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DIPLOMOVÁ PRÁCE

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Některé aspekty nespojité Galerkinovy metody pro řešení konvektivně-difúzních problémů

Katedra Numerické Matematiky

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Název práce: Některé aspekty nespojité Galerkinovy metody pro řešení konvektivně-difuzních problémů

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Abstrakt: V předložené práci se zabýváme stabilitou nespojité časoprostorové Galerkinovy metody, aplikované na nestacionární, nelineární problémy konvekce - difúze. Nespojitá Galerkinova metoda představuje velice efektivní nástroj pro numerické řešení parciálních diferenciálních rovnic, kombinuje výhody metody konečných prvků (polynomiální aproximace vysokého řádu přesnosti) a metody konečných diferencí (nespojité aproximace). Po formulování spojitého problému následuje jeho diskretizace v prostoru i v čase. Ve formulaci nespojité Galerkinovy metody používáme nesymetrickou, symetrickou a neúplnou verzi diskretizace difúzního členu a dále přidáváme do schématu penalizační členy. Ve třetí kapitole následují odhady jednotlivých členů dříve odvozeného přibližného řešení pomocí speciálních norem. Pomocí konceptu diskrétních charakteristických funkcí a diskrétního Gronwallova lemmatu je ukázáno, že analyzované schéma je nepodmíněně stabilní. Na závěr, ve čtvrté kapitole, jsou uvedeny numerické experimenty, které ověřují teoretické výsledky předchozí kapitoly.

Klíčová slova: nelineární problém konvekce - difúze, nespojitá časoprostorová Galerkinova metoda, časová a prostorová diskretizace, stabilita metody, diskrétní charakteristická funkce

Some aspects of the discontinuous Galerkin method for the solution of convection-diffusion problems

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Abstract: In the present work we deal with the stability of the space-time discontinuous Galerkin method applied to non-stationary, nonlinear convection - diffusion problems. Discontinuous Galerkin method is a very efficient tool for numerical solution of partial differential equations, combines the advantages of the finite element method (polynomial approximations of high order of accuracy) and the finite volume method (discontinuous approximations). After the formulation of the continuous problem its discretization in space and time is described. In the formulation of the discontinuous Galerkin method the non-symmetric, symmetric and incomplete version of discretization of the diffusion term is used and there are added penalty terms to the scheme also. In the third chapter are estimated individual terms of the previously derived approximate solution by special norms. Using the concept of discrete characteristic functions and the discrete Gronwall lemma, it is shown that the analyzed scheme is unconditionally stable. At the end, in the fourth chapter, are given some numerical experiments, which verify theoretical results from the previous chapter.

Keywords: nonlinear convection - diffusion problems, space-time discontinuous Galerkin method, space and time discretization, stability of the method, discrete characteristic function

Contents

Introduction

The numerical solution of nonlinear conservation laws, convection–diffusion problems and flow problems requires the application of efficient, robust and accurate methods allowing to overcome various difficulties, as the precise capturing and resolution of boundary and internal layers, shock waves and contact discontinuities. In computational fluid dynamics (CFD) two techniques are used: the finite volume (FV) schemes and stabilized finite element methods (FEM). A survey of FV as well as FE approaches to the numerical simulation of compressible flow can be found, e.g. in [30]. A natural generalization of the FV and FE techniques is the discontinuous Galerkin finite element method (DGFEM), which appears to be very suitable for problems with solutions containing discontinuities and/or steep gradients. The DGFEM uses piecewise polynomial but discontinuous approximations and combines advantages of the FV as well as FE methods. Similarly as in the finite volume method, the DGFEM uses discontinuous approximations and boundary fluxes are evaluated with the aid of a numerical flux. Similarly as in the finite element method, the DGFEM uses higher degree polynomial approximations of solutions, which leads to an accurate resolution in regions, where the solution is smooth.

There is a number of works devoted to theory and applications of the DGFEM. Let us mention, e.g. [2], [3], [5], [6], [7], [13], [18], [19], [22], [23], [30], [33], [35], [36], [37], [38], [42]. For a survey of various discontinuous Galerkin techniques, see, e.g. [11], [12]. The numerical simulation of strongly nonstationary transient problems requires the application of numerical schemes of high order of accuracy both in space and in time. For some applications, the standard Euler schemes or θ-schemes ([22], [38]) are not sufficiently accurate in time. In computational fluid dynamics, Runge-Kutta methods are very popular. Let us mention, for example the well-known Runge-Kutta discontinuous Galerkin methods (see e.g. [14]). They are applicable to the numerical solution of a wide class of problems, including nonlinear conservation laws and nonlinear convection-diffusion problems, but they are conditionally stable. An example of unconditionally stable method is the technique using the backward difference formula (BDF). It was used for the solution of compressible flow, e.g. in [21] and analyzed theoretically in the case of a scalar nonlinear convection-diffusion equation in [25]. However, the stability of the BDF methods is limited to the order of accuracy less than or equal to 6 only. In the paper [4], a time discretization of arbitrary order of parabolic problems was proposed and analyzed. Unfortunately, it is applicable to linear problems only. It appears suitable to use the discontinuous Galerkin discretization with respect to space as well as time for the construction of numerical schemes with high accuracy in space and time for the solution of nonlinear nonstationary problems. The time interval is split into subintervals and on each time level a different space mesh may be used in general. Moreover, the triangulations used for the space discretization may be nonconforming with hanging nodes.

The discontinuous Galerkin time discretization was introduced and analyzed, e.g. in [26] for the solution of ordinary differential equations. In [1], [8], [9], [27], [28], [39] and [40] the solution of parabolic problems is carried out with the aid of conforming finite elements in space combined with the DG time discretization.

See also the monograph [41]. In [31], the space-time DGFE method was analyzed for a linear nonstationary convection-diffusion-reaction problem. The paper [32] is devoted to the theory of error estimates for the space-time DGFE method applied to a nonstationary convection-diffusion problem with a nonlinear convection and linear diffusion. An important tool in the derivation of error estimates was the time Gauss-Radau numerical integration and interpolation.

In paper [16] it is assumed that the diffusion coefficient depends on the sought solution, but we do not allow its degeneration, as happens in some physical models. In comparison to the paper [32], it is necessary to overcome the difficulty in the derivation of the abstract error estimate caused by the fact that the mentioned technique based on the Gauss- Radau numerical quadrature and interpolation is not applicable to the nonlinear diffusion. In this case, the concept of the discrete characteristic function introduced in [8] is used with success. Under the assumption that the triangulations on all time levels are uniformly shape regular, and the exact solution has some regularity properties, error estimates are derived for the space-time DGFE method.

In both papers [32] and [16], for general boundary conditions, it was necessary to consider a CFL-like stability condition in the vicinity of the boundary of the form $\tau \leq Ch_K$ for elements adjacent to the boundary. There is a natural question, if this condition is only technical and necessary for the derivation of error estimates in terms of the sizes of the space and time grids. In this work, we study and analyze this question. As a results we prove the unconditional stability of the space-time discontinuous Galerkin finite element method for the numerical solution of nonstationary convection-diffusion initial-boundary problems with nonlinear convection as well as diffusion.

1. Formulation of the continuous problem

We consider the following nonstationary parabolic problem with nonlinear convection and diffusion: Find $u : Q_T = \Omega \times (0,T) \to \mathbb{R}$ such that

$$\begin{aligned} \frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \operatorname{div}(\beta(u)\nabla u) &= g \quad \text{in } Q_T, & (1.1) \\ u\,|_{\partial\Omega\times(0,T)} &= u_D, & (1.2) \\ u(x,0) &= u^0(x), \quad x \in \Omega. & (1.3) \end{aligned}$$

We assume that $\Omega \subset \mathbb{R}^d, d = 2, 3$, is a bounded polygonal $(d = 2)$ or polyhedral $(d = 2)$ domain, $T > 0$ and

$$\begin{aligned} & f_s \in C^1(\mathbb{R}), \quad s = 1, \ldots, d \\ & g : Q_T \to \mathbb{R}, \\ & u_D : \partial\Omega \times (0,T) \to \mathbb{R}, \\ & u^0 : \Omega \to \mathbb{R}, \end{aligned}$$

are given functions. The functions f_s are called the fluxes of the quantity u in the directions x_s and the function β is the diffusion coefficient, which may depend on u. The continuous differentiability of the fluxes f_s implies that they are locally Lipschitz-continuous in $\mathbb{R}$. In theoretical analysis we shall assume that

$$|f_s'| \le L_f, \quad s = 1, \ldots, d,$$

which imply that the fluxes are Lipschitz continuous with the modul $L_f \ge 0$. Without any loss of generality we can assume that

$$f_s(0) = 0, \quad s = 1, \ldots, d. \qquad (1.4)$$

Moreover let the function β satisfy the conditions

$$\begin{aligned} \beta : \mathbb{R} \to [\beta_0, \beta_1], \quad 0 < \beta_0 < \beta_1 < \infty, & \qquad (1.5) \\ |\beta(u_1) - \beta(u_2)| \le L_\beta |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}, & \qquad (1.6) \end{aligned}$$

which means that the function β has a positive bounded range and β is Lipschitz-continuous with the modul $L_\beta \ge 0$.

We use the following notation of function spaces: Let ω be a bounded domain. Then we define the Lebesgue spaces

$$\begin{aligned} L^\infty(\omega) &= \{\text{measurable functions}\, \varphi;\ \|\varphi\|_{L^\infty(\omega)} = \operatorname{essup}_{x\in\omega} |\varphi(x)| < \infty\}, \\ L^2(\omega) &= \{\text{measurable functions}\, \varphi;\ \|\varphi\|_{L^2(\omega)} = \left(\int_\omega |\varphi(x)|^2 dx\right)^{\frac{1}{2}} < \infty\}, \end{aligned}$$

and the Sobolev space

$$H^k(\omega) = \{\varphi \in L^2(\omega);\ \|\varphi\|_{H^k(\omega)} = \left(\sum_{|\alpha|\le k} ||D^\alpha \varphi||^2_{L^2(\omega)} \right)^{\frac{1}{2}} < \infty\},$$

with the seminorm

$$|\varphi|_{H^k(\omega)} = \left(\sum_{|\alpha|=k} ||D^\alpha \varphi||^2_{L^2(\omega)} \right)^{\frac{1}{2}}.$$

In what follows, by $(\cdot,\cdot)$ we denote the L^2-scalar product, i.e.

$$(\varphi,\psi) = \int_\omega \varphi\psi\, dx \quad \text{for } \varphi,\psi \in L^2(\omega),$$

and by $\|\cdot\|$ (without any subscript) we denote the norm in the space $L^2(\omega)$:

$$\|\varphi\| = (\varphi,\varphi)^{\frac{1}{2}} \quad \text{for } \varphi \in L^2(\omega).$$

Let X be a Banach space with a norm $\|\cdot\|_X$ and a seminorm $|\cdot|_X$ and let s be an integer. Then we define the following Bochner spaces:

$$C([0,T];X) = \{\varphi : [0,T] \to X, \text{ continuous}, \ \|\varphi\|_{C([0,T];X)} = \sup_{t\in[0,T]} \|\varphi\|_X < \infty\},$$

$$L^2(0,T;X)$$
$$= \{\varphi : (0,T) \to X, \text{strongly measurable}, \|\varphi\|^2_{L^2(0,T;X)} = \int_0^T \|\varphi\|^2_X dt < \infty\},$$

$$H^s(0,T;X) = \{\varphi \in L^2(0,T;X);\ \|\varphi\|^2_{H^s(0,T;X)} = \int_0^T \sum_{\alpha=0}^{s} \left\| \frac{\partial^\alpha \varphi}{\partial t^\alpha} \right\|^2_X dt < \infty\},$$

and the corresponding norms as

$$\begin{aligned} |\varphi|_{C([0,T];X)} &= \sup_{t\in[0,T]} |\varphi|_X, \\ |\varphi|_{L^2(0,T;X)} &= \left(\int_0^T |\varphi|^2_X dt \right)^{\frac{1}{2}}, \\ |\varphi|_{H^s(0,T;X)} &= \left(\int_0^T \left| \frac{\partial^s \varphi}{\partial t^s} \right|^2_X dt \right)^{\frac{1}{2}}. \end{aligned}$$

2. Discretization of the problem

2.1 Space semidiscretization

We consider a partition $\mathcal{T}_h$ of the closure $\bar{\Omega}$ of the domain Ω into a finite number of closed d-dimensional simplices with mutually disjoint interiors.

Let $K, K' \in \mathcal{T}_h$. We say that K and K' are neighbouring elements (or neighbours), if the set $\partial K \cap \partial K'$ has positive $(d-1)$-dimensional measure. We say that $\Gamma \subset K$ is a face of K, if it is a maximal connected open subset either of $\partial K \cap \partial K'$, where K' is a neighbour of K, or of $\partial K \cap \partial\Omega$. By $\mathcal{F}_h$ we denote the system of all faces of all elements $K \in \mathcal{T}_h$. Further we distinguish the set of all inner faces: $\mathcal{F}_h^I = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \Omega\}$ and the set of all boundary faces $\mathcal{F}_h^B = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \partial\Omega\}$. Obviously $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^B$.

For each $\Gamma \in \mathcal{F}_h$ we define a unit normal vector $\mathbf{n}_\Gamma$. We assume that for boundary faces $\Gamma \in \mathcal{F}_h^B$ the normal $\mathbf{n}_\Gamma$ has the same orientation as the outher normal to $\partial\Omega$. For inner faces $\Gamma \in \mathcal{F}_h^I$ the orientation of $\mathbf{n}_\Gamma$ is arbitrary, but fixed. For each element $K \in \mathcal{T}_h$ we set $h_K = \mathrm{diam}(K)$.

Over a triangulation $\mathcal{T}_h$ for each integer $k \geq 1$ we define the broken Sobolev space

$$H^k(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^k(K) \quad \forall K \in \mathcal{T}_h\},$$

equipped with the seminorm

$$|v|_{H^k(\Omega,\mathcal{T}_h)} = \left(\sum_{K \in \mathcal{T}_h} |v|^2_{H^k(K)} \right)^{\frac{1}{2}}.$$

For each face $\Gamma \in \mathcal{F}_h^I$ there exist two neighbours $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_h$ such that $\Gamma \subset \partial K_\Gamma^{(L)} \cap \partial K_\Gamma^{(R)}$. We use the convention that $\mathbf{n}_\Gamma$ is the outer normal to the element $K_\Gamma^{(L)}$ and the inner normal to the element $K_\Gamma^{(R)}$. For $v \in H^1(\Omega, \mathcal{T}_h)$ and $\Gamma \in \mathcal{F}_h^I$ we introduce the following notation:

$$\begin{aligned} v|_\Gamma^{(L)} &= \text{the trace of} \quad v|_{K_\Gamma^{(L)}} \quad \text{on}\, \Gamma, \\ v|_\Gamma^{(R)} &= \text{the trace of} \quad v|_{K_\Gamma^{(R)}} \quad \text{on}\, \Gamma, \\ \langle v \rangle_\Gamma &= \frac{1}{2}\left(v|_\Gamma^{(L)} + v|_\Gamma^{(R)} \right), \\ [v]_\Gamma &= v|_\Gamma^{(L)} - v|_\Gamma^{(R)}. \end{aligned} \tag{2.1}$$

For $\Gamma \in \mathcal{F}_h^B$ we define $v|_\Gamma^{(L)}$ in the same way. It is important to notice, that the value $[v]_\Gamma$ depends on the orientation of $\mathbf{n}_\Gamma$, but the value of the product $[v]_\Gamma \mathbf{n}_\Gamma$ is independent of this orientation. We use the notation

$$\begin{aligned} h(\Gamma) &= \frac{h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}}{2} \quad \text{for} \quad \Gamma \in \mathcal{F}_h^I, \\ h(\Gamma) &= h_{K_\Gamma^{(L)}} \quad \text{for} \quad \Gamma \in \mathcal{F}_h^B. \end{aligned} \tag{2.2}$$

Now we introduce the space semidiscretization of problem (1.1) - (1.3). We shall assume, that u is a sufficiently smooth solution of our problem. We choose an arbitrary but fixed $t \in [0,T]$, then we multiply equation (1.1) by a test function $\varphi \in H^2(\Omega, \mathcal{T}_h)$, integrate over an element K and finally sum over all elements $K \in \mathcal{T}_h$. We get

$$\sum_{K\in\mathcal{T}_h} \int_K \frac{\partial u}{\partial t}\varphi \, dx + \sum_{K\in\mathcal{T}_h} \int_K \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s}\varphi \, dx \\ - \sum_{K\in\mathcal{T}_h} \int_K \operatorname{div}(\beta(u)\nabla u)\varphi \, dx = \sum_{K\in\mathcal{T}_h} \int_K g\varphi \, dx. \tag{2.3}$$

At first we modify the second term, representing the convection, using Green's theorem:

$$\sum_{K\in\mathcal{T}_h} \int_K \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s}\varphi \, dx \\ = \sum_{K\in\mathcal{T}_h} \int_{\partial K} \sum_{s=1}^{d} f_s(u) n_s \varphi \, dS - \sum_{K\in\mathcal{T}_h} \int_K \sum_{s=1}^{d} f_s(u) \frac{\partial \varphi}{\partial x_s} \, dx.$$

Here $\mathbf{n} = (n_1, \ldots, n_d)$ denotes the unit outer normal to ∂K. The first term on the right-hand side of the previous equation can be rewritten as

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \sum_{s=1}^{d} f_s(u) n_s \varphi \, dS = \sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \sum_{s=1}^{d} f_s(u) (\mathbf{n}_\Gamma)_s [\varphi]_\Gamma \, dS.$$

By $(\mathbf{n}_\Gamma)_s$, $s = (1, \ldots, d)$ we denote the components of the vector $\mathbf{n}_\Gamma$. The fluxes $\sum_{s=1}^{d} f_s(u)(\mathbf{n}_\Gamma)_s$ through faces Γ will be approximated by the numerical flux H:

$$H(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \mathbf{n}_\Gamma) \approx \sum_{s=1}^{d} f_s(u)(\mathbf{n}_\Gamma)_s.$$

We assume that the numerical flux has the following properties:

(H1) $H(w_1, w_2, \mathbf{n})$ is defined in $\mathbb{R}^d \times B_1$, where $B_1 = \{\mathbf{n} \in \mathbb{R}^d;\ |\mathbf{n}| = 1\}$, and is locally Lipschitz - continuous with respect to w_1, w_2: for each $r > 0$ there exists a constant $L_H(r) \geq 0$ such that

$$|H(w_1, w_2, \mathbf{n}) - H(w_1^*, w_2^*, \mathbf{n})| \leq L_H(r)(|w_1 - w_1^*| + |w_2 - w_2^*|), \\ w_1, w_2, w_1^*, w_2^* \in \mathbb{R},\ |w_1|, |w_2|, |w_1^*|, |w_2^*| < r,\ \mathbf{n} \in B_1$$

(H2) $H(w_1, w_2, \mathbf{n})$ is consistent:

$$H(w_1, w_1, \mathbf{n}) = \sum_{s=1}^{d} f_s(w_1) n_s \quad w_1 \in \mathbb{R},\ \mathbf{n} = (n_1, \ldots, n_d) \in B_1$$

(H3) $H(w_1, w_2, \mathbf{n})$ is conservative:

$$H(w_1, w_2, \mathbf{n}) = -H(w_2, w_1, -\mathbf{n}) \quad w_1, w_2 \in \mathbb{R},\ \mathbf{n} \in B_1.$$

Using some trivial manipulations and the properties of the numerical flux, we can write

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\int_{\partial K}\sum_{s=1}^{d} f_s(u)n_s\,\varphi\,dS \\
&= \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma\sum_{s=1}^{d}\Big(f_s(u_\Gamma^{(L)})(\mathbf{n}_\Gamma)_s\,\varphi|_\Gamma^{(L)} + f_s(u_\Gamma^{(R)})(-\mathbf{n}_\Gamma)_s\,\varphi|_\Gamma^{(R)}\Big)\,dS \\
&\qquad\qquad + \sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma\sum_{s=1}^{d} f_s(u_\Gamma^{(L)})(\mathbf{n}_\Gamma)_s\,\varphi|_\Gamma\,dS \\
&\approx \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma\Big(H(u_\Gamma^{(L)},u_\Gamma^{(R)},\mathbf{n}_\Gamma)\,\varphi|_\Gamma^{(L)} + H(u_\Gamma^{(R)},u_\Gamma^{(L)},-\mathbf{n}_\Gamma)\,\varphi|_\Gamma^{(R)}\Big)\,dS \\
&\qquad\qquad + \sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma H(u_\Gamma^{(L)},u_\Gamma^{(L)},\mathbf{n}_\Gamma)\,\varphi|_\Gamma\,dS \\
&\overset{(H3)}{=} \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma H(u_\Gamma^{(L)},u_\Gamma^{(R)},\mathbf{n}_\Gamma)\,[\varphi]_\Gamma\,dS + \sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma H(u_\Gamma^{(L)},u_\Gamma^{(L)},\mathbf{n}_\Gamma)\,\varphi\,dS
\end{aligned}
$$

To summarize, we showed that

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\int_K\sum_{s=1}^{d}\frac{\partial f_s(u)}{\partial x_s}\varphi\,dx \\
&\approx b_h(u,\varphi) := -\sum_{K\in\mathcal{T}_h}\int_K\sum_{s=1}^{d} f_s(u)\frac{\partial\varphi}{\partial x_s}\,dx + \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma H(u_\Gamma^{(L)},u_\Gamma^{(R)},\mathbf{n}_\Gamma)\,[\varphi]_\Gamma\,dS \\
&\qquad\qquad + \sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma H(u_\Gamma^{(L)},u_\Gamma^{(L)},\mathbf{n}_\Gamma)\,\varphi|_\Gamma\,dS.
\end{aligned}
$$

We can see, that for $\Gamma\in\mathcal{F}_h^B$ we define the boundary value $u_\Gamma^{(R)}$ by extrapolation, i.e. as the value $u_\Gamma^{(L)}$. Similarly as above, now we adjust the third term in (2.3), representing the diffusion. Using Green's theorem we get

$$
\begin{aligned}
&-\sum_{K\in\mathcal{T}_h}\int_K \operatorname{div}(\beta(u)\nabla u)\varphi\,dx \\
&= \sum_{K\in\mathcal{T}_h}\int_K\beta(u)\nabla u\cdot\nabla\varphi\,dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K}\beta(u)\nabla u\cdot\mathbf{n}\,\varphi\,dS.
\end{aligned}
$$

The last term on the right-hand side can be written as follows:

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}\beta(u)\nabla u\cdot\mathbf{n}\,\varphi\,dS$$
$$=\sum_{K\in\mathcal{T}_h}\int_{\partial K}\sum_{k=1}^{d}\beta(u)\frac{\partial u}{\partial x_k}\,\mathbf{n}_k\,\varphi\,dS$$
$$=\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma\sum_{k=1}^{d}\beta(u)\left(\frac{\partial u}{\partial x_k}\right)_\Gamma^{(L)}(\mathbf{n}_\Gamma)_k\,\varphi_\Gamma^{(L)}\,dS$$
$$+\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma\sum_{k=1}^{d}\beta(u)\left(\frac{\partial u}{\partial x_k}\right)_\Gamma^{(R)}(-\mathbf{n}_\Gamma)_k\,\varphi_\Gamma^{(R)}\,dS$$
$$+\sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma\sum_{k=1}^{d}\beta(u)\frac{\partial u}{\partial x_k}\Big|_\Gamma(\mathbf{n}_\Gamma)_k\,\varphi|_\Gamma\,dS.$$

Since we assume that u is sufficiently smooth, for $k=1,2$ we have

$$\left(\beta(u)\frac{\partial u}{\partial x_k}\right)_\Gamma^{(L)}=\left(\beta(u)\frac{\partial u}{\partial x_k}\right)_\Gamma^{(R)}=\left\langle\beta(u)\frac{\partial u}{\partial x_k}\right\rangle_\Gamma.$$

Using this property of u, we can continue in modifying the equation above:

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}\beta(u)\nabla u\cdot\mathbf{n}\,\varphi\,dS$$
$$=\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma\sum_{k=1}^{d}\left\langle\beta(u)\frac{\partial u}{\partial x_k}\right\rangle_\Gamma(\mathbf{n}_\Gamma)_k\,[\varphi]_\Gamma\,dS+\sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma\sum_{k=1}^{d}\beta(u)\frac{\partial u}{\partial x_k}\Big|_\Gamma(\mathbf{n}_\Gamma)_k\,\varphi|_\Gamma\,dS$$
$$=\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma\langle\beta(u)\nabla u\rangle_\Gamma\cdot\mathbf{n_\Gamma}\,[\varphi]_\Gamma\,dS+\sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma\beta(u)\nabla u|_\Gamma\cdot\mathbf{n}_\Gamma\,\varphi|_\Gamma\,dS \tag{2.4}$$

Further, to these terms we add stabilizing terms, which we obtain by formal exchange of variables u and φ in (2.4).
The stabilizing term for the inner faces is

$$\theta\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma\langle\beta(u)\nabla\varphi\rangle_\Gamma\cdot\mathbf{n_\Gamma}\,[u]_\Gamma\,dS. \tag{2.5}$$

We can see, that this term is equal to zero for a sufficiently regular exact solution, because of $[u]_\Gamma=0$. The stabilizing term for boundary faces is

$$\theta\sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma\beta(u)\nabla\varphi|_\Gamma\cdot\mathbf{n_\Gamma}\,u|_\Gamma\,dS. \tag{2.6}$$

This term will be compensated on the basis of condition (1.2) by the expression

$$\theta\sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma\beta(u)\nabla\varphi|_\Gamma\cdot\mathbf{n_\Gamma}\,u_D|_\Gamma\,dS. \tag{2.7}$$

Using (2.4),(2.5),(2.6),(2.7), we define the approximation of the diffusion term in (2.3) as

$$
\begin{aligned}
a_h(u,\varphi) &= \sum_{K\in\mathcal{T}_h}\int_K \beta(u)\nabla u\cdot\nabla\varphi\,dx \qquad (2.8)\\
&- \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \left(\langle\beta(u)\nabla u\rangle_\Gamma\cdot\mathbf{n}_\Gamma\,[\varphi]_\Gamma+\theta\,\langle\beta(u)\nabla\varphi\rangle_\Gamma\cdot\mathbf{n}_\Gamma\,[u]_\Gamma\right)\,dS\\
&- \sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma \left(\beta(u)\nabla u|_\Gamma\cdot\mathbf{n}_\Gamma\,\varphi|_\Gamma+\theta\beta(u)\nabla\varphi|_\Gamma\cdot\mathbf{n}_\Gamma\,u|_\Gamma-\theta\beta(u)\nabla\varphi|_\Gamma\cdot\mathbf{n}_\Gamma\,u_D|_\Gamma\right)\,dS.
\end{aligned}
$$

In this formulation we can choose $\theta=-1,\theta=0$ or $\theta=1$, and get the nonsymmetric (NIPG), incomplete (IIPG) and symmetric (SIPG) variants of the interior and boundary penalty approximation of the diffusion term, respectively. Finally we add to the above discretization the penalty form

$$
J_h(u,\varphi)=c_W\sum_{\Gamma\in\mathcal{F}_h^I}h(\Gamma)^{-1}\int_\Gamma [u]_\Gamma\,[\varphi]_\Gamma\,dS+c_W\sum_{\Gamma\in\mathcal{F}_h^B}h(\Gamma)^{-1}\int_\Gamma u|_\Gamma\,\varphi|_\Gamma\,dS, \qquad (2.9)
$$

where $c_W>0$ is a suitable constant. The first term on the right-hand side vanishes for the exact regular solution u, and the second term is compensated again on the basis of condition (1.2) by the expression

$$
c_W\sum_{\Gamma\in\mathcal{F}_h^B}h(\Gamma)^{-1}\int_\Gamma u_D|_\Gamma\,\varphi|_\Gamma\,dS. \qquad (2.10)
$$

Now, using (2.8) and (2.9), we can define the complete approximation of the of the diffusion term involving penalization as

$$
A_h(u,\varphi)=a_h(u,\varphi)+\beta_0\,J_h(u,\varphi). \qquad (2.11)
$$

Lastly, we define the form representing the right-hand side of (2.3) after the discretization. This form moreover has to contain expression (2.10) multiplied by β_0 due to definition (2.11):

$$
l_h(\varphi)=\sum_{K\in\mathcal{T}_h}\int_K g\varphi\,dx+\beta_0\,c_W\sum_{\Gamma\in\mathcal{F}_h^B}h(\Gamma)^{-1}\int_\Gamma u_D|_\Gamma\,\varphi|_\Gamma\,dS. \qquad (2.12)
$$

2.2 Time discretization

In the time interval $[0, T]$ we shall construct a partition formed by time instants

$$0 = t_0 < t_1 < \ldots < t_M = T,$$

and denote each subinterval by $I_m = (t_{m-1}, t_m)$, $m = 1, \ldots, M$. Then we have

$$[0, T] = \bigcup_{m=1}^{M} \bar{I}_m, \quad \text{where} \quad I_m \cap I_n = \emptyset \quad \text{for} \quad m \neq n.$$

Length of the subintervals is denoted by $\tau_m = t_m - t_{m-1}$, $m = 1 \ldots, M$. Further we shall set $\tau = \max_{m=1,\cdots,M} \tau_m$. For each I_m we consider a partition $\mathcal{T}_{h,m}$ of the closure $\bar{\Omega}$ of the domain Ω into a finite number of closed triangles with mutually disjoint interiors. The partitions $\mathcal{T}_{h,m}$ are in general different for different m.

We shall use similar notation with similar properties as in the previous section, we only add subscript m, because we use different partitions in different time intervals and therefore we get different elements and faces for I_m, $m = 1, \ldots, M$. By $\mathcal{F}_{h,m}$ we denote the system of all faces of all elements $K \in \mathcal{T}_{h,m}$, which consists of the set of all inner faces $\mathcal{F}^I_{h,m}$ and the set of all boundary faces $\mathcal{F}^B_{h,m}$:

$$\mathcal{F}_{h,m} = \mathcal{F}^I_{h,m} \cup \mathcal{F}^B_{h,m}.$$

Each $\Gamma \in \mathcal{F}_{h,m}$ will be associated with a unit normal vector $\mathbf{n}_\Gamma$, which has the same properties as in the previous section. We denote the diameter of elements $K \in \mathcal{T}_{h,m}$ by $h_K = \mathrm{diam}(K)$ and furthermore $h_m = \max_{K \in \mathcal{T}_{h,m}} h_K$, $h = \max_{m=1,\cdots,M} h_m$. By ρ_K we denote the radius of the largest ball inscribed into the element K.

For a function φ defined in $\bigcup_{m=1}^{M} I_m$ we denote

$$\varphi_m^{\pm} = \varphi(t_m\pm) = \lim_{t \to t_m\pm} \varphi(t), \quad \{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-).$$

The last equation expresses jumps of the function φ.

Over a triangulation $\mathcal{T}_{h,m}$, analogously as in the previous section, we define the broken Sobolev spaces

$$H^k(\Omega, \mathcal{T}_{h,m}) = \{v; v \mid K \in H^k(K) \quad \forall K \in \mathcal{T}_{h,m}\},$$

equipped with the seminorm

$$|v|_{H^k(\Omega,\mathcal{T}_{h,m})} = \left(\sum_{K \in \mathcal{T}_{h,m}} |v|^2_{H^k(K)} \right)^{\frac{1}{2}}.$$

In the space $H^1(\Omega, \mathcal{T}_{h,m})$, the following special norm will be used

$$\|\varphi\|_{DG,m} = \left(\sum_{K \in \mathcal{T}_{h,m}} |\varphi|^2_{H^1(K)} + J_{h,m}(\varphi, \varphi) \right)^{\frac{1}{2}}.$$

Similarly as in the previous section for $u, \varphi \in H^2(\Omega, \mathcal{T}_{h,m})$ and $c_W > 0$, we get the forms

$$a_{h,m}(u,\varphi) = \sum_{K\in\mathcal{T}_{h,m}} \int_K \beta(u)\nabla u \cdot \nabla\varphi \, dx \tag{2.13}$$

$$- \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(\langle \beta(u)\nabla u\rangle_\Gamma \cdot \mathbf{n}_\Gamma \, [\varphi]_\Gamma + \theta \, \langle \beta(u)\nabla\varphi\rangle_\Gamma \cdot \mathbf{n}_\Gamma \, [u]_\Gamma \right) dS$$

$$- \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \left(\beta(u)\nabla u|_\Gamma \cdot \mathbf{n}_\Gamma \, \varphi|_\Gamma + \theta\beta(u)\nabla\varphi|_\Gamma \cdot \mathbf{n}_\Gamma \, u|_\Gamma - \theta\beta(u)\nabla\varphi|_\Gamma \cdot \mathbf{n}_\Gamma \, u_D|_\Gamma \right) dS,$$

$$J_{h,m}(u,\varphi) = c_W \sum_{\Gamma\in\mathcal{F}^I_{h,m}} h(\Gamma)^{-1} \int_\Gamma [u]_\Gamma \, [\varphi]_\Gamma dS \tag{2.14}$$

$$+ c_W \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h(\Gamma)^{-1} \int_\Gamma u|_\Gamma \, \varphi|_\Gamma \, dS,$$

$$J^B_{h,m}(u,\varphi) = c_W \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h(\Gamma)^{-1} \int_\Gamma u|_\Gamma \, \varphi|_\Gamma \, dS, \tag{2.15}$$

$$A_{h,m}(u,\varphi) = a_{h,m}(u,\varphi) + \beta_0 \, J_{h,m}(u,\varphi), \tag{2.16}$$

$$b_{h,m}(u,\varphi) = - \sum_{K\in\mathcal{T}_{h,m}} \int_K \sum_{s=1}^d f_s(u) \frac{\partial\varphi}{\partial x_s} \, dx \tag{2.17}$$

$$+ \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \mathbf{n}_\Gamma) \, [\varphi]_\Gamma \, dS + \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(L)}, \mathbf{n}_\Gamma) \, \varphi|_\Gamma \, dS,$$

$$l_{h,m}(\varphi) = \sum_{K\in\mathcal{T}_{h,m}} \int_K g\varphi \, dx + \beta_0 \, C_W \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h(\Gamma)^{-1} \int_\Gamma u_D|_\Gamma \, \varphi|_\Gamma \, dS. \tag{2.18}$$

Let $p, q \geq 1$ be integers. For each $m = 1, \ldots, M$ we define the finite-dimensional space

$$S^p_{h,m} = \{\varphi \in L^2(\Omega); \, \varphi|_K \in P^p(K) \quad \forall K \in \mathcal{T}_{h,m}\}, \tag{2.19}$$

where $P^p(K)$ denotes the space of all polynomials on K of degree less than or equal to p. By Π_m we denote the $L^2(\Omega)$-projection on $S^p_{h,m}$. This means that if $u \in L^2(\Omega)$, then $\Pi_m u \in S^p_{h,m}$ and

$$(\Pi_m u - u, \varphi) = 0 \quad \forall \varphi \in S^p_{h,m}. \tag{2.20}$$

The approximate solution will be sought in the space

$$S^{p,q}_{h,\tau} = \{\varphi \in L^2(Q_T); \, \varphi|_{I_m} = \sum_{i=0}^q t^i \varphi_i \quad \text{with} \quad \varphi_i \in S^p_{h,m}, \, m = 1, \ldots, M\}. \tag{2.21}$$

Definition. *We say that a function U is an approximate solution of problem (1.1) - (1.3), if $U \in S^{p,q}_{h,\tau}$ and satisfies the equation*

$$\int_{I_m} \left(\left(\frac{\partial U}{\partial t}, \varphi \right) + A_{h,m}(U,\varphi) + b_{h,m}(U,\varphi) \right) dt + (\{U\}_{m-1}, \varphi^+_{m-1}) \tag{2.22}$$

$$= \int_{I_m} l_{h,m}(\varphi) \, dt, \quad \forall \varphi \in S^{p,q}_{h,\tau}, \quad m = 1, \ldots, M, \quad \textit{where} \quad U_0^- := \Pi_1 u^0.$$

The exact regular solution u satisfies the identity

$$\int_{I_m} \left(\left(\frac{\partial u}{\partial t}, \varphi \right) + A_{h,m}(u, \varphi) + b_{h,m}(u, \varphi) \right) dt + (\{u\}_{m-1}, \varphi_{m-1}^{+}) \qquad (2.23)$$
$$= \int_{I_m} l_{h,m}(\varphi)\, dt, \quad \forall \varphi \in S_{h,\tau}^{p,q}, \quad \text{with} \quad u(0-) = u(0) = u^0.$$

3. Theoretical analysis

3.1 Basic notation and concepts

In our further considerations we shall work with a system of triangulation $\mathcal{T}_{h,m}$, $m = 1, \ldots, M$, $h \in (0, h_0)$, which has shape regular and locally quasiuniform elements: There exist positive constants c_R and c_Q, independent of K, Γ, m, M and h such that for all $m = 1, \ldots, M$, $h \in (0, h_0)$ it holds

$$\frac{h_K}{\rho_K} \leq c_R, \quad \text{for all} \quad K \in \mathcal{T}_{h,m}, \tag{3.1}$$

$$h_{K_\Gamma^{(L)}} \leq c_Q h_{K_\Gamma^{(R)}}, \quad h_{K_\Gamma^{(R)}} \leq c_Q h_{K_\Gamma^{(L)}} \qquad \text{for all} \quad \Gamma \in \mathcal{F}_{h,m}^I. \tag{3.2}$$

In the analysis we shall use the following, very important inequalities.

Lemma 1. *(Young's inequality) Let a and b nonnegative real numbers and p, q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{3.3}$$

Equality holds if and only if $a^p = b^q$.

We shall also use a special case of this inequality for $p = q = 2$:

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2},$$

which we can for constants $\epsilon, \delta > 0$ rewrite in the form

$$ab = \frac{a}{\sqrt{\epsilon}}\, b\sqrt{\epsilon} \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2} \leq \frac{a^2}{\delta} + \frac{\delta b^2}{4} \leq \frac{a^2}{\delta} + \delta b^2. \tag{3.4}$$

Lemma 2. *(Multiplicative trace inequality) There exists a constant $c_M > 0$ independent of v, h, K and M such that*

$$\|v\|_{L^2(\partial K)}^2 \leq c_M \left(\|v\|_{L^2(K)}\, |v|_{H^1(K)} + h_K^{-1} \|v\|_{L^2(K)}^2 \right), \tag{3.5}$$
$$v \in H^1(K),\ K \in \mathcal{T}_{h,m},\ m = 1, \ldots, M,\ h \in (0, h_0).$$

Lemma 3. *(Inverse inequality) There exists a constant $c_I > 0$ independent of v, h, K and M such that*

$$|v|_{H^1(K)} \leq c_I\, h_K^{-1} \|v\|_{L^2(K)}, \tag{3.6}$$
$$v \in P^p(K),\ K \in \mathcal{T}_{h,m},\ m = 1, \ldots, M,\ h \in (0, h_0).$$

Proof of these lemmas can be found in [10] and [23].

3.2 Summary of results on error estimates

An important tool in the derivation of the error estimates is the $S_{h,\tau}^{p,q}$-interpolation π of functions $v \in H^1(0,T;L^2(\Omega))$, defined as follows:

$$\begin{aligned}
&\text{a)} \quad \pi\, v \in S_{h,\tau}^{p,q}, \\
&\text{b)} \quad (\pi\, v)(t_m-) = \pi_m v(t_m-), \\
&\text{c)} \quad \int_{I_m} (\pi\, v - v, \varphi^*)\, dt = 0 \quad \forall \varphi^* \in S_{h,\tau}^{p,q}, \quad \forall m = 1, \ldots, M.
\end{aligned} \tag{3.7}$$

We are interested in the estimation of the error $e = U - u$ of the method defined by (2.22). It can be written in the form

$$e = \xi + \eta, \tag{3.8}$$

where

$$\xi = U - \pi u \in S_{h,\tau}^{p,q}, \quad \eta = \pi u - u. \tag{3.9}$$

Then, subtracting (2.23) from (2.22) and using (3.8) for each $\varphi \in S_{h,\tau}^{p,q}$, we get

$$\begin{aligned}
&\int_{I_m} \left(\left(\frac{\partial \xi}{\partial t}, \varphi \right) + A_{h,m}(U,\varphi) - A_{h,m}(u,\varphi) \right) dt + (\{\xi\}_{m-1}, \varphi_{m-1}^+) \\
&= \int_{I_m} (b_{h,m}(u,\varphi) - b_{h,m}(U,\varphi))\; dt - \int_{I_m} \left(\frac{\partial \eta}{\partial t}, \varphi \right) dt - (\{\eta\}_{m-1}, \varphi_{m-1}^+).
\end{aligned} \tag{3.10}$$

After estimating each term in (3.10), we get the following abstract error estimates.

Theorem 1. *Let $0 < \tau_m \le c^*$, where c^* is a positive constant independent of m and M. Then there exists a constant $c > 0$ such that the error $e = U - u = \xi + \eta$ satisfies the following estimates:*

$$\begin{aligned}
&\|e_m^-\|^2 + \frac{\beta_0}{2} \sum_{j=1}^m \int_{I_j} \|e\|_{DG,j}^2\, dt \\
&\le c \left(\sum_{j=1}^m \|\eta_j^-\|^2 + \sum_{j=1}^m \int_{I_j} R_j(\eta)\, dt \right) + 2\|\eta_m^-\|^2 + \beta_0 \sum_{j=1}^m \int_{I_j} \|\eta\|_{DG,j}^2\, dt, \\
&m = 1, \ldots, M, \quad h \in (0, h_0),
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&\|e\|_{L^2(Q_T)}^2 \\
&\le c \sum_{m=1}^M \tau_m \left(\|\eta_{m-1}^-\|^2 + \int_{I_m} R_m(\eta)\, dt + \sum_{j=1}^{m-1} \|\eta_j^-\|^2 + \sum_{j=1}^{m-1} \int_{I_j} R_j(\eta)\, dt \right) \\
&\quad + 2\|\eta\|_{L^2(Q_T)}^2, \\
&h \in (0, h_0).
\end{aligned} \tag{3.12}$$

If all meshes $\mathcal{T}_{h,m}$, $m = 1, \ldots, M$ are identical, which means that $\mathcal{T}_{h,m} = \mathcal{T}_h$ for $m = 1, \ldots, M$, then all spaces $S^p_{h,m}$ and forms $a_{h,m}, A_{h,m}, b_{h,m}, J_{h,m}, l_{h,m}$ are also identical: $S^p_{h,m} = S^p_h, a_{h,m} = a_h, A_{h,m} = A_h, b_{h,m} = b_h, J_{h,m} = J_h, l_{h,m} = l_h$ for all $m = 1, \ldots, M$. This implies that the expression $\sum_{j=1}^m \|\eta_j^-\|^2$ does not appear in estimates (3.11) and (3.12), so we get

$$\|e_m^-\|^2 + \frac{\beta_0}{2} \sum_{j=1}^m \int_{I_j} \|e\|^2_{DG,j} dt \tag{3.13}$$
$$\leq c \sum_{j=1}^m \int_{I_j} R_j(\eta) dt + 2\|\eta_m^-\|^2 + \beta_0 \sum_{j=1}^m \int_{I_j} \|\eta\|^2_{DG,j} dt,$$
$$m = 1, \ldots, M, \quad h \in (0, h_0),$$

and

$$\|e\|^2_{L^2(Q_T)} \tag{3.14}$$
$$\leq c \sum_{m=1}^M \tau_m \left(\|\eta_{m-1}^-\|^2 + \int_{I_m} R_m(\eta) dt + \sum_{j=1}^{m-1} \int_{I_j} R_j(\eta) dt \right) + 2\|\eta\|^2_{L^2(Q_T)},$$
$$h \in (0, h_0).$$

On the basis of the abstract error estimates it is possible to obtain error estimates in terms of the size h of the space triangulation and τ of the time partition. We assume that the exact solution is sufficiently regular, namely

$$u \in H^{q+1}(0, T; H^1(\Omega)) \cap C([0, T]; H^{p+1}(\Omega)), \tag{3.15}$$
$$\|\nabla u\|_{L^\infty(\Omega)} \leq c_R \quad \text{for almost every} \quad t \in (0, T), \tag{3.16}$$

and that the meshes are shape regular and locally quasiuniform, i.e they satisfy conditions (3.1), (3.2), and moreover

$$0 < \tau_m \leq c^*, \tag{3.17}$$
$$\tau_m \geq c\, h_m^2, \quad m = 1, \ldots, M, \tag{3.18}$$

where c^* is a positive constant independent of m and M and $c > 0$.

Theorem 2. *Let u be the exact solution of problem (1.1) - (1.3) satisfying the regularity conditions (3.15) and (3.16). Let U be the approximate solution of problem (1.1) - (1.3), obtained by the scheme (2.22). Let conditions (3.1), (3.2),(3.17), (3.18) be satisfied. Then there exists a constant $c > 0$ independent of h, τ, m, M, u, U such that*

$$\|e_m^-\|^2 + \frac{\beta_0}{2} \sum_{j=1}^m \int_{I_j} \|e\|^2_{DG,j} dt \tag{3.19}$$
$$\leq c \left(h^{2p} |u|^2_{C([0,T];H^{p+1}(\Omega))} + \tau^{2q+\gamma} |u|^2_{H^{q+1}(0,T;H^1(\Omega))} \right),$$
$$m = 1, \ldots, M, \quad h \in (0, h_0),$$

and

$$\|e\|^2_{L^2(Q_T)} \leq c \left(h^{2p} |u|^2_{C([0,T];H^{p+1}(\Omega))} + \tau^{2q+\gamma} |u|^2_{H^{q+1}(0,T;H^1(\Omega))} \right), \tag{3.20}$$
$$h \in (0, h_0).$$

Here $\gamma = 0$, *if the condition* $\tau_m \le c_B h_{K_\Gamma^{(L)}}$ *for all* $\Gamma \in \mathcal{F}_{h,m}^B$, $m = 1, \dots, M$, $h \in (0, h_0)$ *holds, and the function* u_D *from the boundary condition (1.2) has a general behavior with respect to* t*. If* u_D *is defined by*

$$u_D = \sum_{j=0}^{q} \psi_j(x) t^j, \quad \textit{where} \quad \psi_j \in H^{p+1/2}(\partial\Omega) \quad \textit{for} \quad j = 0, \dots, q,$$

then $\gamma = 2$ *and the condition* $\tau_m \le c_B h_{K_\Gamma^{(L)}}$ *is not required.*

The proof of these estimates can be found in [16].

If all meshes $\mathcal{T}_{h,m}$, $m = 1, \dots, M$, are identical, then we obtain abstract estimates (3.13) and(3.14) mentioned above. These estimates do not contain the expression $\sum_{j=1}^{m} \|\eta_j^-\|^2$, hence Theorem 2 is valid without assumption (3.18) in the case of identical meshes on all time levels.

Beside error estimates, an interesting and important property of the numerical method is its stability. In the next section we shall be concerned with the stability analysis of method (2.22).

3.3 Analysis of the stability

Our main goal is to show that the approximate solution U of problem (1.1) - (1.3) is estimated by the L^2-norm of g, u^0 and by the $\|\cdot\|_{DGB,m}$-norm of u_D, which is defined as follows:

$$\|u_D\|_{DGB,m} := \left(J^B_{h,m}(u_D,u_D)\right)^{\frac{1}{2}} = \left(c_W \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h^{-1}(\Gamma)\int_\Gamma |u_D|^2 dS\right)^{\frac{1}{2}}. \quad (3.21)$$

This inspires us to define the space $L^2_{DGB}(0,T;L^2(\partial\Omega))$ formed by all functions from the space $L^2(0,T;L^2(\partial\Omega))$, equipped with the norm

$$\|v\|_{L^2_{DGB}(0,T;L^2(\partial\Omega))} = \left(\int_0^T \|v(t)\|^2_{DGB,m}\,dt\right)^{\frac{1}{2}}, \text{for} \quad v\in L^2(0,T;L^2(\partial\Omega)). \quad (3.22)$$

In order to get our basic identity, which will be estimated in this whole section, we substitute $\varphi := U$ into (2.22). We find that

$$\int_{I_m}\left(\left(\frac{\partial U}{\partial t},U\right) + A_{h,m}(U,U) + b_{h,m}(U,U)\right)dt + (\{U\}_{m-1}, U^+_{m-1}) \quad (3.23)$$
$$= \int_{I_m} l_{h,m}(U)\,dt.$$

After using (2.16), we get

$$\int_{I_m}\left(\left(\frac{\partial U}{\partial t},U\right) + a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U) + b_{h,m}(U,U)\right)dt \quad (3.24)$$
$$+(\{U\}_{m-1},\varphi^+_{m-1}) = \int_{I_m} l_{h,m}(U)\,dt.$$

In the following part we estimate the individual terms in (3.24).

Lemma 4. *For the approximate solution U of problem (1.1) - (1.3) we have*

$$\int_{I_m}\left(\frac{\partial U}{\partial t},U\right)dt + (\{U\}_{m-1},U^+_{m-1}) \quad (3.25)$$
$$= \frac{1}{2}\left(\|U^-_m\|^2 - \|U^-_{m-1}\|^2 + \|\{U\}_{m-1}\|^2\right).$$

Proof. A simple calculation yields

$$2\int_{I_m}\left(\frac{\partial U}{\partial t},U\right)dt + 2(\{U\}_{m-1},U^+_{m-1}) \quad (3.26)$$
$$= \int_{I_m}\frac{d}{dt}\|U\|^2 dt + 2(\{U\}_{m-1},U^+_{m-1}).$$

Now using the definition of the time interval $I_m = (t_{m-1},t_m)$, identities $U^\pm_m = U(t_m\pm) = \lim_{t\to t_m\pm} U(t)$, and $\{U\}_{m-1} = U^+_{m-1} - U^-_{m-1}$, we get

$$\int_{I_m}\frac{d}{dt}\|U\|^2 dt + 2(\{U\}_{m-1},U^+_{m-1}) \quad (3.27)$$
$$= \|U^-_m\|^2 - \|U^+_{m-1}\|^2 + 2(U^+_{m-1} - U^-_{m-1}, U^+_{m-1}).$$

Using properties of the scalar product in the space $L^2(\Omega)$, we can rewrite the last term on the right-hand side of (3.27) as follows:

$$
\begin{aligned}
&2(U^+_{m-1} - U^-_{m-1}, U^+_{m-1}) \\
&= (U^+_{m-1} - U^-_{m-1}, U^+_{m-1}) + (U^+_{m-1} - U^-_{m-1}, U^+_{m-1}) \\
&= \|U^+_{m-1}\|^2 - (U^-_{m-1}, U^+_{m-1}) + (U^+_{m-1} - U^-_{m-1}, U^+_{m-1}) \\
&= \|U^+_{m-1}\|^2 - (U^-_{m-1}, U^+_{m-1}) + (U^+_{m-1} - U^-_{m-1}, U^+_{m-1} - U^-_{m-1}) + (U^+_{m-1} - U^-_{m-1}, U^-_{m-1}) \\
&= \|U^+_{m-1}\|^2 - (U^-_{m-1}, U^+_{m-1}) + \|\{U\}_{m-1}\|^2 + (U^+_{m-1}, U^-_{m-1}) + \|U^-_{m-1}\|^2 \\
&= \|U^+_{m-1}\|^2 + \|\{U\}_{m-1}\|^2 + \|U^-_{m-1}\|^2.
\end{aligned}
$$

Now substituting back into (3.27) and (3.26), we get

$$
\begin{aligned}
&\int_{I_m} \frac{d}{dt}\|U\|^2 dt + 2(\{U\}_{m-1}, U^+_{m-1}) \\
&= \|U^-_m\|^2 - \|U^+_{m-1}\|^2 + \|U^+_{m-1}\|^2 + \|\{U\}_{m-1}\|^2 + \|U^-_{m-1}\|^2 \\
&= \|U^-_m\|^2 + \|\{U\}_{m-1}\|^2 + \|U^-_{m-1}\|^2,
\end{aligned}
$$

and finally obtain

$$
2\int_{I_m} \left(\frac{\partial U}{\partial t}, U\right) dt + 2(\{U\}_{m-1}, U^+_{m-1}) = \|U^-_m\|^2 + \|\{U\}_{m-1}\|^2 + \|U^-_{m-1}\|^2.
$$

□

Now we shall estimate other terms from the left-hand side of our basic equation (3.24).

Lemma 5. *Let $\theta = -1$ and*

$$
c_W \geq \frac{4\beta_1^2}{\beta_0^2} c_M (c_I + 1). \tag{3.28}
$$

Then the approximate solution U of problem (1.1) - (1.3) satisfies

$$
\begin{aligned}
&\int_{I_m} (a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U))\, dt \\
&\qquad \geq \frac{\beta_0}{2}\int_{I_m} \|U\|^2_{DG,m} dt - \frac{\beta_0}{2}\int_{I_m} \|u_D\|^2_{DGB,m} dt.
\end{aligned} \tag{3.29}
$$

Proof. First, we rewrite the integrand on the left-hand side using (2.13):

$$
\begin{aligned}
&a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U) \\
&= \sum_{K\in\mathcal{T}_{h,m}} \int_K \beta(U)\nabla U\cdot\nabla U\, dx + \beta_0\, J_{h,m}(U,U) \\
&- \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(\langle\beta(U)\nabla U\rangle_\Gamma\cdot\mathbf{n}_\Gamma\,[U]_\Gamma + \theta\,\langle\beta(U)\nabla U\rangle_\Gamma\cdot\mathbf{n}_\Gamma\,[U]_\Gamma\right) dS \\
&- \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \left(\beta(u)\nabla U|_\Gamma\cdot\mathbf{n}_\Gamma\, U|_\Gamma + \theta\beta(U)\nabla U|_\Gamma\cdot\mathbf{n}_\Gamma\, U|_\Gamma - \theta\beta(U)\nabla U|_\Gamma\cdot\mathbf{n}_\Gamma\, u_D|_\Gamma\right) dS.
\end{aligned}
$$

Using that $\theta = -1$, assumption (1.5) and the definition of the $\|\cdot\|_{DG,m}$-norm, we get

$$
\begin{aligned}
& a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U) \\
& \geq \beta_0 \|U\|^2_{DG,m} - \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_\Gamma \beta(U) \nabla U|_\Gamma \cdot \mathbf{n}_\Gamma\, u_D|_\Gamma\, dS.
\end{aligned} \tag{3.30}
$$

Now we have to estimate the last term on the right-hand side of (3.30). Using properties of the function β and Young's inequality for each $k_1, \delta > 0$ we have

$$
\begin{aligned}
& \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_\Gamma |\beta(U) \nabla U|_\Gamma \cdot \mathbf{n}_\Gamma\, u_D|_\Gamma|\; dS \\
& \leq \beta_1 \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_\Gamma |\nabla U|_\Gamma|\; |u_D|_\Gamma|\; dS \\
& \leq \frac{\beta_1 k_1}{\delta} \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_\Gamma h^{-1}_{K^{(L)}_\Gamma} |u_D|_\Gamma|^2\; dS + \frac{\beta_1 \delta}{k_1} \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_\Gamma h_{K^{(L)}_\Gamma} |\nabla U|_\Gamma|^2\; dS.
\end{aligned}
$$

If we set $\delta := \frac{\beta_0}{\beta_1}$ and use the definition of the form $J^B_{h,m}$, then we obtain

$$
\begin{aligned}
& \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_\Gamma |\beta(U) \nabla U|_\Gamma \cdot \mathbf{n}_\Gamma\, u_D|_\Gamma|\; dS \\
& \leq \frac{\beta_1^2 k_1}{\beta_0 c_W} J^B_{h,m}(u_D, u_D) + \frac{\beta_0}{k_1} \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_{\partial K^{(L)}_\Gamma} h_{K^{(L)}_\Gamma} |\nabla U|_\Gamma|^2\; dS.
\end{aligned}
$$

Now, we express the first term on the right-hand side with the aid of the definition of the $\|\cdot\|_{DGB,m}$-norm (3.21), and to the second term we apply the multiplicative trace inequality. We get

$$
\begin{aligned}
& \frac{\beta_1^2 k_1}{\beta_0 c_W} J^B_{h,m}(u_D, u_D) + \frac{\beta_0}{k_1} \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_{\partial K^{(L)}_\Gamma} h_{K^{(L)}_\Gamma} |\nabla U|_\Gamma|^2\; dS \\
& \leq \frac{\beta_1^2 k_1}{\beta_0 c_W} \|u_D\|^2_{DGB,m} + \frac{\beta_0}{k_1} c_M \sum_{K \in \mathcal{T}_{h,m}} \left(h_K \|\nabla U\|_{L^2(K)} |\nabla U|_{H^1(K)} + \|\nabla U\|^2_{L^2(K)} \right).
\end{aligned}
$$

Taking into account that components of ∇U are elements of $P^p(K)$, the inverse inequality implies that

$$
|\nabla U|_{H^1(K)} \leq c_I h_K^{-1} \|\nabla U\|_{L^2(K)}.
$$

To summarize, we showed that

$$
\begin{aligned}
& \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_\Gamma |\beta(U) \nabla U|_\Gamma \cdot \mathbf{n}_\Gamma\, u_D|_\Gamma|\; dS \\
& \leq \frac{\beta_1^2 k_1}{\beta_0 c_W} \|u_D\|^2_{DGB,m} + \frac{\beta_0}{k_1} c_M (c_I + 1) \sum_{K \in \mathcal{T}_{h,m}} \|\nabla U\|^2_{L^2(K)}.
\end{aligned}
$$

If we use the inequality

$$\sum_{K\in\mathcal{T}_{h,m}} \|\nabla U\|^2_{L^2(K)} \leq \|U\|^2_{DG,m},$$

which obviously follows from the definition of the $\|\cdot\|_{DG,m}$-norm, we get

$$\begin{aligned} &\sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma |\beta(U)\nabla U|_\Gamma \cdot \mathbf{n}_\Gamma \, u_D|_\Gamma| \; dS \\ &\leq \frac{\beta_1^2 k_1}{\beta_0 c_W} \|u_D\|^2_{DGB,m} + \frac{\beta_0}{k_1} c_M(c_I+1)\|U\|^2_{DG,m}. \end{aligned} \tag{3.31}$$

Substituting back to (3.30) and integrating over the interval I_m, we obtain

$$\begin{aligned} &\int_{I_m} (a_{h,m}(U,U) + \beta_0 \, J_{h,m}(U,U)) \, dt \\ &\geq \beta_0 \left(1 - \frac{1}{k_1} c_M(c_I+1)\right) \int_{I_m} \|U\|^2_{DG,m} \, dt - \frac{\beta_1^2 k_1}{\beta_0 c_W} \int_{I_m} \|u_D\|^2_{DGB,m} \, dt. \end{aligned}$$

If we set $k_1 = 2c_M(c_I+1)$ and use assumption (3.28), we finally find that

$$\int_{I_m} (a_{h,m}(U,U) + \beta_0 \, J_{h,m}(U,U)) \, dt \geq \frac{\beta_0}{2} \int_{I_m} \|U\|^2_{DG,m} \, dt - \frac{\beta_0}{2} \int_{I_m} \|u_D\|^2_{DGB,m} \, dt.$$

□

Lemma 6. *Let* $\theta = 0$ *and*

$$c_W \geq \frac{4\beta_1^2}{\beta_0^2} c_M(c_I+1)(c_Q+1). \tag{3.32}$$

Then for the approximate solution U *of problem (1.1) - (1.3) we have*

$$\int_{I_m} (a_{h,m}(U,U) + \beta_0 \, J_{h,m}(U,U)) \, dt \geq \frac{\beta_0}{2} \int_{I_m} \|U\|^2_{DG,m} \, dt. \tag{3.33}$$

Proof. We rewrite the integrand on the left-hand side using (2.13) similarly as in the proof of the previous lemma:

$$\begin{aligned} &a_{h,m}(U,U) + \beta_0 \, J_{h,m}(U,U) \\ &= \sum_{K\in\mathcal{T}_{h,m}} \int_K \beta(U)\nabla U \cdot \nabla U \, dx + \beta_0 \, J_{h,m}(U,U) \\ &- \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma (\langle\beta(U)\nabla U\rangle_\Gamma \cdot \mathbf{n}_\Gamma \, [U]_\Gamma + \theta \, \langle\beta(U)\nabla U\rangle_\Gamma \cdot \mathbf{n}_\Gamma \, [U]_\Gamma) \; dS \\ &- \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma (\beta(u)\nabla U|_\Gamma \cdot \mathbf{n}_\Gamma \, U|_\Gamma + \theta\beta(U)\nabla U|_\Gamma \cdot \mathbf{n}_\Gamma \, U|_\Gamma - \theta\beta(U)\nabla U|_\Gamma \cdot \mathbf{n}_\Gamma \, u_D|_\Gamma) \; dS. \end{aligned}$$

Using that $\theta = 0$, assumption (1.5) and the definition of the $\|.\|_{DG,m}$-norm, we get

$$
\begin{aligned}
& a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U) \qquad (3.34)\\
& \geq \beta_0 \|U\|^2_{DG,m} \\
& -\beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma |\,\langle\nabla U\rangle_\Gamma \cdot \mathbf{n}_\Gamma\,[U]_\Gamma|\,dS - \beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma |\nabla U|_\Gamma \cdot \mathbf{n}_\Gamma\, U|_\Gamma|\,dS \\
& \geq \beta_0 \|U\|^2_{DG,m} \\
& -\beta_1 \left(\sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \frac{|\nabla U_\Gamma^{(L)}| + |\nabla U_\Gamma^{(R)}|}{2}\, |[U]_\Gamma|\,dS + \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma |\nabla U|_\Gamma|\,|U|_\Gamma|\,dS \right).
\end{aligned}
$$

Now applying Young's inequality with $\delta > 0$ separately to the first and the second term above in round brackets and using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ valid for $a, b \in \mathbb{R}$, we obtain

$$
\begin{aligned}
& \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \frac{|\nabla U_\Gamma^{(L)}| + |\nabla U_\Gamma^{(R)}|}{2}\, |[U]_\Gamma|\,dS + \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma |\nabla U|_\Gamma|\,|U|_\Gamma|\,dS \qquad (3.35)\\
& \leq \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \frac{h(\Gamma)}{\delta c_W} \frac{\left(|\nabla U_\Gamma^{(L)}| + |\nabla U_\Gamma^{(R)}|\right)^2}{4}\,dS + \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \frac{\delta c_W}{h(\Gamma)} |[U]_\Gamma|^2\,dS \\
& + \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \frac{h(\Gamma)}{\delta c_W} \left|\nabla U_\Gamma^{(L)}\right|^2\,dS + \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \frac{\delta c_W}{h(\Gamma)} |U|_\Gamma|^2 dS \\
& \leq \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \frac{h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}}{2\delta c_W} \frac{|\nabla U_\Gamma^{(L)}|^2 + |\nabla U_\Gamma^{(R)}|^2}{2}\,dS \\
& + \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \frac{h_{K_\Gamma^{(L)}}}{\delta c_W} \left|\nabla U_\Gamma^{(L)}\right|^2 dS + \delta J_{h,m}(U,U).
\end{aligned}
$$

Using (3.2), we get

$$
\begin{aligned}
& \frac{1}{4\delta c_W} \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}\right) \left(|\nabla U_\Gamma^{(L)}|^2 + |\nabla U_\Gamma^{(R)}|^2\right) dS \qquad (3.36)\\
& + \frac{1}{\delta c_W} \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma h_{K_\Gamma^{(L)}} \left|\nabla U_\Gamma^{(L)}\right|^2 dS + \delta J_{h,m}(U,U) \\
& \leq \frac{c_Q+1}{4\delta c_W} \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(h_{K_\Gamma^{(L)}} |\nabla U_\Gamma^{(L)}|^2 + h_{K_\Gamma^{(R)}} |\nabla U_\Gamma^{(R)}|^2\right) dS \\
& + \frac{1}{\delta c_W} \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma h_{K_\Gamma^{(L)}} \left|\nabla U_\Gamma^{(L)}\right|^2 dS + \delta J_{h,m}(U,U) \\
& \leq \frac{c_Q+1}{\delta c_W} \sum_{K\in\mathcal{T}_{h,m}} \int_{\partial K} h_K |\nabla U|^2\,dS + \delta J_{h,m}(U,U).
\end{aligned}
$$

In the last inequality we used that $c_Q > 0$ and

$$\frac{c_Q+1}{4\delta c_W} \le \frac{c_Q+1}{\delta c_W}, \quad \frac{1}{\delta c_W} \le \frac{c_Q+1}{\delta c_W}.$$

It follows from (3.34)-(3.36) that

$$\begin{aligned} &a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U) \\ &\ge \beta_0 \|U\|^2_{DG,m} - \frac{\beta_1(c_Q+1)}{\delta c_W} \sum_{K\in\mathcal{T}_{h,m}} \int_{\partial K} h_K |\nabla U|^2\, dS - \beta_1 \delta J_{h,m}(U,U). \end{aligned} \tag{3.37}$$

Now we estimate the integral on the right-hand side of (3.37) using the multiplicative trace inequality and the inverse inequality. We find that

$$\begin{aligned} &\int_{\partial K} h_K |\nabla U|^2\, dS = h_K \|\nabla U\|^2_{L^2(\partial K)} \\ &\le h_K c_M (\|\nabla U\|_{L^2(K)} \underbrace{|\nabla U|_{H^1(K)}}_{\le c_I h_K^{-1} \|\nabla U\|_{L^2(K)}} + h_K^{-1} \|\nabla U\|^2_{L^2(K)}) \\ &\le c_M(1+c_I) \|\nabla U\|^2_{L^2(K)} = c_M(1+c_I)|U|^2_{H^1(K)}. \end{aligned}$$

Substituting back to (3.37), we obtain

$$\begin{aligned} &a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U) \\ &\ge \beta_0 \|U\|^2_{DG,m} - \frac{\beta_1 c_M (1+c_I)(c_Q+1)}{\delta c_W} \sum_{K\in\mathcal{T}_{h,m}} |U|^2_{H^1(K)} - \beta_1 \delta J_{h,m}(U,U). \end{aligned} \tag{3.38}$$

If we set $\delta = \frac{\beta_0}{2\beta_1}$, then we find that

$$\begin{aligned} &a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U) \\ &\ge \beta_0 \|U\|^2_{DG,m} - \frac{2\beta_1^2 c_M (1+c_I)(c_Q+1)}{\beta_0 c_W} \sum_{K\in\mathcal{T}_{h,m}} |U|^2_{H^1(K)} - \frac{\beta_0}{2} J_{h,m}(U,U). \end{aligned} \tag{3.39}$$

Using assumption (3.32) for the constant c_W and the definition of the DG-norm, we finally have

$$a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U) \ge \frac{\beta_0}{2} \|U\|^2_{DG,m}. \tag{3.40}$$

Integrating both sides over the interval I_m, we get (3.33). □

Lemma 7. *Let $\theta = 1$ and*

$$c_W \ge \frac{64\beta_1^2}{\beta_0^2} c_M (c_I+1)(c_Q+1). \tag{3.41}$$

Then the approximate solution U of problem (1.1) - (1.3) satisfies

$$\begin{aligned} &\int_{I_m} (a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U))\, dt \\ &\qquad \ge \frac{\beta_0}{2} \int_{I_m} \|U\|^2_{DG,m}\, dt - \frac{\beta_0}{2} \int_{I_m} \|u_D\|^2_{DGB,m}\, dt. \end{aligned} \tag{3.42}$$

Proof. We rewrite the integrand on the left-hand side using (2.13) similarly as in the proof of the previous lemma:

$$
\begin{aligned}
& a_{h,m}(U,U) + \beta_0 \, J_{h,m}(U,U) \\
& = \sum_{K \in \mathcal{T}_{h,m}} \int_K \beta(U) \nabla U \cdot \nabla U \, dx + \beta_0 \, J_{h,m}(U,U) \\
& - \sum_{\Gamma \in \mathcal{F}_{h,m}^I} \int_\Gamma \left(\langle \beta(U) \nabla U \rangle_\Gamma \cdot \mathbf{n}_\Gamma \, [U]_\Gamma + \theta \, \langle \beta(U) \nabla U \rangle_\Gamma \cdot \mathbf{n}_\Gamma \, [U]_\Gamma \right) \, dS \\
& - \sum_{\Gamma \in \mathcal{F}_{h,m}^B} \int_\Gamma \left(\beta(u) \nabla U|_\Gamma \cdot \mathbf{n}_\Gamma \, U|_\Gamma + \theta \beta(U) \nabla U|_\Gamma \cdot \mathbf{n}_\Gamma \, U|_\Gamma - \theta \beta(U) \nabla U|_\Gamma \cdot \mathbf{n}_\Gamma \, u_D|_\Gamma \right) \, dS.
\end{aligned}
$$

Using $\theta = 1$, assumption (1.5) and the definition of the $\| \cdot \|_{DG,m}$-norm, we get

$$
\begin{aligned}
& a_{h,m}(U,U) + \beta_0 \, J_{h,m}(U,U) \\
& \geq \beta_0 \|U\|_{DG,m}^2 \\
& -2\beta_1 \sum_{\Gamma \in \mathcal{F}_{h,m}^I} \int_\Gamma | \langle \nabla U \rangle_\Gamma \cdot \mathbf{n}_\Gamma \, [U]_\Gamma | \, dS - 2\beta_1 \sum_{\Gamma \in \mathcal{F}_{h,m}^B} \int_\Gamma |\nabla U|_\Gamma \cdot \mathbf{n}_\Gamma \, U|_\Gamma| \, dS \\
& -\beta_1 \sum_{\Gamma \in \mathcal{F}_{h,m}^B} \int_\Gamma |\nabla U|_\Gamma \cdot \mathbf{n}_\Gamma u_D|_\Gamma| \, dS \\
& \geq \beta_0 \|U\|_{DG,m}^2 \\
& -2\beta_1 \left(\sum_{\Gamma \in \mathcal{F}_{h,m}^I} \int_\Gamma \frac{|\nabla U_\Gamma^{(L)}| + |\nabla U_\Gamma^{(R)}|}{2} \, |[U]_\Gamma| \, dS + \sum_{\Gamma \in \mathcal{F}_{h,m}^B} \int_\Gamma |\nabla U|_\Gamma| \, |U|_\Gamma| \, dS \right) \\
& -\beta_1 \sum_{\Gamma \in \mathcal{F}_{h,m}^B} \int_\Gamma |\nabla U|_\Gamma| \, |u_D|_\Gamma| \, dS.
\end{aligned}
\tag{3.43}
$$

Expression in round brackets have been already estimated in the proof of the previous lemma. We have

$$
\begin{aligned}
& \sum_{\Gamma \in \mathcal{F}_{h,m}^I} \int_\Gamma \frac{|\nabla U_\Gamma^{(L)}| + |\nabla U_\Gamma^{(R)}|}{2} \, |[U]_\Gamma| \, dS + \sum_{\Gamma \in \mathcal{F}_{h,m}^B} \int_\Gamma |\nabla U|_\Gamma| \, |U| \, dS \\
& \leq \frac{c_Q + 1}{\delta c_W} \sum_{K \in \mathcal{T}_{h,m}} \int_{\partial K} h_K |\nabla U|^2 \, dx + \delta J_{h,m}(U,U) \\
& \leq c_M (1 + c_I) \frac{c_Q + 1}{\delta c_W} \sum_{K \in \mathcal{T}_{h,m}} |U|_{H^1(K)}^2 + \delta J_{h,m}(U,U).
\end{aligned}
\tag{3.44}
$$

It follows from (3.43)-(3.44) that

$$
\begin{aligned}
& a_{h,m}(U,U) + \beta_0 \, J_{h,m}(U,U) \\
& \geq \beta_0 \|U\|_{DG,m}^2 - \frac{2\beta_1 c_M (1 + c_I)(c_Q + 1)}{\delta c_W} \sum_{K \in \mathcal{T}_{h,m}} |U|_{H^1(K)}^2 - 2\beta_1 \delta J_{h,m}(U,U) \\
& -\beta_1 \sum_{\Gamma \in \mathcal{F}_{h,m}^B} \int_\Gamma |\nabla U|_\Gamma| \, |u_D|_\Gamma| \, dS.
\end{aligned}
\tag{3.45}
$$

The last term on the right-hand side can be estimated similarly as in the proof of Lemma 5. For each $k_1 > 0$ we get

$$
\begin{aligned}
&\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma |\nabla U|_\Gamma|\,|u_D|_\Gamma|\,dS && (3.46)\\
&\le \beta_1 k_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma h^{-1}_{K^{(L)}_\Gamma}\,|u_D|_\Gamma|^2\,dS + \frac{\beta_1}{k_1}\sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma h_{K^{(L)}_\Gamma}\,|\nabla U|_\Gamma|^2\,dS\\
&\le \frac{\beta_1 k_1}{c_W} J^B_{h,m}(u_D,u_D) + \frac{\beta_1}{k_1}\sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma h_{K^{(L)}_\Gamma}\,|\nabla U|_\Gamma|^2\,dS\\
&\le \frac{\beta_1 k_1}{c_W}\|u_D\|^2_{DGB,m} + \frac{\beta_1}{k_1} c_M(c_I+1)\sum_{K\in\mathcal{T}^B_{h,m}} \|\nabla U\|^2_{L^2(K)}\\
&\le \frac{\beta_1 k_1}{c_W}\|u_D\|^2_{DGB,m} + \frac{\beta_1}{k_1} c_M(c_I+1)\|U\|^2_{DG,m}.
\end{aligned}
$$

Substituting back to (3.45), we obtain

$$
\begin{aligned}
&a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U) && (3.47)\\
&\ge \beta_0\|U\|^2_{DG,m} - \frac{2\beta_1 c_M(c_I+1)(c_Q+1)}{\delta c_W}\sum_{K\in\mathcal{T}_{h,m}} |U|^2_{H^1(K)} - 2\beta_1\delta J_{h,m}(U,U)\\
&-\frac{\beta_1 k_1}{c_W}\|u_D\|^2_{DGB,m} - \frac{\beta_1}{k_1} c_M(c_I+1)\|U\|^2_{DG,m}.
\end{aligned}
$$

If we set $\delta := \frac{\beta_0}{8\beta_1}$ and $k_1 := 4\frac{\beta_1}{\beta_0} c_M(c_I+1)$, we find that

$$
\begin{aligned}
&a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U) && (3.48)\\
&\ge \beta_0\|U\|^2_{DG,m} - \frac{16\beta_1^2 c_M(c_I+1)(c_Q+1)}{\beta_0 c_W}\sum_{K\in\mathcal{T}_{h,m}} |U|^2_{H^1(K)} - \frac{\beta_0}{4} J_{h,m}(U,U)\\
&-\frac{4\beta_1^2}{\beta_0 c_W} c_M(c_I+1)\|u_D\|^2_{DGB,m} - \frac{\beta_0}{4}\|U\|^2_{DG,m}.
\end{aligned}
$$

Using assumption (3.41) for the constant c_W, we have

$$
\begin{aligned}
&a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U) && (3.49)\\
&\ge \beta_0\|U\|^2_{DG,m} - \frac{\beta_0}{4}\sum_{K\in\mathcal{T}_{h,m}} |U|^2_{H^1(K)} - \frac{\beta_0}{4} J_{h,m}(U,U)\\
&-\frac{\beta_0}{16(c_Q+1)}\|u_D\|^2_{DGB,m} - \frac{\beta_0}{4}\|U\|^2_{DG,m}\\
&\ge \frac{\beta_0}{2}\|U\|^2_{DG,m} - \frac{\beta_0}{2}\|u_D\|^2_{DGB,m}.
\end{aligned}
$$

Finally using the definition of $\|\cdot\|_{DG,m}$-norm and integrating over the interval I_m, we get

$$\int_{I_m} \left(a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U)\right)\, dt \\ \geq \frac{\beta_0}{2}\int_{I_m} \|U\|^2_{DG,m}\, dt - \frac{\beta_0}{2}\int_{I_m} \|u_D\|^2_{DGB,m}\, dt.$$

□

Results of previous three lemmas we can summarize as follows:

Corrolary 1. *Let*

$$c_W \geq \frac{64\beta_1^2}{\beta_0^2} c_M(c_I+1)(c_Q+1) \quad \text{for} \quad \theta = 1\,(SIPG),$$
$$c_W \geq \frac{4\beta_1^2}{\beta_0^2} c_M(c_I+1)(c_Q+1) \quad \text{for} \quad \theta = 0\,(IIPG),$$
$$c_W \geq \frac{4\beta_1^2}{\beta_0^2} c_M(c_I+1) \quad \text{for} \quad \theta = -1\,(NIPG).$$

Then

$$\int_{I_m} \left(a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U)\right)\, dt \\ \geq \frac{\beta_0}{2}\int_{I_m} \|U\|^2_{DG,m}\, dt - \frac{\beta_0}{2}\int_{I_m} \|u_D\|^2_{DGB,m}\, dt. \tag{3.50}$$

Proof. It follows from Lemma 5, Lemma 6 and Lemma 7. □

Lemma 8. *For each $k_2 > 0$ there exists a constant $c_b > 0$ such that for the approximate solution U of the problem (1.1) - (1.3) we have the inequality*

$$\int_{I_m} |b_{h,m}(U,U)|\, dt \leq \frac{\beta_0}{k_2}\int_{I_m} \|U\|^2_{DG,m}\, dt + c_b \int_{I_m} \|U\|^2\, dt. \tag{3.51}$$

(The constant c_b depends on k_2, namely, $c_b = c_1^2 \frac{k_2}{\beta_0}$, where $c_1 > 0$ is independent of k_2.)

Proof. Using (2.17), we get

$$b_{h,m}(U,U) = \underbrace{-\sum_{K\in\mathcal{T}_{h,m}} \int_K \sum_{s=1}^{d} f_s(U)\frac{\partial U}{\partial x_s}\, dx}_{:=\sigma_1} \tag{3.52}$$
$$\underbrace{+ \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma H(U_\Gamma^{(L)}, U_\Gamma^{(R)}, \mathbf{n}_\Gamma)\,[U]_\Gamma\, dS + \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma H(U_\Gamma^{(L)}, U_\Gamma^{(L)}, \mathbf{n}_\Gamma)\, U|_\Gamma\, dS,}_{:=\sigma_2}$$

Then from the Lipschitz-continuity of functions f_s, $s = 1, \dots, d$ with the modul $L_f > 0$, property (1.4) and the discrete Cauchy inequality, we obtain

$$
\begin{aligned}
|\sigma_1| &\le \sum_{K \in \mathcal{T}_{h,m}} \int_K \sum_{s=1}^{d} |f_s(U) - f_s(0)| \left| \frac{\partial U}{\partial x_s} \right| dx \qquad (3.53)\\
&\le L_f \sum_{K \in \mathcal{T}_{h,m}} \int_K \sum_{s=1}^{d} |U| \left| \frac{\partial U}{\partial x_s} \right| dx \\
&\le L_f \sqrt{d}\, \|U\|_{L^2(\Omega)} |U|_{H^1(\Omega, \mathcal{T}_h)}.
\end{aligned}
$$

Now we shall estimate σ_2. Since the fluxes are approximated by the numerical flux H, from $f_s(0) = 0$, $s = 1, \dots, d$, it follows, that $H(0, 0, \mathbf{n}_\Gamma) = 0$. Then we can use the Lipschitz-continuity of the fluxes H, formulated as the property $(H1)$. We get

$$
\begin{aligned}
|\sigma_2| \le\ & L_H \sum_{\Gamma \in \mathcal{F}_{h,m}^I} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)\, |[U]_\Gamma|\, dS \qquad (3.54)\\
+\ & L_H \sum_{\Gamma \in \mathcal{F}_{h,m}^B} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(L)}|)\, |U|_\Gamma^{(L)}|\, dS. \qquad (3.55)
\end{aligned}
$$

Using that $U_\Gamma^{(L)} = U_\Gamma^{(R)}$ for $\Gamma \in \mathcal{F}_{h,m}^B$, Cauchy inequality, definition (2.14) and the relation $h(\Gamma) \le \frac{c_Q+1}{2} h_K$, if $\Gamma \subset \partial K$, we obtain

$$
\begin{aligned}
|\sigma_2| &\le L_H \sum_{\Gamma \in \mathcal{F}_{h,m}^I} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)\, |[U]_\Gamma|\, dS \qquad (3.56)\\
&\quad + L_H \sum_{\Gamma \in \mathcal{F}_{h,m}^B} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)\, |U|_\Gamma^{(L)}|\, dS \\
&\le \frac{L_H}{\sqrt{c_W}} \left(c_W \sum_{\Gamma \in \mathcal{F}_{h,m}^I} \int_\Gamma \frac{[U]_\Gamma^2}{h(\Gamma)}\, dS + c_W \sum_{\Gamma \in \mathcal{F}_{h,m}^B} \int_\Gamma \frac{\left(U|_\Gamma^{(L)}\right)^2}{h(\Gamma)}\, dS \right)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{\Gamma \in \mathcal{F}_{h,m}} h(\Gamma) \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)^2\, dS \right)^{\frac{1}{2}} \\
&\le \frac{L_H}{\sqrt{c_W}} J_{h,m}(U, U)^{\frac{1}{2}} \left(\sum_{\Gamma \in \mathcal{F}_{h,m}} 2h(\Gamma) \int_\Gamma |U_\Gamma^{(L)}|^2 + |U_\Gamma^{(R)}|^2\, dS \right)^{\frac{1}{2}} \\
&\le L_H \sqrt{\frac{c_Q+1}{c_W}}\, J_{h,m}(U, U)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{\Gamma \in \mathcal{F}_{h,m}} h_{K_\Gamma^{(L)}} \int_{\partial K_\Gamma^{(L)} \cap \Gamma} |U_\Gamma^{(L)}|^2 dS + h_{K_\Gamma^{(R)}} \int_{\partial K_\Gamma^{(R)} \cap \Gamma} |U_\Gamma^{(R)}|^2\, dS \right)^{\frac{1}{2}} \\
&\le L_H \sqrt{\frac{c_Q+1}{c_W}}\, J_{h,m}(U, U)^{\frac{1}{2}} \left(\sum_{K \in \mathcal{T}_{h,m}} \int_{\partial K} h_K |U|^2\, dS \right)^{\frac{1}{2}}
\end{aligned}
$$

$$= L_H \sqrt{\frac{c_Q+1}{c_W}}\, J_{h,m}(U,U)^{\frac{1}{2}} \left(\sum_{K\in\mathcal{T}_{h,m}} h_K \|U\|^2_{L^2(\partial K)} \right)^{\frac{1}{2}}.$$

Substituting back to (3.52), using the discrete Cauchy inequality and the definition of the $\|\cdot\|_{DG,m}$-norm, we find that

$$\begin{aligned} |b_{h,m}(U,U)| &\leq L_f\sqrt{d}\,\|U\|\,|U|_{H^1(\Omega,\mathcal{T}_{h,m})} \\ &\qquad + L_H \sqrt{\frac{c_Q+1}{c_W}}\, J_{h,m}(U,U)^{\frac{1}{2}} \left(\sum_{K\in\mathcal{T}_{h,m}} h_K \|U\|^2_{L^2(\partial K)} \right)^{\frac{1}{2}} \\ &\leq \left(L_f^2 d\,\|U\|^2 + L_H^2 \frac{c_Q+1}{c_W} \sum_{K\in\mathcal{T}_{h,m}} h_K \|U\|^2_{L^2(\partial K)} \right)^{\frac{1}{2}} \\ &\qquad \times \left(|U|^2_{H^1(\Omega,\mathcal{T}_{h,m})} + J_{h,m}(U,U) \right)^{\frac{1}{2}} \\ &\leq c\,\|U\|_{DG,m} \left(\|U\| + \left(\sum_{K\in\mathcal{T}_{h,m}} h_K \|U\|^2_{L^2(\partial K)} \right)^{\frac{1}{2}} \right), \end{aligned} \tag{3.57}$$

where $c = \left(\max\{L_f^2\, d,\, L_H^2 \frac{c_Q+1}{c_W}\}\right)^{\frac{1}{2}}$. Furthermore, using the multiplicative trace inequality and the inverse inequality, we get

$$\begin{aligned} &\sum_{K\in\mathcal{T}_{h,m}} h_K \|U\|^2_{L^2(\partial K)} \\ &\leq c_M \sum_{K\in\mathcal{T}_{h,m}} h_K \left(\|U\|_{L^2(K)} |U|_{H^1(K)} + h_K^{-1} \|U\|^2_{L^2(K)} \right) \\ &\leq c_M (c_I + 1) \sum_{K\in\mathcal{T}_{h,m}} \|U\|^2_{L^2(K)} \\ &= c_M (c_I+1) \|U\|^2. \end{aligned}$$

Hence, using this relation and Young's inequality we shall continue in estimating $|b_{h,m}(U,U)|$. We obtain

$$\begin{aligned} |b_{h,m}(U,U)| &\leq c\,\|U\|_{DG,m} \left(\|U\| + \left(\sum_{K\in\mathcal{T}_{h,m}} h_K \|U\|^2_{L^2(\partial K)} \right)^{\frac{1}{2}} \right) \\ &\leq c_1\,\|U\|_{DG,m}\,\|U\| \\ &\leq \frac{\beta_0}{k_2} \|U\|^2_{DG,m} + c_1^2 \frac{k_2}{\beta_0} \|U\|^2 \\ &= \frac{\beta_0}{k_2}\,\|U\|^2_{DG,m} + c_b\,\|U\|^2, \end{aligned}$$

where $c_1 = c(1 + \sqrt{c_M(c_I+1)})$, $k_2 > 0$ and $c_b = c_1^2 \frac{k_2}{\beta_0}$.

Integrating over the interval I_m, we finally have

$$\int_{I_m} |b_{h,m}(U,U)|dt \le \frac{\beta_0}{k_2}\int_{I_m} \|U\|^2_{DG,m} dt + c_b \int_{I_m} \|U\|^2 dt.$$

□

Lemma 9. *For the approximate solution U of problem (1.1) - (1.3) and each $k_3 > 0$ we have*

$$\int_{I_m} |l_{h,m}(U)|\, dt \tag{3.58}$$
$$\le \frac{1}{2}\int_{I_m} \left(\|g\|^2 + \|U\|^2\right) dt + \beta_0 k_3 \int_{I_m} \|u_D\|^2_{DGB,m}\, dt + \frac{\beta_0}{k_3}\int_{I_m} \|U\|^2_{DG,m}\, dt.$$

Proof. From (2.18) we get

$$|l_{h,m}(U)| = \left|(g,U) + \beta_0\, c_W \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h(\Gamma)^{-1} \int_\Gamma u_D|_\Gamma\, U|_\Gamma\, dS\right|.$$

After using the Cauchy inequality for the first term on the right-hand side and applying Young's inequality with $k_3 > 0$ for the second term, we find that

$$\left|(g,U) + \beta_0\, c_W \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h(\Gamma)^{-1} \int_\Gamma u_D|_\Gamma\, U|_\Gamma\, dS\right|$$
$$\le \frac{1}{2}(\|g\|^2 + \|U\|^2) + \beta_0 k_3 \underbrace{C_W \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h^{-1}_{K^{(L)}_\Gamma} \int_\Gamma |u_D|_\Gamma|^2\, dS}_{=\|u_D\|^2_{DGB,m}}$$
$$+\frac{\beta_0}{k_3} \underbrace{c_W \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h^{-1}_{K^{(L)}_\Gamma} \int_\Gamma |U|_\Gamma|^2\, dS}_{\le J_{h,m}(U,U)\le \|U\|^2_{DG,m}}.$$

To summarize, we see that

$$|l_{h,m}(U)| \le \frac{1}{2}(\|g\|^2 + \|U\|^2) + \beta_0 k_3 \|u_D\|^2_{DGB,m} + \frac{\beta_0}{k_3}\|U\|^2_{DG,m},$$

from which we get (3.58) by integrating both sides over the interval I_m. □

Now we have already estimated all terms in our basic identity (3.24). To summarize, we showed that

$$\underbrace{\int_{I_m}\left(\frac{\partial U}{\partial t},U\right)dt+(\{U\}_{m-1},\varphi_{m-1}^+)}_{=\frac{1}{2}\left(\|U_m^-\|^2-\|U_{m-1}^-\|^2+\|\{U\}_{m-1}\|^2\right)}+\underbrace{\int_{I_m}a_{h,m}(U,U)+\beta_0\,J_{h,m}(U,U)\,dt}_{\geq\frac{\beta_0}{2}\int_{I_m}\|U\|_{DG,m}^2\,dt-\frac{\beta_0}{2}\int_{I_m}\|u_D\|_{DGB,m}^2\,dt}$$

$$=\underbrace{\int_{I_m}l_{h,m}(U)\,dt}_{\leq\frac{1}{2}\int_{I_m}(\|g\|^2+\|U\|^2)\,dt+\beta_0k_3\int_{I_m}\|u_D\|_{DGB,m}^2\,dt+\frac{\beta_0}{k_3}\int_{I_m}\|U\|_{DG,m}^2\,dt}-\underbrace{\int_{I_m}b_{h,m}(U,U)\,dt}_{\leq\frac{\beta_0}{k_2}\int_{I_m}\|U\|_{DG,m}^2\,dt+c_b\int_{I_m}\|U\|^2\,dt},$$

which can be rewritten as follows

$$\begin{aligned}&\frac{1}{2}\left(\|U_m^-\|^2-\|U_{m-1}^-\|^2+\|\{U\}_{m-1}\|^2\right)+\frac{\beta_0}{2}\int_{I_m}\|U\|_{DG,m}^2\,dt-\frac{\beta_0}{2}\int_{I_m}\|u_D\|_{DGB,m}^2\,dt\\ &\leq\frac{1}{2}\int_{I_m}\left(\|g\|^2+\|U\|^2\right)\,dt+\beta_0k_3\int_{I_m}\|u_D\|_{DGB,m}^2\,dt+\frac{\beta_0}{k_3}\int_{I_m}\|U\|_{DG,m}^2\,dt\\ &\qquad+\frac{\beta_0}{k_2}\int_{I_m}\|U\|_{DG,m}^2\,dt+c_b\int_{I_m}\|U\|^2\,dt.\end{aligned}$$

After multiplying by two and reorganizing, we find that

$$\begin{aligned}&\|U_m^-\|^2-\|U_{m-1}^-\|^2+\|\{U\}_{m-1}\|^2+\beta_0\left(1-\frac{2}{k_2}-\frac{2}{k_3}\right)\int_{I_m}\|U\|_{DG,m}^2\,dt\\ &\leq\int_{I_m}\left(\|g\|^2+\|U\|^2\right)\,dt+\beta_0(1+2k_3)\int_{I_m}\|u_D\|_{DGB,m}^2\,dt+2c_b\int_{I_m}\|U\|^2\,dt.\end{aligned}$$

By setting $k_2=k_3=8$, $c=\max\{2c_b+1,17\beta_0\}$ and omitting the positive term $\|\{U\}_{m-1}\|^2$ on the left-hand side, we get

$$\begin{aligned}&\|U_m^-\|^2-\|U_{m-1}^-\|^2+\frac{\beta_0}{2}\int_{I_m}\|U\|_{DG,m}^2dt\\ &\leq c\left(\int_{I_m}\|g\|^2\,dt+\int_{I_m}\|U\|^2\,dt+\int_{I_m}\|u_D\|_{DGB,m}^2dt\right).\end{aligned}\tag{3.59}$$

Now our further task is to estimate the term $\int_{I_m}\|U\|^2\,dt$ using g and u_D. Before doing this, we prove an auxiliary lemma:

Lemma 10. *There exists a konstant $c > 0$ such that*

$$a_{h,m}(U,\tilde{U}) + \beta_0 J_{h,m}(U,\tilde{U}) \le c\left(\|U\|^2_{DG,m} + \|\tilde{U}\|^2_{DG,m} + \|u_D\|^2_{DGB,m}\right) \quad (3.60)$$

for all $\tilde{U} \in S^p_{h,m}$.

Proof. From the definition of the forms (2.13) and (2.14) we immediately have

$$\begin{aligned}
a_{h,m}(U,\tilde{U}) &= \sum_{K\in\mathcal{T}_{h,m}} \int_K \beta(U)\nabla U\cdot\nabla\tilde{U}\,dx \\
&- \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(\langle\beta(U)\nabla U\rangle_\Gamma\cdot\mathbf{n}_\Gamma\,[\tilde{U}]_\Gamma + \theta\left\langle\beta(U)\nabla\tilde{U}\right\rangle_\Gamma\cdot\mathbf{n}_\Gamma\,[U]_\Gamma\right) dS \\
&- \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \left(\beta(U)\nabla U|_\Gamma\cdot\mathbf{n}_\Gamma\,\tilde{U}|_\Gamma + \theta\beta(U)\nabla\tilde{U}|_\Gamma\cdot\mathbf{n}_\Gamma\,U|_\Gamma - \theta\beta(U)\nabla\tilde{U}|_\Gamma\cdot\mathbf{n}_\Gamma\,u_D|_\Gamma\right) dS, \\
J_{h,m}(U,\tilde{U}) &= c_W \sum_{\Gamma\in\mathcal{F}^I_{h,m}} h(\Gamma)^{-1}\int_\Gamma [U]_\Gamma\,[\tilde{U}]_\Gamma dS \\
&\quad + c_W \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h(\Gamma)^{-1}\int_\Gamma U|_\Gamma\,\tilde{U}|_\Gamma\,dS.
\end{aligned}$$

Now, using property (1.5) of the function β, Cauchy's and Young's inequality and estimate (3.31) for $k_1 > 0$, similarly as in Lemma 5 we get

$$\begin{aligned}
&a_{h,m}(U,\tilde{U}) + \beta_0\,J_{h,m}(U,\tilde{U}) \le \beta_1 \sum_{K\in\mathcal{T}_{h,m}} \int_K \left(|\nabla U|^2 + |\nabla\tilde{U}|^2\right) dx \\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{c_W}\left(|\nabla U^{(L)}_\Gamma|^2 + |\nabla U^{(R)}_\Gamma|^2\right) + \frac{c_W}{h(\Gamma)}\,|[\tilde{U}]_\Gamma|^2\right) dS \\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{c_W}\left(|\nabla \tilde{U}^{(L)}_\Gamma|^2 + |\nabla \tilde{U}^{(R)}_\Gamma|^2\right) + \frac{c_W}{h(\Gamma)}\,|[U]_\Gamma|^2\right) dS \\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{c_W}|\nabla U|^2 + \frac{c_W}{h(\Gamma)}\,|\tilde{U}|_\Gamma|^2\right) dS \\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{c_W}|\nabla \tilde{U}|^2 + \frac{c_W}{h(\Gamma)}\,|U|_\Gamma|^2\right) dS \\
&+\frac{\beta_1^2 k_1}{\beta_0 c_W}\|u_D\|^2_{DGB,m} + \frac{\beta_0}{k_1}c_M(c_I+1)\|\tilde{U}\|^2_{DG,m} \\
&+\beta_0\,J_{h,m}(U,\tilde{U}).
\end{aligned}$$

After reorganizing and using the definition of the form $J_{h,m}(\cdot,\cdot)$, we find that

$$a_{h,m}(U,\tilde{U}) + \beta_0\, J_{h,m}(U,\tilde{U}) \le \beta_1 \sum_{K\in\mathcal{T}_{h,m}} \int_K \left(|\nabla U|^2 + |\nabla\tilde{U}|^2\right)\, dx$$
$$+\beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \frac{h(\Gamma)}{c_W} \left(|\nabla U_\Gamma^{(L)}|^2 + |\nabla U_\Gamma^{(R)}|^2\right)\, dS + \beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \frac{h(\Gamma)}{c_W} |\nabla U|_\Gamma|^2\, dS$$
$$+\beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \frac{h(\Gamma)}{c_W} \left(|\nabla \tilde{U}_\Gamma^{(L)}|^2 + |\nabla \tilde{U}_\Gamma^{(R)}|^2\right)\, dS + \beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \frac{h(\Gamma)}{c_W} |\nabla \tilde{U}|_\Gamma|^2\, dS$$
$$+\frac{\beta_1^2 k_1}{\beta_0 c_W} \|u_D\|^2_{DGB,m} + \frac{\beta_0}{k_1} c_M(c_I+1)\|\tilde{U}\|^2_{DG,m}$$
$$+\beta_1\, J_{h,m}(\tilde{U},\tilde{U}) + \beta_1\, J_{h,m}(U,U) + \beta_0\, J_{h,m}(U,\tilde{U}).$$

Rewriting $h(\Gamma)$ due to (2.2) and using the inequality $J_{h,m}(U,\tilde{U}) \le J_{h,m}(U,U) + J_{h,m}(\tilde{U},\tilde{U})$, we have

$$a_{h,m}(U,\tilde{U}) + \beta_0\, J_{h,m}(U,\tilde{U}) \le \beta_1 \sum_{K\in\mathcal{T}_{h,m}} \int_K \left(|\nabla U|^2 + |\nabla\tilde{U}|^2\right)\, dx$$
$$+\frac{\beta_1}{2c_W} \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}\right) \left(|\nabla U_\Gamma^{(L)}|^2 + |\nabla U_\Gamma^{(R)}|^2\right)\, dS$$
$$+\frac{\beta_1}{c_W} \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma h_{K_\Gamma^{(L)}} |\nabla U|_\Gamma|^2\, dS$$
$$+\frac{\beta_1}{2c_W} \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}\right) \left(|\nabla \tilde{U}_\Gamma^{(L)}|^2 + |\nabla \tilde{U}_\Gamma^{(R)}|^2\right)\, dS$$
$$+\frac{\beta_1}{c_W} \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma h_{K_\Gamma^{(L)}} |\nabla \tilde{U}|_\Gamma|^2\, dS$$
$$+\frac{\beta_1^2 k_1}{\beta_0 c_W} \|u_D\|^2_{DGB,m} + \frac{\beta_0}{k_1} c_M(c_I+1)\|\tilde{U}\|^2_{DG,m}$$
$$+(\beta_1+\beta_0)\,(J_{h,m}(U,U) + J_{h,m}(\tilde{U},\tilde{U})).$$

After using inequalities (3.2), we get

$$a_{h,m}(U,\tilde{U}) + \beta_0\, J_{h,m}(U,\tilde{U}) \le \beta_1 \sum_{K\in\mathcal{T}_{h,m}} \int_K \left(|\nabla U|^2 + |\nabla\tilde{U}|^2\right)\, dx \tag{3.61}$$
$$+\frac{\beta_1(c_Q+1)}{2c_W} \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(h_{K_\Gamma^{(L)}} |\nabla U_\Gamma^{(L)}|^2 + h_{K_\Gamma^{(R)}} |\nabla U_\Gamma^{(R)}|^2\right)\, dS$$
$$+\frac{\beta_1}{c_W} \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma h_{K_\Gamma^{(L)}} |\nabla U|_\Gamma|^2\, dS$$
$$+\frac{\beta_1(c_Q+1)}{2c_W} \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(h_{K_\Gamma^{(L)}} |\nabla \tilde{U}_\Gamma^{(L)}|^2 + h_{K_\Gamma^{(R)}} |\nabla \tilde{U}_\Gamma^{(R)}|^2\right)\, dS$$

$$
\begin{aligned}
&+\frac{\beta_1}{c_W} \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma h_{K_\Gamma^{(L)}} |\nabla\tilde{U}|_\Gamma|^2\, dS \\
&+\frac{\beta_1^2 k_1}{\beta_0 c_W} \|u_D\|^2_{DGB,m} + \frac{\beta_0}{k_1} c_M(c_I+1)\|\tilde{U}\|^2_{DG,m} \\
&+(\beta_1+\beta_0)\left(J_{h,m}(U,U)+J_{h,m}(\tilde{U},\tilde{U})\right) \\
&\le \beta_1 \sum_{K\in\mathcal{T}_{h,m}} \int_K \left(|\nabla U|^2 + |\nabla\tilde{U}|^2\right) dx + c_1 \sum_{K\in\mathcal{T}_{h,m}} \int_{\partial K} h_K \left(|\nabla U|^2 + |\nabla\tilde{U}|^2\right) dS \\
&+\frac{\beta_1^2 k_1}{\beta_0 c_W} \|u_D\|^2_{DGB,m} + \frac{\beta_0}{k_1} c_M(c_I+1)\|\tilde{U}\|^2_{DG,m} \\
&+(\beta_1+\beta_0)\left(J_{h,m}(U,U)+J_{h,m}(\tilde{U},\tilde{U})\right),
\end{aligned}
$$

where $c_1 = \max\{\frac{\beta_1(c_Q+1)}{2c_W}, \frac{\beta_1}{c_W}\}$.

Now, applying the multiplicative inequality and the inverse inequality, we can estimate the term $\sum_{K\in\mathcal{T}_{h,m}} \int_{\partial K} h_K \left(|\nabla U|^2 + |\nabla\tilde{U}|^2\right) dS$ as follows

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_{h,m}} \int_{\partial K} h_K \left(|\nabla U|^2 + |\nabla\tilde{U}|^2\right) dS \qquad (3.62) \\
&= \sum_{K\in\mathcal{T}_{h,m}} h_K \left(\|\nabla U\|^2_{L^2(\partial K)} + \|\nabla\tilde{U}\|^2_{L^2(\partial K)}\right) \\
&\le c_M \sum_{K\in\mathcal{T}_{h,m}} h_K(\|\nabla U\|_{L^2(K)} \underbrace{|\nabla U|_{H^1(K)}}_{\le c_I h_K^{-1}\|\nabla U\|_{L^2(K)}} + h_K^{-1}\|\nabla U\|^2_{L^2(K)}) \\
&+c_M \sum_{K\in\mathcal{T}_{h,m}} h_K(\|\nabla \tilde{U}\|_{L^2(K)} \underbrace{|\nabla \tilde{U}|_{H^1(K)}}_{\le c_I h_K^{-1}\|\nabla \tilde{U}\|_{L^2(K)}} + h_K^{-1}\|\nabla \tilde{U}\|^2_{L^2(K)}) \\
&\le c_M(1+c_I) \sum_{K\in\mathcal{T}_{h,m}} \left(\|\nabla U\|^2_{L^2(K)} + \|\nabla\tilde{U}\|^2_{L^2(K)}\right) \\
&= c_M(1+c_I) \sum_{K\in\mathcal{T}_{h,m}} \int_K \left(|\nabla U|^2 + |\nabla\tilde{U}|^2\right) dx.
\end{aligned}
$$

From (3.61), (3.62) and the definition of the $\|\cdot\|_{DG,m}$-norm we finally get

$$
\begin{aligned}
&a_{h,m}(U,\tilde{U}) + \beta_0\, J_{h,m}(U,\tilde{U}) \le c_2 \sum_{K\in\mathcal{T}_{h,m}} \int_K \left(|\nabla U|^2 + |\nabla\tilde{U}|^2\right) dx \\
&+\frac{\beta_1^2 k_1}{\beta_0 c_W} \|u_D\|^2_{DGB,m} + \frac{\beta_0}{k_1} c_M(c_I+1)\|\tilde{U}\|^2_{DG,m} \\
&+(\beta_1+\beta_0)\left(J_{h,m}(U,U)+J_{h,m}(\tilde{U},\tilde{U})\right) \\
&\le c\left(\|U\|^2_{DG,m} + \|\tilde{U}\|^2_{DG,m} + \|u_D\|^2_{DGB,m}\right).
\end{aligned}
$$

□

Now we can turn our attention to the estimation of the term $\int_{I_m} \|U\|^2 \, dt$. We shall use the approach based on discrete characteristic functions constructed to U. We define the time instants

$$t_{m-1+l/q} = t_{m-1} + \frac{l}{q}(t_m - t_{m-1}) \quad \text{for} \quad l = 0, \ldots, q,$$

and use the notation

$$U_{m-1+l/q} = U(t_{m-1+l/q}).$$

Analogously as in [16], we can prove, that there exist constants $L_q, M_q > 0$ dependent on q such that

$$\sum_{l=0}^{q} \|U_{m-1+l/q}\|^2 \geq \frac{L_q}{\tau_m} \int_{I_m} \|U\|^2 dt \tag{3.63}$$

$$\|U_{m-1}^+\|^2 \leq \frac{M_q}{\tau_m} \int_{I_m} \|U\|^2 dt, \tag{3.64}$$

For each $y \in I_m$ we define the discrete characteristic function to $U \in S_{h,\tau}^{p,q}$ as $\zeta_y \in S_{h,\tau}^{p,q}$ such that

$$\int_{I_m} (\zeta_y, \varphi) \, dt = \int_{t_{m-1}}^{y} (U, \varphi) \, dt \quad \forall \varphi \in S_{h,\tau}^{p,q-1}, \tag{3.65}$$

$$\zeta_y(t_{m-1}^+) = U(t_{m-1}^+). \tag{3.66}$$

One of the most important properties of the discrete characteristic function is the following inequality

$$\int_{I_m} \|\zeta_y\|_{DG,m}^2 \, dt \leq c_q \int_{I_m} \|U\|_{DG,m}^2 \, dt, \tag{3.67}$$

where the constant $c_q > 0$ depends only on q. Proof of this inequality can be found in [15] and [8].

If $r \geq 0$ is an integer, then by $P^r(I_m)$ we denote the space of all polynomials on I_m of degree less then or equal to r.

Let $\zeta_{t_{m-1+l/q}}$ be a discrete characteristic function to U at the point $t_{m-1+l/q}$ defined by (3.65), (3.66). For brevity we set $\tilde{U}_l = \zeta_{t_{m-1+l/q}}$. The following result holds.

Lemma 11. *There exists a konstant $\tilde{c} > 0$, independent of h, τ, m, l and U such that*

$$\int_{I_m} \|\tilde{U}_l\|^2 \, dt \leq \tilde{c} \int_{I_m} \|U\|^2 \, dt \quad \textit{for all} \quad U \in S_{h,\tau}^{p,q}. \tag{3.68}$$

Proof. We assume $U \in S_{h,\tau}^{p,q}$. Let $\{p_k\}_{k=1}^{q}$ be an orthonormal basis of the space $P^q(I_m)$. We can write the function $U|_{I_m}$ in the form

$$U = \sum_{k=0}^{q} p_k v_k, \quad \text{where} \quad v_k \in S_{h,m}^{p}. \tag{3.69}$$

Then its discrete characteristic function can be expressed in the form

$$\tilde{U}_l = \sum_{k=0}^{q} \tilde{p}_k v_k, \tag{3.70}$$

where $\tilde{p}_k \in P^q(I_m)$ satisfies the conditions

$$\begin{aligned} \int_{I_m} \tilde{p}_k z \, dt &= \int_{t_{m-1}}^{t_{m-1+l/q}} p_k z \, dt \quad \forall z \in P^{q-1}(I_m), \\ \tilde{p}_k(t_{m-1}^+) &= p(t_{m-1}^+). \end{aligned}$$

It follows from [15], Chapter 4, that there exists a constant $\tilde{c}_q > 0$ such that

$$\|\tilde{p}_k\|_{L^2(I_m)} \leq (1 + \tilde{c}_q) \, \|p_k\|_{L^2(I_m)}. \tag{3.71}$$

If we use Fubini's theorem, Cauchy's inequality, the inequalities (3.71) and $\sum_{i,j=1}^{q} a_i a_j \leq (q+1) \sum_{i=1}^{q} a_i^2$ and the orthonormality of the functions p_k, we obtain

$$\begin{aligned}
\int_{I_m} \|\tilde{U}_l\|^2 \, dt &= \sum_{K \in \mathcal{T}_{h,m}} \int_{I_m} \|\tilde{U}_l\|^2_{L^2(K)} \, dt = \sum_{K \in \mathcal{T}_{h,m}} \int_{I_m} \left(\int_K \tilde{U}_l \, \tilde{U}_l \, dx \right) dt \\
&= \sum_{K \in \mathcal{T}_{h,m}} \sum_{k,n=0}^{q} \int_{I_m} \tilde{p}_k \, \tilde{p}_n \, dt \int_K v_k \, v_n \, dx \\
&\leq \sum_{K \in \mathcal{T}_{h,m}} \sum_{k,n=0}^{q} \|\tilde{p}_k\|_{L^2(I_m)} \, \|\tilde{p}_n\|_{L^2(I_m)} \|v_k\|_{L^2(K)} \, \|v_n\|_{L^2(K)} \\
&\leq (1 + \tilde{c}_q)^2 \sum_{K \in \mathcal{T}_{h,m}} \sum_{k,n=0}^{q} \|p_k\|_{L^2(I_m)} \, \|p_n\|_{L^2(I_m)} \|v_k\|_{L^2(K)} \, \|v_n\|_{L^2(K)} \\
&\leq (1 + \tilde{c}_q)^2 (q+1) \sum_{K \in \mathcal{T}_{h,m}} \sum_{k=0}^{q} \|p_k\|^2_{L^2(I_m)} \|v_k\|^2_{L^2(K)} \\
&= (1 + \tilde{c}_q)^2 (q+1) \sum_{K \in \mathcal{T}_{h,m}} \sum_{k=0}^{q} \int_{I_m} p_k^2 \, dt \int_K v_k^2 \, dx \\
&= (1 + \tilde{c}_q)^2 (q+1) \sum_{K \in \mathcal{T}_{h,m}} \sum_{k,n=0}^{q} \int_{I_m} p_k \, p_n \, dt \int_K v_k \, v_n \, dx \\
&= (1 + \tilde{c}_q)^2 (q+1) \sum_{K \in \mathcal{T}_{h,m}} \int_{I_m} \left(\int_K U \, U \, dx \right) dt \\
&= (1 + \tilde{c}_q)^2 (q+1) \sum_{K \in \mathcal{T}_{h,m}} \int_{I_m} \|U\|^2_{L^2(K)} \, dt \\
&= (1 + \tilde{c}_q)^2 (q+1) \int_{I_m} \|U\|^2 \, dt.
\end{aligned}$$

□

Now, for a while, we shall return to our basic inequality (3.24). We shall estimate it in other way using the identity

$$\begin{aligned}
&\int_{I_m} \left(\frac{\partial U}{\partial t}, U\right) \, dt + \left(\{U\}_{m-1}, U_{m-1}^+\right) \qquad (3.72)\\
&= \frac{1}{2}\int_{I_m} \frac{d}{dt}\|U\|^2 \, dt + \left(\{U\}_{m-1}, U_{m-1}^+\right)\\
&= \frac{1}{2}(\|U_m^-\|^2 - \|U_{m-1}^+\|^2) + \left(U_{m-1}^+ - U_{m-1}^-, U_{m-1}^+\right)\\
&= \frac{1}{2}(\|U_m^-\|^2 + \|U_{m-1}^+\|^2) - \left(U_{m-1}^-, U_{m-1}^+\right).
\end{aligned}$$

From (3.24), using (3.72), (3.50), (3.51), (3.58), we get

$$\begin{aligned}
&\underbrace{\int_{I_m} \left(\frac{\partial U}{\partial t}, U\right) \, dt + (\{U\}_{m-1}, \varphi_{m-1}^+)}_{=\frac{1}{2}\left(\|U_m^-\|^2+\|U_{m-1}^-\|^2\right)-(U_{m-1}^-,U_{m-1}^+)} + \underbrace{\int_{I_m} \left(a_{h,m}(U,U) + \beta_0 \, J_{h,m}(U,U)\right) \, dt}_{\geq\frac{\beta_0}{2}\int_{I_m}\|U\|_{DG,m}^2 dt-\frac{\beta_0}{2}\int_{I_m}\|u_D\|_{DGB,m}^2 dt}\\
&= \underbrace{\int_{I_m} l_{h,m}(U) \, dt}_{\leq\frac{1}{2}\int_{I_m}(\|g\|^2+\|U\|^2)\,dt+\beta_0 k_3\int_{I_m}\|u_D\|_{DGB,m}^2 dt+\frac{\beta_0}{k_3}\int_{I_m}\|U\|_{DG,m}^2 dt}\\
&\quad - \underbrace{\int_{I_m} b_{h,m}(U,U) \, dt}_{\leq\frac{\beta_0}{k_2}\int_{I_m}\|U\|_{DG,m}^2 dt+c_b\int_{I_m}\|U\|^2 dt},
\end{aligned}$$

which can be rewritten as follows:

$$\begin{aligned}
&\frac{1}{2}\left(\|U_m^-\|^2 + \|U_{m-1}^-\|^2\right) - (U_{m-1}^-, U_{m-1}^+)\\
&\qquad + \frac{\beta_0}{2}\int_{I_m} \|U\|_{DG,m}^2 dt - \frac{\beta_0}{2}\int_{I_m} \|u_D\|_{DGB,m}^2 dt\\
&\leq \frac{1}{2}\int_{I_m} \left(\|g\|^2 + \|U\|^2\right) \, dt + \beta_0 k_3 \int_{I_m} \|u_D\|_{DGB,m}^2 dt + \frac{\beta_0}{k_3}\int_{I_m} \|U\|_{DG,m}^2 dt\\
&\qquad + \frac{\beta_0}{k_2}\int_{I_m} \|U\|_{DG,m}^2 dt + c_b \int_{I_m} \|U\|^2 dt.
\end{aligned}$$

After multiplying by two and some manipulation, we find that

$$\begin{aligned}
&\|U_m^-\|^2 + \|U_{m-1}^-\|^2 - 2(U_{m-1}^-, U_{m-1}^+) + \beta_0 \left(1 - \frac{2}{k_2} - \frac{2}{k_3}\right) \int_{I_m} \|U\|_{DG,m}^2 dt\\
&\leq \int_{I_m} \left(\|g\|^2 + \|U\|^2\right) \, dt + \beta_0(1 + 2k_3) \int_{I_m} \|u_D\|_{DGB,m}^2 dt + 2c_b \int_{I_m} \|U\|^2 dt.
\end{aligned}$$

By setting $k_2 = k_3 = 8$, $c_1 = 2c_b + 1$, $c_2 = \max\{1, 17\beta_0\}$ and using Young's inequality with any $\delta_1 > 0$ for the term $2(U_{m-1}^-, U_{m-1}^+)$, we finally get

$$\begin{aligned}
&\|U_m^-\|^2 + \|U_{m-1}^-\|^2 + \frac{\beta_0}{2}\int_{I_m} \|U\|_{DG,m}^2 dt \qquad (3.73)\\
&\leq c_1 \int_{I_m} \|U\|^2 \, dt + c_2 \int_{I_m} \left(\|g\|^2 + \|u_D\|_{DGB,m}^2\right) \, dt + \frac{2}{\delta_1}\|U_{m-1}^-\|^2 + 4\delta_1 \|U_{m-1}^+\|^2.
\end{aligned}$$

The last inequality will play an important role in the proof of the following lemma about estimation of $\int_{I_m} \|U\|^2 \, dt$.

Lemma 12. *There exist constants $c, c^* > 0$ such that*

$$\int_{I_m} \|U\|^2 \, dt \le c\, \tau_m \left(\|U_{m-1}^-\|^2 + \int_{I_m} R_m^* \, dt \right), \tag{3.74}$$

where

$$0 < \tau_m \le c^* \tag{3.75}$$

and

$$R_m^* = \|g\|^2 + \|u_D\|^2_{DGB,m}. \tag{3.76}$$

Proof. The proof of this theorem will be divided into two parts.

Part I. Let us consider the case $q = 1$.
From (3.73), (3.63) and (3.64) we get

$$\begin{aligned} &\frac{L_1}{\tau_m} \int_{I_m} \|U\|^2 dt + \frac{\beta_0}{2} \int_{I_m} \|U\|^2_{DG,m} \, dt \\ &\le \left(\tilde{c}_1 + \frac{4M_1\delta}{\tau_m} \right) \int_{I_m} \|U\|^2 \, dt + \tilde{c}_2 \int_{I_m} R_m^* \, dt + \frac{2}{\delta} \|U_{m-1}^-\|^2, \end{aligned} \tag{3.77}$$

with constants $\tilde{c}_1, \tilde{c}_2 > 0$.

If we set $\delta = \frac{L_1}{8M_1}, \tilde{c}_3 = \frac{2}{\delta}$, then under the assumption

$$0 < \tau_m \le c^* := \frac{L_1}{4c_1} \tag{3.78}$$

we find that

$$\begin{aligned} &\frac{L_1}{2\tau_m} \int_{I_m} \|U\|^2 dt + \frac{\beta_0}{2} \int_{I_m} \|U\|^2_{DG,m} \, dt \\ &\le \tilde{c}_2 \int_{I_m} R_m^* \, dt + \tilde{c}_3 \|U_{m-1}^-\|^2, \end{aligned} \tag{3.79}$$

from which the inequality (3.74) follows.

Part II. Now we assume that $q \ge 2, l \in \{1, \ldots, q-1\}$. By (3.65)-(3.67),

$$\int_{I_m} \left(\tilde{U}_l, U' \right) dt = \int_{t_{m-1}}^{t_{m-1+l/q}} (U, U') \, dt \tag{3.80}$$

$$\tilde{U}_l(t_{m-1}^+) = U(t_{m-1}^+) \tag{3.81}$$

$$\int_{I_m} \|\tilde{U}_l\|^2_{DG,m} \, dt \le c_q \int_{I_m} \|U_l\|^2_{DG,m} \, dt. \tag{3.82}$$

Using these relations, we find that

$$\begin{aligned} &\int_{I_m} \left(U', \tilde{U}_l \right) dt + (U_{m-1}^+, (\tilde{U}_l)_{m-1}^+) \\ &= \int_{t_{m-1}}^{t_{m-1+l/q}} (U, U') \, dt + (U_{m-1}^+, (\tilde{U}_l)_{m-1}^+) \\ &= \frac{1}{2} \int_{t_{m-1}}^{t_{m-1+l/q}} \frac{d}{dt} \|U\|^2 \, dt + \|U_{m-1}^+\|^2 \\ &= \frac{1}{2} \left(\|U_{m-1+l/q}^-\|^2 + \|U_{m-1}^+\|^2 \right). \end{aligned} \tag{3.83}$$

Now we return to the definition of the approximate solution (2.22), where by setting $\varphi := \tilde{U}_l \in S^{p,q}_{h,\tau}$ we get

$$\int_{I_m} \left(\frac{\partial U}{\partial t}, \tilde{U}_l \right) dt + (\{U\}^+_{m-1}, (\tilde{U}_l)^+_{m-1}) \tag{3.84}$$
$$= \int_{I_m} \left(-a_{h,m}(U, \tilde{U}_l) - \beta_0 J_{h,m}(U, \tilde{U}_l) - b_{h,m}(U, \tilde{U}_l) + l_{h,m}(\tilde{U}_l) \right) dt.$$

From this, using the identity $(\{U\}_{m-1}, (\tilde{U}_l)^+_{m-1}) = (U^+_{m-1}, (\tilde{U}_l)^+_{m-1}) - (U^-_{m-1}, (\tilde{U}_l)^+_{m-1})$ and (3.83) we have

$$\frac{1}{2} \left(\|U^-_{m-1+l/q}\|^2 + \|U^+_{m-1}\|^2 \right) \tag{3.85}$$
$$\leq \int_{I_m} |a_{h,m}(U, \tilde{U}_l) + \beta_0 J_{h,m}(U, \tilde{U}_l)| \, dt$$
$$+ \int_{I_m} |b_{h,m}(U, \tilde{U}_l)| \, dt + \int_{I_m} |l_{h,m}(\tilde{U}_l)| \, dt + (U^-_{m-1}, (\tilde{U}_l)^+_{m-1}).$$

From (3.81) we know that $U^+_{m-1} = (\tilde{U}_l)^+_{m-1}$. Hence, the last term on the right-hand side can be rewritten as $(U^-_{m-1}, (\tilde{U}_l)^+_{m-1}) = (U^-_{m-1}, U^+_{m-1})$. Furthermore, using Lemma 10, (3.51), (3.58) for $k_1, k_3 > 0$ and Young's inequality with any constant $\delta_2 > 0$, we can continue in estimating the right-hand side of the above inequality. We get

$$\int_{I_m} |a_{h,m}(U, \tilde{U}_l) + \beta_0 J_{h,m}(U, \tilde{U}_l)| \, dt \tag{3.86}$$
$$+ \int_{I_m} |b_{h,m}(U, \tilde{U}_l)| \, dt + \int_{I_m} |l_{h,m}(\tilde{U}_l)| \, dt + (U^-_{m-1}, U^+_{m-1})$$
$$\leq c \int_{I_m} \left(\|U\|^2_{DG,m} + \|\tilde{U}_l\|^2_{DG,m} + \|u_D\|^2_{DGB,m} \right) dt$$
$$+ \frac{\beta_0}{k_1} \int_{I_m} \|\tilde{U}_l\|^2_{DG,m} \, dt + c_b \int_{I_m} \|U\|^2 \, dt$$
$$+ \frac{1}{2} \int_{I_m} \left(\|g\|^2 + \|\tilde{U}_l\|^2 \right) dt + \beta_0 \int_{I_m} \|u_D\|^2_{DG,m} \, dt + \frac{\beta_0}{k_3} \int_{I_m} \|\tilde{U}_l\|^2_{DG,m} \, dt$$
$$+ \frac{\|U^-_{m-1}\|^2}{\delta_2} + \delta_2 \|U^+_{m-1}\|^2.$$

From (3.85) and (3.86), using Lemma 11, we obtain

$$\|U^-_{m-1+l/q}\|^2 + \|U^+_{m-1}\|^2 \tag{3.87}$$
$$\leq c \int_{I_m} \left(\|U\|^2_{DG,m} + \|\tilde{U}_l\|^2_{DG,m} + \|u_D\|^2_{DGB,m} \right) dt$$
$$+ \frac{\beta_0}{k_1} \int_{I_m} \|\tilde{U}_l\|^2_{DG,m} \, dt + c_b \int_{I_m} \|U\|^2 \, dt +$$
$$+ \frac{1}{2} \int_{I_m} \|g\|^2 \, dt + \frac{1}{2} \tilde{c} \int_{I_m} \|U\|^2 \, dt + \beta_0 \int_{I_m} \|u_D\|^2_{DG,m} \, dt + \frac{\beta_0}{k_3} \int_{I_m} \|\tilde{U}_l\|^2_{DG,m} \, dt$$
$$+ \frac{\|U^-_{m-1}\|^2}{\delta_2} + \delta_2 \|U^+_{m-1}\|^2$$
$$\leq \tilde{c}_1 \int_{I_m} \left(\|U\|^2_{DG,m} + \|\tilde{U}_l\|^2_{DG,m} + \|U\|^2 + R^*_m \right) dt + \frac{\|U^-_{m-1}\|^2}{\delta_2} + \delta_2 \|U^+_{m-1}\|^2,$$

where $\tilde{c}_1 = \max\{c, c + \frac{\beta_0}{k_1} + \frac{\beta_0}{k_3}, c_b + \frac{1}{2}\tilde{c}, \frac{1}{2}, \beta_0\}$.
Using inequality (3.82), we have

$$\begin{aligned}
&\|U^-_{m-1+l/q}\|^2 + \|U^+_{m-1}\|^2 \\
&\leq \tilde{c}_2 \int_{I_m} \left(\|U\|^2_{DG,m} + \|U\|^2 + R^*_m\right) dt + \frac{\|U^-_{m-1}\|^2}{\delta_2} + \delta_2\|U^+_{m-1}\|^2.
\end{aligned} \tag{3.88}$$

Now, multiplying (3.88) by $\frac{\beta_0}{4\tilde{c}_2(q-1)}$, summing over all $l = 1, \dots, q-1$ and adding (3.73), we get the following inequality

$$\begin{aligned}
&\tilde{c}_3 \left(\|U^-_m\|^2 + \sum_{l=1}^{q-1} \|U^-_{m-1+l/q}\|^2 + \|U^+_{m-1}\|^2 \right) + \frac{\beta_0}{2} \int_{I_m} \|U\|^2_{DG,m}\, dt \\
&\leq \int_{I_m} \left(\frac{\beta_0}{4}\|U\|^2_{DG,m} + \underbrace{\left(\frac{\beta_0}{4} + c_1\right)}_{:=c_1^*} \|U\|^2 + \underbrace{\left(\frac{\beta_0}{4} + c_2\right)}_{:=c_2^*} R^*_m \right) dt \\
&+ \left(\frac{\beta_0}{4\tilde{c}_2\delta_2} + \frac{2}{\delta_1} \right) \|U^-_{m-1}\|^2 + \left(\frac{\beta_0\delta_2}{4\tilde{c}_2} + 4\delta_1 \right) \|U^+_{m-1}\|^2,
\end{aligned} \tag{3.89}$$

where $\tilde{c}_3 = \min\{\frac{\beta_0}{4\tilde{c}_2(q-1)}, 1\}$.

Using inequalities (3.63) and (3.64), after some manipulations we obtain

$$\begin{aligned}
&\frac{\tilde{c}_3 L_q}{\tau_m} \int_{I_m} \|U\|^2 + \frac{\beta_0}{4} \int_{I_m} \|U\|^2_{DG,m}\, dt \\
&\leq \int_{I_m} \left(\frac{\beta_0\delta_2 M_q}{4\tilde{c}_2\tau_m} + \frac{4\delta_1 M_q}{\tau_m} + c_1^* \right) \|U\|^2 + c_2^* \int_{I_m} R^*_m\, dt \\
&+ \left(\frac{\beta_0}{4\tilde{c}_2\delta_2} + \frac{2}{\delta_1} \right) \|U^-_{m-1}\|^2.
\end{aligned} \tag{3.90}$$

By setting

$$\delta_1 = \frac{\tilde{c}_3 L_q}{16 M_q},\ \delta_2 = \frac{\tilde{c}_3\tilde{c}_2 L_q}{4\beta_0 M_q},\ c_3^* = \frac{\beta_0}{4\tilde{c}_2\delta_2} + \frac{2}{\delta_1},$$

we get

$$\begin{aligned}
&\left(\frac{\tilde{c}_3 L_q}{2\tau_m} - c_1^* \right) \int_{I_m} \|U\|^2 + \frac{\beta_0}{4} \int_{I_m} \|U\|^2_{DG,m}\, dt \\
&\leq c_2^* \int_{I_m} R^*_m\, dt + c_3^* \|U^-_{m-1}\|^2.
\end{aligned} \tag{3.91}$$

If the condition

$$0 < \tau_m \leq c^* := \frac{\tilde{c}_3 L_q}{4c_1^*}$$

is satisfied, then from (3.91) we have

$$\frac{c^* L_q}{4\tau_m} \int_{I_m} \|U\|^2 + \frac{\beta_0}{4} \int_{I_m} \|U\|^2_{DG,m}\, dt \leq c_2^* \int_{I_m} R^*_m\, dt + c_3^* \|U^-_{m-1}\|^2,$$

which already implies (3.74). □

To prove our main theorem about the stability, we will need the following auxiliary lemma.

Lemma 13. *(Discrete Gronwall lemma) Let x_m, a_m, b_m and c_m, where $m = 1, 2, \ldots$, be non-negative sequences and let the sequence a_m nondecreasing. Then, if*

$$\begin{aligned} & x_0 + c_0 \le a_0 \\ & x_m + c_m \le a_m + \sum_{j=0}^{m-1} b_j x_j \quad for m \ge 1, \end{aligned}$$

we have

$$x_m + c_m \le a_m \prod_{j=0}^{m-1} (1 + b_j) \quad for m \ge 0.$$

The proof can be found in [20].

Theorem 3. *Let $0 < \tau_m \le c^*$. Then there exist a constant $c > 0$ such that*

$$\|U_m^-\|^2 + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|U\|_{DG,j}^2 \, dt \le c \left(\|U_0^-\|^2 + \sum_{j=1}^{m} \int_{I_j} R_j^* \, dt \right), \quad (3.92)$$

$$m = 1, \ldots, M, \quad h \in (0, h_0),$$

$$\|U\|_{L^2(Q_T)}^2 \le c \left(\|U_0^-\|^2 + \sum_{m=1}^{M} \int_{I_m} R_m^* \, dt \right), \quad h \in (0, h_0). \quad (3.93)$$

Proof. We begin with the proof of the first inequality. In virtue of the notation $R_m^* = \|g\|^2 + \|u_D\|_{DGB,m}^2$, estimate (3.59) can be written in the form

$$\|U_j^-\|^2 - \|U_{j-1}^-\|^2 + \frac{\beta_0}{2} \int_{I_j} \|U\|_{DG,j}^2 \, dt \le c \int_{I_j} \left(\|U\|^2 + R_j^* \right) \, dt.$$

Now, using (3.74) we obtain

$$\|U_j^-\|^2 + \frac{\beta_0}{2} \int_{I_j} \|U\|_{DG,j}^2 \, dt \quad (3.94)$$

$$\le (1 + c\tau_j) \|U_{j-1}^-\|^2 + c(1 + \tau_j) \int_{I_j} R_j^* \, dt, \quad j = 1, \ldots, M. \quad (3.95)$$

Let $m \ge 1$. Summing (3.95) over all $j = 1, \ldots, m$ and taking into account that $\tau_j < T, j = 1, \ldots, m$, we get

$$\|U_m^-\|^2 + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|U\|_{DG,j}^2 \, dt \le \|U_0^-\|^2 + c \sum_{j=0}^{m-1} \tau_{j+1} \|U_{j-1}^-\|^2 + c \sum_{j=1}^{m} \int_{I_j} R_j^* \, dt.$$

Using the discrete Gronwall theorem for the previous inequality with the setting

$$\begin{aligned} x_0 &= a_0 = \|U_0^-\|^2, c_0 = 0, \\ x_m &= \|U_m^-\|^2, \\ c_m &= \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|U\|^2_{DG,j}\,dt, \\ a_m &= c\sum_{j=1}^{m}\int_{I_m} R_j^*\,dt, \\ b_j &= c\tau_{j+1}, \quad j = 0, 1, \ldots, m, \end{aligned}$$

yields

$$\begin{aligned} &\|U_m^-\|^2 + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|U\|^2_{DG,j}\,dt \\ &\leq c\left(\|U_0^-\|^2 + \sum_{j=1}^{m}\int_{I_j} R_j^*\,dt\right)\prod_{j=0}^{m-1}(1 + c\tau_{j+1}). \end{aligned} \tag{3.96}$$

Furthermore, using the property of the exponential function $1 + c\tau_j \leq \exp(c\tau_j)$, we can write

$$\prod_{j=0}^{m-1}(1 + c\tau_j) = \prod_{j=1}^{m}(1 + c\tau_j) \leq \exp\left(c\sum_{j=1}^{m}\tau_j\right) = \exp(ct_m) \leq \tilde{c} := \exp(cT).$$

This and (3.96) imply that

$$\|U_m^-\|^2 + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|U\|^2_{DG,j}\,dt \leq c\left(\|U_0^-\|^2 + \sum_{j=1}^{m}\int_{I_j} R_j^*\,dt\right),$$

which is (3.92).

Now we turn our attention to the second inequality in the theorem. From (3.74) we have

$$\|U\|^2_{L^2(Q_T)} = \sum_{m=1}^{M}\int_{I_m}\|U\|^2\,dt \leq c\sum_{m=1}^{M}\tau_m\left(\|U_{m-1}^-\|^2 + \int_{I_m} R_m^*\,dt\right)$$

Using (3.92) for $m := m - 1 < M$ we get

$$\begin{aligned} \|U\|^2_{L^2(Q_T)} &\leq c\sum_{m=1}^{M}\tau_m\left(\|U_0^-\|^2 + \sum_{j=1}^{m}\int_{I_j} R_j^*\,dt + \int_{I_m} R_m^*\,dt\right) \\ &\leq c\left(T\|U_0^-\|^2 + \sum_{m=1}^{M}\tau_m\left(\sum_{j=1}^{m}\int_{I_j} R_j^*\,dt\right)\right) \\ &\leq cT\left(\|U_0^-\|^2 + \sum_{m=1}^{M}\int_{I_m} R_m^*\,dt\right) \\ &\leq c\left(\|U_0^-\|^2 + \sum_{m=1}^{M}\int_{I_m} R_m^*\,dt\right). \end{aligned}$$

□

Taking into account that

$$R_m^*(t) = \|g(t)\|^2 + \|u_D(t)\|^2_{DGB,m} \quad \text{for} \quad t \in I_m,$$

we have

$$\int_{I_m} R_m^*(t)\,dt = \int_{I_m} \|g(t)\|^2\,dt + \int_{I_m} \|u_D(t)\|^2_{DGB,m}\,dt.$$

Hence, it follows from (3.92) and (3.93) that

$$\begin{aligned}
&\|U_m^-\|^2 + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|U\|^2_{DG,j}\,dt \\
&\le c\left(\|U_0^-\|^2 + \|g(t)\|^2_{L^2(0,T;L^2(\Omega))} + \|u_D(t)\|^2_{L^2_{DGB,m}(0,T;L^2(\Omega))}\right), \\
&\qquad\qquad m = 1,\dots,M, \quad h \in (0,h_0)
\end{aligned}$$

and

$$\begin{aligned}
&\|U\|^2_{L^2(Q_T)} \\
&\le c\left(\|U_0^-\|^2 + \|g(t)\|^2_{L^2(0,T;L^2(\Omega))} + \|u_D(t)\|^2_{L^2_{DGB,m}(0,T;L^2(\Omega))}\right), \quad h \in (0,h_0).
\end{aligned}$$

As we see, the approximate solution is bounded in the discrete $L^\infty(L^2)$-norm, the energy norm and the $L^2(L^2)$-norm by the data. This means that method (2.22) is stable without the "stability condition" $\tau_m \le c_B h_{K_\Gamma^{(L)}}$, which was necessary for the analysis of the error estimates.

4. Numerical experiments

In this section we verify the theoretical results presented in Section 3. For brevity we consider equation with a linear diffusion: the 2D viscous Burgers equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} + u\frac{\partial u}{\partial x_2} = \epsilon \Delta u + g \quad \text{in} \quad \Omega \times (0,T), \tag{4.1}$$

with a certain initial and Dirichlet boundary condition. We set $\Omega = (0,1)^2, T = 10, \epsilon = 0.1$ and define the function g and the initial and boundary conditions in such a way, that the exact solution has the form

$$u(x_1, x_2, t) = (1 - e^{-10t})\, \hat{u}(x_1, x_2), \tag{4.2}$$

where

$$\begin{aligned} \hat{u}(x_1, x_2) &= 2r^{\alpha} x_1 x_2 (1 - x_1)(1 - x_2) \\ &= r^{\alpha+2} \sin(2\varphi)(1 - x_1)(1 - x_2), \end{aligned} \tag{4.3}$$

where (r, φ) are the polar coordinates ($r = (x_1 + x_2)^{1/2}$) and $\alpha \in \mathbb{R}$ is a constant.

For $t = T = 10$, the solution u differs very little from the function $\hat{u}$. The function $\hat{u}$ is equal to zero on $\partial\Omega$ and its regularity depends on the value of α, namely

$$\hat{u} \in H^{\beta}(\Omega) \quad \forall \beta \in (0, \alpha + 3),$$

where $H^{\beta}(\Omega)$ denotes the Sobolev-Slobodetskii space of functions with fractal-order derivatives (for the definition we refer to [29]).

In the numerical tests presented below we use the value $\alpha = 4$, which gives a function $\hat{u}$ that is sufficiently regular:

$$\hat{u} \in H^{\beta}(\Omega) \quad \text{for} \quad \beta < 7.$$

Furthermore we use also the value $\alpha = -3/2$, which gives

$$\hat{u} \in H^{\beta}(\Omega) \quad \text{where} \quad \beta < 3/2.$$

The problem above is discretized by the NIPG and SIPG space-time discontinuous Galerkin method. Obviously, $f_s(u) = u^2/2$ for $s = 1, 2$. In the definition of the form $b_{h,m}$ we use the upwinding numerical flux, namely

$$H(u, v, \mathbf{n}) = \begin{cases} \sum_{s=1}^{2} f_s(u) n_s, & \text{if } A > 0 \\ \sum_{s=1}^{2} f_s(v) n_s, & \text{if } A \leq 0, \end{cases}$$

where

$$A = \sum_{s=1}^{2} f_s'\left(\frac{u+v}{2}\right) n_s \quad \text{and} \quad \mathbf{n} = (n_1, n_2).$$

We investigate the experimental order of convergence (EOC) of the NIPG and SIPG methods. Let $\|e_{h_1}\|$ and $\|e_{h_2}\|$ be computational errors of numerical solutions obtained on two different meshes $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_2}$, respectively. Then we can define

$$EOC_s := EOC_{space} = \frac{\log(\|e_{h_1}\|/\|e_{h_2}\|)}{\log(h_1/h_2)}. \tag{4.4}$$

To solve this problem, we used program DGMNET, which is a .NET framework developed by a PhD student Martin Hadrva. It uses a semi-implicit scheme, i.e. at each time level inner iterations are performed until a given limit of iteration count or a desired precision is reached. This scheme can be introduced as follows:

If we are on the $n+1$-th time level, i.e. we know the approximation u^n of the exact solution, and want to compute u^{n+1}, then we apply the following algorithm

1. Set $u^{old} = u^n, iter = 0$.
2. Compute the vector $\mathbf{b}$ using the approximation u^n.
3. While $iter < maxiter$ do
   (a) Solve the system $\mathbb{A}\tilde{\mathbf{x}} = \mathbf{b}$.
   (b) Interpret the discrete solution $\tilde{\mathbf{x}}$ as a new approximation u^{new} to the solution u^{n+1}.
   (c) Update the right-hand side $\mathbf{b}$.
   (d) If the difference between u^{new} and u^{old} is sufficiently small, STOP.
   (e) $u^{old} := u^{new}$.
   (f) $iter = iter + 1$.
4. Solve $\mathbb{A}\tilde{\mathbf{x}} = \mathbf{b}$.
5. $u^{n+1} = u^{new}$.

Let us note that the matrix $\mathbb{A}$ is constant. It corresponds to the discretization of the operator $\epsilon\Delta u$. The right-hand side vector $\mathbf{b}$ depends on the nonlinearities and time-dependence of data. As a solver of systems of linear algebraic equations we used the package UMFPACK (Unsymmetric MultiFrontal Method [17]).

4.1 Results on structured meshes

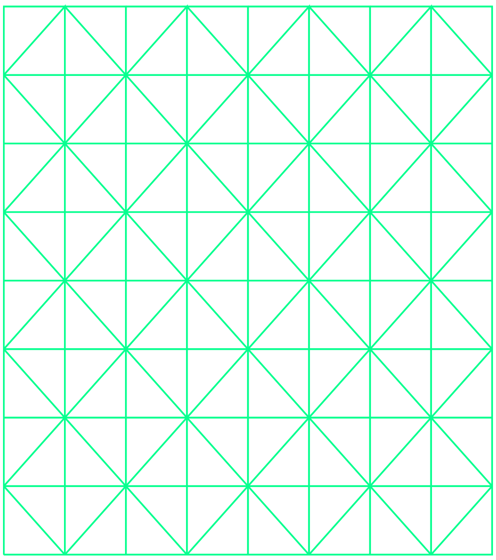

Figure 4.1: structured mesh with 128 elements

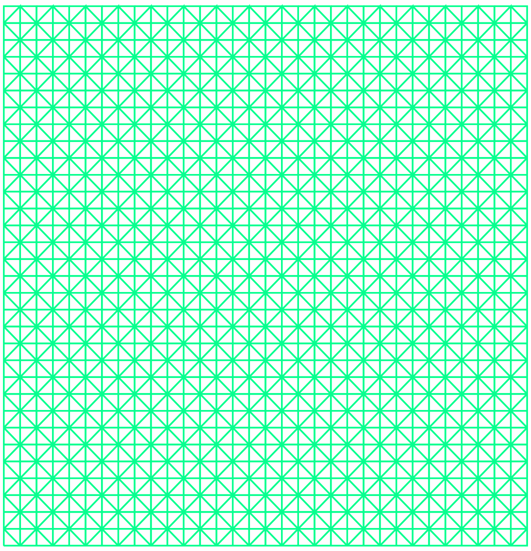

Figure 4.2: structured mesh with 2048 elements

We consider five structured triangular meshes having $128, 288, 512, 1152$ and 2048 elements. Figure 4.1 shows the coarsest mesh and Figure 4.2 shows the finest mesh.

The numerical experiments were carried out using piecewise linear ($p = 1$), quadratic ($p = 2$), and cubic ($p = 3$) elements. We chose time degree $q = 2$ and fixed time step $\tau = 0.025$. In case of the NIPG method we set $c_W = 1$ and for SIPG $c_W = 100$. Tables show the computational errors in the $L^\infty(L^2(\Omega))$-norm along the interval $[0, T]$, $T = 10$, and the corresponding orders of convergence. It is seen, that for a sufficiently regular exact solution (case $\alpha = 4$), for the SIPG method we have optimal order of convergence $O(h^{p+1})$ for $p = 1, 2, 3$.

Table 4.1: Computational errors and the corresponding orders of convergence of the NIPG method for $\alpha = -3/2$

		p=1		p=2		p=3	
Mesh	h	$\lVert e_h \rVert$	EOC_s	$\lVert e_h \rVert$	EOC_s	$\lVert e_h \rVert$	EOC_s
1	1.768E-01	1.238E-02	-	3.711E-03	-	2.139E-03	-
2	1.179E-01	6.773E-03	1.488	1.912E-03	1.636	1.164E-03	1.501
3	8.839E-02	4.337E-03	1.550	1.209E-03	1.593	7.537E-04	1.510
4	5.893E-02	2.287E-03	1.578	6.417E-04	1.562	4.085E-04	1.511
5	4.419E-02	1.449E-03	1.586	4.122E-04	1.539	2.647E-04	1.507

Table 4.2: Computational errors and the corresponding orders of convergence of the NIPG method for $\alpha = 4$

		p=1		p=2		p=3	
Mesh	h	$\lVert e_h \rVert$	EOC_s	$\lVert e_h \rVert$	EOC_s	$\lVert e_h \rVert$	EOC_s
1	1.768E-01	2.173E-03	-	5.508E-04	-	1.803E-05	-
2	1.179E-01	1.023E-03	1.858	2.285E-04	2.170	6.668E-06	2.453
3	8.839E-02	5.921E-04	1.900	1.237E-04	2.132	5.441E-06	0.7065
4	5.893E-02	2.710E-04	1.927	5.287E-05	2.097	4.353E-06	0.5502
5	4.419E-02	1.548E-04	1.946	2.917E-05	2.067		

Table 4.3: Computational errors and the corresponding orders of convergence of the SIPG method for $\alpha = -3/2$

		p=1		p=2		p=3	
Mesh	h	$\lVert e_h \rVert$	EOC_s	$\lVert e_h \rVert$	EOC_s	$\lVert e_h \rVert$	EOC_s
1	1.768E-01	1.244E-02	-	2.699E-03	-	1.281E-03	-
2	1.179E-01	6.640E-03	1.549	1.452E-03	1.530	6.955E-04	1.507
3	8.839E-02	4.256E-03	1.546	9.379E-04	1.519	4.511E-04	1.505
4	5.893E-02	2.280E-03	1.539	5.081E-04	1.512	2.452E-04	1.503
5	4.419E-02	1.468E-03	1.532	3.294E-04	1.507	1.593E-04	1.500

Table 4.4: Computational errors and the corresponding orders of convergence of the SIPG method for $\alpha = 4$

		p=1		p=2		p=3	
Mesh	h	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s
1	1.768E-01	3.833E-03	-	2.170E-04	-	1.189E-05	-
2	1.179E-01	1.772E-03	1.903	6.452E-05	2.992	2.286E-06	4.067
3	8.839E-02	1.011E-03	1.950	2.724E-05	2.998	7.087E-07	4.070
4	5.893E-02	4.543E-04	1.974	8.074E-06	2.999	1.364E-07	4.063
5	4.419E-02	2.565E-04	1.987	3.408E-06	2.999	4.237E-08	4.065

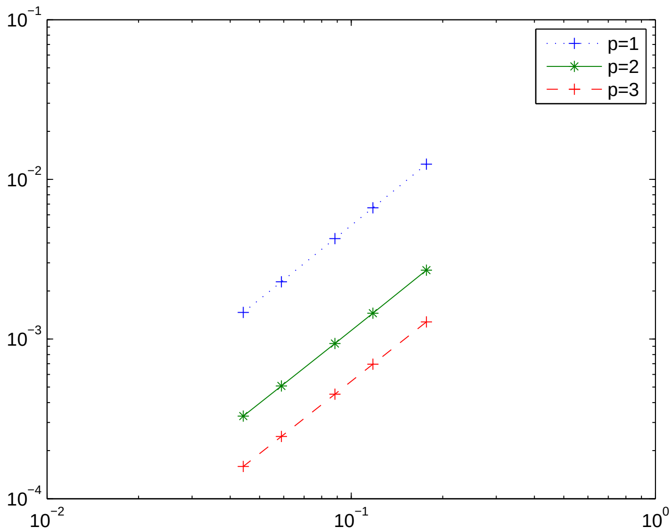

Figure 4.3: Order of convergence for SIPG method in the case $\alpha = -3/2$

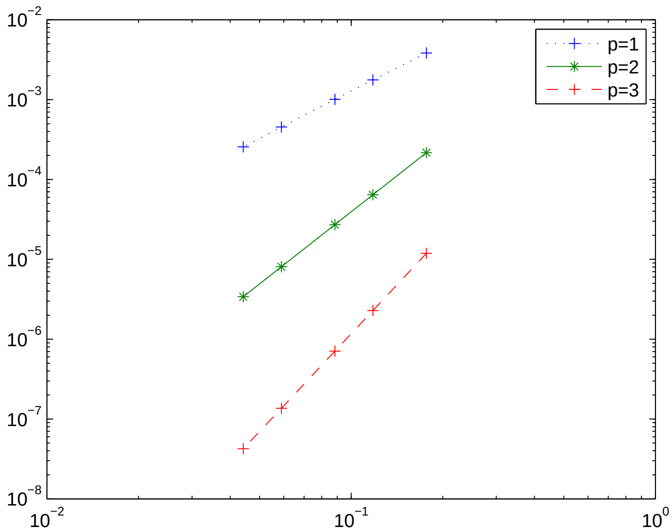

Figure 4.4: Order of convergence for SIPG method in the case $\alpha = 4$

4.2 Results on refined meshes of 1. type

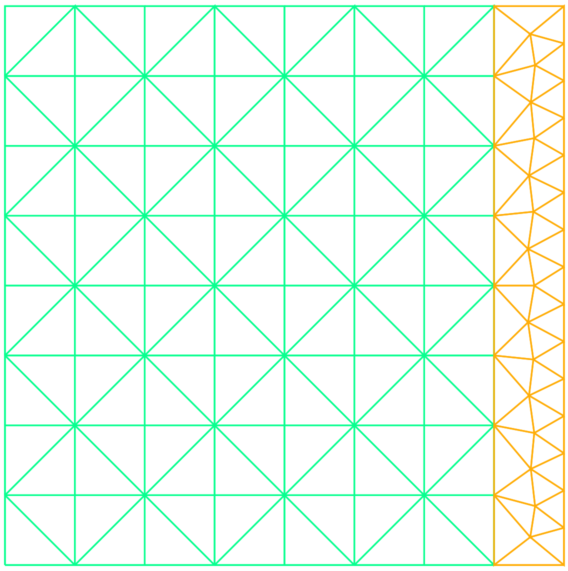

Figure 4.5: refined mesh with 165 elements

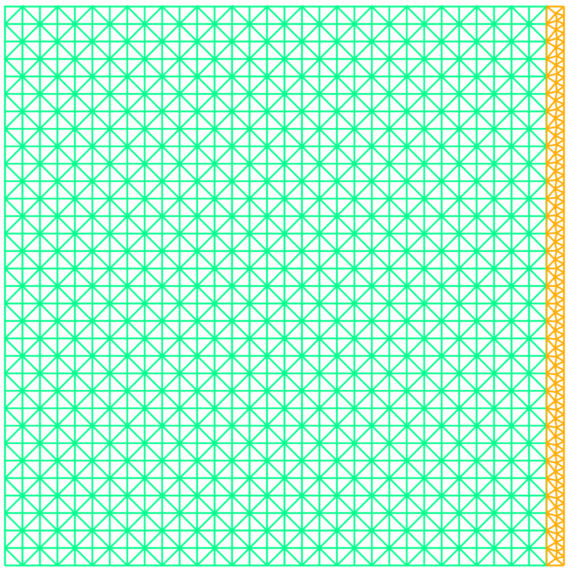

Figure 4.6: refined mesh with 2205 elements

We consider five special triangular meshes having $165, 247, 589, 1372$ and 2205 elements. Figure 4.4 shows the coarsest mesh and Figure 4.5 shows the finest mesh.

The numerical experiments were carried out using piecewise linear ($p = 1$), quadratic ($p = 2$), and cubic ($p = 3$) elements. We chose time degree $q = 2$ and fixed time step $\tau = 0.025$. In case of the NIPG method we set $c_W = 1$ and for SIPG $c_W = 100$. Tables above show the computational errors in the $L^\infty(L^2(\Omega))$-norm along the interval $[0, T]$, and the corresponding orders of convergence. It is seen, that for a sufficiently regular exact solution (case $\alpha = 4$) for the SIPG method we have the optimal order of convergence $O(h^{p+1})$ for $p = 1, 2, 3$.

Table 4.5: Computational errors and the corresponding orders of convergence of the NIPG method for $\alpha = -3/2$

		p=1		p=2		p=3	
Mesh	h	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s
1	1.768E-01	1.075E-02	-	9.711E-03	-	6.569E-03	-
2	1.414E-01	7.673E-03	1.512	6.808E-03	1.592	4.713E-03	1.487
3	8.839E-02	3.771E-03	1.511	3.257E-03	1.569	2.332E-03	1.497
4	5.657E-02	1.926E-03	1.506	1.636E-03	1.543	1.194E-03	1.501
5	4.419E-02	1.330E-03	1.501	1.123E-03	1.525	8.238E-04	1.502

Table 4.6: Computational errors and the corresponding orders of convergence of the NIPG method for $\alpha = 4$

		p=1		p=2		p=3	
Mesh	h	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s
1	1.768E-01	1.291E-03	-	3.564E-04	-	1.115E-05	-
2	1.414E-01	8.569E-04	1.837	2.312E-04	1.939	9.353E-06	0.786
3	8.839E-02	3.562E-04	1.868	9.247E-05	1.950	7.210E-06	0.554
4	5.657E-02	1.523E-04	1.904	3.884E-05	1.943	6.274E-06	0.312
5	4.419E-02	9.478E-05	1.921	2.404E-05	1.943	6.043E-06	0.152

Table 4.7: Computational errors and the corresponding orders of convergence of the SIPG method for $\alpha = -3/2$

		p=1		p=2		p=3	
Mesh	h	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s
1	1.768E-01	2.669E-02	-	6.038E-03	-	2.784E-03	-
2	1.414E-01	1.946E-02	1.416	4.330E-03	1.490	2.003E-03	1.475
3	8.839E-02	9.857E-03	1.447	2.149E-03	1.491	9.985E-04	1.481
4	5.657E-02	5.116E-03	1.469	1.103E-03	1.493	5.141E-04	1.488
5	4.419E-02	3.552E-03	1.478	7.630E-04	1.495	3.556E-04	1.493

Table 4.8: Computational errors and the corresponding orders of convergence of the SIPG method for $\alpha = 4$

		p=1		p=2		p=3	
Mesh	h	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s
1	1.768E-01	2.229E-03	-	1.317E-04	-	6.708E-06	-
2	1.414E-01	1.522E-03	1.708	7.272E-05	2.663	2.956E-06	3.672
3	8.839E-02	6.660E-04	1.759	1.994E-05	2.753	5.026E-07	3.770
4	5.657E-02	2.947E-04	1.827	5.634E-06	2.832	9.017E-08	3.850
5	4.419E-02	1.858E-04	1.869	2.770E-04	2.876	3.446E-08	3.897

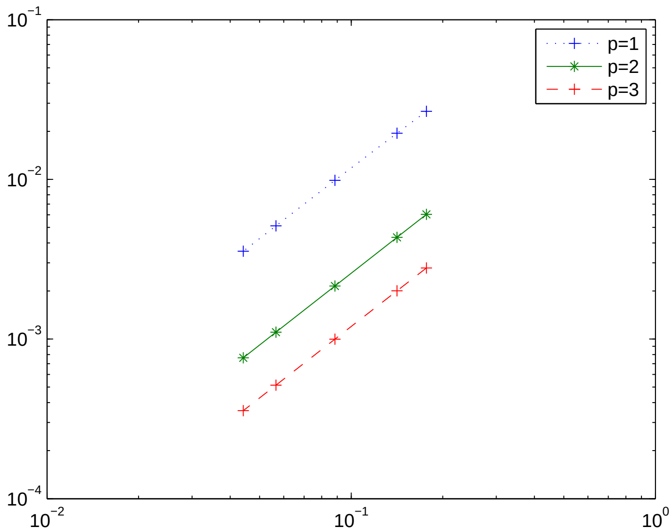

Figure 4.7: Order of convergence for SIPG method in the case $\alpha = -3/2$

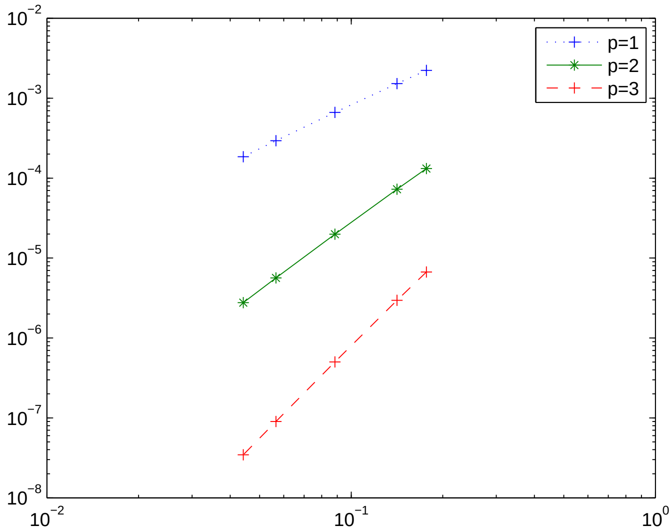

Figure 4.8: Order of convergence for SIPG method in the case $\alpha = 4$

4.3 Results on refined meshes of 2. type

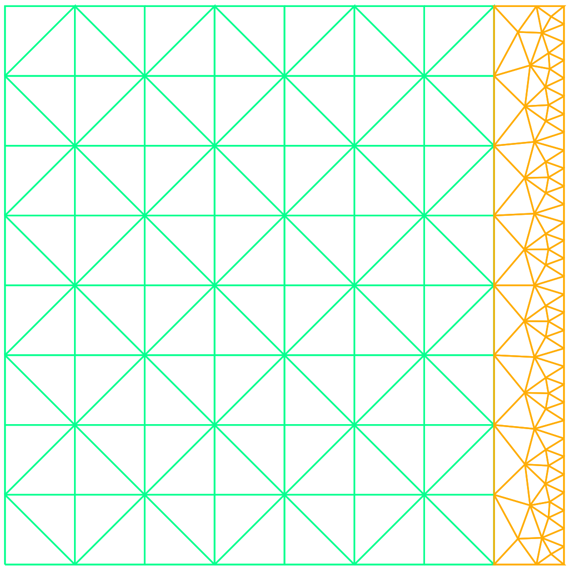

Figure 4.9: refined mesh with 235 elements

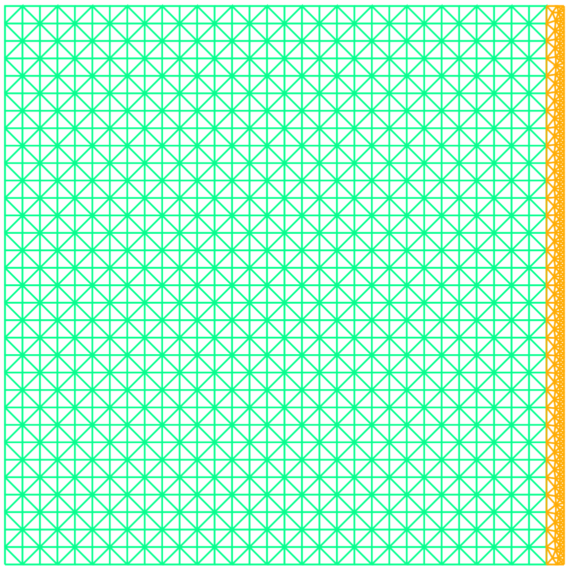

Figure 4.10: refined mesh with 2521 elements

We consider five special triangular meshes having $235, 333, 749, 1622$ and 2521 elements. Figure 4.8 shows the coarsest mesh and Figure 4.9 shows the finest mesh.

The numerical experiments were carried out using piecewise linear ($p = 1$), quadratic ($p = 2$), and cubic ($p = 3$) elements. We chose time degree $q = 2$ and fixed time step $\tau = 0.025$. In case of the NIPG method we set $c_W = 1$ and for SIPG $c_W = 100$. Tables above show the computational errors in the $L^\infty(L^2(\Omega))$-norm along the interval $[0, T]$, and the corresponding orders of convergence. It is seen, that for a sufficiently regular exact solution (case $\alpha = 4$) for the SIPG method we have optimal order of convergence $O(h^{p+1})$ for $p = 1, 2, 3$.

Table 4.9: Computational errors and the corresponding orders of convergence of the NIPG method for $\alpha = -3/2$

		p=1		p=2		p=3	
Mesh	h	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s
1	1.768E-01	1.076E-02	-	9.711E-03	-	6.569E-03	-
2	1.414E-01	7.675E-03	1.513	6.808E-03	1.592	4.713E-03	1.487
3	8.839E-02	3.771E-03	1.512	3.257E-03	1.569	2.332E-03	1.497
4	5.657E-02	1.926E-03	1.506	1.636E-03	1.543	1.193E-03	1.501
5	4.419E-02	1.329E-03	1.501	1.122E-03	1.528	8.237E-04	1.502

Table 4.10: Computational errors and the corresponding orders of convergence of the NIPG method for $\alpha = 4$

		p=1		p=2		p=3	
Mesh	h	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s
1	1.768E-01	1.227E-03	-	3.482E-04	-	1.111E-05	-
2	1.414E-01	8.231E-04	1.789	2.268E-04	1.922	9.337E-06	0.778
3	8.839E-02	3.469E-04	1.838	9.100E-05	1.943	7.205E-06	0.551
4	5.657E-02	1.501E-04	1.877	3.835E-05	1.936	6.272E-06	0.311
5	4.419E-02	9.393E-05	1.900	2.380E-05	1.933	6.042E-06	0.152

Table 4.11: Computational errors and the corresponding orders of convergence of the SIPG method for $\alpha = -3/2$

		p=1		p=2		p=3	
Mesh	h	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s
1	1.768E-01	2.668E-02	-	6.038E-03	-	2.784E-03	-
2	1.414E-01	1.946E-02	1.415	4.330E-03	1.490	2.003E-03	1.475
3	8.839E-02	9.856E-03	1.447	2.149E-03	1.491	9.985E-04	1.481
4	5.657E-02	5.116E-03	1.469	1.103E-03	1.493	5.145E-04	1.486
5	4.419E-02	3.552E-03	1.478	7.629E-04	1.495	3.562E-04	1.489

Table 4.12: Computational errors and the corresponding orders of convergence of the SIPG method for $\alpha = 4$

		p=1		p=2		p=3	
Mesh	h	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s	$\|e_h\|$	EOC_s
1	1.768E-01	2.167E-03	-	1.305E-04	-	6.681E-06	-
2	1.414E-01	1.488E-03	1.685	7.218E-05	2.654	2.948E-06	3.667
3	8.839E-02	6.549E-04	1.746	1.984E-05	2.748	5.019E-07	3.767
4	5.657E-02	2.914E-04	1.814	5.615E-06	2.828	9.011E-08	3.848
5	4.419E-02	1.842E-04	1.858	2.764E-06	2.872	3.440E-08	3.901

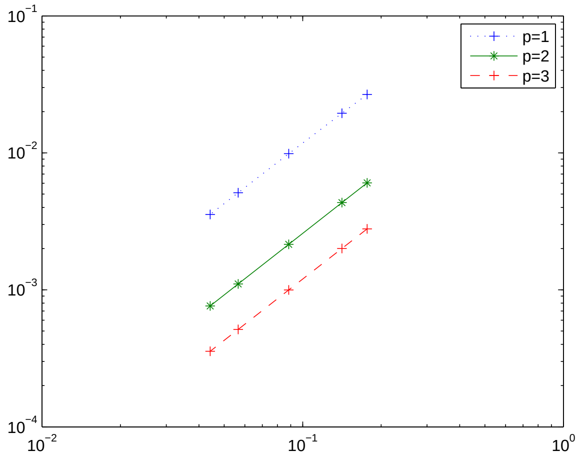

Figure 4.11: Order of convergence for SIPG method in the case $\alpha = -3/2$

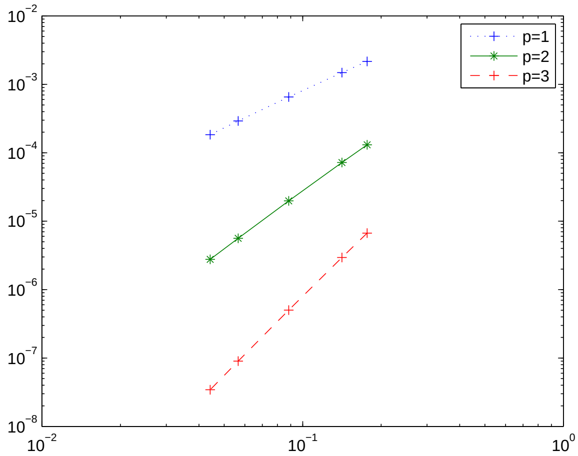

Figure 4.12: Order of convergence for SIPG method in the case $\alpha = 4$

Conclusion

In this thesis we analyzed stability of the discontinuous space-time Galerkin method for non-stationary convection-diffusion problem. In Chapter 1 we formulated the continuous problem with initial and boundary conditions and defined the basic spaces and norms needed in the theoretical analysis.

In Chapter 2 we described the discretization of the problem in space and time and defined the approximate solution. It is assumed that the space meshes may be different on different time levels. In the space discretization we used the nonsymmetric, incomplete and symmetric variants of the interior and boundary penalty approximation of the diffusion terms.

The main results of the thesis is presented in Chapter 3. We estimated individual terms of the approximate solution. Due to the approach based on discrete characteristic functions and discrete Gronwall lemma we proved, that the norm of the approximate solution is estimated by appropriate norms of functions occurring on the right-hand side of initial and boundary conditions and on the right-hand side of the partial differential equation. The result can be interpreted as unconditional stability of the analyzed scheme.

In Chapter 4, we present results of numerical experiments proving the accuracy of the NIPG and SIPG space-time discontinuous Galerkin method. We show the dependence of the error on the size of the space meshes and present the obtained experimental orders of convergence on meshes successively refined in several ways. The time step remains fixed. The results again prove unconditional stability of the method.

Our future task is to carry out further, more complicated numerical experiments.

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