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Tomaszewski's conjecture

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Title: Tomaszewski's conjecture

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Abstract: In 1986, Bogusław Tomaszewski asked the following question: Consider n real numbers $a_1, \dots, a_n$ such that the sum of their squares is 1. Of the 2^n expressions $|\varepsilon_1 a_1 + \cdots + \varepsilon_n a_n|$ with $\varepsilon_i = \pm 1$, can there be more with value > 1 than with value ≤ 1? Apart from being of intrinsic interest in probability, an answer to this conjecture would also have applications in quadratic programming. However, even after more than thirty years the conjecture is still unsolved.

In this thesis we settle a special case of the conjecture – we prove that the conjecture holds for vectors of the form $(\alpha, \delta, \dots, \delta)$ of sufficiently large dimension. This generalizes earlier result which showed that the conjecture holds for vectors of the form $(\delta, \dots, \delta)$.

Keywords: Probability, Tomaszewski's cojnecture, Binomial sums

Contents

1. Introduction

In 1986, Bogusław Tomaszewski asked the following question [Guy, 1986]:

> Consider n real numbers $a_1, \ldots, a_n$ such that $\sum_{i=1}^{n} a_i^2 = 1$. Of the 2^n expressions $|\varepsilon_1 a_1 + \cdots + \varepsilon_n a_n|$ with $\varepsilon_i = \pm 1$, *can there be more with value* > 1 *than with value* ≤ 1*?*

It is conjectured that this is not possible.

In mathematics, it is not rare that a short, elementary question can be quite difficult to answer. Indeed, the Tomaszewski's conjecture falls into this category; after more than 30 years, even though it has received considerable attention, it still remains open.

Let us first give a few reformulations of the original question.

1. *Probability*

   Let $a = (a_1, \ldots, a_n)$ be a real vector with $\|a\|_2 = 1$. Furthermore, let $\varepsilon_1, \ldots, \varepsilon_n$ be independent random variables, each attaining value -1 with probability $1/2$ and value 1 with probability $1/2$. Is it true that

   $$\Pr\left[\left|\sum_{i=1}^{n} \varepsilon_i \cdot a_i\right| \leq 1\right] \geq \frac{1}{2}?$$

   One can observe that the lower bound of $1/2$ is the best we can hope for. For example, this bound is attained for vector $(1/\sqrt{2}, 1/\sqrt{2})$.

2. *Sum partitions*

   The conjecture states that there are many ways to partition a sum $\sum_{i=1}^{n} a_i$ into two roughly equal parts. To be more precise, "many" means at least half of all possible partitions and "roughly equal" means differing by at most 1 given the normalization $\sum_{i=1}^{n} a_i^2 = 1$.

3. *Chebyshev type inequality*

   Let $\varepsilon_1, \ldots, \varepsilon_n$ and a be as before and consider a random variable

   $$X = \sum_{i=1}^{n} \varepsilon_i \cdot a_i.$$

   It is easy to see that $\mathbb{E}[X] = 0$ and $\mathrm{Var}[X] = \|a\|_2 = 1$. The conjecture asserts that X lies within one standard deviation of its mean with probability at least $1/2$. For a general random variable with variance 1 this is not true; consider the random variable attaining values $-1, 0, 1$, each with probability $1/3$. However, the statement is true if we consider a linear combination of mutually independent random variables assuming values -1 and 1 with equal probabilities.

   Note that the Chebyshev inequality yields a lower bound of 0 for this probability. The Hoeffding bound produces even "worse" lower bound of

   $$1 - 2e^{-1/2} \approx -0.213.$$

4. *Unit ball and hyperplanes*

   Consider an n-dimensional unit ball centered at the origin and a hypercube $\{-1,1\}^n$. The conjecture states that whenever we have two parallel supporting hyperplanes of the ball, at least half of the points of the hypercube lie between (or possibly on) them. Indeed, if we take a to be the normal vector of one of the two parallel supporting hyperplanes, the condition $|a \cdot x| \leq 1$ holds exactly when the point x lies between those two hyperplanes.

   A close variant of this reformulation is that if we take a hyperplane passing through the origin, at least half of the points of the hypercube lie within distance one of the hyperplane.

5. *Boolean functions*

   Consider a boolean function $f_a(v)$ on $\{-1,1\}^n$ defined as

$$f_a(v) = \begin{cases} 0 & \text{if } |a \cdot v| > 1, \\ 1 & \text{otherwise.} \end{cases}$$

   Then for every a with $\|a\|_2 = 1$ we have

$$\|f_a\|^2 = \frac{1}{2^n} \sum_{v \in \{-1,1\}^n} f_a(v)^2 \leq \frac{1}{2}.$$

   While by itself this formulation does not bring anything new (after all, the function f_a is just an indicator function of sign vectors that make the sum fall into the desired interval), analysis of Boolean functions has powerful tools that can be used to investigate the problem – we name Fourier analysis for one. For more information about these techniques we direct reader to the book by O'Donnell [2014].

   Von Heymann [2010] also mentions percolation in connection with this formulation, though he does not provide any further details. For more information on connection between boolean functions and percolation we refer the reader to the book by Garban and Steif [2015].

All of the reformulations were already known before. The probabilistic one (1) is folklore; it is the closest one to the original formulation and is probably the most frequent formulation used by authors who tackled the Tomaszewski's conjecture. Holzman and Kleitman [1992] mention variants 2, 3, and 4; to our knowledge this was the first published work that considered other equivalent formulations of the Tomaszewski's conjecture. The reformulation 4 was also used by De et al. [2016] to define the Tomaszewski's constant. Finally, the reformulation 5 was suggested by von Heymann [2010].

For the rest of our work we stick with the variant 1.

1.1 Tomaszewski's constant

Definition 1 (Tomaszewski's constant)**.** For $w \in \mathbb{R}^n$ and $M \subseteq \mathbb{R}$, we define

$$\mathbf{T}(w, M) = \Pr_{\varepsilon \in \{-1,1\}^n}[|w \cdot \varepsilon| \in M].$$

For $W \subseteq \mathbb{R}^n$ and $M \subseteq \mathbb{R}$, we set

$$\mathbf{T}(W, M) = \inf_{w \in W} \mathbf{T}(w, M).$$

Finally, for a sequence $(W_n)_{n=1}^{\infty}$, where each W_n is a subset of $\mathbb{R}^n$, we set

$$\mathbf{T}\Big((W_n)_{n=1}^{\infty}, M\Big) = \inf_{i \in \mathbb{N}} \mathbf{T}(W_i, M).$$

Our interest lies in vectors of unit norm. Let us set $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1\}$ and let $\mathbb{S}$ be the sequence of all $\mathbb{S}^n$. The *Tomaszewski's constant* is defined as $\mathbf{T}(\mathbb{S}, [-1, 1])$; we denote it by $\mathbf{T}$ for short. The conjecture states that $\mathbf{T} = \frac{1}{2}$.

At first glance it is not obvious even whether the constant is positive, though this can be readily shown by the second moment method and is merely an exercise [Alon and Spencer, 2016, Exercise 4.8.2. on p. 64].

1.2 Known results

The partial results can be mainly broken into two types:

- showing weaker lower bound than 1/2,
- showing bound of 1/2 but for restricted set of vectors (i.e., not for every a with $\|a\|_2 = 1$, but only a's of particular form).

1.2.1 Weaker bounds

One of the first results of this type is due to Holzman and Kleitman [1992]; it proves the lower bound of 3/8 on the Tomaszewski's constant. In fact, their results states

$$\Pr[|\varepsilon \cdot a| < 1] \geq \frac{3}{8},$$

provided $a_i < 1$ for each i. This lower bound was the best known until recently. Boppana and Holzman [2017] have proven lower bound of $13/32 = 3/8 + 1/12$; in the same paper they show how to improve this bound to $13/32 + 9 \cdot 10^{-6}$.

Ben-Tal et al. [2002], apparently unaware of the result of Holzman and Kleitman [1992], have shown the lower bound of 1/3 and independently conjectured that the constant is 1/2. Shnurnikov [2012] have proven the lower bound of 0.36, slightly weaker than bound of Holzman and Kleitman [1992], but using different approach. Even though both of these results are weaker than original result of Holzman and Kleitman [1992], their methods were used to improve the result to current best known bound of Boppana and Holzman [2017].

1.2.2 Results for restricted set of vectors

Van Zuijlen [2011] has shown that the conjecture holds for all vectors of the form $(1/\sqrt{n}, \ldots, 1/\sqrt{n})$. Apart from the lower bound of 1/2 the paper also uses the Central Limit Theorem to show that if n goes to infinity, the lower bound goes to $\Phi(1) - \Phi(-1) \approx 0.68$. The result was later generalized by Hendriks and van Zuijlen [2017a] by showing tight bounds on $\mathbf{T}((1/\sqrt{n}, \ldots, 1/\sqrt{n}), [-\xi, \xi])$ for several values of ξ.

The conjecture was also confirmed for small dimension; Hendriks and van Zuijlen [2017b] proved that the conjecture holds for all vectors of dimension up to 9.

Furthermore, Dzindzalieta [2014b] have shown that the conjecture holds if $|a_j| \leq 0.16$ for all j using an appropriate variant of central limit theorem.

1.2.3 Other results

One result that falls outside of the two categories is rather surprising. De et al. [2016] constructed an algorithm that returns an approximation to the Tomaszewski's constant with additive error. More precisely, for every $\varepsilon > 0$ the algorithm returns a value between $\mathbf{T}$ and $\mathbf{T} + \varepsilon$. However, the running time of the algorithm is $2^{\Theta(1/\varepsilon^6)}$. Even if the hidden constant in Θ was 1 (careful reading shows that it is actually higher), the running time for $\varepsilon = 1/10$ would be $2^{1000000}$; a time way beyond capabilities of current computers. This precludes the use of the algorithm as a tool to obtain better results than currently known.

Even if we were able to run such an algorithm, the answer would be a little lacking. While we would obtain a better bound than we are able to prove, the algorithm would give little to no indication *why* does such a bound hold. Still, the methods used are noteworthy; since the Tomaszewski's constant is defined as an infimum over an infinite set, it is a priori not apparent whether an algorithm for computing the constant (even if approximately) exists at all.

Derinkuyu [2004] showed that if ε is taken uniformly at random from the sphere $\{\varepsilon \in \mathbb{R}^n \mid \|\varepsilon\|_2 = \sqrt{n}\}$ then we have

$$\Pr[|a \cdot \varepsilon| \leq 1] \geq \frac{1}{2}.$$

1.3 Applications

The Tomaszewski's constant occurs surprisingly frequently in optimization and operation research. Though in optimization literature it rarely bears Tomaszewski's name and its precise value is often not needed, good lower bounds are of interest. For example, Ben-Tal et al. [2002] used a lower bound of 1/3 as one of the ingredients in the proof of the *approximate S-lemma.*

S-lemma allows us to convert a quadratic program with a single quadratic constraint into a semidefinite program. This is not in general possible for quadratic programs with more quadratic constraints, yet still the program can be at least approximated by a semidefinite program; this is the content of approximate S-lemma. A more detailed treatment of the S-lemma is beyond the scope of this thesis. We refer the reader to work of Ben-Tal et al. [2002] or the survey by Derinkuyu and Pınar [2006].

2. Preliminaries

In this chapter we introduce notation used throughout the thesis.

If a and b are two vectors in $\mathbb{R}^n$ then $a \cdot b$ denotes their *standard inner product* which is defined by

$$a \cdot b = \sum_{j=1}^{n} a_j b_j.$$

The Euclidean norm (also called ℓ_2 norm) of a vector $a \in \mathbb{R}^n$ is denoted by $\|a\|_2$ and is defined by

$$\|a\|_2 = \sqrt{a \cdot a} = \left(\sum_{j=1}^{n} a_j^2 \right)^{1/2}.$$

We denote by $\mathbb{S}^n$ the *n-dimensional unit sphere*, that is,

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1\}.$$

The Gamma function is defined for $x > 0$ as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \, \mathrm{d}t.$$

It is well known that $\Gamma(n) = (n-1)!$ for any positive integer n.

We assume that the reader is already familiar with the basics of probability theory. We now define functions involving the standard normal distribution, as their notation may vary across the literature.

The *probability density function* of the standard normal distribution is denoted by ϕ and is defined as

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

The *cumulative density function* of the standard normal distribution is denoted by Φ and is defined as

$$\Phi(x) = \int_{-\infty}^{x} \phi(t) \, \mathrm{d}t = \Pr[N \leq x].$$

Here N denotes the random variable with the standard normal distribution.

The *complementary cumulative density function* of the standard normal distribution is denoted by Q and defined as

$$Q(x) = 1 - \Phi(x) = \int_{t}^{\infty} \phi(t) \, \mathrm{d}t = \Pr[N \geq x].$$

Finally, we denote the *Mills ratio* of the standard normal distribution by Y. The ratio is defined by the formula

$$Y(x) = \frac{Q(x)}{\phi(x)}.$$

3. Almost uniform case

One of the important cases of the Tomaszewski's conjecture that has been studied before is the *uniform case.* In uniform case we consider only vectors of the form $a = (\delta, \ldots, \delta)$, that is, all entries of the vectors must be the same. Somewhat surprisingly, showing that the Tomaszewski's conjecture holds even for this simple restriction is not trivial. Nevertheless, this case has been resolved by van Zuijlen [2011].

One natural approach for solving the general case of the conjecture might be:

(1) Show that the conjecture holds for the uniform case.

(2) Show that we can transform an arbitrary vector a into a vector a' that is"more uniform" in such a way that $\Pr[|a \cdot \varepsilon| \leq 1] \geq \Pr[|a' \cdot \varepsilon| \leq 1]$. Here, more uniform means that after a finite number of transformations we get a uniform vector.

The step 1 was already resolved as mentioned before. Yet one quickly realizes that the step 2 is hopeless; uniform vectors do not attain the conjectured bound $\frac{1}{2}$ (apart from dimensions 1 and 2). The uniform vectors do not even approach the bound $\frac{1}{2}$. If we denote by $u^{(n)}$ the vector $(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$ then we have even

$$\lim_{n \to \infty} \mathbf{T}(u^{(n)}, [-1, 1]) = \Phi(1) - \Phi(-1) \approx 0.68,$$

by the Central Limit Theorem (see e.g. [Hendriks and van Zuijlen, 2017a] for details). Therefore, it is unlikely that the case for uniform vectors can serve as a "base case" for proving the conjecture in general.

To address this shortcoming, we propose the *almost uniform* case.

Definition 2. A vector $a = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$ is *almost uniform* if it satisfies $a_1 = a_2 = \cdots = a_n$.

Note that we number the entries from zero. This is purely for notational convenience.

Intuitively, one should think that a_0 is large while $a_1, \ldots, a_n$ are small, even though the definition does not require that $|a_0| \geq |a_1|$. Indeed, this is the interesting case; the case where all entries of the vector are small follows from an appropriate variant of the central limit theorem, as pointed out by Dzindzalieta [2014a] (the result is not formally stated as a proposition, see p. 404 and equations (2.3), (2.4) p. 406 in [Dzindzalieta, 2014a]).

To show that the almost uniform case does not have the aforementioned drawback of the uniform case we prove that there exist almost uniform vectors of arbitrarily large dimension that attain the conjectured bound $\frac{1}{2}$.

Proposition 3. *For every odd positive integer n there exists an almost uniform vector $a \in \mathbb{S}^n$ that attains the conjectured Tomaszewski's constant, i.e.,*

$$\mathbf{T}(a, [-1, 1]) = \frac{1}{2}$$

Proof. Suppose we have a vector $a = (a_0, \delta, \dots, \delta)$ with both a_0 and δ positive. If n is odd then

$$\left|\sum_{j=1}^{n} \varepsilon_j a_j\right| = \left|\sum_{j=1}^{n} \varepsilon_j \delta\right| \geq \delta.$$

If $a_0 + \delta > 1$ then the vector a attains the conjectured Tomaszewski's constant; no matter how are the signs $\varepsilon_1, \dots, \varepsilon_n$ chosen, setting $\varepsilon_0 = \operatorname{sgn}\left(\sum_{j=1}^{n} \varepsilon_j \delta\right)$ makes the sum fall outside of the interval $[-1, 1]$. Therefore, for at least for half of all choices of $\varepsilon \in \{-1, 1\}^{n+1}$, we have $|a \cdot \varepsilon| > 1$.

Since a has unit ℓ_2 norm, we have $a_0 = \sqrt{1 - n\delta^2}$. Substituting into $a_0 + \delta > 1$, we obtain

$$\begin{aligned}
\sqrt{1 - n\delta^2} + \delta &> 1, \\
\sqrt{1 - n\delta^2} &> 1 - \delta, \\
1 - n\delta^2 &> 1 - 2\delta + \delta^2, \\
-\delta(n+1) &> -2, \\
\delta &< \frac{2}{n+1}.
\end{aligned}$$

Hence, for any choice $\delta < \frac{2}{n+1}$, the vector $(\sqrt{1 - n\delta^2}, \delta, \dots, \delta)$ attains the conjectured bound $\frac{1}{2}$. □

We mentioned before that the interesting case is the one with a_0 large. However, the case becomes easy when a_0 is *too large*. This is formally stated in the next proposition.

Proposition 4. *Let $a = (a_0, \dots, a_n)$ be a vector of unit ℓ_2 norm satisfying $a_0 \geq \dots \geq a_n \geq 0$. If $a_0 \geq \frac{n-1}{n+1}$ then $\mathbf{T}(a, [-1, 1]) \geq \frac{1}{2}$, i.e., vector a is not a counterexample to Tomaszewski's conjecture.*

Proof. We aim to show that

$$\sum_{j=1}^{n} a_j \leq 1 + a_0.$$

If this condition holds, then for every choice of $\varepsilon_1, \dots, \varepsilon_n$, there is at least one choice of ε_0 such that $|a \cdot \varepsilon| \leq 1$. Denote by $\bar{a}$ the vector $(a_1, \dots, a_n)$ and by $\mathbf{1_n}$ the vector $(1, \dots, 1) \in \mathbb{R}^n$. Then we have

$$\sum_{j=1}^{n} a_j = \mathbf{1_n} \cdot \bar{a} \leq \|\mathbf{1_n}\|_2 \|\bar{a}\|_2 = \sqrt{n}\|\bar{a}\|_2 = \sqrt{n}\sqrt{1 - a_0^2}.$$

Therefore, it is enough to show $\sqrt{n}\sqrt{1 - a_0^2} \leq 1 + a_0$.

From the assumption $a_0 \geq \frac{n-1}{n+1}$, we have

$$\begin{aligned}
\sqrt{n}\sqrt{1 - a_0^2} &\leq \sqrt{n}\sqrt{1 - \left(\frac{n-1}{n+1}\right)^2} = \sqrt{n}\sqrt{\frac{4n}{(n+1)^2}} \\
&= \frac{2n}{n+1} = 1 + \frac{n-1}{n+1} \leq 1 + a_0.
\end{aligned}$$

□

For almost uniform vectors this can be rephrased in the following form.

Corollary 5. *Let $a = (a_0, \delta, \ldots, \delta) \in \mathbb{R}^{n+1}$ be an almost uniform vector of unit ℓ_2 norm with all entries positive. If $\delta \leq \frac{2}{n+1}$ then $\mathbf{T}(a, [-1, 1]) \geq \frac{1}{2}$.*

Proof. Since a has unit norm the relation between a_0 and δ is given by

$$\delta = \sqrt{\frac{1 - a_0^2}{n}}.$$

By assumption of the proposition we have $\delta \leq \frac{2}{n+1}$ which gives

$$\begin{aligned}
\frac{2}{n+1} &\geq \sqrt{\frac{1 - a_0^2}{n}}, \\
\frac{4n}{(n+1)^2} &\geq 1 - a_0^2, \\
a_0^2 &\geq 1 - \frac{4n}{(n+1)^2} = \frac{(n+1)^2 - 4n}{(n+1)^2}, \\
a_0^2 &\geq \frac{(n-1)^2}{(n+1)^2}, \\
a_0 &\geq \frac{n-1}{n+1}.
\end{aligned}$$

The result follows from Proposition 4. □

4. Tools

In this chapter we present the tools required to prove our main theorem

4.1 Littlewood–McKay bounds

The following bound is originally due to Littlewood [1969]. However, Littlewood's proof contained several errors, which were subsequently corrected by McKay [1989].

The bound is formulated as a bound on tail of binomial distribution.

Proposition 6 (Theorem 2, p. 2, [McKay, 1989])**.** *Let p be a real number with $0 \le p \le 1$ and let n, k be positive integers satisfying $pn \le k \le n$. Further define*

$$\begin{aligned} \sigma &= \sqrt{np(1-p)}, \\ x &= \frac{k - pn}{\sigma}, \\ b(k; n, p) &= \binom{n}{k} p^k (1-p)^{n-k}, \\ B(k; n, p) &= \sum_{j=k}^{n} b(k; n, p). \end{aligned}$$

Then we have

$$B(k; n, p) = 2\sigma \binom{n-1}{k-1} Y(x) \exp(E(k; n, p)/\sigma),$$

where

$$0 \le E(k; n, p) \le \min\left\{\sqrt{\frac{\pi}{8}}, \frac{1}{x}\right\}.$$

(Recall the definition of Y from Chapter 2.)

The close relationship between $B(k; n, p)/b(k; n, p)$ and $Y(x)$ should not be surprising, given the central limit theorem. What makes this proposition useful for our purpose is the extremely tight error bound.

By setting $p = 1/2$, it is easy to turn the last proposition into a bound on a sum of binomial coefficients.

Corollary 7. *Let n, k be positive integers such that $\frac{n}{2} < k \le n$. We have*

$$\sum_{j=k}^{n} \binom{n}{j} \le \sqrt{n} \binom{n-1}{k-1} Y(x) \exp\left(\frac{2}{2k-n}\right),$$

where $x = \frac{2k-n}{\sqrt{n}}$.

4.2 Precise Stirling bounds

We also use a variant of Stirling bound on the factorial number with explicit error bounds. This bound is due to Maria [1972], who improved upon earlier result by Robbins [1955].

Proposition 8. *For any positive integer n we have*

$$n! = \sqrt{2\pi n}\,(n/e)^n \cdot \exp(r_n),$$

where

$$\frac{1}{12n + \frac{3}{2(2n+1)}} \leq r_n \leq \frac{1}{12n}.$$

This bound can be turned into slightly weaker but more convenient form.

Corollary 9. *For any positive integer n we have*

$$n! = \sqrt{2\pi n}\,(n/e)^n \cdot \exp(r_n),$$

where

$$\frac{1}{12n} - \frac{1}{192n^3} \leq r_n \leq \frac{1}{12n}.$$

Proof. It is enough to show that

$$\frac{1}{12n + \frac{3}{2(2n+1)}} \geq \frac{1}{12n} - \frac{1}{192n^3}.$$

First, we establish for every real number a and $\Delta > 0$ the inequality

$$\frac{1}{a+\delta} = \frac{1}{a} \cdot \left(\frac{1}{1+\frac{\Delta}{a}}\right) \geq \frac{1}{a} \cdot \left(1 - \frac{\Delta}{a}\right) = \frac{1}{a} - \frac{\Delta}{a^2},$$

where the inequality used was $1/(1+x) \geq (1-x)$, which holds for every $x > -1$.

Using the inequality, we get

$$\frac{1}{12n + \frac{3}{2(2n+1)}} \geq \frac{1}{12n} - \frac{\frac{3}{2(2n+1)}}{(12n)^2} = \frac{1}{12n} - \frac{1}{96n^2(2n+1)} \geq \frac{1}{12n} - \frac{1}{192n^3}.$$

□

Even though the proof of Maria [1972] is formulated for the factorial function, we remark that the bound also holds for the Gamma function and non-integral values. For the proof of such bound see Mortici [2011].

Finally, we can apply this estimate to obtain an upper bound on binomial coefficients.

Corollary 10. *For every integers n, k with $0 < k < n$, we have*

$$\binom{n}{k} \leq \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n}{k(n-k)}} \left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k}$$
$$\cdot \exp\left(\frac{1}{12n} - \frac{1}{12k} - \frac{1}{12(n-k)} + \frac{1}{192k^3} + \frac{1}{192(n-k)^3}\right).$$

5. Main Theorem

In this chapter, we prove the conjecture for almost all cases of the almost uniform vectors – that is when the dimension of vector $a = (a_0, \delta, \ldots, \delta)$ is sufficiently large, where sufficiently large depends on the value of a_0.

Theorem 11 (Main theorem, asymptotic form)**.** *Let a_0 be a real number with $0 \leq a_0 < 1$. There exists an integer n_0 such that for every $n \geq n_0$ the vector $a^{(n)}$ defined as*

$$a^{(n)} = \left(a_0, \sqrt{\frac{1-a_0^2}{n}}, \ldots, \sqrt{\frac{1-a_0^2}{n}}\right)$$

satisfies $\mathbf{T}(a^{(n)}, [-1, 1]) \geq \frac{1}{2}$.

In other words, the theorem states that every almost uniform vector of sufficiently large dimension is not a counterexample to the Tomaszewski's conjecture.

Proof. Let a_0 be fixed as in the statement and let $a^{(n)}$ be the given vector (we show the existence of n_0 later in the proof). Furthermore, let $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ denote a random vector, where each entry attains value -1 or $+1$ with equal probability, independent of every other entry.

To simplify notation, we introduce a few shorthands (significance of some of them will become apparent later):

$$\begin{aligned}
\alpha &:= a_0 \\
\delta &:= a_1 = \cdots = a_n = \sqrt{\frac{1-a_0^2}{n}}, \\
A_\alpha &:= \frac{1-a_0}{\sqrt{1-a_0^2}}, \\
B_\alpha &:= \frac{1+a_0}{\sqrt{1-a_0^2}}.
\end{aligned}$$

Observe the following properties

- $A_\alpha \in (0, 1)$,
- $B_\alpha \in (1, \infty)$, and
- $A_\alpha \cdot B_\alpha = 1$.

We need to show that $\Pr[|a \cdot \varepsilon| \leq 1] \geq \frac{1}{2}$. Due to the symmetry, we can without loss of generality assume that $\varepsilon_0 = 1$. We use this assumption throughout the chapter.

The proof can be split into three parts:

(1) Showing that $\Pr[|a \cdot \varepsilon| > 1]$ is equal to $2^{-n}(S + S')$, where S and S' are sums of binomial coefficients, and

(2) Showing that $2^{-n}(S+S')$ can be bounded from above by

$$\Big(Q(c)+Q(1/c)\Big)(1+o(1))$$

for a particular choice of $c \in (0,1)$ which depends on α only.

(3) Showing that $Q(c)+Q(1/c) < 1/2$ for every $c \in (0,1)$.

The part (1) is shown in the next lemma.

Lemma 12. *Denote by U_n the event that $a^{(n)} \cdot \varepsilon > 1$ and by L_n the event that $a^{(n)} \cdot \varepsilon < -1$.*

Then we have

$$\Pr[U_n] = 2^{-n} \cdot \sum_{j=k_{\min}}^{n} \binom{n}{j},$$

where $k_{\min} = \lfloor \frac{n}{2} + \frac{\sqrt{n}}{2} A_\alpha \rfloor + 1$, and

$$\Pr[L_n] = 2^{-n} \cdot \sum_{j=l_{\min}}^{n} \binom{n}{j},$$

where $l_{\min} = \lfloor \frac{n}{2} + \frac{\sqrt{n}}{2} B_\alpha \rfloor + 1$, and

Proof. Due to our choice of ε_0, we see that the event U_n happens if and only if

$$\sum_{j=1}^{n} a_j \varepsilon_j > 1 - a_0.$$

Denote by k' the number of positive ε_j's with $1 \le j \le n$. Since all a_j are of the value δ, the condition can be rewritten to

$$\begin{aligned}
k'\delta + (n-k)(-\delta) &> 1 - a_0 \\
2k'\delta &> 1 - a_0 + n\delta \\
k' &> \frac{n}{2} + \frac{1-a_0}{2\delta} \\
k' &> \frac{n}{2} + \frac{1-a_0}{2\sqrt{\frac{1-a_0^2}{n}}} \\
k' &> \frac{n}{2} + \frac{\sqrt{n}}{2} \cdot \frac{1-a_0}{\sqrt{1-a_0^2}} = \frac{n}{2} + \frac{\sqrt{n}}{2} A_{a_0}
\end{aligned}$$

Therefore, event U_n occurs if and only if the number of ε_j's with value 1 is strictly greater than

$$\frac{n}{2} + \frac{\sqrt{n}}{2} A_{a_0}.$$

The number $k_{\min}$ is the smallest integer larger than this value.

The number of all choices of $\varepsilon_1, \ldots, \varepsilon_n$ is 2^n and by our argument the number of choices for which the event U_n occurs is

$$\sum_{j=k_{\min}}^{n} \binom{n}{j}.$$

This concludes the proof of the first part of the lemma.

Second part is proven analogously. The event L_n occurs if and only if

$$\sum_{j=1}^{n} a_j \varepsilon_j < -1 - a_0.$$

If we denote by l' the number of negative ε_j with $1 \leq j \leq n$, by the same reasoning we obtain that the condition is equivalent to

$$l' > \frac{n}{2} + \frac{\sqrt{n}}{2} B_{a_0},$$

and hence the event L_n occurs for

$$\sum_{j=l_{\min}}^{n} \binom{n}{j},$$

choices of $(\varepsilon_1, \ldots, \varepsilon_n)$. Again, there are a total of 2^n choices of $(\varepsilon_1, \ldots, \varepsilon_n)$, so the probability of L_n is

$$2^{-n} \cdot \sum_{j=l_{\min}}^{n} \binom{n}{j},$$

as claimed in the statement of the lemma. □

We now get to the most technical part of the proof – part (2).

Lemma 13. *Let c be a real number such that $0 < c \leq \sqrt{n}$. Then we have*

$$\sum_{j=q'}^{n} \binom{n}{j} \leq Q(c) 2^n \left(1 + \frac{2c}{\sqrt{n} - c}\right)^{1/2} \left(1 - \frac{c^2}{n}\right)^{-\frac{c^2}{2}} E(q, n),$$

where

$$q = \frac{n}{2} + \frac{\sqrt{n}}{2} c,$$
$$q' = \lfloor q \rfloor + 1,$$

and

$$E(q, n) = \exp\left(\frac{2}{2q - n} + \frac{1}{12n} - \frac{1}{12q} - \frac{1}{12(n-q)} + \frac{1}{192q^3} + \frac{1}{192(n-q)^3}\right).$$

Note that $E(q, n)$ is simply the product of the error terms from Corollaries 7 and 9.

Proof. To show this lemma we use the tools presented in Section 4. Using the McKay's bound, we have

$$\sum_{j=q'}^{n} \binom{n}{j} \leq \sqrt{n} Y(x) \binom{n-1}{q'-1} E_{McK}(q', n),$$

where $E_{McK}(q', n) = \exp\left(\frac{2}{2k-n}\right)$ is the error term from the McKay bound (Corollary 7) and x is defined as in the McKay bound, i.e., $x = (q' - n/2)/\sigma = (2q' - n)/\sqrt{n}$.

Now notice that the expressions are decreasing in q', since c is positive. Therefore if we replace every occurrence of q' by q, we get an upper bound. Strictly speaking, this is not valid because binomial coefficients are defined only for integer values. However we can extend the definition of the binomial coefficients by using Gamma function, setting

$$\binom{n}{q} = \frac{\Gamma(n+1)}{\Gamma(q+1)\Gamma(n-q+1)}.$$

Expanding further and using Stirling bounds, we obtain

$$\begin{aligned} &\sqrt{n}Y(x)\frac{q}{n}\binom{n}{q}E_{McK}(q,n) \\ &= \sqrt{n}Y(x)\frac{q}{n}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{q(n-q)}}\left(\frac{n}{q}\right)^q\left(\frac{n}{n-q}\right)^{n-q} \\ &\quad \cdot E_{St}(q,n)\cdot E_{McK}(q,n), \end{aligned}$$

where $E_{St}(q,n)$ is the error term from Stirling bound (Corollary 9).

We have the equality

$$x = (2q-n)/\sqrt{n} = c,$$

so $Y(x)$ can be replaced by $Y(c)$. We also denote the product of both error terms as E. Thus, we arrive at

$$\frac{Y(c)}{\sqrt{2\pi}}\sqrt{n}\frac{q}{n}\sqrt{\frac{n}{q(n-q)}}\left(\frac{n}{q}\right)^q\left(\frac{n}{n-q}\right)^{n-q}E(q,n). \tag{5.1}$$

We now show how to bound the expression

$$\left(\frac{n}{q}\right)^q\left(\frac{n}{n-q}\right)^{n-q}.$$

Substituting q into the expression, we get

$$\begin{aligned} \left(\frac{n}{q}\right)^q\left(\frac{n}{n-q}\right)^{n-q} &= \left(\frac{n}{\frac{n}{2}+\frac{\sqrt{n}}{2}c}\right)^{\frac{n}{2}+\frac{\sqrt{n}}{2}c}\left(\frac{n}{\frac{n}{2}-\frac{\sqrt{n}}{2}c}\right)^{\frac{n}{2}-\frac{\sqrt{n}}{2}c} \\ &= \left(\frac{n^2}{\frac{n^2}{4}-\frac{n}{4}c^2}\right)^{n/2}\left(\frac{n-\sqrt{n}c}{n+\sqrt{n}c}\right)^{\frac{\sqrt{n}}{2}c} \\ &= \left(\frac{4}{1-\frac{c^2}{n}}\right)^{n/2}\left(\frac{\sqrt{n}-c}{\sqrt{n}+c}\right)^{\frac{\sqrt{n}}{2}c} \\ &\le \left(\frac{4}{1-\frac{c^2}{n}}\right)^{n/2}e^{-c^2} = 2^n e^{-c^2}\left(1-\frac{c^2}{n}\right)^{-n/2}, \end{aligned}$$

using the inequality given by Lemma 15.

From the well-known inequality $(1-\frac{1}{n})^{n-1} \ge e^{-1}$ we get the inequality $\left(1-\frac{c}{n}\right)^{n-c} \ge e^{-c}$. Using the latter inequality, we finally bound the term

$\left(1-\frac{c^2}{n}\right)^{n/2}$:

$$\left(1-\frac{c^2}{n}\right)^{n/2} = \left(\left(1-\frac{c^2}{n}\right)^{n-c^2}\left(1-\frac{c^2}{n}\right)^{c^2}\right)^{1/2} \geq \left(e^{-c^2}\left(1-\frac{c^2}{n}\right)^{c^2}\right)^{1/2}$$
$$= e^{-\frac{c^2}{2}}\left(1-\frac{c^2}{n}\right)^{\frac{c^2}{2}}.$$

Combining the last two bounds we obtain

$$\left(\frac{n}{q}\right)^q\left(\frac{n}{n-q}\right)^{n-q} \leq 2^n e^{-\frac{c^2}{2}}\left(1-\frac{c^2}{n}\right)^{-\frac{c^2}{2}}.$$

Substituting into (5.1) we get

$$\frac{Y(c)}{\sqrt{2\pi}}\sqrt{n}\frac{q}{n}\sqrt{\frac{n}{q(n-q)}}2^n e^{-\frac{c^2}{2}}\left(1-\frac{c^2}{n}\right)^{-\frac{c^2}{2}} E(q,n). \tag{5.2}$$

Now using the definition $Y(c) = Q(c)/\phi(c) = Q(c)e^{c^2/2}\sqrt{2\pi}$ we can further simplify to

$$Q(c)\sqrt{n}\frac{q}{n}\sqrt{\frac{n}{q(n-q)}}2^n\left(1-\frac{c^2}{n}\right)^{-\frac{c^2}{2}} E(q,n)$$
$$= Q(c)\sqrt{\frac{q}{n-q}}2^n\left(1-\frac{c^2}{n}\right)^{-\frac{c^2}{2}} E(q,n).$$

As the last step, we can rewrite $\frac{q}{n-q}$ to

$$\frac{q}{n-q} = \frac{n+\sqrt{n}c}{n-\sqrt{n}c} = 1+\frac{2c}{\sqrt{n}-c}.$$

which leads to

$$Q(c)2^n\left(1+\frac{2c}{\sqrt{n}-c}\right)^{1/2}\left(1-\frac{c^2}{n}\right)^{-\frac{c^2}{2}} E(q,n).$$

□

The last expression is clearly of order $Q(c)2^n(1+o(1))$.

We can now use the last lemma to bound the probabilities of U_n and L_n. We still need to check the assumptions of the lemma. In the case of U_n we invoke the lemma with $c = A_\alpha \in (0,1)$, so the assumption always holds. In the case of bounding L_n we have to be more careful, as B_α may be arbitrarily large. However, if $B_\alpha^2 > n$, then we have

$$\begin{aligned} B_\alpha^2 = \frac{(1+a_0)^2}{1-a_0^2} &> n,\\ \frac{1+a_0}{1-a_0} &> n,\\ 1+a_0 &> n-na_0,\\ a_0(n+1) &> n-1,\\ a_0 &> \frac{n-1}{n+1}. \end{aligned}$$

But in this case, the conjecture holds trivially due to Proposition 4. Therefore, we can safely assume that $B_\alpha^2 \leq n$ and the assumption of the lemma is satisfied.

Applying the lemma, we get

$$\Pr[U_n] = 2^{-n} \sum_{j=k_{\min}}^{n} \binom{n}{j} \leq Q(A_\alpha) \left(1 + \frac{2A_\alpha}{\sqrt{n} - A_\alpha}\right)^{1/2} \left(1 - \frac{A_\alpha^2}{n}\right)^{-\frac{A_\alpha^2}{2}} E(q, n)$$

and

$$\Pr[L_n] = 2^{-n} \sum_{j=l_{\min}}^{n} \binom{n}{j} \leq Q(B_\alpha) \left(1 + \frac{2B_\alpha}{\sqrt{n} - B_\alpha}\right)^{1/2} \left(1 - \frac{B_\alpha^2}{n}\right)^{-\frac{B_\alpha^2}{2}} E(q, n).$$

We have already observed that $B_\alpha = 1/A_\alpha$. All that remains is to prove that $Q(x) + Q(1/x) < \frac{1}{2}$.

Lemma 14. *Let $x \in (0, \infty)$ be a real number. We have*

$$Q(x) + Q\left(\frac{1}{x}\right) < \frac{1}{2}.$$

Proof. Due to symmetry it is enough to show the statement for $x \in (0, 1]$.

Define the function F as $F(x) = Q(x) + Q\left(\frac{1}{x}\right)$. To prove the statement we show

(i) $\lim_{x \to 0^+} F(x) \leq \frac{1}{2}$, and

(ii) $F'(x) < 0$ for every $x \in (0, 1)$.

Statement (i) is trivial. To prove (ii) we start with

$$\begin{aligned} F'(x) = Q'(x) + \big(Q(1/x)\big)' &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \left(-\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2x^2}} \left(-\frac{1}{x^2}\right)\right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{x^2} e^{-\frac{1}{2x^2}} - e^{-\frac{x^2}{2}}\right). \end{aligned}$$

Therefore, it suffices to show

$$e^{-\frac{1}{2x^2}} < x^2 e^{-\frac{x^2}{2}}.$$

By taking logarithm of both sides and rearranging, we get equivalent inequality

$$\frac{x^2}{2} - \frac{1}{2x^2} < 2 \ln x.$$

Denote by $f(x)$ the left-hand side and by $g(x)$ the right-hand side. Again, to establish the inequality we show that

(a) $f(1) \leq g(1)$, and

(b) $f'(x) > g'(x)$ for every $x \in (0, 1)$.

The statement (a) is trivial. To show (b) we first compute the derivatives:

$$f'(x) = x + \frac{1}{x^3}, \qquad g'(x) = \frac{2}{x}.$$

Finally, by rearranging, we get

$$\begin{aligned} x + \frac{1}{x^3} &> \frac{2}{x} \\ x^4 - 2x^2 + 1 &> 0 \\ (x^2-1)^2 &> 0. \end{aligned}$$

The last inequality holds for all $x \in (0,1)$ and thus the proof of Lemma 14 is finished. □

Putting things together we get

$$\mathbf{T}(a,[-1,1]) = \Pr[U_n] + \Pr[L_n] \le Q(A_\alpha)(1+o(1)) + Q(B_\alpha)(1+o(1)) = (Q(A_\alpha)+Q(B_\alpha))(1+o(1)).$$

Due to Lemma 14 the last expression is at most $\frac{1}{2}$ for n sufficiently large and the proof is finished. □

We finish the chapter by a proof of the auxiliary lemma used during the proof.

Lemma 15. *Let c be a positive real number and x a real number satisfying $x > -c$. Then we have*

$$\frac{x-c}{x+c} \le e^{-\frac{2c}{x}}.$$

Proof. Denote the left-hand side by $f(x)$ and the right-hand side by $g(x)$. To prove the lemma, it is enough to show that

(i) $\lim_{x\to\infty} f(x) \le \lim_{x\to\infty} g(x)$, and

(ii) $f'(x) \ge g'(x)$ on the interval (c,∞).

The statement (i) is easy, as limits of both $f(x)$ and $g(x)$ are clearly 1.
To show (ii), observe that

$$f'(x) = \frac{(x+c)-(x-c)}{(x+c)^2} = \frac{2c}{(x+c)^2} = \frac{2c}{x^2}\cdot\left(1+\frac{c}{x}\right)^{-2} \ge \frac{2c}{x^2}\cdot e^{-\frac{2c}{x}} = g'(x),$$

where we used the inequality $1+y \le e^y$. □

Note that we can alternatively use the inequality $1+x \le e^x$ on

$$\frac{x-c}{x+c} = 1 - \frac{2c}{x+c}.$$

This however leads to a weaker bound of $e^{-\frac{2c}{x+c}}$.

6. Conclusions

6.1 Significance of almost uniform vectors

The significance of the almost uniform vectors was already briefly discussed in Chapter 3. We believe that almost uniform vectors contain a "large" subset of the worst-case vectors (i.e. vectors attaining the conjectured bound 1/2). Recall already mentioned result by Dzindzalieta [2014b] that the conjecture holds true if $|a_j| \leq 0.16$ for all j. This implies that if a vector a is a counterexample to Tomaszewski's conjecture then most of its mass must be concentrated on small number of entries. The almost uniform vectors can be seen as the most well-behaved vectors with this property, therefore it is a natural starting point for extending the known results.

Von Heymann [2010] in his work mentions that the cases close to the conjectured bound are the ones well understood, but the difficulty lies in showing that the other cases will not start to cause trouble in high dimensions. Our main theorem shows that this does not happen in the case of almost uniform vectors.

6.2 Choice of the bounds

We have used a very precise but also somewhat cumbersome bounds for sums of binomial coefficients. This was necessitated by the fact our method does not permit even small constant errors. This means that commonly used bounds are unsuitable.

For example, Chernoff bounds can be used for bounding sum of binomial coefficients as the tail of the binomial distribution corresponds to such a sum. If X denotes a random variable with distribution $\mathrm{Bin}(n, 1/2)$ then Chernoff bound works well for bounding $\Pr[X \geq \mathbb{E}[X] + t]$ for $t \in \omega(\sqrt{n})$. However, we have seen that we need to bound the $\Pr[X \geq \mathbb{E}[X] + c\sqrt{n}]$. For such a case Chernoff bound provides only a constant upper bound (i.e., not improving with increasing n). Another well-known bound

$$\sum_{j=1}^{\lceil qn \rceil} \binom{n}{j} \leq 2^{H(q)n},$$

where $H(q) = -q \log_2 q - (1-q) \log_2(1-q)$ is the *binary entropy of* q also is not precise enough; the bound only achieves accuracy of

$$\log \sum_{j=1}^{\lceil qn \rceil} \binom{n}{j} = (1 + o(1))H(q)n.$$

The McKay bound is the most reasonable bound that we know of and is accurate enough for our approach.

6.3 Improvements of the bounds

We believe that some of the bounds can be improved. One of the most suitable steps to improve is to refine the Lemma 14. If we are able to find an explicit bound for $Q(x)f(x) + Q(1/x)g(1/x)$, where $f(x)$ and $g(x)$ are the $1 + o(1)$ terms from Lemma 13, we might be able to get good bound on n_0 with respect to a_0. In particular, if we get bound better than

$$n_0 \leq \frac{1 + a_0}{1 - a_0}$$

for some a_0 large enough, this would together with Proposition 4 mean that there exist an n_0 independent of a_0 such that all almost uniform vectors of dimension $n \geq n_0$ are not a counterexample to the Tomaszewski's conjecture. The dimensions smaller than n_0 can be checked by computer even for n_0 moderately large. Note that for every dimension n we have to check only finitely many almost uniform vectors. Recall the definitions of $k_{\min}$ and $l_{\min}$ from Lemma 12. If two vectors a, a' have the same $k_{\min}$ and $l_{\min}$ then $\mathbf{T}(a, [-1, 1]) = \mathbf{T}(a', [-1, 1])$. This means that for every dimensions, there are at most quadratic number of cases. Verifying each case would then be equivalent to checking that particular sums of binomial coefficients is small enough. Therefore, it is likely that we would be able to settle the conjecture for all almost uniform vectors.

Finding an explicit bound for $Q(x)f(x) + Q(1/x)g(1/x)$ would be easier if f and g were smaller. We suspect this may be possible too – some of the bounds we used are slightly wasteful and the $1 + o(1)$ terms may be improved.

6.4 Future directions

Motivated by the fact that there are the worst-case vectors among the almost uniform vectors, we tried to find a "uniformization" procedure that moves a vector "closer" to the almost uniform case (in the spirit of 2 in Chapter 3). This proved a difficult task; we did not manage to find meaningful results.

The lower bounds for the general case of Tomaszewski's conjecture usually use some form of case analysis based on the values of the largest entries; this seems inevitable if we want to obtain a tight bound. However, one has to be very careful as the complexity of such proofs tends to get out of hand rather quickly. Finding some kind of uniformization (even if partial) may either simplify the case analysis or show a direction for more appropriate base case.

Another direction might be using tools from analysis of Boolean function. Von Heymann [2010] shows that proving a bound on the total influence of certain Boolean functions provides a lower bound on the Tomaszewski's constant.

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