrt{f(\zeta_c)}, \quad r_{N/S}(\zeta_c) \equiv \int_{\zeta_c/-1}^{1/\zeta_c} \sqrt{q_{\zeta\zeta}}\mathrm{d}\zeta = R\int_{\zeta_c/-1}^{1/\zeta_c} \frac{\mathrm{d}\zeta}{\sqrt{f(\zeta)}}.$$

The limit can be explicitly calculated by l'Hospital rule. For the north pole

$$\lim_{\zeta_c\to+1} \frac{O(\zeta_c)}{r_N(\zeta_c)} = \lim_{\zeta_c\to+1} \frac{2\pi R\sqrt{f(\zeta_c)}}{-R\int_1^{\zeta_c}\sqrt{1/f(\zeta)}\mathrm{d}\zeta} = -\pi \lim_{\zeta_c\to+1} f'(\zeta_c) \quad \Rightarrow \quad \lim_{\zeta_c\to+1} f'(\zeta_c) \overset{!}{=} -2.$$

Analogously for the south pole. The absence of conical singularities requires f to satisfy

$$\lim_{\zeta_c\to\pm1} f'(\zeta_c) = \mp 2. \tag{2.49}$$

Example: To get a better idea about introduced notions, we offer an illustration for a 2-sphere with a deficit angle. The line element in spherical coordinates (θ, φ) is

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + (1-\varphi_d)^2 \sin^2\theta \mathrm{d}\varphi^2,$$

where φ_d denotes the deficit angle. The Killing field is clearly $\phi^a = (\boldsymbol{\partial}/\boldsymbol{\partial}\varphi)^a$ with norm $\Phi^2 = (1-\varphi_d)^2\sin^2\theta$. Circumference O of the sphere at given θ_0 and distance ρ from the north pole to this circle is

$$O = (1-\varphi_d)\sin\theta_0 \int_0^{2\pi} \mathrm{d}\varphi = 2\pi(1-\varphi_d)\sin\theta_0, \quad \rho = \int_0^{\theta_0} \mathrm{d}\theta = \theta_0.$$

Approaching the north pole, the ratio O/ρ reveals the conical singularity

$$\lim_{\theta_0\to0} \frac{O}{\rho} = 2\pi(1-\varphi_d).$$

The coordinate ζ is obtained from the definition (2.43)

$$\frac{\partial\zeta}{\partial\varphi} = 0, \quad \frac{\partial\zeta}{\partial\theta} = -\frac{(1-\varphi_d)}{R^2}\sin\theta,$$

where we used (2.41) and $\det|q| = (1-\varphi_d)^2\sin^2\theta$ therein. The solution clearly is $\zeta = (1-\varphi_d)\cos\theta/R^2 + C$. The second requirement of (2.43) was prescribed to fix the integration constant

$$0 = \int \mathrm{d}\varphi \int \mathrm{d}\theta\, \zeta \sin\theta = 4\pi C \quad \Rightarrow \quad C = 0.$$

[5] More precisely, $r_{N/S}$ is the length of the arc between the pole and the point with coordinate ζ, while the Euclidean radius would be the distance of that point from the axis of symmetry. Such radius, though, cannot be defined using the intrinsic properties of the sphere, i.e. without considering the embedding of the sphere into a hypersurface. However, in the limit $\zeta \to \pm1$ these two notions of radius differ only by higher order terms and drop out from the limit.

The north pole corresponds to $\theta = 0$ and the south pole to $\theta = \pi$, where indeed $\Phi^2 = 0$. Moreover, these are the only points where the norm of the Killing field vanishes. A trivial calculation of the area (2.42) leads to $A = 4\pi(1-\varphi_d) \Rightarrow R = \sqrt{1-\varphi_d}$, so $\zeta = \cos\theta$. The function f which metrics depends on is therefore $f = (1-\varphi_d)\sin^2\theta = (1-\varphi_d)(1-\zeta^2) \Rightarrow f'(\zeta) = -2(1-\varphi_d)\zeta$. The condition (2.49) is satisfied only if $\varphi_d = 0$ as we wanted to demonstrate.

The covariant derivative together with the metrics define the Christoffel symbols and consequently the Riemann tensor, which in the case of two dimensional manifold has only one independent component. This can be equivalently expressed via scalar curvature, from which the geometry of $\mathcal{S}$ can be reconstructed, as it is summarized in the following theorem.

Theorem 15 *Geometry of the manifold $(\mathcal{S}, q_{ab})$ is fully determined by its area A and its scalar curvature $\mathcal{R}$.*

Proof. *We have already mentioned that the metrics q_{ab} depends only on the single function f and one constant R. The constant R is of course defined by the area of $\mathcal{S}$, so we are only supposed to find f from $\mathcal{R}$. The derivation of scalar curvature proceeds by finding the Christoffel symbols ${\Gamma^a}_{bc}$, then the Ricci tensor $\mathcal{R}_{ab}$, which is contracted to the scalar curvature $\mathcal{R} = {\mathcal{R}^a}_a$. It is straightforward but somewhat tedious, so we just summarize intermediate results:*

$$\begin{aligned}
&{\Gamma^\phi}_{\phi\phi} = {\Gamma^\phi}_{\zeta\zeta} = {\Gamma^\zeta}_{\phi\zeta} = 0, \quad {\Gamma^\phi}_{\phi\zeta} = -{\Gamma^\zeta}_{\zeta\zeta} = \frac{f'}{2f}, \quad {\Gamma^\zeta}_{\phi\phi} = -\frac{1}{2}ff', \\
&\mathcal{R}_{\phi\phi} = \frac{1}{2}ff'', \quad \mathcal{R}_{\zeta\zeta} = \frac{f''}{2f}, \quad \mathcal{R}_{\zeta\phi} = 0, \\
&\mathcal{R} = \frac{f''}{R^2}.
\end{aligned} \tag{2.50}$$

Integrating the last expression twice and employing the boundary conditions for f, i.e. $f(-1) = 0$ and $f'(-1) = 2$, one finds

$$f(\zeta) = R^2 \int_{-1}^{\zeta} \mathrm{d}\zeta_1 \int_{-1}^{\zeta_1} \mathcal{R}(\zeta_2)\mathrm{d}\zeta_2 + 2(\zeta+1).$$

■

The coordinates ϕ, ζ allow us to define a "canonical" two-sphere metric. As we have seen in the example, the function f is given on a two-sphere of unit radius by $f_{\mathcal{S}} = 1 - \zeta$. In that spirit, we define

$$q^{\mathcal{S}}_{ab} = -R^2 \left(f_{\mathcal{S}} \mathcal{D}_a\phi\, \mathcal{D}_b\phi + \frac{1}{f_{\mathcal{S}}} \mathcal{D}_a\zeta\, \mathcal{D}_b\zeta \right). \tag{2.51}$$

Let us add a final remark. Throughout this section we have been using convention for the metric signature $(++)$. It was not of much importance, because we dealt only with two dimensional manifolds. If we want to immerse such a manifold into the spacetime with the metric sign convention $(+---)$, the metric (2.48) has to be taken with the opposite sign (what equivalently means $f \mapsto -f$, since f is required positive)

$$q_{ab} = -R^2 \left(f \mathcal{D}_a\phi\, \mathcal{D}_b\phi + \frac{1}{f} \mathcal{D}_a\zeta\, \mathcal{D}_b\zeta \right). \tag{2.52}$$

2.7 Axially symmetric isolated horizons

The formalism of the previous section allows us to describe isolated horizons exhibiting axial symmetry. Moreover, we have a manual how to introduce adapted coordinates in a geometrical way. Taking into account our sign convention for the spacetime metric, the appropriate intrinsic metric of a two sphere is given by (2.52).

The vectors l^a, n^a of an NP tetrad remain untouched by our assumption of axial symmetry, while $m^a, \overline{m}^a$ that are tangent to spherical cuts of the IH can be defined by the coordinate vectors (2.47) via (2.6). Namely,

$$m^a \overset{\mathcal{H}}{=} \frac{1}{R}\sqrt{\frac{f(\zeta)}{2}}\boldsymbol{\partial}^a_\zeta + \frac{\mathrm{i}}{R\sqrt{2f(\zeta)}}\boldsymbol{\partial}^a_\phi, \quad \overline{m}^a \overset{\mathcal{H}}{=} \frac{1}{R}\sqrt{\frac{f(\zeta)}{2}}\boldsymbol{\partial}^a_\zeta - \frac{\mathrm{i}}{R\sqrt{2f(\zeta)}}\boldsymbol{\partial}^a_\phi, \tag{2.53}$$

where we have chosen $X^a = R^{-1}\sqrt{f}\,\boldsymbol{\partial}^a_\zeta$, $Y^a = (R\sqrt{f})^{-1}\boldsymbol{\partial}^a_\phi$ to ensure correct normalization[6]. In agreement with (2.19) we denote

$$\xi \equiv \xi^\zeta \overset{\mathcal{H}}{=} \frac{1}{R}\sqrt{\frac{f(\zeta)}{2}} \tag{2.54}$$

for brevity. Derivative operators (2.22) are therefore reduced to

$$D \overset{\mathcal{H}}{=} \partial_v \quad \Delta \overset{\mathcal{H}}{=} \partial_r, \quad \delta \overset{\mathcal{H}}{=} \overline{\delta} \overset{\mathcal{H}}{=} \xi\partial_\zeta. \tag{2.55}$$

Covariant form of m^a is

$$m_a = g_{ab}m^b \overset{\mathcal{H}}{=} q_{ab}m^b = -\frac{R}{\sqrt{2f}}\mathcal{D}_a\phi - \mathrm{i}R\sqrt{\frac{f}{2}}\mathcal{D}_a\zeta$$

From $\delta - \overline{\delta}$ and equation (2.23d) it is evident that the spin coefficient a is real on the horizon. Actually, it is fully characterized by the geometry of the two-sphere $\mathcal{S}_0$. By direct computation[7]

$$a \equiv m_a\overline{\delta}\overline{m}^a \overset{\mathcal{H}}{=} m^\mu\overline{m}^\nu\partial_\nu\overline{m_\mu} + m^\mu\overline{m}^\nu\overline{m_\rho}\Gamma^\rho_{\mu\nu} \overset{\mathcal{H}}{=} -\frac{1}{2\sqrt{2}R}\frac{f'}{\sqrt{f}} \tag{2.56}$$

Notice $\xi' \equiv \partial_\zeta\xi = -a$. This relation will be useful in the future.

Note: One remark is in place here. Connection components are, in our tetrad language, coefficients in the expansion of directional derivatives of the NP basis vectors $l^a, n^a, m^a, \overline{m}^a$ (see also [2], page 26.). On the sphere $\mathcal{S}_0^2$ the relevant ones are $\delta m^a, \delta\overline{m}^a$. Expanding them into basis, we find

$$\begin{aligned} \delta m^a &= Am^a + B\overline{m}^a \overset{\mathcal{H}}{=} -(\overline{m}_b\delta m^b)\, m^a, \\ \delta\overline{m}^a &= Cm^a + D\overline{m}^a \overset{\mathcal{H}}{=} -(m_b\delta\overline{m}^b)\, m^a = (\overline{m}_b\delta m^b)\, m^a. \end{aligned}$$

[6] We need orthonormal spacelike vectors in (2.6), so $\boldsymbol{\partial}_\zeta, \boldsymbol{\partial}_\phi$ have to be at first normalized to -1.

[7] The first term yields zero, for the second one, the Christoffel symbols can be taken from the proof of the theorem 15.

We immediately see that there is just one independent component, which is precisely the spin coefficient a. It is a consequence of the third equation of (2.55). Hence, the full information about the connection on axially symmetric horizon is encoded in a.

Quantity a is also closely related to the spin-raising operator $\eth$ (and its spin-lowering counterpart $\bar{\eth}$) defined by (A.22), (A.23). These operators act on functions with well-defined spin weight which are regarded as sections of appropriate bundle over the base manifold and define a connection on such bundle. In this sense, $\eth$ is a covariant derivative which is also fully determined by the spin coefficient a which, therefore, gives rise to two different connections over the base manifold $\mathcal{S}^2$.

2.8 Multipole moments of isolated horizons

Isolated horizons that possess a Killing vector field representing axial symmetry can be naturally characterized by a set of numbers, which can be given standard interpretation of mass and angular momentum multipoles. In this section we introduce this description following mainly [17].

At first, we restrict ourselves to isolated horizons without any matter fields. We choose a spherical cut $\mathcal{S}_0$ of a such horizon $\mathcal{H}$. It can be thought as a leaf of foliation $\mathcal{S}_v$ as it has been introduced in the section 2.2. Namely, we choose a null normal l^a defined by the relations (2.11). Projection operator onto leaves of this foliation is obviously $q^a_b = \delta^a_b - l^a n_b$. Geometry of $\mathcal{S}_0$ is fully characterized by the induced metric $\tilde{q}_{ab}$ and (pull-back of) the rotation 1-form $\tilde{\omega}_a$. The forms l_a, n_a annihilate all tangent vectors to $\mathcal{S}_0$ so their pull-back onto $\mathcal{S}_0$ has to vanish identically. Therefore $\tilde{q}_{ab}$ is given just by (2.7) and pull-back of the rotation 1-form is

$$\tilde{\omega}_a = (2\varepsilon n_a - \pi m_a - \overline{\pi m}_a)_{\leftarrow \mathcal{S}_0} = -\pi\, m_a - \bar{\pi}\, \bar{m}_a = \omega_a - 2\varepsilon n_a.$$

Let us choose an another leaf of the foliation $\mathcal{S}_w$. Our construction of coordinates on $\mathcal{H}$ implies that there exists a natural diffeomorphism $\mathfrak{D} : \mathcal{S}_0 \mapsto \mathcal{S}_w$ generated by integral curves of l^a. Since $l^a \overset{\mathcal{H}}{=} \boldsymbol{\partial}^a_v \overset{\mathcal{H}}{=} (1, 0, 0)$ we have

$$\frac{\partial \mathfrak{D}^v}{\partial v} = 1, \quad \frac{\partial \mathfrak{D}^I}{\partial x^I} = 0, \quad \Rightarrow \quad \mathfrak{D} = (w, x^1, x^2).$$

The map $\mathfrak{D}$ induces a tangent map $\mathfrak{D}_\star : T\mathcal{S}_0 \mapsto T\mathcal{S}_w$. It is straightforward to verify that the tangent map is an identity[8]. Hence, we conclude that leaves of the foliation are mutually diffeomorphic, while the geometry $(\tilde{q}_{ab}, \tilde{\omega}_a)$ remains the same.

However, with our choice of the foliation is connected a certain gauge freedom, which we are going to investigate. Define a new coordinate $u = v - F$. Under this transformation, the induced metric remains unchanged $\tilde{q}'_{ab} = \tilde{q}_{ab}$, while the rotation form on $\mathcal{S}^2_0$ transfers to $\tilde{\omega}'_a = \tilde{\omega}_a + 2\varepsilon \nabla_a F$.

Geometry of an isolated horizon is determined by the pair (q_{ab}, ω_a), cf. definition 5. Hence, to define appropriate multipole moments we have to include information coming from both quantities. At the same time we would like to have a coordinate foliation independent characterization of a horizon. To omit this ambiguity in the rotation form

[8]This result in fact follows from our definition of propagation of m^a onto horizon (2.17).

ω_a, it seems natural to consider its curl $\mathcal{D}_{[a}\omega_{b]}$ instead of ω_a itself. From the theorem 8 we know that $\mathcal{D}_{[a}\omega_{b]} = \mathrm{Im}\,\{\Psi_2\}\varepsilon_{ab}$. Another piece of information concerning geometry comes from the metric function f, or equivalently the scalar curvature $\mathcal{R}$ (cf. theorem 15). Similarly to $\mathcal{D}_{[a}\omega_{b]}$, $\mathcal{R}$ is related to the Weyl spinor component Ψ_2, as it summarized in the following lemma.

Lemma 2 *The scalar curvature $\mathcal{R}$ of a spherical section $\mathcal{S}_0^2$ of an isolated horizon without presence of matter fields is related to the tetrad component of the Weyl spinor Ψ_2 via*

$$\mathcal{R} = -4\mathrm{Re}\,\{\Psi_2\}. \tag{2.57}$$

Proof. *The general formula (2.36) for $\mathrm{Re}\,\{\Psi_2\}$ on an (weakly) isolated horizon is reduced to*

$$\mathrm{Re}\,\{\Psi_2\} \overset{\mathcal{H}}{=} a^2 - \xi a' = \frac{1}{2}(\xi^2)'' = \frac{f''}{4R^2} = -\frac{1}{4}\mathcal{R}.$$

where we used (2.50) (in addition, the sign has to be changed because here we use the convention (2.52), where $f \mapsto -f$). ■

Maybe a little surprisingly, the whole information about geometry of $\mathcal{S}_0^2$ in a situation without a matter is encoded in the tetrad component of the Weyl tensor Ψ_2. Generally, when a matter is present on $\mathcal{H}$, the relation (2.57) is modified, but f can be reconstructed from $\mathcal{R}$ anyway. Thus, we are led to define a function on $\mathcal{H}$

$$\Phi_{\mathcal{H}} \equiv \frac{1}{4}\mathcal{R} - \mathrm{i}\,\mathrm{Im}\,\{\Psi_2\}.$$

Without matter fields, it is just $\Phi_{\mathcal{H}} = -\Psi_2$. The function $\Phi_{\mathcal{H}}$ serves to define multipole moments:

Definition 11 Geometric multipole moments $\{I_n\}, \{L_n\}$, $n = 0, 1, 2, \ldots$ *of an axially symmetric isolated horizon $(\mathcal{H}, q_{ab}, \omega_a)$ are defined by*

$$I_n + \mathrm{i}L_n \equiv \oint_{\mathcal{S}_0^2} \Phi_{\mathcal{H}} Y_n^0(\zeta)\,\varepsilon,$$

where Y_n^0 are spherical harmonics defined with respect to canonical two-sphere metric (2.51), $\Phi_{\mathcal{H}}$ is defined above and ε is the metric volume form on a spherical cut $\mathcal{S}_0^2$ of $\mathcal{H}$.

Note: Recall, that the standard spherical harmonics on the unit sphere are defined as

$$Y_l^m(\theta,\varphi) \equiv N e^{\mathrm{i}m\varphi} P_l^m(\cos\theta), \quad l = 0, 1, 2, \ldots;\ m \in \{-l, \ldots, l\}$$

with P_l^m associated Legendre polynomials and a normalization constant

$$N = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}.$$

In this convention, orthogonality reads

$$\oint_{\text{unit sphere}} Y_l^m \overline{Y}_k^n \,\varepsilon = \delta_{lk}\,\delta^{mn}, \quad \Rightarrow \quad \oint_{\mathcal{S}_0^2} Y_l^m \overline{Y}_k^n \,\varepsilon = R^2 \delta_{lk}\,\delta^{mn}$$

Imposing axial symmetry, Y_l^m are independent of φ and we are left with (rescaled) ordinary Legendre polynomials

$$Y_n^0(\theta) \equiv \sqrt{\frac{2n+1}{4\pi}} P_n(\cos\theta), \quad n = 0, 1, 2, \ldots$$

Given a horizon geometry, we may define two sets of numbers $\{I_n\}$ and $\{L_n\}$. It can be shown (see [17]) that these are diffeomorphism invariant. Conversely, one can start with the set $\{I_n, L_n\}$ and then reconstruct the horizon geometry uniquely up to a diffeomorphism. However, multipoles can not be specified completely freely. Some of them already have common value that originate in prescribed boundary conditions of the function f, namely $f(\pm 1) = 0$ and $f'(\pm) = \mp 2$. Then

$$\begin{aligned} I_0 &= \frac{1}{4} \oint_{\mathcal{S}_0^2} \mathcal{R} Y_0^0 \,\varepsilon = \frac{1}{4} \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}\zeta\, R^2 \left(\frac{-f''}{R^2} \right) \frac{1}{\sqrt{4\pi}} = \sqrt{\pi}, \\ I_1 &= \frac{1}{4} \oint_{\mathcal{S}_0^2} \mathcal{R} Y_1^0 \,\varepsilon = -\frac{\sqrt{3\pi}}{4} \int_{-1}^{1} \mathrm{d}\zeta\, f''\zeta \overset{\text{per partes}}{=} \frac{\sqrt{3\pi}}{4} \int_{-1}^{1} \mathrm{d}\zeta\, f' = 0, \end{aligned}$$

where we used $Y_0^0 = \sqrt{1/4\pi}$ and $Y_1^0 = \sqrt{3/4\pi}\,\zeta$. Another condition follows from the relation (2.29) and the Stokes theorem

$$L_0 = -\oint_{\mathcal{S}_0^2} \mathrm{Im}\,\{\Psi_2\} Y_0^0 \,\varepsilon = -\frac{1}{\sqrt{4\pi}} \oint_{\mathcal{S}_0^2} \mathcal{D}_{[a}\omega_{b]}\varepsilon^{ab} = \oint_{\partial\mathcal{S}_0^2} \omega_a \equiv 0.$$

As we shall show bellow, this condition translates to vanishing of angular momentum monopole.

We have already mentioned that I_n, L_n suffice to reconstruct the horizon geometry, but we have not discussed their physical interpretation. From the definition 11, it is clear that I_n, L_n are dimensionless. A suitable rescaling yields mass and angular momentum multipoles, see [17] for details and also [7, 8]:

Definition 12 *Consider an isolated horizon characterized by geometric multipoles* I_m, L_m, $m = 0, 1, 2 \ldots$ *defined above.* Angular momentum *multipole moments are defined by*

$$J_n \equiv \sqrt{\frac{4\pi}{2n+1}} \frac{R^{n+1}}{4\pi G} L_n, \quad n = 0, 1, 2, \ldots$$

where R *is the radius of the horizon and* G *is the gravitational constant. A* mass M *of the horizon is defined by*

$$M \equiv \frac{1}{2GR} \sqrt{R^4 + 4G^2 J_1^2},$$

and finally mass *multipole moments by*

$$M_n \equiv \sqrt{\frac{4\pi}{2n+1}} \frac{M R^n}{2\pi} I_n, \quad n = 0, 1, 2, \ldots$$

Notes: • The zeroth mass moment M_0 always coincide with the horizon mass M. The first one M_1 always vanishes or in other words, the definition 12 places us to the "center of a mass frame". Similarly, as one would expect, the zeroth angular momentum moment J_0 identically vanishes.

• One may wonder, how these multipoles are connected to Geroch-Hansen's multipoles at spatial infinity [18, 19]. Since J_m, M_m are defined locally with respect to the horizon without any reference to surrounding spacetime, one may expect a certain disagreement between the two definitions. In general, it is indeed so. For instance, in the Kerr spacetimes they differ only slightly and coincide as $a/M \mapsto 0$ (M, a are standard mass and angular momentum parameters of the Kerr metric) [17].

3 | Extremal isolated horizons

This chapter is devoted to extremal isolated horizons which are also axially symmetric. The defining condition for an isolated horizon to be extremal is vanishing of the surface gravity $\varkappa_{(\ell)}$ or equivalently $\varepsilon = 0$ (in the gauge where $\bar{\varepsilon} = \varepsilon$). As we already showed in the previous chapter, this property is independent of the choice of a null normal. In the following text we shall show more, namely that the whole geometry of such a horizon is determined uniquely. This is a non-trivial result and hence few remarks are appropriate.

In previous chapter we have also explained why isolated horizons, despite the fact they describe black holes in equilibrium, are in fact compatible with much wider class of spacetimes than the usual Kerr-Newman solution. Recall that in the framework of isolated horizons, the full spacetime is determined by the initial data given on the horizon itself, and on the null hypersurface $\mathcal{N}$ intersecting the horizon in a spherical cut, cf. section 2.5. The former determines the intrinsic geometry of the horizon and its extrinsic curvature, the latter describes the geometry outside. More specifically, data on $\mathcal{N}$ consist of Ψ_4 representing the longitudinal gravitational radiation and ϕ_2 representing the longitudinal part of electromagnetic field. In general, the intrinsic geometry of the horizon can be prescribed arbitrarily (in the sense of section 2.5) and the geometry outside the black hole is distorted by the presence of external matter.

Surprisingly, this fails to be true in the case of *extremal* horizons. As we shall show below, the extremality imposes additional conditions on the intrinsic geometry and these conditions are sufficient to determine the intrinsic geometry completely to a 2-parametric family of solutions. It was first shown in [6] that this family is exactly the family of Kerr-Newman extremal black holes (condition of extremality reduces the 3 parameters of Kerr-Newman metric to two). Their proof contains an implicit assumption that the 2-metric of spherical cuts of the horizon is regular, e.g. free of conical singularities. In the present work we generalize this result by allowing for deficit angles on the north and south poles. This gives as wider class of solutions parametrized by the "radius" of the black hole, its electric and magnetic charges, and two deficit angles.

We proceed as follows. At first, we solve the necessary constrains following [13] and consequently we investigate equation for the metric function f of section 2.6.

3.1 Electromagnetic field and the spin coefficient π

The coefficient π is governed by the equation (2.34). The extremality of the horizon allows one to find a solution explicitly in terms of the metric function f (cf.(2.52)), since the unknown function λ is canceled. Hence, we are left with

$$\xi \partial_\zeta \pi + \pi a \overset{\mathcal{H}}{=} -\pi^2, \quad \text{or explicitly} \quad \pi' - \frac{f'}{2f}\pi \overset{\mathcal{H}}{=} -\frac{\sqrt{2}R}{\sqrt{f}}\pi^2,$$

where we denoted $\partial_\zeta \pi \equiv \pi'$. One can recognize the Bernoulli differential equation [20], which has the general form

$$y' + g(x)y = h(x)y^n$$

for the unknown function $y(x)$ with some general given functions $g(x), h(x)$. This equation can be transformed into a linear one introducing substitution $z(x) = y^{1-n}(x)$, so that

$$z' + (1-n)g(x)z = (1-n)h(x).$$

The general solution is

$$z(x) = y^{1-n} = \left((1-n)\int \mathrm{d}x \left(h(x)e^{(1-n)\int \mathrm{d}x\, g(x)}\right) + z_0\right) e^{-(1-n)\int \mathrm{d}x\, g(x)}. \tag{3.1}$$

In our case, we identify

$$n = 2, \quad g(x) \leftrightarrow -\frac{f'}{2f}, \quad h(x) \leftrightarrow -R\sqrt{\frac{2}{f}}.$$

After substitution to (3.1) and some calculation one obtains

$$\pi \overset{\mathcal{H}}{=} \sqrt{\frac{f}{2}}\frac{1}{R(\zeta + c_\pi)} = \frac{\xi}{\zeta + c_\pi}, \tag{3.2}$$

where c_π is an integration constant connected with integration over ζ and the constant z_0 in the general formula (3.1). In fact, it is fixed by intrinsic geometry and in electro-vacuum spacetimes it might be expressed via electromagnetic dipole and quadrupole moments, see [13] for discussion.

Note: As mentioned above, the integration constant c_π is related to ζ and also π via z_0. Since π is complex, c_π is also in general a complex number even though ζ is real.

Let us have a look at projection ϕ_1 of the electromagnetic tensor. In what follows, we shall assume that the electromagnetic field shares the stationarity and the axial symmetry with the gravitational field. Since $\phi_0 \overset{\mathcal{H}}{=} 0$ the Maxwell equation (A.19a) implies

$$D\phi_1 \overset{\mathcal{H}}{=} 0,$$

so ϕ_1 is time independent. To make *the whole* electromagnetic field time independent, we prescribe

$$D\phi_2 \overset{\mathcal{H}}{=} 0.$$

Under this assumption, equation (A.19b) simplifies to

$$\delta\phi_1 + 2\pi\phi_1 - 2\varepsilon\phi_2 \overset{\mathcal{H}}{=} 0.$$

Moreover, in the situation of extremal horizon we are left with

$$\delta\phi_1 + 2\pi\phi_1 \overset{\mathcal{H}}{=} 0, \quad \text{or} \quad (\zeta + c_\pi)\partial_\zeta\phi_1 + 2\phi_1 \overset{\mathcal{H}}{=} 0$$

the function ϕ_1 can be found easily, multiplying the equation by $(\zeta + c_\pi)$ and using the Leibniz rule one gets

$$\partial_\zeta \left((\zeta + c_\pi)^2 \phi_1 \right) \overset{\mathcal{H}}{=} 0.$$

After direct integration the solution appears in the form

$$\phi_1 \overset{\mathcal{H}}{=} \frac{c_\phi}{(\zeta + c_\pi)^2}, \tag{3.3}$$

where c_ϕ is some complex integration constant and can be related to the total electric charge $Q_{\rm E}$ and magnetic charge $Q_{\rm M}$ of the black hole. For compactness, we introduce the complex charge

$$Q = Q_{\rm E} + {\rm i}\, Q_{\rm M}. \tag{3.4}$$

By the Gauss law, we have[1]

$$Q = \oint_{\mathcal{S}_0^2} \phi_1\, \varepsilon = \frac{4\,\pi_\circ\, R^2\, c_\phi}{c_\pi^2 - 1}. \tag{3.5}$$

Inverting this relation we find

$$c_\phi = \frac{\mathcal{Q}}{4\,\pi_\circ\, R^2}\,(c_\pi^2 - 1). \tag{3.6}$$

Hence, the value of the integration constant c_ϕ is fixed by the total charge of the black hole.

3.2 Uniqueness of extremal horizons

In this section we generalize the result of [21] according to which the intrinsic geometry of an extremal axisymmetric horizon is necessarily isometric to the intrinsic geometry of Kerr-Newman black hole.

We turn our attention to geometry of the horizon, i.e. the function f. Since in (2.37) the spin coefficient μ drops out for the extremal horizon, we find an equation for f

$$\delta\pi \overset{\mathcal{H}}{=} -\pi\bar{\pi} + a\pi - \Psi_2.$$

The imaginary part of this equation is (always) trivially satisfied, we have to take the real part. Substitution for $\delta\pi$ from (2.34) after some manipulations gives

$$\mathrm{Re}\,\{\Psi_2\} \overset{\mathcal{H}}{=} 2(\pi_R a - \pi_I^2),$$

[1]To avoid confusion with the spin coefficient π, in this section we denote the number π by $\pi_\circ$.

where R, I denote real and imaginary parts, resectively, namely $\pi = \pi_R + \mathrm{i}\,\pi_I$. Further, we use (2.36) in the left hand side, the results (3.2), (3.3) and the Leibniz rule to get

$$\frac{1}{2}(\xi^2)'' + (\xi^2)'\frac{\zeta + c_{\pi,R}}{|\zeta + c_\pi|^2} + \xi^2\frac{2c_{\pi,I}^2}{|\zeta + c_\pi|^4} + \frac{|c_{\phi,I}|^2}{|\zeta + c_\pi|^4} = 0.$$

The next step is to use the definition (2.54). After some simplification one arrives at

$$|\zeta + c_\pi|^4 f'' + 2(\zeta + c_{\pi,R})|\zeta + c_\pi|^2 f' + 4\,c_{\pi,I}^2 f + 4R^2|c_\phi|^2 = 0.$$

This equation admits the explicit solution. Imposing the boundary conditions $f(\pm 1) = 0$, we find

$$f(\zeta) = \frac{2\,|c_\phi|^2\,R^2(1-\zeta^2)}{|\zeta + c_\pi|^2(|c_\pi|^2 - 1)}. \tag{3.7}$$

Notice that f formally vanishes for uncharged black holes because of (3.6). However, we have not fixed the integration constant c_π yet. Allowing for the deficit angles around north and south pole, we prescribe [13]

$$f(\pm 1) = \mp\left(2 + \frac{\alpha_\pm}{\pi_\circ}\right), \tag{3.8}$$

where α_+ and α_- are deficit angles on the north and the south poles, respectively. Then, the metric function f reads

$$f(\zeta) = \frac{4\,R^2\,(\alpha_- + 2\pi_\circ)(\alpha_+ + 2\pi_\circ)(1-\zeta^2)}{Q(1-\zeta^2) + 2\,\pi_\circ\,R^2\,(\alpha_+(1-\zeta^2) + \alpha_-(1+\zeta^2) + 4\pi_\circ(1+\zeta^2))}. \tag{3.9}$$

Notice that this function has well-behaved limit for $Q \to 0$, i.e. function f remains non-zero in this limit.

We have proved an important result. Extremal, stationary, axisymmetric horizon is unique in the sense that its metric is necessarily of the form (3.9). Such horizons form a 5-parametric family of solutions. In [21], a similar result has been proved, but there the possibility of the presence of the deficit angles was excluded. The authors of [21] concluded that extremal horizons form a 3-parametric family of solutions isometric to Kerr-Newman black holes. Kerr-Newman black holes are parametrized by the mass, electric and magnetic charges and the angular momentum, but the extremality condition reduces the freedom to 3 parameters. In our setting, we have 5 parameters: radius of the black hole (which replaces the mass), electric and magnetic charges and two deficit angles. Extremality is already taken into account and therefore does not reduce the number of independent parameters. The two additional parameters (compared to Kerr-Newman) presumably correspond to the acceleration parameter of the C-metric (which determines both deficit angles around the axis) and the NUT parameter [12]. Precise interpretation and connection to known exact solutions is the task for our future investigation.

4 Penrose quasi-local mass

One of the longstanding problems in general relativity is to provide a precise definition of energy and angular momentum of a gravitation field. Due to the equivalence principle, energy can not be associated with a particular point of spacetime and the classical interpretation as the first integral of equations of motion following from a symmetry loses its meaning. On the other hand, it seems natural to define this notion quasilocally, i.e. associate conserved charges with a finite spacetime domain. There are several approaches based on different ideas and often yielding different results in the same situations. One of these is the so called Penrose mass. It provides a definition that gives physically reasonable results in many applications but it also suffers from several drawbacks. In the following text, we introduce basic ideas behind the Penrose construction.

In standard field theories on the flat background, the usual way of defining mass, momentum and angular momentum (to be referred to as *charges*) is to apply the (first) Nœther theorem and the symmetries of the Minkowski spacetime. There are essentially two ways of describing the relation between symmetries and conserved charges. Assuming that the dynamics of a given field is governed by the action principle and corresponding Lagrangian, the relativistic covariance of the equations of motion requires that the Lagrangian is invariant under the action of the Poincaré group consisting of 3 spatial rotations, 3 boosts and 4 spacetime translations. The conserved charges are then associated with the generators of Poincaré (Lie) group. Equivalently, the Minkowski spacetime admits 10 independent Killing vectors that are regarded as generators of isometries. The Lie algebra of these Killing vectors is isomorphic to the Lie algebra of the Poincaré group. The requirement of covariance then translates to the statement that the Lagrangian is Lie constant along the Killing vectors and conserved charges are associated with these Killing vectors. Similar construction will work whenever we consider propagation of a field on a fixed background with Killing vectors.

In full general relativity, the situation is significantly more complicated, since the spacetime is a dynamical quantity itself and, in general, does not admit a Killing vector field, i.e. there are no isometries. From the physical point of view, however, any two spacetimes (M, g_{ab}) and $(N, \tilde{g}_{ab})$ related by a diffeomorphism $\phi : M \mapsto N$ such that $\phi^* \tilde{g}_{ab} = g_{ab}$, are regarded as physically equivalent and the diffeomorphism ϕ can be interpreted merely as a coordinate transformation. Hence, values of tensor fields at spacetime points do not have direct physical meaning. In this sense, the equivalence principle prevents us to introduce the *local* notion of the energy.

Nevertheless, we know that, for example, gravitational waves carry the energy, therefore it should be possible to introduce an appropriate notion of energy of gravitational field even in a dynamical situation. It is well known that this can be done unambiguously in the case of asymptotically flat spacetimes representing isolated gravitating sources, as was first shown in pioneering works by Bondi, Metzner, Sachs et al [22, 23, 24] and later formulated in geometrical language by Penrose, Newman, Unti, and others [25, 26, 27, 28]. In this case, the gravitational field decays far from the gravitating sources and vanishes at infinity, so that the isometries of Minkowski spacetime are partially recovered. Relevant group in this case is the Bondi-Metzner-Sachs group which is a (semi-)direct sum of the usual Lorentz group and the infinite-dimensional group of supertranslations. Then, the so-called *Bondi mass* can be defined. It is a global characteristics of the spacetime and, provided that certain energy conditions are satisfied, it is positive and non-increasing in time. The decrease of the Bondi mass amounts to the presence of radiation close to null infinity, so that the Bondi mass measures the loss of energy by radiation.

The drawback of the Bondi amss is that it cannot be associated with a finite domain of the spacetime (it is global rather than quasi-local). On the other hand, thanks to its canonical character, any reasonable quasi-local mass should reduce to that of Bondi in the limit of large spheres [29]. The Penrose mass, which is a subject of this chapter, satisfies this requirement. It also admits a natural Hamiltonian interpretation which makes it a good candidate for the generally acceptable quasi-local mass. Unfortunately, Penrose mass cannot be constructed for a general spacetime and general spacetime domain. Still, it has several interesting properties and, more importantly, it inspired many other approaches to the definition of quasi-local mass based on the Sen connection and Nester-Witten form [30, 31].

4.1 Motivation behind the Penrose construction

An elementary example of relation between field variables and corresponding sources comes from the classical mechanics. Potential of a gravitation field Φ in the Newton's theory satisfies the familiar Poisson equation[1]

$$\Delta\Phi(\mathbf{x}) = 4\pi G\rho(\mathbf{x}), \tag{4.1}$$

where G is gravitational constant, ρ is density of a gravitating mass and $\mathbf{x}$ is radius vector in Cartesian coordinates. Obviously, total mass in a domain Ω enclosed by a surface $\partial\Omega$ is

$$m_\Omega = \int_\Omega \mathrm{d}\mathbf{x}\,\rho(\mathbf{x}). \tag{4.2}$$

The key point is that the equation (4.1) might be used to rewrite (4.2) in terms of the field variable

$$m_\Omega = \frac{1}{4\pi G}\int_\Omega \mathrm{d}\mathbf{x}\,\Delta\Phi(\mathbf{x}) = \frac{1}{4\pi G}\oint_{\partial\Omega} \mathrm{d}\mathbf{S}\cdot\nabla\Phi(\mathbf{x}),$$

where we employed the well known vector identity $\Delta \equiv \nabla\cdot\nabla$ and consequently the Gauss-Ostrogradski theorem. Thus the field equation allows us to determine "charge", which is contained in a particular region of space from the knowledge of the field itself.

[1]In this section we work within SI system of units.

This might be useful in a situation, when one does not know what exactly the charge should be represented by.

Let us move to another motivational example, which is little less trivial and comes from electromagnetism. The Maxwell equations in the special relativity framework can be written in the form

$$\nabla_a F^{ba} = \mu J^b, \quad \nabla_{[a} F_{bc]} = 0 \tag{4.3}$$

where $F_{ab} = \nabla_a A_b - \nabla_b A_a$ is electromagnetic field tensor given by potential A^a, $J^a = (\rho_E/c, \mathbf{j})$ is 4-current composed of electric charge density ρ_E divided by speed of light c and current density $\mathbf{j}$, and finally μ is permeability of vacuum. Furthermore, we introduce the Hodge dual to F_{ab}

$$^\star F_{ab} = \frac{1}{2!} \varepsilon_{abcd} F^{cd}. \tag{4.4}$$

We denoted ε_{abcd} the Levi-Civita tensor. Our convention is $\varepsilon_{0123} = 1$ (in the Cartesian coordinates). For calculation, the following identities are useful

$$\varepsilon^{abcd} \varepsilon_{aklm} = -\delta^{bcd}_{klm} \equiv - \sum_{\pi \in \text{perm.}} \text{sign}(\pi) \delta^b_{\pi(k)} \delta^c_{\pi(l)} \delta^d_{\pi(m)}, \tag{4.5a}$$

$$\varepsilon^{abcd} \varepsilon_{abkl} = -2!\, \delta^{cd}_{kl} = -2(\delta^c_k \delta^d_l - \delta^d_k \delta^c_l), \tag{4.5b}$$

$$\varepsilon^{abcd} \varepsilon_{abck} = -3!\, \delta^d_k \tag{4.5c}$$

$$\varepsilon^{abcd} \varepsilon_{abcd} = -4!, \tag{4.5d}$$

For instance, using (4.5b) an inverse relation to (4.4) can be found

$$F_{ab} = -\frac{1}{2} \varepsilon_{abcd} {}^\star F^{cd}. \tag{4.6}$$

Note: In general, the Hodge operator is isomorphism of k-th and $n-k$-th exterior power of an n-dimensional vector space V, i.e. $\star : \wedge^k(V) \mapsto \wedge^{n-k}(V)$. Substitution (4.4) to (4.6) leads to a special case of an identity

$$\star \circ \star = -\text{id}|_V.$$

Hence, eigenvalues of $\star$ are $\pm \text{i}$. If $\star\omega = \text{i}\omega$ for some form ω, then ω is called *self-dual*, if $\star\omega = -\text{i}\omega$ the form is called *anti-self-dual*.

Furthermore we substitute (4.6) to the Maxwell equations (4.3), after some manipulations using (4.5) we arrive at

$$\nabla_a {}^\star F^{ba} = 0, \quad \nabla_{[a} {}^\star F_{bc]} = 2\mu {}^\star J_{abc},$$

where ${}^\star J_{abc} = \varepsilon_{abcd} J^d$ is Hodge dual to J_a. These equations have formally the same structure as (4.3) except for the presence of the current in the second set of equations instead of the first one.

In a more abstract geometrical language $\nabla_{[a} F_{bc]}$ is de Rham differential $\text{d}\mathbf{F} \equiv \nabla_{[a} F_{bc]}$, so the Maxwell equations can be recast to an elegant version

$$\text{d}\mathbf{F} = 0, \quad \text{d}{}^\star\mathbf{F} = 2\mu {}^\star\mathbf{J}. \tag{4.7}$$

Consider a spacelike domain Ω with smooth boundary $\partial\Omega$ and define quantity

$$Q \equiv \kappa \oint_{\partial\Omega} \mathbf{F} + \mathrm{i}{}^\star\mathbf{F}, \tag{4.8}$$

with some constant κ be to determined later. It is well defined, since both $\mathbf{F}, {}^\star\mathbf{F}$ are two-forms integrated over two-dimensional manifold. Using the general Stokes theorem and (4.7) Q may be simplified to

$$Q = \int_\Omega \mathrm{d}\mathbf{F} + \mathrm{i}\mathrm{d}{}^\star\mathbf{F} = 2\mathrm{i}\mu \int_\Omega {}^\star\mathbf{J}. \tag{4.9}$$

In order to find explicit expression for Q, one has to consider embedding $\mathscr{E} : U \subset \mathbb{R}^3 \mapsto \mathbb{R}^4$, such that $\mathscr{E}(U) = \Omega$ and corresponding pull-back $\mathscr{E}^*({}^\star\mathbf{J})$ on U. We know that pull-back of a form ω in coordinates is $\omega_{a\ldots b} = \partial_a \mathscr{E}^c \ldots \partial_b \mathscr{E}^d \, \omega_{c\ldots d}$. The embedding $\mathscr{E}$ can be chosen as the identity on U, i.e. $\mathscr{E}(x^1, x^2, x^3) = (0, x^1, x^2, x^3)$ which implies $\partial_i \mathscr{E}^a = \delta^a_i$, where now $i = 1, 2, 3$. Then

$$\mathscr{E}^*({}^\star\mathbf{J})_{ijk} = \varepsilon_{abcd}\delta^a_i\delta^b_j\delta^c_k J^d = J^0\varepsilon_{0ijk} = J^0\varepsilon_{ijk} \quad \Rightarrow \quad \mathscr{E}^*({}^\star\mathbf{J}) = 6J^0 \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$$

and the integral (4.9) explicitly

$$\int_\Omega {}^\star\mathbf{J} = \int_U \mathscr{E}^*({}^\star\mathbf{J}) = \frac{6}{c}\int_\Omega \mathrm{d}\mathbf{x}\, \rho_E(\mathbf{x}) \equiv \frac{6}{c} Q_E$$

On the right hand side we recognized a total electric charge Q_E in Ω. If we choose $\kappa = c/(12\mathrm{i}\mu)$, the quantity Q will coincide with Q_E. Analogously to our previous motivational example of mass, we are able to express corresponding source in a spacetime domain directly via integration of the field variable itself. Notice that in both examples, the charges are *not* related to a Noether current connected with a symmetry of the system under investigation. This will be briefly discussed in the next section.

Note: Let us add a final remark here. The Maxwell equations (4.7) are somewhat asymmetric in source terms. Therefore one may think of a generalization on presence of a magnetic charge current

$$\mathrm{d}\mathbf{F} = 2\mu{}^\star\mathbf{J}_\mathbf{M}, \quad \mathrm{d}{}^\star\mathbf{F} = 2\mu{}^\star\mathbf{J}_\mathbf{E}.$$

By the same procedure as above, the integral (4.8) would yield

$$Q \equiv \kappa \oint_{\partial\Omega} \mathbf{F} + \mathrm{i}{}^\star\mathbf{F} \equiv \mathrm{i}Q_M + Q_E,$$

where Q_M is a magnetic charge in the domain Ω.

4.2 Conserved currents of energy-momentum tensor

It is well known that invariance of a particular system under spacetime translations $x'^a = x^a + \xi^a$, where ξ^a is a constant vector, implies, according to the Noether's theorem, conservation of energy-momentum tensor T^{ab}. The conservation law is

$$\nabla_a T^{ab} = 0.$$

Its validity is also assumed in general curved spacetimes, where any matter fields are present. If such a spacetime has additional symmetry represented by a Killing field k^a, one can construct an appropriate conserved current setting $J^a \equiv T^{ab}k_b$. Local conservation law can be inferred from

$$\nabla_a J^a = T^{ab}\nabla_a k_b = 0,$$

since k_a satisfies the Killing equation $\nabla_{(a}k_{b)} = 0$ and T^{ab} is symmetric. Associated charge is defined by

$$Q[k^a] \equiv \frac{1}{3!}\int_\Omega {}^\star\mathbf{J} = \frac{1}{3!}\int_\Omega \varepsilon_{abcd}T^{de}k_e, \tag{4.10}$$

where $Q : \mathcal{K} \mapsto \mathbb{R}$ has to be understood as functional on the Lie algebra of Killing vectors $\mathcal{K}$. The interpretation of the conserved current J^a or the charge Q depends on the nature of a Killing field.

The first observation is that the form ${}^\star\mathbf{J}$ is closed. The exterior differential of ${}^\star\mathbf{J}$ is four-form, so it has to be proportional to the volume form

$$\begin{aligned} \nabla_{[e}\varepsilon_{abc]d}J^d &= \lambda\varepsilon_{eabc} \quad /\varepsilon^{eabc} \\ -3!\delta^e_d\nabla_e J^d &= -4!\lambda \\ \lambda &= \frac{1}{4}\nabla_e J^e = 0. \end{aligned}$$

In the first step we used $\nabla\varepsilon = 0$ and (4.5). Thus, $\mathrm{d}{}^\star\mathbf{J} = 0$. The Poincaré lemma now implies that on a simply connected compact domain Ω there exists a two form $\mathbf{K}$ such that $\mathrm{d}\mathbf{K} = {}^\star\mathbf{J}$. The charge (4.10) can be therefore expressed via surface integral over $\partial\Omega$

$$3!\,Q[k^a] \equiv \int_\Omega \mathrm{d}\mathbf{K} = \int_{\partial\Omega}\mathbf{K}.$$

The form $\mathbf{K}$ is not uniquely defined. Let be $\mathbf{R}$ another two form for which $\mathrm{d}\mathbf{R} = {}^\star\mathbf{J}$. Then $\mathrm{d}(\mathbf{K}-\mathbf{R}) = 0$ so $\mathbf{K}$ is unique up to addition of an exact two-form, or $\mathbf{K} = \mathbf{R} + \mathrm{d}\alpha$. However, $Q[k^a]$ is not affected by this freedom

$$3!\,Q[k^a] = \int_{\partial\Omega}\mathbf{R} + \mathrm{d}\alpha = \int_{\partial\Omega}\mathbf{R} + \int_{\partial\partial\Omega}\alpha = \int_{\partial\Omega}\mathbf{R},$$

where we used that boundary of a boundary is zero chain $\partial\partial\Omega = 0$. So, for a given Killing vector k^a the charge depends only on the surface $\partial\Omega$. Moreover, it can be associated with the whole domain of dependence $\mathscr{D}(\Omega)$ because $Q[k^a]$ is independent of an actual Cauchy hypersurface of $\mathscr{D}(\Omega)$. Choose two such hypersurfaces Ω, Ω'. Since they have both the same boundary, which in a particular time slice coincides with $\partial\Omega$, Q can not change within $\mathscr{D}(\Omega)$ (see figure 4.1). In this sense the charge (4.10) is conserved.

In the Minkowski spacetime $\mathbb{M}^4$, 10 Killing vectors can be found and they constitute a 10 dimensional Lie algebra $\mathcal{K}$ (for details see [29]). Its generators are interpreted as 4-momentum vector P^a and antisymmetric angular momentum tensor $\mathcal{J}^{ab} = x^a\boldsymbol{\partial}^b - x^b\boldsymbol{\partial}^a$ (in the Cartesian coordinates). Every vector of this algebra can be expanded in $k_a = t_\mu P^\mu_a + m_{\mu\nu}(x^\mu\mathbf{d}x^\nu_a - x^\nu\mathbf{d}x^\mu_a)$, where μ,ν are concrete indices. The charge Q can be used to define momentum P_Ω and angular-momentum $\mathcal{J}_\Omega$ of a whole domain Ω by

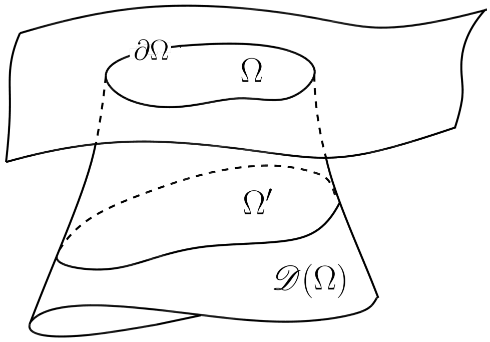

Figure 4.1: Conservation of the charge $Q[k^a]$ within the domain of dependence $\mathscr{D}(\Omega)$.

$Q[k^a] = t_\mu P^\mu_\Omega + m_{\mu\nu}\mathcal{J}^{\mu\nu}_\Omega$. Numbers $P^\mu_\Omega, \mathcal{J}^{\mu\nu}_\Omega$ are coordinate components of corresponding tensor quantities. Mass is obtained by the standard relation $M_\Omega \equiv P^a_\Omega P_{\Omega\,a}$.

This procedure might be applied even in the case, when no Killing vector is present. For a time-like vector u^a the integral

$$\mathcal{Q}[u^a] \equiv \frac{1}{3!}\int_\Omega \varepsilon_{abcd}T^{de}u_e,$$

is interpreted as local energy-momentum density seen by an observer with 4-velocity u^a. However, $\mathcal{Q}$ now *depends* on a particular Cauchy hypersurface. A mass still can be defined reasonably, but in $\mathbb{M}^4$ does not reduce to M_Ω and therefore represents a different concept [29].

4.3 Penrose construction

After some introductory text we proceed to the construction originally presented by R. Penrose [32]. Our starting point is the definition (4.10). In the spirit of the section 4.1 we want to express the charge by a surface integral of a suitable field variable. An appropriate one might seems to be the Ricci tensor R_{ab}, which is directly connected to the energy-momentum tensor T_{ab} via the field equations. However, R_{ab} vanishes in absence of a matter and we want to describe also autonomous gravitation field, e.g waves. Therefore, Penrose suggested to take rather the Riemann tensor R_{abcd}, which does not vanish in such situations[2]. However, R_{abcd} is not a two-form, so it can not be integrated over a two-surface directly. One has to employ an auxiliary function f^{ab}, then

$$Q[k^a] \equiv \frac{1}{3!}\int_\Omega \varepsilon_{abcd}T^{de}k_e = \mathscr{C}\oint_{\partial\Omega} R_{abcd}f^{cd}, \tag{4.11}$$

where $\mathscr{C}$ is a suitable constant, which will be determined later. On the right hand side of (4.11) we use the general Stokes theorem yielding a volume integral

$$\int_\Omega \frac{1}{3!}\varepsilon_{abcd}T^{de}k_e - \mathscr{C}\nabla_{[c}(R_{ab]ed}f^{ed}) = 0.$$

[2]In absence of matter R_{abcd} is, of course, reduced to the Weyl tensor C_{abcd}.

We require validity for any spacetime domain Ω, so the integrand has to vanish identically

$$\frac{1}{3!}\varepsilon_{abcd}T^{de}k_e = \mathscr{C}\nabla_{[c}(R_{ab]ed}f^{ed}) = \mathscr{C}R_{ed[ab}\nabla_{c]}f^{ed} \quad /\varepsilon^{abcg}$$
$$-T^{ge}k_e = \mathscr{C}\varepsilon^{abcg}R_{edab}\nabla_c f^{ed} = \mathscr{C}\varepsilon^{cgab}R_{edab}\nabla_c f^{ed}.$$

We used the Bianchi identity $\nabla_{[c}R_{ab]ed} = 0$ and the symmetry of the Riemann tensor $R_{abcd} = R_{cdab}$ in the first equation and consequently (4.5). It can be further simplified introducing dual Riemann tensor

$$^\star R_{abcd} \equiv \frac{1}{2}\varepsilon_{cd}{}^{ef}R_{abef}.$$

Substitution in the previous formula gives

$$-\mathrm{i}^\star R_{abcd}\nabla^c f^{ab} = \frac{\mathrm{i}}{2\mathscr{C}}T_{de}k^e. \tag{4.12}$$

The next step is to express every quantity by its spinor equivalent. Without loss of generality, the function f^{ab} may be taken antisymmetric, so its spinor decomposition is (see [33] or literature [2, 1, 34])

$$f^{ab} = \omega^{AB}\epsilon^{A'B'} + \overline{\omega}^{A'B'}\epsilon^{AB}, \tag{4.13}$$

where $\omega^{AB} = \omega^{BA}$ is symmetric second rank spinor and ϵ^{AB} is symplectic form on a spinor space $\mathcal{S}^A \otimes \mathcal{S}^B$. Recall that ϵ^{AB} acts as index raising/lowering operator in the following way

$$\kappa_A = \kappa^B\epsilon_{BA} = -\epsilon_{AB}\kappa^B, \quad \kappa^A = \epsilon^{AB}\kappa_B, \quad \forall\kappa \in \mathcal{S},$$

where $\epsilon^{AB} \equiv -(\epsilon_{AB})^{-1}$. So to raise/lower index with plus sign, the indices have to be “adjacent and descending to the right”. It can be also easily proved that

$$\epsilon^{AB}\epsilon_{AB} = \delta_A{}^A = 2. \tag{4.14}$$

Spinor equivalent of the Levi-Civita tensor is ([2], page 76)

$$\varepsilon_{abcd} = \mathrm{i}(\epsilon_{AB}\epsilon_{CD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'}).$$

Using this formula, spinor form of the Hodge dual to f^{cd} can be found

$$^\star f^{ab} = -\mathrm{i}\omega^{AB}\epsilon^{A'B'} + \mathrm{i}\overline{\omega}^{A'B'}\epsilon^{AB}. \tag{4.15}$$

Comparing (4.13) with (4.15) we clearly have

$$\omega^{AB}\epsilon^{A'B'} \overset{\star}{\mapsto} -\mathrm{i}\omega^{AB}\epsilon^{A'B'} \qquad\qquad \overline{\omega}^{A'B'}\epsilon^{AB} \overset{\star}{\mapsto} \mathrm{i}\overline{\omega}^{A'B'}\epsilon^{AB},$$

so $\omega^{AB}\epsilon^{A'B'}$ represents anti-self-dual part and $\overline{\omega}^{A'B'}\epsilon^{AB}$ self-dual part. The make our calculations shorter, we will assume that f^{ab} has only anti-self-dual part, i.e.

$$f^{ab} = \omega^{AB}\epsilon^{A'B'}. \tag{4.16}$$

Furthermore, we need the dual Riemann tensor ([2], page 86)

$$-\mathrm{i}{}^\star R_{abcd} = \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{A'B'C'D'} - \epsilon_{A'B'}\epsilon_{C'D'}\Psi_{ABCD} + \epsilon_{A'B'}\epsilon_{CD}\overline{\Phi}_{ABC'D'} \\ - \epsilon_{AB}\epsilon_{C'D'}\overline{\Phi}_{A'B'CD} + 2\Lambda(\epsilon_{A'B'}\epsilon_{C'D'}\epsilon_{D(A}\epsilon_{B)C} - \epsilon_{AB}\epsilon_{CD}\epsilon_{D'(A'}\epsilon_{B')C'})$$

and Einstein equations (here with zero cosmological constant, see [1], page 235)

$$\Phi_{ABA'B'} + 3\Lambda\epsilon_{AB}\epsilon_{A'B'} = 4\pi G T_{ABA'B'}.$$

The left hand side of (4.12) is then

$$\begin{aligned} \mathrm{LHS} &= -2\epsilon_{C'D'}\Psi_{ABCD}\nabla^{CC'}\omega^{AB} + 2\epsilon_{CD}\Phi_{ABC'D'}\nabla^{CC'}\omega^{AB} + 4\Lambda\epsilon_{C'D'}\epsilon_{D(A}\epsilon_{B)C}\nabla^{CC'}\omega^{AB} \\ &= -2\Psi_{ABCD}\nabla^{C}{}_{D'}\omega^{AB} + 2\Phi_{ABC'D'}\nabla_{D}{}^{C'}\omega^{AB} + 4\Lambda\nabla_{AD'}\omega^{A}{}_{D}, \end{aligned}$$

where we used (4.14) and $\epsilon_{AB}\nabla^{CC'}\omega^{(AB)} = 0$. The right hand side of (4.12) is simply

$$\mathrm{RHS} = \frac{\mathrm{i}}{8\pi G\mathscr{C}}\left(\Phi_{DED'E'}k^{EE'} + 3\Lambda k_{DD'}\right).$$

To cancel unnecessary constants we choose $\mathscr{C} = 1/(16\pi G)$. After same manipulation we finally obtain

$$-\Psi_{ABCD}\nabla^{C}{}_{D'}\omega^{AB} + \Phi_{ABC'D'}\nabla_{D}{}^{C'}\omega^{AB} - \mathrm{i}\Phi_{DED'E'}k^{EE'} \\ + 2\Lambda\nabla_{AD'}\omega^{A}{}_{D} - 3\mathrm{i}\Lambda k_{DD'} = 0. \quad (4.17)$$

Let us first investigate when the part containing the Ricci spinor $\Phi_{ABC'D'}$ vanishes

$$\begin{aligned} \Phi_{ABC'D'}\nabla_{D}{}^{C'}\omega^{AB} - \mathrm{i}\Phi_{DED'E'}k^{EE'} &\overset{!}{=} 0 \\ \Phi_{ABA'D'}\left(\nabla_{D}{}^{A'}\omega^{AB} - \mathrm{i}\epsilon_{D}{}^{B}k^{AA'}\right) &= 0, \quad \text{or} \\ \nabla_{D}{}^{A'}\omega^{AB} - \mathrm{i}\epsilon_{D}{}^{(B}k^{A)A'} &= 0, \end{aligned} \quad (4.18)$$

since we require validity in arbitrary space time (with any $\Phi_{ABA'D'}$). Raising index D and symmetrization in ABD cancels the term proportional to ϵ_{AB}, so finally

$$\nabla^{A'(D}\omega^{AB)} = 0. \quad (4.19)$$

This equation immediately implies that the term proportional to Ψ_{ABCD} in (4.17) is zero. Now, we just have to show that (4.18) is compatible with the rest of (4.17). Renaming index $D \mapsto A$ in (4.18) yields

$$2\nabla_{A}{}^{A'}\omega^{AB} = 2\mathrm{i}\epsilon_{A}{}^{(B}k^{A)A'} = \mathrm{i}(\epsilon_{A}{}^{B}k^{AA'} + \epsilon_{A}{}^{A}k^{BA'}) = 3\mathrm{i}k^{BA'},$$

what is exactly the part of (4.17) containing Λ. To conclude, the necessary condition, which the spinor analogue of the auxiliary function f^{ab} has to satisfy is (4.19) (or (4.18) respectively).

In (4.19) we obtained an example of the so called *twistor equation* for a second rank symmetric spinor ω^{AB}. Its variations appear throughout physics in different contexts ([34, 35]). The general definition is as follows:

Definition 13 *The* twistor equation *for a fully symmetric spinor of k-th valence* $\omega^{AB\ldots C} \in \mathcal{S}^A \otimes \mathcal{S}^B \otimes \cdots \otimes \mathcal{S}^C$ *is*

$$\nabla^{(A}_{A'}\omega^{BC\ldots D)} = 0.$$

After finding the condition that ω^{AB} has to satisfy, we can proceed to calculation of explicit formula for the charge $Q[k^a]$. Denote $r_{ab} \equiv R_{abcd}f^{cd}$, $\mathscr{E} : \Sigma \mapsto \partial\Omega$ embedding of the surface $\partial\Omega$ and $\{y^i\}, i = 1, 2$ coordinates on Σ. Since $r_{ab} \equiv \mathbf{r}$ is two-form it can be written as $\mathbf{r} = r_{\mu\nu}\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu$ in some coordinates x^μ, $\mu = 1, 2, 3, 4$. Its pull-back onto Σ reads

$$\begin{aligned}\mathscr{E}^*(\mathbf{r}) &= r_{\mu\nu} \circ \mathscr{E}(\mathbf{y})\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}\mathbf{d}y^i \wedge \mathbf{d}y^j = r_{\mu\nu} \circ \mathscr{E}(\mathbf{y})\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}\epsilon_{ij}\,\mathbf{d}y^1 \wedge \mathbf{d}y^2\\ &= r_{\mu\nu} \circ \mathscr{E}(\mathbf{y})\varepsilon^{\mu\nu}\mathbf{d}y^1 \wedge \mathbf{d}y^2\end{aligned}$$

where we recognized the surface form[3] ε_{ab}. Thus, we have

$$Q[k^a] = \frac{1}{16\pi G}\oint_{\partial\Omega} R_{abcd}f^{cd} = \frac{1}{16\pi G}\int_\Sigma R_{\mu\nu\rho\sigma}f^{\rho\sigma}\varepsilon^{\mu\nu}\mathrm{d}(y^1, y^2).$$

The integral on the left is understood in the Lebesgue sense. The surface two-form in our NP tetrad of chapter 2 is given by (2.8). Its spinor equivalent may be found easily using definition of NP tetrad vector m^a via a spin basis $m^a = o^A\bar{\iota}^{A'}$

$$\varepsilon_{ab} = -\mathrm{i}(o_{(A}\iota_{B)}\epsilon_{A'B'} - \bar{o}_{(A'}\bar{\iota}_{B')}\epsilon_{AB}). \tag{4.20}$$

To find $R_{abcd}f^{cd}\varepsilon^{ab} = R_{abcd}f^{ab}\varepsilon^{cd}$ explicitly, we employ the spinor equivalent of the Riemann tensor which is given by (A.6) and the relation (4.16). At first, the form $R_{abcd}f^{ab}$ after a simple calculation is

$$R_{abcd}f^{ab} = 2\Psi_{ABCD}\omega^{AB}\epsilon_{C'D'} + 2\Phi_{ABC'D'}\omega^{AB}\epsilon_{CD} + 4\Lambda\omega_{CD}\epsilon_{D'C'}.$$

Contracting it further with ε^{cd} using (4.20) yields

$$R_{abcd}f^{ab}\varepsilon^{cd} = -4\mathrm{i}\left(\Psi_{ABCD}\omega^{AB}o^C\iota^D - \Phi_{ABC'D'}\omega^{AB}\bar{o}^{C'}\bar{\iota}^{D'} + 2\Lambda\omega_{AB}o^A\iota^B\right). \tag{4.21}$$

Terms proportional to the Weyl and Ricci spinor may be expanded in the spin basis via (A.9) and (A.11) respectively. The horrible looking expressions are radically reduced with the help of defining formulae of a spin basis

$$o_Ao^A = \iota_A\iota^A = 0, \quad o_A\iota^A = -o^A\iota_A = 1$$

Then, the first two terms are simplified to

$$\begin{aligned}\Psi_{ABCD}o^C\iota^D &= \Psi_3 o_A o_B - \Psi_2(o_A\iota_B + \iota_A o_B) + \Psi_1\iota_A\iota_B,\\ \Phi_{ABC'D'}\bar{o}^{C'}\bar{\iota}^{D'} &= \Phi_{21}o_Ao_B - \Phi_{11}(o_A\iota_B + \iota_Ao_B) + \Phi_{01}\iota_A\iota_B.\end{aligned}$$

[3] Here ϵ_{ij} denotes permutation symbol, c.f. (2.41).

Since the spinor ω^{AB} is symmetric, there exists univalent spinors $\eta^A, \vartheta^A \in \mathcal{S}^A$ such that $\omega^{AB} = \eta^{(A}\vartheta^{B)}$ (see [2]). Expansion into the basis is

$$\begin{aligned} \eta^A &= \eta^0 o^A + \eta^1 \iota^A, \quad \vartheta^A = \vartheta^0 o^A + \vartheta^1 \iota^A, \\ \omega^{AB} =& \eta^0\vartheta^0 o^A o^B + \eta^0\vartheta^1 o^A \iota^B + \eta^1\vartheta^0 \iota^A o^B + \eta^1\vartheta^1 \iota^A \iota^B. \end{aligned}$$

Substituting to (4.21) after some process we finally arrive at

$$R_{abcd} f^{ab} \varepsilon^{cd} = 4\mathrm{i} \left(\eta^0\vartheta^0(\Phi_{01} - \Psi_1) + (\eta^0\vartheta^1 + \eta^1\vartheta^0)(\Phi_{11} + \Lambda - \Psi_2) + \eta^1\vartheta^1(\Phi_{21} - \Psi_3) \right)$$

and explicit formula for the charge $Q[k^a]$ in terms of tetrad components of the Riemann tensor and spin basis components of ω^{AB}

$$Q[\eta, \vartheta] = \frac{\mathrm{i}}{4\pi G} \oint_{\partial\Omega} \left(\eta^0\vartheta^0(\Phi_{01} - \Psi_1) + (\eta^0\vartheta^1 + \eta^1\vartheta^0)(\Phi_{11} + \Lambda - \Psi_2) + \eta^1\vartheta^1(\Phi_{21} - \Psi_3) \right). \tag{4.22}$$

So far we have not discussed possible solutions of the twistor equation (4.19). It turns out that it does *not* posses a non-trivial solution in a general spacetime. Actually this is not surprising, because we started with a spacetime, where at least one Killing field k^a is present. The relation (4.18) tells us that ω^{AB} plays a role of a "potential" for $k^{AA'}$. So, if we want to achieve working construction in a general spacetime, requirements on ω^{AB} have to be weakened. Penrose suggested that ω^{AB} should satisfy only tangential projections of (4.19) to $\partial\Omega$ in a suitable chosen tetrad. However, obtained equations are under-determined and another additional assumption is needed. Further, it is assumed that both spinors η and ϑ are solutions of univalent twistor equation, i.e.

$$\nabla^{(A}_{\ A'}\eta^{B)} = 0, \quad \nabla^{(A}_{\ A'}\vartheta^{B)} = 0. \tag{4.23}$$

The charge (4.22) is then defined in terms of these solutions. The necessary condition for this construction to work is that (4.23) has exactly four linearly independent solutions that can be identified with four components of the 4-momentum. We return to discussion of this problem at the end of the chapter 5.

5 | Twistor equation

The main purpose of this chapter is to get some insight into a problem of defining the Penrose mass on a general isolated horizon (IH). The Penrose mass, as it was introduced in the chapter 4, requires to solve the twistor equation for univalent spinor ω^A,

$$\nabla_{A'}^{(A}\omega^{B)} = 0. \tag{5.1}$$

It has been already mentioned that this equation does not have a solution in a general spacetime. However, an isolated horizon is a prominent spacetime region and one may be interested in the exact conditions that has to be satisfied to get a non-trivial solution. Thus, we first derive the most general equations governing our problem and then spend some time investigating appropriate conditions of integrability. Later, we provide a time dependent solution of (5.1) on the isolated horizon and finish our discussion summarizing the result of the general approach to the problem.

5.1 Twistor equation on an isolated horizon

Our starting point is the twistor equation (5.1). In order to find a solution on an isolated horizon $\mathcal{H}$, we employ NP formalism and adapted coordinates of the section 2.2. For that purpose, we introduce a spinor basis $\{o^A, \iota^A\}$ and expand the spinor ω^A into it

$$\omega^A = \omega^0 o^A + \omega^1 \iota^A. \tag{5.2}$$

Notes: • Following standard conventions of the NP formalism, we expand contravariant spinors in the form (5.2), i.e. into a spinor dyad (o^A, ι^A). Covariant spinors, on the other hand, are expanded into the dual dyad $(-\iota_A, o_A)$. In particular, spinor $\pi_{A'}$ to introduced shortly will have an expansion in the form

$$\pi_{A'} = \pi_{1'}\,\bar{o}_{A'} - \pi_{0'}\,\bar{\iota}_{A'}. \tag{5.3}$$

• Notice that ω^0 and ω^1 have spin weights $-1/2$ and $+1/2$, respectively, in the sense of section A.3.

Corresponding NP equations to (5.1) are then obtained by projections onto the spin basis, which yields differential equations for components ω^0 and ω^1, e.g.

$$\begin{aligned}
o_A o_B o^{A'} \nabla_{A'}^{A} \omega^B &= -o_B l^a \nabla_a (\omega^0 o^B + \omega^1 \iota^B) = -\omega^0 o_B D o^B - D\omega^1 - \omega^1 o_B D\iota^B \\
&= -D\omega^1 + \kappa\omega^0 + \varepsilon\omega^1, \\
o_A \iota_B o^{A'} \nabla_{A'}^{(A} \omega^{B)} &= \frac{1}{2}(o_A \iota_B + \iota_A o_B) o^{A'} \nabla_{A'}^{A} \omega^B = -\frac{1}{2}\iota_B D\omega^B - \frac{1}{2} o_B \overline{\delta}\omega^B \\
&= -\frac{1}{2}\left(D\omega^0 - \omega^0 \iota_B D o^B - \omega^1 \iota_B D\iota^B - \omega^0 o_B \overline{\delta} o^B - \overline{\delta}\omega^1 - o_B \overline{\delta}\iota^B \omega^1\right) \\
&= -\frac{1}{2}\left(D\omega^0 - \overline{\delta}\omega^1 + \varepsilon\omega^0 + \pi\omega^1 + \rho\omega^0 + \alpha\omega^1\right),
\end{aligned}$$

Where we used $l^a = o^A o^{A'}$, $\overline{m}^a = \iota^A o^{A'}$ and the definition of spin coefficients (A.3a). The other equations can be obtained in the same fashion. The full set of equations reads

$$\begin{aligned}
D\omega^1 &= \kappa\omega^0 + \varepsilon\omega^1, \\
\delta\omega^1 &= \sigma\omega^0 + \beta\omega^1, \\
-D\omega^0 + \overline{\delta}\omega^1 &= (\varepsilon + \rho)\omega^0 + (\alpha + \pi)\omega^1, \\
-\delta\omega^0 + \Delta\omega^1 &= (\beta + \tau)\omega^0 + (\mu + \gamma)\omega^1, \\
\overline{\delta}\omega^0 &= -\alpha\omega^0 - \lambda\omega^1, \\
\Delta\omega^0 &= -\gamma\omega^0 - \nu\omega^1.
\end{aligned} \tag{5.4}$$

In the case of the isolated horizon $\mathcal{H}$, some of the spin coefficients are zero on $\mathcal{H}$ and some are zero identically in the whole spacetime, namely

$$\kappa \overset{\mathcal{H}}{=} \rho \overset{\mathcal{H}}{=} \sigma \overset{\mathcal{H}}{=} \varepsilon - \overline{\varepsilon} \overset{\mathcal{H}}{=} 0, \quad \gamma = \nu = \tau = 0, \quad \alpha + \overline{\beta} = \pi, \quad \alpha - \overline{\beta} = a, \quad \mu = \overline{\mu}. \tag{5.5}$$

The system of equations (5.4) in the neighborhood of IH simplifies to

$$\begin{aligned}
D\omega^1 &= \kappa\omega^0 + \varepsilon\omega^1, && (5.6a) \\
\delta\omega^1 &= \sigma\omega^0 + \beta\omega^1, && (5.6b) \\
-D\omega^0 + \overline{\delta}\omega^1 &= (\varepsilon + \rho)\omega^0 + (\alpha + \pi)\omega^1, && (5.6c) \\
-\delta\omega^0 + \Delta\omega^1 &= \beta\omega^0 + \mu\omega^1, && (5.6d) \\
\overline{\delta}\omega^0 &= -\alpha\omega^0 - \lambda\omega^1, && (5.6e) \\
\Delta\omega^0 &= 0. && (5.6f)
\end{aligned}$$

Using the general commutation relations (A.4) subsequently on ω^0, ω^1 and employing (5.6) plus the Ricci identities (A.13) one arrives at integrability conditions for the twistor equation in a sufficiently small neighborhood of $\mathcal{H}$:

$$
\begin{aligned}
&[\Delta, D]\omega^0 : \\
&(\pi - a)\delta\omega^0 \overset{\mathcal{H}}{=} -\frac{1}{2}\omega^0(\pi\overline{\pi} - a\overline{a} - \pi\overline{a} + \overline{\pi}a + 2\Psi_2 + 4\Psi_3) - \omega^1(\lambda\overline{\pi} - \lambda\overline{a} + 3\Psi_3)\,, \qquad (5.7)\\
&[\Delta, D]\omega^1 : \\
&(\overline{\pi} - \overline{a})\overline{\delta}\omega^1 \overset{\mathcal{H}}{=} -\frac{1}{2}\omega^0(\overline{\pi} - \overline{a}) + \frac{1}{2}\omega^1\left((\overline{\pi} - \overline{a})(\pi - a) - 3\Psi_2\right)\,, \qquad (5.8)\\
&[\delta, \Delta]\omega^1 : \\
&\delta^2\omega^0 \overset{\mathcal{H}}{=} -\overline{\pi}\delta\omega^0 + \overline{\lambda}\,\overline{\delta}\omega^1 - \frac{1}{4}\omega^0(2\delta\overline{\pi} - 2\delta\overline{a} - \overline{\pi}^2 + \overline{a}^2 + \Phi_{02}) \\
&\qquad - \omega^1\left(\delta\mu + \frac{1}{2}\overline{\lambda}(\pi + a) + \overline{\pi}\mu + \Phi_{12}\right)\,. \qquad (5.9)
\end{aligned}
$$

The other commutators are trivially satisfied (e.g. $[D,\delta]\omega^1 \overset{\mathcal{H}}{=} 0$) or bring no new information, because the derivatives $\delta\omega^0, \overline{\delta}\omega^1$ can not be at the present stage expressed fully from (5.6) (as for instance $[\delta,\Delta]\omega^0$). On the other hand, the first two equations can be considered as supplementary ones to (5.6). The third one should be consistent with them, as the remaining commutators should be, consequently. Unfortunately, these equations are far too complicated to yield reasonable predictions. Hence, we leave them as they are and further restrict ourselves only to an axially symmetric horizon.

The twistor equation in a neighborhood of an isolated horizon can be written explicitly in coordinates using the coordinate expansion of directional derivatives (2.22):

$$
\begin{aligned}
\partial_v\omega^1 + U\partial_r\omega^1 + X^I\partial_I\omega^1 &= \kappa\omega^0 + \varepsilon\omega^1,\\
\Omega\partial_r\omega^1 + \xi^I\partial_I\omega^1 &= \sigma\omega^0 + \beta\omega^1,\\
\overline{\Omega}\partial_r\omega^1 + \overline{\xi}^I\partial_I\omega^1 - \partial_v\omega^0 - U\partial_r\omega^0 - X^I\partial_I\omega^0 &= (\varepsilon + \rho)\omega^0 + (\alpha + \pi)\omega^1,\\
\partial_r\omega^1 - \Omega\partial_r\omega^0 - \xi^I\partial_I\omega^0 &= \beta\omega^0 + \mu\omega^1,\\
\overline{\Omega}\partial_r\omega^0 + \overline{\xi}^I\partial_I\omega^0 &= -\alpha\omega^0 - \lambda\omega^1,\\
\partial_r\omega^0 &= 0.
\end{aligned}
\qquad (5.10)
$$

5.2 Axially symmetric case

Imposing axial symmetry of the spacetime, we can introduce coordinates (ζ, ϕ) on spherical cuts of $\mathcal{H}$ adapted to this symmetry in correspondence with the section 2.6. Axial symmetry eliminates ϕ-dependence of all quantities. Practically, we set $\partial_\phi F \equiv 0$ for any spacetime function F.

Let us first analyze the full set of equations (5.6) under this assumption. The simplest way to deal with our task is to find an expansion of the derivative ∂_ϕ in terms of the NP derivatives and coordinate components of NP vectors. Let us recall (2.22), or specifically

$$\delta = \Omega\partial_r + \xi^\zeta\partial_\zeta + \xi^\phi\partial_\phi, \quad \overline{\delta} = \overline{\Omega}\partial_r + \overline{\xi}^\zeta\partial_\zeta + \overline{\xi}^\phi\partial_\phi.$$

Using these two equations and $\Delta = \partial_r$ we derive

$$0 \overset{!}{=} \left(\overline{\xi}^\zeta \xi^\phi - \xi^\zeta \overline{\xi}^\phi\right) \partial_\phi = \overline{\xi}^\zeta \delta - \xi^\zeta \overline{\delta} - \left(\overline{\xi}^\zeta \Omega - \xi^\zeta \overline{\Omega}\right) \Delta, \quad \text{or}$$
$$\overline{\xi}^\zeta \delta - \xi^\zeta \overline{\delta} = \left(\overline{\xi}^\zeta \Omega - \xi^\zeta \overline{\Omega}\right) \Delta. \tag{5.11}$$

On an IH it reduces to

$$\xi^\zeta \overset{\mathcal{H}}{=} \overline{\xi}^\zeta, \quad \delta - \overline{\delta} \overset{\mathcal{H}}{=} 0.$$

For the sake of simplicity, we denote $\eta \equiv \xi/\overline{\xi}$, where $\xi \equiv \xi^\zeta$ (cf. (2.54)). Of course, $\eta \overset{\mathcal{H}}{=} 1$.

Substitution (5.11) to (5.6f) immediatelly gives

$$\left(\delta - \eta\overline{\delta}\right) \omega^0 = 0. \tag{5.12}$$

Further, we want to implement (5.11) into the equations (5.6). To do so, we multiply (5.6d) by $\overline{\xi}$ and (5.6e) by ξ. After adding the obtained equations and some simplification using (5.12), we arrive at

$$\Delta\omega^1 = (\beta - \eta\alpha)\omega^0 + (\mu - \eta\lambda)\omega^1.$$

In the same way, we multiply (5.6b) by $\overline{\xi}$, equation (5.6c) by $-\xi$ and (5.6d) by $\left(-\overline{\xi}^\zeta \Omega + \xi^\zeta \overline{\Omega}\right)$. After some manipulations, one gets

$$D\omega^0 = \left(\sigma - \eta(\varepsilon + \rho) + (\eta\overline{\Omega} - \Omega)(\beta - \eta\alpha)\right) \omega^0 + \left(\beta - \eta(\alpha + \pi) + (\eta\overline{\Omega} - \Omega)(\mu - \eta\lambda)\right) \omega^1.$$

The full set of equations (5.6) in the axially symmetric case is

$$D\omega^1 = \kappa\omega^0 + \varepsilon\omega^1, \tag{5.13a}$$
$$\delta\omega^1 = \sigma\omega^0 + \beta\omega^1, \tag{5.13b}$$
$$\begin{aligned} D\omega^0 = {} & \left(\sigma - \eta(\varepsilon + \rho) + (\eta\overline{\Omega} - \Omega)(\beta - \eta\alpha)\right) \omega^0 + \\ & + \left(\beta - \eta(\alpha + \pi) + (\eta\overline{\Omega} - \Omega)(\mu - \eta\lambda)\right) \omega^1, \end{aligned} \tag{5.13c}$$
$$\Delta\omega^1 = (\beta - \eta\alpha)\omega^0 + (\mu - \eta\lambda)\omega^1, \tag{5.13d}$$
$$\overline{\delta}\omega^0 = -\alpha\omega^0 - \lambda\omega^1, \tag{5.13e}$$
$$\Delta\omega^0 = 0. \tag{5.13f}$$

At first sight, it does not seem any simpler than the original equations. However, an advantage of this full form is that one can derive integrability conditions in terms of only ω^0, ω^1 since all derivatives are given as their linear combinations. In what follows, we do not analyze the integrability conditions in general, but we rather restrict ourselves at first only to the horizon, where $\Omega = 0, \eta = 1$ and analyze them afterwards. Moreover, there is no r-dependence of quantities on $\mathcal{H}$, so we are left just with

$$\begin{aligned} D\omega^1 &\overset{\mathcal{H}}{=} \varepsilon\omega^1, \\ \delta\omega^1 &\overset{\mathcal{H}}{=} \beta\omega^1, \\ D\omega^0 &\overset{\mathcal{H}}{=} -\varepsilon\omega^0 + (\beta - \alpha - \pi)\omega^1, \\ \delta\omega^0 &\overset{\mathcal{H}}{=} -\alpha\omega^0 - \lambda\omega^1. \end{aligned}$$

In order to solve this set of equations, we rewrite them in the coordinates (cf. (5.10)),

$$\partial_v \omega^1 \overset{\mathcal{H}}{=} \varepsilon \omega^1, \tag{5.14a}$$

$$\xi \partial_\zeta \omega^1 \overset{\mathcal{H}}{=} \beta \omega^1, \tag{5.14b}$$

$$\partial_v \omega^0 \overset{\mathcal{H}}{=} -\varepsilon \omega^0 + (\beta - \alpha - \pi)\omega^1, \tag{5.14c}$$

$$\xi \partial_\zeta \omega^0 \overset{\mathcal{H}}{=} -\alpha \omega^0 - \lambda \omega^1. \tag{5.14d}$$

5.2.1 Solving the twistor equation

The starting point of this section are the equations (5.14). The first thing one may think of is time independence of ω^0, ω^1. At first sight it seems reasonable, since all non-zero spin coefficients are time-independent on an IH (cf. theorems 9, 10 and 11). However, for non-extremal black holes ($\varepsilon \neq 0$) this assumption is consistent only with the trivial solution $\omega^0, \omega^1 \overset{\mathcal{H}}{=} 0$. Assuming $\partial_v \omega^{0,1} \overset{\mathcal{H}}{=} 0$ and $\varepsilon \neq 0$ the first equation (5.14a) immediately implies $\omega^1 \overset{\mathcal{H}}{=} 0$. Using this result in (5.14c) we get $\omega^0 \overset{\mathcal{H}}{=} 0$. So at this level, a solution of (5.14) in the non-extremal case *must* be time dependent. For the extremal black hole $\varepsilon \overset{\mathcal{H}}{=} 0$ only tangential projection of the twistor equation (5.14b), (5.14d) remain. This situation will be analyzed later.

Let us rather look at compatibility of this equations and what it implies. The spin coefficient ε is constant on $\mathcal{H}$ so the differentiation (5.14a) with respect to ζ yields

$$\partial_\zeta \partial_v \omega^1 = \varepsilon \partial_\zeta \omega^1 = \frac{\varepsilon\beta}{\xi}\omega^1.$$

On the other hand, differentiating (5.14b) with respect to v one gets

$$\xi \partial_v \partial_\zeta \omega^1 = \beta \partial_v \omega^1 = \beta\varepsilon\omega^1,$$

where we used $\partial_v \beta \overset{\mathcal{H}}{=} 0$. Since partial derivatives commute (on smooth functions), equating the two second derivatives one obtains trivially $0 = 0$. Thus (5.14a) and (5.14b) are compatible. Differentiation of (5.14c), (5.14d) respectively gives

$$\begin{aligned}\xi \partial_\zeta \partial_v \omega^0 &= \alpha\varepsilon\omega^0 + \omega^1(\lambda\varepsilon + \beta^2 - \alpha\beta - \pi\beta + \xi\partial_\zeta(\beta - \alpha - \pi))\\ \xi \partial_v \partial_\zeta \omega^0 &= \alpha\varepsilon\omega^0 + \omega^1(\alpha^2 + \alpha\pi - \alpha\beta - \lambda\varepsilon).\end{aligned}$$

Compatibility requires

$$\omega^1(2\lambda\varepsilon + \beta^2 - \alpha^2 - \pi\beta - \alpha\pi + \xi\partial_\zeta(\beta - \alpha - \pi)) = 0. \tag{5.15}$$

Since $\xi\partial_\zeta \overset{\mathcal{H}}{=} \delta$, the derivatives are governed by the Ricci identities (A.13h) and (A.13q)

$$\xi\partial_\zeta(\beta - \alpha - \pi) \overset{\mathcal{H}}{=} -\alpha\overline{\alpha} - \beta\overline{\beta} + 2\alpha\beta + \Psi_2 - \Lambda - \Phi_{11} - 2\varepsilon\lambda + 2\alpha\pi.$$

Substituting back to (5.15) and expressing everything in terms of π and a gives

$$\omega^1(\frac{1}{4}(\pi - \overline{\pi})^2 - a^2 + \Psi_2 - \Lambda - \Phi_{11}) = 0.$$

Obviously, we recognize two possibilities:

1. $\omega^1 = 0$

2. $\frac{1}{4}(\pi - \overline{\pi})^2 - a^2 + \Psi_2 - \Lambda - \phi_{11} = 0, \quad \wedge \quad \omega^1 \neq 0$

If the first one is satisfied, the twistor equation does not restrict spacetime geometry. The equations (5.14a), (5.14b) holds trivially, so we are left with

$$\partial_v \omega^0 \overset{\mathcal{H}}{=} -\varepsilon \omega^0, \tag{5.16a}$$

$$\xi \partial_\zeta \omega^0 \overset{\mathcal{H}}{=} -\alpha \omega^0. \tag{5.16b}$$

Simple integration yields solution

$$\omega^0(v, \zeta) \overset{\mathcal{H}}{=} \omega^0_{(0)}\, e^{-\varepsilon v - \int \mathrm{d}\tilde{\zeta} \frac{\alpha}{\xi}},$$

where integration runs through interval $[-1, \zeta]$ (cf. lemma 1) and $\omega^0_{(0)}$ is a (complex) integration constant. Furthermore, we can substitute for ξ, express α by (2.26) and use the explicit formula (2.56) for a in the integral

$$\int \mathrm{d}\tilde{\zeta} \frac{\alpha}{\xi} = \frac{1}{2} \int \mathrm{d}\tilde{\zeta} \frac{\pi + a}{\xi} = \frac{R}{\sqrt{2}} \int \mathrm{d}\tilde{\zeta} \frac{\pi}{\sqrt{f}} - \frac{1}{4} \int \mathrm{d}\tilde{\zeta} \frac{f'}{f} = \frac{R}{\sqrt{2}} \int \mathrm{d}\tilde{\zeta} \frac{\pi}{\sqrt{f}} - \ln\left(f^{1/4}\right).$$

Thus, the solution acquires the form

$$\omega^0(v, \zeta) \overset{\mathcal{H}}{=} \omega^0_{(0)} f^{1/4}\, e^{-\varepsilon v - \frac{R}{\sqrt{2}} \int \mathrm{d}\tilde{\zeta} \frac{\pi}{\sqrt{f}}}.$$

Clearly, it represents an exponential damping in time v, while in its spatial part a certain freedom in the form of λ via π remains. Recall that π is governed by the Ricci identity

$$\xi \partial_\zeta \pi \overset{\mathcal{H}}{=} 2\varepsilon\lambda - \pi^2 - \pi a. \tag{5.17}$$

On each slice v =const. ω^0 is constant. Expressing the integration constant in terms of solution on $\mathcal{S}^2_0$ yields

$$\omega^0(v, \zeta) \overset{\mathcal{H}}{=} \omega^0|_{\mathcal{S}_0}\, e^{-\varepsilon v}.$$

In the limit $v \to \infty$ we get $\omega^0 \to 0$. The time dependence of ω^0 would reflect itself in the Penrose's charge (4.22). Its physical interpretation is not straightforward, especially when the solution decays with time. However, in the case with $\omega^1 = 0$ the charge is automatically zero, because on every IH it holds $\Phi_{01} \overset{\mathcal{H}}{=} \Psi_1 \overset{\mathcal{H}}{=} 0$.

We stay with this integrability condition for a little more time and look at the special case of extremal horizon, when $\varepsilon \overset{\mathcal{H}}{=} 0$. The explicit form of π allows us to evaluate ω^0. The corresponding integral is

$$-\frac{R}{\sqrt{2}} \int_{-1}^{\zeta} \mathrm{d}\tilde{\zeta} \frac{\pi}{\sqrt{f}} = -\frac{1}{2} \int_{-1}^{\zeta} \mathrm{d}\tilde{\zeta} \frac{1}{\tilde{\zeta} + c_\pi} = -\frac{1}{2} \ln\left(\frac{\zeta + c_\pi}{c_\pi - 1}\right).$$

Thus,

$$\omega^0(\zeta) \overset{\mathcal{H}}{=} \omega^0_0 f^{1/4} \sqrt{\frac{c_\pi - 1}{\zeta + c_\pi}}. \tag{5.18}$$

The second condition of integrability, when $\omega^1 \neq 0$, is a direct restriction on spacetime, since a, π are (at least in principle) known functions.

In the case of electrovacuum spacetimes the projections of the Ricci tensor are simply $\Phi_{ab} = \phi_a \overline{\phi_b}$ and $\Lambda = 0$. Assuming this situation we are left with

$$\frac{1}{4}(\pi - \overline{\pi})^2 - a^2 + \Psi_2 - |\phi_1|^2 = 0.$$

Taking real and imaginary parts separately we have

$$\mathrm{Re}: \quad -\pi_I^2 - a^2 + \mathrm{Re}\{\Psi_2\} - |\phi_1|^2 \overset{\mathcal{H}}{=} 0, \tag{5.19}$$

$$\mathrm{Im}: \quad \mathrm{Im}\{\Psi_2\} \overset{\mathcal{H}}{=} 0, \tag{5.20}$$

where $\pi = \pi_R + \mathrm{i}\pi_I$. However, we already know, that the real and imaginary part of Ψ_2 are determined by (2.36). In our case of axial symmetry these are reduced to

$$\mathrm{Re}\{\Psi_2\} \overset{\mathcal{H}}{=} a^2 - \xi a' + |\phi_1|^2$$
$$\mathrm{Im}\{\Psi_2\} \overset{\mathcal{H}}{=} -\mathrm{Im}\{\xi\pi' + \xi'\pi\}.$$

Taking into account (5.20) the second equation can be solved,

$$\mathrm{Im}\{(\xi\pi)'\} = 0 \quad \Rightarrow \quad \pi_I = \frac{K_I}{\xi} = K_I R\sqrt{\frac{2}{f}},$$

with some real integration constant K_I. With the knowledge of π_I the equation (5.19) is the aforementioned restriction on spacetime geometry. Substituting $\mathrm{Re}\{\Psi_2\}$ into (5.19) gives

$$-\pi_I^2 + \xi\xi'' = 0, \quad \text{or} \quad \xi'' = K_I^2\xi^{-3}, \tag{5.21}$$

which is the equation for f. This equation has form

$$y''(x) = Ax^n y^m,$$

for a function $y(x)$ and some constant A, and can be found in tables [36] under the name Emden-Fowler equation. Solution for $n = 0, m \neq -1$ in general is

$$x = \pm \int \mathrm{d}y \left(\frac{2A}{1+m}y^{1+m} + C_1\right)^{-1/2} + C_2,$$

where $C_1, C_2 \in \mathbb{R}$ are integration constants. In our case $m = -3$, so after a simple calculation one arrives at

$$y = \sqrt{\frac{A}{C_1} + C_1(x - C_2)^2},$$

or in terms of our variables

$$\xi = \sqrt{\frac{K_I^2}{C_1} + C_1(\zeta - C_2)^2}, \quad \text{or} \quad f = 2R^2\left(\frac{K_I^2}{C_1} + C_1(\zeta - C_2)^2\right).$$

The function f has to vanish at the poles, where $\zeta = \pm 1$, and its derivative has to be $f' \underset{\zeta\to\pm1}{\longrightarrow} \mp 2$ which fixes the integration constants C_1, C_2:

$$\begin{aligned} 0 \stackrel{!}{=} f(\pm 1) &= 2R^2\left(\frac{K_I^2}{C_1} + C_1(\pm 1 - C_2)^2\right), \\ \mp 2 \stackrel{!}{=} f'(\pm 1) &= 4R^2 C_1(\pm 1 - C_2). \end{aligned}$$

The second condition implies $C_1 = -1/(2R^2), C_2 = 0$. When we plug it into the first one, we find $K_I^2 = -1/(2R^2)$. However, K_I has to be real, contradicting our result. On the other hand, the requirement imposed on the derivative f', which expresses the absence of conical singularities, can be relaxed. Then, the first two equations fix C_1, C_2, while the second two fix deficit angles at the poles. For C_1, C_2 we get

$$C_2 = 0, \quad C_1^2 = -K_I^2 \quad \Rightarrow \quad C_1 = K_I = 0.$$

For $K_I = 0$ the imaginary part of π is zero. The equation (5.21) transforms to

$$\xi'' = 0 \quad \Rightarrow \quad \xi = U\zeta + V, \quad \text{or} \quad f = 2R^2(U\zeta + V), \quad U, V \in \mathbb{R}.$$

Vanishing of f at poles gives

$$\left.\begin{aligned} 0 \stackrel{!}{=} f(1) = 2R^2(U+V)^2, \\ 0 \stackrel{!}{=} f(-1) = 2R^2(-U+V)^2 \end{aligned}\right\} \Rightarrow U = V = 0.$$

Thus, introduction of conical singularities is not sufficient for the existence of a non-trivial solution.

To conclude our computation, we state a summarizing lemma.

Lemma 3 *Let the time dependent projections of a twistor equation* (5.14) *for univalent spinor ω^A be given on a non-extremal axially symmetric isolated horizon $(\mathcal{H}, q_{ab})$. Then, the only non-trivial solution is*

$$\omega^0(v,\zeta) \stackrel{\mathcal{H}}{=} \omega^0_{(0)} f^{1/4}\, e^{-\varepsilon v - \frac{R}{\sqrt{2}}\int \mathrm{d}\tilde{\zeta}\frac{\pi}{\sqrt{f}}}, \qquad \omega^1(v,\zeta) = 0.$$

Other possibilities, where $\omega^1 \neq 0$ are excluded by integrability conditions of the twistor equation. Moreover, this solution yields zero Penrose's charge.

5.2.2 Solving the twistor equation – tangential projections

As we saw in the previous section, time dependence of the twistor equation has only solutions that yield zero Penrose charge. Hence, time dependence itself is incompatible with physical requirements, so in what follows we proceed in the way originally suggested by Penrose – we look only at the tangential projections of the twistor equation to a 2-surface whose mass has to be found. Resulting set of equations is known as the *2-surface twistor equation.* Concerning an isolated horizon, the surface is the two sphere $\mathcal{S}_0$. A basis of its tangent vectors can be taken as $m^a, \overline{m}^a$ as discussed at the end of the section 2.6. We already have the required equations at hand. They are simply

$$\xi \partial_\zeta \omega^1 \overset{\mathcal{H}}{=} \beta \omega^1, \tag{5.22}$$

$$\xi \partial_\zeta \omega^0 \overset{\mathcal{H}}{=} -\alpha \omega^0 - \lambda \omega^1. \tag{5.23}$$

Analysis of these equations is analogous to the one that was already done. But now, they are fully independent, so no conditions of integrability need to be verified.

Both equations (5.22), (5.23) are linear differential equations, the second one in addition with the right hand side. The general solution reads

$$\begin{aligned}
\omega^1(\zeta) &= \omega^1_0 e^{\int \mathrm{d}\zeta \, \frac{\beta}{\xi}}, \\
\omega^0(\zeta) &= \left(\omega^0_0 - \int \mathrm{d}\zeta \, \frac{\lambda \omega^1}{\xi} e^{\int \mathrm{d}\zeta \, \frac{\alpha}{\xi}} \right) e^{-\int \mathrm{d}\zeta \, \frac{\alpha}{\xi}}.
\end{aligned}$$

Using relations (2.26) one can simplify the integrals in the exponents:

$$\int \mathrm{d}\zeta \, \frac{\beta}{\xi} = \frac{1}{2} \int \mathrm{d}\zeta \, \frac{\overline{\pi}}{\xi} + \ln(\xi^{1/2}), \quad -\int \mathrm{d}\zeta \, \frac{\alpha}{\xi} = -\frac{1}{2} \int \mathrm{d}\zeta \, \frac{\pi}{\xi} + \ln(\xi^{1/2}),$$

then ω^1, ω^0 can be recast into[1]

$$\begin{aligned}
\omega^1(\zeta) &= \omega^1_0 f^{1/4} e^{\frac{1}{2} \int \mathrm{d}\zeta \, \frac{\overline{\pi}}{\xi}}, \\
\omega^0(\zeta) &= \omega^0_0 \left(1 - C \int \mathrm{d}\zeta \, \frac{\lambda \omega^1}{\xi} f^{-1/4} e^{\frac{1}{2} \int \mathrm{d}\zeta \, \frac{\pi}{\xi}} \right) f^{1/4} e^{-\frac{1}{2} \int \mathrm{d}\zeta \, \frac{\pi}{\xi}}.
\end{aligned} \tag{5.24}$$

To work out explicit solution, one has to provide the spin coefficient λ, which represents free data on an IH.

Let us for simplicity consider a situation, where $\lambda = 0$ (e.g. Schwarzschild spacetime). The spin coefficient π is governed by the same equation as in the case of extremal horizons, so it admits solution (3.2). Substituting to (5.24) after a simple calculation we obtain

$$\begin{aligned}
\omega^1(\zeta) &= \omega^1_0 f^{1/4}(\zeta) \sqrt{\zeta + \overline{c}_\pi}, \\
\omega^0(\zeta) &= \frac{\omega^0_0 f^{1/4}(\zeta)}{\sqrt{\zeta + c_\pi}}.
\end{aligned} \tag{5.25}$$

If we add a condition of extremality $\varkappa_{(\ell)} = 0$, the function $f(\zeta)$ is unique and has the closed form (3.7). To conclude, for extremal horizons with $\lambda = 0$ it is possible to find an explicit solution of the 2-surface twistor equation.

We should add a final remark here. In order to solve the system even in the case when $\lambda \neq 0$ one is forced to decompose λ and μ into the spherical harmonics $Y^0_n(\zeta)$ similarly to multipole decomposition of the isolated horizon geometry introduced in the section 2.8. One would then expect to obtain some form of Penrose mass multipoles. However, the problem turns out to be a little more difficult then it appears and will be examined in the future work.

[1] Here we hide all constants to ω^1_0, ω^0_0 and C, but C is in fact fully determined by the first two.

5.3 The 2-surface twistor space

Without going into much details, let us return to general discussion about solutions of the twistor equation at the end of the section 4.3.

The twistor equation was originally introduced by Penrose in order to find explicit solutions for equations of motion of fields with arbitrary spin in terms of the contour integrals which are today known as *Penrose transform* [37]. Notice that (5.1) is equivalent to

$$\nabla^{A}_{A'}\omega^{B} = -\mathrm{i}\,\epsilon^{AB}\,\pi_{A'}, \tag{5.26}$$

for some spinor $\pi_{A'}$. In the flat spacetime, this equation admits a solution in the form

$$\omega^{A} = \omega^{A}_{0} - \mathrm{i}\,x^{AA'}\,\pi_{A'},$$

where ω^{A}_{0} is constant spinor playing the role of an integration constant and $\pi_{A'}$ is also necessarily constant. Hence, the space of solutions of the twistor equation can be coordinatized by two constant spinors . By a *twistor* we mean the pair

$$\mathsf{Z}^{\alpha} \equiv (\omega^{A}, \pi_{A'}),$$

where spinors ω^{A} and $\pi_{A'}$ transform under a translation of the origin of the coordinates by

$$\omega^{A} \mapsto \omega^{A} - \mathrm{i}\,x^{AA'}\,\pi_{A'}, \quad \pi_{A'} \mapsto \pi_{A'}.$$

The dual twistor to Z^{α} is defined

$$\mathsf{Z}_{\alpha} \equiv (\omega_{A}, \pi^{A'}).$$

The space of the solutions of (4.23) in the form of twistors has natural structure of a vector space. For that reason it will be referred to as the *twistor space*. Moreover, we can introduce the *norm*

$$\Sigma = \mathsf{Z}^{\alpha}\,\bar{\mathsf{Z}}_{\alpha} = \omega^{A}\,\bar{\pi}_{A} + \bar{\omega}^{A'}\,\pi_{A'} \equiv 2\,s, \tag{5.27}$$

where s can be interpreted as the helicity of a massless field [34, 38, 39] and complex conjugate twistor is defined by $\overline{\mathsf{Z}^{\alpha}} \equiv \bar{\mathsf{Z}}_{\alpha} = (\overline{\pi}_{A}, \overline{\omega}^{A'})$. In the flat spacetime, Σ is constant throughout the spacetime, i.e. it is invariant under the shift of the origin. Thus, Σ is a well-defined norm on the twistor space.

Projecting (5.26) onto NP tetrad adapted to IH, analogously to the section 5.1, one finds

$$\delta\omega^{1} = \beta\,\omega^{1}, \qquad \delta\omega^{0} = -\alpha\,\omega^{0} - \lambda\,\omega^{1}, \tag{5.28}$$

$$\delta\omega^{1} = \alpha\,\omega^{1} - \mathrm{i}\,\pi_{0'}, \qquad \delta\omega^{0} = -\beta\,\omega^{0} - \mu\,\omega^{1} - \mathrm{i}\,\pi_{1'}, \tag{5.29}$$

where we already imposed axial symmetry and used the relations (5.5). The former pair of equations determines spinor ω^{A}, while the latter pair can be regarded as a definition of $\pi_{A'}$. As shown in [35], the charge integral (4.22) can be rewritten in terms of $\pi_{A'}$ as

$$Q = -\frac{\mathrm{i}}{4\,\pi\,G}\oint_{\mathcal{S}^2_0}\left(\pi^{(1)}_{0'}\,\pi^{(2)}_{1'} + \pi^{(1)}_{1'}\,\pi^{(2)}_{0'}\right)\varepsilon,$$

where superscripts (1) and (2), respectively, denote two independent solutions of the 2-surface twistor equation.

The existence of solutions of elliptic system (5.28) is a delicate question and requires deeper topological considerations which we only briefly sketch here. For details, see [35, 37, 32, 30, 31, 29].

The operators appearing in (5.28) together form the so-called 2-twistor operator which is one of the irreducible parts of the Sen connection adapted to a topological 2-sphere (the other irreducible part being the Witten operator which can be used to prove several energy positivity theorems [40]). This operator can be regarded as a mapping between bundles of quantities with appropriate spin weights, schematically,

$$T : E \mapsto F. \tag{5.30}$$

Roughly speaking, E is the bundle on which the components of ω^A live, while F is the space where the image of the 2-twistor operator lives. We can choose an arbitrary Hermitian inner product on the product bundle $E \otimes F$, then one can define the adjoint operator $T^\dagger : F \mapsto E$. The solution of the twistor equation belongs to $\ker T$, the kernel of T. Similarly, we introduce the co-kernel $\operatorname{coker} T$ as the kernel of the adjoint $T^\dagger$. Finally, the *index* of the operator T is defined as

$$\operatorname{ind} T = \dim \ker T - \dim \operatorname{coker} T. \tag{5.31}$$

In other words, index gives the difference of number of independent solutions of the 2-twistor equation and the adjoint equation $T^\dagger \omega = 0$.

One of the most important results in mathematical physics is called the *Atiyah-Singer index theorem*, see, e.g. [41] for a slight introduction. Loosely speaking, the theorem asserts that index of an elliptic operator is a topological invariant and does not depend neither on the metric or the connection. Index of the twistor operator has been calculated by Baston [42] who has shown that

$$\operatorname{ind} T = 4(1 - g), \tag{5.32}$$

where g is the genus of the 2-surface ($g = 0$ for the sphere). Hence, if the kernel of $T^\dagger$ is trivial, 2-twistor equation has exactly 4 solutions which can be identified with the components of the 4-momentum. This is indeed a generic case, i.e. it holds in many physically relevant situations. Nevertheless, it is not true in general and if the $\operatorname{coker} T$ is not trivial, twistor equation has too many solutions.

For the extraction of components of the 4-momentum from the Penrose charge integral (4.22), one needs an inner product on the twistor space. Such inner product is naturally provided by the twistor norm (5.27) which is constant in the Minkowski space, but not in general curved spacetime. If the 2-surface with spherical topology can be embedded into a flat spacetime, while preserving its intrinsic and extrinsic curvatures, it is called *non-contorted*. Therefore, for a non-contorted surface one can use the flat spacetime twistor norm to defined the Penrose mass and the Atiyah-Singer index theorem guarantees that the set of solutions of the twistor equation is the same for both the original non-contorted surface and its embedding.

The situation is much less clear for contorted surfaces which do not admit such embedding. There various suggestions how to circumvent the failure of original Penrose's construction in this case, see [29] for a general review and [43] for technical details and further references.

In our setting, we aimed to investigate the relation between Penrose's mass (and its modifications) to Ashtekar multipole moments, since we expect they both describe the same properties of black hole (mass, angular momentum). Because of the complexity of the underlying theory and difficulty of calculations, we did not finish this part of the project. As a partial result we have derived the twistor norm for axisymmetric isolated horizons in the form

$$\Sigma = \mathrm{i}(\bar{\pi} - \pi) \left(\omega^0 \,\overline{\varpi}^{1'} + \omega^1 \,\overline{\varpi}^{0'} \right) + \mathrm{i}\, |\omega^1|^2 \left(\overline{\lambda} - \lambda \right). \tag{5.33}$$

This is a surprising and confusing result. Indeed, for a Schwarzschild black hole, where the Penrose's construction is known to work, this reduces to zero. Hence, it seems impossible to use this norm in order to have a well-defined inner product on the space of 2-twistors. Perhaps, the reason is the degeneracy of the horizon (being a null hypersurface), while the original calculation of the mass by Tod was based on spheres of symmetry surrounding the black hole, including singularity and the horizon. For the calculation there, standard Schwarzschild coordinates have been employed which do not penetrate the horizon and therefore these surfaces do not feel the degeneracy of the induced metric on the horizon. Moreover, the Penrose mass is known to give standard Schwarzschild mass for sphere surrounding the singularity and zero otherwise. The only sign of the presence of singularity in our formalism is the spin coefficient μ describing the expansion of ingoing null congruence n^a. In order that the surface be trapped, μ must be strictly negative. However, from our expression for the norm, the spin coefficient μ has dropped out. In other words, another possibility why we obtain degenerate norm might be that our 2-surface actually does not feel the presence of singularity. We leave better understanding of this situation for future work.

6 | Conclusion

The recently developed formalism of isolated horizons was an important tool throughout our work. Therefore a significant part of the thesis was dedicated to the study of necessary notions and existing results. We started with the basic definitions of isolated horizons, which provide a quasi-local generalization of even horizons and describe black holes in an equilibrium. Instead of detailed calculation and proofs of theorems we stated their precise formulation in order to make our assumptions clear. The corresponding details can be found in the cited literature. Construction of adapted coordinates and the choice of a Newman-Penrose tetrad is especially emphasized since it is important for understanding of introduced ideas. Further we merely restricted ourselves to the axially symmetric horizons characterized by the presence of an axial Killing field ϕ^a. This property was used to define a preferred coordinate system in an invariant way. The general discussion ends with the definition of multipole moments of the isolated horizon, which provide an equivalent characterization of its (intrinsic) geometry. Moreover, they can be given the interpretation similar to the standard mass and angular multipoles.

In the subsequent chapter we stayed a little more time with isolated horizons and looked at a special case of the extremal horizon defined by vanishing of its surface gravity $\varkappa_{(\ell)}$. We showed, following [13], that the electromagnetic field and the spin coefficient π entering the game are fully determined. In addition, we were also able to prove the uniqueness of the geometry, represented by the metric function f, in this setting. It is a non-trivial result and a generalization of [6]. The function f turned out to depend on 5 parameters: radius (representing the mass), electric and magnetic charges and two deficit angles, which might be connected with the acceleration parameter of the C-metric or a NUT parameter. Precise interpretation and connection to the known exact spacetimes of the Einstein's theory of relativity will be studied in the future.

Afterwards, we turned our attention to the construction of the Penrose charge. A short motivation was included to make the basic ideas behind this construction more familiar to the reader. Since it was one of our main interests, more detailed calculations accompany the main text. Here, we also briefly explained the connection to the Nœther currents associated with energy-momentum tensor.

After the derivation of explicit formula for the Penrose charge we proceeded to solving the twistor equation on the isolated horizon that turn out to be the crucial point. We worked out the general equations governing this problem and discussed integrability conditions for them. In what followed we again restricted ourselves to the case of axial symmetry and found an explicit time-dependent solution. However, this solution yielded just a zero Penrose charge and hence was not suitable to construct mass or angular momentum. Thus, we chose to follow the initial Penrose's idea to consider only tangential projections of the twistor equation to a surface which mass has to be be found. Here, we gave the general result and, as and example, calculated the solution fully in the case, where the spin coefficient has an explicit form.

We concluded our work by a discussion of possible solutions of the Penrose charge and problems that appeared during our investigation.

A | Newman-Penrose formalism

A.1 Gravitational field

We offer a brief summary of definitions and equations which are used in the Newmann-Penrose formalism.
Directional derivatives:

$$D = \ell^a \nabla_a, \qquad \Delta = n^a \nabla_a, \qquad \delta = m^a \nabla_a, \qquad \bar{\delta} = \bar{m}^a \nabla_a. \tag{A.1}$$

Decomposition of covariant derivative:

$$\nabla_a = g_a^b \nabla_b = \ell_a \, \Delta + n_a \, D - m_a \, \bar{\delta} - \bar{m}_a \, \delta. \tag{A.2}$$

Spin coefficients:

$$\begin{aligned}
\kappa &= m^a D\ell_a = o^A D o_A, & \tau &= m^a \Delta \ell_a = o^A \Delta o_A, \\
\sigma &= m^a \delta \ell_a = o^A \delta o_A, & \rho &= m^a \bar{\delta} \ell_a = o^A \bar{\delta} o_A, \\
\pi &= n^a D \bar{m}_a = \iota^A D \iota_A, & \nu &= n^a \Delta \bar{m}_a = \iota^A \Delta \iota_A, \\
\lambda &= n^a \bar{\delta} \bar{m}_a = \iota^A \bar{\delta} \iota_A, & \mu &= n^a \delta \bar{m}_a = \iota^A \delta \iota_A, \\
\varepsilon &= \frac{1}{2} \left[n^a D \ell_a - \bar{m}^a D m_a \right] = \iota^A D o_A, & \beta &= \frac{1}{2} \left[n^a \delta \ell_a - \bar{m}^a \delta m_a \right] = \iota^A \delta o_A, \\
\gamma &= \frac{1}{2} \left[n^a \Delta \ell_a - \bar{m}^a \Delta m_a \right] = \iota^A \Delta o_A, & \alpha &= \frac{1}{2} \left[n^a \bar{\delta} \ell_a - \bar{m}^a \bar{\delta} m_a \right] = \iota^A \bar{\delta} o_A,
\end{aligned} \tag{A.3a}$$

The operators (A.1) acting on a scalar function obey this commutation relations:

$$D\delta - \delta D = (\bar{\pi} - \bar{\alpha} - \beta) D - \kappa \Delta + (\bar{\rho} - \bar{\varepsilon} + \varepsilon) \delta + \sigma \bar{\delta}, \tag{A.4a}$$

$$\Delta D - D\Delta = (\gamma + \bar{\gamma}) D + (\varepsilon + \bar{\varepsilon}) \Delta - (\bar{\tau} + \pi) \delta - (\tau + \bar{\pi}) \bar{\delta}, \tag{A.4b}$$

$$\Delta \delta - \delta \Delta = \bar{\nu} D + (\bar{\alpha} + \beta - \tau) \Delta + (\gamma - \bar{\gamma} - \mu) \delta - \bar{\lambda} \bar{\delta}, \tag{A.4c}$$

$$\delta \bar{\delta} - \bar{\delta} \delta = (\mu - \bar{\mu}) D + (\rho - \bar{\rho}) \Delta + (\bar{\alpha} - \beta) \bar{\delta} - (\alpha - \bar{\beta}) \delta. \tag{A.4d}$$

Transport equations:

$$
\begin{aligned}
D\ell^a &= (\varepsilon+\bar\varepsilon)\,\ell^a - \bar\kappa\, m^a - \kappa\,\bar m^a, && \text{(A.5a)}\\
\Delta\ell^a &= (\gamma+\bar\gamma)\,\ell^a - \bar\tau\, m^a - \tau\,\bar m^a, && \text{(A.5b)}\\
\delta\ell^a &= (\bar\alpha+\beta)\,\ell^a - \bar\rho\, m^a - \sigma\,\bar m^a, && \text{(A.5c)}\\
Dn^a &= -\,(\varepsilon+\bar\varepsilon)\,n^a + \pi\, m^a + \bar\pi\,\bar m^a, && \text{(A.5d)}\\
\Delta n^a &= -\,(\gamma+\bar\gamma)\,n^a + \nu\, m^a + \bar\nu\,\bar m^a, && \text{(A.5e)}\\
\delta n^a &= -(\bar\alpha+\beta)n^a + \mu\, m^a + \bar\lambda\,\bar m^a, && \text{(A.5f)}\\
Dm^a &= \bar\pi\,\ell^a - \kappa\, n^a + (\varepsilon-\bar\varepsilon)\,m^a, && \text{(A.5g)}\\
\Delta m^a &= \bar\nu\,\ell^a - \tau\, n^a + (\gamma-\bar\gamma)\,m^a, && \text{(A.5h)}\\
\delta m^a &= \bar\lambda\,\ell^a - \sigma\, n^a + (\beta-\bar\alpha)\,m^a, && \text{(A.5i)}\\
\bar\delta m^a &= \bar\mu\,\ell^a - \rho\, n^a + \left(\alpha-\bar\beta\right)m^a. && \text{(A.5j)}
\end{aligned}
$$

The Riemann tensor can be decomposed as follows:

$$
\begin{aligned}
R_{abcd} = {}& \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar\Psi_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}\\
&+ \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \bar\Phi_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'}\\
&+ 2\Lambda\left(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'}\right)
\end{aligned} \tag{A.6}
$$

The first part is the the totally symmetric Weyl spinor Ψ_{ABCD}, which corresponds to the Weyl tensor C_{abcd}. The symmetric Ricci spinor $\Phi_{ABC'D'}$ is equivalent to the trace-free part of the Ricci tensor and the scalar Λ is related to the scalar curvature R by

$$\Lambda = \frac{1}{24}R. \tag{A.7}$$

The five (complex) tetrad components of the Weyl spinor are

$$
\begin{aligned}
\Psi_0 &= C_{abcd}l^a m^b l^c m^d &&= \Psi_{ABCD}\, o^A o^B o^C o^D, && \text{(A.8a)}\\
\Psi_1 &= C_{abcd}l^a n^b l^c m^d &&= \Psi_{ABCD}\, o^A o^B o^C \iota^D, && \text{(A.8b)}\\
\Psi_2 &= C_{abcd}l^a m^b \bar m^c n^d &&= \Psi_{ABCD}\, o^A o^B \iota^C \iota^D, && \text{(A.8c)}\\
\Psi_3 &= C_{abcd}l^a n^b \bar m^c n^d &&= \Psi_{ABCD}\, o^A \iota^B \iota^C \iota^D, && \text{(A.8d)}\\
\Psi_4 &= C_{abcd}\bar m^a n^b \bar m^c n^d &&= \Psi_{ABCD}\, \iota^A \iota^B \iota^C \iota^D. && \text{(A.8e)}
\end{aligned}
$$

Then the spinor basis expansion of Ψ_{ABCD} is

$$
\begin{aligned}
\Psi_{ABCD} = {}& \Psi_4 o_A o_B o_C o_D - \Psi_3(o_A o_B o_C \iota_D + o_A o_B \iota_C o_D + o_A \iota_B o_C o_D + \iota_A o_B o_C o_D)\\
&+ \Psi_2(o_A o_B \iota_C \iota_D + o_A \iota_B o_C \iota_D + \iota_A o_B o_C \iota_D + o_A \iota_B \iota_C o_D + \iota_A o_B \iota_C o_D\\
&+ \iota_A \iota_B o_C o_D) - \Psi_1(o_A \iota_B \iota_C \iota_D + \iota_A o_B \iota_C \iota_D + \iota_A \iota_B o_C \iota_D + \iota_A \iota_B \iota_C o_D)\\
&+ \Psi_0 \iota_A \iota_B \iota_C \iota_D
\end{aligned} \tag{A.9}
$$

The trace-less Ricci tensor has the following components, which are 3 real and 3 complex:

$$
\begin{aligned}
\Phi_{00} &= -\frac{1}{2}R_{ab}l^a l^b && = \Phi_{ABA'B'}o^A o^B \bar{o}^{A'}\bar{o}^{B'}, && \text{(A.10a)}\\
\Phi_{01} &= -\frac{1}{2}R_{ab}l^a m^b && = \Phi_{ABA'B'}o^A o^B \bar{o}^{A'}\bar{\iota}^{B'}, && \text{(A.10b)}\\
\Phi_{02} &= -\frac{1}{2}R_{ab}m^a m^b && = \Phi_{ABA'B'}o^A o^B \bar{\iota}^{A'}\bar{\iota}^{B'}, && \text{(A.10c)}\\
\Phi_{11} &= -\frac{1}{4}R_{ab}\left(l^a n^b + m^a \bar{m}^b\right) && = \Phi_{ABA'B'}o^A \iota^B \bar{o}^{A'}\bar{\iota}^{B'}, && \text{(A.10d)}\\
\Phi_{12} &= -\frac{1}{2}R_{ab}n^a m^b && = \Phi_{ABA'B'}o^A \iota^B \bar{\iota}^{A'}\bar{\iota}^{B'}, && \text{(A.10e)}\\
\Phi_{22} &= -\frac{1}{2}R_{ab}n^a n^b && = \Phi_{ABA'B'}\iota^A \iota^B \bar{\iota}^{A'}\bar{\iota}^{B'}. && \text{(A.10f)}
\end{aligned}
$$

The three remaining components can be obtained via the condition $\Phi_{ij} = \bar{\Phi}_{ji}$. Spinor basis expansion of $\Phi_{ABC'D'}$ is

$$
\begin{aligned}
\Phi_{ABC'D'} = {} & \Phi_{22}\, o_A o_B \bar{o}_{C'}\bar{o}_{D'} - \Phi_{12}\,(o_A\iota_B + \iota_A o_B)\bar{o}_{C'}\bar{o}_{D'} + \Phi_{02}\, \iota_A\iota_B\bar{o}_{C'}\bar{o}_{D'}\\
& + \Phi_{21}\, o_A o_B(\bar{o}_{C'}\bar{\iota}_{D'} + \bar{\iota}_{C'}\bar{o}_{D'}) + \Phi_{11}\,(o_A\iota_B + \iota_A o_B)(\bar{o}_{C'}\bar{\iota}_{D'} + \bar{\iota}_{C'}\bar{o}_{D'})\\
& - \Phi_{01}\, \iota_A\iota_B(\bar{o}_{C'}\bar{\iota}_{D'} + \bar{\iota}_{C'}\bar{o}_{D'}) + \Phi_{20}\, o_A o_B\bar{\iota}_{C'}\bar{\iota}_{D'}\\
& - \Phi_{10}\,(o_A\iota_B + \iota_A o_B)\bar{\iota}_{C'}\bar{\iota}_{D'} + \Phi_{00}\iota_A\iota_B\bar{\iota}_{C'}\bar{\iota}_{D'}
\end{aligned}
\tag{A.11}
$$

The Ricci identities in the spinor formalism:

$$
\begin{aligned}
\nabla_{A'(A}\nabla^{A'}_{B)}\xi_C &= \Psi_{ABCD}\xi^D - 2\Lambda\xi_{(A}\epsilon_{B)C},\\
\nabla_{A(A'}\nabla^{A}_{B')}\xi_C &= \Phi_{CDA'B'}\xi^D.
\end{aligned}
\tag{A.12}
$$

Projections onto the spin basis read:

$$D\rho - \bar{\delta}\kappa = \rho^2 + (\epsilon + \bar{\epsilon})\,\rho - \kappa\left(3\alpha + \bar{\beta} - \pi\right) - \tau\bar{\kappa} + \sigma\bar{\sigma} + \Phi_{00}, \tag{A.13a}$$

$$D\sigma - \delta\kappa = (\rho + \bar{\rho} + 3\varepsilon - \bar{\varepsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0, \tag{A.13b}$$

$$D\tau - \Delta\kappa = \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + (\varepsilon - \bar{\varepsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 + \Phi_{01}, \tag{A.13c}$$

$$D\alpha - \bar{\delta}\varepsilon = (\rho + \bar{\varepsilon} - 2\varepsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\varepsilon - \kappa\lambda - \bar{\kappa}\gamma + (\varepsilon + \rho)\pi + \Phi_{10}, \tag{A.13d}$$

$$D\beta - \delta\varepsilon = (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\varepsilon})\beta - (\mu + \gamma)\kappa - (\bar{\alpha} - \bar{\pi})\varepsilon + \Psi_1, \tag{A.13e}$$

$$D\gamma - \Delta\varepsilon = (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta - (\varepsilon + \bar{\varepsilon})\gamma - (\gamma + \bar{\gamma})\varepsilon + \tau\pi - \nu\kappa + \Psi_2 - \Lambda + \Phi_{11}, \tag{A.13f}$$

$$D\lambda - \bar{\delta}\pi = (\rho - 3\varepsilon + \bar{\varepsilon})\lambda + \bar{\sigma}\mu + (\pi + \alpha - \bar{\beta})\pi - \nu\bar{\kappa} + \Phi_{20}, \tag{A.13g}$$

$$D\mu - \delta\pi = (\bar{\rho} - \varepsilon - \bar{\varepsilon})\mu + \sigma\lambda + (\bar{\pi} - \bar{\alpha} + \beta)\pi - \nu\kappa + \Psi_2 + 2\Lambda, \tag{A.13h}$$

$$D\nu - \Delta\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\pi - (3\varepsilon + \bar{\varepsilon})\nu + \Psi_3 + \Phi_{21}, \tag{A.13i}$$

$$\Delta\lambda - \bar{\delta}\nu = -(\mu + \bar{\mu} + 3\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_4, \tag{A.13j}$$

$$\Delta\mu - \delta\nu = -(\mu + \gamma + \bar{\gamma})\mu - \lambda\bar{\lambda} + \bar{\nu}\pi + (\bar{\alpha} + 3\beta - \tau)\nu - \Phi_{22}, \tag{A.13k}$$

$$\Delta\beta - \delta\gamma = (\bar{\alpha} + \beta - \tau)\gamma - \mu\tau + \sigma\nu + \varepsilon\bar{\nu} + (\gamma - \bar{\gamma} - \mu)\beta - \alpha\bar{\lambda} - \Phi_{12}, \tag{A.13l}$$

$$\Delta\sigma - \delta\tau = -(\mu - 3\gamma + \bar{\gamma})\sigma - \bar{\lambda}\rho - (\tau + \beta - \bar{\alpha})\tau + \kappa\bar{\nu} - \Phi_{02}, \tag{A.13m}$$

$$\Delta\rho - \bar{\delta}\tau = (\gamma + \bar{\gamma} - \bar{\mu})\rho - \sigma\lambda + (\bar{\beta} - \alpha - \bar{\tau})\tau + \nu\kappa - \Psi_2 - 2\Lambda, \tag{A.13n}$$

$$\Delta\alpha - \bar{\delta}\gamma = (\rho + \varepsilon)\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma - \Psi_3, \tag{A.13o}$$

$$\delta\rho - \bar{\delta}\sigma = (\bar{\alpha} + \beta)\rho - (3\alpha - \bar{\beta})\sigma + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01}, \tag{A.13p}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + (\rho - \bar{\rho})\gamma + (\mu - \bar{\mu})\varepsilon - \Psi_2 + \Lambda + \Phi_{11}, \tag{A.13q}$$

$$\delta\lambda - \bar{\delta}\mu = (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21}. \tag{A.13r}$$

The spinor form of the Bianchi identities is

$$\nabla^D_{B'}\Psi_{ABCD} = \nabla^{A'}_A\Phi_{BCA'B'} + \epsilon_{C(A}\,\nabla_{B)B'}\Lambda - \frac{3}{2}\,\epsilon_{AB}\,\nabla_{CB'}\Lambda. \tag{A.14}$$

Projecting these equations onto the spin basis leads to the Bianchi identities in the NP formalism:

$$\begin{aligned} D\Psi_1 - \bar{\delta}\Psi_0 - D\Phi_{01} + \delta\Phi_{00} = {}& (\pi - 4\alpha)\Psi_0 + 2(2\rho + \varepsilon)\Psi_1 - 3\kappa\Psi_2 + 2\kappa\Phi_{11} - (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} \\ & - 2\sigma\Phi_{10} - 2(\bar{\rho} + \varepsilon)\Phi_{01} + \bar{\kappa}\Phi_{02}, \end{aligned} \tag{A.15a}$$

$$\begin{aligned} D\Psi_2 - \bar{\delta}\Psi_1 + \Delta\Phi_{00} - \bar{\delta}\Phi_{01} + 2D\Lambda = {}& -\lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02} \\ & + (2\gamma + 2\bar{\gamma} - \bar{\mu})\Phi_{00} - 2(\alpha + \bar{\tau})\Phi_{01} - 2\tau\Phi_{10}, \end{aligned} \tag{A.15b}$$

$$\begin{aligned} D\Psi_3 - \bar{\delta}\Psi_2 - D\Phi_{21} + \delta\Phi_{20} - 2\bar{\delta}\Lambda = {}& -2\lambda\Psi_1 + 3\pi\Psi_2 + 2(\rho - \varepsilon)\Psi_3 - \kappa\Psi_4 + 2\mu\Phi_{10} - 2\pi\Phi_{11} \\ & - (2\beta + \bar{\pi} - 2\bar{\alpha})\Phi_{20} - 2(\bar{\rho} - \varepsilon)\Phi_{21} + \bar{\kappa}\Phi_{22}, \end{aligned} \tag{A.15c}$$

$$\begin{aligned} D\Psi_4 - \bar{\delta}\Psi_3 + \Delta\Phi_{20} - \bar{\delta}\Phi_{21} = {}& -3\lambda\Psi_2 + 2(\alpha + 2\pi)\Psi_3 + (\rho - 4\varepsilon)\Psi_4 + 2\nu\Phi_{10} - 2\lambda\Phi_{11} \\ & - (2\gamma - 2\bar{\gamma} + \bar{\mu})\Phi_{20} - 2(\bar{\tau} - \alpha)\Phi_{21} + \bar{\sigma}\Phi_{22}, \end{aligned} \tag{A.15d}$$

$$\Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 \\ + (\bar{\rho} + 2\varepsilon - 2\bar{\varepsilon})\Phi_{02} + 2\sigma\Phi_{11} - 2\kappa\Phi_{12} - \bar{\lambda}\Phi_{00} + 2(\bar{\pi} - \beta)\Phi_{01}, \quad \text{(A.15e)}$$

$$\Delta\Psi_1 - \delta\Psi_2 - \Delta\Phi_{01} + \bar{\delta}\Phi_{02} - 2\delta\Lambda = \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 \\ - \bar{\nu}\Phi_{00} + 2(\bar{\mu} - \gamma)\Phi_{01} + (2\alpha + \bar{\tau} - 2\bar{\beta})\Phi_{02} + 2\tau\Phi_{11} - 2\rho\Phi_{12}, \quad \text{(A.15f)}$$

$$\Delta\Psi_2 - \delta\Psi_3 + D\Phi_{22} - \delta\Phi_{21} + 2\Delta\Lambda = 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 + \sigma\Psi_4 \\ - 2\mu\Phi_{11} - \bar{\lambda}\Phi_{20} + 2\pi\Phi_{12} + 2(\beta + \bar{\pi})\Phi_{21} + (\bar{\rho} - 2\varepsilon - 2\bar{\varepsilon})\Phi_{22}, \quad \text{(A.15g)}$$

$$\Delta\Psi_3 - \delta\Psi_4 - \Delta\Phi_{21} + \bar{\delta}\Phi_{22} = 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4 - 2\nu\Phi_{11} \\ - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} + 2(\gamma + \bar{\mu})\Phi_{21} + (\bar{\tau} - 2\bar{\beta} - 2\alpha)\Phi_{22}, \quad \text{(A.15h)}$$

$$D\Phi_{11} - \delta\Phi_{10} + \Delta\Phi_{00} - \bar{\delta}\Phi_{01} + 3D\Lambda = (2\gamma + 2\bar{\gamma} - \mu - \bar{\mu})\Phi_{00} + (\pi - 2\alpha - 2\bar{\tau})\Phi_{01} \\ + (\bar{\pi} - 2\bar{\alpha} - 2\tau)\Phi_{10} + 2(\rho + \bar{\rho})\Phi_{11} + \bar{\sigma}\Phi_{02} + \sigma\Phi_{20} - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21}, \quad \text{(A.15i)}$$

$$D\Phi_{12} - \delta\Phi_{11} + \Delta\Phi_{01} - \bar{\delta}\Phi_{02} + 3\delta\Lambda = (2\gamma - \mu - 2\bar{\mu})\Phi_{01} + \bar{\nu}\Phi_{00} - \bar{\lambda}\Phi_{10} \\ + 2(\bar{\pi} - \tau)\Phi_{11} + (\pi + 2\bar{\beta} - 2\alpha - \bar{\tau})\Phi_{02} + (2\rho + \bar{\rho} - 2\bar{\varepsilon})\Phi_{12} + \sigma\Phi_{21} - \kappa\Phi_{22}, \quad \text{(A.15j)}$$

$$D\Phi_{22} - \delta\Phi_{21} + \Delta\Phi_{11} - \bar{\delta}\Phi_{12} + 3\Delta\Lambda = \nu\Phi_{01} + \bar{\nu}\Phi_{10} - 2(\mu + \bar{\mu})\Phi_{11} - \lambda\Phi_{02} - \bar{\lambda}\Phi_{20} \\ + (2\pi - \bar{\tau} + 2\bar{\beta})\Phi_{12} + (2\beta - \tau + 2\bar{\pi})\Phi_{21} + (\rho + \bar{\rho} - 2\varepsilon - 2\bar{\varepsilon})\Phi_{22}. \quad \text{(A.15k)}$$

A.2 Electromagnetic field

Electromagnetic field is described by the electromagnetic tensor F_{ab} or its spinor equivalent ϕ_{AB}:

$$F_{ab} = \phi_{AB}\epsilon_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB}. \quad \text{(A.16)}$$

Projections of ϕ_{AB} on a spin basis (NP tetrad) are

$$\phi_0 \equiv F_{ab}l^a m^b = \phi_{AB}o^A o^B, \quad \text{(A.17a)}$$

$$\phi_1 \equiv \frac{1}{2}F_{ab}\left[l^a n^b - m^a \bar{m}^b\right] = \phi_{AB}o^A \iota^B, \quad \text{(A.17b)}$$

$$\phi_2 \equiv F_{ab}\bar{m}^a n^b = \phi_{AB}\iota^A \iota^B, \quad \text{(A.17c)}$$

The source free Maxwell equations are equivalent to the spin-1 zero-rest-mass equation

$$\nabla^A_{A'}\phi_{AB} = 0. \quad \text{(A.18)}$$

Projecting this onto the spin basis we obtain Maxwell equations in the NP formalism:

$$D\phi_1 - \bar{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2, \tag{A.19a}$$
$$D\phi_2 - \bar{\delta}\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\varepsilon)\phi_2, \tag{A.19b}$$
$$\Delta\phi_0 - \delta\phi_1 = (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2, \tag{A.19c}$$
$$\Delta\phi_1 - \delta\phi_2 = \nu\phi_0 - 2\mu\phi_1 + (2\beta - \tau)\phi_2. \tag{A.19d}$$

The Ricci tensor in the electrovacuum spacetime is given by

$$\Phi_{mn} = \phi_m \bar{\phi}_n, \quad \Lambda = 0, \tag{A.20}$$

what are also the Einstein equations for electrovacuum spacetimes.

A.3 Spin transformation

The spin transformation is defined as rotation in the plane spanned by $m^a, \overline{m}^a$,

$$m^a \mapsto e^{\mathrm{i}\chi}\, m^a, \quad \bar{m}^a \mapsto e^{-\mathrm{i}\chi}\, \bar{m}^a, \tag{A.21}$$

where χ is arbitrary real function. The scalar quantity η is said to have the *spin weight s* provided

$$\eta' = e^{\mathrm{i}\,s\,\chi}\,\eta.$$

under transformation (A.21). The NP operators δ and $\bar{\delta}$ contain vectors $m^a, \overline{m}^a$ and do not preserve the spin weight. If η has spin weight s, we have

$$\delta'\eta' = e^{\mathrm{i}(s+1)\chi}\left(\delta\eta + \mathrm{i}\,s\,\eta\,\delta\chi\right).$$

This suggests that $\delta\eta$ could have the weight $n+1$, but there is an inhomogeneous term proportional to $\delta\chi$. However, this term can be eliminated defining a new operator ð by

$$ð\eta \equiv \delta\eta + s(\bar{\alpha} - \beta)\eta = \delta\eta + s\bar{a}\eta, \tag{A.22}$$

which transforms yet homogeneously:

$$ð\eta \mapsto e^{\mathrm{i}(s+1)\chi}\, ð\eta.$$

So if η has spin weight s, $ð\eta$ has spin weight $s+1$. Therefore ð acts as a spin-raising operator. Analogously we define

$$\bar{ð}\eta = \bar{\delta}\eta - s(\alpha - \bar{\beta})\eta = \delta\eta - sa\eta. \tag{A.23}$$

The spin weights of the Newman-Penrose quantities are summarized in the table A.1.

−2	−1	0	1	2
λ	ν	ρ	κ	σ
	π	μ	τ	
	ϕ_2	ϕ_1	ϕ_0	
Ψ_4	Ψ_3	Ψ_2	Ψ_1	Ψ_0

(A.24)

Table A.1: Spin weights of the Newman-Penrose scalars.

A.4 Lorentz transformations

The transformation between two spin basis is described by a simplectomorphism[1], which reflect itself as a Lorentz transformation acting on vectors. In this section we give a summary of transformation properties of NP quantities under a particular simplectomorphism known as a null rotation about l^a

$$(o'^A, \iota'^A) = (o^A, \iota^A + co^A), \quad c \text{ is complex function.}$$

For other transformations see appendix B in [2]. Transformation of NP tetrad

$$l'^a = l^a, \quad m'^a = m^a + \bar{c}l^a, \quad n'^a = n^a + cm^a + \bar{c}\,\overline{m}^a + c\bar{c}l^a$$

The NP scalars transforms as

$$\begin{aligned}
\kappa' &= \kappa \\
\varepsilon' &= \varepsilon + c\kappa \\
\sigma' &= \sigma + \bar{c}\kappa \\
\rho' &= \rho + c\kappa \\
\tau' &= \tau + c\sigma + \bar{c}\rho + c\bar{c}\kappa \\
\alpha' &= c\alpha\varepsilon + c\rho + c^2\kappa \\
\beta' &= \beta + c\sigma + \bar{c}\varepsilon + c\bar{c}\kappa \\
\pi' &= \pi + 2c\varepsilon + c^2\kappa + Dc \\
\gamma' &= \gamma + \bar{c}\alpha + c(\tau + \beta) + c\bar{c}(\rho + \varepsilon) + c^2\sigma + c^2\bar{c}\kappa \\
\lambda' &= \lambda + c\pi + 2c\alpha + c^2(\rho + 2\varepsilon) + c^3\kappa + cDc + \bar{\delta}c \\
\mu' &= \mu + 2c\beta + \bar{c}\pi + c^2\sigma + 2c\bar{c}\varepsilon + c^2\bar{c}\kappa + \bar{c}Dc + \delta c \\
\nu' &= \nu + c(2\gamma + \mu) + \bar{c}\lambda + c^2(\tau + 2\beta) + c\bar{c}(\pi + 2\alpha) \\
&\quad + c^3\sigma + c^2\bar{c}(\rho + 2\varepsilon) + c^3\bar{c}\kappa + \Delta c + c\delta c + \bar{c}\bar{\delta}c + c\bar{c}Dc
\end{aligned} \tag{A.25}$$

For transformations of Riemann and Ricci tensor components see again [2].

[1] A simplectomorphism is a linear map that preserves the simplectic structure of a spinor space.

Bibliography

[1] R. Penrose and W. Rindler, *Spinors and space-time. Vol. 1: Two-spinor calculus and relativistic fields.* Cambridge University Press, Cambridge, 1984.

[2] J. Stewart, *Advanced general relativity.* Cambridge University Press, 1993.

[3] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics.* Cambridge University Press, 2004.

[4] B. Krishnan, "Quasi-local black hole horizons," in *Springer Handbook of Spacetime*, pp. 527–555, Springer, 2014.

[5] A. Ashtekar, C. Beetle, and J. Lewandowski, "Geometry of generic isolated horizons," *Classical and Quantum Gravity*, vol. 19, no. 6, p. 1195, 2002.

[6] J. Lewandowski and T. Pawłowski, "Neighborhoods of isolated horizons and their stationarity," *Classical and Quantum Gravity*, vol. 31, no. 17, p. 175012, 2014.

[7] A. Ashtekar, S. Fairhurst, and B. Krishnan, "Isolated horizons: Hamiltonian evolution and the first law," *Phys. Rev. D*, vol. 62, 10 2000.

[8] A. Ashtekar, C. Beetle, and J. Lewandowski, "Mechanics of rotating isolated horizons," *Phys. Rev. D*, vol. 64, 07 2001.

[9] B. Krishnan, "The spacetime in the neighborhood of a general isolated black hole," *Classical and quantum gravity*, vol. 29, no. 20, p. 205006, 2012.

[10] A. Flandera, "Geometry of isolated horizons," *arXiv:gr-qc/1611.02215*, 2016.

[11] R. Geroch and J. B. Hartle, "Distorted black holes," *Journal of Mathematical Physics*, vol. 23, no. 4, pp. 680–692, 1982.

[12] J. B. Griffiths and J. Podolský, *Exact space-times in Einstein's general relativity.* Cambridge University Press, 2009.

[13] N. Gürlebeck and M. Scholtz, "Meissner effect for axially symmetric charged black holes," *Phys. Rev. D*, vol. 97, p. 084042, Apr 2018.

[14] L. Evans, *Partial Differential Equations.* Graduate studies in mathematics, American Mathematical Society, 2010.

[15] A. Ashtekar and B. Krishnan, "Isolated and dynamical horizons and their applications," *Living Reviews in Relativity*, vol. 7, p. 10, Dec 2004.

[16] I. Rácz, "Stationary black holes as holographs," *Classical and Quantum Gravity*, vol. 24, no. 22, p. 5541, 2007.

[17] A. Ashtekar, J. Engle, T. Pawlowski, and C. V. D. Broeck, "Multipole moments of isolated horizons," *Class. Quantum Grav.*, vol. 21, pp. 2549–2570, 04 2004.

[18] R. Geroch, "Multipole Moments. II. Curved Space," *Journal of Mathematical Physics*, vol. 11, no. 8, p. 2580, 1970.

[19] R. O. Hansen, "Multipole moments of stationary space-times," *Journal of Mathematical Physics*, vol. 15, no. 1, p. 46, 1974.

[20] I. Kluvánek, L. Mišík, and M. Švec, *Matematika II.* Edition of theoretical literature, SVTL, 1965.

[21] J. Lewandowski and T. Pawlowski, "Extremal isolated horizons: a local uniqueness theorem," *Classical and Quantum Gravity*, vol. 20, no. 4, p. 587, 2003.

[22] H. Bondi, "Gravitational waves in general relativity," *Nature*, vol. 186, no. 4724, p. 535, 1960.

[23] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, "Gravitational waves in general relativity. VII. Waves from Axi-Symmetric isolated systems," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 269, no. 1336, pp. 21–52, 1962.

[24] R. K. Sachs, "Gravitational waves in general relativity VIII. Waves in asymptotically flat space-time," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 270, no. 1340, pp. 103–126, 1962.

[25] R. Penrose, "Zero rest-mass fields including gravitation: Asymptotic behaviour," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 284, no. 1397, pp. 159–203, 1965.

[26] E. T. Newman and T. W. J. Unti, "Behavior of Asymptotically Flat Empty Spaces," *Journal of Mathematical Physics*, vol. 3, no. 5, 1962.

[27] E. T. Newman and R. Penrose, "An Approach to Gravitational Radiation by a Method of Spin coefficients," *Journal of Mathematical Physics*, vol. 3, no. 3, p. 566, 1962.

[28] R. Penrose, "Republication of: Conformal treatment of infinity," *General Relativity and Gravitation*, vol. 43, no. 3, pp. 901–922, 2011.

[29] L. B. Szabados, "Quasi-local energy-momentum and angular momentum in gr: A review article," *Living Reviews in Relativity*, vol. 7, p. 4, Mar 2004.

[30] L. B. Szabados, "Two-dimensional sen connections in general relativity," *Classical and Quantum Gravity*, vol. 11, no. 7, p. 1833, 1994.

[31] L. B. Szabados, "Two-dimensional Sen connections and quasi-local energy–momentum," *Classical and Quantum Gravity*, 1999.

[32] R. Penrose, "Quasi-local mass and angular momentum in general relativity," *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 381, no. 1780, pp. 53–63, 1982.

[33] D. Matejov, *Investigation of chaotic dynamical systems using the methods of Riemannian geometry.* bachelor thesis, Faculty of Mathematics and Physics, Charles University, Prague, 2016.

[34] R. Penrose and W. Rindler, *Spinors and space-time. Vol. 2: Spinor and twistor methods in space-time geometry.* Cambridge University Press, Cambridge, 1986.

[35] S. A. Huggett and K. P. Tod, *An Introduction to Twistor Theory.* London Mathematical Society Student Texts, Cambridge University Press, 2 ed., 1994.

[36] A. Polyanin and V. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations.* CRC Press, 1995.

[37] R. S. Ward and R. O. Wells, *Twistor Geometry and Field Theory.* Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1991.

[38] L. Hughston, *Twistors and Particles.* Lecture Notes in Mathematics; 709, Springer-Verlag, 1979.

[39] R. Penrose and M. A. H. MacCallum, "Twistor theory: An approach to the quantisation of fields and space-time," 1973.

[40] E. Witten, "A new proof of the positive energy theorem," *Comm. Math. Phys.*, vol. 80, no. 3, pp. 381–402, 1981.

[41] P. Shanahan, *The Atiyah-Singer Index Theorem: An Introduction.* Lecture Notes in Mathematics, Springer Berlin Heidelberg, 1978.

[42] R. J. Baston, "The index of the 2-twistor equations," *Twistor newsletters*, vol. 17, 1984.

[43] K. P. Tod, "More on Penrose's quasi-local mass," *Classical and Quantum Gravity*, vol. 3, no. 6, p. 1169, 1986.