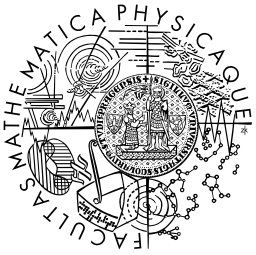

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BACHELOR THESIS

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Thermodynamics of Black Holes. Entropy and Information

Institute of Theoretical Physics

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Study programme:	Physics
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I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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I dedicate this thesis to my supervisor Giovanni Acquaviva and to my advisor Ana Alonso Serrano whose help has been absolutely invaluable to me and goes well beyond the scope of this work. I am very grateful for the opportunity to spend some time with them.

Title: Thermodynamics of Black Holes. Entropy and Information

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Abstract: The aim of the thesis is to provide a review of black hole thermodynamics and its relation with concepts of entropy and physical information. We start by deriving the four laws of black hole thermodynamics in the context of classical general relativity. To supplement this, we use semiclassical limit of quantum mechanics to show that black holes radiate and have non-zero thermodynamic temperature. In the second part of the thesis we describe the concepts of the Shannon and von Neumann entropy and of physical information. Lastly, we discuss the use of these concepts in the context of black hole mechanics.

Keywords: general relativity, black hole thermodynamics, physical information

Contents

Introduction

The black hole thermodynamics plays an important role in modern theoretical physics as it simultaneously involves gravity, quantum mechanics and physics of information. There are still many open issues, most notably the black hole information paradox and the interpretation of the black hole entropy. At the same time, the formula for the black hole entropy found by Bekenstein and Hawking is widely accepted to be correct and the ability to retrieve it is often used to test proposed theories of quantum gravity.

The purpose of this thesis is to give an overview of the theory and rederive certain important results. We will start by giving a brief chronology of the development of the black hole thermodynamics. In chapter 1 we shall discuss the black hole analogues of the four laws of thermodynamics and derive them in a manner valid for stationary axisymmetric spacetimes fully described by general relativity. Then we will use semiclassical limit of the quantum mechanics to show that black holes radiate. Chapter 2 will be focused on the connection between entropy and information. We will introduce concepts of the Shannon and von Neumann entropy, discuss the interpretation of the black hole entropy and the black hole information paradox. Lastly, we will briefly outline several ideas we consider exploring in the future.

Unless otherwise explicitly specified, we will use the geometric units with $c = G = \hbar = k_B = 1$. All sign conventions and definitions of relativistic quantities are in accordance with Misner et al. [2017].

History of Black Hole Thermodynamics

First examples of thermodynamic considerations involving black holes appeared in 1960s. One of them is the Penrose process, a method of extraction of energy from a black hole described by R. M. Floyd and Roger Penrose. They also showed that this process cannot lower the energy of the black hole under a certain limit called the irreducible mass. In effect, they determined the amount of work which can be extracted from a black hole. Robert Geroch suggested a process involving a black hole which would allow to convert heat into work with a hundred percent efficiency Bekenstein [1973]. He proposed to lower a box filled with black body radiation from infinity to black hole's event horizon, where its energy as measured from the infinity becomes zero. The box then emits some amount of radiation into the black hole and is pulled back to infinity. By lowering the box, work equal to its mass is obtained. The work necessary to retrieve the box is lower by the amount of radiation released. The energy of the radiation is thus converted into work with a hundred percent efficiency. This suggestion was somewhat alarming as it would be in dispute with the second law of thermodynamics. Also, a more fundamental problem with the second law in a spacetime which includes a black hole was pointed out by John A. Wheeler. If any matter falls inside a black hole, its entropy will no longer be measurable. One can either imagine that the entropy vanishes and the second law is thus broken, or assume that the entropy is still present under the horizon. Then, however, the second law becomes experimentally unverifiable Wald [2001].

In early 1970s important similarities between the equilibrium mechanics of black holes and standard thermodynamics were discovered, suggesting answers to the aforementioned paradoxes. The key concept of black hole entropy as a quantity proportional to its event horizon area was introduced by Jacob D. Bekenstein in two articles Bekenstein [1972, 1973]. Bekenstein himself mentions that J. M. Greif considered the idea of black hole entropy in his thesis as early as in 1969 but didn't suggest any specific formula for it Bekenstein [1973]. The idea of connecting the black hole's entropy to its horizon area is motivated by the area increase theorem proved by Stephen W. Hawking in 1970. It states that, if some fairly reasonable conditions are obeyed, the area of the black hole horizon cannot decrease in any process. This is a quality shared also by the common entropy. Apart from a heuristic derivation of a formula for black hole entropy, Bekenstein's article also included a derivation of an expression for the change of mass of a Kerr-Newman black hole analogous to the first law of thermodynamics and, most importantly, a statement of the generalized second law of thermodynamics. This law states that the sum of the black hole entropy and the common entropy in black hole's exterior can never decrease. This idea is then supported by several model examples in which the generalized second law is showed to hold even for the most unfavorable cases. Bekenstein also dealt with the Geroch process, demonstrating that when the minimal necessary dimensions of the box (enforced by the wavelengths of the radiation in it) are considered, the generalized second law holds.

In 1973 James M. Bardeen, Brandon Carter and Hawking provided a more general version of the first law's analogue and introduced the zeroth and the third law Bardeen et al. [1973]. The zeroth law stated that a quantity known as the surface gravity is constant all over the horizon of a stationary black hole. It also appeared (only multiplied by a constant) in both the Bekenstein and Bardeen-Carter-Hawking formulation of the first law in a product with the entropy. The third law states that the surface gravity cannot be reduced to zero by any finite sequence of operations. Thus the surface gravity plays the same role as the thermodynamic temperature in standard thermodynamics. However, as was argued by Bardeen, Carter and Hawking as well as by Bekenstein, no energy can escape a black hole fully described by general relativity Misner et al. [2017]. Therefore, a black hole cannot radiate and its thermodynamic temperature must be zero Bardeen et al. [1973]. Any similarities between thermodynamics and black hole mechanics thus appeared to be purely formal.

This changed in 1974 when Hawking performed an analysis of a free quantum field in the curved spacetime described by the Schwarzschild metric Hawking [1975]. He discovered that in this picture the black hole radiates with a perfect black body spectrum corresponding to thermodynamic temperature equal to the surface gravity divided by 2π Wald [2001]. At this point, all quantities involved in the description of the black hole equilibrium mechanics were physically equivalent to respective quantities used in thermodynamics description (surface gravity and temperature, mass and energy, angular momentum, angular frequency, charge, electric potential). Therefore, it was assumed that the nature of the black hole entropy should be the same as of the common entropy. This gave rise to the belief that the entropy formula for the black hole should be retrievable by counting its quantum degrees of freedom. It was first attempted by Gibbons and Hawking in

1977 Wald [2001].

Since then, there have been numerous attempts to evaluate black hole entropy using various theories of quantum gravity. Calculations within the weak coupling limit of the string theory yielded the exact reproduction of the formula for black hole entropy but only for certain classes of extremal and nearly extremal black holes Wald [2001]. Loop quantum gravity managed to give geometric arguments for the finiteness of black hole entropy and its proportionality to the horizon area Wald [2001].

1. The Four Laws of Black Hole Thermodynamics

1.1 Mathematical Introduction

We will begin by introducing (and in certain cases proving) several fundamental results that will be useful for our derivation of the four laws of black hole mechanics.

As is typical for general relativity, we will work with pseudo-Riemannian manifold (M, g), where M is a four-dimensional differentiable manifold and g denotes a symmetric metric tensor with a signature $(-1, 1, 1, 1)$. The covariant derivative of vector $v \in M$ equals:

$$v^{\lambda}{}_{;\mu} = v^{\lambda}{}_{,\mu} + \Gamma^{\lambda}{}_{\mu\nu} v^{\nu}, \tag{1.1}$$

where comma stands for a partial derivative, Greek letters denote indices from the set $\{0, 1, 2, 3\}$ and we use the Einstein summation convention ($v_\mu v^\mu \equiv \sum_{\mu=0}^{3} v_\mu v^\mu$). $\Gamma^{\lambda}{}_{\mu\nu}$ is the affine connection defined by the Christoffel symbols:

$$\Gamma^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2} g^{\kappa\lambda} (g_{\kappa\mu,\nu} + g_{\nu\kappa,\mu} - g_{\mu\nu,\kappa}). \tag{1.2}$$

The Riemann tensor $R^{\mu}{}_{\nu\kappa\lambda}$, the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R are then defined by the following relations:

$$R^{\mu}{}_{\nu\kappa\lambda} v_\mu \equiv v_{\nu;\kappa\lambda} - v_{\nu;\lambda\kappa}, \tag{1.3}$$

$$R^{\mu}{}_{\nu\kappa\lambda} = \Gamma^{\mu}{}_{\nu\lambda,\kappa} - \Gamma^{\mu}{}_{\nu\kappa,\lambda} + \Gamma^{\mu}{}_{\kappa\sigma}\Gamma^{\sigma}{}_{\nu\lambda} - \Gamma^{\mu}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\kappa}, \tag{1.4}$$

$$R_{\mu\nu} \equiv R^{\mu}{}_{\nu\mu\lambda}, \tag{1.5}$$

$$R \equiv R^{\mu}{}_{\mu}. \tag{1.6}$$

1.1.1 Killing Vector Fields

One of the most important mathematical objects we will work with are the Killing vectors. For every isometry a spacetime exhibits we can define orbits along which the local metric does not change. The tangent vectors to these orbits are known as the Killing vectors and all such vectors corresponding to one isometry constitute a Killing vector field. A diffeomorphism is an isometry if and only if the corresponding Lie derivative equals zero everywhere in the spacetime. From this we can directly get a defining equation for a Killing vector field ξ^μ called the Killing equation Adamson:

$$\pounds_\xi g_{\mu\nu} = g_{\mu\nu;\kappa}\xi^\kappa + g_{\kappa\nu}\xi_{\mu}{}^{;\kappa} + g_{\mu\kappa}\xi_{\nu}{}^{;\kappa} = 0, \tag{1.7}$$

$$\xi^{\mu}{}_{;\nu} = -\xi^{\nu}{}_{;\mu}. \tag{1.8}$$

By using equation 1.8 and the definition of the Riemann tensor we may get two other important identities:

$$\xi_{\nu;[\kappa\lambda]} = \frac{1}{2}(\xi_{\nu;\kappa\lambda} - \xi_{\nu;\lambda\kappa}) = \frac{1}{2} R^{\mu}{}_{\nu\kappa\lambda}\xi_\mu, \tag{1.9}$$

$$\xi^{\mu;\nu}{}_{\nu} = -\xi^{\nu;\mu}{}_{\nu} = \xi^{\nu}{}_{;\nu}{}^{\mu} - \xi^{\nu;\mu}{}_{\nu} = R_{\kappa}{}^{\nu}{}_{\nu}{}^{\mu}\xi^\kappa = -R_{\kappa}{}^{\mu}\xi^\kappa = -R^{\mu}{}_{\kappa}\xi^\kappa. \tag{1.10}$$

We will be mostly working within a stationary axisymmetric asymptotically flat spacetime. A typical example is the Kerr-Newman spacetime which includes a rotating charged black hole and is empty otherwise. We can also add any form of energy outside of the black hole as long as its distribution is axisymmetric and does not change in time. For every such spacetime there exists a unique time translational Killing vector field ξ^μ which near infinity is timelike and normalized to $\xi^\mu\xi_\mu = -1$. Furthermore, there is a unique rotational Killing vector field $\tilde{\xi}^\mu$ normalized to $\tilde{\xi}_\mu\tilde{\xi}^\mu = 1$ Bardeen et al. [1973]. Any linear combination of Killing vectors with constant coefficients is clearly again a Killing vector. Especially, for a Kerr-Newman black hole with an angular velocity on the event horizon, Ω_H, a vector field:

$$l^\mu = \xi^\mu + \Omega_H\tilde{\xi}^\mu, \tag{1.11}$$

tangent to the null geodesic generators of the horizon is a Killing vector field.

1.1.2 Energy-Momentum Tensor and Einstein Equations

The Einstein equations of course play fundamental role in general relativity. Here we derive their form most advantegous for our calculations and impose some necessary constraints on their right hand side, the energy-momentum tensor. Since we are not interested in cosmological scales we will use the Einstein equations with the cosmological constant set to zero:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{1.12}$$

By taking the trace of the equation we can rewrite it into a form which contains the trace of the energy-momentum tensor T instead of the scalar curvature R:

$$R^\nu{}_\nu - \frac{1}{2}Rg^\nu{}_\nu = 8\pi T^\nu{}_\nu,$$

$$R = -8\pi T, \tag{1.13}$$

$$\tag{1.14}$$

and substituting back in equation 1.13:

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right). \tag{1.15}$$

The energy-momentum tensor is often required to obey various so-called *energy conditions* which assure that any solution of the Einstein equation has certain physically reasonable properties. We will assume the Dominant Energy Condition stating that for any timelike vector field v^μ and any null or timelike vector field w^μ it holds that $T_{\mu\nu}v^\mu v^\nu \geq 0$ and $T_{\mu\nu}w^\nu$ is a non-spacelike vector. The quantity $T_{\mu\nu}v^\mu v^\nu$ can be interpreted as a local energy density which is thus required to be non-negative. The requirement imposed on $T_{\mu\nu}w^\nu$ effectively prohibits energy from being observed to travel faster than light. Let us note that this requirement especially holds for the null vector field l^μ defined by the equation 1.11.

1.1.3 Metrics

We shall derive the laws of black hole thermodynamics for the case of a stationary, axisymmetric, asymptotically flat spacetime with a black hole in its center. The metric of such spacetime can be expressed in the Boyer-Lindquist coordinates:

$$\mathrm{d}s^2 = \left(-\frac{\Delta\Sigma}{\mathcal{A}} + \frac{\mathcal{A}}{\Sigma}\omega^2 \sin^2\theta\right)\mathrm{d}t^2 - 2\frac{\mathcal{A}}{\Sigma}\omega\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\varphi + \frac{\mathcal{A}}{\Sigma}\sin^2\theta\,\mathrm{d}\varphi^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2. \tag{1.16}$$

Where:

$$\Delta = r^2 - 2Mr + a^2 + Q^2, \tag{1.17}$$

$$\Sigma = r^2 + a^2\cos^2\theta, \tag{1.18}$$

$$\mathcal{A} = \left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta, \tag{1.19}$$

$$\omega = a\frac{2Mr - Q^2}{\mathcal{A}}, \tag{1.20}$$

$$a = \frac{J}{M}. \tag{1.21}$$

M has the physical interpretation of the total mass present in the spacetime, Q is the total electric charge and J the total angular momentum. These coordinates are well suited for our intentions because Killing vector fields ξ^μ and $\tilde{\xi}^\mu$ are expressed in them as constant unit vectors $(1,0,0,0)$ and $(0,0,0,1)$ respectively. It can be easily seen that for very large values of the coordinate r the Boyer-Lindquist coordinates are well approximated by the Schwarzschild coordinates which describe a vacuum spherically symmetric spacetime:

$$\begin{aligned}\mathrm{d}s^2 &= -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{1}{1-\frac{2M}{r}}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2 = \\ &= -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{1}{1-\frac{2M}{r}}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2.\end{aligned} \tag{1.22}$$

1.2 The Second Law

Bekenstein's original suggestion of the concept of black hole entropy was based on two pillars. One is the assumption that the second law of thermodynamics should be valid for the universe as a whole, even if it includes black holes. The second one is the tendency of the event horizon area to increase when the black hole absorbs matter. For a Schwarzschild black hole is the horizon area A proportional to its total mass M, namely $A = 16\pi M^2$. For other types of black holes the situation is more complicated. However, to reduce, for example, a Kerr black hole's horizon area, a particle of mass m falling inside it would need to have angular momentum $J > m^2$ (oriented in the same direction as the black hole's angular momentum).

Two years before Bekenstein wrote his article, Hawking had proven a general theorem stating that for a predictable black hole fully described by general relativity its event horizon area can never decrease. We will follow a derivation of this result presented by Adamson.

1.2.1 The Area Increase Theorem

Let us study the behaviour of a null hypersurface denoted by η. We will choose a null geodesic generator of η with the affine parameter λ and tangent null vector field k^μ.

We define expansion θ of a point belonging to the generator:

$$\theta \equiv k^\mu_{\ ;\mu}. \tag{1.23}$$

It can be understood as a spatial divergence of two nearby geodesics measured by an observer which moves alongside one of them. Since η is normal to its null geodesics generators, θ also gives the local change in the size of an area element of η. To completely describe the evolution of geodesics we need two more quantities. The first one is a shear tensor $\sigma_{\mu\nu}$ which accounts for a tendency of some geodesics to move further than others with a small change of λ. The second is a twist tensor $\omega_{\mu\nu}$ which describes the rotation of an area element of η. Similarly to their equivalents for solid bodies, these tensors are both traceless with the shear tensor being symmetric and twist tensor antisymmetric. The most general local evolution of geodesics can be described by tensor $B_{\mu\nu}$:

$$B_{\mu\nu} = \frac{1}{2}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} = k_{\mu;\nu}, \tag{1.24}$$

where $h_{\mu\nu} = g_{\mu\nu} + k_\mu N_\nu + N_\mu k_\nu$ is the projection of the metric on hypersurface η. Here N^μ is a null vector normalized to $k_\mu N^\mu = -1$. It is introduced because standard projection $g_{\mu\nu} + k_\mu k_\nu$ does not work for a null hypersurface as $(g_{\mu\nu} + k_\mu k_\nu)k^\mu = k_\nu \neq 0$ Kar and SenGupta [2007].

If we want to describe the behaviour of $B_{\mu\nu}$ along the geodesics, we can just use the geodesic equation:

$$\begin{aligned} B_{\mu\nu;\kappa}k^\kappa = k_{\mu;\nu\kappa}k^\kappa = k_{\mu;\kappa\nu}k^\kappa + R^\lambda_{\ \mu\nu\kappa}k_\lambda k^\kappa = \\ = (k_{\mu;\kappa}k^\kappa)_{;\nu} - k_{\mu;\kappa}k^\kappa_{\ ;\nu} + R^\lambda_{\ \mu\nu\kappa}k_\lambda k^\kappa = -B_{\mu\kappa}B^\kappa_{\ \nu} + R^\lambda_{\ \mu\nu\kappa}k_\lambda k^\kappa. \end{aligned} \tag{1.25}$$

We used only the definition of the Riemann tensor and the geodesic equation for k^μ. We can now take the trace of the result and use equation 1.24 and properties of the shear and twist tensors to modify it:

$$\begin{aligned} B^\mu_{\ \mu;\kappa}k^\kappa &= -B_{\mu\kappa}B^{\kappa\mu} + R^{\lambda\mu}_{\ \ \mu\kappa}k_\lambda k^\kappa, \\ \theta_{;\kappa}k^\kappa &= -\frac{1}{2}\theta^2 - \sigma_{\mu\kappa}\sigma^{\kappa\mu} - \omega_{\mu\kappa}\omega^{\kappa\mu} - R_{\lambda\kappa}k^\lambda k^\kappa, \\ \frac{\mathrm{d}\theta}{\mathrm{d}\lambda} &= -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu. \end{aligned} \tag{1.26}$$

This is known as the Raychaudhuri equation. We can modify the term with the Ricci tensor using equation 1.15 and the normalization of k^μ:

$$R_{\mu\nu}k^\mu k^\nu = 8\pi\left(T_{\mu\nu}k^\mu k^\nu - \frac{1}{2}T\right) \tag{1.27}$$

If the dominant energy condition is satisfied, the right hand side of equation 1.27 must be non-negative. Since the shear tensor is spatial, the term $\sigma_{\mu\nu}\sigma^{\mu\nu}$ is

also non-negative. Furthermore, there exists an evolution equation for $\omega_{\mu\nu}$ which allows a solution $\omega_{\mu\nu} = 0$ Kar and SenGupta [2007]. Thus, it holds:

$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}\lambda} &\leq -\frac{1}{2}\theta^2, \\ \int_{\theta_0}^{\theta} \frac{\mathrm{d}\theta'}{(\theta')^2} &\leq -\int_0^{\lambda} \frac{\mathrm{d}\lambda'}{2}, \\ \frac{1}{\theta(\lambda)} &\geq \frac{1}{\theta_0} + \frac{\lambda}{2}. \end{aligned} \tag{1.28}$$

If $\theta_0 < 0$, the right hand side of the equation 1.28 will equal zero in some point of the geodesic. Therefore, the left hand side must also be equal to zero in some point, which corresponds to an infinite expansion. Such a result seems intuitively unphysical and can be shown to contradict some known black hole properties. The expansion of the horizon is hence non-negative along any null geodesic generator of hypersurface η. Since we may choose η to be the event horizon of a black hole and the expansion corresponds to the change of size of an area element of η, we have shown that the horizon area can never decrease.

The area increase theorem is restricted to black holes fully described by general relativity and therefore, it does not exclude a possibility for the horizon area to decrease during processes in which the microscopic structure of the black hole becomes relevant.

1.2.2 Black Hole Entropy

The event horizon area A of a black hole can be expressed in terms of its *irreducible mass* M_{ir}, namely $A = 16\pi M_{ir}^2$. The irreducible mass is defined as mass which cannot be extracted from the black hole by any classical process (such as the Penrose process). Because of the area increase theorem, the irreducible mass cannot decrease. For a thermodynamic system, the energy which cannot be transformed into work is equal to the product of temperature and entropy. The irreducible mass can also be understood as the part of the total black hole energy which cannot be transformed into work. Furthermore, the irreducible mass of a system of black holes is in fact smaller than the sum of their irreducible masses as they can emit gravitational waves while merging, reducing their total mass (but never the total area). Thus, even though the total mass of a Schwarzschild black hole equals its irreducible mass (this is obvious from the relation between the irreducible mass and the horizon area) and no energy can be extracted from it, two Schwarzschild black holes can still emit energy in the form of gravitational waves while they merge. Similarly, by putting two thermodynamic systems which are separately in thermal equilibrium in contact, further work can be extracted from them Bekenstein [1973]. In conclusion, there are some similarities between the irreducible mass of a black hole and the entropy of a thermodynamic system and if one wishes to construct an expression for the black hole entropy, irreducible mass is a logical starting point.

We can try to define black hole entropy as a monotonically increasing function of its horizon area (or, equivalently, irreducible mass). Two options are probably the most reasonable: a square root, which would give the entropy as a linear function of irreducible mass, and a linear function. However, a square root of

the horizon area of a Schwarzschild black hole formed by merging of two other Schwarzschild black holes is lower than the sum of square roots of their horizon areas. Therefore, entropy proportional to a square root would decrease in such process. Bekenstein thus decided to construct an expression for entropy which is proportional to the black hole's horizon area (we will use standard units in the following expressions):

$$S = \eta \frac{k_B c^3}{G\hbar} \alpha. \tag{1.29}$$

where α is the *rationalized horizon area* (horizon area A divided by 4π) and η is dimensionless constant. Factor c^3/G is just a transformation from geometric to standard units. The Planck constant was included in the formula mainly because its presence assures that the entropy is expressed in the correct units, namely $[S] = J.K^{-1}$. Bekenstein noted that Planck constant is in fact present in formulas for entropy of many systems, which are otherwise described by classical physics, as a result of the fact that entropy can be understood as a count of possible microscopic states of a system which are of a quantum nature. He speculated that the same reasoning may apply to black hole entropy and the appearance of the Planck constant is therefore quite natural. The constant η can be roughly estimated (as it can not be calculated exactly without considering the underlying microscopic theory) by letting a particle fall into a black hole and then comparing the increase of rationalized area with the loss of information about particle existence (numerically this equals $\ln 2$ as we will see in the chapter about entropy and information). Bekenstein showed that the rationalized area increase $\mathrm{d}\alpha$ in such a case is equal to at least $2b\mu$, where b is particle's proper radius and μ its rest mass. Proper radius b must clearly be greater than or equal to both the particle's Compton wavelength $\hbar/\mu$ and its gravitational radius 2μ Bekenstein [1973]. This gives the lower limit $\Delta\alpha_{min} = 2\hbar$, which yields:

$$\eta = \frac{\ln 2}{2}. \tag{1.30}$$

The Bekenstein estimate of black hole entropy becomes:

$$S = \frac{k_B c^3 \ln 2}{8\pi G\hbar} A. \tag{1.31}$$

Hawking determined a proportionality constant different from Bekenstein's. His formula for black hole entropy is:

$$S = \frac{k_B c^3}{4G\hbar} A \tag{1.32}$$

The numerical factor here is almost ten times larger than in Bekenstein's original formula. The reason for this are different definitions of the black hole temperature. Bekenstein was working in the framework of classical general relativity in which the effective thermodynamic temperature of the black hole equals zero. He used the expression $\hbar\kappa/2\eta k_B$ (here κ is black hole's surface gravity which we will define later) as an analogue of the black hole temperature. By considering quantum mechanical effects Hawking later showed that the thermodynamic temperature of the black hole is in fact $\hbar\kappa/2\pi k_B$ Hawking [1975]. Thus, both Bekenstein and Hawking arrived to the same value of the product of temperature

and entropy but, as they considered different expressions for temperature, their formulas for entropy differ.

It is interesting to express the entropy of a Schwarzschild black hole of a mass M in SI units in terms of solar mass $M_\odot$:

$$S \approx 1,45.10^{54} \left(\frac{M}{M_\odot}\right)^2 J.K^{-1}. \tag{1.33}$$

The entropy of the Sun is in the order of $10^{35} J.K^{-1}$ [quote Bekenstein 1973]. This makes it obvious that the entropy per unit mass of a black hole is by many orders of magnitude greater than for stars.

After washing out of initial conditions, a Kerr-Newman black hole is fully described by its mass, charge and angular momentum Misner et al. [2017]. There are many possible ways a black hole with a given values of these three quantities can be formed, for example by a collapse of a normal star, a collapse of a neutron star, a collision of gravitational waves, a collision of two black holes, etc. However, once a black hole is formed and initial conditions are washed out, these cases become undistinguishable. The black hole entropy can be then interpreted as a measure of inaccessibility of information about its specific internal configuration.

We now have an expression for the black hole entropy which, at least in the context of classical general relativity, is non-decreasing and tends to increase when the black hole absorbs matter. It is also independent of the configuration of matter inside the event horizon. Since it is in principle possible to measure black hole's mass, angular momentum and charge and use them to compute its event horizon area, the black hole entropy can be, at least in principle, determined by an external observer. We will soon see that it also plays the role of the common entropy in the first law of black hole thermodynamics.

This line of reasoning led Bekenstein to formulation of the generalized second law of thermodynamics: The common entropy in the black hole exterior plus the black hole entropy never decreases Bekenstein [1973].

We shall return to this law and to the interpretation of the black hole entropy in chapter 2.

1.3 The First Law

1.3.1 Bekenstein's Derivation for a Kerr-Newman Black Hole

For a Kerr-Newman black hole it is possible to derive an equation analogous to the first law of thermodynamics Bekenstein [1973]. We start by taking the rationalized area α of such a black hole with mass M, angular momentum J and charge Q. It is possible to see that:

$$\alpha = r_+^2 + a^2, \tag{1.34}$$

$$\alpha = 2Mr_+ - Q^2, \tag{1.35}$$

where

$$r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2} \tag{1.36}$$

are values of the radial coordinate corresponding to the event horizons of a Kerr-Newman black hole. We can obtain a formula for the difference of mass δM between two slightly different Kerr-Newman black holes by varying equation 1.35:

$$\delta\alpha = 2(r_+\delta M + M\delta r_+ - Q\delta Q). \tag{1.37}$$

We now separately evaluate δr_+:

$$\delta r_+ = \delta M + \frac{2}{r_+ - r_-}\left(M\delta M - a\left(\frac{\delta J}{M} + \frac{a}{M}\delta M - Q\delta Q\right) - Q\delta Q\right). \tag{1.38}$$

By substituting this result into equation 1.37, we get:

$$\delta\alpha = 2\left(r_+\delta M + M\delta M + \frac{2}{r_+ - r_-}(M^2\delta M - a\delta J + a^2\delta M - MQ\delta Q) - Q\delta Q\right).$$

Our goal is to get a formula for the change of mass. To achieve it we will put all terms containing δM to the left side:

$$\begin{aligned}
2\left(r_+ + M + \frac{2}{r_+ - r_-}(M^2 + a^2)\right)\delta M &= \\
= \delta\alpha + 2\frac{2}{r_+ - r_-}\left(a\delta J + \left(M + \frac{r_+ - r_-}{2}\right)Q\delta Q\right) & \\
\frac{2}{r_+ - r_-}(r_+(r_+ - r_-) + M(r_+ - r_-) + 2M^2 + 2a^2)\delta M &= \\
= \delta\alpha + \frac{4}{r_+ - r_-}(a\delta J + r_+ Q\delta Q) & \\
\frac{2}{r_+ - r_-}(2M(r_+ - r_-) + 2M^2 - 2Q^2 - 2a^2 + 2M^2 + 2a^2)\delta M &= \\
= \delta\alpha + \frac{4}{r_+ - r_-}(a\delta J + r_+ Q\delta Q). &
\end{aligned} \tag{1.39}$$

Using now equation 1.35 to express M, we will arrive to the final form of the first law:

$$\delta M = \frac{r_+ - r_-}{4\alpha}\delta\alpha + \frac{a}{\alpha}\delta J + \frac{r_+ Q}{\alpha}Q\delta Q. \tag{1.40}$$

If we now compare equation 1.40 with its thermodynamical analogue, we see that if the change in black hole's mass corresponds to the change in energy, then the term $(a/\alpha)\delta J + (r_+ Q/\alpha)Q\delta Q$ is equivalent to the work done on the black hole in order to change its charge and angular momentum. Specifically, a/α represents the rotational angular frequency of the black hole and $(r_+ Q/\alpha)Q$ is the electric potential on the outer event horizon. Since $\delta\alpha$ is proportional to the black hole's entropy, the term $(r_+ - r_-)/4\alpha$ (multiplied by a certain constant) is analogous to black hole temperature.

Temperature defined in this way is non-negative, just like the thermodynamic temperature, and equals zero only for extremal black holes. However, the term temperature can not be successfully applied to a black hole fully described by general relativity, because such a black hole does not radiate and thus can not be in thermal equilibrium with black body radiation at any non-zero temperature. Therefore, in the context of classical general relativity, this analogy should be regarded as purely formal.

1.3.2 Bardeen-Carter-Hawking Formulation

In their article Bardeen et al. [1973] Bardeen, Carter and Hawking derived the first law of black hole dynamics in a more general way which holds for any stationary axisymmetric asymptotically flat spacetime which includes a black hole. They presented both integral and differential form of the first law. We will derive both versions in detail following the approach used in the original article.

The Integral Formula

Since the first law is basically a statement of the law of energy conservation, our first task is obtain an expression for the total energy contained in the spacetime. We will start from equation 1.10 for the second covariant derivative of the time translational Killing vector and integrate both its sides over hypersurface Σ which we choose to be spacelike and tangent to the axisymmetric Killing vector field. Then we can use the fact that $\xi^{\mu;\nu}$ is antisymmetric and transform the left hand side to an integral over a two-surface boundary of hypersurface Σ. This boundary is composed of the event horizon of the black hole and some other two-surface that constitues an outer boundary. Using the fact that all components of the Ricci tensor are equal to zero in vacuum, the outer boundary can be chosen arbitrarily far away. The most convenient choice is a spacelike, asymptotically flat two-surface $\partial\Sigma_\infty$ located at infinity and tangent to the Killing vector field $\tilde{\xi}^\mu$:

$$\int_\Sigma \xi^{\mu;\nu}{}_{;\nu}\,\mathrm{d}\Sigma_\mu = -\int_\Sigma R^\mu{}_\nu \xi^\nu\,\mathrm{d}\Sigma_\mu, \tag{1.41}$$

$$\int_{\partial\Sigma} \xi^{\mu;\nu}\,\mathrm{d}\Sigma_{\mu\nu} = -\int_\Sigma R^\mu{}_\nu \xi^\nu\,\mathrm{d}\Sigma_\mu, \tag{1.42}$$

$$\int_{\partial\Sigma_\infty} \xi^{\mu;\nu}\,\mathrm{d}\Sigma_{\mu\nu} = -\int_\Sigma R^\mu{}_\nu \xi^\nu\,\mathrm{d}\Sigma_\mu - \int_{\partial B} \xi^{\mu;\nu}\,\mathrm{d}\Sigma_{\mu\nu}. \tag{1.43}$$

We can modify the first integral on the right hand side of equation 1.43 by using equation 1.15:

$$\begin{aligned}\int_\Sigma R^\mu{}_\nu \xi^\nu\,\mathrm{d}\Sigma_\mu &= 8\pi\int_\Sigma \left(T^\mu{}_\nu - \frac{1}{2}T g^\mu{}_\nu\right)\xi^\nu\,\mathrm{d}\Sigma_\mu = \\ &= 8\pi\int_\Sigma \left(T^\mu{}_\nu - \frac{1}{2}T\delta^\mu_\nu\right)\xi^\nu\,\mathrm{d}\Sigma_\mu.\end{aligned} \tag{1.44}$$

If we introduce a radial vector normalized to unity $\rho^\mu = \frac{1}{\sqrt{g_{rr}}}(0,1,0,0)$ we can write surface element $\mathrm{d}\Sigma_{\mu\nu}$ at infinity in the following way:

$$\begin{aligned}\int_{\partial\Sigma_\infty} \xi^{\mu;\nu}\,\mathrm{d}\Sigma_{\mu\nu} &= \int_{\partial\Sigma_\infty} \xi^{\mu;\nu}\frac{1}{\sqrt{-g_{tt}}} g_{\alpha\mu}g_{\beta\nu}\xi^{[\alpha}\rho^{\beta]}\,\mathrm{d}\Sigma = \\ &= \int_{\partial\Sigma_\infty} g^{rr}\xi^t{}_{;r}\frac{1}{\sqrt{-g_{tt}}} g_{tt}g_{rr}\xi^t\rho^r\,\mathrm{d}\Sigma\end{aligned} \tag{1.45}$$

We are working in an asymptotically flat spacetime that can be expressed in terms of the Boyer-Lindquist metric which converges to the Schwarzschild metric far away from the energy sources. Then, we can conveniently evaluate the covariant derivative of the time translational Killing vector field ξ^μ in the Schwarzschild

coordinates:

$$\xi^t{}_{;r} = \xi^t{}_{,r} + g^{tt}\Gamma_{ttr}\xi^t = 0 + \frac{1}{2}g^{tt}\left(g_{tt,r} + g_{rt,t} - g_{tr,t}\right)\xi^t = g^{tt}\frac{1}{2}g_{tt,r}\xi^t = \\ = -\frac{1}{2}\frac{1}{\left(1-\frac{2M}{r}\right)}\left(-\frac{2M}{r^2}\right)\xi^t = \frac{1}{\left(1-\frac{2M}{r}\right)}\frac{M}{r^2}\xi^t \qquad (1.46)$$

The area element $\mathrm{d}\Sigma$ of the two-surface Σ_∞ is simply the area element of a two-sphere in a flat spacetime. The integral on the left hand side of equation 1.43 can now be easily evaluated for a very large constant radius r which is then sent to infinity:

$$\int_{\partial\Sigma_\infty}\frac{M}{r^2}\xi_t\xi^t\,\mathrm{d}\Sigma = -\lim_{r\to\infty}\int_0^{2\pi}\int_0^{\pi}\frac{M}{r^2}r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = -4\pi M \qquad (1.47)$$

With this result, equation 1.43 becomes:

$$M = \int_\Sigma\left(2T^\mu_{\ \nu} - T\delta^\mu_\nu\right)\xi^\nu\,\mathrm{d}\Sigma_\mu + \frac{1}{4\pi}\int_{\partial B}\xi^{\mu;\nu}\,\mathrm{d}\Sigma_{\mu\nu}. \qquad (1.48)$$

The first term on the right hand side is now fully determined by properties of the matter outside the black hole. The second term can be interpreted as the total energy of the black hole which is still possible to divide into several separate contributions.

We will start by obtaining an expression for total angular momentum J of the spacetime. To this end, we will integrate equation 1.10, this time for axisymmetric Killing vector field $\tilde{\xi}^\mu$, over the same hypersurface Σ we have used so far. We shall again change the left hand side to the integration over the boundary and use the Einstein equations for the right hand side. Also, we can use the fact that hypersurface Σ was chosen so that $\tilde{\xi}^\mu\,\mathrm{d}\Sigma_\mu = 0$. In the end we acquire:

$$\int_{\partial\Sigma_\infty}\tilde{\xi}^{\mu;\nu}\,\mathrm{d}\Sigma_{\mu\nu} = -8\pi\int_\Sigma\left(T^\mu_{\ \nu} - \frac{1}{2}T\delta^\mu_{\ \nu}\right)\tilde{\xi}^\nu\,\mathrm{d}\Sigma_\mu - \int_{\partial B}\tilde{\xi}^{\mu;\nu}\,\mathrm{d}\Sigma_{\mu\nu} = \\ = -8\pi\int_\Sigma T^\mu_{\ \nu}\tilde{\xi}^\nu\,\mathrm{d}\Sigma_\mu - \int_{\partial B}\tilde{\xi}^{\mu;\nu}\,\mathrm{d}\Sigma_{\mu\nu}. \qquad (1.49)$$

We will explicitly evaluate the left hand side integral. This time we need to use the Boyer-Lindquist coordinates and surface element $\mathrm{d}\Sigma_{\mu\nu}$ has to be written as $\tau_{[\mu}\rho_{\nu]}\,\mathrm{d}\Sigma$, where vector $\tau^\mu = \frac{1}{\sqrt{-g_{tt}+\omega^2 g_{\varphi\varphi}}}(\xi^\mu + \omega\tilde{\xi}^\mu)$ is normalized to unity and

orthogonal to $\tilde{\xi}^\mu$. By performing the integration, we obtain:

$$\begin{aligned}
\int_{\partial\Sigma_\infty} \tilde{\xi}^{\mu;\nu}\tau_{[\mu}\rho_{\nu]}\,\mathrm{d}\Sigma &= \int_{\partial\Sigma_\infty} \frac{1}{2}g^{\beta\mu}\left(g_{\beta\gamma,\nu}+g_{\nu\beta,\gamma}-g_{\gamma\nu,\beta}\right)\tilde{\xi}^\gamma g_{\alpha\mu}\tau^\alpha\rho^\nu\,\mathrm{d}\Sigma = \\
&= \frac{1}{2}\lim_{r\to\infty}\int_0^{2\pi}\int_0^\pi \left(\frac{\partial g_{t\varphi}}{\partial r}+\omega\frac{\partial g_{\varphi\varphi}}{\partial r}\right)\sqrt{\frac{\Delta}{\Sigma}}\sqrt{\frac{\mathcal{A}}{\Sigma\Delta}}\sqrt{\Sigma}\sqrt{\frac{\mathcal{A}}{\Sigma}}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \\
&= \frac{1}{2}\lim_{r\to\infty}\int_0^{2\pi}\int_0^\pi \left(\frac{-\Sigma+2r^2}{\Sigma^2}+\frac{r}{\mathcal{A}}\frac{\Sigma\mathcal{A}_{,r}-2r\mathcal{A}}{\Sigma^2}\right)\frac{\mathcal{A}}{\Sigma}2J\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \\
&= J\lim_{r\to\infty}\int_0^{2\pi}\int_0^\pi \left(-\mathcal{A}+r\mathcal{A}_{,r}\right)\frac{1}{\Sigma^2}\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = J\lim_{r\to\infty}\int_0^{2\pi}\int_0^\pi (-(r^2+a^2)^2+ \\
&\quad +(r^2-2Mr+a^2)a^2\sin^2\theta+(r^2+a^2)4r^2-(2r^2-2Mr)a^2\sin^2\theta) \\
\frac{1}{\Sigma^2}\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi &= J\lim_{r\to\infty}\int_0^{2\pi}\int_0^\pi \left(3r^4+2a^2r^2-a^4+a^4\sin^2\theta-a^2r^2\sin^2\theta\right)
\end{aligned} \tag{1.50}$$

$$\frac{1}{\Sigma^2}\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = 8\pi J. \tag{1.51}$$

The result is trivial in the limit of r approaching infinity but it does in fact hold for any value of r Modak and Samanta [2012]. Showing this requires performing a tedious but straightforward integration of rational functions.

After substituting $8\pi J$ to its left hand side, equation 1.49 yields:

$$J = -\int_\Sigma T^\mu{}_\nu\tilde{\xi}^\nu\,\mathrm{d}\Sigma_\mu - \frac{1}{8\pi}\int_{\partial B}\tilde{\xi}^{\mu;\nu}\,\mathrm{d}\Sigma_{\mu\nu}. \tag{1.52}$$

The first term on the right hand side equals the total angular momentum of the matter outside the black hole. Since result $8\pi J$ does not depend on the radial coordinate, the second term gives exactly the black hole's angular momentum J_H:

$$J_H = -\frac{1}{8\pi}\int_{\partial B}\tilde{\xi}^{\mu;\nu}\,\mathrm{d}\Sigma_{\mu\nu}. \tag{1.53}$$

If we now rewrite integral $\frac{1}{4\pi}\int_{\partial B}\xi^{\mu;\nu}\,\mathrm{d}\Sigma_{\mu\nu}$ in terms of Killing vector field l^μ (defined by equation 1.11) and black hole's angular momentum J_H, equation 1.48 takes form:

$$M = \int_\Sigma \left(2T^\mu{}_\nu - T\delta^\mu{}_\nu\right)\xi^\nu\,\mathrm{d}\Sigma_\mu + \frac{1}{4\pi}\int_{\partial B} l^{\mu;\nu}\,\mathrm{d}\Sigma_{\mu\nu} + 2\Omega_H J_H. \tag{1.54}$$

Tensor $\mathrm{d}\Sigma_{\mu\nu}$ can be expressed on the horizon as $l_{[\mu}n_{\nu]}\,\mathrm{d}A$, where n^ν is a null vector orthogonal to the horizon and normalized so that $n_\mu l^\mu = -1$ and $\mathrm{d}A$ is the surface element of the horizon. We then define $\kappa = -l_{\mu;\nu}n^\mu l^\nu$ and use it to rewrite the second term on the right hand side of equation 1.54:

$$\begin{aligned}
\frac{1}{4\pi}\int_{\partial B} l^{\mu;\nu}\,\mathrm{d}\Sigma_{\mu\nu} &= \frac{1}{4\pi}\int_{\partial B} l_{\mu;\nu}l^{[\mu}n^{\nu]}\,\mathrm{d}A = \\
&= \frac{1}{4\pi}\int_{\partial B}\frac{1}{2}l_{\mu;\nu}(l^\mu n^\nu - l^\nu n^\mu)\,\mathrm{d}A = \\
&= \frac{1}{4\pi}\int_{\partial B}\frac{1}{2}(-2)l_{\mu;\nu}l^\nu n^\mu\,\mathrm{d}A = \frac{1}{4\pi}\int_{\partial B}\kappa\,\mathrm{d}A.
\end{aligned} \tag{1.55}$$

If we define complex conjugate null vectors m^μ, $\bar{m}^\mu$ lying on the horizon and normalized so that $m_\mu \bar{m}^\mu = 1$, they will form a null tetrad together with vectors l^μ, n^μ. Then the metric on the horizon can be expressed:

$$g_{\mu\nu} = -l_\mu n_\nu - n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu. \tag{1.56}$$

Requirement $g_{\mu\nu;\lambda} = 0$ yields:

$$n_{\mu;\lambda} = n_\mu n^\nu l_{\nu;\lambda}, \tag{1.57}$$

$$l_{\mu;\lambda} = l_\mu l^\nu n_{\nu;\lambda}. \tag{1.58}$$

Let us introduce a four-vector $a^\mu = \frac{l_\nu l^{\mu;\nu}}{l_\lambda l^\lambda}$ and a scalar $V = \sqrt{l_\rho l^\rho}$. Then for a particle infinitesimally close to the horizon Va tends to κ:

$$Va = V\sqrt{a_\mu a^\mu} = \sqrt{l_\rho l^\rho}\sqrt{\frac{l_\nu l^{\mu;\nu} l^\kappa l_{\mu;\kappa}}{l_\lambda l^\lambda l_\sigma l^\sigma}} = \sqrt{\frac{l_\nu l^\mu l_\lambda n^{\lambda;\nu} l^\kappa l_\mu l^\rho n_{\rho;\kappa}}{l_\sigma l^\sigma}} = \\ = \sqrt{l_\nu l_\lambda n^\lambda n_\mu l^{\mu;\nu} l^\kappa l^\rho n_\rho n^\sigma l_{\sigma;\kappa}} = \kappa. \tag{1.59}$$

Equations 1.57 and 1.58 were repeatedly used to obtain the last three equalities. The four-velocity u^μ of a particle rigidly corotating with the black hole just outside its horizon can be written as $u^\mu = \frac{l^\mu}{\sqrt{l_\lambda l^\lambda}}$. The corresponding acceleration is $u^\mu{}_{;\nu} u^\nu = \frac{l_\nu l^{\mu;\nu}}{l_\lambda l^\lambda} = a^\mu$ and its magnitude is a. If we now multiply this by V, we convert the result from acceleration per unit of the proper time of the particle to acceleration per unit of the coordinate time. Since we are working in an asymptotically flat spacetime, the coordinate time has the meaning of a proper time for an observer standing still at infinity. Thus, κ on the horizon equals the acceleration of a particle corotating with the black hole just outside its event horizon as measured by such an observer. In this sense κ has the meaning of the *surface gravity* of the black hole, which is a term commonly used for it.

We will now show that κ is constant over the horizon. To do this we will prove that κ does not change along the vector field m^μ and is therefore indeed constant on the horizon:

$$\kappa_{;\mu} m^\mu = (-l^\lambda{}_{;\nu} n^\lambda l^\nu)_{;\mu} m^\mu = -l^\lambda{}_{;\nu\mu} n^\lambda l^\nu m^\mu - l^\lambda{}_{;\nu} n^\lambda{}_{;\mu} l^\nu m^\mu - l^\lambda{}_{;\nu} n^\lambda l^\nu{}_{;\mu} m^\mu. \tag{1.60}$$

We can use the first Bianchi identity and equations 1.8 and 1.9 to rewrite the first term of the previous expression:

$$R_{\kappa\mu\nu\lambda} l^\kappa = \left(-R_{\kappa\lambda\mu\nu} - R_{\kappa\nu\lambda\mu}\right) l^\kappa = -l^\lambda{}_{;\mu\nu} + l^\lambda{}_{;\nu\mu} - l^\nu{}_{;\lambda\mu} + l^\nu{}_{;\mu\lambda} = \\ = 2l^\lambda{}_{;\nu\mu} - l^\lambda{}_{;\mu\nu} + l^\lambda{}_{;\nu\mu} = 2l^\lambda{}_{;\nu\mu} - R_{\kappa\mu\nu\lambda} l^\kappa.$$

Thus, we obtain:

$$l^\lambda{}_{;\nu\mu} = R_{\kappa\mu\nu\lambda} l^\kappa. \tag{1.61}$$

With the help of equations 1.57 and 1.58 we can show that only the term with $l^\lambda{}_{;\nu\mu}$ remains in the equation 1.60, as the other two terms yield zero together:

$$-l^\lambda{}_{;\nu} n^\lambda{}_{;\mu} l^\nu m^\mu = -l^\lambda{}_{;\nu} n^\lambda n^\kappa l_{\kappa;\mu} l^\nu m^\mu = \kappa l_{\mu;\nu} n^\mu m^\nu, \tag{1.62}$$

$$-l^\lambda{}_{;\nu} n^\lambda l^\nu{}_{;\mu} m^\mu = -l^\lambda{}_{;\nu} n^\lambda l^\nu l^\kappa n_{\kappa;\mu} m^\mu = \kappa n_\kappa n^\nu l_{\nu;\mu} l^\kappa m^\mu = -\kappa l_{\mu;\nu} n^\mu m^\nu. \tag{1.63}$$

In total we acquired:

$$\kappa_{;\mu} m^\mu = -R_{\kappa\lambda\nu\mu} l^\kappa m^\lambda l^\nu n^\mu. \tag{1.64}$$

Thanks to equation 1.56 and antisymmetry of $l_{\mu;\nu}$ we get:

$$0 = l^{\mu;\nu} g_{\mu\nu} = l^{\mu;\nu} \left(-l_\mu n_\nu - n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu\right) = 2 l^{\mu;\nu} m_\mu \bar{m}_\nu. \tag{1.65}$$

We now differentiate this identity in the direction of m^λ:

$$\begin{aligned} 0 = \left(l_{\mu;\nu} m^\mu \bar{m}^\nu\right)_{;\lambda} m^\lambda = R_{\kappa\mu\nu\lambda} l^\kappa m^\mu \bar{m}^\nu m^\lambda + l_{\mu;\nu} \left(m^\mu \bar{m}^\nu\right)_{;\lambda} m^\lambda = \\ = R_{\kappa\mu\nu\lambda} l^\kappa m^\mu \bar{m}^\nu m^\lambda. \end{aligned} \tag{1.66}$$

This expression can be further modified:

$$\begin{aligned} -R_{\kappa\lambda} l^\kappa m^\lambda = g^{\mu\nu} R_{\kappa\mu\nu\lambda} l^\kappa m^\lambda = \\ = \left(-l^\mu n^\nu - n^\mu l^\nu + m^\mu \bar{m}^\nu + \bar{m}^\mu m^\nu\right) R_{\kappa\mu\nu\lambda} l^\kappa m^\lambda = \\ = \left(-n^\mu l^\nu + m^\mu \bar{m}^\nu\right) R_{\kappa\mu\nu\lambda} l^\kappa m^\lambda. \end{aligned} \tag{1.67}$$

The last step uses antisymmetries of the Riemann tensor and its result can be further rewritten using the first Bianchi identity:

$$\begin{aligned} \left(-n^\mu l^\nu + m^\mu \bar{m}^\nu\right) R_{\kappa\mu\nu\lambda} l^\kappa m^\lambda = R_{\kappa\mu\nu\lambda} l^\kappa m^\mu \bar{m}^\nu m^\lambda - \\ -n^\mu l^\nu \left(-R_{\kappa\lambda\mu\nu} - R_{\kappa\nu\lambda\mu}\right) l^\kappa m^\lambda = R_{\kappa\mu\nu\lambda} l^\kappa m^\mu \bar{m}^\nu m^\lambda + R_{\kappa\lambda\mu\nu} l^\kappa m^\lambda n^\mu l^\nu. \end{aligned} \tag{1.68}$$

$$\begin{aligned} R_{\kappa\mu\nu\lambda} l^\kappa m^\mu \bar{m}^\nu m^\lambda = -R_{\kappa\lambda} l^\kappa m^\lambda + R_{\kappa\lambda\nu\mu} l^\kappa m^\lambda l^\nu n^\mu = \\ = -R_{\kappa\lambda} l^\kappa m^\lambda - \kappa_{;\mu} m^\mu = 0. \end{aligned} \tag{1.69}$$

This yields an expression for the covariant derivative of κ in the direction of m^μ:

$$\kappa_{;\mu} m^\mu = -R_{\kappa\lambda} l^\kappa m^\lambda. \tag{1.70}$$

Using equation 1.15 and orthogonality of m^μ, l^μ we now have:

$$\kappa_{;\mu} m^\mu = -8\pi \left(T_{\kappa\lambda} + \frac{1}{2} T g_{\kappa\lambda}\right) l^\kappa m^\lambda = -8\pi T_{\kappa\lambda} l^\kappa m^\lambda. \tag{1.71}$$

The vector $T_{\kappa\lambda} l^\kappa$ is non-spacelike, because l^κ is a null vector and we expect the Dominant Energy Condition to be obeyed. We consider again the Raychaudhuri equation 1.26. For a stationary black hole expansion, shear and twist of the event horizon vanish (the horizon does not change alongside its null geodesic generator). Taking into account that k^μ for our situation equals l^μ, equation 1.26 is reduced to:

$$0 = R_{\kappa\lambda} l^\kappa l^\lambda = 8\pi \left(T_{\kappa\lambda} - \frac{1}{2} T g_{\kappa\lambda}\right) l^\kappa l^\lambda = 8\pi T_{\kappa\lambda} l^\kappa l^\lambda. \tag{1.72}$$

In the last step we have used that on the horizon l^μ is a null vector. Then it follows that $T_{\kappa\lambda} l^\kappa$ is either zero or parallel to l^λ, and thus $T_{\kappa\lambda} l^\kappa m^\lambda = 0$. In conclusion, we obtain $\kappa_{;\mu} m^\mu = 0$ and we have proved that κ is constant on the horizon.

We can now rewrite the equation 1.55 to its final form:

$$M = \int_S \left(2T^{\mu}_{\ \nu} - T\delta^{\mu}_{\nu}\right) \xi^{\nu} \, \mathrm{d}\Sigma_{\mu} + 2\Omega_H J_H + \frac{\kappa}{4\pi} A. \tag{1.73}$$

This is the integral formula for the first law of black hole thermodynamics. The nature of this expression is analogous to the Euler equation of standard thermodynamics which expresses the entire internal energy of a thermodynamic system as a function of thermodynamic variables describing it. Similarly, equation 1.73 gives an expression for the total energy contained in the spacetime. The first term on the right hand side contains all the energy contributions from the matter outside the black hole. To further modify this term, we would have to specify the energy-momentum tensor of the matter. Remaining terms give contributions from the black hole parameters. The term $2\Omega_H J_H$ represents black hole's total rotational energy. Since A is proportional to the black hole entropy, it is tempting to view the last term as analogous to the term θS (θ being the thermodynamic temperature) in the Euler equation. In section 1.6 we shall demonstrate that this is indeed the case (in fact, it equals $2\theta S$) but, as was argued in subsection 1.3.1, such interpretation is impossible in the context of classical general relativity.

The Differential Formula

Our next goal is to obtain a result analogous to the Bekenstein formula 1.40 by varying equation 1.73. For simplicity, as well as to get a clearer physical interpretation of contributions from the outside matter, we will restrict ourselves to a case of a black hole surrounded by a perfect fluid whose distribution is stationary and axisymmetric. We may describe such fluid by its energy density $\varepsilon(n, s)$, where n is the particle number density and s is the entropy density. Then temperature θ, chemical potential μ and pressure p are defined by the standard thermodynamic relations:

$$\theta = \frac{\partial \varepsilon}{\partial s}, \tag{1.74}$$

$$\mu = \frac{\partial \varepsilon}{\partial n}, \tag{1.75}$$

$$p = \mu n + \theta s - \epsilon. \tag{1.76}$$

The energy-momentum tensor can be written as:

$$T_{\mu\nu} = (\varepsilon + p) v_{\mu} v_{\nu} + p g_{\mu\nu}, \tag{1.77}$$

$$v^{\mu} = \frac{1}{\sqrt{-u_{\lambda} u^{\lambda}}} u^{\mu}, \tag{1.78}$$

$$u^{\mu} = \xi^{\mu} + \Omega \tilde{\xi}^{\mu}. \tag{1.79}$$

Here v^{μ} is the unit vector tangent to the flow lines. The four-velocity of the fluid can be expressed as $u^{\mu} = u^{t}(\xi^{\mu} + \Omega\tilde{\xi}^{\mu})$, otherwise the stationarity and axial symmetry of the spacetime would have been lost. Then Ω is the angular velocity of the fluid. The angular momentum, entropy and number of particles of the fluid

can be expressed as:

$$J = -\int_S T^{\mu}{}_{\nu}\tilde{\xi}^{\nu}\,\mathrm{d}\Sigma_{\mu}, \tag{1.80}$$

$$S = \int_S s v^{\mu}\,\mathrm{d}\Sigma_{\mu}, \tag{1.81}$$

$$N = \int_S n v^{\mu}\,\mathrm{d}\Sigma_{\mu}. \tag{1.82}$$

When comparing two slightly different solutions, there is a certain freedom in choosing in which points are they meant to correspond. We choose hypersurfaces Σ, the event horizons, and Killing vector fields ξ^{μ}, $\tilde{\xi}^{\mu}$ to be the same in both solutions. We then have:

$$\delta\xi^{\mu} = \delta\tilde{\xi}^{\mu} = 0, \tag{1.83}$$

$$\delta g_{\mu\nu} = h_{\mu\nu} = -g_{\mu\kappa}g_{\nu\lambda}\delta g^{\kappa\lambda}, \tag{1.84}$$

$$\delta\xi_{\mu} = h_{\mu\nu}\xi^{\nu}, \tag{1.85}$$

$$\delta\tilde{\xi}_{\mu} = h_{\mu\nu}\tilde{\xi}^{\nu}, \tag{1.86}$$

$$\delta l^{\mu} = \tilde{\xi}^{\mu}\delta\Omega_H, \tag{1.87}$$

$$\delta l_{\mu} = \delta(g_{\mu\nu}l^{\nu}) = h_{\mu\nu}l^{\nu} + \tilde{\xi}_{\nu}\delta\Omega_H. \tag{1.88}$$

Since the event horizons are the same, vector fields l^{μ}, n^{μ} normal to the horizon in the first solution are parallel with their counterparts in the second solution:

$$l_{[\mu}\delta l_{\nu]} = \frac{1}{2}(l_{\mu}\delta l_{\nu} - l_{\nu}\delta l_{\mu}) \propto \frac{1}{2}(l_{\mu}l_{\nu} - l_{\nu}l_{\mu}) = 0, \tag{1.89}$$

$$n^{[\mu}\delta n^{\nu]} = 0. \tag{1.90}$$

For the same reason the Lie derivative of δl_{μ} with respect to l^{ν} is zero:

$$\pounds_l \delta l_{\mu} = (\delta l_{\mu})_{;\nu}l^{\nu} + l^{\nu}{}_{;\mu}\delta l_{\nu} = 0. \tag{1.91}$$

We shall use these identities to evaluate the variation of κ:

$$\begin{aligned}\delta\kappa &= -n^{\mu}l^{\nu}(\delta l_{\mu})_{;\nu} - l_{\mu;\nu}n^{\mu}\delta l^{\nu} - l_{\mu;\nu}l^{\nu}\delta n^{\mu} = \\ &= n^{\mu}l^{\nu}(\delta l_{\nu})_{;\mu} + l_{\nu;\mu}n^{\mu}\delta l^{\nu} + l_{\nu;\mu}l^{\nu}\delta n^{\mu} = \\ &= \frac{1}{2}(l^{\nu}\delta l_{\nu} + l_{\nu}\delta l^{\nu})_{;\mu}n^{\mu} + l^{\nu}{}_{;\mu}l_{\nu}\delta n^{\mu}.\end{aligned}$$

The next step uses equation 1.87 for the variation of vector l^{μ}:

$$\begin{aligned}&\frac{1}{2}l^{\nu}n^{\mu}(\delta l_{\nu})_{;\mu} + \frac{1}{2}l^{\nu}{}_{;\mu}n^{\mu}\delta l_{\nu} + \frac{1}{2}\tilde{\xi}^{\nu}{}_{;\mu}l_{\nu}n^{\mu}\delta\Omega_H + \\ &+\frac{1}{2}l_{\nu;\mu}n^{\mu}\delta(g^{\nu\lambda}l_{\lambda}) + l^{\nu}{}_{;\mu}l_{\nu}\delta n^{\mu} = \\ &= \frac{1}{2}l^{\nu}n^{\mu}(\delta l_{\nu})_{;\mu} + l^{\nu}{}_{;\mu}n^{\mu}\delta l_{\nu} + \frac{1}{2}\tilde{\xi}^{\nu}{}_{;\mu}l_{\nu}n^{\mu}\delta\Omega_H - \frac{1}{2}l^{\rho}{}_{;\mu}n^{\mu}h_{\rho\sigma}l^{\sigma} + \\ &+l^{\nu}{}_{;\mu}l_{\nu}\delta n^{\mu}.\end{aligned}$$

We will rewrite the fourth term using equation 1.88:

$$\frac{1}{2}l^\nu n^\mu(\delta l_\nu)_{;\mu} + l^\nu{}_{;\mu}n^\mu\delta l_\nu + \frac{1}{2}\tilde{\xi}^\nu{}_{;\mu}l_\nu n^\mu\delta\Omega_H - \frac{1}{2}l^\rho{}_{;\mu}n^\mu\delta l_\rho + \\ + \frac{1}{2}\tilde{\xi}_\rho l^\rho{}_{;\mu}n^\mu\delta\Omega_H + l^\nu{}_{;\mu}l_\nu\delta n^\mu = \frac{1}{2}l^\nu n^\mu(\delta l_\nu)_{;\mu} + l^\nu{}_{;\mu}n^\mu\delta l_\nu + \\ + \frac{1}{2}(\tilde{\xi}^\nu l_\nu)_{;\mu}n^\mu\delta\Omega_H + \frac{1}{2}l^\rho n^\mu(\delta l_\mu)_{;\rho} + l^\nu{}_{;\mu}l_\nu\delta n^\mu. \tag{1.92}$$

Considering equation 1.91 we get:

$$\frac{1}{2}(\delta l_\nu)_{;\mu}(l^\nu n^\mu + l^\mu n^\nu) + l^\nu{}_{;\mu}(n^\mu\delta l_\nu + l_\nu\delta n^\mu) + \tilde{\xi}^\nu{}_{;\mu}l_\nu n^\mu\delta\Omega_H = \tag{1.93}$$

$$= \frac{1}{2}(\delta l_\nu)_{;\mu}(l^\nu n^\mu + l^\mu n^\nu) + \tilde{\xi}^\nu{}_{;\mu}l_\nu n^\mu\delta\Omega_H. \tag{1.94}$$

Since δl^μ is proportional to l^μ on the horizon, $(\delta l_\mu)_{;\nu}m^\mu\bar{m}^\nu = 0$. Therefore, using expression 1.56 for the metric we get:

$$\delta\kappa = \frac{1}{2}(\delta l_\nu)_{;\mu}(-g^{\mu\nu}) + \tilde{\xi}^\nu{}_{;\mu}l_\nu n^\mu\delta\Omega_H = -\frac{1}{2}(\delta(g_{\nu\lambda}l^\lambda))^{;\nu} + \tilde{\xi}^\nu{}_{;\mu}l_\nu n^\mu\delta\Omega_H = \\ = -\frac{1}{2}(h_{\nu\lambda}l^\lambda)^{;\nu} - \frac{1}{2}(\delta l^\lambda)_{;\lambda} + \tilde{\xi}^\nu{}_{;\mu}l_\nu n^\mu\delta\Omega_H = -\frac{1}{2}h_{\nu\lambda}{}^{;\nu}l^\lambda - \frac{1}{2}h_{\nu\lambda}l^{\lambda;\nu} + \\ + \tilde{\xi}^\nu{}_{;\mu}l_\nu n^\mu\delta\Omega_H = -\frac{1}{2}h_{\nu\lambda}{}^{;\nu}l^\lambda + \tilde{\xi}^\nu{}_{;\mu}l_\nu n^\mu\delta\Omega_H. \tag{1.95}$$

We have assembled all the tools necessary to vary equation 1.73 which we will express in the form:

$$M = \int_\Sigma \left(2T^\mu{}_\nu + \frac{1}{8\pi}R\delta^\mu_\nu\right)\xi^\nu\,\mathrm{d}\Sigma_\mu + 2\Omega_H J_H + \frac{\kappa}{4\pi}A. \tag{1.96}$$

We first vary the scalar curvature R:

$$\delta R = \delta(g^{\nu\lambda}R^\mu{}_{\nu\mu\lambda}) = R_{\nu\lambda}\delta(g^{\nu\lambda}) + g^{\nu\lambda}\delta(R^\mu{}_{\nu\mu\lambda}) = -R_{\nu\lambda}h^{\nu\lambda} + \\ + g^{\nu\lambda}\delta(R^\mu{}_{\nu\mu\lambda}). \tag{1.97}$$

Secondly, we perform the variation of the Riemann tensor:

$$\delta(R^\mu{}_{\nu\mu\lambda}) = \delta\Gamma^\mu{}_{\nu\lambda,\mu} - \delta\Gamma^\mu{}_{\nu\mu,\lambda} + \Gamma^\mu{}_{\mu\sigma}\delta\Gamma^\sigma{}_{\nu\lambda} + \Gamma^\sigma{}_{\nu\lambda}\delta\Gamma^\mu{}_{\mu\sigma} - \Gamma^\mu{}_{\lambda\sigma}\delta\Gamma^\sigma{}_{\nu\mu} - \\ - \Gamma^\sigma{}_{\nu\mu}\delta\Gamma^\mu{}_{\lambda\sigma}. \tag{1.98}$$

While the connection itself does not transform like tensor, the varition of the non-tensor term in its transformation is equal to zero. The variation of the connection is therefore a tensor Misner et al. [2017] and we can expand its covariant derivative in the standard way. Then we may observe:

$$(\delta\Gamma^\mu{}_{\nu\lambda})_{;\mu} = \delta\Gamma^\mu{}_{\nu\lambda,\mu} + \Gamma^\mu{}_{\mu\sigma}\delta\Gamma^\sigma{}_{\nu\lambda} - \Gamma^\sigma{}_{\nu\mu}\delta\Gamma^\mu{}_{\sigma\lambda} - \Gamma^\sigma{}_{\lambda\mu}\delta\Gamma^\mu{}_{\nu\sigma}, \\ (\delta\Gamma^\mu{}_{\nu\mu})_{;\lambda} = \delta\Gamma^\mu{}_{\nu\mu,\lambda} + \Gamma^\mu{}_{\lambda\sigma}\delta\Gamma^\sigma{}_{\nu\mu} - \Gamma^\sigma{}_{\nu\lambda}\delta\Gamma^\mu{}_{\sigma\mu} - \Gamma^\sigma{}_{\lambda\mu}\delta\Gamma^\mu{}_{\nu\sigma}, \\ \delta(R^\mu{}_{\nu\mu\lambda}) = (\delta\Gamma^\mu{}_{\nu\lambda})_{;\mu} - (\delta\Gamma^\mu{}_{\nu\mu})_{;\lambda}. \tag{1.99}$$

We can rewrite this result in terms of the variation of the metric:

$$(\delta\Gamma^{\mu}{}_{\nu\lambda})_{;\mu} - (\delta\Gamma^{\mu}{}_{\nu\mu})_{;\lambda} = -h^{\mu\sigma}{}_{;\mu}\Gamma_{\sigma\nu\lambda} + g^{\sigma\mu}(\delta\Gamma_{\sigma\nu\lambda})_{;\mu} + h^{\mu\sigma}{}_{;\lambda}\Gamma_{\sigma\nu\mu} - \\ -g^{\sigma\mu}(\delta\Gamma_{\sigma\nu\mu})_{;\lambda}. \quad (1.100)$$

We will use the Christoffel symbols to rewrite the terms that include the variation of Γ:

$$g^{\sigma\mu}(\delta\Gamma_{\sigma\nu\lambda})_{;\mu} - g^{\sigma\mu}(\delta\Gamma_{\sigma\nu\mu})_{;\lambda} = \frac{1}{2}g^{\sigma\mu}[(h_{\sigma\nu,\lambda} + h_{\lambda\sigma,\nu} - \\ -h_{\nu\lambda,\sigma})_{;\mu} - (h_{\sigma\nu,\mu} + h_{\mu\sigma,\nu} - h_{\nu\mu,\sigma})_{;\lambda}]. \quad (1.101)$$

The variation of the Ricci tensor contracted with the metric can be compactly written as $2g^{\mu\sigma}g^{\nu\lambda}h_{\sigma[\mu;\nu]\lambda}$, which can be showed by expanding this desired result:

$$g^{\mu\sigma}g^{\nu\lambda}h_{\sigma\mu;\nu\lambda} = g^{\mu\sigma}g^{\nu\lambda}((h_{\sigma\mu,\nu})_{;\lambda} - \Gamma^{\rho}{}_{\sigma\nu}h_{\rho\mu;\lambda} - \Gamma^{\rho}{}_{\mu\nu}h_{\sigma\rho;\lambda}), \\ 2g^{\mu\sigma}g^{\nu\lambda}h_{\sigma[\mu;\nu]\lambda} = g^{\mu\sigma}g^{\nu\lambda}((h_{\sigma\mu,\nu})_{;\lambda} - \Gamma^{\rho}{}_{\sigma\nu}h_{\rho\mu;\lambda} - (h_{\sigma\nu,\mu})_{;\lambda} + \Gamma^{\rho}{}_{\sigma\mu}h_{\rho\nu;\lambda}). \quad (1.102)$$

When we use the metrics to raise indices and rename some of them we can indeed observe:

$$g^{\nu\lambda}\delta(R^{\mu}{}_{\nu\mu\lambda}) = -2h^{\mu}{}_{[\mu;\nu]}{}^{\nu}. \quad (1.103)$$

In order to vary term $\int_\Sigma (1/8\pi)R\delta^{\mu}{}_{\nu}\xi^{\nu}\,\mathrm{d}\Sigma_{\mu}$ we also need to evaluate $\delta\left(\xi^{\mu}\,\mathrm{d}\Sigma_{\mu}\right) = \delta\left(\xi^{\mu}\sqrt{-\tilde{g}}\zeta_{\mu}\right)$, where ζ^{μ} is the unit vector normal to hypersurface Σ and $\tilde{g}_{\mu\nu} = g_{\mu\nu} - \zeta_{\mu}\zeta_{\nu}$ is a projection of the metric on Σ. Since both hypersurface Σ and Killing vector ξ^{μ} are unaffected by the variation we only need to find $\delta\sqrt{-\tilde{g}}$. We can use the following identity valid for a derivative of the determinant of any non-singular matrix M Wald [1984]:

$$\mathrm{Tr}\left(\frac{\mathrm{d}A}{\mathrm{d}\lambda}A^{-1}\right) = \frac{1}{\det A}\frac{\mathrm{d}}{\mathrm{d}\lambda}(\det A) \quad (1.104)$$

We will expand expression $\delta\sqrt{-\tilde{g}}$ and then apply equation 1.104:

$$\delta\sqrt{-\tilde{g}} = \frac{\partial\sqrt{-\tilde{g}}}{\partial g^{\mu\nu}}h^{\mu\nu} = \frac{\partial\sqrt{-\tilde{g}}}{\partial\tilde{g}_{\alpha\beta}}\frac{\partial\tilde{g}_{\alpha\beta}}{\partial g^{\mu\nu}}h^{\mu\nu} = \frac{1}{2\sqrt{-\tilde{g}}}\frac{\partial(-\tilde{g})}{\partial\tilde{g}_{\alpha\beta}}\frac{\partial\tilde{g}_{\alpha\beta}}{\partial g^{\mu\nu}}h^{\mu\nu} = \\ = \frac{1}{2\sqrt{-\tilde{g}}}(-\tilde{g})\tilde{g}^{\alpha\beta}\frac{\partial\tilde{g}_{\alpha\beta}}{\partial g^{\mu\nu}}h^{\mu\nu}. \quad (1.105)$$

We now have to deal with the term $\frac{\partial\tilde{g}_{\alpha\beta}}{\partial g^{\mu\nu}}$. This will require one more auxiliary result:

$$0 = \frac{\partial\delta^{\alpha}_{\beta}}{\partial g^{\mu\nu}} = \frac{\partial(g^{\alpha\gamma}g_{\gamma\beta})}{\partial g^{\mu\nu}} = g^{\alpha\gamma}\frac{\partial g_{\gamma\beta}}{\partial g^{\mu\nu}} + g_{\gamma\beta}\frac{\partial g^{\alpha\gamma}}{\partial g^{\mu\nu}}, \\ g^{\alpha\gamma}\frac{\partial g_{\gamma\beta}}{\partial g^{\mu\nu}} = -g_{\gamma\beta}\frac{1}{2}(\delta^{\alpha}_{\mu}\delta^{\gamma}_{\nu} + \delta^{\alpha}_{\nu}\delta^{\gamma}_{\mu}), \\ \frac{\partial g_{\alpha\beta}}{\partial g^{\mu\nu}} = -\frac{1}{2}(g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu}). \quad (1.106)$$

Substituting the definition of $\tilde{g}_{\alpha\beta}$ into equation 1.105 and using orthogonality of $\mathrm{d}\Sigma_\mu$ and ζ^μ and equation 1.106 yields the final result:

$$\delta\sqrt{-\tilde{g}} = \frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}^{\alpha\beta}\frac{\partial(g_{\alpha\beta}-\zeta_\alpha\zeta_\beta)}{\partial g^{\mu\nu}}h^{\mu\nu} = \frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}^{\alpha\beta}$$
$$\left(-\frac{1}{2}(g_{\alpha\mu}g_{\beta\nu}+g_{\alpha\nu}g_{\beta\mu}) - \zeta_\alpha\frac{\partial\zeta_\beta}{\partial g^{\mu\nu}} - \zeta_\beta\frac{\partial\zeta_\alpha}{\partial g^{\mu\nu}}\right)h^{\mu\nu} = -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}h^{\mu\nu} \qquad (1.107)$$

Since Σ is unaffected by the variation, $\tilde{g}_{\mu\nu}h^{\mu\nu}$ can be replaced by $g_{\mu\nu}h^{\mu\nu}$. Thus, the variation of $\xi^\mu\,\mathrm{d}\Sigma_\mu$ equals $-\frac{1}{2}g_{\mu\nu}h^{\mu\nu}\xi^\mu\,\mathrm{d}\Sigma_\mu$.

In total, the variation of $\int_\Sigma(1/8\pi)R\delta^\mu{}_\nu\xi^\nu\,\mathrm{d}\Sigma_\mu$ equals:

$$\delta\int_\Sigma\frac{1}{8\pi}R\delta^\mu{}_\nu\xi^\nu\,\mathrm{d}\Sigma_\mu = -\frac{1}{8\pi}\int_S\left(\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)h^{\mu\nu}+2h^\mu{}_{[\mu;\nu]}{}^\nu\right)\xi^\lambda\,\mathrm{d}\Sigma_\lambda. \quad (1.108)$$

We will rewrite the last term in equation 1.108 in terms of variations of mass, angular momentum and black hole's surface gravity. First we need to obtain one auxiliary result by modifying the last term in equation 1.108. Using equations 1.83, 1.85 and 1.8 (the fact that equation 1.8 holds even for partial derivatives, which is easy to check, will also be used) we will vary identity 1.7:

$$\begin{aligned}0 &= g_{\mu\nu;\lambda}\xi^\lambda + g_{\lambda\nu}\xi^\lambda{}_{;\mu} + g_{\lambda\mu}\xi^\lambda{}_{;\nu},\\ 0 &= \delta(g_{\mu\nu,\lambda}\xi^\lambda - \Gamma^\kappa{}_{\mu\lambda}g_{\kappa\nu}\xi^\lambda - \Gamma^\kappa{}_{\nu\lambda}g_{\mu\kappa}\xi^\lambda + g_{\lambda\nu}\xi^\lambda{}_{,\mu} + g_{\lambda\nu}\Gamma^\lambda{}_{\kappa\mu}\xi^\kappa +\\ &+ g_{\lambda\mu}\xi^\lambda{}_{,\nu} + g_{\lambda\mu}\Gamma^\lambda{}_{\kappa\nu}\xi^\kappa) = h_{\mu\nu,\lambda}\xi^\lambda + h_{\lambda\nu}\xi^\lambda{}_{,\mu} + h_{\lambda\mu}\xi^\lambda{}_{,\nu} = h_{\mu\nu;\lambda}\xi^\lambda +\\ &+ h_{\lambda\nu}\xi^\lambda{}_{;\mu} + h_{\lambda\mu}\xi^\lambda{}_{;\nu}. \qquad (1.109)\end{aligned}$$

We can now rewrite term $2h^\mu{}_{[\mu;\nu]}{}^\nu\xi^\lambda$:

$$2h^\mu{}_{[\mu;\nu]}{}^\nu\xi^\lambda = (2h^\mu{}_{[\mu;\nu]}\xi^\lambda)^{;\nu} - 2h^\mu{}_{[\mu;\nu]}\xi^{\lambda;\nu}.$$

Expansion of the second term yields:

$$-2h^\mu{}_{[\mu;\nu]}\xi^{\lambda;\nu} = 2h^\mu{}_{[\mu;\nu]}\xi^{\nu;\lambda} = (h^\mu{}_{\mu;\nu}\xi^\nu - h^\mu{}_{\nu;\mu}\xi^\nu)^{;\lambda} - 2h^\mu{}_{[\mu;\nu]}{}^\lambda\xi^\nu.$$

We will now take a look at each term of this result separately:

$$\begin{aligned}(h^\mu{}_{\mu;\nu}\xi^\nu - h^\mu{}_{\nu;\mu}\xi^\nu)^{;\lambda} &= (-3h_{\mu\nu}\xi^{\mu;\nu} - (h^\mu{}_\nu\xi^\nu)_{;\mu})^{;\lambda} = (0-(\delta\xi^\mu)_{;\mu})^{;\lambda} = 0,\\ -2h^\mu{}_{[\mu;\nu]}{}^\lambda\xi^\nu &= -h^\mu{}_{\mu;\nu}{}^\lambda\xi^\nu + h^\mu{}_{\nu;\mu}{}^\lambda\xi^\nu = -h_\mu{}^{\mu;\lambda}{}_\nu\xi^\nu + h^\mu{}_{\nu;\mu}{}^\lambda\xi^\nu.\end{aligned}$$

The first term is now already in a form suitable for our goal. We can further modify the second term using equation 1.109:

$$\begin{aligned}h^\mu{}_{\nu;\mu}{}^\lambda\xi^\nu &= (h^\mu{}_{\nu;\mu}\xi^\nu)^{;\lambda} - h^\mu{}_{\nu;\mu}\xi^{\nu;\lambda} = 0 - (h^\mu{}_\nu\xi^{\nu;\lambda})_{;\mu} + h_{\mu\nu}\xi^{\nu;\lambda\mu} =\\ &= (h^\lambda{}_\nu\xi^{\nu;\mu} + h^{\lambda\mu}{}_{;\nu}\xi^\nu)_{;\mu} + h_{\mu\nu}\xi^{\nu;\lambda\mu} = h_\mu{}^{\lambda;\mu}{}_\nu\xi^\nu + \left(-\left(h_\nu{}^\lambda\xi^\mu\right)_{;\nu} + h^{\lambda\mu}{}_{;\nu}\xi^\nu\right)_{;\mu} +\\ &+ h_{\mu\nu}\xi^{\nu;\lambda\mu}.\end{aligned}$$

All terms except the first one yield zero together:

$$\begin{aligned}&\left(-\left(h_\nu{}^\lambda\xi^\mu\right)_{;\nu} + h^{\lambda\mu}{}_{;\nu}\xi^\nu\right)_{;\mu} + h_{\mu\nu}\xi^{\nu;\lambda\mu} = R^{\sigma\mu}{}_{\nu\mu}h_\sigma{}^\lambda\xi^\nu +\\ &+ R^{\sigma\lambda}{}_{\nu\mu}h^\mu{}_\sigma\xi^\nu + h_{\mu\nu}R_\sigma{}^{\mu\lambda\nu}\xi^\sigma - h_\nu{}^\lambda R_{\sigma\mu}{}^{\nu\mu}\xi^\sigma = 0.\end{aligned}$$

We used symmetries of the Riemann tensor and Killing vectors, equation 1.61 and also the identities:

$$h^{\mu}{}_{\mu;\nu\lambda} - h^{\mu}{}_{\mu;\lambda\nu} = R^{\sigma}{}_{\mu\nu\lambda} h_{\sigma}{}^{\mu} + R^{\sigma}{}_{\mu\nu\lambda} h^{\mu}{}_{\sigma} = 0, \tag{1.110}$$

$$h^{\lambda\mu}{}_{;\nu\mu} - h^{\lambda\mu}{}_{;\mu\nu} = R^{\sigma\mu}{}_{\nu\mu} h_{\sigma}{}^{\lambda} + R^{\sigma\lambda}{}_{\nu\mu} h^{\mu}{}_{\sigma}. \tag{1.111}$$

In total we have shown:

$$2h^{\mu}{}_{[\mu;\nu]}{}^{\nu}\xi^{\lambda} = \left(2h^{\mu}{}_{[\mu;\nu]}\xi^{\lambda} - 2h^{\mu}{}_{[\mu;\lambda]}\xi^{\nu}\right)_{;\nu}. \tag{1.112}$$

This expression is antisymmetric, allowing us to transform the integral over hypersurface Σ to an integral over its boundary:

$$\begin{aligned} -\frac{1}{8\pi}\int_{\Sigma} 2h^{\mu}{}_{[\mu;\nu]}{}^{\nu}\xi^{\lambda}\,\mathrm{d}\Sigma_{\lambda} &= -\frac{1}{4\pi}\int_{\Sigma}\left(h^{\mu}{}_{[\mu;\nu]}\xi^{\lambda} - h^{\mu}{}_{[\mu;\lambda]}\xi^{\nu}\right)_{;\nu}\mathrm{d}\Sigma_{\lambda} = \\ &= -\frac{1}{4\pi}\int_{\partial\Sigma}\left(h_{\mu}{}^{[\mu;\nu]}\xi^{\lambda} - h_{\mu}{}^{[\mu;\lambda]}\xi^{\nu}\right)\mathrm{d}\Sigma_{\lambda\nu}. \end{aligned} \tag{1.113}$$

The boundary is again formed by two-surfaces $\partial\Sigma_{\infty}$ and ∂B identical to those in equation 1.43. In this case we can again rely on the Schwarzschild coordinates while evaluating the integral over $\partial\Sigma_{\infty}$:

$$\begin{aligned} -\frac{1}{4\pi}\int_{\partial\Sigma_{\infty}}\left(h_{\mu}{}^{[\mu;\nu]}\xi^{\lambda} - h_{\mu}{}^{[\mu;\lambda]}\xi^{\nu}\right)\mathrm{d}\Sigma_{\lambda\nu} = \\ -\frac{1}{4\pi}\int_{\partial\Sigma_{\infty}}\left(h_{\mu}{}^{[\mu;\nu]}\xi^{\lambda} - h_{\mu}{}^{[\mu;\lambda]}\xi^{\nu}\right)\frac{1}{\sqrt{-g_{tt}}}\xi_{[\lambda}\rho_{\nu]}\,\mathrm{d}\Sigma = \\ = \lim_{r\to\infty}\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{1}{2}\left(-h^{\mu}{}_{\mu;r} + h^{\mu}{}_{r;\mu}\right)r^2\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta = \\ = \lim_{r\to\infty}\frac{1}{8\pi}\int_0^{\pi}\int_0^{2\pi} g^{\mu\nu}\left(-h_{\nu\mu;r} + h_{\nu r;\mu}\right)r^2\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta. \end{aligned} \tag{1.114}$$

The only quantity in the Schwarzschild metric influenced by the variation that we consider is total mass M present only in terms g_{tt}, g_{rr}:

$$\begin{aligned} \lim_{r\to\infty}\frac{1}{8\pi}\int_0^{\pi}\int_0^{2\pi} g^{\mu\nu}\left(-h_{\nu\mu;r} + h_{\nu r;\mu}\right)r^2\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta = \\ = \lim_{r\to\infty}\frac{1}{8\pi}\int_0^{\pi}\int_0^{2\pi}\left(-g^{tt}h_{tt;r} - g^{rr}h_{rr;r} + g^{rr}h_{rr;r}\right)r^2\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta = \\ = \lim_{r\to\infty}\frac{1}{8\pi}\int_0^{\pi}\int_0^{2\pi}\left(1 - \frac{2M}{r}\right)\left(-\frac{2\delta M}{r^2}\right)r^2\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta = -\delta M. \end{aligned} \tag{1.115}$$

Since h_{tt} is proportional to $1/r$ and relevant Christoffel symbols are proportional to $1/r^2$, we could take advantage of the limit $r\to\infty$ and write $h_{tt;r} = h_{tt,r}$ as the remaining terms are proportional to $1/r^3$. We will now deal with the integral over the horizon, once again using $\mathrm{d}\Sigma_{\mu\nu} = l_{[\mu}n_{\nu]}\,\mathrm{d}A$:

$$\begin{aligned} -\frac{1}{4\pi}\int_{\partial B}\left(h_{\mu}{}^{[\mu;\nu]}\xi^{\lambda} - h_{\mu}{}^{[\mu;\lambda]}\xi^{\nu}\right)\mathrm{d}\Sigma_{\lambda\nu} = \\ = -\frac{1}{8\pi}\int_{\partial B}\left(h_{\mu}{}^{[\mu;\nu]}\xi^{\lambda} - h_{\mu}{}^{[\mu;\lambda]}\xi^{\nu}\right)\left(l_{\lambda}n_{\nu} - l_{\nu}n_{\lambda}\right)\mathrm{d}A = \\ = \frac{1}{4\pi}\int_{\partial B}\left(-h_{\mu}{}^{[\mu;\nu]}l^{\lambda} + \Omega_H h_{\mu}{}^{[\mu;\nu]}\tilde{\xi}^{\lambda}\right)\left(l_{\lambda}n_{\nu} - l_{\nu}n_{\lambda}\right)\mathrm{d}A = \\ = \frac{1}{4\pi}\int_{\partial B}\left(\frac{1}{2}h_{\mu}{}^{\nu;\mu}l_{\nu} + \Omega_H h_{\mu}{}^{[\mu;\nu]}\tilde{\xi}^{\lambda}\left(l_{\lambda}n_{\nu} - l_{\nu}n_{\lambda}\right)\right)\mathrm{d}A = \\ = -\frac{A}{4\pi}\delta\kappa - 2J_H\delta\Omega_H + \frac{1}{4\pi}\int_{\partial B}\Omega_H h_{\mu}{}^{[\mu;\nu]}\tilde{\xi}^{\lambda}\,\mathrm{d}\Sigma_{\lambda\nu}. \end{aligned} \tag{1.116}$$

We have used equation 1.95 for the variation of the surface gravity and the definition of the black hole angular momentum (equation 1.53). The integral of $(1/4\pi)\Omega_H h_\mu{}^{\mu;\nu}\tilde{\xi}^\lambda$ over 2-surface $\partial\Sigma_\infty$ can be written as:

$$\frac{1}{4\pi}\int_{\partial\Sigma_\infty}\Omega_H h_\mu{}^{[\mu;\nu]}\tilde{\xi}^\lambda\,\mathrm{d}\Sigma_{\lambda\nu} = \frac{1}{4\pi}\int_{\partial\Sigma_\infty}\Omega_H h_\mu{}^{[\mu;\nu]}g_{t\varphi}\,\mathrm{d}\Sigma. \tag{1.117}$$

The dominant terms of the integrand tend to zero as $1/r$ and the integral thus equals zero. We can use it to return to the integration over hypersurface Σ and take advantage of the fact that equation 1.112 holds for any Killing vector field (it is a direct consequence of the Killing equation):

$$\frac{1}{4\pi}\int_{\partial B}\Omega_H h_\mu{}^{[\mu;\nu]}\tilde{\xi}^\lambda\,\mathrm{d}\Sigma_{\lambda\nu} = \frac{1}{4\pi}\int_{\partial S}\Omega_H h_\mu{}^{[\mu;\nu]}\tilde{\xi}^\lambda\,\mathrm{d}\Sigma_{\lambda\nu} = \tag{1.118}$$

$$= \frac{1}{8\pi}\int_S \Omega_H h^\mu{}_{[\mu;\nu]}{}^\nu\tilde{\xi}^\lambda\,\mathrm{d}\Sigma_\lambda = 0. \tag{1.119}$$

The last integral equals zero because hypersurface Σ was chosen to be tangent to vector field $\tilde{\xi}^\mu$. If we now use the Einstein equations we get the final result:

$$\delta\int_\Sigma \frac{1}{8\pi}R\delta^\mu{}_\nu\xi^\nu\,\mathrm{d}\Sigma_\mu = -\int_\Sigma T_{\mu\nu}h^{\mu\nu}\xi^\lambda\,\mathrm{d}\Sigma_\lambda - \delta M + \frac{A}{4\pi}\delta\kappa + 2J_H\delta\Omega_H. \tag{1.120}$$

The variation of the energy-momentum tensor term yields:

$$\begin{aligned}
\delta\int_\Sigma 2T^\mu{}_\nu\xi^\nu\,\mathrm{d}\Sigma_\mu &= -2\int_\Sigma \Omega\delta(T^\mu{}_\nu\tilde{\xi}^\nu\,\mathrm{d}\Sigma_\mu) + 2\int_\Sigma\delta(T^\mu{}_\nu u^\nu\,\mathrm{d}\Sigma_\mu) = \\
&= 2\int_\Sigma\Omega\delta\,\mathrm{d}J + 2\int_\Sigma\delta(((\varepsilon+p)v^\mu v_\nu + pg^\mu{}_\nu)u^\nu\,\mathrm{d}\Sigma_\mu) = \\
= 2\int_\Sigma\Omega\delta\,\mathrm{d}J + 2\int_\Sigma\delta(p\xi^\mu\,\mathrm{d}\Sigma_\mu) + 2\int_\Sigma u^\nu\delta\left((\mu n+\theta s)\frac{1}{-g_{\kappa\lambda}u^\kappa u^\lambda}u_\nu\xi^\mu\,\mathrm{d}\Sigma_\mu\right) = \\
= 2\int_\Sigma\Omega\delta\,\mathrm{d}J + 2\int_\Sigma\delta((\mu n+\theta s+p)\xi^\mu\,\mathrm{d}\Sigma_\mu) + \int_\Sigma(\mu n+\theta s)h_{\kappa\lambda}v^\kappa v^\lambda\xi^\mu\,\mathrm{d}\Sigma_\mu = \\
= 2\int_{\mathbf{S}}\Omega\delta\,\mathrm{d}J + 2\int_\Sigma\bar{\theta}\delta\,\mathrm{d}\Sigma + 2\int_\Sigma\bar{\mu}\delta\,\mathrm{d}N + \int_\Sigma T^{\kappa\lambda}h_{\kappa\lambda}\xi^\mu\,\mathrm{d}\Sigma_\mu.
\end{aligned} \tag{1.121}$$

The first equality uses the definition of fluid four-velocity u^μ. We also repeatedly used orthogonality of $\tilde{\xi}^\mu$ and $\mathrm{d}\Sigma_\mu$. The final expression is given in the quantities measured by an observer at the asymptotic infinity, while the quantities present in the energy-momentum tensor are measured locally by an observer co-moving with the fluid. To account for this, we have introduced red-shifted chemical potential $\bar{\mu}$:

$$\bar{\mu} = \sqrt{-g_{\kappa\lambda}u^\kappa u^\lambda}\mu \tag{1.122}$$

and red-shifted temperature $\bar{\theta}$:

$$\bar{\theta} = \sqrt{-g_{\kappa\lambda}u^\kappa u^\lambda}\theta. \tag{1.123}$$

We have also defined the change of the fluid angular momentum crossing $\mathrm{d}\Sigma_\mu$:

$$\delta\,\mathrm{d}J = -\delta(T^\mu{}_\nu\tilde{\xi}^\nu\,\mathrm{d}\Sigma_\mu), \tag{1.124}$$

the change in the number of particles crossing $\mathrm{d}\Sigma_\mu$:

$$\delta\,\mathrm{d}N = \delta\left(n\frac{1}{\sqrt{-g_{\kappa\lambda}u^\kappa u^\lambda}}\xi^\mu\,\mathrm{d}\Sigma_\mu\right), \tag{1.125}$$

and finally the change in entropy crossing $\mathrm{d}\Sigma_\mu$:

$$\delta\,\mathrm{d}S = \delta\left(s\frac{1}{\sqrt{-g_{\kappa\lambda}u^\kappa u^\lambda}}\xi^\mu\,\mathrm{d}\Sigma_\mu\right). \tag{1.126}$$

By putting results of both variations together we acquire the differential mass formula:

$$\delta M = \frac{\kappa}{8\pi}\delta A + \int_S \Omega\delta\,\mathrm{d}J + \int_S \bar{\theta}\delta\,\mathrm{d}S + \int_S \bar{\mu}\delta\,\mathrm{d}N + \Omega_H\delta J_H \tag{1.127}$$

The first two terms describe the changes in the black hole's energy. We will soon demonstrate that in the case of the Kerr spacetime the are equivalent to the terms present in the Bekenstein formula 1.40. The remaining three terms are standard thermodynamic contributions from the perfect fluid outside the black hole. The first one gives the heat transfered to the fluid, the second one work performed on the fluid to change the number of particles and the third one the work done on the fluid to change its angular momentum. All these contributions are measured from the asymptotic infinity.

Formulas for a Kerr black hole

It is instructive to explicitly show how the formulas found by Bardeen, Carter and Hawking look for a vacuum case, a Kerr black hole. In the integral formula, the energy-momentum tensor term becomes zero and the expression is reduced to:

$$M = 2\Omega_H J_H + \frac{\kappa}{4\pi}A. \tag{1.128}$$

Here M and $J_H = Ma$ are parameters in the Boyer-Lindquist coordinates as given by equation 1.16. By analysing the orbits of stationary observers around a Kerr black hole, it can be shown that $\Omega_H = \omega(r_+)$ Misner et al. [2017]. The horizon area A is $4\pi\left(r_+^2 + a^2\right)$. Using these relations and equation 1.128, we can express κ as a function of the parameters of the Kerr metric:

$$\begin{aligned}\kappa = 4\pi\frac{M - 2\Omega_H J_H}{A} = M\frac{1 - 2\omega(r_+)a}{r_+^2 + a^2} = \frac{1 - \frac{4Ma^2r_+}{\mathcal{A}(r_+)}}{2r_+} = \\ = \frac{r_+^2 - a^2}{2r_+\left(r_+^2 + a^2\right)} = \frac{r_+^2 - r_+r_-}{2r_+\alpha} = \frac{r_+ - r_-}{2\alpha}.\end{aligned} \tag{1.129}$$

During the calculation we used equations 1.34 - 1.36 with black hole's charge Q equal to zero.

The differential formula for a Kerr black hole takes form:

$$\delta M = \Omega_H\delta J_H + \frac{\kappa}{8\pi}\delta A, \tag{1.130}$$

as all terms describing the fluid outside the black hole are clearly equal to zero. We can insert expressions for Ω_H and κ:

$$\delta M = \frac{a}{\alpha}\delta J + \frac{r_+ - r_-}{16\pi\alpha}\delta A = \frac{a}{\alpha}\delta J + \frac{r_+ - r_-}{4\alpha}\delta\alpha. \tag{1.131}$$

This is the Bekenstein formulation of the first law for a Kerr black hole (with $Q = 0$). Both versions of the first law of black hole thermodynamics thus give the same result for the case of a Kerr black hole for which they are both valid.

1.4 The Zeroth Law

As a part of the derivation of the integral formula we have demonstrated that surface gravity κ is constant on the event horizon of a stationary black hole. Since we have already seen that the role of κ in the first law is analogous to that of the thermodynamic temperature in standard thermodynamics, this gives us the analogy of the zeroth law of thermodynamics stating that for a system in thermal equilibrium the temperature is constant throughout the system. In fact, the actual physical temperature of a black hole (corresponding to the distribution of the black body radiation it emits) was shown by Hawking to be Wald [2001]:

$$\frac{\kappa}{2\pi}. \tag{1.132}$$

This result stems from quantum mechanical effects and it will be discussed in section 1.6.

1.5 The Third Law

The third law was postulated by Bardeen, Carter and Hawking Bardeen et al. [1973] in the following form: It is impossible by any procedure, no matter how idealized, to reduce black hole's surface gravity κ to zero by a finite sequence of operations.

We can see that this statement is almost identical to Nernst's formulation of the standard third law, only the thermodynamic temperature is replaced by the surface gravity. For the expression of κ for a Kerr black hole (equation 1.129, the same expression holds for a Kerr-Newman black hole) we can easily check that κ is non-negative and equal to zero only in case of an extremal black hole $r_+ = r_- = M$. Furthermore, let us assume we want to reduce κ to zero by throwing particles inside a black hole. The most efficient approach is to throw them in a way which maximalizes the increase in the black hole's angular momentum for the given mass of a particle. Then it can be shown that the decrease of κ per particle thrown in will slow down as κ tends to zero Bardeen et al. [1973]. Israel later provided a proof of this law Israel [1986].

On the other hand, it seems impossible to acquire the equivalent of the Planck's formulation of the third law. It states that when the temperature of any thermodynamic system reaches the absolute zero the entropy acquires some arbitrary constant value independent of system's macroscopic parameters (which is usually set to zero). However, it is easy to calculate that entropy of an extremal black hole is πM^2 which clearly depends on black hole's mass.

In case that $M^2 < a^2$ or $M^2 < a^2 + Q^2$ for a Kerr and Kerr-Newman black hole respectively, the values of r_+ and r_- become imaginary, the event horizon no longer exists and the singularity in the center of a black hole can interact with the rest of the universe. Such a case is known as a *naked singularity*. The presence of naked singularities in our universe would mean, at least in the context of general relativity, that the spacetime would no longer be asymptotically predictable (i.e., a solution for a Cauchy problem ceases to be determined by the initial data once information coming from the singularity, which cannot be predicted by general relativity, can influence it). Asymptotic predictability is necessary for many important results about black holes, including the area increase theorem Bardeen et al. [1973]. In 1965, Penrose suggested the *cosmic censorship* hypothesis stating that naked singularities cannot appear under physically realistic conditions Misner et al. [2017]. Whether this hypothesis holds remains one of the most important unanswered questions of relativistic physics. The third law of black hole thermodynamics excludes the possibility of a non-extremal black hole ever becoming extremal, much less a naked singularity, and is thus be an important argument in favour of the cosmic censorship, though a possibility that a naked singularity can be created in a different way would remain.

1.6 Hawking Radiation and Black Hole Temperature

As was already mentioned, no satisfactory interpretation of black hole temperature exists in the context of classical general relativity. To assign a non-zero thermodynamic temperature to an object, we require it to come to thermodynamic equilibrium with an infinite thermal bath of the same temperature, with the object radiating the same amount of heat that is receiving. This is of course impossible for a classical black hole, as nothing can escape its event horizon. Therefore, the only reasonable value of its thermodynamic temperature is zero. However, in 1974 Hawking showed that, when quantum effects are taken into account, black holes radiate with a spectrum corresponding to a black body of temperature $\kappa/2\pi$. He acquired this result by analyzing a quantum field on a classical background of the Schwarzschild solution Hawking [1975]. We shall use a different approach developed by Parikh and Wilczek Parikh and Wilczek [2000] in which WKB (Wentzel-Kramers-Brillouin) semiclassical limit of quantum theory is considered and Hawking radiation is treated as tunneling of particles out of the black hole.

Our aim is to analyze tunneling of a positive energy particle outwards from a Schwarzschild black hole's event horizon along a radial null geodesics or, equivalently, tunneling of a negative energy particle inside the horizon. To do so, a coordinate system nonsingular on the horizon is required. We will use a convenient transformation of the Schwarzschild coordinates 1.22, the Gullstrand-Painlevé coordinates:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + 2\sqrt{\frac{2M}{r}}\,\mathrm{d}t\,\mathrm{d}r + \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2. \tag{1.133}$$

The time coordinate t is obtained from the Schwarzschild time coordinate t_S by

the following transformation:

$$t = t_S + 2M\sqrt{2Mr} + 2M \ln \frac{\left|\sqrt{(}r) - \sqrt{(}2M)\right|}{\sqrt{(}r) + \sqrt{(}2M)}. \tag{1.134}$$

The spatial coordinates are left unchanged. The line element given by equation 1.133 is stationary (but no longer static) and non-singular through the horizon (the singular behavior on the horizon is moved to the definition of the time coordinate). This property makes the Gullstrand-Painlevé coordinates more suitable for evaluating physical processes close to the horizon. The radial null geodesics in this coordinates are given by the equation:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm 1 - \sqrt{\frac{2M}{r}}. \tag{1.135}$$

Here plus sign corresponds to an outgoing and minus sign to an ingoing geodesic.

While the tunneling approach to Hawking radiation may seem natural at first, we are soon confronted with a question: what is the barrier a particle tunnels through? A typical situation for quantum tunneling involves two classical turning points separated by a classically forbidden region of finite size. However, a pair of particles, one with a positive energy and one with negative, can be created inside the event horizon arbitrarily close to it and once the positive energy particle gets just outside the horizon it can classically travel away from the black hole. For the negative energy particle the same argument applies just with the inside and outside of the horizon exchanged. It thus appear that the separation distance between the classical turning points is zero Parikh [2004]. The situation changes if we explicitly enforce the conservation of energy. Then during the emission process the black hole's total energy is lowered by the energy of the particle emitted and its horizon radius decreases accordingly. The gap between the original and new position of the radius is then the classicaly forbidden region between turning points through which the particle must tunnel. In this sense is the tunneling process itself responsible for the existence of the barrier. It was shown by Kraus and Wilczek that, if the total ADM mass in the spacetime is fixed and the mass of the black hole is allowed to change, a particle of energy ω travels along the geodesics described by equation 1.133, where we substitute $(M - \omega)$ for M Kraus and Wilczek [1995].

It can be easily shown that the tunneling rate in WKB approximation is proportional to $e^{-2\,\mathrm{Im}\,S}$, where S stands for the action. To get the tunneling rate (without the pre-factor) we need to evaluate imaginary part of the action over a radial null geodesic between the starting point r_{in} and the ending point r_{out}. We will consider a positive energy particle with zero angular momentum (an s-wave particle). The imaginary part of the action equals:

$$\mathrm{Im}\, S = \mathrm{Im} \int_{r_{in}}^{r_{out}} p_r \,\mathrm{d}r = \mathrm{Im} \int_{r_{in}}^{r_{out}} \int_0^{p_r} \mathrm{d}p'_r \,\mathrm{d}r. \tag{1.136}$$

We now multiply and divide the integrand by two sides of the Hamilton equation $\mathrm{d}r/\,\mathrm{d}t = \mathrm{d}H/\,\mathrm{d}p'_r$, change the integration variable from momentum to energy, use equation 1.135 for an outgoing geodesic (with M replaced by $(M - \omega)$) to

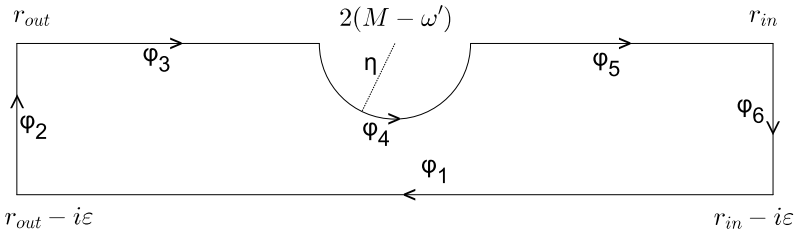

Figure 1.1: The closed curve over which the imaginary part of the action is evaluated

substitute for $\mathrm{d}r/\,\mathrm{d}t$ and change the order of integration:

$$\begin{aligned}\operatorname{Im} S = \operatorname{Im}\int_{r_{in}}^{r_{out}}\int_0^{p_r}\frac{1}{\dfrac{\mathrm{d}r}{\mathrm{d}t}}\frac{\mathrm{d}H}{\mathrm{d}p_r'}\,\mathrm{d}p_r'\,\mathrm{d}r = \operatorname{Im}\int_{r_{in}}^{r_{out}}\int_M^{M-\omega}\frac{1}{1-\sqrt{\frac{2M'}{r}}}\,\mathrm{d}M'\,\mathrm{d}r = \\ = -\operatorname{Im}\int_0^{\omega}\int_{r_{in}}^{r_{out}}\frac{1}{1-\sqrt{\frac{2(M-\omega')}{r}}}\,\mathrm{d}r\,\mathrm{d}\omega'.\end{aligned} \tag{1.137}$$

To evaluate the imaginary part of this integral we will change ω' to $\omega' - i\varepsilon$, where ε is an arbitrarily small positive number. The minus sign of the imaginary term ensures that positive energy solutions decay in time. Imaginary part of the integral can now be evaluated using methods of complex analysis. We will integrate function $\left(1-\sqrt{2\left(M-\omega'\right)/z}\right)^{-1}$ of a complex variable z over closed curve φ displayed in figure 1.1. Since the function is holomorphic in the interior of φ, the integral is zero by the residue theorem. We want to obtain the imaginary part of the integral over line φ_1. Integrals over lines φ_2, φ_6 can be made arbitrarily small by lowering the value of ε. Integrals over lines φ_3, φ_5 are real for any non-zero η. Therefore, the imaginary part of the integral over φ_1 is equal to minus the imaginary part of the integral over half-circle φ_4. In the limit of η approaching zero this integral equals $i\pi Res_{z=2(M-\omega')}\left(1-\sqrt{2\left(M-\omega'\right)/z}\right)^{-1}$. The residue is equal to:

$$\begin{aligned}Res_{z=2(M-\omega')}\frac{1}{1-\sqrt{\frac{2(M-\omega')}{z}}} = \frac{1}{\dfrac{\mathrm{d}\left(1-\sqrt{\frac{2(M-\omega')}{z}}\right)}{\mathrm{d}z}} = \frac{1}{\frac{1}{2}\sqrt{\frac{2(M-\omega')}{z^3}}} = \\ = 4\left(M-\omega'\right).\end{aligned} \tag{1.138}$$

Provided that r_{in} is greater than r_{out} we have in total:

$$\operatorname{Im} S = \operatorname{Im}\int_0^{\omega} i\pi 4\left(M-\omega'\right)\mathrm{d}r = \int_0^{\omega}\pi 4\left(M-\omega'\right)\mathrm{d}\omega' = 4\pi\omega\left(M-\frac{\omega}{2}\right). \tag{1.139}$$

If we instead evaluate the integral without changing the order of integration, we may get the values of r_{in} and r_{out}:

$$\operatorname{Im} S = \operatorname{Im}\int_{r_{in}}^{r_{out}}\int_M^{M-\omega}\frac{1}{1-\sqrt{\frac{2M'}{r}}}\,\mathrm{d}M'\,\mathrm{d}r. \tag{1.140}$$

We shall construct a closed curve completely analogous to figure 1.1 and use the same line of reasoning. The residue yields:

$$Res_{z=\frac{r}{2}} \frac{1}{1-\sqrt{\frac{2z}{r}}} = \frac{1}{\frac{\mathrm{d}\left(1-\sqrt{\frac{2z}{r}}\right)}{\mathrm{d}z}} = \frac{1}{-\frac{1}{2}2\sqrt{\frac{2}{zr}}} = -r. \tag{1.141}$$

And the integral is:

$$\mathrm{Im}\, S = -\,\mathrm{Im} \int_{r_{in}}^{r_{out}} i\pi r \,\mathrm{d}r = \int_{r_{in}}^{r_{out}} \pi r \,\mathrm{d}r = 4\pi\omega \left(M - \frac{\omega}{2}\right). \tag{1.142}$$

We can thus see that $r_{in} = 2M$ and $r_{out} = 2\left(M - \omega\right)$ and the condition $r_{in} > r_{out}$ holds. This physically means that the particle is created just outside the original horizon position and then propagates inwards and stops just outside the new position of the horizon. An analogous calculation can be performed for a negative energy particle tunneling inside the horizon. In this case the radial null geodesic has a minus sign and the total spacetime energy M term is replaced by $M + \omega$ rather than by $M - \omega$. The calculation yields the same tunneling rate. Technically, both positive and negative energy channel contribute to the black hole radiation and their emission amplitudes should be added together before calculating the total emission rate (square of the amplitude) but this only concerns the pre-factor. In any case, the emission rate Γ in the WKB limit is proportional to $e^{-2\,\mathrm{Im}\, S}$:

$$\Gamma \sim e^{-2\,\mathrm{Im}\, S} \sim e^{-8\pi\omega\left(M-\frac{\omega}{2}\right)}. \tag{1.143}$$

Note that for a Schwarzschild black hole of mass M the change of entropy corresponding to the lowering of its mass by ω equals $\Delta S_{BH} = -8\pi\omega\left(M - \frac{\omega}{2}\right)$ and the emission rate is proportional to $e^{\Delta S_{BH}}$ Parikh and Wilczek [2000]. In terms of statistical physics, this results corresponds to the radiation being described by a microcanonical ensemble with entropy $-\Delta S_{BH}$. In this treatment the outgoing radiation carries away exactly the amount of entropy lost by the black hole. However, if we neglect the term proportional to ω^2 (whose presence is a result of the requierement of energy conservation) we get exactly thermal spectrum of radiation corresponding to the Hawking temperature of the Schwarzschild black hole, $\kappa/2\pi = 1/8\pi M$. We shall return to this result in the discussion of the black hole information paradox.

Let us now briefly discuss how justified are the assumptions we imposed to get this result. The semiclassical limit can be assumed to be reasonable as when one traces the radiation back to the event horizon it is increasingly blue-shifted and exactly on the horizon the blue-shift becomes infinite. This allows us to treat the radiation as being composed of point-like particles, thus justifying the semiclassical limit. The assumption of stationary spacetime may appear problematic but at least in case $\omega << M$ seems reasonable. In any case, the same assumption is made by Hawking's original derivation of the black hole radiation [quote Hawking 1974]. However, two other problems remain. The calculation explicitly relies on the choice of the Gullstrand-Painlevé coordinate system. This is questionable considering the general covariance principle of the general relativity. Furthermore, only null radial geodesics are considered. One would imagine that all possible paths of tunneling must be taken into account to get the correct tunneling rate. While these questions remain unresolved in the context of

Parikh-Wilczek approach, later developed method using Hamilton-Jacobi equation Padmanabhan [2004] has been able to replicate the results discussed above without relying on a specific coordinate system or propagation trajectory. Also, it allows generalization for non-stationary black holes Vanzo et al. [2011]. A review of Hamilton-Jacobi approach is unfortunately beyond the scope of this work.

2. Entropy as Information

So far we have written the expression for black hole entropy (equation 1.32) but we have barely touched the subject of its interpretation. We shall now remedy this omission, starting by introducing two definitons of entropy which turn out to be rather suitable in the context of the black hole physics.

2.1 Shannon and von Neumann Entropy

2.1.1 Shannon Entropy

Claude Shannon reached his definition of the entropy examining properties of communication channels. In his 1948 paper he gave answers to two important questions, the first one being: how much redundant information is contained in a message transmitted by an ideal (noiseless) channel? The second one considers the information redundancy from a different point of view: how much redundant information must be transmitted over a noisy channel to allow precise decoding of the message? Pursuing these questions led Shannon to the formulation of the noiseless coding theorem and the noisy channel coding theorem Preskill [2018]. While discussion of these theorems exceeds the scope of this work, we will at least show how the analysis of communication gives rise to the Shannon entropy formula.

Consider a message in a form of a n-character string transmitted by a noiseless channel. Let us assume that the message is generated by a source which chooses to transmit the character x with probability $p(x)$ independently of other transmitted characters. Now, the probability that we receive a specific message $(x_1, x_2, ..., x_n)$ equals:

$$p(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} p(x_i). \tag{2.1}$$

Let us now consider that there are k different characters we can include in the message (it is useful to imagine just a binary code, in which 0 is included with probability p and 1 with probability $(1-p)$). If n is very large we may expect, thanks to the law of large numbers, that a typical string will include character x_i about $np(x_i)$ times and that the amount of strings which differ from the typical ones will become negligible Preskill [2018]. Number N of possible typical strings is given as a simple permutation with repetition:

$$N = \frac{n!}{\prod_{i=1}^{k} (np(x_i))!}. \tag{2.2}$$

If we rewrite this as $e^{\ln N}$ and use the fact that both n and all $np(x_i)$ are very large (no matter how small probabilities $p(x_i)$ are, we are able to find n large enough for this to be true), we can invoke the Stirling formula for natural logarithm of a

factorial and get:

$$\ln N = \ln\left(\prod_{i=1}^{k}\left(np\left(x_i\right)\right)!\right) \approx -n\sum_{i=1}^{k} p\left(x_i\right)\ln\left(p\left(x_i\right)\right), \tag{2.3}$$

$$N \approx \exp\left(-n\sum_{i=1}^{k} p\left(x_i\right)\ln\left(p\left(x_i\right)\right)\right) \equiv \exp\left(nS\right). \tag{2.4}$$

We have defined the Shannon entropy:

$$S = -\sum_{i=1}^{k} p\left(x_i\right)\ln\left(p\left(x_i\right)\right). \tag{2.5}$$

Unlike Shannon, we use natural logarithms instead of logarithms with base 2 Shannon [1948]. This is more common in the context of standard thermodynamics. The difference is only in a multiplicative constant.

Considerations made above constitute the essence of the Shannon noiseless coding channel theorem. To decode the message without risking a non-negligible error we now only need a code which assigns a unique character to any possible typical string. Thus, the code needs to distinguish about $2^{(nS)}$ different messages occuring with nearly equal probability. This allows us to compress a message to just slightly more than $nS\log_2 e$ characters. It's simple to see that $S\log e = 1$ only for the uniform distribution $p\left(x_i\right), \forall i \in \{1, 2, ..., k\}$. In any other case $nS\log_2 e < n$ and the final message is shorter. In this sense, the Shannon entropy quantifies how much ignorant are we about the content of the message, how big a part of it do we need to reliably reconstruct it Preskill [2018]. The whole argument can of course be made in a formally correct way and it can be shown to be indeed the most efficient way to shorten the message Shannon [1948]. However, we only followed it to get an idea about the origin of the definition of the Shannon entropy.

Let us now consider two correlated sources of information X, Y working in the same way as the one introduced above, with numbers of characters k_x, k_y and character occurence probabilities $p\left(x_i\right)$, $p\left(y_i\right)$. Thanks to the correlation between sources, by reading the message generated by the source Y one can acquire some knowledge about the message generated by the source X. Denoting the entire n-character (we will again expect n to be very large) messages by vectors $\vec{x}$, $\vec{y}$ we can express conditional probability $p\left(\vec{x}|\vec{y}\right)$ that the message sent by X is $\vec{x}$, once we have determined the the message sent by Y to be $\vec{y}$, using Bayes theorem Preskill [2018]:

$$p\left(\vec{x}|\vec{y}\right) = \frac{p(\vec{x},\vec{y})}{p\left(\vec{y}\right)}. \tag{2.6}$$

Here $p(\vec{x},\vec{y}) = \prod_{i=1}^{n} p\left(x_i, y_i\right)$ is the probability of joint message $(\vec{x},\vec{y})$. The right hand side of the equation 2.6 can be rewritten in terms of the Shannon entropy:

$$p\left(\vec{x}|\vec{y}\right) \approx \frac{2^{-nS(X,Y)\log_2 e}}{2^{-nS(Y)\log_2 e}}. \tag{2.7}$$

It is thus reasonable to define conditional entropy $S(X|Y)$:

$$p\left(\vec{x}|\vec{y}\right) \equiv 2^{-nS(X|Y)\log_2 e}. \tag{2.8}$$

From equations 2.7, 2.8 we see that:

$$S(X|Y) = S(XY) - S(Y). \tag{2.9}$$

And analogously:

$$S(Y|X) = S(XY) - S(X). \tag{2.10}$$

Conditional entropy $S(X|Y)$ quantifies how long a string is needed to specify $\vec{x}$ once $\vec{y}$ is known. In this sense $I(X;Y) \equiv S(X) - S(X|Y)$ represents the amount of information about message $\vec{x}$ acquired by reading message $\vec{y}$. This information turns out to be symmetric:

$$\begin{aligned} I(X;Y) = S(X) - S(X|Y) = S(X) + S(Y) - S(XY) = \\ = S(Y) - S(Y|X) = I(Y;X). \end{aligned} \tag{2.11}$$

By reading message $\vec{y}$ one learns the same amount of information about $\vec{x}$ as one does about $\vec{y}$ by reading $\vec{x}$. Quantity $I(X;Y)$ is called the mutual information and it quantifies the level of correlation between sources X and Y Preskill [2018]. If the sources are independent, $S(X|Y) = S(X)$ and $I(X;Y)$ equals zero. This is of course reasonable behaviour. If message $\vec{y}$ does not depend on message $\vec{x}$ it cannot provide any insight about its content.

On the first glance any connection between the Shannon and thermodynamic entropy may not be apparent and the very existence of such connection is actually questioned by some authors Wüthrich [2017]. Following an argument presented in Adami [2004], we will attempt to show that even the thermodynamic entropy can be regarded as being related to the information we have about the system we study. Consider a thermodynamic system in which we are only interested in positions and momenta of the particles that constitute it (disregarding other possible degrees of freedom). The maximal thermodynamic entropy of the system is given by the volume it occupies in the phase space:

$$S_{max} = \ln\left(\frac{\Delta q \Delta p}{(2\pi)^n}\right). \tag{2.12}$$

Here $\Delta q \Delta p$ gives volume in the phase space and n is the number of degrees of freedom of the system. If we now gain the knowledge of a Hamiltonian which describes our system and measure its thermodynamic temperature θ, the thermodynamic entropy will be reduced to that of a canonical ensemble:

$$S(p,q) = -\sum_p \rho(p,q) \ln \rho(p,q), \tag{2.13}$$

$$\rho(p,q) = \frac{1}{Z} e^{-\frac{\langle H(p,q)\rangle}{\theta}}. \tag{2.14}$$

with Z being the partition function and $\langle H(p,q)\rangle$ the mean value of the Hamiltonian, i.e., system's total energy. The sum goes over all momenta in the given phase volume element. We have used both theory (determining the Hamiltonian) and experiment (measuring temperature) to acquire information about our system and by doing so we reduced system's entropy, which, before learning anything about it, we had to assume to be given by the equation 2.12. We may express the information we gained as the decrease of entropy:

$$I = S_{max} - S(p,q) = \ln\left(\frac{\Delta q \Delta p}{(2\pi)^n}\right) - \sum_p \left(\ln Z + \ln\left(\frac{\langle H(p,q)\rangle}{\theta}\right)\right). \tag{2.15}$$

In this way the thermodynamic entropy can be regarded as related to information in a way which resembles the Shannon entropy Adami [2004].

2.1.2 Density Matrices

The Shannon entropy is in fact just a classical limit of the von Neumann entropy, which was conceived as early as in 1927 Preskill [2018]. To define this entropy we first need to introduce the density matrix formalism.

Consider an ensemble of n quantum systems, with system i being described by a ket $|\psi_i\rangle$. Let us denote the probability to find the state $|\psi_i\rangle$ by p_i, with a normalization $\sum_{i=1}^{n} p_i = 1$. Such an ensemble is called a mixed state and we define its density matrix ρ:

$$\rho \equiv \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|. \tag{2.16}$$

Several properties of a density matrix can be easily proven:

1. $\mathrm{Tr}\rho = 1$.
2. $0 \leq \mathrm{Tr}\rho^2 \leq 1$, and $\mathrm{Tr}\rho^2 = 1$ only for a pure state (a state fully described by a single ket $|\psi\rangle$ or density matrix $\rho_{pure} = |\psi\rangle\langle\psi|$).
3. ρ is self-adjoint: $\rho^\dagger = \rho$.
4. All eigenvalues of ρ are non-negative, ρ is thus non-negative-definite.
5. Expectation value of an observable A, $\langle A\rangle = \sum_{i=1}^{n} p_i \langle\psi_i|A|\psi_i\rangle$, can be computed using the density matrix: $\langle A\rangle = \mathrm{Tr}(A\rho)$.

Thanks to the property 2. the trace of ρ^2 can be regarded as a measure of how mixed is a state Bertlmann [2008], with value 1 corresponding to a pure and 0 to a maximally mixed state. Let us now turn to the interpretation of elements of a density matrix. For diagonal elements ρ_{ii} we have ($\{|\Phi_i\rangle\}$ is a basis of the Hilbert space we work with):

$$\rho_{ii} = \langle\Phi_i|\rho|\Phi_i\rangle = \sum_{j=1}^{n} p_j \langle\Phi_i|\psi_j\rangle\langle\psi_j|\Phi_i\rangle = \sum_{j=1}^{n} p_j |\langle\Phi_i|\psi_j\rangle|^2. \tag{2.17}$$

Therefore, elements ρ_{ii} are simply probabilities of finding the system in a state $|\Phi_i\rangle$. Non-diagonal elements $\rho_{ik} = \sum_{j=1}^{n} p_j \langle\Phi_i|\psi_j\rangle\langle\psi_j|\Phi_k\rangle$ are called coherences. These can be always eliminated by the choice of a basis in which the density matrix is diagonal Porter.

To find the time evolution of a density matrix we will differentiate it with respect to time, using Schrödinger equation $i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$ (H is system's Hamiltonian) and its complex conjugate:

$$\frac{\partial}{\partial t}\rho = \sum_{i=1}^{n} p_i \left(\frac{\partial}{\partial t}|\psi\rangle\langle\psi| + |\psi\rangle\frac{\partial}{\partial t}\langle\psi|\right) = \sum_{i=1}^{n} p_i \left(\frac{1}{i}H|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|\frac{1}{i}H\right),$$

$$i\frac{\partial}{\partial t}\rho = [H, \rho]. \tag{2.18}$$

The square brackets denote a commutator, $[H, \rho] \equiv H\rho - \rho H$. Equation 2.18 represents a quantum analogy of the classical Liouville equation, the difference is only in the presence of the imaginary unit (and Planck's constant which we set to 1) and a commutator instead of a Poisson bracket.

We will now turn our attention to density matrices of bipartite states. Consider a pure state AB described by a ket $|\Psi\rangle$. Let the system AB be composed of a system A with orthonormal basis $\{\Phi_{Ai}\}$ and a system B with orthonormal basis $\{\Phi_{Bj}\}$. Typically state A might represent a quantum system we study and state B its surrounding. Density matrix ρ_{AB} of state AB can be written as:

$$\rho_{AB} = |\Psi\rangle\langle\Psi| = \sum_{ijkl} \alpha_{ij}\alpha^*_{kl}|\Phi_{Ai}\Phi_{Bj}\rangle\langle\Phi_{Ak}\Phi_{Bl}|. \tag{2.19}$$

Here α_{ij} are complex numbers and $|\Psi\rangle = \sum_{ij} \alpha_{ij}|\Phi_{Ai}\Phi_{Bj}\rangle$. If we wish to acquire density matrices ρ_A, ρ_B of parts of the system AB, we can do so by evaluating partial traces of ρ_{AB}:

$$\rho_A = Tr_B\rho_{AB} = \sum_{ijklm} \alpha_{ij}\alpha^*_{kl}\langle\Phi_{Bm}|\Phi_{Ai}\Phi_{Bj}\rangle\langle\Phi_{Ak}\Phi_{Bl}|\Phi_{Bm}\rangle = \\ = \sum_{ik} |\Phi_{Ai}\rangle\langle\Phi_{Ak}| \sum_j \alpha_{ij}\alpha^*_{kj} \tag{2.20}$$

$$\rho_B = Tr_A\rho_{AB} = \sum_{jl} |\Phi_{Bj}\rangle\langle\Phi_{Bl}| \sum_i \alpha_{ij}\alpha^*_{il}. \tag{2.21}$$

By performing the Schmidt decomposition we can find bases $\{\Phi'_{Ai}\}$, $\{\Phi'_{Bj}\}$ such that Preskill [2018]:

$$|\Psi\rangle = \sum_{ij} \sqrt{p_i}|\Phi'_{Ai}\Phi'_{Bj}\rangle. \tag{2.22}$$

In this case we have diagonal density matrices ρ_A, ρ_B equal to:

$$\rho_A = \sum_i p_i|\Phi'_{Ai}\rangle\langle\Phi'_{Ai}|, \tag{2.23}$$

$$\rho_B = \sum_i p_i|\Phi'_{Bi}\rangle\langle\Phi'_{Bi}|. \tag{2.24}$$

Eigenvalues of both partial density matrices are therefore equal.

2.1.3 Von Neumann Entropy

We can now use the density matrix to define entropy of a quantum state:

$$S \equiv -\operatorname{Tr}(\rho \ln \rho). \tag{2.25}$$

This is the entropy formula proposed by John von Neumann von Neumann [1927]. If we choose an orthonormal basis $\{|\Phi_i\rangle\}$ in which ρ is diagonal and use Taylor series expansion for logarithm, we get:

$$S = -\sum_j \langle\Phi_j| \sum_i p_i|\Phi_i\rangle\langle\Phi_i| \ln(p_i|\Phi_i\rangle\langle\Phi_i|)|\Phi_j\rangle = \tag{2.26}$$

$$= -\sum_j \sum_i p_i\delta_{ij}\langle\Phi_i| \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(p_i)^k(|\Phi_i\rangle\langle\Phi_i|)^k|\Phi_j\rangle = \tag{2.27}$$

$$= -\sum_j \sum_i p_i\delta_{ij} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(p_i)^k\delta_{ij} = -\sum_j p_j \ln p_j. \tag{2.28}$$

The last expression gives the Shannon entropy, which thus is just a limit of the von Neumann entropy for the case of an orthonormal basis which diagonalize ρ. In this case all states $|\Phi_i\rangle$ can be unambiguously distinguished and the system behaves like a classical ensemble with probability p_i assigned to each $|\Phi_i\rangle$ Preskill [2018]. In other words the coherences vanish. Let us now list some of the most important properties of von Neumann entropy (unless otherwise stated their proofs are trivial):

1. The entropy of a pure state equals zero.

2. The entropy does not change during a unitary change of basis $S(\rho) = S(U\rho U^{-1})$.

3. For ρ with d non-zero eigenvalues the maximum value of entropy is $\ln d$ which is achieved if and only if all the eigenvalues are equal. This is analogous to the Shannon entropy being maximized by the uniform distribution.

4. The entropy is concave: for any $\{\lambda_i\}_{i=1}^n$, $\sum_{i=1}^n \lambda_i = 1$ and $\{\rho_i\}_{i=1}^n$ it holds $S\left(\sum_{i=1}^n \lambda_i \rho_i\right) \geq \sum_{i=1}^n \lambda_i S(\rho_i)$. This is a consequence of the concavity valid also for the Shannon entropy. Physically, it means that if we have less information about the state's density matrix (we do not know that it is a linear combination of matrices ρ_i), the entropy is greater.

5. For a bipartite state AB the entropy is subadditive: $S(AB) \leq S(A)+S(B)$. Both sides are equal only if states A, B are uncorrelated and thus $\rho_{AB} = \rho_A \bigotimes \rho_B$.

6. For a pure bipartite state AB composed of two states A, B we have $S(A) = S(B)$ as both density matrices have the same eigenvalues (equations 2.23, 2.24).

7. Inspired by the classical case, we define the mutual information: $I(A;B) = S(A) + S(B) - S(AB)$. Thanks to subadditivity of the entropy the mutual information is non-negative. It equals zero only for uncorrelated states A, B.

8. The entropy satisfies triangle inequality: $S(AB) \leq |S(A) - S(B)|$. To see this, consider a pure state ABC. From property 6. it follows that $S(A) = S(BC)$ and $S(C) = S(AB)$ and property 5. gives $S(A) \leq S(B) + S(C) = S(B) + S(AB)$. Thus $S(AB) \geq S(A) - S(B)$. By interchanging A and B in this argument we in total get the desired inequality Preskill [2018].

9. Apart from subadditivity the entropy satisfies even the strong subadditivity condition: $I(A;BC) \geq I(A;B)$ The proof of this inequality is rather complicated. However, we can see that the strong subadditivity is a reasonable property: it states that correlation between system A and combined system BC is at least as strong as correlation between A and B Preskill [2018].

While many of these properties are shared by both the Shannon and the von Neumann entropy, triangle inequality $S(AB) \leq |S(A) - S(B)|$ valid for the von Neumann entropy is much weaker than inequalities $S(AB) \geq S(A)$ and $S(AB) \geq S(B)$ for the Shannon entropy. The Shannon entropy of a combined

system is always greater than or equal to the entropy of any of its subsystem. No such property holds for von Neumann entropy. For example, consider a pure state AB divided into two mutually entangled states A, B. Then $S(AB) = 0$ but $S(A) = S(B) > 0$. The explanation lies in the quantum entanglement, a phenomenom with no classical analogue. When we measure entire system AB, we can obtain the maximal amount of information allowed by the laws of quantum mechanics. The entropy of AB thus has its minimal value, zero. However, if we for example only measure system A, we are unable to access information contained in correlations between systems A and B. This information is inaccessible by measuring A and B separately and can only be acquired by examining entire system AB Preskill [2018].

2.2 Interpretation of Black Hole Entropy

When a black hole's mass increases, its temperature becomes lower. Therefore, black holes have negative heat capacity which prevents them from achieving thermodynamic equilibrium with a surrounding infinite thermal bath Adami [2004]. Thus, if we wish to give the black hole entropy some physical interpretation, we need a non-equilibrium concept of entropy. The information entropy, either the von Neumann or the Shannon in the classical context, appears to be a logical choice as no notion of equilibrium is needed to define it. The idea to understand black hole entropy as the Shannon entropy goes back to the Bekenstein's original article Bekenstein [1973]. The article includes several model examples which demonstrate that the change of the sum of a black hole's entropy and the Shannon entropy of the matter outside the black hole satisfy the generalized second law of thermodynamics. We will present probably the most instructive one of these examples which involves a harmonic oscillator.

Let us consider a non-relativistic linear harmonic oscillator made of two balls of equal rest mass $\frac{m}{2}$ which are connected by a nearly massless spring. We will denote the frequency of oscillations by ω. This oscillator is enclosed into a spherical box mantained at a constant temperature θ. Hamiltonian of such harmonic oscillator is:

$$H = \frac{1}{m}p^2 + \frac{1}{4}m\omega^2(\Delta y)^2, \tag{2.29}$$

where Δy is the difference of the distance between the balls from its mean value. Thermal distribution of its energy states is assumed, with the probability for the oscillator to be found in its n-th state being:

$$p_n = \frac{1 - e^{-\frac{\omega}{\theta}}}{1 - e^{-\frac{N\omega}{\theta}}} e^{\frac{\omega}{\theta}} e^{-\frac{n\omega}{\theta}}. \tag{2.30}$$

We will now calculate the Shannon entropy of the system. By substituting equation 2.30 into the Shannon formula 2.5, integrating with respect to ω and then again differentiating the result we can easily get:

$$S = \frac{\omega}{\theta}\frac{1}{e^{\frac{\omega}{\theta}} - 1} - \ln\left(1 - e^{-\frac{\omega}{\theta}}\right). \tag{2.31}$$

The mean energy of oscillations is equal to:

$$\langle E \rangle = \sum_{n=0}^{N} p_n \left(n + \frac{1}{2}\right)\omega = \left(\frac{1}{e^{\frac{\omega}{\theta}} - 1} + \frac{1}{2}\right)\omega. \tag{2.32}$$

To test the generalized second law we want the entropy of the system falling into the black hole to be maximal for the given mean energy. We can use the following simple argument to see that the thermal distribution indeed satisfies this condition. If we want to slightly change the probability distribution from $\{p_n\}$ to $\{p_n + \delta_n\}$ we have a boundary condition given by the normalization of the total probability to one:

$$\sum_{n=1}^{N} \delta_n = 0. \tag{2.33}$$

If we also wish to keep the mean energy constant we have another boundary condition:

$$\sum_{n=1}^{N} \delta_n n = 0. \tag{2.34}$$

Using these conditions we can determine the change of the entropy:

$$\Delta S = -\sum_{n=1}^{N} (p_n + \delta_n) \ln (p_n + \delta_n) + \sum_{n=1}^{N} p_n \ln p_n = -\sum_{n=1}^{N} p_n \ln p_n -$$
$$-\sum_{n=1}^{N} \delta_n \ln p_n - \sum_{n=1}^{N} p_n \frac{\delta_n}{p_n} + O\left(\delta_n^2\right) + \sum_{n=1}^{N} p_n \ln p_n = -\sum_{n=1}^{N} \delta_n \ln p_n + O\left(\delta_n^2\right). \tag{2.35}$$

For the thermal distribution we have:

$$\Delta S_T = -\sum_{n=1}^{N} \delta_n \ln \left(\left(1 - e^{-\frac{\omega}{\theta}}\right) e^{-\frac{n\omega}{\theta}}\right) + O\left(\delta_n^2\right) = \frac{\omega}{\theta} \sum_{n=1}^{N} \delta_n n -$$
$$- \ln \left(1 - e^{-\frac{\omega}{\theta}}\right) \sum_{n=1}^{N} \delta_n + O\left(\delta_n^2\right) = O\left(\delta_n^2\right). \tag{2.36}$$

We showed that if we slightly change the probability distribution from a thermal one, the entropy will remain constant to the linear order. The thermal distribution therefore locally extremizes the entropy. Furthermore, both the Shannon and the von Neumann entropy are concave as we discussed in the subsection 2.1.3. A local extreme of a concave functional can be identified as its global maximum. In conclusion, our choice of thermal distribution is justified.

As mentioned in section 1.2.2 for a spherical box with non-zero radius b and proper mass μ falling into a black hole is the minimal increase of the horizon's rationalized area equal to Bekenstein [1973]:

$$\Delta\alpha_{min} = 2\mu b. \tag{2.37}$$

To calculate the minimal increase of the black hole's entropy, the minimal necessary radius b of the box must therefore be determined. It is clearly at least one half of the mean value of the distance between balls. The mean value must in turn be larger than thermal oscillations of the distance Δy. Using the virial theorem and equation 2.29 we get:

$$\frac{1}{8} m\omega^2 (\Delta y)^2 = \frac{1}{2} \langle E \rangle,$$
$$(\Delta y)^2 = \frac{4}{m\omega^2} \langle E \rangle. \tag{2.38}$$

Therefore:

$$b \geq \frac{1}{\sqrt{m\omega}}\sqrt{\langle E\rangle}. \tag{2.39}$$

Since $\mu \geq m + \langle E\rangle$ we have for the black hole entropy increase:

$$S_{BH} = \frac{A}{4} = 2\pi\mu b \geq 2\pi(m + \langle E\rangle)\frac{1}{\sqrt{m\omega}}\sqrt{\langle E\rangle}. \tag{2.40}$$

If we neglect the entropy of the black body radiation inside the box (a reasonable assumption for low temperatures) we can write down the expression for the left hand side of the generalized second law:

$$\Delta(S_{BH} + S_C) \geq 2\pi(m + \langle E\rangle)\frac{1}{\sqrt{m\omega}}\sqrt{\langle E\rangle} - \frac{\omega}{\theta}\frac{1}{e^{\frac{\omega}{\theta}} - 1} + \ln\left(1 - e^{-\frac{\omega}{\theta}}\right), \tag{2.41}$$

and find its minimum with respect to ω/θ for $(\omega/\theta)_{min} = 2\pi(m(\langle E\rangle)^{-1}+1)\sqrt{m^{-1}(\langle E\rangle)}$. The corresponding minimal change in entropy equals:

$$\Delta(S_{BH} + S_C)_{min} = \left(\frac{\omega}{\theta}\right)_{min} + \ln\left(1 - e^{-\left(\frac{\omega}{\theta}\right)_{min}}\right). \tag{2.42}$$

The condition of non-relativistic oscillator yields $m >> \langle E\rangle$. Therefore, it holds $\left(\frac{\omega}{\theta}\right)_{min} >> 2\pi$ and $\Delta(S_{BH} + S_C)_{min} > 0$. In conclusion, the generalized second law is obeyed under described conditions. This calculation of course does not prove the generalized second law. However, since a general proof has not yet been given, an example showing that it can indeed work is of some importance. It also demonstrate that inserting the Shannon (or, more generally, von Neumann) entropy into the generalized second law together with the black hole entropy yields reasonable results.

The crucial problem is identification of the degrees of freedom which are responsible for the black hole entropy. There are several attempts to achieve this in the context of quantum gravity theories, some of their results were briefly mentioned in the introduction.

Apart from quantum gravity approaches, there are attempts to describe the black hole entropy as the entanglement entropy resulting from the quantum correlations between the exterior and interior of the black hole. The black hole's interior and exterior together form a pure state but each of them separately is in a mixed state (such situation is discussed at the end of the subsection 2.1.3). Then the entropy can be shown to be indeed proportional to the horizon area. We can get this result even if we do not consider a black hole but just any imaginary sphere in a spacetime with a free massless quantum field present [quote Srednicky 1993]. If we restrict our measurements only to the sphere's exterior, or only to its interior, we get entropy proportional to the sphere's area (recall that the entropy in the exterior equals the entropy in the interior if the whole system is in a pure state). One problem with this approach lies in the proportionality constant: it is given as a square of the ultraviolet cutoff wavelength multiplied by some numerical constant. If the ultraviolet cutoff is chosen to be of the order of the Planck mass ($M_{Pl} = \sqrt{\hbar c/G}$) and an appropriate numerical factor is chosen we indeed get the black hole entropy formula in the standard units. However, there is no known way to determine the numerical factor and it was even argued

that it should in fact depend on number N of particle species present Wald [2001]. Then the entropy formula would not hold universally for any black hole as the numerical factors might differ. One idea to remedy this involves N-dependence of the gravitational constant G in the context of quantum gravity Wald [2001]. The N-dependences could then cancel each other out and in total yield an entropy formula independent of N.

Another approach attributes the black hole entropy to its thermal atmosphere. We can imagine it as a free massless quantum field just outside the event horizon. In a rough estimate, the entropy of such a field is proportional to θ^3. However, this temperature is of course the one measured locally by an observer co-moving with the atmosphere, whereas Hawking temperature $\kappa/2\pi$ is measured by an observer in the infinity. The conversion factor is $(V)^{-1}$ with $V = \sqrt{l_\rho l^\rho}$ (l^ρ is a Killing vector defined by equation 1.11). Since vector l^ρ becomes null on the horizon the locally measured temperature diverges as we approach it. In other words, as the redshift on the horizon becomes infinite, locally measured frequency can be arbitrarily high and there is an arbitratrily large number of modes. This problem can be averted by imposing an ultraviolet cutoff on the locally measured frequencies. If we again choose a wavelength cutoff of the order of the Planck mass we get the same result for the entropy as in the previously discussed entanglement treatment. It can be shown that most of the total entropy is confined in a shell around the horizon whose thickness is of the order of the Planck length. Since the atmosphere outside the black hole is in thermal state we can even interpret its von Neumann entropy as the standard thermodynamic entropy (though this could be just a result of using a very idealized model). This approach have the same problems as the entanglement one: the cutoff frequency and the numerical factor in the entropy formula cannot be evaluated and are just guessed to give the correct result Wald [2001].

It is interesting to note where the degrees of freedom associated with the black hole entropy are localized in all the different approaches we mentioned. In both the entanglement and thermal atmosphere picture they are associated with standard degrees of freedom of quantum fields just outside the black hole. In the loop quantum gravity approach the degrees of freedom reside directly on the horizon Wald [2001].

The problem of the interpretation of black hole entropy remains unsolved. This section contains only a very brief discussion of a few selected approaches that are being considered and is in no way a complete review of the field.

2.3 Black Hole Information Paradox

It is easy to imagine that matter which forms a black hole was originally entangled with matter which remains outside the black hole. Then the interior and exterior of the black hole together are in a pure quantum state, while each of them separately is in a mixed state. Now consider a formation of an isolated black hole which is then let to completely evaporate by the process of Hawking radiation. Are the quantum correlations restored during evaporation? If the answer is no, then the original spacetime without black holes whose matter was in a pure state evolves to a spacetime which again does not contain black holes but

now the matter in it is in a mixed state. Thus, evolution of an isolated system we considered was non-unitary, which is in conflict with known laws of quantum physics. In terms of information, we started with a pure state whose von Neumann entropy was zero, meaning we had the maximum possible amount of information about it, and in the end we have a state with a positive entropy S. We thus lost the amount of information $\Delta I = -S$. However, the entropy of an isolated system does not change during its evolution. We can derive this result from the quantum analogue of Liouville theorem (equation 2.18). Therefore, loss of information contradicts known laws of quantum physics. Since the derivation of the Liouville theorem assumes unitary evolution, this is in fact again a result of the loss of unitarity. Because of this apparent contradiction, the possibility of information loss during black hole evaporation is referred to as the black hole information paradox.

If we assume that a black hole radiates with exactly thermal spectrum corresponding to the Hawking temperature $\kappa/2\pi$ until the moment it completely evaporates, the quantum correlations will indeed be lost. To see this, we will turn to the information statement of the paradox and to the semiclassical treatment for a Schwarzschild black hole described in section 1.6. As we discussed there, the thermal spectrum assumption corresponds to the neglection of the term proportional to the square of radiated energy ω in the imaginary action which yields tunneling rate proportional to $e^{-8\pi M\omega}$. Then the outgoing radiation can be described by a microcanonical ensemble with the entropy $8\pi M\omega$. The decrease of black hole entropy is however smaller, equal to $\Delta S_{BH} = -8\pi\omega\left(M - \frac{\omega}{2}\right)$. Therefore, the radiation carries less information than is lost from the black hole, the difference being $\Delta I = -4\pi\omega^2$. We can see that if the black hole is indeed allowed to completely evaporate in this manner, the loss of information (equivalently the loss of unitarity) does occur.

Many possible resolutions of this paradox have been suggested. Among them is the idea that black hole is not allowed to evaporate completely and all the missing information would be stored in some remnant, or that the late stages of the evaporation are no longer thermal, allowing to restore the correlations before the black hole completely disappears. Another approach is to suggest that a loss of unitarity indeed occurs in the complete theory of quantum gravity Wald [2001].

The semiclassical treatment of Hawking radiation we presented actually also offers a possibility to resolve the information paradox. When the energy conservation is enforced during the evaporation process and term proportional to ω^2 is not neglected, the entropy of the radiation is exactly equal to the decrease of black hole's entropy and no loss of information occurs Parikh and Wilczek [2000]. However, it should be pointed out that while the semiclassical approach can be used to gain some insight about the nature of the process, it is inherently inconsistent and any attempt to deduce fundamental conclusions from it should be regarded with caution. In any case, the black hole information paradox still awaits a widely accepted resolution.

2.4 Entropy Bounds

In the introduction we mentioned the Geroch's process in which a box containing some matter is dropped into a black hole. If the box can be lowered arbitrarily

close to the event horizon before it is dropped, no increase in the horizon's area occurs, black hole's entropy remains unchanged and the entropy of the box disappears. This would not only break the standard second law of thermodynamics but even the Bekenstein's generalized second law. Bekenstein's response to this was that dimensions of the box cannot be arbitrarily small and thus there exists limit on how close to the event horizon we can lower it. In his original article he applied this approach specifically to a box containing black body radiation and demonstrated that the generalized second law remains valid in this case Bekenstein [1973]. Later he suggested a universal bound on the size of a system with entropy S and total energy E Bekenstein [1981], stating:

$$\frac{S}{E} \leq 2\pi R, \tag{2.43}$$

where R is a characteristic dimension of the system (in the simplest case of a sphere its radius). It is easy to see that this bound effectively states that a system whose total energy is E cannot have greater entropy than a Schwarzschild black hole of the same energy. Later, an alternative entropy bound was suggested Wald [2001]:

$$S \leq \frac{A}{4}, \tag{2.44}$$

with A being the area of systems boundary.

It is actually easy to devise a counter-example to both entropy bounds. In the case of a system containing N species of non-interacting particles, all either bosons or fermions, the entropy is proportional to $N^{\frac{1}{4}}R^{\frac{3}{4}}E^{\frac{3}{4}}$ and for a large enough N it exceeds both bounds. However, N must be extremely big to achieve this Wald [2001] and physical relevance of such a counter-example is thus questionable.

The question whether some entropy bound exists is still unanswered. It is also unknown whether any such bound is necessary for the validity of the generalized second law of thermodynamics, as alternative resolutions to the Geroch's process problem have been suggested.

2.5 Possible Developments

The aim of this subsection is to briefly introduce two open ideas which we hope to explore in the future. So far, these ideas were not followed upon and the outline presented here is thus very basic.

Entropy and Stability The first idea involves examining possible connections between the entropy and stability of a black hole. For a given mass M and radius R of the event horizon, a charged (with a charge Q), spherically symmetrical (Reissner-Nordström) black hole has the lowest entropy of all Kerr-Newman black holes, namely $S = \pi(2MR - Q^2)$, and, generally, black hole with greater charge have lower entropy (for fixed M, R). At the same time, charged black holes are believed to be very difficult to form because repulsive forces between the charges of the same sign work against attractive gravitational forces. Furthermore, for a given mass M, an extremal black hole has the lowest possible entropy, $S = \pi M^2$, and a change of an ordinary black hole to an extremal one is explicitly prohibited

by the third law. Since, in the context of standard thermodynamics, the principle of maximum entropy is a very powerful concept it might be interesting to look at charged and extremal black holes from this perspective.

Thermodynamic Interaction Between Black Hole and Matter The second idea is to look at the interaction between a black hole and surrounding matter, say an accretion disk, from the perspective of two thermodynamic systems exchanging energy. If nothing else, this might be interesting in order to explore the notion of negative heat capacity of black holes. One easily foreseeable problem is with the black hole energy. Since it is of gravitational nature, there is no invariant local definition of it. However, such a problem can hopefully be overcome by choosing some definition of local gravitational energy advantageous for our case.

Conclusion

In the first chapter we have presented derivations of several important results of classical black hole thermodynamics, most notably of the event horizon area increase theorem and of the zeroth and first law of black hole thermodynamics. These laws are analogous to their standard thermodynamic versions, the zeroth law stating that a quantity called surface gravity is constant throughout the event horizon of a stationary black hole (this makes the surface gravity analogous to temperature). The first law is basically a statement of energy conservation which gives a formula for the change of black hole's mass. We have also discussed the line of reasoning which led Bekenstein to the formula for black hole entropy directly proportional to the event horizon area and to the generalized second law of thermodynamics. This law states that the sum of black hole entropy and common entropy in the universe can never decrease. Finally, we postulated the third law of black hole thermodynamics, stating that the surface gravity cannot be reduced to zero by any finite sequence of operations. This is analogous to the Nernst's formulation of the third law in standard thermodynamics. Then, after noting incompleteness of classical black hole thermodynamics caused by the fact that black holes fully described by general relativity cannot radiate and their thermodynamic temperature is therefore absolute zero, we used the semiclassical WKB limit of quantum mechanics to demonstrate that black holes in fact radiate. We used this to argue that black holes in fact have non-zero thermodynamic temperature proportional to their surface gravity. Unlike the original derivation of black hole radiation performed by Hawking in the context of quantum field theory, the semiclassical approach describes the black hole radiation as a tunneling effect and gives a correction to its thermal spectrum.

Our principal aim in the second chapter has been gaining some insight about the nature of the black hole entropy. Showing that black holes have non-zero thermodynamic temperature might one lead to believe that the black hole entropy is also in fact of thermodynamic nature. However, a black hole has negative heat capacity (it becomes colder when it absorbs matter) and thus cannot achieve thermal equilibrium with a surrounding infinite heat bath. Since the thermodynamic entropy is defined only for the case of thermal equilibrium, interpreting black hole's entropy as thermodynamic one is problematic. Therefore, we turned our attention to non-equilibrium definitions of entropy, the von Neumann entropy defined for quantum states and its classical limit, the Shannon entropy. We noted the connection between these definitions of entropy and information, as the entropy is in effect a measure of our ignorance about the system we study. After introducing these definitions of entropy, we have examined a model example of a harmonic oscillator falling into a black hole. We showed that the increase of black hole entropy is greater than the Shannon entropy of the oscillator and the generalized second law thus holds in this case. Then we have discussed several more recent developments, namely some attempts to identify the degrees of freedom contributing to the black hole entropy, the black hole information paradox and the entropy bounds. Lastly, we have outlined two ideas related to black hole thermodynamics we consider pursuing in the future, namely the black hole stability and thermodynamic description of the interaction between the black hole

and matter.

In total, the thesis presents relatively exhausting review of classical black hole thermodynamics, including detailed derivations of all the important results. The discussion of the black hole radiation and the entire chapter 2 are limited by the fact that we do not consider the quantum field theory results. Nevertheless, we managed to give a basic overview of the quantum effects important for black hole thermodynamics.

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