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Enumeration of polyomino fillings

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Abstract: We prove two new results about 0-1-fillings of skew diagrams avoiding long increasing and decreasing chains. In the first half of the thesis, we show that for a large class of skew diagrams, there is a bijection between sparse fillings avoiding an increasing chain of fixed length and sparse fillings avoiding a decreasing chain of the same length. In the second half, we extend a known inequality between the number of sparse 0-1-fillings of skew diagrams avoiding an increasing chain of length 2 and a decreasing chain of length 2 to all 0-1-fillings.

Keywords: polyomino filling skew diagram

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Introduction

A permutation π is said to be *contained* in a permutation σ, if some rows and columns of the permutation matrix of σ can be removed to obtain the permutation matrix of π, otherwise σ *avoids* π. A set of permutations closed under containment is called a *permutation class*, and it is a *singleton class* if, in addition, it can be described as the set of all permutations avoiding a single permutation π. Such a singleton class is denoted by $\mathrm{Av}(\pi)$. The study and enumeration of pattern-avoiding permutations and permutation classes has now been an active field for several decades. One of the important concepts studied in this field is *Wilf-equivalence.* Two permutation classes are *Wilf-equivalent*, if for every positive integer n they contain the same number of permutations of length n.

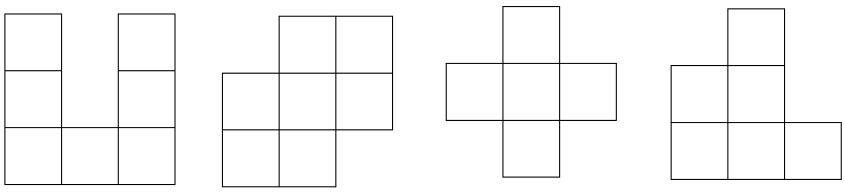
The search for new Wilf-equivalent classes has led to the investigation of a stronger kind of equivalence. Consider a Ferrers diagram whose every cell is filled with a 0 or a 1 such that there is at most one 1 in every row and every column. Such a filling of the Ferrers diagram is called *sparse* and it *contains* a permutation σ, if some rows and columns of the Ferrers diagram can be removed to obtain the permutation matrix of σ, otherwise it *avoids* σ. We then say that two singleton classes $\mathrm{Av}(\pi)$ and $\mathrm{Av}(\sigma)$ are *shape-Wilf-equivalent* if for every Ferrers diagram the number of sparse fillings avoiding π is the same as the number of sparse fillings avoiding σ. The most notable example of shape-Wilf-equivalence was found by Backelin, West and Xin [1], who have shown that for any k, the classes $\mathrm{Av}(12\cdots k)$ and $\mathrm{Av}(k(k-1)\cdots 21)$ are shape-Wilf-equvalent. Later, Krattenthaler [2] found a nice and simpler bijective proof of this result, and his work, in return, was further generalized by Rubey [3] who extended the bijection to more general diagrams.

In the present thesis, we prove two new results about 0-1-fillings of *skew diagrams*, which are diagrams obtained as difference of two Ferrers diagrams one of which contains the other. The thesis consists of three chapters. In the first chapter, we introduce all of the necessary terminology and notation. In the second chapter, we define a subclass of skew diagrams, the *simple skew diagrams*, and make use of the approaches of Krattenthaler and Rubey to construct a bijection between sparse fillings of a given simple skew diagram avoiding $12\cdots k$ and sparse fillings avoiding $k\cdots 21$. In the final chapter we generalise a result of Jelínek [4] and show that for every skew diagram, there at least as many general fillings avoiding 12 as there are fillings avoiding 21.

1. Preliminaries

1.1 Polyominoes

A *cell* is a unit square whose vertices lie on lattice points of the $\mathbb{Z}^2$ plane. We then define a *polyomino* as a finite set of cells. When referring to *coordinates* of a cell, we will use the $\mathbb{Z}^2$ coordinates of its bottom left corner. A polyomino is *convex* if for any two of its cells in the same row or column the polyomino also contains every cell in between. Two columns of a polyomino are *comparable* if the set of row coordinates of one column is contained in the set of row coordinates of the other. We say that a polyomino is *intersection-free* if any two of its columns are comparable. Note that this is equivalent to having any two of its rows comparable. A *moon polyomino* is a convex, intersection-free polyomino. A polyomino is *top-justified* if the top ends of all of its columns are in the same row and it is *left-justified* if the left ends of all of its rows are in the same column. Similarly we define a *bottom-justified* or a *right-justified* polyomino. A moon polyomino is a *Ferrers diagram* if it is top-justified or bottom justified and in addition either left-justified or right-justified.



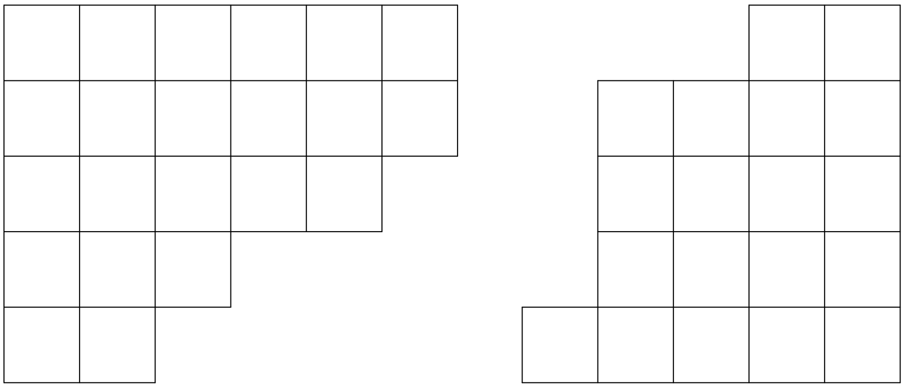
(a) non-convex (b) convex, not intersection-free (c) moon polyomino (d) bottom-justified moon polyomino

Figure 1.1: Examples of polyominoes

A *NW Ferrers diagram* (standing for *northwest*) is top-justified and left-justified. A *SE Ferrers diagram* (standing for *southeast*) is bottom-justified and right-justified. We can represent both kinds of Ferrers diagrams by sequences of characters R (meaning a step right) and U (meaning step up), describing the right-up border or the up-right border of *NW* Ferrers diagrams or *SE* Ferrers diagrams respectively. For example, the *NW* Ferrers diagram in Figure 1.2(a) is represented by the sequence $RRURURRURUU$ and the *SE* Ferrers diagram in Figure 1.2(b) is represented by the sequence $URUUURRURR$. We call the sequences of characters R and U the *R-U sequences* and if a Ferrers diagram F is represented by an *R-U* sequence w, we say that F *is of type* w.

Given a convex polyomino P, we define its *height* $h(P)$ as the number of its nonempty rows. Similarly, we define its *width* $w(P)$ as the number of nonempty columns of P.

Sometimes we will need to speak about the boundary of a convex polyomino, considered as a closed convex subset of the plane. We call the boundary of a convex polyomino P the *border* of P. If two adjacent sides of a cell of $\mathbb{Z}^2$ lie on the border of P, the corner of the cell in which the two sides meet is a *turn* on the border. A line along the border between two adjacent turns of M is a *segment*.



(a) *NW* Ferrers diagram (b) *SE* Ferrers diagram

Figure 1.2: Examples of Ferrers diagrams

For example, the border of the polyomino in Figure 1.1(b) consists of 8 turns and 8 segments.

1.2 Partitions

A *partition* of length n is a finite weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Each of the numbers λ_i is called a *part* of λ and the *size* of λ is the sum of its parts. Such a partition λ also represents a *NW* Ferrers diagram with column heights $\lambda_1, \ldots, \lambda_n$. For example, the diagram in Figure 1.2(a) is represented by a partition $(5, 5, 4, 3, 3, 2)$. A partition $\mu = (\mu_1, \ldots, \mu_m)$ is *contained* in $\lambda = (\lambda_1, \ldots, \lambda_n)$ if $m \leq n$ and $\mu_i \leq \lambda_i$ for $1 \leq i \leq m$. The *union* of partitions $\mu = (\mu_1, \ldots, \mu_m)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the partition $\nu = (\nu_1, \ldots, \nu_k)$ such that $k = \max(m, n)$ and $\nu_i = max(\mu_i, \lambda_i)$. Finally, the *transpose* of a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the partition $\lambda^T = (\lambda_1^T, \ldots, \lambda_t^T)$ such that $t = \lambda_1$ and λ_i^T is equal to the number of parts of λ greater than or equal to i. Notice that transposing a partition is equivalent to reflecting the associated Ferrers diagram over the $y = x$ line.

1.3 Skew diagrams

A skew diagram is a *NW* Ferrers diagram with a smaller *NW* Ferrers diagram cut out of its north-west corner. This is represented by a pair of partitions (λ, μ) such that μ is contained in λ.

Let S be a skew diagram and let C be one of its columns. We define the operation of *deleting* the column C from S as removing the cells of C and shifting all cells of S right of C one cell to the left. Similarly we define the operation of deleting a row R from S as removing the cells of R from S and shifting all cells of S below R one cell upward. A skew diagram S is a *subdiagram* of a skew diagram T or is *contained* in T if S can be created from T by deleting some rows and

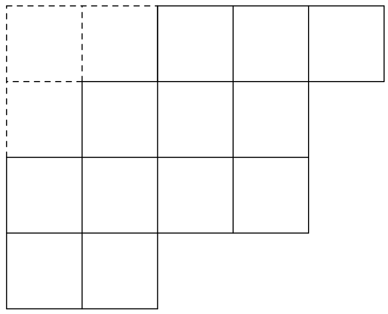


Figure 1.3: The skew diagram $((4,4,3,3,1),(2,1))$

columns. If S is not containted in T, we say that T *avoids* S. Note that both the class of skew diagrams and the class of Ferrers diagrams are closed under the operations of deleting a row or a column, i.e. deleting a row or a column from a skew diagram always results in a skew diagram and in addition, if the diagram is a Ferrers diagram, it remains a Ferrers diagram.

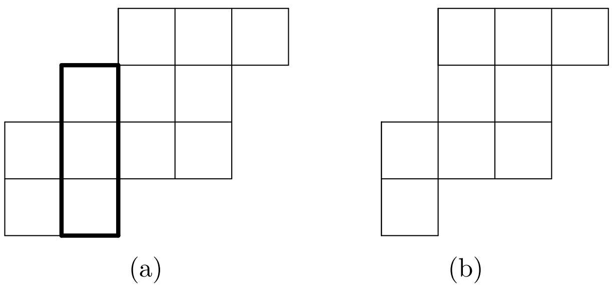


Figure 1.4: Deleting a column from a skew diagram

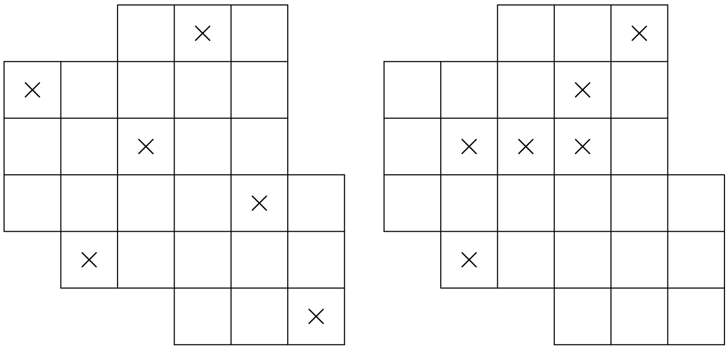
A skew diagram is clearly bounded by two paths leading from its lower left and to its upper right corner and consisting only of steps up and to the right. We shall call them the *upper border* and the *lower border* of the diagram.

For convenience, we will only consider *connected* skew diagrams, which satisfy the additional condition that their upper and lower borders only meet in the lower left and the upper right corner and nowhere in between. All of the presented results about connected skew diagrams can be extended to general skew diagrams without any effort.

1.4 Fillings of polyominoes

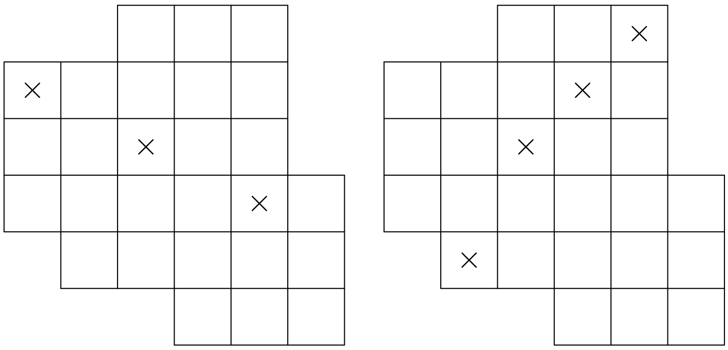
The main theme of this thesis is filling cells of polyominoes with integers and then counting fillings having certain properties. A *0-1-filling* of a polyomino is an assignment of either a 0 or a 1 to each cell of the polyomino. A *sparse filling* is a 0-1-filling, such that there is at most one 1 in each row and in each column. In figures, we will often represent 0's in such fillings by empty cells and 1's by crosses. In addition, if a subset of a polyomino contains only 0's, we shall say that it is *empty*, otherwise it is *nonempty*.

Let P be a polyomino filled with a 0-1-filling and let $c_1, c_2, \ldots, c_l$ be l of its cells filled with a 1. These cells form a NE*-chain (pronounced "northeast chain")*



(a) A sparse filling (b) A non-sparse filling

Figure 1.5: Examples of 0-1-fillings



(a) A SE-chain of length 3 (b) Not a NE-chain of length 4

Figure 1.6: Examples of chains in a polyomino with a 0-1-filling

of length l in P if for $1 < i \leq l$ each c_i is strictly northeast of c_{i-1} and the smallest rectangle containing all l cells is entirely contained in P. Similarly the cells form a *SE-chain (pronounced "southeast chain") of length* l, if for $1 < i \leq l$ each c_i is strictly southeast of c_{i-1} and again the smallest rectangle containing all the cells is entirely contained in P. We then say that P *contains* a NE-chain (or SE-chain) of length l if there is at least one occurence of such a chain in P. Otherwise we say that P *avoids* a NE-chain (or SE-chain) of length l. For example, in Figure 1.6(a) the smallest rectangle containing the three nonzero entries are contained in the polyomino, so the 1's do form a SE-chain of length 3. However, in Figure 1.6(b) the northwest corner of the smallest rectangle containing the four nonempty cells is not a part of the polyomino, so this is not a NE-chain of length 4. However both the lower three 1's and the upper three 1's form a NE-chain of length 3.

In the following chapter we deal with sparse fillings of skew diagrams avoiding long NE-chains and SE-chains. To describe the sets of fillings in question easily, we will use the notation introduced by Rubey [3].

Definition 1.1 ([3, Definition 5.2])**.** Let P be a moon polyomino or a connected skew diagram, l a positive integer and $\boldsymbol{r}$, $\boldsymbol{c}$ sequences of 0's and 1's of lengths $h(P)$ and $w(P)$ respectively. Then $\mathcal{F}^{NE}(P, l, \boldsymbol{r}, \boldsymbol{c})$ is the set of all sparse fillings of P such that

- there is a 1 in the i-th row of P if and only if $r_i = 1$,

- there is a 1 in the i-th column of P if and only if $c_i = 1$,
- the length of the longest NE-chain in the filling is equal to l.

Similarly we define $\mathcal{F}^{SE}(P, l, \boldsymbol{r}, \boldsymbol{c})$ as the set of all sparse fillings of P with the length of the longest SE-chain equal to l and having 1's exactly in the rows and columns prescribed by $\boldsymbol{r}$ and $\boldsymbol{c}$. Note that since we are dealing with sparse fillings, the number of nonempty rows and columns prescribed by $\boldsymbol{r}$ and $\boldsymbol{c}$ must be the same for any fillings to exist.

2. A bijection between fillings of special skew diagrams

2.1 Main result

Let S^{forb} be the forbidden skew diagram $((3,3,2),(1))$.

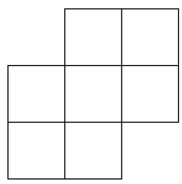


Figure 2.1: S^{forb}

Lemma 2.1. *A skew diagram S avoids S^{forb} if and only if for every cell c of S at least one of the following conditions is satisfied:*

- *either the set of cells of S to the left and above c (including the cells in the same row or column as c) forms a rectangle, or*
- *the set of cells of S to the right and below of c (including the cells in the same row or column as c) forms a rectangle.*

Proof. If S does contain S^{forb}, then the cell that represents the middle cell of an occurence of S^{forb} violates both conditions. On the other hand, if a cell c of S violates both conditions, we find an occurence of S^{forb} in S with this cell as its middle cell. Let A be the maximal rectangle contained in S which has c as its northeast corner and let B be the maximal rectangle contained in S which has c as its southwest corner. Then the 7 cells in the corners of these two rectangles form an occurence of S^{forb}. □

We can now formulate the main result of this chapter.

Theorem 2.2. *Let S be a skew diagram avoiding S^{forb}, let $l \geq 1$ be an integer and let $\boldsymbol{r}$ and $\boldsymbol{c}$ be sequences of 0's and 1's of lengths $w(S)$ and $h(S)$ respectively. Then there is a bijection between the sets $\mathcal{F}^{NE}(S,l,\boldsymbol{r},\boldsymbol{c})$ and $\mathcal{F}^{SE}(S,l,\boldsymbol{r},\boldsymbol{c})$.*

To prove this theorem, we will utilize other known bijection theorems on Ferrers diagrams and moon polyominoes described in sections 2.3 and 2.4, but first, we will prove a useful equivalent characterization of skew diagrams avoiding S^{forb}.

2.2 Simple skew diagrams

Let S be a skew diagram and let a vertical line leading between two adjacent columns of S divide it into two skew diagrams S_1 and S_2. Then we say that

S is a *vertical concatenation* of S_1 and S_2 and we write $S = S_1|^v S_2$. Similarly we define *horizontal concatenation* and write $T = T_1|^h T_2$. Note that there are mutiple ways how to concatenate two skew diagrams.

We say that S is a *simple skew diagram* if it can be written as

$$S = F_1|^v G_1|^h F_2|^v \cdots |^h F_n|^v G_n,$$

where all F_i are *NW* Ferrers diagrams, all G_i are *SE* Ferrers diagrams and all F_i and G_i are nonempty with the possible exception of G_n. We call this representation of S the *simple representation.*

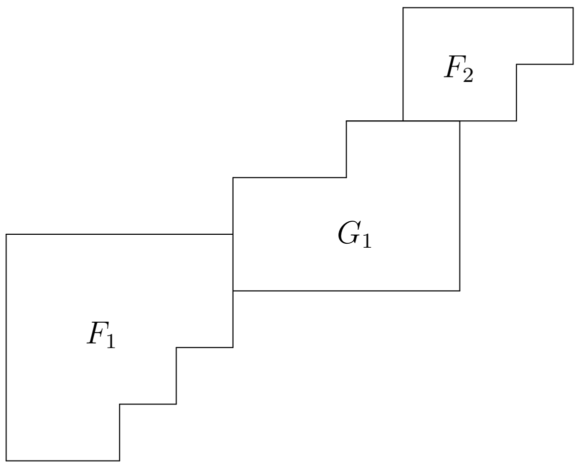


Figure 2.2: A simple skew skew diagram

The key lemma follows.

Lemma 2.3. *A skew diagram is simple if and only if it does not contain S^{forb}.*

Proof. In a simple skew diagram S every cell is in either a *NW* Ferrers subdiagram or a *SE* Ferrers subdiagram and thus satisfies at least one of the conditions of Lemma 2.1. Therefore S avoids S^{forb}.

We now prove that every skew diagram avoiding S^{forb} is simple by induction on the number of turns on the upper border of a skew diagram. If there is only one turn on the upper border of a skew diagram, then it is necessarily a *NW* Ferrers diagram and there is nothing to prove. Let S be a skew diagram whose upper border has at least two turns.

Let u_1 and r_1 be the first two segments of the upper border of S, starting from the lower left corner, with u_1 going up and r_1 going to the right. Together with the corresponding part of the lower border they bound a *NW* Ferrers diagram F which can be separated by a vertical line v from the rest of S, as we can see in Figure 2.3.

Next, consider the cell c attached to the upper right corner of F as indicated in Figure 2.3. Since S avoids S^{forb} and the upper right corner of c is a turn of the border of S, we get by Lemma 2.1 that the part of S to the left and below of c must form a rectangle, bounded from below by the lower border segment r_2 and from the right by the lower border segment u_2. These two segments of the lower border, together with the corresponding part of the upper border of S bound a

SE Ferrers diagram G, which can again be separated from the rest of S by a horizontal line h.

Finally, let S' be the part of S above the line h. Then S' is a skew diagram which definitely has fewer turns on its upper border than S, so the induction applies and we get that S' can be written as

$$S' = F_1|^v G_1|^h F_2|^v \cdots |^h F_n|^v G_n,$$

where $F_1, \dots, F_n$ are nonempty NW Ferrers diagrams and $G_1, \dots, G_n$ are SE Ferrers diagrams. Together with the constructed diagrams F and G we get that

$$S = F|^v G|^h F_1|^v G_1|^h F_2|^v \cdots |^h F_n|^v G_n,$$

and thus S is simple.

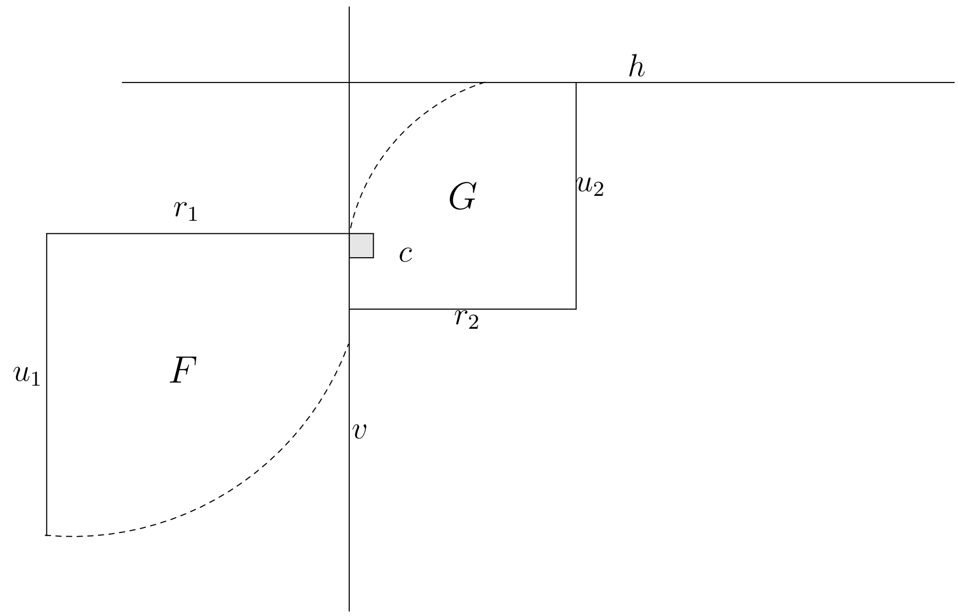


Figure 2.3

□

2.3 The bijection on Ferrers diagrams

Krattenthaler [2] proved Theorem 2.2 for the special case of Ferrers diagrams using the growth diagram construction developed by Britz and Fomin [5]. We briefly introduce growth diagrams and reformulate the results of this approach in our setting. For convenience, we will describe the construction for NW Ferrers diagrams only, however, the modification for SE diagrams is obvious and we shall use it without stating it properly.

Let F be a Ferrers diagram filled with a sparse filling. We will inductively label every corner of every cell of F by a partition, obtaining a *growth diagram*. We start by labeling the corners on the left and upper borders by the empty partition $\emptyset$. If a cell already has all corners except the lower-right corner labeled by partitions λ, μ, and ν as in Figure 2.4, we construct the remaining partition ρ by the following *forward local* rules:

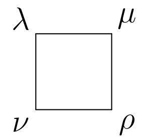


Figure 2.4: A cell of a growth diagram

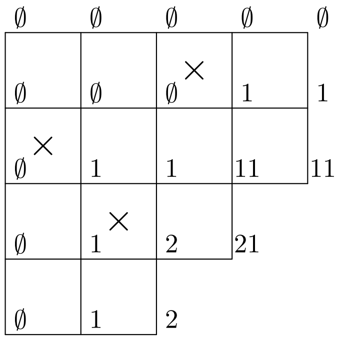


Figure 2.5: An example of a growth diagram

(F1) If the cell does not contain a 1 and $\lambda = \mu = \nu$, then $\rho = \lambda$.

(F2) If the cell does not contain a 1 and $\mu \neq \nu$, then $\rho = \mu \cup \nu$.

(F3) If the cell does not contain a 1, $\lambda \subsetneq \mu = \nu$ and λ and μ differ in their i-th part, then we obtain ρ by increasing the $i+1$-th part of μ by 1.

(F4) If the cell contains a 1, then necessarily $\lambda = \nu = \mu$ and we obtain ρ by increasing the first part of λ by 1.

Note that the equality of partitions in (F4) is implied by the fact that there is no 1 in the row of cells to the left or in the column of cells above the considered cell.

Let $w = w_1 w_2 \cdots w_k$ be an *R-U* sequence. An *oscillating tableau of type w and shape $\emptyset/\emptyset$* is a sequence of partitions $(\lambda^0, \lambda^1, \dots, \lambda^k)$ with the following properties:

- $\lambda^0 = \lambda^k = \emptyset$
- For $1 \leq i \leq k$, the sizes of λ^{i-1} and λ^i differ by at most 1.
- If $w_i = R$, then $\lambda^{i-1} \subseteq \lambda^i$.
- If $w_i = U$, then $\lambda^{i-1} \supseteq \lambda^i$.

Remarkably, both the filling and the entire growth diagram of a Ferrers diagram of type w can be reconstructed from the sequence of partitions labeling the corners of its south-east border, which clearly is an oscillating tableau of type w and shape $\emptyset/\emptyset$. For more details we refer the reader to [2, Section 2]. The main consequence of this construction is the following theorem.

Theorem 2.4 ([2, Theorem 1]). *Let F be a NW Ferrers diagram of type $w = w_1w_2\dots w_k$. The mapping, which to each sparse filling of F assigns the oscillating tableau $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^k = \emptyset)$ constructed as shown above, is a bijection between the set of all sparse fillings of F and the set of all oscillating tableaux of type w and shape $\emptyset/\emptyset$. In addition, this mapping has the following properties:*

- *$\lambda^{i-1} \subsetneq \lambda^i$ if and only if there is a 1 in the column above the corners labeled by λ^{i-1} and λ^i.*
- *$\lambda^i \subsetneq \lambda^{i-1}$ if and only if there is a 1 in the row to the left of the corners labeled by λ^{i-1} and λ^i.*

For example, the filling of the Ferrers diagram in Figure 2.5 is mapped to the oscillating tableau $(\emptyset, 1, 2, 2, 21, 11, 11, 1, \emptyset)$.

The very useful fact about growth diagrams is that the lengths of both longest NE-chains and SE-chains in a Ferrers diagram can be deduced from partitions labeling the corners.

Theorem 2.5 ([2, Theorem 2]). *Let F be a Ferrers diagram with a sparse filling and a growth diagram constructed accordingly and let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition labeling the southwest corner of a cell c. Let R be the maximal rectangle contained in F with the cell c in its southwest corner. Then the longest NE-chain contained in R has length k and the longest SE-chain contained in R has length λ_1.*

Using growth diagrams and the theorems above it is now easy to prove Theorem 2.2 for Ferrers diagrams.

Theorem 2.6 ([2, Theorem 3]). *Let F be a Ferrers diagram and $l \geq 1$ an integer. Let $\boldsymbol{r}$ and $\boldsymbol{c}$ be sequences of 0's and 1's of length $w(F)$ and $h(F)$ respectively. Then there is a bijection between the sets $\mathcal{F}^{NE}(F, l, \boldsymbol{r}, \boldsymbol{c})$ and $\mathcal{F}^{SE}(F, l, \boldsymbol{r}, \boldsymbol{c})$.*

Proof. For given sequences $\boldsymbol{r}$ and $\boldsymbol{c}$ let F' be the diagram obtained from F by deleting the rows with a 0 entry in $\boldsymbol{r}$ and columns with a 0 entry in $\boldsymbol{c}$. Since the deleted rows and columns are always empty in the considered fillings, it is enough to construct a bijection between the sets $\mathcal{F}^{NE}(F', l, \mathbf{1}, \mathbf{1})$ and $\mathcal{F}^{SE}(F', l, \mathbf{1}, \mathbf{1})$ where $\mathbf{1}$ is a sequence of only 1's and this bijection is trivially extended to the sets in question.

Choose a filling of F' from $\mathcal{F}^{NE}(F', l, \mathbf{1}, \mathbf{1})$. The Theorem 2.4 assigns to this filling an oscillating tableau $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^k = \emptyset)$ and by Theorem 2.5 every partition λ^i has at most l parts with equality occuring at least once. Now consider the oscillating tableau $(\emptyset = (\lambda^0)^T, (\lambda^1)^T, \dots, (\lambda^k)^T = \emptyset)$ obtained by transposing every partition of the original tableau. We use the Theorem 2.4 again to assign a sparse filling of F' to the transposed tableau. Since the transposition does not change the size of a partition, it is true that $(\lambda^{i-1})^T$ and $(\lambda^i)^T$ differ by 1 in size for each $1 \leq i \leq k$ and therefore the obtained filling of F' contains a 1 in each row and each column. In addition, each $(\lambda^i)^T$ satisfies $(\lambda^i)^T_1 \leq l$ with equality occuring at least once and so by Theorem 2.5 the length of the longest SE-chain in F' is equal to l. Therefore, the obtained filling is in $\mathcal{F}^{SE}(F', l, \mathbf{1}, \mathbf{1})$. □

2.4 A bijection on moon polyominoes

In his work, Rubey [3] proves more general bijective results about fillings of moon polyominoes. In particular, he shows that permuting the columns of a moon polyomino in any way such that it remains a moon polyomino does not change the number of 0-1 fillings having a fixed length of the longest NE-chain and prescribed number of 1's in each row. Here we formulate a part of his results which will be useful in our efforts.

Given a finite sequence $s = (s_1, s_2, \dots, s_n)$ and a permutation σ of length n, we denote by σs the sequence $(s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(n)})$. In addition, given a moon polyomino M and a permutation π of length $w(M)$, we denote by σM the polyomino created by permuting the columns of M according to π. We will also need a stronger version of Definition 1.1.

Definition 2.7 ([3, Definition 5.2]). Let M be a moon polyomino, $\boldsymbol{r}$, $\boldsymbol{c}$ sequences of 0's and 1's of lengths $h(M)$ and $w(M)$ respectively and Λ a mapping which assigns to every maximal rectangle R in M a positive integer $\Lambda(R)$. Then $\mathcal{F}^{NE}(M, \Lambda, \boldsymbol{r}, \boldsymbol{c})$ is the set of all sparse fillings of P such that

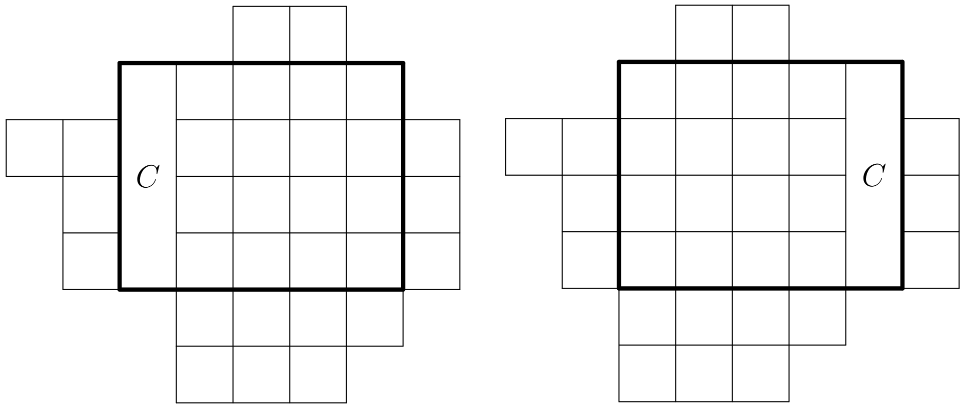
- there is a 1 in the i-th row of P if and only if $r_i = 1$,
- there is a 1 in the i-th column of P if and only if $c_i = 1$,
- for every maximal rectangle R the length of the longest NE-chain in the filling of R is equal to $\Lambda(R)$.

Note that in a moon polyomino, every maximal rectangle is uniquely determined by its height and width. Indeed, consider two maximal rectangles of the same height in a moon polyomino M. Since the columns of the two rectangles are comparable, they must in fact span the same rows, and since M is convex, they must be contained in each other and thus be identical.

Theorem 2.8 ([3, Theorem 5.3]). *Let M be a moon polyomino and R be a maximal rectangle in M such that the column of M containing the leftmost column C of R has the same height as C. Let σ be the permutation of columns of M which moves the column C to the right end of R and shifts the other columns intersecting R one spot to the left. Then the sets of maximal rectangles of M and σM coincide and for any Λ, $\boldsymbol{r}$ and $\boldsymbol{c}$ there is a bijective map which maps every filling in $\mathcal{F}^{NE}(M, \Lambda, \boldsymbol{r}, \boldsymbol{c})$ to a filling in $\mathcal{F}^{NE}(\sigma M, \Lambda, \boldsymbol{r}, \boldsymbol{\sigma c})$.*

Of course, since the mapping described by the theorem is bijective, the inverse mapping is also a bijection and therefore we may transform moon polyominoes by moving the rightmost column of a maximal rectangle instead of the leftmost. Furthermore, due to symmetry we can also use this result for moving rows instead of columns.

We follow up by using the theorem above to prove a bijection between sparse fillings of Ferrers diagrams which satisfy two different sets of additional constraints along with having the length of the longest NE-chain equal to $l \geq 1$. Consider a NW Ferrers diagram F with at least n longest columns and at least m longest rows and let A be the rectangle consisting of the n leftmost columns of F and let B be the rectangle consisting of top m rows of F. Let $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$



(a) M (b) σM

Figure 2.6: The operation described by Theorem 2.8

and $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)$ be two weakly increasing sequences of nonnegative integers satisfying $a_n \leq l$ and $b_m \leq l$. Given any sequences $\boldsymbol{r}$ and $\boldsymbol{c}$ as in Definition 1.1, we define the following sets of sparse fillings of F:

- $\mathcal{F}^{NE}_{in}(F, l, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$ is the set of all fillings from $\mathcal{F}^{NE}(M, l, \boldsymbol{r}, \boldsymbol{c})$ satisfying for every $1 \leq i \leq n$ that the length of the longest NE-chain inside the rectangle consisting of i *rightmost* columns of A is equal to a_i and for every $1 \leq j \leq m$ and the length of the longest NE-chain inside the rectangle consisting of j *bottom* rows of B is equal to b_j,

- $\mathcal{F}^{NE}_{out}(F, l, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$ is the set of all fillings from $\mathcal{F}^{NE}(M, l, \boldsymbol{r}, \boldsymbol{c})$ satisfying for every $1 \leq i \leq n$ that the length of the longest NE-chain inside the rectangle consisting of i *leftmost* columns of A is equal to a_i and for every $1 \leq j \leq m$ the length of the longest NE-chain inside the rectangle consisting of j *top* rows of B is equal to b_j.

Similarly we define the sets $\mathcal{F}^{SE}_{in}(F, l, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$ and $\mathcal{F}^{SE}_{out}(F, l, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$ with the constraints imposed the same way as above but on lengths of SE-chains instead of NE-chains. We now show that the two described sets of fillings are actually of the same size.

Lemma 2.9. *Let F be a NW Ferrers diagram with the sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ of constraints on fillings as described above. Then there is a bijection between the sets $\mathcal{F}^{NE}_{in}(F, l, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$ and $\mathcal{F}^{NE}_{out}(F, l, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$.*

Proof. We start by modifying the diagram F by adding some new rows and columns, adjusting $\boldsymbol{r}$ and $\boldsymbol{c}$ appropriately by assigning 0's to the new rows and columns, which will therefore always remain empty. The modification is as follows:

- for i iterating from n to 1, attach a new row of length i below the i rightmost columns of A,

- for j iterating from m to 1, attach a new column of length j to the right of the bottom j rows of B.

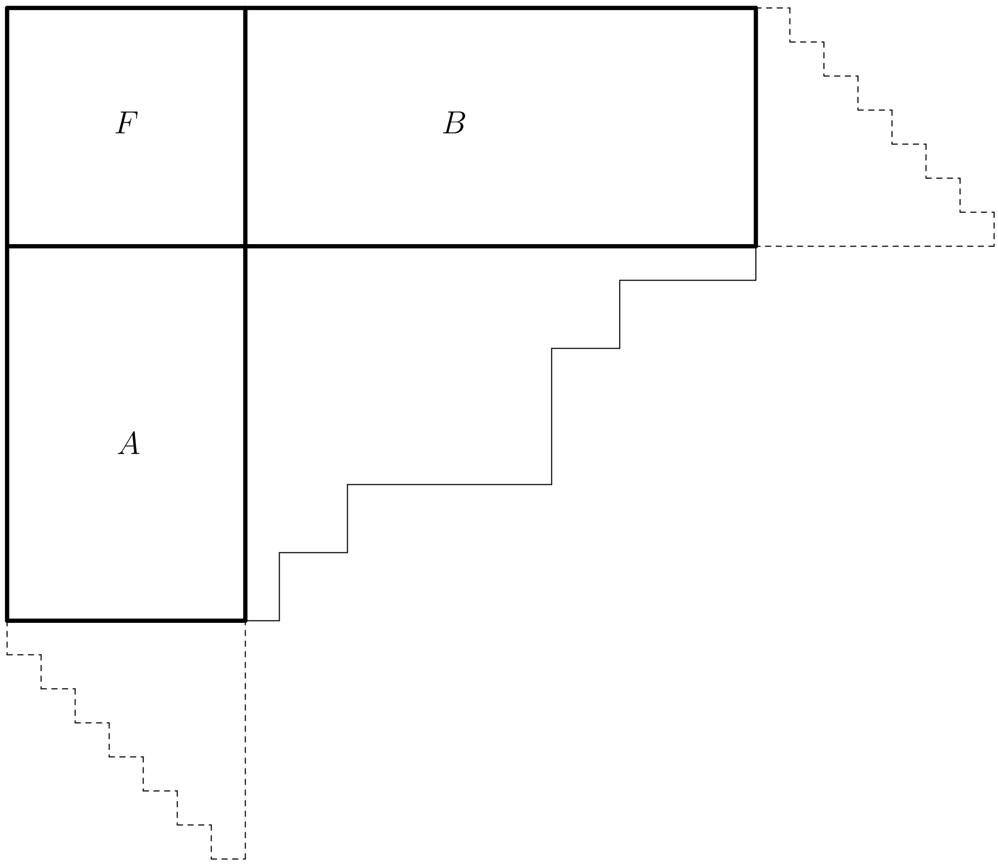


Figure 2.7: The moon polyomino M_1 after modifying F

As a result of this modification we obtain a moon polyomino M_1 (see Figure 2.7) and sequences $\boldsymbol{r^1}$ and $\boldsymbol{c^1}$.

We continue by defining a family $\mathcal{L}$ of mappings assigning a positive integer to every maximal rectangle of M_1. A mapping Λ belongs to $\mathcal{L}$ if and only if the following conditions are met:

(a) For $1 \leq i \leq n$, let A_i be the unique maximal rectangle of width i in M_1. Then $\Lambda(A_i) = a_i$.

(b) For $1 \leq j \leq m$, let B_j be the unique maximal rectangle of height j in M_1. Then $\Lambda(B_j) = b_j$,

(c) for any other maximal rectangle R it is true that $\Lambda(R) \leq l$

(d) There is at least one maximal rectangle R such that $\Lambda(R) = l$.

Let $\mathcal{F}'$ be the set of fillings of M_1 obtained as a union of $\mathcal{F}(M_1, \Lambda, \boldsymbol{r}^1, \boldsymbol{c}^1)$ for every $\Lambda \in \mathcal{L}$. Then $\mathcal{F}'$ is the set of fillings of M_1 in which the longest NE-chain has length l and in addition for every $1 \leq i \leq n$ the longest NE-chain in the rectangle A_i has length a_i and for every $1 \leq j \leq m$ the longest NE-chain in the rectangle B_j has length b_j. But since the newly attached rows and columns are always empty, the chains are always contained in the original diagram F and therefore the set $\mathcal{F}'$ is in 1-to-1 correspondence with the set $\mathcal{F}^N E_{in}(F, l, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}$.

Next we perform the transformation described in Theorem 2.8 several times to obtain a Ferrers diagram F_1:

- for every i iterating from n to 1, move the leftmost column of A_i to the rightmost end of the rectangle,
- for every j iterating from m to 1, move the top row of B_j to the bottom of the rectangle.

This creates the Ferrers diagram F_1 as illustrated by the Figure 2.8. Let σ be the permutation of columns and π be the permutation of rows which created F_1 from M_1. By Theorem 2.8 we get that for every $\Lambda \in \mathcal{L}$ there is a bijection between $\mathcal{F}^{NE}(M_1, \Lambda, \boldsymbol{r}^1, \boldsymbol{c}^1)$ and $\mathcal{F}^{NE}(F_1, \Lambda, \pi\boldsymbol{r}^1, \sigma\boldsymbol{c}^1)$. Let $\mathcal{F}''$ be the set of fillings of F_1 obtained as a union of $\mathcal{F}^{NE}(F_1, \Lambda, \pi\boldsymbol{r}^1, \sigma\boldsymbol{c}^1)$ for every $\Lambda \in \mathcal{L}$. Overall we obtain a bijection between the sets $\mathcal{F}'$ and $\mathcal{F}''$. Finally, notice that, similarly as the fillings of $\mathcal{F}'$ correspond to the fillings of $\mathcal{F}^{NE}_{in}(F, l, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$, also the fillings of $\mathcal{F}''$ correspond to the fillings of $\mathcal{F}^{NE}_{out}(F, l, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$, which completes the proof.

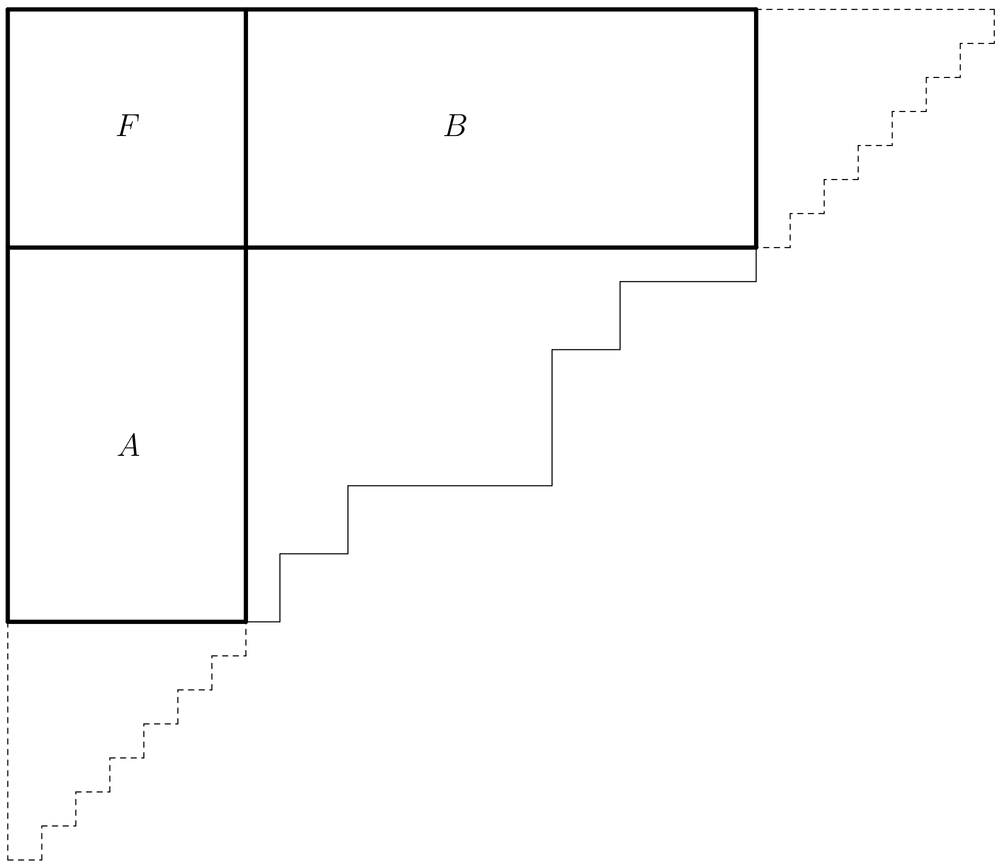


Figure 2.8: The Ferrers diagram F_1 after applying Theorem 2.8

□

2.5 Proof of Theorem 2.2

We start by using the growth diagram construction of Section 2.3 to prove an additional useful bijection between sets of constrained fillings described in Section 2.4.

Lemma 2.10. *Let F be a NW Ferrers diagram with at least n longest columns and at least m longest rows, $l \geq 1$ an integer, $\boldsymbol{r}$ and $\boldsymbol{c}$ sequences prescribing nonempty rows and columns of F, and $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_m)$ weakly increasing sequences of integers less than equal to l. Then there is a bijection between the sets $\mathcal{F}^{NE}_{out}(F, l, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$ and $\mathcal{F}^{SE}_{out}(F, l, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$.*

Proof. Consider a filling of F from $\mathcal{F}^{NE}_{out}(F, l, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$ and build its growth diagram, obtaining the corresponding oscillating tableau $(\lambda^0, \lambda^1, \dots, \lambda^k)$. The constraints $\boldsymbol{a}$ together with Theorem 2.5 imply that for $1 \leq i \leq n$ the partition λ^i has exactlya_i parts. Similarly the constraints $\boldsymbol{b}$ imply that for $1 \leq j \leq m$ the partition λ^{k-j} has exactly b_j parts. Applying the transformation described in Theorem 2.6 to the filling of F, we obtain a filling of $\mathcal{F}^{SE}(F, l, \boldsymbol{r}, \boldsymbol{c})$. In addition, since the oscillating tableau corresponding to this new filling is obtained by transposing the oscillating tableau for the original filling, we get for $1 \leq i \leq n$ that $(\lambda^i)^T_1 = a_i$ and so the longest SE-chain in the rectangle consisting of i leftmost columns of F has length a_i by Theorem 2.5. Similarly we get for $1 \leq j \leq m$ that $(\lambda^{k-j})^T_1 = b_j$ and so the longest SE-chain in the rectangle consisting of top j rows of F has length b_j by Theorem 2.5. Therefore the obtained filling belongs to $\mathcal{F}^{SE}_{out}(F, l, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$. $\square$

Finally we have all we need to prove the main result.

Proof of Theorem 2.2. Lemma 2.3 implies that it is enough to prove the theorem for simple skew diagrams. Let

$$S = F_1|^v G_1|^h F_2|^v \cdots |^h F_n|^v G_n$$

be the simple representation of S. The main idea of the proof is now to apply Lemma 2.9 to individual Ferrers diagrams in the simple representation of S using constraints constructed based on neighbouring diagrams.

Choose any filling from $\mathcal{F}^{NE}(S, l, \boldsymbol{r}, \boldsymbol{c})$. Consider the diagram F_i and let $\boldsymbol{r}^{F_i}$ and $\boldsymbol{c}^{F_i}$ be the sequences describing which rows and columns of F_i currently contain a 1. The current filling of F_i of course belongs to $\mathcal{F}^{NE}(F_i, l, \boldsymbol{r}^{F_i}, \boldsymbol{c}^{F_i})$. Let n be the number of columns of F_i which are connected to a column of G_{i-1} (if it exists, otherwise set n to zero). Let m be the number of rows of F_i which are connected to a row of G_i. We define the following rectangles:

- $A^{G_{i-1}}$ is the rectangle consisting of the n rightmost columns of G_{i-1},
- A^{F_i} is the rectangle consisting of the n leftmost columns of F_i,
- B^{F_i} is the rectangle consisting of the top m rows of F_i,
- B^{G_i} is the rectangle consisting of the bottom m rows of G_i.

We define constraints $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_m)$ for fillings of F_i as follows:

- if x is the length of the longest NE-chain contained in j rightmost columns of A^{F_i}, set a_j to x,
- if y is the length of the longest NE-chain contained in the bottom j rows of B^{F_i}, set b_j to y.

From the way we defined $\boldsymbol{a}$ and $\boldsymbol{b}$ it is now clear that the filling of F_i belongs to $\mathcal{F}_{in}^{NE}(F_i, l, \boldsymbol{r}^{F_i}, \boldsymbol{c}^{F_i}, \boldsymbol{a}, \boldsymbol{b})$. We now use the bijective map of Lemma 2.9 to transform the filling of F_i into a filling from $\mathcal{F}_{out}^{NE}(F_i, l, \boldsymbol{r}^{F_i}, \boldsymbol{c}^{F_i}, \boldsymbol{a}, \boldsymbol{b})$ and to this filling we apply the bijective transformation of Lemma 2.10, obtaining a filling from $\mathcal{F}_{out}^{SE}(F_i, l, \boldsymbol{r}^{F_i}, \boldsymbol{c}^{F_i}, \boldsymbol{a}, \boldsymbol{b})$.

This way we transform the filling of every F_i and every G_i in the simple representation of S. The individual Ferrers diagrams now avoid SE-chains longer than l, so it remains to prove that the fillings of rectangles connecting any two neighbouring diagrams avoid them as well. Consider the diagrams F_i and G_i. Any SE-chain longer than l in $F_i|^v G_i$ would have to be contained in the rectangle R consisting of the rectangles B^{F_i} and B^{G_i}. Let $\boldsymbol{a}^{F_i}$ and $\boldsymbol{b}^{F_i}$ be the constraints for the filling of F_i as constructed above and let $\boldsymbol{a}^{G_i}$ and $\boldsymbol{b}^{G_i}$ be the same constraints for the filling of G_i. Suppose for the sake of a contradiction that there is a SE-chain of length $l+1$ inside R and that k 1's of the chain are contained in the top j rows of B^{F_i} and the remaining $l+1-k$ 1's of the chain are contained in the bottom $m-j$ rows of B^{G_i}, thus $k \leq b_j^{F_i}$ and $l+1-k \leq b_{m-j}^{G_i}$, giving $b_j^{F_i} + b_{m-j}^{G_i} \geq l+1$. On the other hand $b_j^{F_i}$ is the length of the longest NE-chain contained in the bottom j rows of B^{F_i} in the original filling and $b_{m-j}^{G_i}$ is the length of the longest NE-chain contained in the top $m-j$ rows of B^{G_i} in the original filling and therefore $b_j^{F_i} + b_{m-j}^{G_i} \leq l$ and a contradiction is obtained. Therefore the resulting filling of S belongs to $\mathcal{F}^{SE}(S, l, \boldsymbol{r}, \boldsymbol{c})$ and since it was obtained using bijective transformations, the proof is finished. $\square$

3. General 0-1-fillings avoiding a chain of length 2

Given any skew diagram S, it is known that there are at least as many sparse 0-1-fillings of S avoiding an NE-chain of length 2 as there are sparse 0-1-fillings of S avoiding an SE-chain of length 2. In this chapter, we will extend this result to general 0-1-fillings.

Recall the skew diagram S^{forb} defined in the previous chapter. We denote by $S^{forb}(132)$ the diagram S^{forb} associated with a sparse 0-1-filling of a 132 pattern, as shown in the Figure 3.1. We then say that a filling of a skew diagram S *contains* $S^{forb}(132)$ if we can obtain $S^{forb}(132)$ by removing some columns and rows of S and replacing some entries 1 by entries 0. Otherwise the filling of S *avoids* $S^{forb}(132)$.

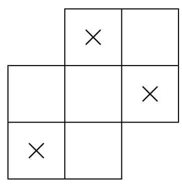


Figure 3.1: $S^{forb}(132)$

The following theorem is a simple consequence of a known result due to Jelínek [4, Lemmas 29 and 30].

Theorem 3.1. *Let S be a skew diagram. Then the number of sparse 0-1-fillings of S avoiding an NE-chain of length 2 and $S^{forb}(132)$ is equal to the number of sparse 0-1-fillings of S avoiding an SE-chain of length 2.*

The goal of this chapter is to prove the following stronger claim.

Theorem 3.2. *Let S be a skew diagram. Then there is a bijection between general 0-1-fillings of S avoiding a NE-chain of length 2 and $S^{forb}(132)$ and general 0-1-fillings of S avoiding a SE-chain of length 2.*

Given a skew diagram S with a total of N cells, we assign labels $c_1, c_2, \ldots, c_N$ to every cell starting from the lower left corner, iterating over rows from the bottom to the top of S and labeling cells in a row from left to right, as indicated in Figure 3.2.

We associate with every cell c a three-part piecewise linear curve $l(c)$ consisting of the ray going from the upper-right corner of c to the left, the right border of c and the ray going from the lower-right corner of c to the right. The curve $l(c_i)$ of a cell c_i of a skew diagram S clearly divides it into two parts, as illustrated in Figure 3.2.

We continue by defining a special set of fillings for every cell of a skew diagram. Let N be the number of cells of a skew diagram S and let i be an integer between 1 and N. We define the set $\mathcal{G}_i(S)$ as the set of all 0-1-fillings of S which satisfy the following conditions:

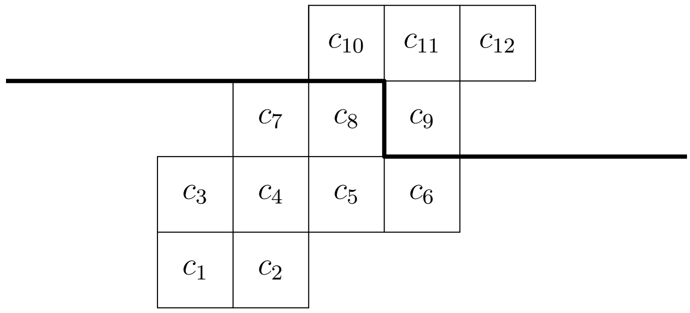


Figure 3.2: A skew diagram with cell labels and the curve $l(c_8)$

(a) there is no occurence of a SE-chain of length 2 with both 1's above $l(c_i)$,

(b) there is no occurence of a SE-chain of length 2 with the upper 1 above $l(c_i)$ and the lower 1 below $l(c_i)$,

(c) there is no occurence of a SE-chain of length 2 with the upper 1 in the same row as c_i and the lower 1 strictly right of c_i,

(d) there is no occurence of a NE-chain of length 2 entirely below $l(c_i)$,

(e) there is no occurence of $S^{forb}(132)$ entirely below $l(c_i)$.

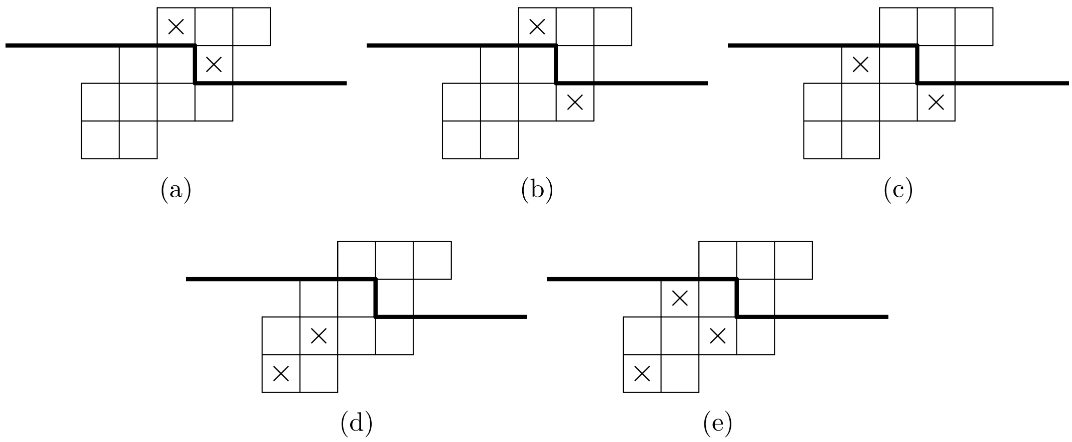


Figure 3.3: Forbidden patterns of $\mathcal{G}_8(S)$

Note that $\mathcal{G}_1(S)$ is the set of all 0-1-fillings of S avoiding a SE-chain of length 2 and $\mathcal{G}_N(S)$ is the set of all 0-1-fillings of S avoiding a NE-chain of length 2 and $S^{forb}(132)$. Therefore, to prove Theorem 3.2 it is enough to construct a chain of bijections between consecutive sets $\mathcal{G}_i(S)$ and $\mathcal{G}_{i+1}(S)$, which is done by the following lemma.

Lemma 3.3. *Let S be a skew diagram with N cells and let $1 \leq i < N$. Then there is a bijection between fillings in $\mathcal{G}_i(S)$ and $\mathcal{G}_{i+1}(S)$.*

Proof. First, consider the case where the cells c_i and c_{i+1} are not in the same row, i.e. c_{i+1} is the first cell of a new row and c_i is the last cell of the previous row. In this case we will show that the sets $\mathcal{G}_i(S)$ and $\mathcal{G}_{i+1}(S)$ are identical.

Choose any filling of S from $\mathcal{G}_i(S)$. Suppose that the selected filling contains an occurence of a pattern forbidden in $\mathcal{G}_{i+1}(S)$. Since $l(c_i)$ and $l(c_{i+1})$ create the same division of S into two parts except for the cell $c_{i+1}(S)$, the occurence of a forbidden pattern must use the cell c_{i+1} filled with a 1, otherwise it would also be forbidden in $\mathcal{G}_i(S)$. Since c_{i+1} starts a row, it cannot be used to break the conditions (a), (b) or (d) of $\mathcal{G}_{i+1}(S)$. Also since the rest of the row starting with c_{i+1} is above $l(c_{i+1})$, the condition (e) cannot be broken either. Therefore the supposed occurence of a forbidden pattern must break the condition (c) of $\mathcal{G}_{i+1}(S)$. However, such an occurence would break the condition (b) of $\mathcal{G}_i(S)$, which is a contradiction, and so the chosen filling is also in $\mathcal{G}_{i+1}(S)$.

Now choose any filling of S from $\mathcal{G}_{i+1}(S)$ and again suppose that it breaks at least one condition of $\mathcal{G}_i(S)$. Using similar arguments as before we can show that this must be the condition (b) and a 1 in the cell c_{i+1} is used as the upper 1 of the SE-chain. Then this SE-chain breaks the condition (c) of $\mathcal{G}_{i+1}(S)$ and the contradiction is reached again, showing that indeed $\mathcal{G}_i = \mathcal{G}_{i+1}$.

The second and more interesting case is when the cells c_i and c_{i+1} are adjacent cells in a row. We will divide each of the sets $\mathcal{G}_i S$ and $\mathcal{G}_{i+1}(S)$ into two disjoint parts and construct bijections between corresponding pairs. Let R be the row of all cells strictly left of c_{i+1}, let C be the column of all cells strictly below c_{i+1} and let A be the rectangle consisting of all cells that are strictly below R and left of C. Note that A, R, C and c_{i+1} form the rectangle M which is the maximal rectangle contained in S with c_{i+1} as its northeast corner. Therefore, in any occurence of a NE-chain with the upper 1 in the cell c_{i+1}, the lower 1 is inside the rectangle A.

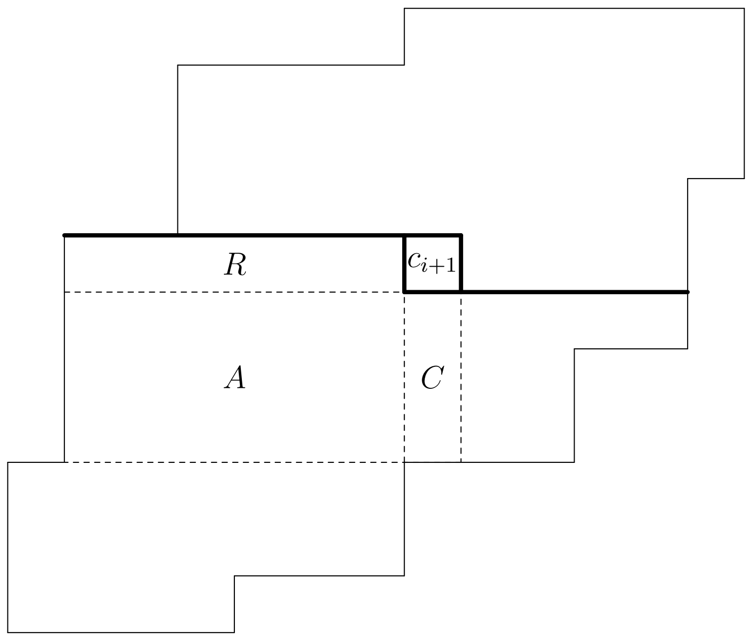


Figure 3.4: The maximal rectangle M with c_{i+1} in the northeast corner

We divide $\mathcal{G}_i(S)$ into two disjoint sets as follows:

- $\mathcal{G}_i^1(S)$ contains the fillings in which there is either no 1 in c_{i+1} or no 1 inside A,
- $\mathcal{G}_i^2(S)$ contains the fillings with a 1 in c_{i+1} and at least one 1 inside A.

We divide $\mathcal{G}_{i+1}(S)$ into five disjoint sets as follows:

- $\mathcal{G}^1_{i+1}(S)$ contains the fillings in which there is either no 1 inside R or no 1 inside C,
- $\mathcal{G}^2_{i+1}(S)$ contains the fillings in which both R and C are nonempty.

First of all we show that, similarly as in the first part of the proof, the sets $\mathcal{G}^1_i(S)$ and $\mathcal{G}^1_{i+1}(S)$ in fact contain the same fillings, so we may use the identity map between them. Choose a filling from $\mathcal{G}^1_i(S)$. Clearly, if c_i and c_{i+1} are adjacent cells, a filling that satisfies the conditions of $\mathcal{G}_i(S)$ also satisfies the conditions (a), (b) and (c) of $\mathcal{G}_{i+1}(S)$. The condition (d) can only be broken by a NE-chain of length 2 with the upper 1 in the cell c_{i+1}, but there is no 1 in that cell in the fillings of $\mathcal{G}^1_i(S)$. Finally, if there is an occurence of $S^{forb}(132)$ in the filling that breaks the condition (e) of $\mathcal{G}_{i+1}(S)$ but not of $\mathcal{G}_i(S)$, it must be the case that the upper right corner of the occurence is the cell c_{i+1} and therefore there is a 1 in R and a 1 in C and so the condition (c) of $\mathcal{G}_i(S)$ is broken. Therefore the selected filling is in $\mathcal{G}^1_{i+1}$. By reverting the arguments we easily get also the opposite inclusion and so $\mathcal{G}^1_i(S) = \mathcal{G}^1_{i+1}(S)$.

For the fillings of $\mathcal{G}^2_i(S)$ we perform the following transformation f into a filling of $\mathcal{G}^2_{i+1}(S)$:

1. Label all nonempty columns of A from left to right as $C_1, C_2, \dots C_k$.
2. If C is nonempty, then R is empty and we move the 1 from the cell c_{i+1} to the cell of R above the column C_1 and finish.
3. If C is empty, then replace the filling of C by the filling of C_k and for $1 \leq j < k$ replace the filling of C_{j+1} by the filling of C_j. Replace the filling of C_1 by all zeros.
4. If there is a 1 in the cell of R above the column C_1, finish if $k = 1$ or move the 1 from c_{i+1} to the cell of R above the column C_2 if $k > 1$.
5. Finally if there is no 1 in the cell of R above the column C_1, move the 1 from c_{i+1} to this cell.

We continue by showing that the result of the transformation f satisfies all five conditions of $\mathcal{G}_{i+1}(S)$.

(a) Since we only modified entries below $l(c_{i+1})$, the condition (a) is satisfied.

(b) The condition (b) could only be broken by an SE-chain with its lower 1 inside A or C, but the upper 1 of this chain would have formed an SE-chain with the 1 originally in the cell c_{i+1}, breaking the condition (a) of $\mathcal{G}_i(S)$. Therefore, the condition (b) is satisfied.

(c) The condition (c) could only be broken by an SE-chain with its upper 1 in R or c_{i+1}, but the lower 1 of this chain together with the 1 originally in c_{i+1} would break the condition (b) of $\mathcal{G}_i(S)$.

(d) The condition (d) could be broken in one of the following ways:

- There is a NE-chain with the upper 1 in R or c_{i+1} and lower 1 in A. After the performed transformation all 1's in R are stricly left of or right above the column C_1, and a 1 remains in c_{i+1} only if A ends up empty, so this case cannot occur.
- There is a NE-chain with the upper 1 in A and lower 1 outside A. In this case suppose that there is a NE-chain with the upper 1 being the lowest 1 in the column C_j. The column C_j was nonempty also before the transformation and it was either the same or it contained the current filling of C_{j+1}, so the lowest 1 in C_j was not higher than after the transformation, therefore there was a NE-chain to begin with, which is a contradiction.
- There is a NE-chain with the upper 1 in C. If the filling of C has not changed, the NE-chain was there before the transformation also. Otherwise C now contains the filling of C_k. If the lower 1 of the NE-chain is strictly left of the column C_k, it forms a NE-chain with a 1 in C_k in the original filling. Otherwise it is below the rectangle A and so it forms a NE-chain with the 1 in the cell c_{i+1} in the original filling.

(e) If the condition (e) is broken, the occurence of $S^{forb}(132)$ must have its upper SE-chain contained in M and the lower 1 is outside of M strictly southwest of it. Let u, v be the two cells inside M and let w be the cell outside M as illustrated in Figure 3.5. Let B be the maximal rectangle contained in S with the cell w as its southwest corner. S Since there is a 1 in the cell w, all entries in the intersection of M and B are zero both before and after the transformation of the filling. The cell u is strictly above the rectangle B and the cell v is strictly right of the rectangle B. Now we observe an important property of the described transformation: any row or column of M was nonempty before the transformation if and only if it is nonempty after the transformation. This is obvious for rows because we shift nonzero entries only in the horizontal direction. If the column C_1 is empty after the transformation, then the cell of R right above C_1 always contains a 1. If the column C is nonempty after the transformation, either it was nonempty before the transformation or there was a 1 in c_{i+1} before the transformation. For other columns the discussion is straightforward. We can use this property to deduce that there was a 1 in the same column of M as u and a 1 in the same row of M as v before the transformation. Since these have to be outside the rectangle B, they form an occurence of $S^{forb}(132)$ together with the cell w in the original filling and a contradiction is reached. Therefore the condition (e) of $\mathcal{G}_{i+1}(S)$ could not have been broken by the transformation either.

We have shown that the transformation f indeed transforms a filling of $\mathcal{G}_i^2(S)$ into a filling of $\mathcal{G}_{i+1}^2(S)$. Next we describe the transformation g transforms a filling of $\mathcal{G}_{i+1}^2(S)$ into a filling of $\mathcal{G}_i^2(S)$. Note that g simply reverts the steps of the transformation f.

1. If there is exactly one entry 1 in R and the column of A below this entry is nonempty, move this entry to c_{i+1} and finish.
2. Otherwise there are either at least two 1's in R or the column below the single entry 1 is empty. In both cases we can choose the rightmost empty

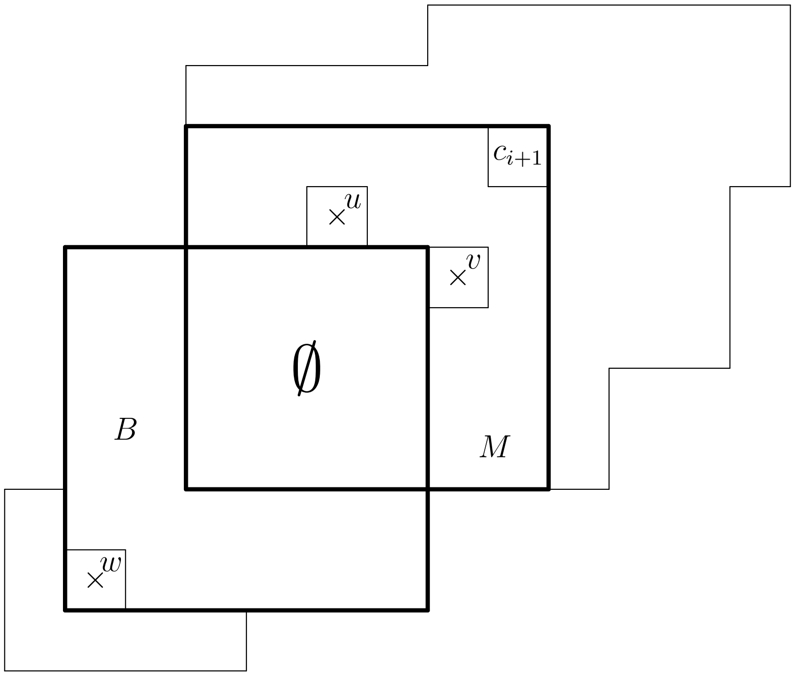


Figure 3.5: An occurence of $S^{forb}(132)$ in the cells u, v and w

column of A such that there is a 1 in R above it. We call this column C_1. Label all nonempty columns of A from left to right as $C_2, C_3, \ldots, C_k$.

3. For each $1 \leq j < k$ copy the filling of C_{j+1} to C_j, copy the filling of C to C_k and replace the filling of C by all zeros.

4. If $k = 1$ finish. If $k > 2$ and there is a 1 in R above C_2, this is the rightmost 1 in R, move it to c_{i+1} and finish.

5. Finally if the entry 1 in R above C_1 is the rightmost 1 in R, move this 1 to c_{i+1}.

Next we choose any filling from $\mathcal{G}_{i+1}(S) \setminus \mathcal{G}^1_{i+1}(S)$, transform it using the described transformation g and show that it satisfies all five conditions for the fillings of $\mathcal{G}_i(S)$.

(a) The condition (a) could only be broken by a SE-chain with the lower 1 in the cell c_{i+1}, but then there is a SE-chain present in the original filling with the same upper 1 and the lower 1 in the column C violating the condition (b) of $\mathcal{G}_{i+1}(S)$, which is not possible.

(b) All 1's that were moved in the transformation were moved to the left except the 1 in the cell c_{i+1} which is above $l(c_i)$. Therefore if the condition (b) is broken after the transformation, it must be broken in the original filling as well.

(c) The condition (c) could only be broken by a SE-chain with the upper 1 in R and the lower 1 in C, since otherwise it would have been in the original filling. But after the transformation, either R or C is empty.

(d) Suppose that after the transformation there is a NE-chain below $l(c_i)$. Then due to the properties of the transformation one of them is inside of M, and the other is outside of M. Label the cell with the 1 inside M as v and the cell outside of M as u. Again we will make use of the fact that emptiness and nonemptiness of rows and columns of M is preserved by the transformation. If the column of S in which lies u intersects M, then the cell in the column of M in which lies v which was nonempty in the original filling creates a NE-chain with u. Similarly if the row of S in which lies u intersects M, then the cell in the row of M in which lies v which was nonempty in the original filling creates a NE-chain with u. Therefore assume that neither the column or the row in which lies u intersect M, thus u lies strictly southwest of M. Consider the maximal rectangle B contained in S with u for its southwest corner. Clearly the intersection of M and B was empty before the transformation and it contains at least the 1 in the cell u now. Let z be the cell in the same column of M as v which was nonempty before the transformation and let w be the cell in the same row of M as v which was nonempty before the transformation. Then both z and w must lie outside of B and therefore they form an occurence of $S^{forb}(132)$ together with u in the original filling.

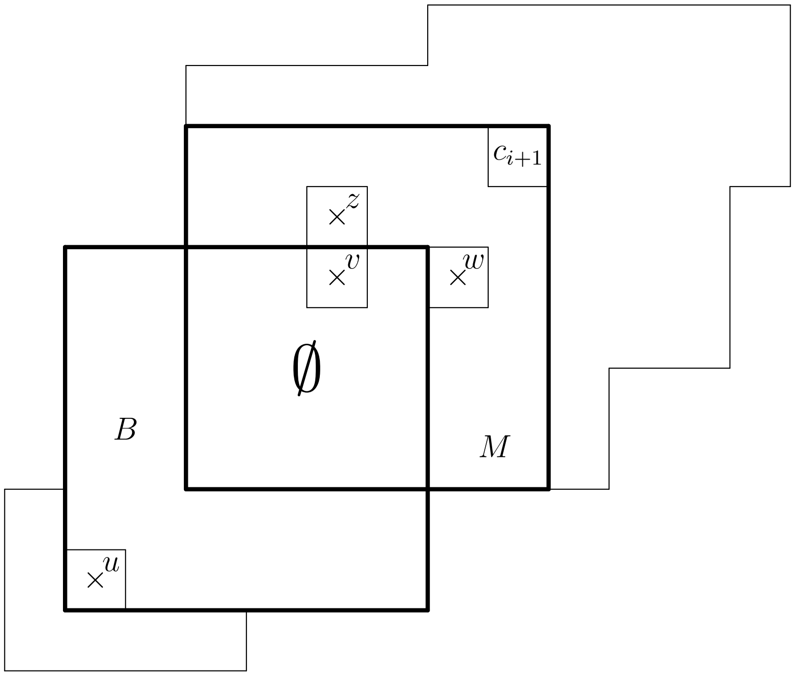


Figure 3.6: An occurence of $S^{forb}(132)$ in the cells u, w and z

(e) The condition (e) can be verified easily using the same approach as in the discussion of the map f.

Overall we have shown that f maps $\mathcal{G}_i^2(S)$ into $\mathcal{G}_{i+1}^2(S)$ and that g maps $\mathcal{G}_{i+1}^2(S)$ into $\mathcal{G}_i^2(S)$. In addition, since the transformations are carefully constructed so that one performs the exact opposite of the other, we get that $fg = \text{id}$ and $gf = \text{id}$, which implies that $g = f^{-1}$ and f is the bijection we were looking for.

□

Proof of Theorem 3.2. Let N be the number of cells of S. From Lemma 3.3 we immediately get that there is a bijection between the fillings of $\mathcal{G}_1(S)$ and $\mathcal{G}_N(S)$. As discussed above, $\mathcal{G}_1(S)$ is exactly the set of fillings avoiding a SE-chain of length 2 and $\mathcal{G}_N(S)$ is exactly the set of fillings avoiding $S^{forb}(132)$ and a NE-chain of length 2, which completes the proof. □

Conclusion

Skew diagrams lack some of the important properties of moon polyominoes (e.g. comparability) which makes the fillings of skew diagrams behave in more complicated ways and thus not as much progress has been made in the past years regarding fillings of skew diagrams. The aim of the present thesis is to extend the current knowledge of skew diagrams and perhaps find new approaches to proving facts about them.

This thesis presents two original results which were the product of the research conducted during author's master studies. In the first half of the thesis, we deal with sparse fillings of skew diagrams and attempt to prove results similar to what is known about sparse fillings of Ferrers diagrams. Theorem 2.2 is a partial result and an initial step towards proving a more general hypothesis, which could be used to prove new enumerative results about singleton classes, similar to the work of Backelin, West and Xin [1].

Hypothesis. *Given any skew diagram S and an integer $l \geq 1$, the number of sparse fillings of S avoiding a NE-chain of length l is greater or equal to the number of sparse fillings of S avoiding a SE-chain of length l.*

We have shown that for many skew diagrams, the inequality holds, even with equality. The author hopes that this work will be useful in the future attempts to prove the general result.

In the second half of the thesis some progress was made considering general 0-1-fillings of skew diagrams instead of just sparse fillings. The proof of Theorem 3.2 is a direct generalisation of Theorem 3.1 and introduces an interesting iterative proof method which might also find its use elsewhere in the field.

Bibliography

[1] J. Backelin, J. West, and G. Xin. Wilf-equivalence for singleton classes. *Advances in Applied Mathematics*, 38:133–148, 2007.

[2] C. Krattenthaler. Growth diagrams, and increasing and decreasing chains in fillings of Ferrers shapes. *Advances in Applied Mathematics*, 37:404–431, 2006.

[3] M. Rubey. Increasing and decreasing sequences in fillings of moon polyominoes. *Advances in Applied Mathematics*, 47:57–87, 2011.

[4] V. Jelínek. *Wilf-Type Classifications.* PhD thesis, Charles University in Prague, 2008.

[5] T. Britz and S. Fomin. Finite posets and Ferrers shapes. *Advances in Mathematics*, 158:86–127, 2001.