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Report on the PhD dissertation "Reasoning with Inconsistent Information", submitted by Adam Přenosil (Charles University)

This dissertation is a contribution to the field of non-classical logics, i.e. formal systems of reasoning that deviate from the classical paradigm. Namely, the thesis presents a deep study of the Belnap-Dunn logic and its extensions. The history of this family of logics started in the 1960s, when Dunn focused on a 4-valued system BD that had been previously introduced by Belnap and Anderson in the context of their study of relevance logic. The 4-valued structure allowed to consider, besides the classical "True" and "False", two additional truth-values: "Both" and "Neither", which would respectively account for situations of inconsistent or insufficient information. Other well-known 3-valued logics (such as Kleene's logic K or Priest's Logic of Paradox LP), as well as classical logic itself, can be seen as extensions of BD. However, despite their appeal as "logics of inconsistency", there had been very little research aiming at a systematic description of the family of all possible extensions of BD. The most prominent exception to this rule were the works of Rivieccio et al that introduced the Exactly True Logic ETL and realized the existence of an infinite number of logics above BD. The present dissertation collects the results of the efforts of the author during his PhD studies by offering a systematic description of the family of extensions of BD, known as "super-Belnap logics", thus filling an important gap in the knowledge of non-classical logics.

The methods employed to achieve this goal mostly should be classified as belonging to Abstract Algebraic Logic (AAL). This area of mathematical logic studies the relation between deductive systems and their algebraic semantics. Abstracting from the well-known Lindenbaum-Tarski proof of completeness of classical logic with respect to the semantics given of Boolean algebras, AAL has become in the last four decades a deep mathematical theory presented in numerous research papers and a few celebrated monographs. Arguably, its main achievement has been the discovery of precise mathematical formulations of the relationships between logical proof and algebraic notions of consequence. However, when applied to the logic BD and its extensions, the methods of AAL immediately face very serious difficulties. Indeed, AAL shows its best results when applied to logics whose link to their algebraic semantics is somewhat similar to the connection between classical logic and Boolean algebras. Several instances of such similarity have given rise to different well-behaved classes of non-classical logics: algebraizable, equivalential, protoalgebraic, selfextensional, or Fregean logics, among others. However, as we learn in this thesis, the extensions of BD (with the obvious exceptions of classical and the trivial logic) do not belong to any of these families. Therefore, the standard methods of AAL turn out to be insufficient for a study of such logics on inconsistency. At the face of such a difficulty, one might have balked and turned back, abandoning the goal altogether or searching for a completely different approach. This is not the case of this dissertation. Indeed, the author has taken the opportunity to develop a handful of new notions and techniques, completely in the spirit of AAL but previously unknown, which have expanded the field and endowed it with the necessary means to deal with BD and its extensions.

The thesis starts with a clearly written introduction, that states the main goals of the dissertation, gives credit to previous works, and offers a structured presentation of the main contents, followed by a reasonably self-contained chapter with the necessary preliminaries on universal algebra and AAL. Chapter 2 and 3 introduce the main characters of the story: De Morgan algebras (with a focus on their most prominent examples, that is, the finite algebras with two, three and four values) and the logics they define, which had been widely studied and used in the literature (Belnap-Dunn logic BD, the Logic of Paradox LP, Kleene's logic K, Kleene's logic of order KO, and the Exactly True Logic ETL) together with their basic properties (Hilbert-style axiomatization, completeness, translations, normal forms, filters). Chapter 4 begins the presentation of original material with the aforementioned new contributions to the general theory of AAL. Indeed, the author introduces the notions of antitheorem or antiaxiom (sets of formulas that can never be satisfied in non-trivial models of a given logic), explosive extensions (those obtained by adding antitheorems), explosive part of an extension L (the largest explosive extension under L of the base logic), shows that explosive extensions for a completely distributive complete sublattice of the lattice of extensions and describes those given by products of matrices. Chapter 5 and 6 achieve one of the main goals of the dissertation: a description of the lattice of super-Belnap logics. We are presented with appropriately chosen axioms, rules and antiaxioms, several infinite chains of logics, and then we are given a study of their unions and intersections in the lattice, their explosive parts, and their completeness theorems, which eventually yields to the promised description.

Having presented the whole family of extensions of BD, now the author is ready to start what can be seen as the second part of the thesis: the mathematical and metalogical study of such logics. Indeed, Chapter 7 starts fulfilling this goal by discovering and exploiting a striking connection with graph theory. Building on a previously known duality for De Morgan algebras, the author extends it to De Morgan matrices and shows how finite reduced matrix models of BD are in one-to-one correspondence with triples given by two finite graphs and a natural number. This is used to show the correspondence of subfamilies of finitary super-Belnap logics with particular classes of finite graphs. Antivarieties of De Morgan algebras are introduced as classes axiomatized by negative universal clauses (i.e. disjunctions of negated equations) and the author proves that there is a continuum of such classes and, hence, a continuum of finitary super-Belnap logics. Chapter 8 reviews the behavior of extensions of BD with respect to usual metalogical properties and well-studied classes of logics in AAL. The picture outlined above regarding their wild nature is here confirmed: protoalgebraicity fails for all logics (with the only exceptions of classical and trivial logic), only some specific regions of the lattice are truth-equational or assertional logics, and also only few exceptions are Fregean or even self-extensional. Moreover, the authors shows in which part of the lattice the algebraic counterpart of the logics is a (quasi)variety. Chapter 9 complements the previous algebraic study of the family with a presentation of a proof-theoretic approach. This is another highly original part of the thesis, in which the logics are given Gentzen-style calculi which allow to show that they form a family orthogonal, in a way, to that of substructural logics. Indeed, while the latter are characterized by the loss of contraction, weakening of exchange, here these rules are kept while the other two structural rules (identity and cut) are free to vary. Coherently with the previous introduction of antitheorems and explosive extensions, here the author develops the notions of antiadmissible rule and antistructural completeness. We are given also a normal form for proofs, a corresponding normalization theorem, and a study of the interpolation property.

Finally, Chapters 10 and 11 conclude the dissertation with a discussion of several possible, and still interesting, modifications of the considered framework. Namely, the author considers the variation of the family when truth-constants are included in the language, multiple-conclusion versions of the logics, expansions with the propositional truth operator Delta, and expansions with two separated truth predicates for exact truth and non-falsity.

The dissertation is very nicely written, with clear goals and motivations, with a didactic style that carefully guides the reader from the basics to the new advanced results, and accurate detailed proofs (as far as this opponent had the time to check). As main merits let us stress the following:

1. Completeness of the work: The author has chosen a well-defined problem inside the field of non-classical logics (that is, describe and study the family of extensions of Belnap-Dunn logic) and has pursued it in virtually all possible aspects. We have been given a picture of the structure of such lattice of logics, a powerful mathematical tool to describe parts of it (in terms of graph theory), an exhaustive study of their metalogical properties and their proof theory, and, even, all reasonable variations of the framework.

2. Development of the general theory: After realizing that the solution of the problem would not be possible by simple routine application of known techniques, instead of shying away, the author has resolved to develop the necessary mathematical tools to achieve his goal. This is witnessed by valuable new contributions to AAL, the remarkable connection with another field of mathematics (i.e. with graph theory) and the development of a specific kind of proof theory for such peculiar family of logics.

Many of the results have been presented in several journal and conference papers, some of them already published or accepted for publication, which partially confirm their correction and importance.

As minor points of criticism one should mention the lack of a term index at the end of the dissertation (which should definitely be added if in the future the author considers the publication of his work as a monograph) and the ill-chosen misleading title, which, as the author admits in the preface, is not really descriptive of the content of the thesis.

Therefore, for all the mentioned reasons, I can undoubtedly conclude that the submitted dissertation:

a. meets all the standards required for a doctoral dissertation,
b. can be recommend for public defence, and
c. it should be graded as «Pass».

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