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KOMBINATORIKA FILTRŮ NA PŘIROZENÝCH ČÍSLECH
COMBINATORICS OF FILTERS ON THE NATURAL NUMBERS

Bakalářská práce

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Abstrakt

Práce se věnuje kombinatorickým vlastnostem filtrů na přirozených číslech. Obsahuje úvod do problematiky definovatelnosti filtrů a jejich kombinatoriky, definice základních typů filrů: P-filtr, Q-filtr, Rapid filtr; uspořádání: Rudin-Kiesler, Rudin-Blass, Katětov a Tukey; konstrukce filtrů; základní definice z kombinatoriky na ω; úvod do deskriptivní teorie množin, topologie a základní výsledky.

Abstract

The work is devoted to combinatorial properties of filters on natural numbers as an introduction and motivation to the definability of the filters and its combinatorics. The work contains definitions of basic filter types: P-filter, Q-filter, Rapid filter; orders: Rudin-Kiesler, Rudin-Blass, Katětov and Tukey; filter constructions; basic definitions related to combinatorics on ω; introduction to basic descriptive set theory and topology and some specific results.

Contents

Introduction

The goal of this work is to show the Mazur theorem from [9] as a bridge between topology and combinatorics. In Chapters I and II there are basic definitions related to combinatorics on ω. Chapter III contains an introduction to topology and basic descriptive set theory. Chapter IV focuses on Mazur's specific result.

The concept of *ultrafilter* is important concept and the theory of *definability* plays important role here. It developes the topological hierarchy which classifies the sets over real numbers $\mathbb{R}$. As the real number it is possible to take the points from *Cantor space* and an ultrafilter could be regarded as a subspace of Cantor space.

The natural numbers $\mathbb{N}$ is the set $\{0, 1, 2, ...\}$.

Set theory is a domain of mathematical logic that studies sets. Georg Cantor created this theory as the theory of actual infinity, now commonly based on *ZFC* (the Zermelo-Fraenkel axioms with the axiom of choice). Informally set theory is the theory of the membership relation $\in$.

$x \in A$ means that x is a member of the set A.
$x \notin A$ means that x is not a member of the set A.

The set theoretic version of numbers is folowing: the finite ordinals begin with the empty set $\emptyset$, which is folowed by $\{\emptyset\}$, the set containing empty set, $\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, ... Every ordinal is the set of the previous ordinals. The first infinite ordinal number (the first after all natural numbers) is denoted ω. After ω it is possible to count other transifinte numbers. In the set theory there are coded two kind of numbers: ordinal number and cardinal number which are same if they are finite. The size comparison of infinite sets using the subset relation doesn't work so assume folowing appropriate definition:

1.1 Definition. The set X is *strictly larger* than Y, denoted $X \succ Y$, if there exists one-to-one function from Y into X and there is no map from Y onto X. $\diamond$

In ZFC there is the *Power set* axiom which says there exists the set of all subsets of any set X denoted $\mathcal{P}(X)$.

1.2 Theorem. $\mathcal{P}(X) \succ X$ *The power set of any set is strictly larger then the set.*

Proof. There is a one-to-one function from X to $\mathcal{P}(X)$: $f(x) = \{x\}$. Assume towards contradiction, let there is an onto map $f : X \to \mathcal{P}(X)$.

Consider the set $A = \{Y \in X \mid Y \notin f(Y)\}$. A is a member of $\mathcal{P}(X)$, so there must be some element $z \in X$ such that $f(z) = A$. There are two cases:

If $z \in A$, then $z \notin f(z) = A$, a contradiction.
If $z \notin A$, then $z \in A$ by definition of A, again a contradiction.

□

Informally from Theorem 1.2 it folllows that there are infinite many sizes of sets (cardinalities). The first infinite cardinal $\aleph_0$ (the first after all natural numbers), denoted by the Hebrew letter aleph, is the size of the set $\mathbb{N}$. The next cardinal numbers are $\aleph_1$, $\aleph_2$, $\aleph_3$, ... The sets with cardinalities $\aleph_1$ and larger are called *uncountable* sets.

In ZFC is not provable which cardinality equals to the cardinality of $\mathcal{P}(\omega)$. By $\mathcal{P}(\omega) \succ \omega$, the cardinality of $\mathcal{P}(\omega)$ is not $\aleph_0$. This question can be assumed as the additional axiom $2^{\aleph_0} = \aleph_1$ which is called the *Continuum hypothesis.* For this, the size of continuum $2^{\aleph_0}$ is abbreviated $\mathfrak{c}$, and the first uncountable cardinal $\aleph_1$ (the first uncountable ordinal ω_1). The continuum could mean $\mathbb{R}$, *Cantor space* 2^ω, $[\omega]^\omega$ or Baire space ω^ω. These spaces are essentially the same: after removal of at most a countable set from each space, there exists a homeomorphism between the modified spaces.

Chapter I

In this chapter there is an introduction of basic definitions and facts related to the concept of *filter*. Filter formalizes the notion of bigness.

2.1 Filters

2.1 Definition (Filter on a set)**.** A *filter* on a set X is a collection $\mathcal{F}$ of subsets of X such that:

1. $X \in \mathcal{F}$;
2. if $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
3. if $A, B \subseteq X$, $A \in \mathcal{F}$, and $A \subseteq B$, then $B \in \mathcal{F}$.

If, moreover, the following holds:

4. $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$ for all $A \subseteq X$.

Then $\mathcal{F}$ is called *ultrafilter*.

A filter $\mathcal{F}$ is *proper* if $\emptyset \notin \mathcal{F}$. Only proper filters are considered. A filter $\mathcal{F}$ is *principal* if there is an $x \in X$ such that $\mathcal{F} = \{A \subseteq X \mid x \in A\}$. Non-principal ultrafilter is called *free*. ◇

2.2 Observation. *Principal filter is ultrafilter.*

2.3 Proposition. *An ultrafilter is principal if and only if it contains a finite set.*

Proof. The right direction is obvious so we prove other implication. Assume finite $A \in \mathcal{U}$. Let B is $\subseteq$-minimal subset of A from the sets in $\mathcal{U}$. If B is not a singleton set then let $x \in B$ and because $\mathcal{U}$ is ultrafilter then $\{x\} \in \mathcal{U}$ or $B \setminus \{x\} \in \mathcal{U}$ so B is not minimal. □

2.4 Definition. A filter $\mathcal{F}$ is *Fréchet filter* on a infinite set X if

$$\mathcal{F} = \{A \subseteq X \mid |X \setminus A| < \omega\}.$$

◇

2.5 Proposition. *A filter extends the Fréchet filter if the intersection of all its members is empty.*

Proof. If $\bigcap \mathcal{F} = A$ and $a \in A$, then $X \setminus \{a\} \notin \mathcal{F}$, so $\mathcal{F}$ can't contain Fréchet filter. $\square$

2.6 Observation. *If A is a nonempty family of filters over X, then* $\bigcap A$ *is a filter over X.*

Proof. Assume aiming toward contradiction $\bigcap A$ is not a filter. Assume, for example, that there are some $b \supseteq a$ such that $b \notin \bigcap A$ and $a \in \bigcap A$. Then for any filter from A, for all $b \supseteq a$ is satisfied $b \in \mathcal{F}$, contradiction. The other filter properties works similarly. $\square$

2.7 Observation. *If A is a* $\subset$*-chain of filters over X, then* $\bigcup A$ *is a filter over X.*

Proof. If $\bigcup A$ is not a filter. For example $a, b \in \bigcup A$ and $a \cap b \notin \bigcup A$, then there is some filter $\mathcal{F} \in A$ for which $a, b \in \mathcal{F}$ and $a \cap b \in \mathcal{F}$, contradiction. $\square$

2.8 Observation. *If* $\mathcal{F}$ *is a filter and* $X \in \mathcal{F}$*, then* $\mathcal{P}(X) \cap \mathcal{F}$ *is a filter over X.*

Proof. For any $A, B \subseteq X$ in filter $\mathcal{F}$ there is $A \cap B \subseteq X$ in filter $\mathcal{F}$. For any $A, B \subseteq X$ in filter $\mathcal{F}$ and $A \subseteq B$, then $B \subseteq X$. $\square$

2.9 Definition (Finite intersection property FIP)**.** A nonempty system E of sets has the *Finite intersection property*, FIP; if for every $n \in \omega$ and every family $e_0, ..., e_n \in E$ is true:

$$e_0 \cap ... \cap e_n \neq \emptyset.$$

$\diamond$

2.10 Observation. *Every* $E \subseteq \mathcal{P}(X)$ *with the FIP can be extended to a proper filter.*

Proof. $\mathcal{F}$ is defined: $\mathcal{F} = \{A \subseteq X \mid \exists n \in \omega \exists e_0, ..., \exists e_n \in E(e_0 \cap ... \cap e_n \subseteq A)\}$.

$\mathcal{F}$ is closed under intersection, i.e. that for $A, B \in \mathcal{F}$ there is $A \cap B \in \mathcal{F}$ because if

$$e_0 \cap ... \cap e_n \subseteq A \text{ and } f_0 \cap ... \cap f_m \subseteq B$$

then

$$e_0 \cap ... \cap e_n \cap f_0 \cap ... \cap f_m \subseteq A \cap B$$

$\square$

2.11 Lemma. *A filter $\mathcal{F}$ over X is an ultrafilter if and only if it is maximal in the order $\subseteq$.*

Proof. Let $\mathcal{U}$ is ultrafilter. For contradiction, there is a $\mathcal{F} \supset \mathcal{U}$ so there is some $A \in \mathcal{F} \setminus \mathcal{U}$. $\mathcal{U}$ is ultrafilter so $X \setminus A \in \mathcal{U}$. Then $X \setminus A \in \mathcal{F}$ and $A \in \mathcal{F}$ is contradiction. For other side assume $\mathcal{F}$ is a filter that is not an ultrafilter. To find $\mathcal{F}' \supset \mathcal{F}$: Let $B \subseteq X$ be such that neither B nor $X \setminus B$ is in $\mathcal{F}$. Consider the family $\mathcal{G} = \mathcal{F} \cup \{B\}$, $\mathcal{G}$ has the finite intersection property because if $A \in \mathcal{F}$, then $A \cap B \neq \emptyset$, otherwise there is $A \subseteq X \setminus B$ and $X \setminus B \in \mathcal{F}$. If $A_1, ..., A_n \in \mathcal{F}$, we have $A_1 \cap .. \cap A_n \in \mathcal{F}$ and so

$$B \cap A_1 \cap .. \cap A_n \neq \emptyset$$

$\mathcal{G}$ has finite intersection property, so there is a filter $\mathcal{F}' \supseteq \mathcal{G}$.

Since $B \in \mathcal{F}' \setminus \mathcal{F}$, $\mathcal{F}$ is not maximal. □

The *Axiom of choice* implies following useful theorem.

2.12 Theorem (Zorn's lemma)**.** *If X is a partially ordered set such that every chain in X has an upper bound, then X contains a maximal element.*

2.13 Theorem (Tarski's Ultrafilter Theorem)**.** *Every filter can be extended to an ultafilter*

Proof (taken from [6]). Let $\mathcal{F}_0$ be a filter. $P = \{\mathcal{F} \mid \mathcal{F}_0 \subseteq \mathcal{F}$ and $\mathcal{F}$ is filter$\}$. $\langle P, \subset \rangle$ is partially ordered set. Let C be a chain in P, then $\bigcup C$ is a filter by Observation 2.7 and an upper bound of C in P. By Zorn's lemma there exists a maximal element $\mathcal{U}$ in P. This $\mathcal{U}$ is an ultrafilter by Lemma 2.11. □

A filter $\mathcal{F}$ over S is countably complete (σ-complete) if it is closed under countable intersections. Every principal filter is closed under arbitrary intersections.

2.14 Definition (Filter Base)**.** A filter *Base* over a set X is a collection $\mathcal{B}$ of subsets of X such that:

1. if $A \in \mathcal{B}$ and $A' \in \mathcal{B}$, then $A \cap A' \in \mathcal{B}$;
2. $\mathcal{B} \neq \emptyset$ and $\emptyset \notin \mathcal{B}$.

Given a filter base $\mathcal{B}$, the filter generated by $\mathcal{B}$ is defined as the smallest filter containing $\mathcal{B}$. Every filter is also a filter base. ◇

Let X be a non-empty set and C be a non-empty subset of X. Then $\{C\}$ is a filter base. The filter generated by C (i.e., the collection of all subsets of X containing C) is called the filter generated by C.

2.15 Definition. An ultrafilter $\mathcal{U}$ is a *uniform* ultrafilter on X if $|A| = |X|$ for every $A \in \mathcal{U}$. ◇

2.16 Definition (Filter Generators)**.** The set S is said to *generate* a filter $\mathcal{F}$ (or it is called a set of *filter generators* of $\mathcal{F}$) if the family all finite intersections of elements of S forms a filter base of $\mathcal{F}$. ◇

For the answer how many ultrafilters are possible on ω it it useful to define following concept.

2.17 Definition. A family $\mathcal{C} \subseteq \mathcal{P}(\omega)$ is *uniformly independent* on ω if for any distinct sets $X_1, ..., X_n, Y_1, ..., Y_m \in \mathcal{C}$

$$|X_1 \cap ... \cap X_n \cap (\omega \setminus Y_1) \cap ... \cap (\omega \setminus Y_m)| = \omega.$$

It means that for all finite boolean combinations of distinct sets the intersection has cardinality ω. ◇

We first prove the following lemma.

2.18 Lemma. *There exist continuum sized uniformly independent family of subsets of* ω.

Proof (taken from [6]). Let Fin be the set of all finite subsets of ω and let

$$A = \{\langle F, F' \rangle \mid F \in Fin \text{ and } F' \subseteq Fin \text{ and } |F'| \in Fin\}.$$

The size of $Fin \times Fin^{<\omega}$ is ω, so $|A| = \omega$. We will construct the independent family on A. For each $X \subseteq \omega$, let

$$A_X = \{\langle F, F' \rangle \in A \mid F \cap X \in F'\}$$

and let

$$\mathcal{C} = \{A_X \mid X \subseteq \omega\}$$

If X and Y are distinct subsets of ω, then $A_X \neq A_Y$. For example, if $n \in X$ but $n \notin Y$, then let $F = \{n\}$, $F' = \{F\}$, and $\langle F, F' \rangle \in A_X$ and $\langle F, F' \rangle \notin A_Y$, so $|\mathcal{C}| = 2^\omega$. To show that $\mathcal{C}$ is uniformly independent, let $X_1, ..., X_n, Y_1, ..., Y_m$ be distinct subsets of ω. For each $i \leq n$ and each $j \leq m$, let $a_{ij} \in \omega$ such that either $a_{ij} \in X_i \setminus Y_j$ or $a_{ij} \in Y_j \setminus X_i$. Now let $F \in Fin$ such that

$\{a_{ij} \mid i \leq n$ and $j \leq m\} \subseteq F$, $\forall i \leq n, j \leq m(F \cap X_i \neq F \cap Y_j)$ and if $F' = \{F \cap X_i \mid i \leq n\}$, then

$$\forall i \leq n \langle F, F' \rangle \in A_{X_i},$$

$$\forall j \leq m \langle F, F' \rangle \notin A_{Y_j},$$

then,

$$|A_{X_1} \cap \cdots \cap A_{X_n} \cap (\omega \setminus A_{Y_1}) \cap \cdots \cap (\omega \setminus A_{Y_m})| = \omega.$$

□

2.19 Theorem (Pospíšil). [1] *The number of uniform ultrafilters on ω is 2^{2^ω}*

Proof (taken from [6]). Let $\mathcal{C}$ be an uniformly independent family of subsets of ω. For every function $f : \mathcal{C} \to \{0, 1\}$, consider this family of subsets of ω:

$$G_f = \{X \mid \omega \setminus X \mid \ \leq \omega\} \cup \{X \mid f(X) = 1\} \cup \{\omega \setminus X \mid f(X) = 0\}$$

The family G_f has the finite intersection property, and so there exists an ultrafilter D_f such that $D_f \supseteq G_f$. D_f is uniform. If $f \neq g$, then for some $X \in C, f(X) \neq g(X)$; e.g. $f(X) = 1$ and $g(X) = 0$ and then $X \in D_f$ while $\omega \setminus X \in D_g$. So there are 2^{2^ω} distinct uniform ultrafiters over ω.[6] □

[1] *Bedřich Pospíšil (1912-1944) was arrested by the Gestapo and sentenced to three years in a concentration camp, from where he returned on May 17, 1944 but be soon succumbed to the consequences of long imprisonment.*

Chapter II

It is not obvious that all non-principal ultrafilters are not the same (up to permutation of ω). A cardinality argument shows that they can't be same. There are too many ultrafilters and not enough permutations so that there are non-isomorphic non-principal ultrafilters on ω. It is an interesting problem to find the properties that distinguish them.

The analysis of different orders on the set of all ultrafilters on ω gives some view on complex structure of this set. There is a ordering of the ultrafilters which says that $\mathcal{U}$ is less than $\mathcal{V}$ if it is a quotient of $\mathcal{V}$ under some mapping of the natural numbers.

Let define following useful order concepts.

3.1 Definition. A *quasiorder* is a set with a transitive reflexive relation $\leq$. ◇

3.2 Definition. A *partial order* is antisymetric quasiorder. ◇

3.3 Definition. A partial order is *directed* if for any two members there is another member above both. ◇

3.4 Definition. A subset $A \subseteq X$ of partially ordered set $\langle X, \leq \rangle$ is *cofinal* if $\forall x \in X \exists a \in A (x \leq a)$. ◇

3.5 Definition. A subset $A \subseteq X$ of partially ordered set $\langle X, \leq \rangle$ is *bounded* if $\exists x \in X \forall a \in A (a \leq x)$. ◇

3.6 Observation. *If $\langle X, \leq \rangle$ is directed order, and $A \subseteq X$ is cofinal, then A is directed.*

Proof. For any two $a, b \in A$ there is another $c \in X$ above. The cofinality gives some $d \in A$ above c. From transitivity $a, b \leq d$. □

3.7 Definition. A function $f : X \to Y$ is cofinal if the image of each cofinal subset of X is cofinal in Y. ◇

3.8 Definition (Tukey)**.** [13] A partial ordering $\langle Y, \leq_Y \rangle$ is *Tukey reducible* to a partial ordering $\langle X, \leq_X \rangle$, $X \leq_T Y$, if there is a cofinal function $f : Y \to X$. ◇

3.1 Orders on filters on ω

3.9 Definition (image of a filter under a function $f : \omega \to \omega$)**.** For $f \in {}^{\omega}\omega$ and a filter $\mathcal{V} \subseteq \mathcal{P}(\omega)$ let

$$f(\mathcal{V}) = \{x \subseteq \omega \mid \exists y \in \mathcal{V} f[y] \subseteq x\}.$$

$\diamond$

3.10 Observation. $f(\mathcal{V}) = \{x \subseteq \omega \mid f^{-1}[x] \in \mathcal{V}\}$

3.11 Observation. *If $\mathcal{V} \subseteq \mathcal{P}(\omega)$ is an ultrafilter over ω, then $\mathcal{U} = f(\mathcal{V})$ is also an ultrafilter over ω.*

Proof. Since $f^{-1}[\omega] = \omega$, so $\omega \in \mathcal{U}$, and since $f^{-1}[\emptyset] = \emptyset$, so $\emptyset \notin \mathcal{U}$.

If $x \subseteq x'$ and $x \in f(\mathcal{V})$, then $f[y] \subseteq x$ for some $y \in \mathcal{V}$, and therefore $f[y] \subseteq x'$, which shows that $x' \in f(\mathcal{V})$.

If $x, x' \in f(\mathcal{V})$, then $f^{-1}[x], f^{-1}[x'] \in \mathcal{V}$, and since $\mathcal{V}$ is a filter, $f^{-1}[x] \cap f^{-1}[x'] \in \mathcal{V}$. Since $f^{-1}[x \cap x'] \in \mathcal{V}$ we get $x \cap x' \in f(\mathcal{V})$.

If $x \notin f(\mathcal{V})$, then $f^{-1}[x] \notin \mathcal{V}$, and $\omega \setminus f^{-1}[x] \in \mathcal{V}$, then $f^{-1}[\omega] \setminus f^{-1}[x] \in \mathcal{V}$, and $f^{-1}[\omega \setminus x] \in \mathcal{V}$, so $\omega \setminus x \in \mathcal{V}$. $\mathcal{U}$ is ultrafilter. $\square$

3.12 Lemma. *If $\mathcal{U}$ is ultrafilter and $f(\mathcal{U}) = \mathcal{U}$, then $\{n \mid f(n) = n\} \in \mathcal{U}$, i.e. f is identity on a set in $\mathcal{U}$.*

Proof. Let $A = \{n \mid f(n) = n\}$, $B = \{n \mid f(n) < n\}$, and $C = \{n \mid f(n) > n\}$. $f^{(n)}$ denotes n-th iteration of f.

If $B \in \mathcal{U}$, let $B_n = \{m \mid \forall n' < n(f^{(n')}(m) \in B)$ and $f^{(n)}(m) \notin B\}$.

$$B = \bigcup_{1 \leq n} B_n$$

One of $B_E = \bigcup_{1 \leq n} B_{2n}$ and $B_O = \bigcup_{1 \leq n} B_{2n+1}$ is in $\mathcal{U}$ because $\mathcal{U}$ is ultrafilter.

If $B_E \in \mathcal{U}$, then $f[B_E] \in \mathcal{U}$, and if $B_O \in \mathcal{U}$, then $f[B_O] \in \mathcal{U}$, so both cases are impossible, $B \notin \mathcal{U}$

If $C \in \mathcal{U}$, let $C_n = \{m \mid \forall n' < n(f^{(n')}(m) \in C)$ and $f^{(n)}(m) \notin C\}$.

$$C = \bigcup_{1 \leq n} C_n.$$

Same as for B, $C \notin \mathcal{U}$. Let $C^c = \omega \setminus C$ and $C^c_0 = \{n \in C^c \mid n \notin f[C^c]\}$, $C^c_n = \{m \in C^c \mid \forall n' < n(m \in f^{(n')}[C^c_0])$ and $m \notin f^{(n)}[C^c_0]\}$, so $C^c \notin \mathcal{U}$, then $A \in \mathcal{U}$. $\square$

3.13 Definition (Rudin-Keisler order, [7])**.** Let $\mathcal{F}$, $\mathcal{G}$ be filters. If there is a function $f : \omega \to \omega$ such that $A \in \mathcal{F}$ if and only if $f^{-1}[A] \in \mathcal{G}$, then $\mathcal{F} \leq_{RK} \mathcal{G}$. $\diamond$

3.14 Definition. $\mathcal{F} \equiv_{RK} \mathcal{G}$ if and only if $\mathcal{F} \leq_{RK} \mathcal{G}$ and $\mathcal{G} \leq_{RK} \mathcal{F}$. $\diamond$

Ultrafilters that are RK equivalent are said to be isomorphic. There are several partial orders on *isomorphism types* of ultrafilters in the following definitions. The given isomorphism type means the set of all isomorphic ultrafilters.

3.15 Observation. *If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on ω and $\forall A \in \mathcal{V}(f[A] \in \mathcal{U})$, then f witnesses that $\mathcal{U} \leq_{RK} V$.*

Proof. Let $B \in \mathcal{U}$, for contradiction let $f^{-1}[B] \notin \mathcal{V}$, then $\omega \setminus f^{-1}[B] \in \mathcal{V}$, so $f^{-1}[\omega \setminus B] \in \mathcal{V}$, then $f[f^{-1}[\omega \setminus B]] \subseteq \omega \setminus B \in \mathcal{U}$, then $B \notin \mathcal{U}$.

The other side, let $f^{-1}[A] \notin \mathcal{V}$, then $\omega \setminus f^{-1}[A] \in \mathcal{V}$, and $f^{-1}[\omega \setminus A] \in \mathcal{V}$, so $f[f^{-1}[\omega \setminus A]] \subseteq \omega \setminus A \in \mathcal{U}$, and then $A \notin \mathcal{U}$. $\square$

The relation $\leq_{RK}$ is a quasiorder since the relation is not antisymmetric. Transitivity is given by the function compositions.

3.16 Definition (Katětov order, [7])**.** Let $\mathcal{F}$, $\mathcal{G}$ be filters. If there is a function $f : \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{G}$, for all $A \in \mathcal{F}$ then $\mathcal{F} \leq_K \mathcal{G}$. $\diamond$

As noted in [5], the Katětov order was introduced by Miroslav Katětov[2] together with the Rudin-Keisler order.

On ultrafilers the Rudin-Keisler and Katětov orders are the same. Katětov equivalence is defined in the same way as RK-equivalence.

3.17 Observation. *If $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{F} \leq_K \mathcal{G}$.*

We consider the following variant of Katětov order defined above.

3.18 Definition (Katětov-Blass order, [7])**.** Let $\mathcal{F}$, $\mathcal{G}$ be filters. If there is a finite-to-one function $f : \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{G}$, for all $A \in \mathcal{F}$ then $\mathcal{F} \leq_{KB} \mathcal{G}$. $\diamond$

3.19 Definition (Rudin-Blass order, [8])**.** Let $\mathcal{F}$, $\mathcal{G}$ be filters. If there is a finite-to-one function $f : \omega \to \omega$ such that $A \in \mathcal{F}$ if, and only if $f^{-1}[A] \in \mathcal{G}$, then $\mathcal{F} \leq_{RB} \mathcal{G}$. $\diamond$

[2] *Since 1953 to 1957 he was rector of Charles University in Prague.*

An ultrafilter can be considered as a partial ordering by reverse inclusion. So $\langle \mathcal{U}, \supseteq \rangle$ is a directed partial ordering.

3.20 Definition (Tukey order)**.** Let $\mathcal{U}$, $\mathcal{V}$ be ultrafilters. If there is a cofinal function $f : \mathcal{V} \to \mathcal{U}$, then $\mathcal{U} \leq_T \mathcal{V}$. ◇

3.21 Observation. *Let $\mathcal{U}$, $\mathcal{V}$ are ultrafilters. If $\mathcal{U} \leq_{RK} \mathcal{V}$, then $\mathcal{U} \leq_T \mathcal{V}$.*

Proof. Let $\forall A \in \mathcal{V}(f[A] \in \mathcal{U})$. The Tukey function $f'(A) = f[A]$ for all $A \in \mathcal{V}$. If $\mathcal{B}$ is cofinal in $\mathcal{V}$, then $\forall A \in \mathcal{V} \exists B \in \mathcal{B}(A \supseteq B)$, and then $\forall A \in \mathcal{V} \exists B \in \mathcal{B}(f[A] \supseteq f[B])$, so f' is cofinal and $\mathcal{U} \leq_T \mathcal{V}$. □

Tukey ordering on ultrafilters is a weakening of Rudin-Keisler ordering. The Tukey equivalence class of an ultrafilter is called its Tukey type.

The folowing standard definitions are taken from unpublished notes of my advisor:

3.22 Definition (Fubini product)**.** Let $\mathcal{F}$, $\mathcal{G}$ be filters on ω. $\mathcal{F} \times \mathcal{G} = \{A \subseteq \omega \times \omega \; |\{n \; |A(n) \in \mathcal{G}\} \in \mathcal{F}\}$ where $A(n)$ is vertical section at n; $A^x(n) = \{m \; |(n, m) \in A\}$. ◇

3.23 Definition (F-sum)**.** If $\{\mathcal{F}_s \; |s \in S\}$ is a set of filters and $\mathcal{F}$ is a filter on S. Then the $\mathcal{F}$-sum of the filters is

$$\mathcal{F} - \sum_{s \in S} \mathcal{F}_s = \{A \subseteq \bigcup_{s \in S} \{s\} \times S_s \; |\{s \; |A_x(s) \in \mathcal{F}_s\} \in \mathcal{F}\}$$

◇

3.24 Definition (Free-product filter)**.** Let $\mathcal{F}$, $\mathcal{G}$ be filters on ω.

$$\mathcal{F} \otimes \mathcal{G} = \{(A, B) \; |A \in \mathcal{F} \text{ and } B \in \mathcal{G}\}$$

◇

3.2 Standard combinatorial properties

Let us define special sorts of ultrafilters. The first combinational property of filters is a generalization of the standard P-point property of ultrafilters.

3.25 Definition (P-filter)**.** A filter $\mathcal{F}$ is *P-filter* if for every (descending: $A_0 \supseteq A_1 \supseteq A_2 ...$) countable sequence $\langle A_n \in \mathcal{F} \; |n < \omega \rangle$ of elements of $\mathcal{F}$ there exists $X \in \mathcal{F}$ such that $X \subseteq^* A_n$ (for all $n < \omega$ $X \setminus A_n$ is finite).

Non-principal ultrafilters which are P-filters are called P-points (weakly selective). A point of topological space is a P-point if its neighbourhoods filter is closed under countable intersections. ◇

3.26 Definition (P-ultrafilter)**.** An ultrafilter $\mathcal{U}$ is *P-ultrafilter* (weakly selective) if for all factoring $\dot{\bigcup}_{n<\omega} X_n = \omega$ is satisfied one of the following items:

1. $\exists n < \omega (X_n \in \mathcal{U})$;
2. $\exists X \in \mathcal{U}\ \forall n\ (|X \cap X_n| < \omega)$.

◇

3.27 Observation. *An ultrafilter $\mathcal{U}$ is P-ultrafilter if and only if*

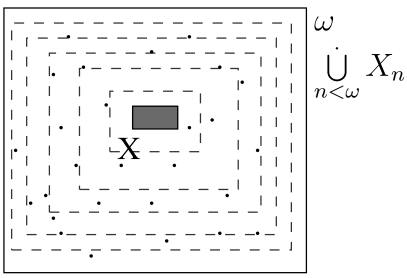
$$\forall f : \omega \to \omega\ \exists X \in \mathcal{U}\ (f \restriction X \text{ is finite-to-one or constant}).$$

Proof. Let there is a factoring $\langle X_n \rangle_{n\in\omega}$. The factoring can be translated to function f satisfying $f(x) = n \Leftrightarrow x \in X_n$ and vice versa. Then there exists $X \in \mathcal{U}$. $f \restriction X$ is constant if and only if $\exists n < \omega(X_n \in \mathcal{U})$. $f \restriction X$ is finite-to-one if and only if $\forall n\ (|X \cap X_n| < \omega)$. □

3.28 Observation. *The definitions of P-point ultrafilter and P-ultrafilter are equivalent.*

Proof. Let there is a factoring $\langle X_n \rangle_{n\in\omega}$. If some set $X_n \in \mathcal{U}$, it is finished. If no partition is in the ultrafilter, let there is an enumeration of theirs complements: $\langle X'_n \mid X'_n = \omega \setminus X_n \text{ for } n \in \omega\rangle$. For this set exists $X \in \mathcal{U}$, and for every $n \in \omega$, $|X \cap X'_n| < \omega$.

The other direction, let $\langle A_n \in \mathcal{U} \mid n < \omega\rangle$ is a sequence in $\mathcal{U}$. Without loss of generality the sequence is strictly decreasing, and $A_0 = \omega$. If $\mathcal{U}$ contains the intersection, it is finished. If not, let consider the factoring defined $X_n = A_n \setminus A_{n+1}$ ilustrated on the following picture.



No part this factoring of ω is in $\mathcal{U}$ since if $X_n \in \mathcal{U}$ then $X_n \cap A_{n+1} = \emptyset \in \mathcal{U}$. There is some $X \in \mathcal{U}$ where $|X \cap A_n| < \omega$. Proof by induction, $X \subseteq A_0$. Suppose $X \subseteq^* A_n$. $X \cap A_{n+1} = (X \cap A_n) \setminus X_n$, since $X_n \cap X$ is finite, then $X \cap A_n =^* X \cap A_{n+1}$, so $X \subseteq^* A_{n+1}$. □

3.29 Definition (Q-filter)**.** A filter $\mathcal{F}$ is *Q-filter* if for every partition P of ω into finite sets there is a selector $A \in \mathcal{F}$, i.e. $\forall p \in P(|A \cap p| = 1)$. $\diamond$

3.30 Definition (Rapid-filter)**.** A filter $\mathcal{F}$ is *Rapid-filter* if for each function $h : \omega \to \omega$, there is $A \in \mathcal{F}$ with $|A \cap h(n)| \leq n$ for every $n < \omega$. $\diamond$

Chapter III

This chapter presents the filters on ω in the context of their topological properties. It means to identify filters on ω with subsets of Cantor space 2^ω.

4.1 Topology

In classical topology the points of a space are primitive objects and open sets are defined as sets of points (point-set topology).

4.1 Definition (Topological space)**.** A *Topological space* is an ordered pair $\langle X, \tau \rangle$, where X is a set and $\tau \subseteq \mathcal{P}(X)$ such that:

1. $\emptyset, X \in \tau$;
2. if $\mathcal{A} \subseteq \tau$, then $\bigcup \mathcal{A} \in \tau$;
3. if $A, B \in \tau$, then $A \cap B \in \tau$.

The collection τ is called *topology*. Members of the topology are called *open* sets. A set is called *closed* if its complement is open. As noted in [12], the idea behind this definition, at least for the standard spaces, is that an open set is one which contains no point of its boundary. For instance, in 2-dimensional euclidean space, an open disc, meaning the set of points having distance strictly less than some fixed number from a fixed point, forms an open set. Another way to explain this is that wherever in the set it is possible to move a little in any direction, and stay in the set. For the closed disc moving any distance may leave the set.

Though the definition of closed as the complement of open, it is possible for a set to be both closed and open. In this case the set is called *clopen*. Obvious examples of clopen sets in all spaces are $\emptyset$ and X, but there may be many more clopen sets than that. The more clopen sets are in the more disconnected spaces.

◇

4.2 Definition (Neighbourhood)**.** N_x is *neighbourhood* of $x \in X$ if there is an open set O containing x such that $O \subseteq N_x$. If N_x is open, we call it open neighbourhood O_x. ◇

4.3 Observation. *Directly from definition, the system of closed sets contains X and $\emptyset$ and is closed under arbitrary intersections and finite unions (De Morgan's laws).*

4.4 Lemma. *The set A is open, if and only if* $\forall x \in A\ \exists N_x(N_x \subseteq A)$.

Proof. The right direction is obvious. Let $\forall x \in A(N_x \subseteq A)$, so

$$S = \bigcup \{N_x \mid x \in A\}$$

is open and $\forall x \in S(N_x \subseteq S)$, then $A \subseteq S$. $\forall x \in S(N_x \subseteq A)$, then $S \subseteq A$, then $A = S$ and A is open. □

4.5 Definition (Interior)**.** If Y is a subset of X, let int(Y) be the union of open sets contained in Y.

$$\text{int}(Y) = \bigcup \{O \in \tau \mid O \subseteq Y\}$$

◇

4.6 Definition (Closure)**.** Let $\overline{Y}$ be the intersection of all closed sets containing Y.

$$\overline{Y} = \bigcap \{C \mid C \text{ is closed and } Y \subseteq C\}$$

◇

4.7 Observation. *int(Y) is the greatest open set contained in Y and $\overline{Y}$ is the smallest closed set containing Y in the ordering under inclusion.*

4.8 Definition. Set $D \subseteq X$ is *dense* in (X, τ) if $\overline{D} =$ X. ◇

4.9 Definition. Set $\mathcal{B} \subseteq \mathcal{P}(X)$ is topology *base* if:

1. for $U, V \in \mathcal{B}$ and $x \in U \cap V$ then $\exists W \in \mathcal{B}(x \in W \subseteq U \cap W)$;
2. $\forall x \in X\ \exists U \in \mathcal{B}(x \in U)$.

◇

4.10 Definition (Compactness)**.** $\langle X, \tau \rangle$ is *compact* if every open cover of X has a finite subcover, where C is an open cover if $C \subseteq \tau$ and $\bigcup C = X$.

Conversely if F is a system of closed sets and has FIP then $\bigcap F$ is non-empty. ◇

4.11 Definition. $\langle X, \tau \rangle$ is *locally compact* if every point x has a compact neighbourhood. ◇

4.12 Definition (Filter converges to x)**.** Let $\mathcal{F}$ be a filter on X and $x \in X$. We say that the filter converges to x, or that x is a limit of $\mathcal{F}$, if all $N_x \subseteq \mathcal{F}$. ◇

4.13 Example. *Fréchet filter $\mathcal{F}$ in discrete topology on ω is a non-convergent filter: the singleton set $\{n\}$ cannot belong to $\mathcal{F}$.*

4.14 Definition (Hausdorff space)**.** A *Hausdorff space* is a topological space with a separation property: any two distinct points can be separated by disjoint open sets. ◇

4.15 Observation. *Singleton set is closed in Hausdorff space.*

Proof. Let there be a point x in space X. For any point y different from x there is an open neigbourhood N_y not containing x. So

$$\bigcup_{y \in X \setminus \{x\}} N_y = X \setminus \{x\}$$

is open. □

4.16 Lemma. *X is Hausdorff space if every filter has at most one limit.*

Proof. Suppose X is Hausdorff and let $x \neq y$. Then there are neighbourhoods U and V of x and y respectively with $U \cap V = \emptyset$. No filter contains both U and V, and so no filter can converge to both x and y. Hence all filters have at most one limit.

Conversely, suppose that x and y do not have disjoint neighbourhoods. Then $N_x \cup N_y$ forms a subbase for a filter which converges to both x and y. So if every filter has at most one limit then X is Hausdorff. □

So requiring X to be Hausdorff is equivalent to requiring unique limits. In Hausdorf space $\lim_{\mathcal{F}} = x$ means x is unique limit of $\mathcal{F}$. Note that not all filters have a limit.

4.17 Definition (Regular space)**.** A *regular space* is a topological space with a separation property: Any point and closed set can be separated by disjoint open sets. ◇

4.18 Definition (Normal space)**.** A *normal space* is a topological space with a separation property: Any two distinct closed sets can be separated by disjoint open sets. ◇

4.19 Definition (Continuous function)**.** Let $\langle X, \tau \rangle$, $\langle Y, \sigma \rangle$ be topological spaces and $f : X \to$ Y is function. f is *continuous* if for every open set U in Y, $f^{-1}[U]$ is open in X. ◇

4.20 Observation. *A topological space is normal if and only if for every open set U and every closed $C \subseteq U$, there is an open set V such $C \subseteq V \subseteq \overline{V} \subseteq U$.*

4.21 Fact. *A closed subset of a compact space is compact.*

4.22 Fact. *A compact subset of a Hausdorff space is closed.*

4.23 Fact. *The continuous image of closed set in the compact space is closed.*

4.24 Definition (Metric space)**.** A *Metric space* is an ordered pair $\langle X, \rho \rangle$, where X is a set and $\rho : X^2 \to \mathbb{R}$. ρ is called *metric* if it has folowing properties:

1. $\rho(x, y) \leq 0$ for all $x, y \in X$;
2. $\rho(x, y) = 0$ if and only if $x = y$;
3. $\rho(x, y) = \rho(y, x)$;
4. $\rho(x, z) \leq \rho(x, y) + \rho(x, z)$ for all $x, y, z \in X$.

◇

4.25 Theorem (Urysohn's lemma)**.** [3] *Let $\langle X, \tau \rangle$ be normal space and F, H be closed sets such that $F \cap H = \emptyset$, then exists a continuous function which separates F and H.*

4.26 Definition (Product topology)**.** Let $\langle X_i, \tau_i \rangle$ be topological spaces for $i \in I, I \neq \emptyset$. Consider space $\langle \prod_{i \in I} X_i, \tau_\varphi \rangle$ where O is a basic open set in τ_φ if and only if there is some finite $J \subseteq I$ and open sets $V_j \in \tau_j$ for $j \in J$ such that

$$O = \bigcap \{\pi_j^{-1}[V_j] \mid j \in J\},$$

where π_j is projection of $\prod_{i \in I} X_i$ on the X_j component. ◇

A base for the product topology consists of all finite intersections of cylinders so the projections are continuous and it is preferable in contrast to natural box topology.

4.27 Theorem (Tychonoff)**.** *If each space X_i is compact, then $\prod_{i \in I} X_i$ is compact.*

[3] *Urysohn's lemma has useful applications. For example Urysohn Metrization Theorem. If X is a normal space with a countable basis, then there is the continuous function from X to [0, 1] to assign numerical coordinates to the points of X and obtain an embedding of X into $\mathbb{R}^\omega$. From this, every countable normal space is a metric space.*

4.2 Cantor space

The product space 2^ω (all functions from the set ω to the discrete space whose only members are 0 and 1, with the product topology) is called the *Cantor space*.

4.28 Definition (Cantor space)**.** *Cantor space* is countable product of two point space with discrete topology. ◇

The following is a corollary of the Tychonoff theorem.

4.29 Observation. *Cantor space is compact.*

4.30 Definition (Standard metric in Cantor space)**.** $\rho(x, y) = 2^{-(r-1)}$ where $r = min\{n \mid x_n \neq y_n\}$ ◇

For $A \subseteq \omega$, $a \in [A]^{<\omega}$ and $b \in [\omega \setminus A]^{<\omega}$, $[a, b] = \{X \in 2^\omega \mid a \subseteq X$ and $b \cap X = \emptyset\}$; $[a, b]$ is basic clopen sets in 2^ω.

4.31 Observation. *The intersection is continuous function* $\cap : 2^\omega \times 2^\omega \to 2^\omega$.

Proof. Pre-image $\cap^{-1}[[a, b]] = \{(A, B) \mid A \cap B \in [a, b]\}$ and $\cap^{-1}[[a, b]] = \{(A, B) \mid a \subseteq A \cap B$ and $b \subseteq (\omega \setminus A) \cup (\omega \setminus B)\}$. For all A, B, if $A \cap B \in [a, b]$, then $(A, B) \in [a, b \setminus A] \times [a, b \setminus B]$, so every point from $\cap^{-1}[[a, b]]$ is contained in the clopen neighbourhood. For showing that $[a, b \setminus A] \times [a, b \setminus B] \subseteq \cap^{-1}[[a, b]]$, let some $(P_0, P_1) \in [a, b \setminus A] \times [a, b \setminus B]$, then $a \subseteq P_0$, $a \subseteq P_1$, $(b \setminus A) \cap P_0 = \emptyset$ and $(b \setminus B) \cap P_1 = \emptyset$, so $a \subseteq P_0 \cap P_1$ and $b \subseteq (\omega \setminus P_0) \cup (\omega \setminus P_1)$. □

4.32 Observation. *The union is continuous function* $\cup : 2^\omega \times 2^\omega \to 2^\omega$.

4.33 Theorem. *A topological space X is compact if and only if every ultrafilter on X converges to at least one point.*

Proof. Suppose that X is compact, and let $\mathcal{U}$ be an ultrafilter on X. Then $\mathcal{U}$ has FIP, since it is closed under finite intersections, and $\emptyset \notin \mathcal{U}$. Compactness causes that there is some point $x \in \bigcap_{B \in \mathcal{U}} \overline{B}$. This means that every open neighbourhood of x meets every $B \in \mathcal{U}$. Let N_x be an open neighbourhood of x. Since no member of $\mathcal{U}$ is disjoint from N_x, in particular $X \setminus N_x \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, it must be that $N_x \in \mathcal{U}$. This proves that $\mathcal{U}$ converges to x.

For the converse, suppose that every ultrafilter converges and let F be a family of subsets of X that has FIP. Then F generates a filter, which can be extended to an ultrafilter $\mathcal{U}$. By assumption, $\mathcal{U}$ converges to some point x. Consider $B \in F$. Since $\mathcal{U}$ converges to x, every neighbourhood of x meets B. This says exactly that $x \in \overline{B}$, so, since this is true of every $B \in F$, so $x \in \bigcup_{B \in F} \overline{B}$. This proves that X is compact. □

4.34 Definition (P-point)**.** A point x in topological space X is called a *P-point* if the intersection of countably many neighbourhoods of x contains a neighbourhood of x. ◇

4.35 Definition (Weak P-point)**.** A point x in a topological space that is not an accumulation point of any countable subset of the space is called a *weak P-point*. Every P-point is a weak P-point. ◇

4.3 Definable sets

Descriptive set theory clasifies subsets of a topological space according to the complexity of their definitions. *Borel hierarchy* is used to describe classes of subsets of $\mathbb{R}$, Baire space or Cantor space, etc. Level one consists of all open (Σ_1^0) and closed (Π_1^0) sets, and levels 2, 3, 4, ... are obtained by taking countable unions and intersections of the sets on the previous level. More complex definable sets are *projective sets*, those obtained from Borel sets by the operation of continuous image and complementation.

4.36 Definition (F_σ)**.** A set $A \subseteq \mathbb{R}$ is F_σ[4] if it is a countable union of closed sets. The class is denoted Σ_2^0 in logical notation. ◇

4.37 Definition (G_δ)**.** A set $A \subseteq \mathbb{R}$ is G_δ[5] if it is a countable intersection of open sets. The class is denoted Π_2^0 in logical notation. ◇

The next levels are $F_{\sigma\delta}$, it is a countable intersections of F_σ. And $G_{\delta\sigma}$, it is a countable unions of G_δ.

4.38 Example. *Consider real numbers with the usual topology.* $\mathbb{Q} = \bigcup_{n \in \omega} \{q_n\}$ *is* F_σ *and the complement of* $\mathbb{Q}$ *must be* G_σ *set.*

[4]F_σ *comes from French: The F stands for fermé, meaning "closed,"while the sigma stands for somme, meaning "sum."*

[5]G_δ *comes from German: The G stands for Gebiet, meaning "area," while the delta stands for Durchschnitt, meaning "intersection."*

4.4 Meager sets

Meager set (or a set of first category) is a set that, considered as a subset of a topological space, is in a precise sense small or negligible.

4.39 Definition (Nowhere dense set)**.** Given a topological space X, a subset A of X is *nowhere dense* if for every non-empty open set O there is a non-empty open set $O' \subseteq O$ such that $O' \cap A = \emptyset$. ◇

A subset B of X is nowhere dense if there is no neighbourhood on which B is dense: for any nonempty open set U in X, there is a nonempty open set V contained in U such that V and B are disjoint.

4.40 Definition (Meager set)**.** Given a topological space X, a subset A of X is *meager* (the first category) if it can be expressed as the union of countably many nowhere dense subsets of X. ◇

The rational numbers are meager as a subset of $\mathbb{R}$. The Cantor set is meager as a subset of $\mathbb{R}$, but not as a space, since it is complete metric space.

4.41 Definition (Baire space)**.** A topological space is called a *Baire space* if the complements of meager sets in X are dense. ◇

4.42 Lemma. *A topological space is Baire if and only if the intersection of countable many open dense sets in X is dense in X.*

Proof. Assume the space X is not Baire, so there is is a meager set M, such that $X \setminus M$ is not dense. Assume that M is open. $M = \bigcup_{n\in\omega} A_n$; A_n are nowhere dense, and $\bigcap_{n\in\omega} X \setminus A_n$ is not dense. $X \setminus \overline{A_n}$ is open dense, so the intersection of countable many open dense sets is not dense.
For the other direction, let there be open dense sets A_n and $\bigcap_{n\in\omega} A_n$ is not dense, then there exists open set O and $\bigcap_{n\in\omega} A_n \cap O = \emptyset$. $A_n = O \cap (X \setminus A_n)$ is nowhere dense and $X \setminus \bigcup_{n\in\omega} A_n$ is not dense. □

4.43 Theorem (Baire category theorem)**.** *Every locally compact Hausdorff space $\langle X, \tau\rangle$ is Baire.*

Proof (taken from [12]). Let there be countable many open dense sets:

$$\mathcal{D} = \{D_{n\in\omega} \in \tau \mid D_n \text{ is dense}\},$$

and open set O, so $O \cap D_0$ is not empty, then there exists open set O_0,

$$\overline{O_0} \subseteq O \cap D_0,$$

by the regularity of locally compact Hausdorff space. Inductively there exists

$$\overline{O_{n+1}} \subseteq O_n \cap D_n.$$

$\bigcap\limits_{n\in\omega} \overline{O_n}$ has FIP and by the local compactness is not empty.

$$\bigcap_{n\in\omega} \overline{O_n} = \bigcap_{n\in\omega} O_n \subseteq \bigcap \mathcal{D} \cap O,$$

so $\bigcap \mathcal{D}$ is dense. □

4.5 Filters and convergence

Standard limit (convergence) of a sequence $\langle x_{n\in\omega} \mid x_n \in \mathbb{R}\rangle$ is defined:

$$\lim_{n\to\infty} x_n = a \text{ if } \forall\varepsilon\exists n_0\forall n > n_0(|a_n - a| < \varepsilon)$$

The notion of the filter convergence is a generalization of the classical notion of the convergence of a sequence. Let $\mathcal{N}_a$ be a set of all open neighbourhoods of a. $\mathcal{N}_a$ has following properties:

1. $X \in \mathcal{N}_a$;
2. if $A \in \mathcal{N}_a$ and $B \in \mathcal{N}_a$, then $A \cap B \in \mathcal{N}_a$;
3. if $A, B \subseteq \mathcal{N}_a$, $A \in \mathcal{N}_a$, and $A \subseteq B$, then $B \in \mathcal{N}_a$;
4. $\emptyset \notin \mathcal{N}_a$.

The neighbourhood satisfies the filter properties and is called a *neighbourhood filter*.

4.44 Definition. $\mathcal{F}$-$\lim x_n = a$ if $\forall A \in \mathcal{N}_a(\{n \mid x_n \in A\} \in \mathcal{F})$, for $\langle x_n \mid n \in \omega\rangle$. [6] ◇

[6] *Filter convergence was formulated by Henri Cartan around 1937 and explored by Bourbaki in the 1940s.*

In other words for all neighbourhoods A of the point a almost all sequence members are in $\mathcal{N}_a$. Standard limit definition is equivalent to $\mathcal{F}$-lim where $\mathcal{F}$ is Fréchet filter.

4.45 Observation. *Let S be a sequence $\langle x_{n\in\omega} \,|x_n \in \mathbb{R}\rangle$ and a its limit point. Then $a \in \overline{\{x_n \,|n < \omega\} \setminus \{a\}}$. Let $A = \{X \subseteq \omega \mid \lim\limits_{n\in X} x_n = a\}$, if A is non-empty, A is closed under union and subsets.*

Chapter IV

In this chapter we present Mazur's result on the relation between submeasures and F_σ ideals on ω.

5.1 Ideals and filters

5.1 Definition (Ideal over a set)**.** An *ideal* over a set X is a collection $\mathcal{I}$ of subsets of X such that:

1. $\emptyset \in \mathcal{I}$;
2. if $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$;
3. if $A, B \subseteq X$, $A \in \mathcal{I}$, and $A \subseteq B$, then $A \in \mathcal{I}$.

Given an ideal $\mathcal{I}$, $\mathcal{I}^*$ is the dual filter, consisting of complements of the sets in $\mathcal{I}$. Similarly, if $\mathcal{F}$ is a filter on X, $\mathcal{F}^*$ denotes the dual ideal.

$$\mathcal{I}^* = \{A \subseteq X \mid X \setminus A \in \mathcal{I}\}$$

◇

Duality between ideals and filters allows to examine only one of this concepts which is in some particular situation better. The sentences could be transformed using De Morgan's laws.

The ideal convergence is dual to the filter convergence. The sequence $\langle x_n \mid n \in \omega\rangle$ is $\mathcal{I}$-convergent to a if $\forall\varepsilon > 0$ $(\{n \in \omega \mid \varepsilon \leq |x_n - a|\} \in \mathcal{I})$, so I-lim $x_n = a$. If $\mathcal{I} = Fin$, then $\mathcal{I}$-convergence is equivalent to standard convergence.

5.2 Definition (P-ideal)**.** A ideal $\mathcal{I}$ is *P-ideal* if for every (increasing: $A_0 \subseteq A_1 \subseteq A_2 ...$) countable sequence $\langle A_i \in \mathcal{I} \mid i \in \omega\rangle$ of elements of $\mathcal{I}$ there exists $B \in \mathcal{I}$ such that $B \supseteq^* A_i$ for all $n < \omega$. $A_i \setminus B$ is finite. ◇

5.2 Submeasure

A measure on a set is a function which assigns a positive number to each suitable subset of given set. The measure is intuitively interpreted as size.

5.3 Definition. A submeasure on ω is a function $\varphi : \mathcal{P}(\omega) \to \mathbb{R}_0^+ \cup \{\infty\}$ satisfying:

1. $\varphi(\emptyset) = 0$;
2. if $A \subseteq B$ then $\varphi(A) \leq \varphi(B)$;
3. $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$.

To avoid trivialities, let $\varphi(A) < \infty$ for all finite subsets of ω. ◇

5.4 Definition. If φ submeasure satisfies $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap \{1, ..., n\})$, then φ is called a *lower semicontinuous submeasure* (lscsm). ◇

5.5 Definition. $Fin(\varphi) = \{A \subseteq \omega \mid \varphi(A) < \infty\}$, called a *finite ideal* of φ. ◇

5.6 Observation. *If φ is lscsm, then $Fin(\varphi)$ is an F_σ ideal.*

Proof. $Fin(\varphi) = \bigcup_{m\in\omega} \{A \subseteq \omega \mid \varphi(A) \leq m\}$. For φ lscsm is equal to

$$\bigcup_{m\in\omega} \{A \subseteq \omega \mid \lim_{n\to\infty} \varphi(A \cap \{1, ..., n\}) \leq m\}$$

and

$$\bigcup_{m\in\omega} \bigcap_{n\in\omega} \{A \subseteq \omega \mid \varphi(A \cap \{1, ..., n\}) \leq m\},$$

so $\varphi(A \cap \{1, ..., n\}) \leq m$ is finite union of closed sets, then $Fin(\varphi)$ is F_σ. □

5.7 Definition. Let X be a topological space. The function $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ is *lower semicontinuous* if and only if $\forall r \in \mathbb{R}(\{A \in X \mid f(A) \leq r\}$ is closed). ◇

5.8 Definition. $Exh(\varphi) = \{A \subseteq \omega \mid \lim_{n\to\infty} \varphi(A \setminus \{1, ..., n\}) = 0\}$, called the *exhaustive ideal* of φ. (Trivially by definition 5.3 $Exh(\varphi)$ is ideal.) ◇

5.9 Observation. *If φ is lscsm, then $Exh(\varphi) \subseteq Fin(\varphi)$.*

Proof. Let $A \in Exh(\varphi)$ then $\lim_{n\to\infty} \varphi(A \setminus \{1, ..., n\}) = 0$, so there is some n_0 which satisfies $\varphi(A \setminus \{1, ..., n_0\}) < \infty$. From definition 5.3.3

$$\varphi(A) \leq \varphi(A \setminus \{1, ..., n_0\}) + \varphi(A \cap \{1, ..., n_0\}),$$

so $\varphi(A) < \infty$. □

5.10 Observation. *If φ is lscsm, then $Exh(\varphi)$ is an $F_{\sigma\delta}$ P-ideal.*

Proof. Let $F_{m,n} = \{A \subseteq \omega \mid \varphi(A \setminus \{1, ..., m\}) \leq \frac{1}{n}\}$, $F_{m,n}$ is closed set, then

$$Exh(\varphi) = \bigcap_{n\in\omega} \bigcup_{m\in\omega} F_{m,n}$$

Let $\langle A_i \in \mathcal{I} \mid i \in \omega\rangle$ is in $Exh(\varphi)$, then let have a sequence

$$\langle n_i \mid \varphi(A_i \setminus \{1, ..., n_i\}) \leq \frac{1}{2^{n+1}}\rangle,$$

and $B = \bigcup_{i\in\omega}(A_i \setminus \{1, ..., n_i\})$, so $A_i \setminus B$ is finite.

For any n there exists k

$$\varphi(\bigcup_{i\leq n} A_i \setminus \{1, ..., k\}) \leq \frac{1}{2^{n+1}},$$

so for any n $\varphi(B \setminus \{1, ..., k\}) \leq \frac{1}{2^n}$, then $B \in Exh(\varphi)$. □

5.11 Definition. A set $A \subseteq \mathcal{P}(\omega)$ is *hereditary* if it is closed under subsets. ◇

5.12 Lemma. *For any hereditary F_σ set H there exists a family $\{F_n \mid n \in \omega\}$ of hereditary closed sets such that $H = \bigcup_{n\in\omega} F_n$ and $F_n \subseteq F_{n+1}$ for $n \in \omega$.*

Proof. Let $H = \bigcup_{n\in\omega} D_n$ where D_n is closed for $n \in \omega$.

$$F_n = \{A \cap B \mid A \in \bigcup_{k\leq n} D_k \text{ and } B \in \mathcal{P}(\omega)\}.$$

F_n is closed because it is continuous image of closed sets in compact space. So F_n is hereditary closed set. □

5.13 Theorem (Mazur)**.** *Let $\mathcal{I}$ be an ideal on ω. Then $\mathcal{I}$ is an F_σ if and only if there is a lscsm φ such that $\mathcal{I} = Fin(\varphi)$.([9])*

The idea of the proof is to define such sets with the indexes satisfying the submeasure conditions.

Proof. For right direction of equivalence let have a F_σ-ideal $\mathcal{I}$.

$$\mathcal{I} = \bigcup_{n\leq\omega} D_n,$$

where each D_n are closed sets.

$$\mathcal{I} = \bigcup_{n \leq \omega} F'_n,$$

where each F'_n is hereditary closed and $F'_n \subseteq F'_{n+1}$ for each n. Let define $F_0, F_1, F_2, ...$ inductively:

1. $F_0 = F'_0$;
2. $F_{n+1} = \{A \cup B \mid A, B \in F_n\} \cup F'_{n+1}$.

$\{A \cup B \mid A, B \in F_n\}$ is closed because it is continuous image of closed sets in compact space. For every $A \in Fin$ there is $\varphi(A) = min(\{n \mid A \in F_n\})$ which satisfies:

3. $\varphi(\emptyset) = 0$;
4. $A \subseteq B \Rightarrow \varphi(A) \leq \varphi(B)$;
   Let $\varphi(A) > \varphi(B)$, then $\exists n(A \notin F_n$ and $B \in F_n)$ where F_n is hereditary, so $A \nsubseteq B$.
5. $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$;
   Let $\varphi(A \cup B) > \varphi(B) + \varphi(B)$, then
   $\exists m \exists n(A \cup B \notin F_{m+n}$ and $A \in F_m$ and $B \in F_n)$. Then $A \notin F_{m+n}$ and $B \notin F_{m+n}$.
6. φ is lscsm.

So we can extend φ to $\overline{\varphi} : \mathcal{P}(\omega) \to \mathbb{R}$:

$$\overline{\varphi} = \lim_{n\to\infty} \varphi(A \cap \{1, ..., n\})$$

and $\mathcal{I} = \{A \subseteq \omega \mid \lim_{n\to\infty} \varphi(A \cap \{1, ..., n\}) < \infty\}$.

For the proof of the left direction there is a submeasure $\varphi : Fin \to \mathbb{R}_0^+ \cup \{\infty\}$, so for every n let

$$F_n = \{A \subseteq \omega | \ \forall k \in \omega \ (\varphi(A \cap \{1, ..., k\}) \leq n)\},$$

so

$$F_n = \bigcup_{k \leq \omega} \{A \subseteq \omega | \ \varphi(A \cap \{1, ..., k\}) \leq n\}.$$

For fixed k the set is a finite union of basic clopen sets, so F_n is closed and $\mathcal{I} = \bigcup_{n \leq \omega} F_n$. $\mathcal{I}$ is hereditary, closed under finite unions and $\omega \notin \mathcal{I}$. □

Following examples shows some ideals on the countable sets.

5.14 Example. $\mathcal{I}_{\frac{1}{n}} = \{A \subseteq \omega \mid \sum\limits_{n \in A} \frac{1}{n} < \infty\}$ *is* F_σ *P-ideal where submeasure* φ *is defined:* $\varphi(A) = \sum\limits_{n \in A} \frac{1}{n}$

5.15 Example. $\mathcal{I}_{Fin^\omega} = \{A \in 2^{\omega \times \omega} \mid \forall n \in \omega((\{n\} \times \omega) \cap A \text{ is finite})\}$

5.16 Example. $\mathcal{I}_{nwd} = \{A \subseteq \mathbb{Q} \mid A \text{ is nowhere dense in } \mathbb{R}\}$ *is neither a P-ideal nor* F_σ*.*

5.17 Example ([11])**.** $\mathcal{I}_1 = \{A \in 2^{\omega \times \omega} \mid \exists n \in \omega(A \subseteq n \times \omega)\}$

5.18 Theorem (Solecki, [11])**.** $\mathcal{I}$ *is an analytic P-ideal if and only if there is a lscsm* φ *such that* $\mathcal{I} = Exh(\varphi)$*.*

Analytic P-ideals are $F_{\sigma\delta}$. This is, in fact, a corollary of the previous theorem.

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