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The thesis is a throughout study of the extensions of Belnap-Dunn logic ($\mathcal{BD}$), which are known as *super-Belnap* logics. The most well-know of such extensions are the strong three-valued logic of Kleene ($\mathcal{K}$), the Logic of Paradox ($\mathcal{LP}$) of Graham Priest and the Exactly true Logic ($\mathcal{ETL}$) introduced by Andreas Pietz and Umberto Rivieccio. The notion of logic here is that of a logic as a consequence relation invariant under substitutions (or structural), which is the notion of logic studied in abstract algebraic logic. The methods used in the thesis are the methods of algebraic logic together with important findings by the author that relate the finite matrix models of super-Belnap logics with finite graphs. This allows to use some results on the theory of graphs as the Erdö's Girth-Chromatic Number Theorem.

The thesis studies several aspects of super-Belnap logics, like axiomatization by Hilbert calculi, completeness, their classification into the hierarchies considered in abstract algebraic logic, axiomatizations by sequent calculi, interpolation property, and expansions by a truth operator. The thesis also addresses, and I see it as the main topic of the thesis, the structure of the lattice Ext$\mathcal{BD}$ of all super-Belnap logic, whose top element is classical logic, as well as the structure of the lattice of all finitary super-Belnap logics. The thesis contains a wealth of important results that I hope to see published soon.

The most studied lattices of extensions of logics are the lattice of the extensions of the normal modal logic K, the lattice of the extensions of some other modal logics like $S4$, and the lattice of superintuitionistic logics. Since these logics are algebraizable (in an appropriate presentation) the study of them can be reduced to the study of the lattice of subvariaties of the variety of modal algebras, the lattice of subvariaties of the variety of interior algebras and to that of subvarieties of the variety of Heyting algebras. These lattices can be studied by the standard tools of universal algebra. In the case of the extensions of Belnap-Dunn logic we have a completely different situation. The super-Belnap logics are not algebraizable for many reasons, the most basic because they are not protoalgebraic except for classical logic. This implies that the relation of them with their algebraic counterparts is not as strong as it is for algebraizable logics. In fact, as shown in the thesis, many super-Belnap logics have the same algebraic counterpart, for example all the three logics in the interval $[\mathcal{BD}, \mathcal{ETL}]$ have as algebraic counterpart the variety of De Morgan algebras. Consequently, to study the lattice Ext$\mathcal{BD}$ one need to devise different methods. One of the main merits of the thesis is the finding of very original methods to delve into the structure of Ext$\mathcal{BD}$.

The first three chapters of the dissertation are devoted to introduce and recall the basic notions needed in a systematic and clear way. Chapter 2 concerns De Morgan lattices and De Morgan algebras and Chapter 3 presents the basic known results on Belnap-Dunn logic and the logics $\mathcal{K}$, $\mathcal{LP}$, $\mathcal{ETL}$; also the Kleene logic of the order $\mathcal{KO}$, which is the infimum of $\mathcal{K}$ and $\mathcal{LP}$ in the lattice of extensions of Belnap-Dunn logic, is studied. The reduced matrix models are described, as well as the subdirectly irreducible ones.

In Chapter 4 the author introduces a new notion, that of an *explosive extension* of a logic. A logic $\mathcal{L}$ is an explosive extension of a logic $\mathcal{L}'$ if $\mathcal{L}$ is obtainable from $\mathcal{L}'$ by considering sets $\{\Delta_i : i \in I\}$ and for each $i \in I$ adding to $\mathcal{L}'$ the rules $\sigma[\Delta_i] \vdash Fm$ for every substitution σ to obtain $\mathcal{L}$. To every super-Belnap logic $\mathcal{L}$ it is associated its explosive part, namely the largest explosive extension of $\mathcal{BD}$ below $\mathcal{L}$. The explosive extensions of $\mathcal{BD}$ are shown to form a completely distributive complete sublattice of the lattice of extensions of $\mathcal{BD}$ and in which the join is given by the union. Chapter 5 is devoted to prove completeness theorems for certain crucial logics in the lattice of extensions of $\mathcal{BD}$ that play a central role in the study of splittings of Ext$\mathcal{BD}$.

Chapter 6 deals with the structure of the lattice of super-Belnap logics. Several splittings of it are described, which are given by logics whose completeness is studied in Chapter 5. Also some results on the lattice of explosive extensions of $\mathcal{BD}$ are proved.

The lattice of finitary extensions of $\mathcal{BD}$ is studied in Chapter 7. Using Cornish and Fowler duality for De Morgan algebras, to every finite reduced matrix model M of $\mathcal{BD}$, the algebra reduct of which is a finite De Moran algebra, is associated a dual frame with a distinguished set (that corresponds to the filter of the matrix). This frame is reduced in a sense defined in the thesis. It is shown how to associate with it two graphs and a natural number and a procedure to obtain from them a reduced matrix model isomorphic with M is given. In this way a correspondence between classes of finite reduced matrix models of $\mathcal{BD}$ and classes of triples given by two finite graphs and a natural number is established. This allows to describe the finitary extensions of $\mathcal{BD}$ by classes of triples $\langle G, H, k\rangle$, where G and H are graphs and k a natural number, that enjoy certain closure conditions. Using this description better results are obtained to describe certain intervals of the lattice of finitary extensions of $\mathcal{BD}$; in particular a graph-theoretic description of the lattices of the finitary explosive extensions of $\mathcal{ETL}$ and of $\mathcal{BDL}$ is obtained. Using the results on graphs it is shown that there is a non-finitary explosive extension of $\mathcal{ETL}$ and using the Erdö's Girth-Chromatic Number Theorem it is proved that for $n \geq 2$ the expansions $\mathcal{ECQ}_n$ and $\mathcal{ETL}_n$ of $\mathcal{BDL}$ are not complete w.r.t any finite set of finite matrices.

In Chapter 8 the super-Belnap logicss are classified in the Leibniz and Frege hierarchies studied in abstract algebraic logic. Except for classical logic, as we already mentioned, all the super-Belnap logics are non-protoalgebraic. The truth-equational super-Belnap logics are the extensions of $\mathcal{LP}$ or of $\mathcal{ETL}$, being the extensions of the latter the assertional super-Belnap logics. The only selfextensional super-Belnap logics are $\mathcal{BD}$, $\mathcal{KO}$ and classical logic; and classical logic is the only Fregean. Also the strong versions of the super-Belnap logics are described in Chapter 8.

Chapter 9 is devoted to provide sequent calculi for several super-Belnap logics and to the study of some of their proof-theoretic properties. The calculi operate on sequents that are pairs of multisets $(\Gamma \rhd \Delta)$. The topic of interpolation is also addressed in this chapter. It is shown that if a super-Belnap logic has interpolation then it is $\mathcal{BD}$ or $\mathcal{LP}$ or extends the logic $\mathcal{ECQ}$. Moreover it is proved that there is a continuum of finitary super-Belnap logics with interpolation as well as a continuum without it.

Chapter 10 has some miscellaneous results on variants of the notion of super-Belnap logic of the thesis. The language for super-Belnap logics considered in the dissertation has the two propositional constants $\mathtt{t}$ and $\mathtt{f}$. In many papers on Belnap-Dunn logic and extensions the language lacks these constants. The comparison is discussed and one sees that a part from the fragments without $\mathtt{t}$ and $\mathtt{f}$ of the super-Belnap logics there are four new ones. Another variant in the presentation of super-Belnap logics is as multiple-conclusion logics instead as single-conclusion. This variant is discussed in section 10.2. In section 10.3 we find a discussion of some universal strict Horn theories without equality of classes of matrices given by taking a de Morgan algebra and instead of a set of distinguished elements two such sets. The discussion is in the spirit of logic, namely in the spirit of the approach to universal strict Horn theories without equality using the methods of abstract algebraic logic.

Finally, in Chapter 7 the language of super-Belnap logics is expanded with a unary operation symbol. The De Morgan algebra $\mathbf{DM_4}$ is expanded by a unary operation that maps the top element $\mathtt{t}$ and the element $\mathtt{b}$ to $\mathtt{b}$ and the remaining elements to $\mathtt{f}$, obtaining the algebra $\mathbf{DM}_4^{\Delta}$. Then the variety $\mathsf{DMA}\Delta$ generated by this algebra is considered as well as the varieties generated by its three proper subalgebras. Several algebraic results on these varieties are obtained. For example an isomorphism between the congruences and the so-called Δ-filters for each algebra in $\mathsf{DMA}\Delta$, the characterization of the subdirectly irreducible elements, etc. The section concludes with a study of the logic $\mathcal{BD}\Delta$ of the matrix $\langle \mathbf{DM}_4^{\Delta}, \{\mathtt{t}, \mathtt{b}\}\rangle$. It is shown that they are algebraizable and that has four non-trivial axiomatic extensions. Moreover, a Hilbert calculus for it is given. Finally, a variant of $\mathcal{BD}\Delta$ is given in the language of distributive lattices with two unary operations and without negation.

The thesis is well organized and well written, although sometimes it is a bit hard to read: one would appreciate some more detail in some proofs.

Summary: This is an outstanding thesis. It constitutes a very significant step in the study of super-Belnap logics in that it does not consist in the the study of some particular super-Belnap logics and some of their neighbors as has been done in the literature, but in a global study of the lattices of all super-Belnap logics and of all finitary super-Belnap logics. To attain this goal new methods and notions are introduced. I found many of the ideas to tackle the problems addressed in the thesis very imaginative and cleaver. In particular the results on Chapter 7 relating the lattice of finitary super-Belnap logics with a lattice of classes of triples given by two graphs and a natural number and the notion (novel to me) of explosive extension of a logic. The thesis, as I already said, contains a wealth of important results and several papers publishable in hight-quality journals can be written using the material presented in it.

The thesis fully meets the standards customarily required of a doctoral dissertation and due to its quality I recommend the dissertation for a public defence. My assessment is a grade of "Pass", and I recommend the thesis be accepted for as a doctoral dissertation in Logic.

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