FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

BACHELOR THESIS

Adam Zaplatílek

Convexity in normed linear spaces and more general spaces

Department of Mathematical Analysis

Supervisor of the bachelor thesis:	prof. RNDr. Luboš Pick, CSc., DSc.
Study programme:	Mathematics
Study branch:	General Mathematics

Prague 2018

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In Prague date 18. 5. 2018 Adam Zaplatílek

I wish to express my sincere thanks to my supervisor, professor Pick, for expert consultations, valuable advice and motivations.

I must also express my very profound gratitude to my family without whom this thesis would never have been conceivable.

Title: Convexity in normed linear spaces and more general spaces

Author: Adam Zaplatílek

Department: Department of Mathematical Analysis

Supervisor: prof. RNDr. Luboš Pick, CSc., DSc., Department of Mathematical Analysis

Abstract: We study questions concerning convexity and the existence of the nearest point for a given set in spaces equipped with either a norm, or with a more general functional, namely a quasinorm or an $\alpha-$norm. We characterize convexity in a Hilbert space. We investigate relations between convexity and properties of the distance function.

Keywords: convexity, Chebyshev set, $\alpha-$norm, quasinorm

Contents

1. Introduction

Convex set is a subset of a vector space that contains the line segments between every two its points. Chebyshev set is a subset of a normed linear space that, for every point outside of it, possesses a unique nearest point inside.

One of the most basic statements in the theory of Hilbert spaces is the theorem that every nonempty and closed convex set is a Chebyshev set. While the inverse implication is still unclear, there are few proofs with additional assumptions.

In 1969 Edgar Asplund [1] proved that every Chebyshev set in a Hilbert space with continuous metric projection is convex. It will be used to prove our characterisation of convexity.

In this thesis we study the convexity in various spaces. We begin with normed linear spaces and then try to find similar statements in spaces with weaker forms of norm-like mapping.

2. Convexity in normed linear spaces

The aim of this chapter is to show some basic properties of convexity in normed linear spaces. Our intention is to find out what convexity has to say about Chebyshev sets and, in addition, which properties of its metric projection are secured.

In the first part of this chapter we define basics, while the second part is left for statements.

One important result is the characterisation of convexity in a Hilbert space. It gives us somewhat elegant sight of convex sets that is not usually included in basic courses of Mathematical analysis.

2.1 Definitions

In this section we clarify few necessary definitions and try to keep them as general as possible.

We first define the norm on a vector space. A vector space equipped with a norm is called the normed linear space.

Definition 1. Let X be a vector space over the field $\mathbb{F}$ of real or complex numbers. A mapping $\|\cdot\| : X \to [0,\infty)$ is called *norm* on X, if for all $a \in \mathbb{F}$ and $x, y \in X$

$$\begin{aligned} \|x\| = 0 \;&\Leftrightarrow\; x = 0, \\ \|ax\| \;&=\; |a|\|x\|, \\ \|x+y\| \;&\leq\; \|x\| + \|y\|. \end{aligned}$$

Next we denote the nonexpansiveness as a special case of Lipschitz property.

Definition 2. A 1-Lipschitz mapping is called *nonexpansive*.

Now we define the convex set using line segments. The line segment could be easily defined as the convex hull of both of its end points, but that would not ease the situation.

Definition 3. Let X be a vector space and K be a subset of X. We shall say that K is *convex* if the line segment $[x,y] \subseteq K$ for all $x, y \in K$. Here $[x,y]$ stands for the set

$$[x,y] \;=\; \Big\{\lambda x + (1-\lambda)y \in X : \lambda \in [0,1]\Big\}.$$

We shall now state the definition of the Chebyshev set.

Definition 4. A set C in a normed linear space X is called a *Chebyshev set*, if for each point x in X there is a unique nearest point $p(x)$ in C, i.e.

$$\|x-y\| > \|x-p(x)\| \;\text{ if } y \in X \text{ and } y \neq p(x).$$

In this case, we call the function $x \to p(x)$ the *metric projection* onto C.

Clearly, every Chebyshev set is nonempty and closed.

2.2 Convexity and Chebyshev sets

In this section we show some equivalent conditions of convexity in a Hilbert space.

Let us begin with one rather technical lemma. The proof is geometrical with use of orthogonal projection and Pythagorean theorem.

Lemma 1. *Let $H = (H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Suppose $x, y, z \in H$. Then*

$$\|x - z\| \;=\; \|x - y\| + \|y - z\| \tag{2.1}$$

if and only if y lies on the line segment $[x, z]$.

Proof. First suppose $y \in [x, z]$. Since

$$y = \lambda x + (1 - \lambda) z$$

for some $\lambda \in [0, 1]$, it holds that

$$\|x - y\| \;=\; \|x - \lambda x - (1 - \lambda) z\| \;=\; (1 - \lambda)\|x - z\|$$

and

$$\|y - z\| \;=\; \|\lambda x + (1 - \lambda) z - z\| \;=\; \lambda \|x - z\|.$$

Furthermore,

$$\|x - z\| \;=\; \|x - y\| + \|y - z\|.$$

Let the equation (2.1) hold. For $x = z$ we get

$$\|x - z\| \;=\; \|x - y\| + \|y - z\| \;=\; 0.$$

Hence,

$$\|x - y\| \;=\; \|y - z\| \;=\; 0$$

and

$$y \;=\; x \;=\; z \;\in\; [x, z].$$

Suppose $x \neq z$. Let $p(y)$ stand for the orthogonal projection of y onto the line

$$L \;:=\; x + \alpha(z - x), \quad \alpha \in \mathbb{R}.$$

It can be written as

$$p(y) \;=\; x + \lambda(z - x)$$

for some $\lambda \in \mathbb{R}$. Since

$$\|w - y\| \;\geq\; \|w - p(y)\|$$

for each $w \in L$, and from (2.1), we see that

$$\|x - z\| \;=\; \|x - p(y)\| + \|p(y) - z\|. \tag{2.2}$$

Next we substitute for $p(y)$. It holds that

$$\begin{aligned}
\|x - z\| \;&=\; \|x - \big(x + \lambda(z - x)\big)\| + \|x + \lambda(z - x) - z\| \\
&=\; \|\lambda(x - z)\| + \|(1 - \lambda)(x - z)\| \\
&=\; |\lambda|\|x - z\| + |1 - \lambda|\|x - z\|.
\end{aligned}$$

Hence, $\lambda \in [0,1]$ and $p(y) \in [x,z]$. Pythagorean theorem gives us that

$$\|x-y\|^2 \;=\; \|x-p(y)\|^2 + \|p(y)-y\|^2. \tag{2.3}$$

It follows that

$$\|x-y\| \;\geq\; \max\Big(\|x-p(y)\|, \|p(y)-y\|\Big)$$

and, by symmetry,

$$\|z-y\| \;\geq\; \max\Big(\|z-p(y)\|, \|p(y)-y\|\Big).$$

Now from (2.1) and (2.2) we get

$$\|x-y\| \;=\; \|x-p(y)\|.$$

Finally we substitute in (2.3)

$$\begin{aligned}\|x-y\|^2 &= \|x-p(y)\|^2 + \|p(y)-y\|^2 \\ &= \|x-y\|^2 + \|p(y)-y\|^2,\end{aligned}$$

which gives us

$$\|p(y)-y\| \;=\; 0.$$

Moreover,

$$y \;=\; p(y) \;\in\; [x,z].$$

□

A corollary published by Edgar Asplund [1] says that every Chebyshev set in a Hilbert space with continuous metric projection is convex. It will be used to prove the next theorem.

Theorem 2. *Let K be a nonempty and closed subset of a Hilbert space H. The following are equivalent:*

(i) *K is convex.*

(ii) *K is a Chebyshev set with nonexpansive metric projection.*

(iii) *There exists a continuous operator $p : H \to K$ of closest point in K.*

Proof. For the proof that (i) ⇒ (ii) choose x_0, y_0 arbitrary in H. A well-known theorem says that every nonempty closed convex set in a Hilbert space is also a Chebyshev set. Let p be the metric projection onto K. The nonexpansiveness is left to show. If $p(x_0) = p(y_0)$, we are done. Suppose $p(x_0) \neq p(y_0)$. Now every point x in H can be written as

$$x \;=\; p(x_0) + z + \lambda\Big(p(x_0) - p(y_0)\Big)$$

for some $z \in H$ orthogonal to the vector $(p(x_0) - p(y_0))$ and $\lambda \in \mathbb{R}$. Let us use such notation to define next two sets

$$\begin{aligned}H_{x_0} &:= \Big\{p(x_0) + z + \lambda\Big(p(x_0) - p(y_0)\Big) \;:\; z \in H;\; \Big\langle z,\; p(x_0) - p(y_0)\Big\rangle = 0;\; \lambda \geq 0\Big\}, \\ H_{y_0} &:= \Big\{p(y_0) + z + \lambda\Big(p(y_0) - p(x_0)\Big) \;:\; z \in H;\; \Big\langle z,\; p(x_0) - p(y_0)\Big\rangle = 0;\; \lambda \geq 0\Big\}.\end{aligned}$$

We show next that x_0 lies in H_{x_0}. Looking for a contradiction, first suppose $x_0 \in H \setminus (H_{x_0} \cup H_{y_0})$. It follows that

$$x_0 \; = \; p(x_0) + z + \lambda\Big(p(x_0) - p(y_0)\Big)$$

for some fixed $z \in H$ orthogonal to the vector $(p(x_0) - p(y_0))$ and $\lambda \in (-1, 0)$. Its orthogonal projection

$$\widehat{p}(x_0) \; := \; p(x_0) + \lambda\Big(p(x_0) - p(y_0)\Big) \; \in \; [p(x_0), p(y_0)]$$

is its unique best approximation onto the line

$$L \; := \; p(x_0) + \alpha\Big(p(x_0) - p(y_0)\Big), \quad \alpha \in \mathbb{R}.$$

By convexity of K the line segment $[p(x_0), p(y_0)]$ lies in K. We have found a better approximation of x_0 onto K. Supposing $x_0 \in H_{y_0}$, we find $p(y_0)$ being better approximation than $p(x_0)$, which also leads to a contradiction. Since

$$\|x_0 - y_0\| \; \geq \; \operatorname{dist}(H_{x_0}, H_{y_0}) \; = \; \|p(x_0) - p(y_0)\|,$$

p is nonexpansive.

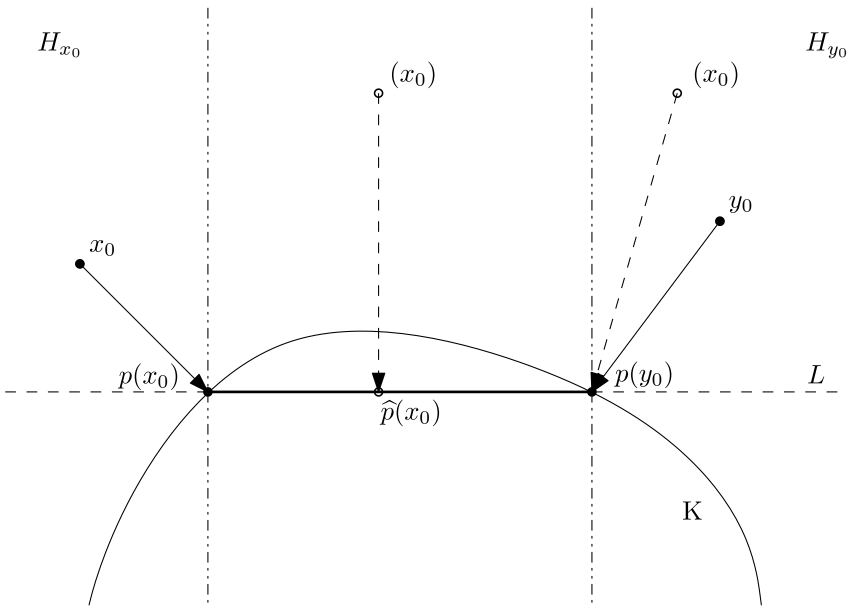

Figure 2.1: Metric projection is nonexpansive

Implication (ii) $\Rightarrow$ (iii) is trivial.

For the last implication (iii) $\Rightarrow$ (i) we show the unicity of such operator. Space H possesses a continuous operator of closest point in K. Let p be such operator. Suppose $x_0 \in H$ has two closest points $p(x_0)$ and $q(x_0)$ in K.

Trivially, they are of the same distance $d > 0$ from x_0. Also there is no point of K lying in the interior of $\mathrm{B}(x_0, d)$, but every point z on the open line segment $(x_0, q(x_0))$ has its closest point in K lying in the closed ball $\mathrm{B}(z, \|z - q(x_0)\|)$.

By Lemma 1, sets $\mathrm{B}(z, \|z - q(x_0)\|)$ and $\mathrm{S}(x_0, d)$ contradicts in just one point $q(x_0)$. Therefore,

$$q(x_0) = p(z) \quad \text{for each } z \text{ in } (x_0, q(x_0)],$$

and so

$$\lim_{\substack{z \to x_0 \\ z \in (x_0, q(x_0)]}} p(z) = q(x_0).$$

By continuity of p, we get $p(x_0) = q(x_0)$. Hence, K is a Chebyshev set with continuous metric projection and, by mentioned corollary from Asplund [1], K is convex. □

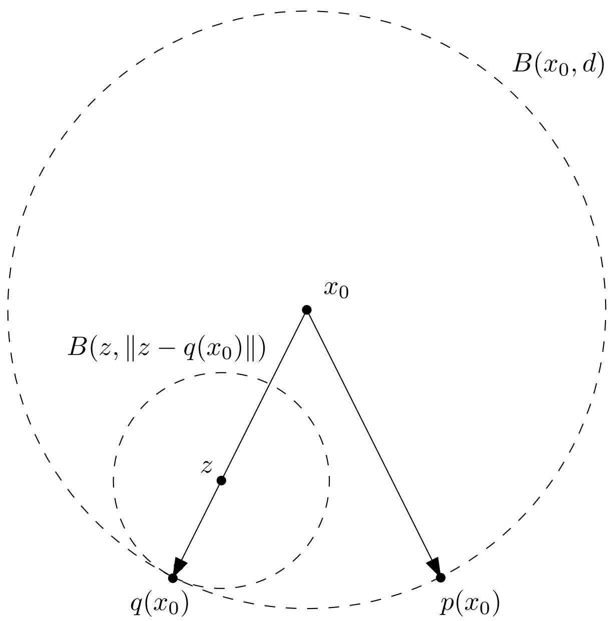

Figure 2.2: The unicity of continuous metric projection

In 1935, Motzkin [4] showed that every Chebyshev set in a finite dimensional Hilbert space is convex. Together with Theorem 2 we get the following corollary.

Corollary 3. *Every Chebyshev set in a finite dimensional Hilbert space possesses nonexpansive metric projection.*

Proof. The proof follows directly from Theorem 2 and Motzkin [4]. □

3. Convexity in vector spaces with $\alpha-$norm and quasinorm

In this chapter we focus on convexity in spaces with weaker form of norm-like mapping. The aim is to either confirm or contradict the basic statements that apply in normed spaces.

In the first part of this chapter we focus on definitions and preliminaries, while in the second part we attempt to write down some statements.

3.1 Definitions and preliminaries

Let us begin this section with the definitions of the quasinorm and the $\alpha-$norm. A vector space equipped with one of these norm-like mappings is considered a topological space with topology generated by open balls (with respect to the corresponding quasinorm or $\alpha-$norm). Such space we call a quasinormed vector space or an $\alpha-$normed vector space, respectively.

Definition 5. Let X be a vector space over the field $\mathbb{F}$ of real or complex numbers. A mapping $\|\cdot\| : X \longrightarrow [0,\infty)$ is called a *quasinorm* on X, if for all $a \in \mathbb{F}$ and $x, y \in X$

$$\begin{aligned} \|x\| = 0 \ &\Leftrightarrow \ x = 0, \\ \|ax\| \ &= \ |a|\|x\|, \\ \|x+y\| \ &\leq \ M\Big(\|x\| + \|y\|\Big) \quad \text{for some } M \geq 1. \end{aligned}$$

Definition 6. Let X be a vector space over the field $\mathbb{F}$ of real or complex numbers. Let $\alpha \in (0,1]$, then a mapping $\|\cdot\| : X \longrightarrow [0,\infty)$ is called an *$\alpha-$norm* on X, if for all $a \in \mathbb{F}$ and $x, y \in X$

$$\begin{aligned} \|x\| = 0 \ &\Leftrightarrow \ x = 0, \\ \|ax\| \ &= \ |a|\|x\|, \\ \|x+y\|^\alpha \ &\leq \ \|x\|^\alpha + \|y\|^\alpha. \end{aligned}$$

Remark 1. Every $\alpha-$norm is automatically a quasinorm (see e.g. Kuncová [3]) but not vice versa.

Next we define the midpoint convexity.

Definition 7. Let X be a vector space and K be a subset of X. We shall say that K is *midpoint convex* if $\frac{x+y}{2} \in K$ for all $x, y \in K$.

And we stay with convexity for one more definition.

Definition 8. Let K be a nonempty convex subset of a vector space X. We say that a function $f : K \to \mathbb{R}$ is *convex* if

$$f\Big(\lambda x + (1-\lambda)y\Big) \ \leq \ \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in K$ and all $\lambda \in [0,1]$.

3.2 Convexity and distance functions

In this section we test whether some of the statements published by Fletcher and Moors [2] still hold in spaces with a quasinorm or an $\alpha-$norm. At the end we gather the results in one final theorem.

Let us start with the proposition that the distance function $\mathrm{d}(x, K)$ is nonexpansive (in a normed linear space). The following example speaks for itself.

Example 1. Suppose $M > 1$. Let $X := (\mathbb{R}^2, \|\cdot\|)$ denote the Cartesian plane equipped with a quasinorm defined as

$$\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = \begin{cases} M|x_1| & \text{if } x_2 = 0 \\ |x_1| + |x_2| & \text{otherwise.} \end{cases}$$

Define $K := \{\binom{0}{0}\}$. Then the distance function $\mathrm{d}(\cdot, K)$ is discontinuous.

Proof. First observe that

$$\mathrm{d}(x, K) = \inf\left\{\|x-y\| : y \in K\right\} = \inf\left\{\|x-y\| : y = \tbinom{0}{0}\right\} = \|x\|, \quad x \in X.$$

Next we verify that the mapping $\|\cdot\| : X \to [0, \infty)$ is truly a quasinorm. For every $x, y \in X$ and $a \in \mathbb{R}$ it holds that

1) $$\|x\| = 0 \Leftrightarrow \big(x_1 = 0 \wedge x_2 = 0\big) \Leftrightarrow x = \tbinom{0}{0},$$

2) $$\|ax\| = \left\|\tbinom{ax_1}{ax_2}\right\| = |a|\|x\|,$$

3) $$\begin{aligned} \|x+y\| &\le M\big(|x_1+y_1| + |x_2+y_2|\big) \\ &\le M\big(|x_1| + |x_2| + |y_1| + |y_2|\big) \\ &\le M\big(\|x\| + \|y\|\big). \end{aligned}$$

The discontinuity of the quasinorm is left to show. Let us, for every $k \in \mathbb{N}$, define $x^k := \binom{1}{1/k}$ and $x := \binom{1}{0}$. It holds that

$$\|x^k - x\| = |0| + \left|\frac{1}{k}\right| \xrightarrow{k\to\infty} 0,$$

while

$$\|x^k\| = |1| + \left|\frac{1}{k}\right| \xrightarrow{k\to\infty} 1 \quad \wedge \quad \|x\| = M > 1.$$

Therefore, by Heine's theorem for topological spaces, the distance function $\mathrm{d}(\cdot, K)$ is discontinuous. □

Next we try to prove the statement saying that every closed midpoint convex set is also convex. But first we prove one short lemma.

Lemma 4. *Let X denote a quasinormed vector space. Suppose $x, y \in X$. The mapping*

$$T : \lambda \longmapsto \lambda x + (1-\lambda)y, \quad \lambda \in [0, 1],$$

is continuous.

Proof. For $\lambda_1, \lambda_2 \in [0,1]$ it holds that

$$\begin{aligned}\|T(\lambda_1) - T(\lambda_2)\| &= \|\lambda_1 x + (1-\lambda_1)y \; - \; \lambda_2 x - (1-\lambda_2)y\| \\ &= \|(\lambda_1 - \lambda_2)x - (\lambda_1 - \lambda_2)y\| \\ &= |\lambda_1 - \lambda_2|\|x - y\| \\ &\leq M|\lambda_1 - \lambda_2|\Big(\|x\| + \|y\|\Big),\end{aligned}$$

where $M > 1$ is the constant for the quasinorm. Since T is Lipschitz, it is also continuous. □

Now we can formulate a statement from Fletcher-Moors [2]. Together with Lemma 4 their proof shall hold in any quasinormed vector space.

Proposition 5. *Let X denote a quasinormed vector space. Suppose $K \subseteq X$ is closed. Then K is convex if and only if K is midpoint convex.*

Proof. Clearly, every convex set is also midpoint convex, and the empty set is both convex and midpoint convex. So suppose $K \subseteq X$ is a nonempty closed midpoint convex set. Let $x, y \in K$. Consider the mapping $T : [0,1] \to X$ defined by

$$T(\lambda) \; := \; \lambda x + (1-\lambda)y.$$

Let

$$U \; := \; \Big\{\lambda \in [0,1] : T(\lambda) \notin K\Big\}.$$

Since — by Lemma 4 — T is continuous and K is closed, it follows that U is open. If $U = \emptyset$, then $[x,y] \subseteq K$ and we are done, so suppose otherwise. Thus, there exist distinct $\lambda_1, \lambda_2 \in [0,1] \backslash U$ such that $(\lambda_1, \lambda_2) \subseteq U$. Therefore, $\frac{\lambda_1+\lambda_2}{2} \in (\lambda_1, \lambda_2) \subseteq U$, and so

$$\frac{T(\lambda_1) + T(\lambda_2)}{2} \; = \; T\Big(\frac{\lambda_1 + \lambda_2}{2}\Big) \notin \; K,$$

but this is impossible since $T(\lambda_1), T(\lambda_2) \in K$ and K is midpoint convex. Hence, it must be the case that $U = \emptyset$, which implies that K is convex. □

And, since — by Remark 1 — every $\alpha-$norm is also a quasinorm, we get the following corollary.

Corollary 6. *Let X denote an $\alpha-$normed vector space. Suppose $K \subseteq X$ is closed. Then K is convex if and only if K is midpoint convex.*

Proof. The proof follows directly from Proposition 5 and Remark 1. □

Another statement, which we will disprove in a quasinormed vector space, says that a nonempty closed set K is convex if and only if the distance function $\mathrm{d}(\cdot, K)$ is convex. We use the quasinormed vector space from the previous example.

Example 2. Suppose $M > 2$. Let $X := (\mathbb{R}^2, \|\cdot\|)$ denote the Cartesian plane equipped with a quasinorm defined as

$$\left\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\| \; = \; \begin{cases} M|x_1| & \text{if } x_2 = 0 \\ |x_1| + |x_2| & \text{otherwise.} \end{cases}$$

Define $K := \{\binom{0}{0}\}$. The distance function $\mathrm{d}(\cdot, K)$ is not convex, while the set K is nonempty, convex and closed.

Proof. Clearly, K is nonempty, convex and closed. In Example 1 we showed that X is truly a quasinormed vector space and that the distance function from K is equal to the quasinorm. The nonconvexity of the quasinorm is left to show. Suppose $x = \binom{1}{1}$, $y = \binom{1}{-1}$ and $\lambda = \frac{1}{2}$. It holds that

$$\|\lambda x + (1-\lambda)y\| \;=\; \left\|\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\right\| \;=\; \left\|\begin{pmatrix}1/2\\0\end{pmatrix}\right\| \;=\; \frac{1}{2}M \;>\; 1,$$

while

$$\lambda\|x\| + (1-\lambda)\|y\| \;=\; \frac{1}{2}\left\|\begin{pmatrix}1\\1\end{pmatrix}\right\| + \frac{1}{2}\left\|\begin{pmatrix}1\\-1\end{pmatrix}\right\| \;=\; \frac{1}{2}\,|2| + \frac{1}{2}\,|2| \;=\; 1.$$

Therefore, the distance function for K is not convex. □

We see that the equivalence can not hold. One of the implications is left to clarify, though. With an inspiration from Fletcher, Moors [2] we prove the remaining relation between convexities of a closed set and the corresponding distance function.

Proposition 7. *Let K be a nonempty closed subset of a quasinormed vector space X. If the distance function for K is convex, the set K is convex.*

Proof. Let $M > 1$ denote the constant for the quasinorm. Let $x, y \in K$ and $\lambda \in [0,1]$. Then

$$0 \;\leq\; d\Big(\lambda x + (1-\lambda)y,\, K\Big) \;\leq\; M\Big(\lambda\, \mathrm{d}(x,\, K) + (1-\lambda)\, \mathrm{d}(y,\, K)\Big) \;=\; 0.$$

Since K is closed, $\lambda x + (1-\lambda)y \in K$, which implies that K is convex. □

Hereafter, by Remark 1, we get the following corollary.

Corollary 8. *Let K be a nonempty closed subset of an $\alpha-$normed vector space X. If the distance function for K is convex, the set K is convex.*

Proof. The proof follows directly from Proposition 7 and Remark 1. □

Next we focus on $\alpha-$norms. First of all, an $\alpha-$norm is continuous.

Proposition 9. *Let $\alpha \in (0,1]$. Let X denote an $\alpha-$normed vector space. Then the $\alpha-$norm is continuous.*

Proof. Let $x \in X$. The triangle inequality for the $\alpha-$norm gives us that

$$\|x\|^\alpha \;=\; \|x - y + y\|^\alpha \;\leq\; \|x-y\|^\alpha + \|y\|^\alpha$$

for each $y \in X$. Hence,

$$\|x\|^\alpha - \|y\|^\alpha \;\leq\; \|x-y\|^\alpha.$$

By symmetry, it holds that

$$\Big|\|x\|^\alpha - \|y\|^\alpha\Big| \;\leq\; \|x-y\|^\alpha.$$

Therefore, the mapping $x \mapsto \|x\|^\alpha$ is continuous. And, since the mapping $t \mapsto t^{1/\alpha}$ is continuous on $[0,\infty)$, theirs composite mapping $x \mapsto \|x\|$ is also continuous. □

We see that the properties of an $\alpha-$norm are sufficient for the continuity. Let us have a brief look at the convexity in the following example.

Example 3. Let $X = (\mathbb{R}^2, \|\cdot\|)$ denote the Cartesian plane equipped with a $\frac{1}{2}-$norm defined as

$$\left\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\| = \left(|x_1|^{\frac{1}{2}} + |x_2|^{\frac{1}{2}}\right)^2.$$

Define $K := \{\binom{0}{0}\}$. The distance function $\mathrm{d}(\cdot, K)$ is not convex, while the set K is nonempty, convex and closed.

Proof. First we show that the mapping $\|\cdot\| : X \to [0, \infty)$ is truly a $\frac{1}{2}-$norm. For every $x, y \in X$ and $a \in \mathbb{R}$ it holds that

1) $$\|x\| = 0 \Leftrightarrow \left(x_1 = 0 \wedge x_2 = 0\right) \Leftrightarrow x = \binom{0}{0},$$

2) $$\|ax\| = \left\|\binom{ax_1}{ax_2}\right\| = \left(|a|^{\frac{1}{2}}\left(|x_1|^{\frac{1}{2}} + |x_2|^{\frac{1}{2}}\right)\right)^2 = |a|\|x\|,$$

3) $$\begin{aligned} \|x + y\|^{\frac{1}{2}} &= \left(\left(|(x+y)_1|^{\frac{1}{2}} + |(x+y)_2|^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} \\ &= |x_1 + y_1|^{\frac{1}{2}} + |x_2 + y_2|^{\frac{1}{2}} \\ &\leq |x_1|^{\frac{1}{2}} + |x_2|^{\frac{1}{2}} + |y_1|^{\frac{1}{2}} + |y_2|^{\frac{1}{2}} \\ &= \|x\|^{\frac{1}{2}} + \|y\|^{\frac{1}{2}}. \end{aligned}$$

Similarly to the previous examples, the distance function from K is equal to the $\frac{1}{2}-$norm, and K is nonempty, convex and closed. The nonconvexity is left to show. Suppose $x = \binom{1}{0}$, $y = \binom{0}{1}$ and $\lambda = \frac{1}{2}$. It holds that

$$\|\lambda x + (1-\lambda)y\| = \left\|\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\right\| = \left\|\begin{pmatrix}1/2\\1/2\end{pmatrix}\right\| = \left(\left|\frac{1}{2}\right|^{\frac{1}{2}} + \left|\frac{1}{2}\right|^{\frac{1}{2}}\right)^2 = 2,$$

while

$$\lambda\|x\| + (1-\lambda)\|y\| = \frac{1}{2}\left\|\begin{pmatrix}1\\0\end{pmatrix}\right\| + \frac{1}{2}\left\|\begin{pmatrix}0\\1\end{pmatrix}\right\| = \frac{1}{2}\left(|1|^{\frac{1}{2}}\right)^2 + \frac{1}{2}\left(|1|^{\frac{1}{2}}\right)^2 = 1.$$

Therefore, the distance function for K is not convex. □

Here follows a theorem summarizing some of the results of this chapter.

Theorem 10. *Every $\alpha-$norm on every vector space is continuous, yet there exist nonconvex $\alpha-$norm and discontinuous, nonconvex quasinorm on $\mathbb{R}^2$.*

Proof. The proof follows directly from Examples 1, 2, 3 and Proposition 9. □

Bibliography

[1] Asplund, E. (1969). Čebyšev sets in Hilbert space. *Trans. Amer. Math. Soc.*, 144:235–240.

[2] Fletcher, J. and Moors, W. B. (2015). Chebyshev sets. *J. Aust. Math. Soc.*, 98(2):161–231.

[3] Kuncová, K. (2009). Lorentz spaces. *Bachelor Thesis, Prague.*

[4] Motzkin, T. S. (1935). Sur quelques propriétés caractéristiques des ensembles convexes. *Rend. Acad. dei Lincei (Roma)*, 21:562–567.