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Examiner Report on the Doctoral Thesis "Treewidth, Extended Formulations of CSP and MSO Polytopes, and their Algorithmic Applications" Submitted by Mgr. Martin Koutecký

1 Contents of the Thesis

At the heart of the doctoral thesis of Mr. Koutecký lies a fresh look at a number of topics from theoretical computer science that have been investigated for quite some time, but that appear rather unrelated at first sight: constraint satisfaction problems (CSPs), monadic second-order logic (MSO logic), tree width, and polytopes. While combinations of these topics have been studied before – indeed, the combination of MSO logic and tree width is a cornerstone of the whole field of algorithmic metatheorems –, all four topics are woven together in the thesis, resulting in a number of surprising connections.

The thesis starts with a detailed introduction that gives not only a motivation, but makes an effort to explain the main concepts and tools used in the rest of the thesis in enough depth so that the reader can understand the main theorems, which are already stated in the introduction. It is also the job of this introductory chapter to provide an overview of how the different topics relate. Following the introduction, a chapter on preliminaries summarizes all mathematical concepts and tools needed in later chapters in a rather concise manner.

The core of the thesis consists of five chapters that address the above-mentioned topics in different combinations. More precisely, the first three of these chapters tackle, first, the combination of CSPs, polytopes, and tree width; second, the combination of MSO logic, polytopes, and tree width; and third, of all of them. The fourth and fifth of the main chapters then focus on applications and extensions of the obtained results: first, on extensions of MSO logic and, second, on shifted combinatorial optimization.

The thesis ends with the presentation of a number of open problems.

1.1   Towards a Master MSO-CSP Theorem

As just mentioned, Chapters 3 to 5 of the thesis all revolve around different ways of combining constraint satisfaction problems, monadic second-order logic, tree width, and polytopes; culminating in a "master theorem" that connects all of these topics.

The first step, taken in Chapter 3, is the casting of CSPs as extended formulations. While it is standard to describe CSP solution sets using linear program relaxations (LP relaxations), the point made in the thesis is that every CSP solution set has an extended formulation (where there are additional dimensions) that is an integral polytope whose size depends exponentially on the tree width of the CSP and linearly on its size. The author demonstrates some applications of this approach by showing that a number of graph problems studied in the literature can be solved using linear programs on extended formulations whose sizes depend on the tree width of the graphs.

Chapter 4 is of a similar spirit as Chapter 3, but this time the solution sets are not described using constraints (as in CSPs), but using monadic second-order logic (MSO logic). Just as for CSPs, one can use polytopes to describe the solution sets of MSO formulas, and just as for CSPs, there exist extended formulations whose sizes depend computably on the tree width of the input structures and linearly on the structure size. Furthermore, these formulations are efficiently computable. Like Chapter 3, the chapter also ends with some applications, including a proof of an optimization version of Courcell's Theorem based entirely on solving an LP on the extended formulations of MSO polytopes.

Chapter 5, finally, connects the ideas and results of Chapters 3 and 4. The connections between the ideas from these two chapters are first established by showing that the main results of Chapter 3 follow from the main results of Chapter 4 *and vice versa*. During this back-and-forth reduction of the main ideas, the author arrives at a theorem, dubbed a bit grandiosely "Master MSO-CSP Theorem," that shows that the solution polytopes resulting from *joint* descriptions using both MSO logic and CSPs at the same time still allow extended formulations of a size depending mainly on the tree width.

1.2   Beyond Standard MSO and Linear Programming

While Chapters 3 to 5 form a rather tightly interconnected conceptual framework, culminating in the just-mentioned Master MSO-CSP Theorem, Chapters 6 and 7 are stand-alone chapters that build on the earlier results, but each has a specific focus that is not directly related to the rest of the thesis.

In Chapter 6, extensions of MSO logic are studied; namely extensions where local and/or global constraints on the size of set variables (so-called cardinality constraints) may be given. It is known that allowing certain combinations of such constraints makes MSO logic much more

powerful and, hence, the problems that can be formulated using it much harder. In the chapter, the "line of tractability" resulting from the different possible extensions is traced: A number of hardness results are proved, but also a number of tractability results (in the framework of fixed-parameter tractability theory) are given. This chapter also includes a closer look at graph parameters other than tree width such as neighborhood diversity.

Chapter 7 is perhaps removed furthest from the previous chapters since it studies a different kind of basic problem, so-called *shifted combinatorial optimization* problems. This setting provides a framework for describing problems using highly non-linear integer optimization problems on explicitly or implicitly given polytopes. In its most general form, the framework is so powerful that very hard problems can easily be described, but under certain conditions on the problem formulation, upper bounds such as XP or FPT or even P can be obtained. The connection to the previous chapters is made mainly by the use of MSO logic in a variant where the polytopes are described using MSO logic rather than being given explicitly.

2 Scientific Contributions of the Thesis

2.1 Main Results

The thesis abounds with numerous lemmas, corollaries, and theorems that are proved rigorously. There are, however, a number of core results that merit a longer discussion.

The Size of an Extended Formulation of the CSP Polytope

The first main result of the paper is Theorem 3.0.1, which states that every constraint satisfaction problem instance I has an extended formulation $P(I)$ of size $O(D^\tau n)$ where τ is the tree width of the instance, D is the domain size and n is the universe size. Furthermore, an LP describing $P(I)$ can be constructed in time linear in its size.

This result is quite beautiful since it allows us to replace CSPs, which are a technically rather "messy" way of describing problems, by polytopes and linear programs. As shown in the thesis, this result allows us to solve numerous classical problems simply by running an LP solver on the polytopes constructed in the theorem.

This is not to say that it always desirable to switch from a CSP formulation to an LP formulation. For instance, in the context of space-bounded computations (which I am very interested in), solving an LP is *not* an easy thing and the original, CSP-based problem formulations may be better suited to algorithms that may only use little space or that must work in parallel. Nevertheless, I agree with the author that Theorem 3.0.1 provides a very elegant way of transforming CSP-based description of problems into the clean setting of polytopes.

Apart from the elegance of the theorem statement and its many applications, I also liked the proof of the theorem. The author shows how a tree decomposition of the CSP instance can be

turned into a set of linear inequalities and I have a strong feeling that this proof idea may be applicable in other situations. There is, however, a point where I could not follow, namely in the paragraph right after the end of the proof: It is claimed that "we immediately obtain that **f** is an integral vector," while from the definition of this vector in Lemma 3.1.1 I could only conclude that it is a multiple of $1/M$. Clearly, I am missing something and "we immediately obtain" should be replaced by a more detailed argument.

The Size of an Extended Formulation of the MSO Polytope

The main result of Chapter 4 is a symmetric statement to the main result of Chapter 3, namely Theorem 4.3.1. This theorem states that, just like the CSP polytope, the MSO polytope of a formula and a graph also has a size that depends (exponentially or worse) on the tree width of the graph and linearly on the size of the graph.

Just like the result on the CSP polytope, this Theorem 4.3.1 is also a very elegant way of turning descriptions in one formalism (MSO logic) into a conceptually simpler formalism (polytopes and linear programs). The power of this approach is demonstrated by Theorem 4.5.3, which is a version of Courcelle's Theorem and which follows easily from Theorem 4.3.1. (The same word of caution that I added above also applies here, however: For space-bounded or parallel computations the linear-programm-based formulations are far less useful. I do not see, for instance, how the Logspace Version of Courcelle's Theorem would follow from Theorem 4.3.1. However, this is neither claimed by the author, nor is this setting in the scope of the thesis.)

The proof of Theorem 4.3.1 follows a different route than that of Theorem 3.0.1: The proof is based on a Feferman–Vaught-style argument, an elegant method of proving Courcelle's Theorem without using (tree) automata theory. Using Feferman–Vaught arguments usually comes at the cost of significant technical and notational challenges and this thesis is no exception as is also acknowledged by the author when we writes "The following lemma really only says this simple fact; we encourage the reader not be frightened by the notation." Well, I was not really frightened, but felt that the definition of glue products is overly complicated (especially since the author advertised them as a sort of necessary new tool). As I understood the definition of the glue product, it is essentially the same as the product studied by Margot, but instead of insisting the we glue along the last coordinates, we specify the sets of coordinates that we wish to glue explicitly. I had the feeling that Margot's simpler form of gluing is sufficient if one allows a reordering of the coordinates at appropriate times, which – to me – seem to be a trivial operation. If I am missing some complications here, I suggest pointing them out in the thesis.

The Master MSO-CSP Theorem

As the its name suggest, the author considers Theorem 5.2.13 to be the main achievement of his thesis. Indeed, this theorem is a powerful fusion of the previous results in a single formalism. Both results from the previous chapters as well as results in later chapters can all be derived

directly from this theorem. In some sense, this main theorem is a by-product of the author's efforts to formulate Courcelle's Theorem using CSP polytopes and vice versa.

While this theorem is the most general theorem in the thesis (and, thus, also the most "powerful" one), it is also the most "technical" one with a large number of assumptions, a large number of concepts, and larger number of parameters that have to be taken into account. This makes applying the theorem rather hard in practice.

Results on Extensions of MSO Logic by Cardinality Constraints

In Chapter 6, Mr. Koutecký demonstrates that while his Master MSO-CSP Theorem may be technically challenging, applications of this theorem lead to much easier results. For instance, Theorem 6.2.1 states on input of an MSO formula $\varphi(X_1, \ldots, X_m)$ with global cardinality constraints, an n-vertex graph G, and m weight vectors, we can find a minimal-weight assignment to the X_i with $G \models \varphi(X_1, \ldots, X_m)$ in time $n^{f(|\varphi|, \tau)}$ where τ is the tree width of G and f is a computable function. Phrased differently, the weighted MSO model checking problem with global cardinality constraints is in XP with respect to tree width and formula size. The proof of this theorem relies heavily on an application of the Master MSO-CSP Theorem.

In Section 6.2.1, Mr. Koutecký shows that Theorem 6.2.1 has a number of rather nice applications since a number of well-known problems can be described in terms on the weighted MSO model checking problem with global cardinality constraints and, thus, these problems are shown to lie in XP. In the same place (Theorem 6.2.3), the author also claims that he can derive from Theorem 6.2.1 that a number of problems (equitable r-coloring, equitable connected r-partition, r-balanced partition, and graph motif) lie in FPT. I do not see how this follows: What ever way I try to formulate the problems as weighted MSO model checking problems, I always get an XP algorithm when I apply Theorem 6.2.1. Some clarification is needed here.

The theorems in Section 6.3 nicely complement the previous results by considering a different graph parameter, but proof-wise these results are more "stand-alone" results whose proofs are not integrated into the greater framework of the thesis.

Shifted Combinatorial Optimization

Even more strongly than Section 6.3, Chapter 7 is conceptually independent of the rest of the thesis as it discusses a new concept ("shifts" in combinatorial optimizations). Nevertheless, this chapter, too, builds on the results of the previous chapters and uses the Master MSO-CSP Theorem to prove a rather general result on a shifted version of the MSO partitioning problem (Theorem 7.2.2).

2.2 Originality and Importance of Main Results

The thesis includes three kinds of results that are scientifically new: New results that settle known open problems, results that unify and extend previous results in new ways, and results

that make new statements about specific computational problems.

The first new main result is Theorem 3.0.1 on the size of an integral extended formulation of CSP problems. Not only is this an elegant, new result, but there have been previous, published, failed attempts at proving it. I congratulate the author on this result and the beautiful proof.

Second, the thesis contains results like the main theorems of Chapters 4 and 5, which have not been open problems in the literature, but which provide a new, unified framework to describe computational problems and to derive their complexity easily. The theorems have the flavor of algorithmic metatheorems, but they are actually not quite "algorithmic" in nature. Rather, as I described earlier, they allow us to transform problem phrasings using "powerful, but messy" formulations into the "simple, clean" formulation of linear programs and polytopes. I find this a very interesting and original approach.

Lastly, the thesis contains a number of results for concrete computational problems that result from applying the general framework to concrete MSO formulas. Some of the resulting statements about the concrete problems are most likely new, but theses results are not really at the heart of the thesis.

3   Creativity of the Scientific Work and Overall Appraisal

With his thesis, Mr. Koutecký has demonstrated impressively his ability to do highly creative scientific research, to make new discoveries, to use and integrate mathematical concepts from different areas in order to prove new results, and, finally, to present his findings in a precise, clear, correct, and readable format.

I congratulate him on his thesis and on his contribution to the scientific community.

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