DOCTORAL THESIS

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Stationary fields in black-hole space-times

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Motivated by modelling of astrophysical black holes surrounded by accretion structures, as well as by theoretical interest, we study two methods how to obtain, within stationary and axisymmetric solutions of general relativity, a metric describing the black hole encircled by a thin ring or a disc. The first is a suitable perturbation of a Schwarzschild black hole. Starting from the seminal paper by Will [1974], we showed that it is possible to express the Green functions of the problem in a closed form, which can then be employed to obtain, e.g., a reasonable linear perturbation for a black hole surrounded by a thin finite disc. In the second part we tackle the same problem using the Belinski-Zakharov generating algorithm, showing/confirming that in a stationary case its outcome is unphysical, yet at least obtaining a modest new result for the (static) “superposition” of a Schwarzschild black hole with the Bach–Weyl ring.

Keywords: general relativity, black hole, perturbations, Belinski-Zakharov method
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<tr>
<td>BH</td>
<td>Black hole.</td>
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<td>BZM</td>
<td>Belinski–Zakharov method.</td>
</tr>
<tr>
<td>CTB metric</td>
<td>Carter–Thorne–Bardeen form of metric (1.6)</td>
</tr>
<tr>
<td>EFE</td>
<td>Einstein field equations.</td>
</tr>
<tr>
<td>GF</td>
<td>Green function.</td>
</tr>
<tr>
<td>l.h.s.</td>
<td>Left hand side (of equation).</td>
</tr>
<tr>
<td>ODE</td>
<td>Ordinary differential equations.</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial differential equation.</td>
</tr>
<tr>
<td>QED</td>
<td>What was to be demonstrated (Quod erat demonstrandum).</td>
</tr>
<tr>
<td>r.h.s.</td>
<td>Right hand side (of equation).</td>
</tr>
<tr>
<td>WLOG</td>
<td>Without loss of generality.</td>
</tr>
<tr>
<td>WLP metric</td>
<td>Weyl–Lewis–Papapetrou form of metric (1.5)</td>
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<td>ZAMO</td>
<td>Zero angular momentum observer.</td>
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List of Symbols

Coordinates

\( x^\mu \)  
Generic spacetime coordinates. \( (x^0 \text{ being time-like}) \)

\( t, \phi, \rho, z \)  
Cylindric spacetime coordinates.

\( t, \phi, \theta, r \)  
Spherical spacetime coordinates.

\( \zeta = \rho + iz \)  
Complex spatial coordinate used in associated linear problems.

General Symbols

\( \circ_\mu \)  
One-form.

\( \circ_{(\mu)} \)  
One-form, expressed using local tetrad.

\( \circ^\mu \)  
Vector.

\( \circ_{(\mu)} \)  
Vector, expressed using local tetrad.

\( \circ^\mu \cdot_\mu = \sum_{\mu=1}^4 \circ^\mu \cdot_\mu \)  
Einstein’s summation rule.

\( \varepsilon_{\mu_1...\mu_n} \)  
Levi-Civita symbol. Totally antisymmetric, \( \varepsilon_{1...n} = +1 \).

\( \circ_{[\mu_1...\mu_n]} \)  
Antisymmetric part of tensor.

\( \circ_{(\mu_1...\mu_n)} \)  
Symmetric part of tensor.

\( \Gamma^\alpha_{\beta\gamma} = g^{\alpha\mu} \left[ g_{\beta\gamma,\mu} - \frac{1}{2} g_{\alpha\beta,\mu} \right] \)  
Christoffel’s symbol.

\( i \)  
Imaginary unit, \( i^2 = -1 \).

\( \overline{\circ} \)  
Complex conjugate of \( \circ \).

\( \{\circ\}_k = \frac{1}{k!} \frac{\partial^k \circ}{\partial x^k} \bigg|_{x=0} \)  
Coefficient of \( x^k \) in Taylor expansion of \( \circ \).

\( \delta_{jk} = \begin{cases} 
1 & j = k \\
0 & j \neq k 
\end{cases} \)  
Kronecker’s delta symbol.

\( Lin[...] \)  
(Suitable) linear combination of given object.

\( O(f) \)  
Function behaves (up to multiplication constant) like \( f \) in given limit.

\( \mathbb{Z} \)  
Integers.

\( \mathbb{Z}_0^+ \)  
Non-negative integers.

\( \mathbb{N} = \mathbb{Z}_0^+ \)  
Natural numbers (positive integers).

\( \mathbb{R} \)  
Real numbers.

\( \mathbb{C} \)  
Complex numbers.

\( \mathbb{S} \)  
Real part.

\( \mathbb{I} \)  
Imaginary part.
(Differential) Operators

\[ \cdot \quad \text{Dot product. As in Euclidean cylindric / spherical coordinates.} \]

\[ \text{Tr} \quad \text{Trace of matrix.} \]

\[ \text{det} \quad \text{Determinant of matrix.} \]

\[ d \quad \text{Total differential.} \]

\[ \frac{\partial \circ \mu}{\partial x^\mu} \quad \text{Partial derivative by } x^\mu. \]

\[ \frac{\partial \circ \mu}{\partial x^\mu} \quad \text{Covariant derivative by } x^\mu. \]

\[ \mathcal{L}_\xi \quad \text{Lie derivative (along vector field } \xi). \]

\[ \nabla = (\frac{\partial}{\partial \rho}, \frac{\partial}{\partial z}) \quad \text{Gradient. As in Euclidean cylindric / spherical coordinates.} \]

\[ \nabla \cdot = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{\partial}{\partial z} \quad \text{Divergence. As in Euclidean cylindric / spherical coordinates.} \]

\[ \Delta = \nabla \cdot \nabla \quad \text{Laplace operator.} \]

Special functions

\[ P_n(x) \quad n\text{-th Legendre polynomial.} \]

\[ Q_n(x) \quad \text{Legendre function of second kind.} \]

\[ C_n^{(\lambda)}(x) \quad n\text{-th Gegenbauer polynomial.} \]

\[ D_n^{(\lambda)}(x) \quad \text{Gegenbauer function of second kind, see } [C.3]. \]

\[ _2F_1 \left( \frac{a, b}{c}; x \right) \quad \text{Gauss hypergeometric function.} \]

\[ \Gamma(x) = \int_0^\infty t^x e^{-t} dt \quad \text{(Complete) Gamma function.} \]

\[ \Gamma(x, z) = \int_0^z t^x e^{-t} dt \quad \text{Incomplete gamma function.} \]

\[ \gamma(x, z) = \int_0^1 t^x e^{-t} dt \quad \text{Incomplete gamma function.} \]

\[ K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad \text{Complete elliptic integral of the first kind.} \]

\[ E(k) = \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}} dx \quad \text{Complete elliptic integral of the second kind.} \]

\[ \Pi(\alpha, k) = \int_0^1 \frac{dx}{(1-\alpha x^2)\sqrt{(1-x^2)(1-k^2x^2)}} \quad \text{Complete elliptic integral of the third kind.} \]

\[ H(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x < 0 \end{cases} \quad \text{Heaviside (step) function.} \]

\[ \text{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt \quad \text{Sine integral.} \]
Scalars

\(\kappa\)  
Surface gravity.

\(s\)  
Line element (interval).

\(g\)  
Determinant of metric tensor.

\(\nu (\nu_k)\)  
Gravitational potential (metric function). And its expansion into \(\lambda\)-series.

\(\omega (\omega_k)\)  
Dragging (metric function). And its expansion into \(\lambda\)-series.

\(\zeta (\zeta_k)\)  
Metric function. And its expansion into \(\lambda\)-series.

\(k\)  
Metric function (in section 1.1) or argument of Elliptic integral (in the rest of the thesis).

\(A\)  
Metric function.

\(B\)  
Metric function.

\(\Omega = \frac{d\phi}{dt}\)  
Angular velocity.

\(\Omega_H\)  
Angular velocity of the (black-hole) horizon.

\(v = \rho B e^{-2\nu} (\Omega - \omega)\)  
Linear velocity (with respect to ZAMO).

\(R = R_{\mu}^{\mu}\)  
Scalar curvature.

\(T = T_{\mu}^{\mu}\)  
Trace of energy-momentum tensor.

\(\epsilon\)  
Rest mass-energy density.

\(p\)  
Pressure.

\(\sigma\)  
Surface rest mass-energy density (precisely \(\sigma e^{\nu-\lambda}\) denotes surface rest mass-energy density).

\(P\)  
Azimuthal pressure (precisely \(Pe^{\nu-\lambda}\) denotes the azimuthal pressure).

\(\sigma_{\pm}\)  
Surface rest mass-energy density of streams going in positive / negative direction. (with respect to \(\phi\))

\(\delta\)  
Dirac’s delta-distribution.

\(\mathcal{E}\)  
Ernst potential.

\(a\)  
Angular momentum.

\(b\)  
Radius of the ring.

\(V\)  
Newtonian gravitational potential.

\(m\)  
Perturbation or ring mass.

\(N\)  
Number of solitons. Also used as normalization constant in section 3.1

\(\mathcal{M}\)  
Total mass of the space-time.

\(\mathcal{M}_D, \mathcal{M}_H\)  
Mass of the disc and black-hole.

\(\mathcal{J}\)  
Total angular momentum of the space-time.

\(\mathcal{J}_D, \mathcal{J}_H\)  
Angular momentum of the black-hole.
Vector (fields)

\[ \eta^\mu = \frac{\partial x^\mu}{\partial t} \quad \text{Time Killing vector field.} \]
\[ \xi^\mu = \frac{\partial x^\mu}{\partial \phi} \quad \text{Axial Killing vector field.} \]
\[ \chi^\mu = \eta^\mu + \Omega H \xi^\mu \quad \text{Killing field normal to black-hole horizon.} \]
\[ u^\mu \quad \text{Fluid velocity (one stream).} \]
\[ u^\mu_\pm \quad \text{Fluid velocity of streams going in positive / negative direction. (with respect to } \phi) \]

Tensors

\[ g_{\mu\nu} \quad \text{Metric tensor (with signature -+++)}. \]
\[ g_{ab} = t - \phi \text{ part of metric.} \]
\[ R_{\beta\gamma\delta} = 2 \Gamma^\alpha_{\beta[\delta,\gamma]} + 2 \Gamma^\gamma_{\beta[\delta} \Gamma^\delta_{\gamma]} \quad \text{Riemann tensor.} \]
\[ R_{\mu
u} = R^\alpha_{\mu\alpha\nu} \quad \text{Ricci tensor.} \]
\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad \text{Einstein’s tensor.} \]
\[ T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu + pg_{\mu\nu} \quad \text{Energy-momentum tensor. (Definition corresponds to perfect fluid.)} \]

Other symbols

\[ \lambda \quad \text{Expansion parameter (proportional to disc / ring mass) in chapter 2.} \]
\[ \lambda \quad \text{Spectral parameter of inverse scattering methods.} \]
\[ = \sqrt{\frac{K - \zeta}{K + \zeta}} \quad \text{In the case of associated linear problem of Ernst equation (chapter 1).} \]
\[ = \frac{\alpha - z}{\sqrt{(\alpha - z)^2 + \rho^2}} \quad \text{In the case of Belinski-Zakharov method (chapters 1 and 3).} \]
\[ \Psi \quad \text{Matrix solution of the associated linear problems,} \]
\[ \mathcal{C}, \mathcal{C}_\circ \quad \text{Integration path (or it’s } \circ \text{ part), see section 3.1} \]
Objects generating a very strong gravity (neutron stars, black-holes) play an interesting role in the most energetic astrophysical sources like active galactic nuclei, X-ray binaries or gamma-ray bursts. In these systems, they strongly interact with matter and fields (such an interaction actually must happen in order that their systems be observable). Should the surrounding be also taken into account when trying to analytically model their gravitational fields, it becomes in itself a challenge. Namely, such strong sources have to be treated within general relativity rather than by Newtonian physics — and general relativity is non-linear, so it is typically very difficult to find the gravitational field (space-time) of a compound system. Models used in astrophysics usually neglect the gravitational effect of the surroundings, and in most cases the strong-gravity object really dominates the field by orders of magnitude. However, this dominance may not be true for some components of the field (simply because of symmetries of the system) and mainly for higher derivatives of the potential/metric. These higher derivatives (space-time curvature) are in turn crucial for stability of motion of the surrounding matter, so, because of its own self-gravity (non-linear effect of sources on themselves), the actual surrounding matter can assume quite a different configuration than the test, non-gravitating one.

In some very symmetric situations, (some of) the Einstein field equations reduce to a linear problem and the exact solution is almost as easy as in the Newtonian case. This is for example the case for static and axially symmetric configurations. However, for the modelling of astrophysical systems, this class of space-times is too restricted, because it does not involve rotation — a typical ingredient of celestial systems (often even necessary as a support against gravity). It is thus desirable to proceed to stationary (but non-static) settings where at least a stationary rotation is “permitted”. Stationary and axially symmetric models can indeed well approximate most of the astrophysical systems.

The step from static to stationary space-times is quite non-trivial, as indicated by the history of exact solutions of Einstein’s equations itself. The famous Schwarzschild solution [Schwarzschild 1916], historically the first solution of general relativity, was found about one month after publication of this theory by Einstein. It describes any spherically symmetric vacuum region of space-time. However, it turned out to be a pioneering solution, because it can describe a black-hole — a hardly imaginable, intrinsically dynamical object separated from the outer world by a horizon. It required some 50 years for this theoretical construction to be accepted by the (astro)physical community. And about the same time was needed for a generalization of the Schwarzschild solution to a rotating one: the latter was found by Kerr in 1963 [Kerr 1963]. It is the same year when quasars were discovered, today “canonically” explained in terms of power driven by supermassive Kerr-type black-holes.

Shortly after the Schwarzschild solution, Weyl [Weyl 1917] showed that in fact all static and axially symmetric (electro-)vacuum space-times can be described by a simple (“Weyl”) metric containing just two unknown functions, provided that one works in a suitable coordinates which were later also given his name. Still more importantly, one of those unknown functions i) has quite a clear physical
meaning, namely it is a counter-part of the Newtonian gravitational potential, and, even ii) like in the Newtonian theory, it is given by Laplace (Poisson) equation, which is linear, so a superposition of partial solutions can be applied. The second metric function can be found by a line integral if the potential is already known. The Weyl metrics will be important as a starting point of this thesis. (See chapter 1.3.)

Several metrics belonging to the Weyl class were found soon using the Newtonian analogy — for example the Bach–Weyl solution [Bach and Weyl 1922] for a circular thin ring (which will be important below), or, later, the Morgan–Morgan solution [Morgan and Morgan 1969] for families of circular thin discs. However, it also turned out that a simple transfer of results from the Newtonian realm may not always yield an expected outcome (see section (1.4.2)).

When speaking about stationary and axially symmetric space-times, it is almost automatically supposed that besides the “independence on time and azimuth” they also have another important geometrical property — their meridional plane, locally spanned by directions orthogonal to both basic symmetries, is integrable, i.e. existing as a global surface. This property is standardly called “orthogonal transitivity” (see section 1.1) and the metrics without this property are only very little known. Physically, the orthogonal transitivity means that the source elements have to move only in the directions of the time and axial symmetry (circular motion, hence “circular” space-times), namely they must not perform a solenoidal motion (although a combination of such motions could keep the time and axial symmetries).

An important development in the history of the above circular space-times was the reformulation of the problem in terms of a complex potential by Ernst [Ernst 1968]. This enabled to encode the problem in a very compact form and stimulated further study in the field. This lead to discoveries of new classes of solutions, e.g. the Tomimatsu–Sato ones [Tomimatsu and Sato 1973] which provide a certain generalization of the Kerr solution. Even more importantly, the Ernst formulation lead to the proof that the stationary axisymmetric problem is completely integrable and boosted the effort to find transformations under which the equation remains satisfied, i.e. of the “generation techniques” by which one can transform one its solution into another one of the same symmetry class. These typically start from a linear problem associated with the original field equations, which is more promising for solution than the original non-linear formulation. In this thesis, we will specifically mention some results of the so-called inverse-scattering methods, see e.g. [Neugebauer and Meinel 2003] and [Belinskiii and Verdaguer 2004].

Both of the above mentioned generating methods can be used to generate new solutions by inserting “solitons” into the known metrics. Usually such attempts result in a metric with unwanted singularities, but one remarkable result was achieved in this way — the metric for a rigidly rotating disc of dust (see Neugebauer and Meinel 1994 and Neugebauer and Meinel 1995). In chapter 3 we will employ the inverse-scattering method discovered by Belinski & Zakharov (surveyed in Belinskiii and Verdaguer 2004) with the aim to obtain a metric for a black-hole surrounded by a ring or disc. Many papers were interested in BZM applied on Weyl space-times — mentioning few of them Chaudhuri and Das 1997, Zellerin and Semerák 2000 and Semerák 2002 (this one also discussed
non-physical properties of obtained solution similarly to the examined by section (3.3). Inspired by article [2011] de Castro and Letelier Bach–Weyl ring is going to be used as seed metric.

Leaving aside numerical approaches, there is still another option how to tackle Einstein (or in fact any difficult) equations: perturbation techniques. These are not appropriate for any purposes, but can at least provide solutions which are “close” to some known exact solutions. Clearly our problem of a heavy and compact centre surrounded by a light source is a very good candidate for such an approach, the more that the centre is a black-hole whose metric is known and simple (at least in the Schwarzschild case). In particular, we will follow the results by [1974] Will who proposed a method how to perturb a Schwarzschild metric by a slowly rotating and light thin circular ring. Also mentioned should be the famous Teukolsky equation [1973] which “masters” (even non-stationary) perturbations of a Kerr black-hole due to physical fields of different kinds (different spins).

Recently the primary target of approximation techniques has been non-stationary strong-field systems, in particular collisions of compact objects as a source of gravitational waves. There, their results are being matched to those of numerical treatment. Actually, numerical methods can tackle almost any situation (without particular symmetries) and in some cases factually represent the only option. On the other hand, analytical results still have their importance, because they allow to study whole families of solutions, dependence of their properties on parameters, stability, etc., better than numerical “shooting”.

Plan of the thesis

This thesis consists of three main chapters. The first chapter is of a purely review type and introduces basic concepts necessary to study stationary and axisymmetric black-holes perturbed by some surrounding matter — the metric of circular space-times (section 1.1), a suitable energy-momentum tensor (1.2), the form of the Einstein equations (1.3), an overview of several basic solutions of the given class (1.4), and some information on two basic generation techniques (1.5).

The second chapter deals with the application of a perturbative approach to circular space-times. An expansion analogous to the one suggested by [1974] Will is introduced, which leads to the set of linear differential equations whose fundamental solution can be found. The complete solution can however be achieved only for sufficiently simple background metric; regarding the astrophysical motivation, it is natural to choose the Schwarzschild black-hole as the background metric (section 2.2.2), plus appropriate boundary conditions (section 2.3). Will found the Green functions for the two crucial metric functions of this problem (section 2.4.2). Our own contribution centers on finding the closed form of these Green functions (see 2.4.6), which is much more suited for further work (e.g. for treating extended sources, where the Green functions have to be integrated) and for numerical evaluation. In section 2.5, we employ the above closed-form Green functions to derive the first-order perturbation of the Schwarzschild black-hole due to a simple type of finite thin disc, including its interpretation and several illustrations. The correspondence between the ring of matter as a source and the Green functions is examined in section 2.4.4 and the basic physical properties of
the perturbed-solution disc are checked in section 2.5.3.

In the third chapter we mention some results we obtained using generation techniques, namely the Belinskii–Zakharov inverse-scattering method. In particular, the spectral function $F$ is calculated in section 3.1, which opens the possibility to generate new metrics. We specifically consider a two–soliton solution. In section 3.2 we derive expressions for the generated metric and discuss its asymptotic behaviour. Unfortunately, it turns out that without special conditions the metric is not continuous across the equatorial plane (section 3.3). The solution is regular in a static case, but in the stationary one only the continuity of the metric itself can be ensured (normal derivatives of the metric functions have jumps in the equatorial plane, which indicates the presence of a kind of supporting surface). An example of the generated stationary metric, together with its problems just mentioned, is given in section 3.4.

Many “technical” parts of calculations are shifted to (three) appendices. Appendix A just shows transformation between two metric forms used in this thesis and appendix B brings expressions for the corresponding connection coefficients (Christoffel symbols). Most important is the last appendix C where the properties of orthogonal polynomials are examined, crucial for putting the Green functions of our perturbation problem to the closed form; they include some theorems which may not be well known. In particular, the Gegenbauer functions (those of the first-kind being commonly known as Gegenbauer polynomials) and the corresponding, ultraspherical differential equation (C.1) are discussed, because they are important for the more involved, “dragging” part of the problem. Originally the matter presented in this appendix was supposed to become a separate, purely mathematical paper (supporting the paper Čížek and Šemerák [2017]), but I have been unable to finish it in time. However, the whole thesis can be understood even without reading this long appendix, assuming the reader is familiar with the Gegenbauer polynomials and trusts the crucial relation (C.104) (it is of course possible to check the final results ex post, without going into details of their derivation).
1. Circular space-times

1.1 Metric

In this part we summarize basic properties of space-times which are stationary, axially symmetric and orthogonally transitive. Of many general-relativity textbooks which cover this subject, we follow here Stephani et al. [2003], chapter 19.

Stationary axisymmetric space-times are characterized by the existence of two commuting Killing vector fields, one of them ($\eta^\mu$) time-like ($\eta^\mu \eta_\mu < 0$) and the other one ($\xi^\mu$) space-like ($\xi^\mu \xi_\mu > 0$). Usually it is also required that the integral lines of $\eta^\mu$ be closed in analogy with Euclidean (Minkowski) space. This assumption does not affect Einstein’s equations which only determine local curvature (not global topology), but it is important for the interpretation of results. Namely, without it the plane-wave solutions are included as well. (These correspond to different boundary conditions and some quantities — for example total angular momentum — lose their meaning.)

Commutativity of the vector fields $\eta^\mu$ and $\xi^\mu$ implies that they are surface-forming. Metric can be expected simple if the corresponding integral surfaces are covered by coordinates ($t$ and $\phi$) directly given by parameters of the Killing symmetries, $\eta^\alpha = \partial x^\mu / \partial t$, $\xi^\alpha = \partial x^\mu / \partial \phi$, with $t$ being time and $\phi$ azimutal angle.

In addition, stationary axisymmetric space-times are almost “by default” assumed to satisfy another important property called orthogonal transitivity: the bivector orthogonal to both $\eta^\mu$ and $\xi^\mu$ (spanning the meridional plane) must be surface-forming as well. The necessary and sufficient condition for this property reads (Kundt and Trümper [1966])

$$\eta^\mu R_{\mu[\alpha \eta_\beta \xi_\gamma]} = 0 = \xi^\mu R_{\mu[\alpha \xi_\beta \eta_\gamma]} \quad (1.1)$$

and through the Einstein equations it can also be rewritten in terms of the energy-momentum tensor,

$$\eta^\mu T_{\mu[\alpha \eta_\beta \xi_\gamma]} = 0 = \xi^\mu T_{\mu[\alpha \xi_\beta \eta_\gamma]} \quad (1.2)$$

Simply speaking, one requires invariance under the transformation ($t \rightarrow -t, \phi \rightarrow -\phi$), which physically means that the source elements have to move only in Killing directions, not within the meridional plane. Hence the term circular space-times. Thanks to orthogonal transitivity, the remaining two coordinates ($x^2$ and $x^3$) can be chosen to cover the meridional planes (locally orthogonal to both Killing vector fields) and the metric takes form

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{t\phi} & 0 & 0 \\ g_{t\phi} & g_{\phi\phi} & 0 & 0 \\ 0 & 0 & g_{22} & g_{23} \\ 0 & 0 & g_{23} & g_{33} \end{pmatrix}, \quad (1.3)$$

Similarly for the axial symmetry and $\phi$.

\[ L_{\eta}g_{\mu\nu} = 2\eta_{(\mu,\nu)} = 0 \Rightarrow 0 = g_{\mu\nu,\alpha} \eta^\alpha + g_{\mu\alpha} \eta^\alpha_{,\nu} + g_{\nu\alpha} \eta^\alpha_{,\mu} = g_{\mu\nu,\alpha}. \]
where all the components \((g_{tt}, g_{t\phi}, g_{\phi\phi}, g_{22}, g_{23}\) and \(g_{33}\)) depend only on the coordinates \(x^2\) and \(x^3\).

The remaining freedom in the choice of \(x^2\) and \(x^3\) can be fixed in various ways, usually according to additional features of the given specific solution. One of generic options follows from the fact that all two-dimensional Riemannian manifolds are locally conformally flat and can thus be covered by “Cartesian”-type coordinates isotropically. This type of coordinates is called Weyl and the metric assumes in them the form

\[
g_{\mu\nu} = \begin{pmatrix}
g_{tt} & g_{t\phi} & 0 & 0 \\
g_{t\phi} & g_{\phi\phi} & 0 & 0 \\
0 & 0 & e^{2k} & 0 \\
0 & 0 & 0 & e^{2k}
\end{pmatrix},
\]

(1.4)

where \(k\) is a metric function parametrizing \(g_{22} = g_{33}\).

Another ambiguity arises in expressing the \((t, \phi)\) part of the metric tensor, the two mostly used “extreme” cases being the Weyl–Lewis–Papapetrou (WLP) form

\[
ds^2 = -e^{2\nu} (dt + Ad\phi)^2 + \rho^2 B^2 e^{-2\nu} d\phi^2 + e^{2\kappa-2\nu} (d\rho^2 + dz^2) = \]

\[
= -e^{2\nu} (dt + Ad\phi)^2 + r^2 \sin^2 \theta B^2 e^{-2\nu} d\phi^2 + e^{2\kappa-2\nu} (dr^2 + r^2 d\theta^2),
\]

(1.5)

and the “Carter–Thorne–Bardeen” (CTB) form\(^2\)

\[
ds^2 = -e^{2\nu} dt^2 + \rho^2 B^2 e^{-2\nu} (d\phi - \omega dt)^2 + e^{2\kappa-2\nu} (d\rho^2 + dz^2) = \]

\[
= -e^{2\nu} dt^2 + r^2 \sin^2 \theta B^2 e^{-2\nu} (d\phi - \omega dt)^2 + e^{2\kappa-2\nu} (dr^2 + r^2 d\theta^2),
\]

(1.6)

where cylindrical coordinates \((\rho, z)\) or spherical coordinates \((r, \theta)\) are employed instead of generic \(x^2\) and \(x^3\)\(^3\). The metrics contain 4 unknown functions (as in the original case \(1.4\)) — \(\nu, \omega, \zeta\) and \(B\). Transformations between the two sets of metric functions can be found in appendix A and the corresponding Christoffel symbols are given in appendix B. In both cases, the function \(\nu\) can be interpreted as gravitational potential, \(\omega\) represents angular velocity of azimuthal rotational dragging, \(\zeta\) describes scaling of the space-like 2-metric and \(B\) is a defect of circumference with respect to the radius \(\rho\). In a static limit \((\omega = 0)\) both metrics reduce to the Weyl form

\[
ds^2 = -e^{2\nu} dt^2 + \rho^2 B^2 e^{-2\nu} d\phi^2 + e^{2\kappa-2\nu} (d\rho^2 + dz^2) = \]

\[
= -e^{2\nu} dt^2 + r^2 \sin^2 \theta B^2 e^{-2\nu} d\phi^2 + e^{2\kappa-2\nu} (dr^2 + r^2 d\theta^2).
\]

(1.8)

### 1.2 Energy-momentum tensor

The sources orbiting around some centre can be kept on their orbits due to hoop stresses and by the centrifugal effect. The source we consider in this thesis is

\[\text{This was used, for example, by Bardeen and Wagoner [1971] and Will [1974], the articles followed closely in perturbation part of this thesis.}

\[\text{The coordinates are connected by relations}
\]

\[
r = \sqrt{\rho^2 + z^2}, \quad \cos \theta = \frac{z}{\sqrt{\rho^2 + z^2}}, \quad \rho = r \sin \theta, \quad z = r \cos \theta.
\]

(1.7)
a circular thin disc (or infinitesimal ring) which bears neither electric charge nor current. Following astrophysical motivation, we rather adhere to the second limit possibility and interpret the source as an orbiting particles or fluid. Two basic pictures are now possible: a single-component fluid, having certain surface density, orbiting with some suitable orbital velocity and in general generating some (suitable) pressure in the azimuthal direction; or a combination of two non-interacting (pressureless), dust components with suitable surface densities and orbiting on counter-rotating circular geodesics. Below we give some details on these two interpretations.

Note also that for an infinitesimally thin disc encircling symmetrically a black-hole, the energy-momentum tensor vanishes everywhere but "across" the disc, hence it can be included as a part of boundary conditions for vacuum field equations (see section 1.3.2). These are naturally chosen so that the space-time is reflection symmetric with respect to the equatorial plane (fixed by the disc), and this symmetry is also expected to hold for all the metric functions.

Besides the $T_\mu^\nu$ itself, it is useful to define a surface energy-momentum tensor $S_\mu^\nu$ by

$$T_\nu^\mu = S_\nu^\mu e^{2 \nu - 2 \delta(z)} \Rightarrow \int_{-\infty}^{+\infty} T_\nu^\mu g_{zz} dz = \int_{0-}^{0+} T_\nu^\mu g_{zz} dz \equiv S_\nu^\mu$$ (1.9)

in order to separate the distributional part of source and work with finite quantities.

### 1.2.1 Single-component perfect fluid

Energy-momentum tensor of a perfect fluid reads

$$T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu + pg_{\mu\nu},$$ (1.10)

where $\epsilon$ denotes the rest mass-energy density, $p$ is pressure and $u^\mu$ is the 4-velocity of fluid elements (assumed to be a time-like vector and subject to normalization $u^\mu u_\mu = -1$). In the dust limit, $p = 0$. To satisfy the assumptions of stationarity, axial symmetry and circularity, the fluid velocity is restricted to the $t$ and $\phi$ directions,

$$u^\mu = (u^t, u^\phi, 0, 0) = u^t (1, \Omega, 0, 0) = u^t \eta^\mu + u^\phi \xi^\mu,$$ (1.11)

$\Omega = d\phi/dt = u^\phi/u^t$ denoting the angular velocity of the fluid. Clearly $u_{[\alpha \xi \beta \eta]} = 0$ and the property of orthogonal transitivity holds.

Provided that $(S_\phi^\phi - S_t^t)^2 + 4S_t^t S_\phi^\phi \geq 0$, the energy-momentum tensor can be diagonalized to

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + pw^\mu w^\nu, \quad S^{\mu\nu} = \sigma u^\mu u^\nu + Pw^\mu w^\nu,$$ (1.12)

where $w^\mu$ denotes total velocity of fluid, $w^\mu$ stands for spatial vector orthogonal to fluid velocity (still belonging to the $t-\phi$ surface), $\epsilon$ and $p$ are mass-energy density of the fluid and azimuthal pressure and $\sigma$ and $P$ describes surface mass-energy
Vector field \( w^\mu \) is given by
\[
 w^\mu = \left( \frac{u^\phi}{\rho B}, -\frac{u^t}{\rho B}, 0, 0 \right), \quad w_\mu = \rho B \left( -u^\phi, u^t, 0, 0 \right) = u^t \rho B \left( -\Omega, 1, 0, 0 \right). \tag{1.14}
\]

### 1.2.2 Two-component dust source

Another possibility how to interpret the disc is a combination of two non-interacting, dust components orbiting on counter-rotating circular geodesics. In this case the energy-momentum tensor reads
\[
 T^\mu_\nu = \epsilon_+ u^\mu_+ u^\nu_+ + \epsilon_- u^\mu_- u^\nu_-, \quad S^\mu_\nu = \sigma_+ u^\mu_+ u^\nu_+ + \sigma_- u^\mu_- u^\nu_- \tag{1.15}
\]
where \( u^\mu_\pm \) denote orbital velocities of the “co-rotating” and “counter-rotating” stream, \( \epsilon_\pm \) are the mass-energy densities of the streams and \( \sigma_\pm \) are the corresponding surface mass-energy densities as introduced above.

Because the dust streams are not interacting with each other (and themselves), their orbital velocities must obey geodesic equation
\[
 0 = u^\mu_+ u^\nu_+ g_{\nu\sigma} + u^\mu_- u^\nu_- g_{\nu\sigma} = \left( u^\nu_\pm \right)^2 \left( \Gamma^\mu_{\nu\sigma} + 2\Omega_\pm \Gamma^\mu_{\nu t} + \Omega_\pm^2 \Gamma^\mu_{\nu\phi} \right)
\]
\[
 = -\frac{1}{2} \left( u^\nu_\pm \right)^2 g^{pp} \left( g_{tt,\rho} + 2g_{\phi\rho,\rho} \Omega_\pm + g_{\phi\phi,\rho} \Omega_\pm^2 \right), \quad \Rightarrow
\]
\[
 \Omega_\pm = -\frac{g_{\phi,\rho}}{g_{\phi,\rho}} \pm \sqrt{\frac{\left( g_{tt,\rho} \right)^2}{g_{\phi,\rho} g_{\phi,\rho}} - \frac{g_{tt,\rho}}{g_{\phi,\rho}}}, \tag{1.16}
\]
where \( \Omega_\pm = d\phi/dt = u^\phi_\pm / u^t_\pm \) denotes angular velocity of the \( \pm \) stream.

### 1.2.3 Relation between one-component and two-component interpretations

Because the energy-momentum tensor must come out the same in both cases, equations (1.15) and (1.12) can be used to express relations between single- and two-component quantities.

\[
 \sigma - P = \sigma_+ + \sigma_- - P_+ + P_-, \quad \sigma^2 + P^2 = \sigma_+^2 + \sigma_-^2 + 2\sigma_+ \sigma_- \left( u^\mu_+ u^\mu_- \right)^2,
\]
\[
 \sigma = \sigma_+ \left( u^\mu_+ u^\mu_- \right)^2 + \sigma_- \left( u^\mu_- u^\mu_+ \right)^2. \tag{1.17}
\]

\(^4\)The relationship between quantities is given by
\[
 \epsilon = e^{2\nu - 2\zeta} \delta(z), \quad P = pe^{2\nu - 2\zeta} \delta(z). \tag{1.13}
\]
To be precise \( e^{\nu - \zeta} \) and \( p e^{\nu - \zeta} \) denotes surface mass-energy density and azimuthal pressure.

\(^5\)The quotation marks indicate that in general one has to be careful what these terms mean: usually they are understood in the sense of the \( \phi \) coordinate, but actually it is more appropriate to take them “with respect to the geometry”, i.e. with respect to zero-angular-momentum orbits.

\(^6\)Many such relations can be found; we only list here the simplest ones, following from the comparison of \( S^\mu_\nu g_{\mu\nu} \), \( S^\mu_\nu S_{\mu\nu} \) and \( S^\mu_\nu u_\mu u_\nu \).
1.2.4 Linear velocity

For the interpretation purposes, it is also useful to define the linear velocity with respect to zero-angular-momentum observer (ZAMO) which orbits, in the Killing directions, with $\Omega = \omega$,

$$v \equiv \rho Be^{-2\nu} (\Omega - \omega).$$  \hfill (1.18)

In terms of this quantity, the fluid “bulk” four-velocity $u^\mu$ can be written

$$\left( u^\mu \right)^2 = \frac{e^{-2\nu}}{1 - v^2}, \quad \Rightarrow \quad u^\mu = \left( \frac{e^{-\nu}}{\sqrt{1-v^2}}, \frac{e^{-\nu}\Omega}{\sqrt{1-v^2}}, 0, 0 \right).$$  \hfill (1.19)

It is also useful to endow the ZAMO family with a tetrad adapted to the global coordinates,

$$e(t) = e^{-\nu} dt + \omega e^{-\nu} d\omega, \quad e(\phi) = \frac{e^{\nu}}{\rho B} d\phi, \quad e(\rho) = e^{\nu - \mu} d\rho, \quad e_z = e^{\nu - \mu} dz.$$  \hfill (1.20)

In this frame the perfect-fluid energy-momentum tensor (1.10) has only five non-zero components,

$$T(t)(t) = \frac{\epsilon + p v^2}{1 - v^2}, \quad T(\phi)(\phi) = \frac{p + \epsilon v^2}{1 - v^2},$$

$$T(t)(\phi) = \frac{\epsilon + p v - 1}{\sqrt{1-v^2} \sqrt{1-v^2}}, \quad T(\rho)(\rho) = T(z)(z) = p.$$  \hfill (1.21)

Analogous relations hold for the corresponding surface energy-momentum tensor.

Linear velocity can be used in the relations between single- and two-component interpretations. Projections of the velocities are given by

$$u_\pm u_\alpha = \frac{vv_\pm - 1}{\sqrt{1-v^2} \sqrt{1-v^2}}, \quad u_\pm u_\alpha = \frac{v_\pm - v}{\sqrt{1-v^2} \sqrt{1-v^2}},$$

$$g_{\alpha\beta} u_\alpha u_\beta = \frac{v_+ v_- - 1}{\sqrt{1-v_+^2} \sqrt{1-v_-^2}},$$  \hfill (1.22)

where $v_\pm$ denote linear velocities corresponding to the two dust streams. Equations (1.17) then read

$$\sigma - P = \sigma_+ + \sigma_-, \quad \sigma^2 + P^2 = \sigma_+^2 + \sigma_-^2 + 2\sigma_+ \sigma_- \frac{(1 - v_+ v_-)^2}{(1 - v_+^2)(1 - v_-^2)},$$

$$\sigma = \sigma_+ \frac{(vv_+ - 1)^2}{(1 - v^2)(1 - v_+^2)} + \sigma_- \frac{(vv_- - 1)^2}{(1 - v^2)(1 - v_-^2)}.$$  \hfill (1.23)

1.3 Einstein field equations

Einstein field equations link the geometry with the energy-momentum distribution. We assume the reader is familiar with this central point of general relativity and only focus on their form specific to circular space-times and to our particular black-hole–disc problem.
1.3.1 Equations for metric functions

For the metric (1.6) and for the perfect-fluid energy-momentum tensor (1.10) the Einstein equations read (see, e.g., Bardeen and Wagoner [1971])

\[
\nabla \cdot \left( B \nabla \nu \right) - \frac{1}{2} \rho^2 B^2 e^{-4\psi} \nabla \omega \cdot \nabla \omega = 4\pi B e^{2\zeta - 2\nu} \left[ (\epsilon + p) \frac{1 + \nu^2}{1 - \nu^2} + 2p \right],
\]

(1.24)

\[
\nabla \cdot \left( \rho^2 B^3 e^{-4\psi} \nabla \omega \right) = -16\pi \rho B^2 e^{2\zeta - 4\nu} (\epsilon + p) \frac{\nu}{1 - \nu^2},
\]

(1.25)

\[
\nabla \cdot (\rho \nabla B) = -16\pi \rho B e^{2\zeta - 2\nu} p,
\]

(1.26)

and

\[
0 = \left( \frac{B_{,\rho}}{B} + \frac{1}{\rho} \right) \zeta_{,\rho} - \frac{B_{,z}}{B} \zeta_{,z} - \frac{B_{,\rho\rho}}{\rho B} + \frac{B_{,zz}}{2B} - \left( \nu_{,\rho} \right)^2 + \left( \nu_{,z} \right)^2 + \frac{1}{4} \rho^3 B^2 e^{-4\psi} \left[ (\omega_{,\rho})^2 - (\omega_{,z})^2 \right],
\]

(1.27)

\[
0 = \left( \frac{B_{,\rho}}{B} + \frac{1}{\rho} \right) \zeta_{,z} + \frac{B_{,z}}{B} \zeta_{,\rho} - \frac{B_{,\rho\rho}}{\rho B} + \frac{B_{,zz}}{2B} - 2\nu_{,\rho} \nu_{,z} + \frac{1}{2} \rho^3 B^2 e^{-4\psi} \omega_{,\rho} \omega_{,z}.
\]

(1.28)

The latter group of equations can be used to determine the function \( \zeta \), assuming that \( B, \nu \) and \( \omega \) are already known. Condition for integrability of (1.27) and (1.28) reads

\[
0 = \frac{2\rho \left( \rho B_{,\rho} \nu_{,z} + B \nu_{,z} - \rho B_{,z} \nu_{,\rho} \right) RHS_{1.24}}{B^2 2\rho B B_{,\rho} + \rho^2 (B_{,\rho})^2 + (B_{,z})^2} - \rho \left( \rho B_{,\rho} \omega_{,z} + B \omega_{,z} - \rho B_{,z} \omega_{,\rho} \right) RHS_{1.25} + \frac{2}{B^2 2\rho B B_{,\rho} + \rho^2 (B_{,\rho})^2 + (B_{,z})^2} + \text{Terms proportional to (powers and derivatives of) } RHS_{1.26},
\]

(1.29)

where \( RHS_{1.24} \) denotes right-hand sides of the corresponding equations. (In the vacuum case, \( \epsilon = p = 0 \) and they all vanish.) The condition is fulfilled if vacuum equations (1.24), (1.25) and (1.26) hold. Supplemented with appropriate boundary conditions, \( \zeta \) can be expressed in the form of a line integral, see (2.35) and (2.34).

Moreover, assuming dust source (\( p = 0 \)), (1.26) for \( B \) can be satisfied in many ways. Physically the choice of \( B \) corresponds to a coordinate transformation, the relation between coordinates corresponding to some generic \( B \) and the ones corresponding to \( B = 1 \) (denoted \( \tilde{\rho}, \tilde{z} \)) reading

\[
\tilde{\rho} = \rho B, \quad \tilde{z} = \pm \int \rho B_{,z} d\rho - (\rho B)_{,\rho} dz.
\]

(1.30)

where the sign can be chosen arbitrarily. Integrability condition for \( \tilde{z} \) holds as a consequence of (1.26). The simplest solution \( B = 1 \) is usually being employed, but it is not ideal in the presence of a black-hole, as we will see in section 1.4.1.

\footnote{Except the choice \( B = \rho^{-1} \) which leads to plane-wave solutions rather than to axially symmetric ones.}
In the static case ($\omega = 0$), equations (1.24)–(1.28) become much simpler. Equation (1.25) is satisfied trivially and (1.26)’s left-hand side reduces to a two-dimensional Laplace operator, with known solutions. With $B$ known, equation (1.24) reduces to a linear partial differential equation (in the vacuum case and for $B = 1$, it becomes a Laplace equation in three dimensions), so superposition of solutions can be applied. The knowledge of $B$ and $\nu$ is then used to solve (1.27) and (1.28), see (2.35) and (2.34).

1.3.2 Boundary conditions

Assuming the existence of a thin equatorial disc, there are four different areas to treat specifically: spatial infinity, symmetry axis, black-hole horizon and equatorial plane.

Spatial infinity

We assume the space-time is asymptotically flat, actually, the black-hole–disc configuration is supposed to be the only source present, we do not consider a cosmological “background”, we do not take into account cosmological constant or gravitational radiation. So the space-time should approach a flat one towards spatial infinity, which for our metric components implies

$$g_{tt} \to -1, \quad g_{t\phi} \to 0, \quad g_{\phi\phi} \to \rho^2, \quad g_{zz} \to 1.$$  \hspace{1cm} (1.31)

In terms of individual metric functions of the (1.5) metric, this means

$$\nu \to 0, \quad A \to 0, \quad B \to 1, \quad \zeta \to 0,$$  \hspace{1cm} (1.32)

while the functions parametrizing the (1.6) form should behave like

$$\nu \to 0, \quad \omega \to 0, \quad B \to 1, \quad \zeta \to 0$$  \hspace{1cm} (1.33)

(all the limits represent $\sqrt{\rho^2 + z^2} \to \infty$).

Axis

To avoid a conical singularity, the $z = \text{const}$ surfaces have to be locally flat on the axis, which is ensured by

$$\rho \to 0+ \quad \Rightarrow \quad \zeta \to \ln B.$$  \hspace{1cm} (1.34)

Also, derivative by $\rho$ of all the metric coefficients has to vanish at the axis, otherwise string-like singular source would appear there. This requirement implies

$$\rho = 0 \quad \Rightarrow \quad \nu_\rho = B_\rho = \omega_\rho = \zeta_\rho = 0.$$  \hspace{1cm} (1.35)

Black-hole horizon

The black-hole horizon has a number of special properties valid for any circular space-time. In particular, the dragging angular velocity ($\omega_H$) and the surface
gravity $\kappa$ have to be constant over the horizon. The latter is defined by (see Wald [1984] (12.5.14))

$$\kappa^2 = -\frac{1}{2} \chi^\alpha \chi^\beta \chi_{\alpha \beta} = e^{4\nu - 2\zeta} \left[ \nabla \nu \cdot \nabla \nu - \frac{1}{4} e^{-4\nu} \rho^2 B^2 \nabla \omega \cdot \nabla \omega \right],$$

(1.36)

where $\chi^a = \eta^a + \omega H \xi^a$ stands for a Killing field normal to the horizon. Both of these quantities ($\omega H$ and $\kappa$) thus should be finite.

Since there should be no matter present at the very (stationary) horizon, equation (1.36) can be simplified there, using Einstein equation (1.24), to

$$\kappa^2 = e^{6\nu - 2\zeta} \cdot \left( B \nabla e^{-2\nu} \right).$$

(1.37)

Also the horizon, in order not to be degenerate, should have finite (and non-vanishing) azimuthal and latitudinal circumferences. These conditions are ensured by non-degeneracy of $B \rho e^{-2\nu}$ and $e^{2\zeta - 2\nu}$.

Equatorial plane

Boundary conditions at the equatorial plane in both vacuum and disc regions can be found using (1.24), (1.25) and (1.26). No additional conditions are necessary for $\zeta$ because its value is fixed on the axis and equations (1.27) and (1.28) are of the first order.

Because the source is located in the equatorial plane only and it is singular ($\delta$-like) in the $z$ direction, there must be discontinuity in the first derivatives of the metric functions. As already mentioned above, we also require reflection symmetry with respect to the equatorial plane. This leads to

$$\frac{\partial f}{\partial z} \bigg|_{z=0-} = -\frac{\partial f}{\partial z} \bigg|_{z=0+},$$

(1.38)

$$\frac{\partial^2 f}{\partial z^2} \bigg|_{z=0} = \left( \frac{\partial f}{\partial z} \bigg|_{z=0+} - \frac{\partial f}{\partial z} \bigg|_{z=0-} \right) \delta(z) = 2 \frac{\partial f}{\partial z} \bigg|_{z=0+} \delta(z),$$

(1.39)

where $f$ stands for an arbitrary function. It also confirms the assumption that we can treat the vacuum case in the same way – without the $\delta$-like source, both one-sided derivatives have to vanish. Because of the discontinuity on the equatorial plane, the directions of derivatives must be carefully distinguished. Due to the reflection symmetry it is sufficient to know the solution in the “upper” half-space $z \geq 0$. Unless stated otherwise, the first derivatives in the equatorial plane are understood as $\partial / \partial z \big|_{z=0+}$.

Finally, in the dust case the boundary condition reads

$$\nu, z \big|_{z=0+} = 2\pi\sigma \frac{1 + v^2}{1 - v^2}, \quad B, z \big|_{z=0} = 0,$$

8Meaning these expressions are finite and not equal to zero there.

9 Even more exotic boundary conditions could be considered, including discontinuity of the metric itself. However, our astrophysically motivated thin disc is just an idealization of an accretion disc with small but finite thickness, in which case the metric is continuous (including its first and second derivatives). Denoting $l$ the characteristic thickness of the disc, the first (second) derivatives of metric behave like $l^{-1}$ ($l^{-2}$) as $l \to 0+$. With discontinuous metric we would encounter more strange objects (generating products and derivatives of $\delta$ distribution, in particular) and the field equations would be even problematic to interpret.

20
\[ \omega_{z|z=0+} = -8\pi\sigma \frac{\Omega - \omega}{1 - v^2} = -8\pi\sigma \frac{v e^{2\nu}}{\rho B (1 - v^2)}. \] (1.40)

### 1.3.3 Ernst equation

One reformulation of the field equations for the (electro-)vacuum case should be mentioned — the Ernst equation (see \textsc{Ernst} [1968]). It can be derived using the WLP form of the metric (1.5), usually choosing \( B = 1 \). The WLP-metric analogue of equation (1.25) reads

\[
(\rho^{-1} e^{4\nu} A_{,\rho} + (\rho^{-1} e^{4\nu} A_{,z})_{,z} = 0
\] (1.41)

and can be regarded as an integrability condition for the twist potential \( \mathcal{A} \),

\[
A_{,\rho} = \rho e^{4\nu} A_{,z}, \quad A_{,z} = -\rho e^{4\nu} A_{,\rho}.
\] (1.42)

Introducing the (complex) Ernst potential

\[
\mathcal{E} = e^{2\nu} + i\mathcal{A},
\] (1.43)

equations (1.24) and (1.25) can be represented by the Ernst equation

\[
\Re \mathcal{E} \triangle \mathcal{E} = \nabla \mathcal{E} \cdot \nabla \mathcal{E},
\] (1.44)

where \( \Re \) denotes a real part. The remaining field equations (1.27) and (1.28) then read

\[
\zeta_{,\rho} = \frac{\rho}{(\mathcal{E} + \overline{\mathcal{E}})} \left( \mathcal{E}_{,\rho} \overline{\mathcal{E}}_{,\rho} - \mathcal{E}_{,z} \overline{\mathcal{E}}_{,z} \right), \quad \zeta_{,z} = \frac{\rho}{(\mathcal{E} + \overline{\mathcal{E}})} \left( \mathcal{E}_{,\rho} \overline{\mathcal{E}}_{,\rho} + \mathcal{E}_{,z} \overline{\mathcal{E}}_{,z} \right),
\] (1.45)

where \( \overline{f} \) stands for complex conjugate of \( f \).

### 1.4 Some important solutions

#### 1.4.1 Schwarzschild black-hole

Leaving aside “trivial” Minkowski metric, the simplest solution of Einstein equations is the Schwarzschild metric (\textsc{Schwarzschild} [1916]). It describes any vacuum spherically symmetric (region of) space-time without the cosmological constant (Birkhoff’s theorem; \textsc{Birkhoff} [1923] and \textsc{Jebsen} [1921]). In the extreme case when there is no \( T_{\mu\nu} \) anywhere except the very centre (“point-like source”), the solution represents an isolated static and spherically symmetric black-hole. This will below be the object of our perturbation effort.

The best known form of the metric

\[
ds^2 = -\left( 1 - \frac{2M}{R} \right) dt^2 + \frac{dR^2}{1 - \frac{2M}{R}} + R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2
\] (1.46)

employs Schwarzschild coordinates, in particular, we denote by \( R \) the Schwarzschild radius (different from the “isotropic” radius \( r \) in (1.8), see below); the single metric parameter \( M \) represents mass of the black-hole. Such a metric, however, does
not fit in the form (1.4) and we will rather use the metric written in isotropic coordinates,

\[ ds^2 = -\left(\frac{2r - M}{2r + M}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\right). \] (1.47)

The metric functions of (1.5), (1.6) and (1.8) appear in these coordinates as

\[ \nu = \ln\left(\frac{2r - M}{2r + M}\right), \quad \omega = A = 0, \quad B = 1 - \frac{M^2}{4r^2}, \quad \zeta = \ln\left(1 - \frac{M^2}{4r^2}\right). \] (1.48)

See figure 1.1 for illustration of behaviour of the metric coefficients. In this form, all the metric functions are independent on both angular coordinates and the black-hole horizon is represented as a sphere with radius \( r = M/2 \). The only drawback is the complexity of the metric function \( B \).

On the other hand, with the simplest possible choice \( B = 1 \), the Schwarzschild metric reads

\[ ds^2 = -\frac{d_+ - 2M}{d_+ + 2M} dt^2 + \rho^2 \frac{d_+ + 2M}{d_+ - 2M} d\phi^2 + \frac{d_+^2 - 4M^2}{d_+^2 - d_-^2} (d\rho^2 + dz^2), \] (1.49)

where

\[ d_\pm = \sqrt{\rho^2 + (z + M)^2} \pm \sqrt{\rho^2 + (z - M)^2} \] (1.50)

and the metric functions are given by

\[ \nu = \frac{1}{2} \ln\left(\frac{d_+ - 2M}{d_+ + 2M}\right), \quad \zeta = \frac{1}{2} \ln\left(\frac{(d_+ + 2M)^2}{d_+^2 - d_-^2}\right), \quad B = 1. \] (1.51)

Such form is however less suitable for perturbation, because it leads to a more complicated form of the field equations.

The function \( \nu \) assumes a simple form when expressed in terms of infinite
Figure 1.2: Orbital velocities of circular geodesics about a Schwarzschild black-hole (of mass $M = 1$). No such orbits exist in the dotted part (it would require superluminal speed). Left: Angular velocity $\Omega$. Right: Linear velocity (with respect to ZAMO, i.e. static observer) $v$.

A series of Legendre polynomials $P_n(x)$\textsuperscript{10}

$$\nu = -\sum_{n=0}^{\infty} \frac{1}{2n+1} \left( \frac{M}{r} \right)^{2n+1} P_{2n}(\cos \theta). \quad (1.53)$$

This however does not help in perturbative approach much, because multiplication of Legendre polynomials leads to Clebsh–Gordan coefficients and lengthy sums of terms.

Finding of the ring- or disc-like solutions requires orbital velocity of circular orbits in the equatorial plane. Angular velocity is given by (1.16),

$$\Omega = \sqrt{-g_{\mu \nu}} g_{\phi \phi} = \frac{8\sqrt{Mr^3}}{(2r+M)^3}, \quad (1.54)$$

and the corresponding linear velocity is

$$v = rB e^{-2\nu} \Omega = \frac{2\sqrt{rM}}{2r-M}. \quad (1.55)$$

See figure 1.2 for illustration.

### 1.4.2 Chazy–Curzon metric

To demonstrate pitfalls of a straightforward Newtonian analogy, it is useful to check Chazy–Curzon metric (Chazy [1924] and Curzon [1924]). Newtonian gravitational potential of a point mass reads $V = -M/r$. Because $\nu$ in (1.8) denotes the corresponding quantity (also given by Laplace equation), it could have been

\textsuperscript{10} We follow notation of chapter 14 of DLMF. Legendre polynomial can be, for example, expressed by formula (14.7.13) there:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \quad (1.52)$$
expected that the field of a “point mass” in general relativity would have very similar properties. However, there also appears the second metric function $\zeta$ which often deforms (literarily) the picture considerably. Actually, in this case the full metric reads

$$ds^2 = -e^{-2M/r}dt^2 + e^{2M/r}\left[r^2 \sin^2 \theta d\phi^2 + e^{-3M^2 \sin^2 \theta/r^2} (dr^2 + r^2 d\theta^2)\right]$$ (1.56)

and the solution as a whole is even not spherically symmetric ($\zeta$ function does not possess this symmetry). The “correct” solution of the spherically symmetric problem is the Schwarzschild solution recalled in previous section.

### 1.4.3 Bach–Weyl ring

Another example of distorted Newtonian analogy is the Bach–Weyl ring (see Bach and Weyl [1922]). It describes the field generated by a thin (singular) circular ring in the equatorial plane. In the Newtonian case the gravitational potential reads

$$V(\rho, z) = -\frac{m}{2\pi} \int_0^{2\pi} \frac{1}{\sqrt{\rho^2 + z^2 + b^2 - 2\rho b \sin \phi}} d\phi =$$

$$= -\frac{2mK\left(\sqrt{\frac{4\rho b}{(\rho+b)^2+z^2}}\right)}{\pi \sqrt{(\rho + b)^2 + z^2}},$$ (1.57)

where $m$ denotes the ring mass, $b$ is its radius and $K(x)$ stands for the complete elliptic integral of the first kind. The corresponding relativistic picture is, again, distorted by the second metric function $\zeta$: it is described by the metric (1.8) with

$$\nu = -\frac{2m}{\pi \sqrt{(\rho + b)^2 + z^2}} K(k),$$

$$B = 1,$$

$$\zeta = \frac{m^2}{4\pi^2 b^2} \left[\frac{(b^2 - \rho^2 - z^2)(2 - k^2)}{(\rho^2 + z^2 + b^2)(1 - k^2)} E(k)^2 + 4K(k)E(k) - \frac{(3b^2 + \rho^2 + z^2)(2 - k^2)}{(\rho^2 + z^2 + b^2)} K(k)^2\right],$$ (1.58)

where $E(k)$ stands for the complete elliptic integral of the second kind and

$$k = \sqrt{\frac{4\rho b}{(\rho + b)^2 + z^2}}$$ (1.59)

is the argument of the elliptic functions. The metric functions are plotted in figure 1.3. Distortion due to the metric function $\zeta$ is clearly visible. (See Basovnik and Semerák [2016] for a detailed study of this solution and a comparison with other singular-ring ones.)

The Bach–Weyl ring plays an important role in this thesis, in particular, it represents the source corresponding to the Green functions of our perturbation procedure.
Figure 1.3: Metric functions for the Bach–Weyl ring with mass $m = 1$ and radius $b = 3$. Left: $\nu$. Displayed levels are $-0.7, -0.6, \ldots, -0.2$. Right: $\zeta$. Displayed levels are $-1.0, -0.9, \ldots, -0.1$.

1.4.4 Kerr black-hole

Another example of a physically significant black-hole solution is the Kerr metric [Kerr [1963]], in the Boyer–Lindquist coordinates (generalization of the Schwarzschild ones) having the form

$$ds^2 = \left(1 - \frac{2MR}{R^2 + a^2 \cos^2 \vartheta}\right) \left(dt + \frac{2MaR \sin^2 \vartheta}{R^2 - 2MR + a^2 \cos^2 \vartheta} d\phi\right)^2 + \frac{(R^2 + a^2 \cos^2 \vartheta)(R^2 - 2MR + a^2) \sin^2 \vartheta}{R^2 - 2MR + a^2 \cos^2 \vartheta} d\vartheta^2 + \left(R^2 - 2MR + a^2 \cos^2 \vartheta\right) \left(d\vartheta^2 + \frac{dR^2}{R^2 - 2MR + a^2}\right).$$

(1.60)

where $R$ denotes radius, $M$ is mass of the black-hole and $a$ is its angular momentum. Indeed, the metric is stationary (not static), so it allows rotation. The metric can be written in the WLP form (1.5) using the coordinate transformation $\rho = \sqrt{R^2 - 2MR + a^2 \sin \vartheta}$, $z = (R - M) \cos \vartheta$. (1.61)

The metric functions come out as

$$\nu = \frac{1}{2} \ln \left(\frac{R^2 - 2MR + a^2 \cos^2 \vartheta}{R^2 + a^2 \cos^2 \vartheta}\right),$$

$$A = \frac{2MaR \sin^2 \vartheta}{R^2 - 2MR + a^2 \cos^2 \vartheta},$$

$$B = 1,$$

$$\zeta = \frac{1}{2} \ln \left(\frac{(R^2 - 2MR + a^2 \cos^2 \vartheta)^2}{[(R - M)^2 + (a^2 - M^2) \cos^2 \vartheta] (R^2 + a^2 \cos^2 \vartheta)}\right),$$

(1.62)

where $R$ and $\vartheta$ are now considered as functions of $\rho$ and $z$ given by implicit relations (1.61).

The Kerr space-time can be generated by thin discs — an infinite differentially (counter-)rotating one Bičák and Ledvinka [1993] or a rigidly rotating finite one Neugebauer and Meinel [2003], but in contrast to the Schwarzschild black-hole no real internal solution has been found so far.

\footnote{Inverse transformation is not mentioned here because it is quite cumbersome.}
1.5 Associated linear problem

For a circular space-time, Einstein’s equations (namely the Ernst equation) are known to be completely integrable. However, it is not easy to solve them directly from scratch, so in more generic cases “generating techniques” are being employed. These are mathematical procedures which transform some known, “seed” solution of the problem into another one of the same (circular) class. Examples of such techniques are Bäcklund transformations and inverse-scattering methods (see [Stephani et al., 2003, chapter 10] for quick overview). Most of the (infinite families of) metrics obtained by generating techniques still remain to be interpreted, and it is very probable that almost none of them has a reasonable physical sense. In this thesis we nevertheless include a section on a possible usage of the inverse-scattering method.

Generating techniques are mostly not applied directly to the field equations, but rather to an associated linear problem. There are two basic ways how to find this. The one involving Ernst equation (1.44) (e.g. Neugebauer and Meinel [1994], Neugebauer and Meinel [1995] who found the solution for rigidly rotating disc in this manner) and the Belinskii–Zakharov method (great review can be found in Belinskii and Verdaguer [2004]).

1.5.1 Linear problem associated with the Ernst equation

Let \( \Psi \) be a \( 2 \times 2 \) matrix depending on some (“spectral”) parameter \( \lambda \) and on spatial coordinates \( \rho \) and \( z \). It is useful to introduce a complex coordinate \( \zeta = \rho + iz \) and consider the matrix \( \Psi \) as the function of \( \lambda, \zeta \) and \( \bar{\zeta} \). Let the matrix \( \Psi \) solve the pair of linear differential equations (originally, including even Maxwell equations, given by Neugebauer and Kramer [1983])

\[
\frac{\partial \Psi}{\partial \zeta} = \left( \begin{array}{cc} B & 0 \\ 0 & A \end{array} \right) + \lambda \left( \begin{array}{cc} 0 & B \\ A & 0 \end{array} \right) \Psi, \tag{1.63}
\]

\[
\frac{\partial \Psi}{\partial \bar{\zeta}} = \left( \begin{array}{cc} \bar{A} & 0 \\ 0 & \bar{B} \end{array} \right) + \frac{1}{\lambda} \left( \begin{array}{cc} 0 & \bar{A} \\ \bar{B} & 0 \end{array} \right) \Psi, \tag{1.64}
\]

where the spectral parameter is usually expressed (in terms of still another parameter \( K \)) as

\[
\lambda = \sqrt{\frac{K - i\zeta}{K + i\zeta}} \tag{1.65}
\]

and \( A \) and \( B \) are complex functions of physical coordinates \( (\rho, z) \) alone (they do not depend on \( K \) or \( \lambda \)). If these functions are related to the Ernst potential by

\[
A = \frac{\mathcal{E}_\zeta}{2\Re \mathcal{E}}, \quad B = \frac{\bar{\mathcal{E}}_{\bar{\zeta}}}{2\Re \mathcal{E}}, \tag{1.66}
\]

then the integrability condition of \( 1.63 \) and \( 1.64 \) is exactly the Ernst equation \( 1.44 \).

Hence, the solution of the above linear differential matrix equations yields the solution of the Ernst equation and, up to a quadrature, metric functions \( \nu \) and \( \omega \) as well. Problem is that the coefficients of the differential equations \( 1.63 \) and \( 1.64 \) are not a priory known. In some special cases it is possible
to overcome this difficulty using suitable normalization and a special behaviour of spectral parameter on the axis \((\lambda = 1)\). A considerable amount of work on this formulation has been done by the Jena GR group (see e.g. Neugebauer and Meinelt [2003] for a survey).

### 1.5.2 Belinskii–Zakharov method

The Belinskii–Zakharov inverse-scattering method, originally published in Belinskii and Zakharov [1978], assumes the metric has the form

\[
ds^2 = g_{ab}dx^a dx^b + f(\rho^2 + dz^2),
\]

where \(g_{ab}\) denotes its Killing \((t, \phi)\) part. They, as well as \(f\), depend on \(\rho\) and \(z\) only. In addition, the determinant of the 2-metric is assumed to amount to

\[
\det g_{ab} = -\rho^2.
\]

Such a condition is satisfied by metrics (1.5) and (1.6), if \(B = 1\) is chosen. Introducing \(2 \times 2\) matrices

\[
U \equiv \rho g_x g^{-1}, \quad V \equiv \rho g_z g^{-1},
\]

where \(g^{-1}\) denotes the inverse of \(g\) and multiplication is the ordinary matrix multiplication, the Einstein equations (1.24) and (1.25) take the form

\[
U_{,\rho} + V_{,z} = 0.
\]

The remaining two equations — (1.27) and (1.28) — can be written as

\[
\frac{f_{,\rho}}{f} = -\frac{1}{\rho} + \frac{1}{4\rho} \text{Tr} (U^2 - V^2),
\]

\[
\frac{f_{,z}}{f} = \frac{1}{2\rho} \text{Tr} (UV).
\]

Again, (1.69) represents the integrability condition of these two equations. Therefore, \(f\) can be obtained by quadrature provided that \(U\) and \(V\) are known.

Equation (1.69) seems simple at first glance. However, the matrices \(U\) and \(V\) are not independent. For example, using the relation

\[
\frac{\partial}{\partial z} \text{Tr} \left( \frac{U}{\rho} \right) = \frac{\partial}{\partial \rho} \text{Tr} \left( \frac{V}{\rho} \right),
\]

one can find that

\[
\frac{\partial}{\partial z} \text{Tr} \left( \frac{U}{\rho} \right) = \frac{\partial}{\partial \rho} \text{Tr} \left( \frac{V}{\rho} \right).
\]

In order to solve the equation (1.69), Belinskii and Zakharov introduced an additional (spectral) parameter \(\lambda\) and differential operators

\[
\hat{D}_1 = \frac{\partial}{\partial z} - \frac{2\lambda^2}{\lambda^2 + \rho^2} \frac{\partial}{\partial \lambda}, \quad \hat{D}_2 = \frac{\partial}{\partial \rho} + \frac{2\lambda \rho}{\lambda^2 + \rho^2} \frac{\partial}{\partial \lambda}
\]

and a generating matrix \(\Psi\) (again \(2 \times 2\)) which satisfies the linear system

\[
\hat{D}_1 \Psi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2} \Psi, \quad \hat{D}_2 \Psi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2} \Psi.
\]

The normalization of \(\Psi\) is fixed by boundary condition

\[
\Psi(\rho, z, \lambda = 0) = g(\rho, z)
\]

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which is in agreement with (1.68).

Knowing the solution of (1.74) (let us denote it by $\Psi_0$ and the corresponding original/seed 2-metric by $g_0$), the new solution of the system (1.74) can be found by prescription

$$\Psi(\rho, z, \lambda) = \chi(\rho, z, \lambda)\Psi_0(\rho, z, \lambda), \quad g(\rho, z) = \chi(\rho, z, 0)g_0(\rho, z),$$

where $\chi$ is the so-called dressing matrix. It has to satisfy

$$\hat{D}_1 \chi = \frac{\rho U - \lambda V}{\rho^2 + \rho^2} \chi - \chi \frac{\rho V_0 - \lambda U_0}{\rho^2 + \rho^2} \chi - \chi \frac{\rho U_0 - \lambda V_0}{\rho^2 + \rho^2},$$

$$\hat{D}_2 \chi = \frac{\rho U - \lambda V}{\rho^2 + \rho^2} \chi - \chi \frac{\rho U_0 - \lambda V_0}{\rho^2 + \rho^2} \chi - \chi \frac{\rho U_0 - \lambda V_0}{\rho^2 + \rho^2},$$

where $U_0$ and $V_0$ are defined by (1.68) with the seed metric $g_0$ plugged in.

There are several restrictions which the matrix $\chi$ has to satisfy (see [Belinskii and Verdaguer, 2004, ch. 1.3] for details). In particular, its determinant has to be $\det \chi = 1$ in order that the generated metric have correct determinant ($\det g = -\rho^2$). However, it may be useful to omit this condition in the procedure and normalize the obtained metric only later (we will denote the renormalized quantities by the superscript $^{\text{ph}}$ (meaning physical). For the so-called soliton solutions when the dressing matrix contains only simple poles in the $\lambda$ plane, it can be expressed in algebraic terms. It can finally be used to find a new metric,$$
g_{ab} = (g_0)_{ab} - (g_0)_{ac}(g_0)_{bd} \sum_{k,l=1}^N \Pi^{(k)}(m^{(k)})(m^{(l)})_{ab} \mu(k)\mu(l),$$

$$f = C_N f_0 \prod_{k=1}^N \frac{\mu^2(k)}{\mu^2(k) + \rho^2} \det \Gamma^{(k)},$$

where

$$\mu(k) = \alpha_k - z + \sqrt{(\alpha_k - z)^2 + \rho^2},$$

$$m_a^{(k)} = C_b^{(k)} \left[\Psi(\rho, z, \mu(k))\right]^{-1}_{ab},$$

$$\Gamma^{(k)}(l) = \frac{m_a^{(k)} m_b^{(l)}(g_0)_{ab}}{\rho^2 + \mu(k)\mu(l)},$$

$$\Pi^{(k)}(l) = \left[\Gamma^{(k)}(l)\right]^{-1} \Leftrightarrow \sum_{m=1}^N \Gamma^{(k)}(m)\Pi^{(m)}(l) = \delta^{(k)}(l).$$

The indices $a, b, c, d \in \{t, \phi\}$ and $k, l, m \in \{1, 2, \ldots, N\}$, where $N$ denotes the number of “solitons”. The parentheses are used to distinguish between the soliton and coordinate indices. (Einstein’s summation rule still applies, i.e. when the same index appears twice it is summed over. The constants $\alpha(k)$ correspond to the position of the solitons on the axis and the constants $C_a^{(k)}$ encode physical quantities like black-hole mass, angular momentum, etc. (See section (3.2) for a discussion of the two-soliton solution.) Both these sets of constants can be chosen arbitrarily, as well as the integration constant $C_N$ which fixes scaling of the $\rho$ and $z$ coordinates.
Concerning the physical parameters of the solution, it can be shown that

\[
\det g = \frac{(-1)^N \rho^{2N}}{\prod_{k=1}^N \mu_{(k)}^2} \det g_0
\]

and so \([\text{Belinskii and Verduguer}, 2004, (1.110)]\)

\[
g_{\text{ph}} = \frac{\rho}{\sqrt{|\det g|}} g,
\]

\[
f_{\text{ph}} = C_N^{\text{ph}} f_0 \frac{\prod_{k=1}^N \mu_{(k)}^{N+1}}{\rho^{N+2} \prod_{k=2}^N \prod_{l=1}^{N-1} [\mu_{(k)} - \mu_{(l)}]} \det \Gamma_{(k)(l)}.
\]
2. Perturbative approach

The pair of equations (1.24) and (1.25) is, due to their non-linearity, hard to solve. A usual way how to proceed is the perturbation one: start from some known exact solution, make slight perturbation of some appropriate quantities (“$X + \delta X$”), substitute the decomposition into the equations, subtract the $X$-part which is satisfied exactly and linearize the rest in $\delta X$. For a sufficiently simple original, exact “background”, one can hope the linear equations could be solved better than the original, non-linear ones. However, general relativity is non-linear and to include its effects (namely, the “back reaction” of the source on itself, i.e. “self-gravitation”) properly, it is desirable to extend the perturbation order beyond the first, linear one. One thus arrives at the solution of the field equations in terms of series expansions of the relevant quantities in terms of some small parameter(s).

In particular, this chapter tackles, in a perturbative way, the problem of a slowly rotating and light annular thin disc encircling a black-hole. The system is studied within the class of circular space-times and the target of perturbation is a Schwarzschild black-hole. In order that the disc’s mass and angular-momentum effect can be considered as a perturbation, one should restrict to

$$\frac{m}{M} \frac{1}{1 - \frac{v^2}{1 - \frac{2M}{r}}} \ll 1,$$

where $m$ denotes the disc rest mass, $M$ is the black-hole mass, $v$ stands for a (maximal) linear velocity of the dust and $r$ stands for radius (of the inner rim) of the disc. Physically, this assumption of course means weak self-gravitation of the disc (non-linear effect of the dust to its own gravitational field) and should ensure that the perturbation series may be cut after just several terms. (For astrophysical purposes, this should certainly be enough, because it is likely that a bigger error arises as a consequence of the model idealizations anyway.)

Perturbations of black-hole space-times are a “classical” GR problem, recently mainly studied in a non-stationary regime in connection with collisions of compact objects and a consequent gravitational emission. Much weaker interest has been focused on stationary perturbations, so one has to mainly search in older literature. Actually, the methods derived by Bardeen and Wagoner \cite{Bardeen:1971} and Will \cite{Will:1974} can still be taken as a starting point.

In this chapter we follow closely our recent paper Čížek and Semerák \cite{Cizek:2017} (conference proceedings Čížek and Semerák \cite{Cizek:2009}, Čížek \cite{Cizek:2011} and Čížek and Semerák \cite{Cizek:2012} can be mentioned for some preliminary results).

2.1 Expansion in perturbing mass

Let the expansion parameter be denoted by $\lambda$. Provided that the series (2.3) converges, $\lambda$ need not be specified further, important is that the r.h. sides of the field equations (1.24)–(1.28) are proportional to it, see (2.4) and (2.5). One suitable suggestion could be

$$\lambda = \frac{m}{M} \frac{1}{1 - \frac{v^2}{1}}.$$
The metric functions are expanded as

\[ \nu = \sum_{k=0}^{\infty} \nu_k \lambda^k, \quad \omega = \sum_{k=0}^{\infty} \omega_k \lambda^k, \quad \zeta = \sum_{k=0}^{\infty} \zeta_k \lambda^k, \tag{2.3} \]

where \( \nu_k, \omega_k \) and \( \lambda_k \) are functions of \( \rho \) and \( z \). Clearly \( \nu_0, \omega_0, \zeta_0 \) and \( B \) are metric function describing the background (vacuum) solution. Let us remind that the metric function \( B \) need not be expanded, because in our case (thin disc or ring) the equation \([1.26]\) can be solved exactly — it is a homogeneous linear partial differential equation (PDE). The surface rest mass density of our source is given by

\[ \sigma = \lambda e^\nu \tau, \quad \epsilon = \lambda \tau e^{3\nu - 2\zeta} \delta(z), \tag{2.4} \]

where \( \tau \) describes the disc mass distribution and the term \( e^\nu \) ensures that the total rest mass of the disc, given by

\[ \mathcal{M} = \int_\text{disc} \epsilon \sqrt{g_{\phi\phi} g_{\rho\rho} z \rho} \, d\phi d\rho dz = \int_\text{disc} \epsilon \rho B e^{2\nu - 3\zeta} \, d\phi d\rho dz = 2\pi \lambda \int_{\rho^-}^{\rho^+} \tau \rho B d\rho, \tag{2.5} \]

depends on \( \lambda \) linearly; finally, \( \rho_- \) and \( \rho_+ \) are inner and outer radii of the disc. Equations \([1.24]\) and \([1.25]\) for \( \nu \) and \( \omega \) lead to the following equations for individual powers of \( \lambda \):

\[ \nabla \cdot (B \nabla \nu_k) - \frac{\rho^2 B^2}{2} \sum_{j=0}^{k/2} \sum_{l=0}^{k/2} \{ e^{-4\nu} \} j-l, l \cdot \nabla \omega_l = 4\pi B \tau \delta(z) \left\{ e^{\nu} \frac{1 + \nu^2}{1 - \nu^2} \right\}, \tag{2.6} \]

\[ \sum_{j=0}^{k} \nabla \cdot \left( \rho^2 B^3 \{ e^{-4\nu} \} j \nabla \omega_{k-j} \right) = -16\pi \rho B^2 \tau \delta(z) \left\{ \frac{v e^{-\nu}}{1 - \nu^2} \right\}, \tag{2.7} \]

where

\[ \{ f \}_k = \left. \frac{1}{k!} \frac{\partial^k f}{\partial \lambda^k} \right|_{\lambda=0} \tag{2.8} \]

is the \( \lambda^k \)-term arising from the Taylor expansion of \( f \) and \( k \geq 1 \). (In particular, the \( k = 0 \) case corresponds to vanishing r.h. sides of \([2.6]\) and \([2.7]\), i.e. to \( \nu \) and \( \omega \) given by vacuum equations \([1.24]\) and \([1.25]\) and thus reducing to a given background metric. Finally, the linear velocity \( v \) is given by the metric functions and their derivatives.

The above perturbation equations can be solved iteratively. If \( \nu_j \) and \( \omega_j \) are known for all \( 0 \leq j < k \), equations \([2.6]\) and \([2.7]\) can be written as

\[ \nabla \cdot (B \nabla \nu_k) - 2\rho^2 B^2 e^{-4\nu_0} \nu_k \nabla \omega_0 \cdot \nabla \omega_0 - \rho^2 B^2 e^{-4\nu_0} \nabla \omega_k \cdot \nabla \omega_0 = \]

\[ = \frac{BR_k}{\rho^2 + z^2}, \tag{2.9} \]

\[ \nabla \cdot \left( \rho^2 B^3 e^{-4\nu_0} \nabla \omega_k \right) - 4\nabla \cdot \left( \rho^2 B^3 e^{-4\nu_0} \nu_k \nabla \omega_0 \right) = \]

\[ = \frac{M^4 \rho^2 S_k}{B (\rho^2 + z^2)^{3/2}} \tag{2.10} \]
where $R_k$ and $S_k$ stand for the already known “source terms”,

$$
R_k = \frac{\rho^2 + z^2}{B} \left[ \frac{\rho^2 B^2}{2} \sum_{j=0}^{\infty} \left\{ e^{-4\nu} \right\}_{k-l-j} \nabla \omega_j \cdot \nabla \omega_l + \frac{\rho^2 B^2}{2} \sum_{j=1}^{k-1} e^{-4\nu} \nabla \omega_j \cdot \nabla \omega_{k-j} 
+ 4\pi B \tau \delta(z) \left\{ e^{-\nu} \frac{1 + v^2}{1 - v^2} \right\}_{k-1} \right], (2.11)$$

$$
S_k = -\frac{B(\rho^2 + z^2)^3}{M^4 \rho^2} \left[ \sum_{j=1}^{k-1} \nabla \cdot \left( \rho^2 B^3 \left\{ e^{-4\nu} \right\}_j \nabla \omega_{k-j} \right) + 16\pi \rho B^2 \tau \delta(z) \right] \left\{ \frac{ve^{-\nu}}{1 - v^2} \right\}_{k-1}. (2.12)
$$

Equations (2.9) and (2.10) are easier to solve due to their linearity. Thanks to a possible superposition, it is sufficient to find a vector basis (represented by combinations of functions $\nu_k$ and $\omega_k$) of the homogenous pair of equations, possibly supplemented by particular solutions incorporating the r.h. sides. However, even the latter pair of linear equations may be hard to solve. The situation is easier when the initial metric is static (i.e. $\omega = 0$): the equations (2.9) and (2.10) decouple then. This is one of the reasons for considering the Schwarzschild black-hole as the starting background.

### 2.2 Multipole expansion: Spherical harmonics

#### 2.2.1 Methods of solving linear PDE

Linear PDE are usually being solved by expansion (with respect to some set of functions of coordinates) or by using a Green function. The latter case can be applied to linear differential equations only. Assume the equation takes the form

$$\hat{O}f = R(x^1, x^2, \ldots x^N), (2.13)$$

where $\hat{O}$ denotes some linear differential operator, $R$ some source term and $x^1, \ldots x^N$ are coordinates. The boundary conditions have the form

$$\hat{D}f \bigg|_{\partial \Sigma} = 0, (2.14)$$

where $\hat{D}$ stands for some other linear differential operator and $\partial \Sigma$ is the border of the domain $\Sigma$. Green function $G$ is defined then as a particular solution of (2.13) with a $\delta$-like r.h. side and satisfying the boundary conditions (2.14), i.e.

$$\hat{O}G(x^1, \ldots, x^N, y^1, \ldots, y^N) = \delta(y^1 - x^1)\delta(y^2 - x^2)\ldots \delta(y^N - x^N), (2.15)$$

$$\hat{D}G \bigg|_{\partial \Sigma} = 0. (2.16)$$

(Despite its name, the Green function is defined using distributions. The term fundamental solution, used in a more general sense, may be more appropriate.) Equation (2.13) is solved by

$$f(x^1, \ldots, x^N) = \int_{\Sigma} R(y^1, \ldots, y^N)G(x^1, \ldots, x^N, y^1, \ldots, y^N)dy^1 \ldots dy^N. (2.17)$$
This clearly satisfies the boundary conditions \((2.14)\). In the case of a compact source (thin disc in our case), the integral on the r.h. side of \((2.17)\) simplifies considerably. The problem is then to find the Green function itself, i.e. a solution of the corresponding differential equation with the special r.h. side and given boundary conditions.

Fundamental systems of ordinary differential equations (ODEs) can be found straightforwardly, but for PDEs no generic direct procedure is known. We are going to find the Green functions of \((2.9)\) and \((2.10)\) using series expansion and, ex post, summation of the obtained series (see section \(2.4.6\)). Series expansion, used in the paper [Will 1974] we start from, reduces PDE to a set of ODEs. In a general case, these equations are not independent and their solution is not simple. However, they can decouple in some cases if using an expansion into eigenfunctions of a suitable part of the original differential equation.

An important attribute of the series solution is its convergence speed. Generally the series converge exponentially fast, but one has to be much more careful in the case of singular sources (spatially less than three-dimensional). In our case of the thin disc (spatially a two-dimensional source) the convergence is still power-law and breaks down just at some particular points (analogue of the Gibbs phenomenon occurs due to the discontinuities). The convergence of the multipole expansion towards the exact Green function is illustrated, for a linear perturbation by a massive thin ring, in figures \(2.2\) and \(2.3\). Convergence problems also appear in numerical simulations using spectral methods (see, for example, the survey Grandclément and Novak 2009). They can sometimes be eliminated by a clever choice of coordinates. For example, Ansorg et al. 2003 obtained better behaviour in the oblate spheroidal coordinates

\[
\rho = a \cosh u \cos v, \quad z = a \sinh u \sin v, \quad (2.18)
\]

where \(a\) is an arbitrary fixed parameter describing the outer rim of the disc. Also toroidal coordinates, employed in Ansorg et al. 2009, can be used. For our purpose, even more suitable (for obtaining smooth boundary conditions) seem to be the flat-ring coordinates given by

\[
\rho = a \sin v \sqrt{\cos^2 u + k^2 \sin^2 u \sin^2 v}, \quad z = a \cos v \sqrt{1 - k^2 \sin^2 u \cos^2 u \sin^2 v}, \quad (2.19)
\]

where \(a\) and \(k\) are arbitrary fixed parameters describing the inner and outer rims of the disc (see Kuchel et al. 1987). Unfortunately, these do not fit well into the series expansion. Both coordinate systems mentioned are sketched in figure 2.1.

2.2.2 Choice of background metric

As already mentioned, we will start from a static background metric in order to decouple the set of equations \((2.9)\) and \((2.10)\). Among the black-hole solutions, the simplest is the Schwarzschild metric which, in the limit case (vacuum everywhere), describes an isolated, spherically symmetric black-hole (section \(1.4.1\)).

Although the Schwarzschild metric can be written in canonical Weyl coordinates \(\rho, z\) and the field equations look quite simple in them, such a possibility naturally involves the \(B = 1\) choice. This means that the black-hole horizon degenerates to a finite segment of the symmetry axis \((\rho = 0, z \in (-M, +M))\),
which is not an ideal option for treating the respective boundary conditions and for physical interpretation. In isotropic coordinates $r, \theta$, we saw the horizon is a sphere located at radius $r = M/2$. This is given by the choice

$$B = 1 - \frac{M^2}{4r^2},$$

where $M$ denotes the black-hole mass. Since the function $B$ is not subject to perturbation, the horizon remains on $r = M/2$ even after perturbation. The remaining background-metric functions read

$$\nu_0 = \ln \left(\frac{2r - M}{2r + M}\right), \quad \omega_0 = 0, \quad \zeta_0 = \ln \left(1 - \frac{M^2}{4r^2}\right).$$

Substituting into the equations (2.9) and (2.10), we have

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[(4r^2 - M^2) \frac{\partial \nu_k}{\partial r}\right] + \frac{4r^2 - M^2}{4r^4 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial \nu_k}{\partial \theta}\right] = \frac{BR_k}{r^2},$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[\frac{(2r + M)^7}{2^6 r^2 (2r - M)} \frac{\partial \omega_k}{\partial r}\right] + \frac{(2r + M)^7}{2^8 r^6 (2r - M) \sin^3 \theta} \frac{\partial}{\partial \theta} \left[\sin^3 \theta \frac{\partial \omega_k}{\partial \theta}\right] = \frac{M^4 S_k}{r^4 B}.$$
where \( x \) a is dimensionless radius given by
\[
x \equiv \frac{r}{M} \left( 1 + \frac{M^2}{4r^2} \right),
\]  
(2.27)
equations (2.22) and (2.23) read
\[
\sum_{j=0}^{\infty} \left\{ \frac{d}{dx} \left[ (x^2 - 1) \frac{d\nu_{kj}}{dx} \right] - j(j+1) \nu_{kj} \right\} P_j(\cos \theta) = R_k, \tag{2.28}
\]
\[
\sum_{j=0}^{\infty} \left\{ (x^2 - 1) \frac{d}{dx} \left[ (x+1)^4 \frac{d\omega_{kj}}{dx} \right] - j(j+1)^2 \omega_{kj} \right\} C_j^{(3/2)}(\cos \theta) = S_k, \tag{2.29}
\]
where \( r \) is now considered a function of \( x \).

Using the orthogonality of polynomials \( P_j(\cos \theta) \) and \( C_j^{(3/2)}(\cos \theta) \) (section C.2.2), the equations can be separated to yield
\[
\frac{d}{dx} \left[ (x^2 - 1) \frac{d\nu_{kj}}{dx} \right] - j(j+1) \nu_{kj} = R_{kj}, \tag{2.30}
\]
\[
(x^2 - 1) \frac{d}{dx} \left[ (x+1)^4 \frac{d\omega_{kj}}{dx} \right] - j(j+3)(x+1)^4 \omega_{kj} = S_{kj}, \tag{2.31}
\]
where \( R_{kj} \) and \( S_{kj} \) are coefficients of expansion of the r.h. sides of (2.28) and (2.29), i.e.
\[
R_{kj} = \frac{2j+1}{2} \int_{-1}^{1} R_k(x, \cos \theta) P_j(\cos \theta) d(\cos \theta), \tag{2.32}
\]
\[
S_{kj} = \frac{2j+3}{2(j+1)(j+2)} \int_{-1}^{1} S_k(x, \cos \theta) C_j^{(3/2)}(\cos \theta) \sin^2 \theta d(\cos \theta). \tag{2.33}
\]

### 2.2.3 Metric function \((\zeta)\)

As already mentioned in section 1.3.1 the metric function \( \zeta \) can be obtained by quadrature. Boundary condition on the axis (1.34) suggests a suitable integration path — along the axis to a given radius and then along the contour of constant radius (in our case all this path is through vacuum). Rewriting (1.27) and (1.28) using (2.20) gives
\[
\zeta_{,\theta} = \frac{r \left( 1 + \frac{M^2}{4r^2} \right) \left[ 2B \nu_{,\theta} - \frac{1}{2} \omega_{,\theta} \sin^2 \theta B^3 e^{-4\nu} \right]}{(1 + \frac{M^2}{4r^2})^2 + B^2 \cot^2 \theta}
\]
\[
+ \frac{r^2 \cot \theta \left[ \frac{M^2}{4r^2} - B^2 \left( \nu_{,r}^2 - \frac{\nu_{,\theta}^2}{r^2} \right) + \frac{1}{2} \omega_{,r}^2 \left( \nu_{,r} - \frac{\nu_{,\theta}}{r} \right) \right] r^2 \sin^2 \theta B^4 e^{-4\nu}}{(1 + \frac{M^2}{4r^2})^2 + B^2 \cot^2 \theta}
\]
and the \( \zeta \) function itself is obtained by prescription
\[
\zeta = \ln B + \int_{0}^{\theta} \zeta_{,\theta} d\theta. \tag{2.35}
\]
For \( \nu \) and \( \omega \) known, the calculation of \( \zeta \) is just a “technical” issue and will not be investigated in detail. Exact expressions for \( \zeta \) are anyway quite complicated and not very useful, while the integral (2.35) usually requires numerical evaluation.

Relation (2.34) simplifies considerably on the horizon

\[
\zeta,|_{r=\frac{M}{\nu}} = -\frac{2}{\nu} \delta \nu,|_{r=0}, \quad \zeta,|_{r=\frac{M}{\omega}} = -\frac{2}{\omega} \delta \omega,|_{r=0},
\]

(2.36)

where \( \delta \zeta = \sum_{j=1}^{\infty} \nu_j \lambda_j \), i.e. perturbation of gravitational potential. The fact that radial derivatives of dragging (see page 51) vanishes there was used as well. Finally \( \zeta \) itself reads

\[
\zeta,|_{r=\frac{M}{\nu}} = \ln B + 2 \delta \nu,|_{r=\frac{M}{\nu}} - 2 \delta \omega,|_{r=\frac{M}{\omega}}.
\]

(2.37)

The \( B \) function is here kept as it describes diverging part of \( \zeta \).

### 2.3 Boundary conditions

**Spatial infinity**

Boundary conditions at spatial infinity are given by (1.33). When expanded according to (2.3) and (2.26), they read

\[
\nu_{kj} \rightarrow 0, \quad \omega_{kj} \rightarrow 0,
\]

(2.38)
as \( x \rightarrow \infty \).

**Axis**

Boundary conditions on the symmetry axis (1.34) hold as a consequence of the dependence of angular part on \( \cos \theta \). In our spherical-type coordinates, vanishing of the \( \rho \)-derivative at the axis is equivalent to vanishing of the derivative with respect to \( \theta \),

\[
\frac{\partial}{\partial \rho}|_{\rho=0} \quad \frac{1}{\rho} \frac{\partial}{\partial \theta}|_{\theta=0,\pi} = 0.
\]

(2.39)
The expected dependence of the angular part on \( \cos \theta \) implies

\[
\frac{\partial f(\cos \theta)}{\partial \theta}|_{\theta=0,\pi} = -f'(\cos \theta) \sin \theta|_{\theta=0,\pi} = 0,
\]

(2.40)
where \( f(x) \) is an arbitrary function and \( f'(x) \) is its derivative with respect to the argument. In the last equality it has been assumed that \( f'(1) \) is bounded. In the case of polynomials (whether Legendre or Gegenbauer) such assumption is satisfied.
Black-hole horizon

Unperturbed metric has the horizon located at sphere \( r = M/2 \) corresponding to the singularity of metric function \( B \), which is not subjected to the perturbation scheme. It can be expected that coordinates of horizon stay the same 1.

Using orthogonality of Legendre \((C.14)\) and Gegenbauer polynomials \((C.15)\), the boundary conditions on horizon can be separated. Constancy of dragging reads

\[
\omega_{kj}\big|_{r = \frac{M}{2}} = 0, \tag{2.41}
\]

where \( j > 1 \) (\( \omega_{k0} \) can be arbitrary).

Condition of non–degenerate azimuthal circumference reads

\[
B\rho e^{-\nu}\bigg|_{r = \frac{M}{2}} = 2M \sin \theta e^{-\sum_{j=1}^{\infty} \nu_j \lambda_j}\bigg|_{r = \frac{M}{2}} \tag{2.42}
\]

implying finiteness of \( \nu_j \) on the horizon so \( \nu_{jk} \) must not diverge there. Physically meaningful latitudinal circumference is ensured by this as well. Given \((2.37)\) it can be concluded that

\[
\zeta - \nu\big|_{r = \frac{M}{2}} = \ln(4) + \delta\nu\big|_{r = \frac{M}{2}} - 2 \delta\nu\bigg|_{r = \frac{M}{2}, \theta = 0}, \tag{2.43}
\]

Finiteness of perturbation of gravitational potential \( \delta\nu \) on the horizon finishes sketch of the proof of non–degeneracy.

Constancy surface gravity, due to non-linearity of the equations \((1.36)\) and \((1.37)\) is not analysed here explicitly.

Equatorial plane

There are two ways how to treat a thin source located in the equatorial plane.

1. Consider it as a limit case of source that is spatially extended (see sections \[1.2\] and \[1.3.1\]). The reflection symmetry is of course assumed.

2. Include it by prescribing jumps of the normal derivative of the metric functions there (see section \[1.3.2\]). If the source is absent, the normal derivatives vanish — see \((1.40)\).

We will use the first view for a calculation of the metric and the second one for the analysis of properties of the solution. Actually, the r.h. sides of \((2.6)\) and \((2.7)\) involve the equatorial boundary conditions automatically since the source terms are \( \delta \)-like and they do show reflection symmetry. In terms of the orthogonal polynomials, the necessary (and sufficient) condition for reflection symmetry is, regarding their parity \((C.12)\), that the odd expansion coefficients has to vanish,

\[
\nu_{k(2j+1)} = 0, \quad \omega_{k(2j+1)} = 0, \tag{2.44}
\]

where \( j \in \mathbb{Z}_0^+ \).

---

1To prove that it is necessary to calculate all orders of expansions. Being it not done in this thesis the position of horizon remains assumption.

The results from low order of mass expansion (see Green functions \((2.81)\) and \((2.96)\)) indicates that nothing dramatic takes place and the assumption holds.
2.4 Green functions

2.4.1 Solving the expanded field equations

Fundamental solution

Fundamental solutions of (2.30) and (2.31) are already presented in the paper Will [1974]. They can be written in the form

\[ \nu_{jk} = \text{Lin} [P_k(x), Q_k(x)]; \]
\[ \omega_{jk} = \text{Lin} [F_k(x), G_k(x)]; \]

(2.45)

(2.46)

where \( \text{Lin}[] \) means a suitable linear combination, \( P_k, Q_k \) denote Legendre functions of the first (second) kind and

\[ F_k(x) = 2F_1 \left( \frac{-k, k + 3, 1 + x}{4}; \frac{1}{2} \right), \]
\[ G_k(x) = F_k(x) \int_x^{\infty} \frac{dx'}{(1 + x')^4 [F_k(x')]^2}; \]

(2.47)

(2.48)

\( 2F_1 \left( a, b; c; x \right) \) denotes the Gauss hypergeometric function,

\[ 2F_1 \left( a, b; c; x \right) = \sum_{k=0}^{\infty} \frac{\Gamma(a + k)\Gamma(b + k)\Gamma(c)x^k}{\Gamma(a)\Gamma(b)\Gamma(c + k)k!}, \]

(2.49)

where \( \Gamma(x) \) is the Gamma function

\[ \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt. \]

(2.50)

Note that \( k \in \mathbb{Z}_0^+ \) ensures that the Legendre function of the first kind \( P_k \) and \( F_k(x) \) are polynomials of degree \( k \). Asymptotic \( (x \to \infty) \) behaviour of the fundamental solutions is given by

\[ P_k(x) \approx x^k, \quad Q_k(x) \approx x^{-k-1}, \quad F_k(x) \approx x^k, \quad G_k(x) \approx x^{-k-3}, \]

(2.51)

and the behaviour at the horizon \( (x \to 1+) \) by

\[ P_k(x) = 1, \quad Q_k(x) \approx \frac{1}{2} \ln(x - 1), \quad F_k(x) = \delta_{k0}, \quad G_k(x) \approx \text{const.} \neq 0, \]

(2.52)

where \( \delta_{kj} \) is the usual Kronecker symbol. See [DLMF] ch. 14.8 or section C.2.3 for the properties of Legendre functions. The behaviour of \( F_k \) and \( G_k \) follows from the definition of the Gauss hypergeometric function ([DLMF], (15.4.24)) and of the function \( G_k \). In the last case we have actually used an alternative definition

\[ G_k(x) = \frac{(-1)^k(k + 2)!(k + 3)!}{48(2k + 3)!} \left( \frac{1 + x}{2} \right)^{-k-3} 2F_1 \left( k, k + 3; 2 \frac{2}{1 + x} \right), \]

(2.53)

which can be obtained thanks to [DLMF] (15.10.6) and regarding that only one fundamental solution (unique up to a multiplicative constant) vanishes when \( x \to \infty \) (the multiplicative constant can be fixed by comparing the leading-order terms of the hypergeometric function and of (2.48).
Source terms

Let us denote by $G_{\nu}(x, x', \theta, \theta')$ and $G_{\omega}(x, x', \theta, \theta')$ the Green functions of equations (2.22) and (2.23), respectively, and by

$$R_k = S_k = \delta(x' - x)\delta(\cos \theta' - \cos \theta),$$

the corresponding source terms, with $(x', \theta')$ representing the position of the source. (Note that differential operators on the l.h.s. sides of the perturbation equations — and thus also the Green functions — do not depend on the perturbation order $k$, whereas the source terms do depend on it.) Together with (2.32) and (2.33) we have

$$R_{kj} = 2j + 1 \int_{-1}^{1} \delta(x' - x)\delta(\cos \theta' - \cos \theta)P_j(\cos \theta)d(\cos \theta) =$$

$$= \frac{2j + 1}{2} P_j(\cos \theta')\delta(x' - x), \quad (2.55)$$

$$S_{kj} = \frac{2j + 3}{2(j + 1)(j + 2)} \int_{-1}^{1} \delta(x' - x)\delta(\cos \theta' - \cos \theta)C_{j}^{(3/2)}(\cos \theta) \sin^2 \theta d(\cos \theta) =$$

$$= \frac{2j + 3}{2(j + 1)(j + 2)} C_{j}^{(3/2)}(\cos \theta') \sin^2 \theta' \delta(x' - x), \quad (2.56)$$

where up to now [Will 1974, (50)] has been followed (except for his expansion in horizon angular velocity).

2.4.2 Green functions

The Green functions can be expanded into spherical harmonics analogously to (2.26),

$$G_{\nu}(x, x', \theta, \theta') = \sum_{n=0}^{\infty} G_{\nu n}(x, x', \theta')P_n(\cos \theta), \quad (2.57)$$

$$G_{\omega}(x, x', \theta, \theta') = \sum_{n=0}^{\infty} G_{\omega n}(x, x', \theta')C_{n}^{(3/2)}(\cos \theta). \quad (2.58)$$

They have to obey (2.30) and (2.31) with the r.h.s. sides (2.55) and (2.56). The latter, however, are $\delta$-like, so $G_{\nu n}$ and $G_{\omega n}$ are, up to a scaling factor, Green’s functions of (2.30) and (2.31). Since the source vanishes almost everywhere (except the point $x = x'$ where it is singular), the solution of (2.30) and (2.31) has the form

$$G_{\nu j}(x, x', \theta') = \begin{cases} A_{\nu < j}(x', \theta')P_j(x) + B_{\nu < j}(x', \theta')Q_j(x) & \text{when } x \leq x' \\ A_{\nu > j}(x', \theta')P_j(x) + B_{\nu > j}(x', \theta')Q_j(x) & \text{when } x \geq x' \end{cases}, \quad (2.59)$$

$$B_{\omega < j}(x, x', \theta, \theta') = \begin{cases} A_{\omega < j}(x', \theta')F_j(x) + B_{\omega < j}(x', \theta')G_j(x) & \text{when } x \leq x' \\ A_{\omega > j}(x', \theta')F_j(x) + B_{\omega > j}(x', \theta')G_j(x) & \text{when } x \geq x' \end{cases}, \quad (2.60)$$

where $A_{\omega \nu}$ and $B_{\omega \nu}$ are suitable functions of the source position $(x')$. The overlap of intervals in (2.59) and (2.60) is intentional: the equations are of the second
order and the Green functions should be continuous (discontinuity is only in the first derivatives), so both definitions have to coincide at \( x = x' \).

Green functions has to also obey boundary conditions. Those at spatial infinity, combined with (2.51), give

\[ A_{\nu>j} = A_{\omega>j} = 0, \]  

where \( j \in \mathbb{Z}^+ \). On the horizon, uniformity of the angular velocity requires \( \omega_{kj} = 0 \) (see (2.41)) for \( j > 0 \) and (2.52) yields

\[ B_{\omega<j} = 0, \]  

where \( j \in \mathbb{N} \) (while \( B_{\omega<0} \) can be chosen arbitrarily). Also there is requirement of finiteness of horizon circumference implying regularity of perturbation of \( \nu \) there. So \( Q_j(x) \) must not be present, i.e.

\[ B_{\nu<j} = 0, \]  

for all \( j \in \mathbb{Z}^+ \). Finally, one has to take care of the first derivatives in order to get \( \delta(x - x') \). Analogously to (1.39), it can be shown that

\[ \frac{\partial^2 f}{\partial x^2} \bigg|_{x=x'} = \left[ \frac{\partial f}{\partial x} \bigg|_{x=x'} - \frac{\partial f}{\partial x} \bigg|_{x=x'}^- \right] \delta(x - x'). \]  

(2.64)

The above requirements fix the unknown functions \( A_{\nu\nu} \) and \( B_{\nu\nu} \) and the expanded Green functions read

\[ G_{\nu j}(x, x', \theta') = -\frac{2j + 1}{2} P_j(\cos \theta') \sin \theta' P_j(x_<) Q_j(x_), \]  

(2.65)

\[ G_{\omega j}(x, x', \theta') = -\frac{2j + 3}{2(j + 1)(j + 2)} C_{ij}^{(3/2)}(\cos \theta') \sin^2 \theta' F_j(x_<) G_j(x_), \]  

(2.66)

where

\[ x_< = \min(x, x'), \quad x_> = \max(x, x'). \]  

(2.67)

Note that \( G_{\omega j} \) has one degree of freedom related to \( B_{\omega<0} \) on top of (2.66) — see discussion in the next section.

### 2.4.3 Angular velocity of the horizon

Special attention deserves the term in (2.66) which arises from freedom in the choice of \( B_{\omega<0} \) and which has been omitted. Consider first the function

\[ -B_{\omega<0} \int_0^3 C_{ij}^{(3/2)}(\cos \theta') \sin^2 \theta' F_0(x') G_0(x) = -B_{\omega<0} \frac{1}{4} \sin^2 \theta' \frac{1}{(x + 1)^3}. \]  

(2.68)

It is smooth everywhere (including spatial infinity, horizon and ring radius) and solves the homogeneous equation (2.31) (with \( j = 0 \)). Hence,

\[ \tilde{G}_{\omega 0}(x, x', \theta') = G_{\omega 0}(x, x', \theta') - B_{\omega<0} \frac{\sin^\varepsilon \theta'}{\Delta((\tilde{\theta} + \infty)^3}. \]  

(2.69)
is also a Green function of (2.31), just with a different value of angular momentum of the horizon. A solution with general r.h. side can be obtained by

\[ \tilde{\omega}_{k0} = \int \int S_{k0}(x', \theta') \tilde{G}_{\omega 0}(x, x', \theta') dx' d(\cos \theta') = \]

\[ = \int \int S_{k0}(x', \theta') G_{\omega 0}(x, x', \theta') dx' d(\cos \theta') - \int \int S_{k0}(x', \theta') B_{\omega<0} \frac{\sin^2 \theta'}{4(x + 1)^3} dx' d(\cos \theta') = \]

\[ = \int \int S_{k0}(x', \theta') G_{\omega 0}(x, x', \theta') dx' d(\cos \theta') + \frac{J_k}{(x + 1)^3}, \quad (2.70) \]

where the constants \( J_k \) are proportional to \( B_{\omega<0} \) and to the mass distribution. These constants clearly adjust the horizon angular velocity and can be interpreted as fixing the own angular momentum of the black-hole (note in passing that the part of the Green function corresponding to \( B_{\omega<0} \) is smooth across the source). More precisely, the angular velocity of the horizon is determined by

\[ \omega_H = \sum_{k=0}^{\infty} \omega_{Hk} \lambda^k, \quad \omega_{Hk} = \int S_{k0}(x', \theta') \frac{\sin^2 \theta'}{4(x' + 1)^3} dx' d\theta' + \frac{J_k}{8}. \quad (2.71) \]

Our choice differs from the original paper [Will 1974]. Will used additional expansion parameter \( \beta = k \omega_H \) (besides the “mass” perturbation denoted there by \( \gamma \)) rather than considering the constants \( J_k \) (he did not mention the freedom in the horizon angular momentum). In any case, the introduction of constants \( J_k \) serves the same purpose and, at least by my opinion, is simpler to work with. (Note that interaction of angular momentum of the disc and of the black-hole may however be present in higher orders of the mass perturbation (2.3), so there one would have to be more careful.

In order to fix the ambiguity in the black-hole angular momentum, we put \( J_k = 0 \) in the rest of this thesis. Effectively it means that the black-hole angular momentum is kept zero in perturbation, so the black-hole is just being “carried along” by dragging induced by the external source. (We will confirm this explicitly after deriving a specific solution for a simple finite thin disc.)

### 2.4.4 Linear perturbation due to a thin ring

Green functions of the equations (2.28) and (2.29) have a clear physical interpretation — they describe linear perturbation of due to a circular thin (singular) ring of matter. We assume this ring is located in the equatorial plane, \( \theta' = \pi/2 \), on some radius \( r = r' \) (a ring placed off the equatorial plane would require some kind of supporting struts in order to remain stationary, otherwise it would be pulled towards this plane). Two further parameters are sufficient to describe it — its mass \( m \) and its linear orbital velocity \( v \) (we note that the total mass of the system will be denoted by \( M \)). The velocity should equal a local Keplerian orbital velocity, which for a linear perturbation of a Schwarzschild black-hole is given by (1.55) (namely, a correction due to self-gravity of the ring is of the second or higher order in \( \lambda \)). The ring mass density is described by

\[ \epsilon = \frac{m}{2 \pi \rho} \delta(r - r') \delta(z), \quad (2.72) \]
Figure 2.2: Linear perturbation of a Schwarzschild black-hole (having mass $M = 1$) by a ring of matter (located at $r = 3$). Metric functions are calculated using series truncated to 30 orders of the “multipole” expansion. Left: Perturbation of gravitational potential ($\nu_1$). Right: Perturbation of dragging ($\omega_1$).

so one can conclude that

$$\nu_1(x, \theta) \propto G_\nu \left(x, x', \theta, \frac{\pi}{2}\right), \quad \omega_1(x, \theta) \propto G_\omega \left(x, x', \theta, \frac{\pi}{2}\right)$$

(line in second equation we have already set $J_1 = 0$). The multiplicative constant can be fixed by combining (2.72), (1.55), (2.4), (2.11) and (2.12):

$$\nu_1(x, \theta) = \frac{2mM^2 x'(x' + 1)^2 \sqrt{(x')^2 - 1}}{(r')^{3/2}(x'^2 - 2)} G_\nu \left(x, x', \theta, \frac{\pi}{2}\right),$$

$$\omega_1(x, \theta) = -\frac{4m\sqrt{M r'}(2r' + M)(x' + 1)^3(x' - 1)^2}{(r')^4(x'^2 - 2)} G_\omega \left(x, x', \theta, \frac{\pi}{2}\right),$$

where $r'$ is now considered to be a function of $x'$, and $\lambda = 1$ (more precisely, $m\lambda$ being mass of the ring).

### 2.4.5 Convergence

Important aspect of every perturbation scheme is its convergence. Indeed, in our case the series (2.57) and (2.58) do not behave very well and their truncation may be an issue. This is apparent in figure 2.2 where $\nu_1$ and $\omega_1$ for the ring perturbation are calculated using the first 30 terms of the multipole expansion. As expected, convergence is the worst close to the radius of the source and, besides the very source (in the equatorial plane), it shows itself particularly at the axis. In tables 2.1 and 2.2 the convergence is illustrated in a numerical detail.

In order to estimate the convergence speed, we combine (2.65) and the asymptotic behaviour from section C.2.3 to obtain a lower bound on the convergence speed of (2.57),

$$|G_{\nu j}| = \frac{2j + 1}{2} |P_j(\cos \theta)P_j(x_<)Q_j(x_>)P_j(0)| \approx \frac{1}{j} \left(\frac{x_< + \sqrt{x_<^2 - 1}}{x_> + \sqrt{x_>^2 - 1}}\right)^{1+j} \left[f(x, x', \theta) + O\left(\frac{1}{j}\right)\right],$$
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Table 2.1: Convergence of the Green function for gravitational potential $G_\nu$. The situation corresponds to a ring located in the equatorial plane on radius $x' = 3$. $N$ denotes the number of terms in the truncated expansion.
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Table 2.2: Convergence of the Green function for the dragging angular velocity $G_\omega$. The situation corresponds to a ring located in the equatorial plane on radius $x' = 3$. $N$ denotes the number of terms in the truncated expansion.
where \( f(x, x', \theta) \) denotes some suitable function independent of \( j \). Except at the radius of the source \((x \neq x')\), the series converge very well (exponentially fast). The problem occurs at the radius of the source \((x = x' \Rightarrow x_\leq = x_\geq)\). The exponential convergence disappears (the base of the exponential part at (2.77) becomes 1) and only \(1/j\) remains, which is not sufficient to ensure convergence. A bit more precise (and lengthy) analysis shows that the actual divergence is only bound to the axis. Off the axis, the Abel–Dirichlet criterion reveals that the series (2.57) converges conditionally (due to alternating signs of its terms). This behaviour is known from spectral methods in general as already mentioned (see Grandclément and Novak [2009]).

In passing, one might think this behaviour is simply due to the singularity of the ring. It is not so. In the thin-disc case\(^2\) (i.e. with a distribution of matter described by regular \( R_k \))

\[
|\nu_j| < \frac{2j+1}{2} \left| \int_{x_{R-}}^{x_{R+}} R_{kj}(x_R)P_j(\min(x, x_R))Q_j(\max(x, x_R)) \times \right.
\]

\[
\left. \times P_j(\cos \theta)P_j(0)dx_R \right| < \frac{1}{j^2} \left( \frac{x'_\leq + \sqrt{(x'_\leq)^2 - 1}}{x'_\leq + \sqrt{(x'_\leq)^2 - 1}} \right)^{\frac{1}{2} + j} \left[ f(x, x_\leq, x_\geq, \theta) + O\left(\frac{1}{j}\right) \right], \tag{2.78}
\]

where \( R_{kj} \) are given by (2.32), \( x_{R\pm} \) denotes the inner/outer rim of the disc, and \( x'_\leq \) and \( x'_\geq \) are defined by

\[
x'_\leq = \begin{cases} x & \text{when } x \leq x_\geq, \\ x_{R+} & \text{otherwise} \end{cases}, \quad x'_\geq = \begin{cases} x_{R-} & \text{when } x \leq x_{R-}, \\ x & \text{otherwise} \end{cases} \tag{2.79}
\]

and \( f(x, x_\leq, x_\geq, \theta) \) denotes suitable \( j \)-independent function. Again, convergence is exponentially fast outside the radii of the disc. Within these radii the convergence is only like \( 1/j^2 \), i.e. a power-law behaviour like for the ring. In

\^2\text{Let us sketch the proof using } \nu_j \text{ assuming } x > x_{R+}. \text{ Other cases are analogous (up to possible split of integration interval into two). Asymptotic relation (C.16), (C.18) and (C.19) can be used to give bound to } |\nu_j| \text{ in the form}

\[
|\nu_j| \rightarrow \frac{1}{j\pi \sqrt{\sin \theta} \sqrt{x'^2 - 1}} \cos \left( \left( \frac{1}{2} + j \right) \theta \right) \cos \left( \left( j + \frac{1}{2} \right) \frac{\pi}{2} \right) \frac{1}{(x + \sqrt{x'^2 - 1})^{j+\frac{1}{2}}} \times \\
\left| \int_{x_{R-}}^{x_{R+}} \frac{1}{x_{R-}^2 - 1} \left( x_R + \sqrt{x_{R-}^2 - 1} \right)^{\frac{1}{2} + j} dx_R \right| < \\
\left| \frac{f'(x, x_\leq, x_\geq, \theta) \max(|R_{kj}|)}{j \left( x + \sqrt{x'^2 - 1} \right)^{j+\frac{1}{2}}} \right| \left( \frac{x_{R-}^2 + \sqrt{x_{R-}^2 - 1}}{x_{R-}^2 + \sqrt{x_{R-}^2 - 1}} \right) \frac{x_{R-} + \sqrt{x_{R-}^2 - 1}}{x_{R+} - \sqrt{x_{R+}^2 - 1}} \left( x_{R+} + \sqrt{x_{R+}^2 - 1} \right)^{\frac{1}{2} + j} \left| f(x, x_\leq, x_\geq, \theta) \right| < \\
\]

where \( t = x_R + \sqrt{x_{R-}^2 - 1} \), \( f(x, x_\leq, x_\geq, \theta) \) and \( f'(x, x_\leq, x_\geq, \theta) \) are suitable \( j \)-independent function and \( j \) is assumed to be sufficiently large.
Figure 2.3: Linear perturbation of a Schwarzschild black-hole (having mass \( M = 1 \)) by a ring of matter (located at \( r = 3 \)). \textit{Left}: Perturbation of gravitational potential \((\nu_1)\). \textit{Right}: Perturbation of dragging \((\omega_1)\).

Contrast to the ring perturbation, the (power-law) convergence is absolute here, but, on the other hand, it spans over a region with finite volume (compared to just a shell \( r = r' \) for ring).

A similar analysis can be performed for linear perturbation of dragging \( \omega_1 \). It is quite a lengthy technical exercise, with the same results — exponential convergence outside the radii of the source and a power-law one (like \( j^{-1} \) or \( j^{-2} \)) where the matter is present (in the equatorial plane).

The calculation can also be challenging from the numerical point of view. \( Q_j(x) \) and \( G_j(x) \), when expressed in terms of elementary functions, can be split into two terms, which are orders of magnitude larger than the final result and almost precisely cancel each other. This circumstance can be healed either by using many digits of precision or by a suitable representation of the problematic functions. Many rapidly converging recipes are known for Legendre functions of the second kind and (2.53) helps with the evaluation of \( G_j(x) \).

### 2.4.6 Closed form of Green functions

Surprisingly, (2.57) and (2.57) can be expressed in a closed form. The main arguments and proofs are given in appendix C (C.104). (Although crucial for our results, this technical part would not look "organic" if included in this chapter.) As an advertisement, we present here a counter-part of figure 2.2 computed using the closed-form Green functions — see figure 2.3. Improvement of convergence is clearly seen in smoothness of the contours. Formulas used to produce these plots are given in the rest of this section.
Green function for gravitational potential

Green function for gravitational potential \( \mathcal{G}_\nu \), as given by (2.57) and (2.65), already has the form (C.104), so

\[
\mathcal{G}_\nu = -\frac{1}{2} \sum_{j=0}^\infty (2j+1) P_j(x_<)Q_j(x_>)P_j(\cos \theta)P_j(\cos \theta') = \text{(2.80)}
\]

\[
= -\int_{\cos(\theta-\theta')} \frac{d\xi}{\pi \sqrt{x^2 + (x')^2 + \xi^2 - 1 - 2\xi xx' \sqrt{\cos(\theta - \theta') - \xi} \left[\cos(\theta + \theta') - \xi\right]}} \text{,}
\]

where the hypergeometric function is expressed in terms of elementary functions and simplification was done using assumption \( x, x' > 1 \). The notation of \( x_< \) is in accordance with (2.67) and in the integral it has been dropped due to its symmetry (with respect to the exchange of \( x_< \)). Further calculation reveals

\[
\mathcal{G}_\nu = -\frac{K(k)}{\pi \sqrt{xx' - \cos \theta \cos \theta'}^2 - \left(\sqrt{(x^2 - 1)((x')^2 - 1)} - \sin \theta \sin \theta'\right)^2}, \quad \text{(2.81)}
\]

where

\[
k^2 = \frac{4\sqrt{(x^2 - 1)((x')^2 - 1)} \sin \theta \sin \theta'}{(xx' - \cos \theta \cos \theta')^2 - \left(\sqrt{(x^2 - 1)((x')^2 - 1)} - \sin \theta \sin \theta'\right)^2}. \quad \text{(2.82)}
\]

These expressions simplify considerably when \( B = 1 \). Let us denote the corresponding coordinates, respectively spherical and cylindrical, by \((t, \phi, \tilde{r}, \tilde{\theta})\) and \((t, \phi, \tilde{\rho}, \tilde{z})\). In accordance with (1.30), the transformation reads

\[
\tilde{\rho} = \rho \left[1 - \frac{M^2}{4(\rho^2 + z^2)}\right], \quad \tilde{z} = z \left[1 + \frac{M^2}{4(\rho^2 + z^2)}\right], \quad \text{(2.83)}
\]

and the Green function (2.81) is given by

\[
\mathcal{G}_\nu = -\frac{MK(k)}{\pi \sqrt{\rho^2 + (\tilde{\rho})^2 + \left(\tilde{z} - \tilde{z}'\right)^2}}, \quad \text{(2.84)}
\]

Analogously, (2.82) takes the form

\[
k^2 = \frac{4\tilde{\rho}\tilde{\rho}'}{(\tilde{\rho} + (\tilde{\rho})^2 + \left(\tilde{z} - \tilde{z}'\right)^2)}. \quad \text{(2.85)}
\]

In fact, such a simple form is not surprising. The result is actually well known from the Newtonian theory. Given \( B = 1 \) and static background \((\omega_0 = 0)\), equation (2.6) becomes a Laplace equation. Its Green function for point charge is well known \((1/(4\pi |\vec{r} - \vec{r}'|))\). Therefore, the Green function adapted to axially symmetric source (i.e. ring) reads

\[
\begin{align*}
\mathcal{G}'_\nu &= -\frac{1}{4\pi} \int_{\text{ring}} \frac{|d\vec{r}'|}{|\vec{r} - \vec{r}'|} = -\frac{1}{4\pi} \int \frac{\tilde{\rho}'}{\sqrt{\tilde{\rho}^2 + (\tilde{\rho}')^2 + (\tilde{z} - \tilde{z}')^2 - 2\tilde{\rho}\tilde{\rho}' \cos \phi}} \, d\phi =
\end{align*}
\]

Assuming \( \cos(\theta - \theta') \geq \cos(\theta + \theta') \). Otherwise integration bounds have to be interchanged.

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\[
\rho' - \frac{\rho'}{\pi \sqrt{(\rho + \rho')^2 + (z - z')^2}} K(k) = - \frac{\rho'}{\pi \sqrt{(\rho + \rho')^2 + (z - z')^2}} K(k). \tag{2.86}
\]

It follows that
\[
G_{\nu} = \frac{M}{\rho'} G_{\nu}' = - \frac{M}{\rho'} \int_{\text{ring}} \frac{|d\rho'|}{|r' - r'|}. \tag{2.87}
\]

The difference in normalization is due to the slightly different normalization factor at the delta functions on the r.h. side of the field equation.

**Green function for dragging**

At first glance, no such simple way exists in the case of dragging, since the corresponding fundamental system is given in terms of the Gauss hypergeometric functions. Curiously enough, they can be expressed using the Gegenbauer function according to

\[
F_k(x) = \frac{12(-1)^k}{k(k+1)^2(k+2)^2(k+3)} \hat{O}_x C_k^{(3/2)}(x), \tag{2.88}
\]

\[
G_k(x) = \frac{(-1)^k}{12} \hat{O}_x D_k^{(3/2)}(x), \tag{2.89}
\]

where

\[
\hat{O}_x \equiv \frac{d}{dx} \left( x - 1 \right) ^2 = \frac{d}{dx} \left( x - 1 \right) ^2 = \frac{d}{dx} \left( x - 1 \right) ^2 \tag{2.90}
\]

and \( k \geq 1 \) for the purpose of (2.88) \((k \in \mathbb{Z}^+ \text{ in the case of (2.89)})\). The formula (2.88) can be proven by a straightforward calculation. The formula (2.89) is a bit more tricky. First, it solves the homogenous equation (2.31). It is also clear that it vanishes at infinity. Therefore, (2.89) has to be proportional to the \( G_k(x) \) given by (2.48) (the second solution \( F_k(x) \) does not vanish asymptotically). Finally, the multiplicative constant can be fixed by comparing the leading terms at \( x \to \infty \).

Combining this knowledge with (2.58) and (2.66) brings the desired alternative formulation of the Green function for dragging,

\[
G_{\omega} = - \sum_{j=0}^{\infty} \frac{2j + 3}{2(j + 1)(j + 2)} \sin^2 \theta' F_j(x_<) G_j(x_>) C_j^{(3/2)}(\cos \theta) C_j^{(3/2)}(\cos \theta') =
\]

\[
= \Delta - \hat{O}_{x_>} \frac{d}{dx_<} (x_< - 1)^2 \sum_{j=1}^{\infty} \frac{j + \frac{3}{2}}{j(j + 1)^3(j + 2)^3(j + 3)} D_j^{(3/2)}(x_>) \times \]

\[
\times \frac{d}{dx_<} C_j^{(3/2)}(x_<) C_j^{(3/2)}(\cos \theta) C_j^{(3/2)}(\cos \theta') =
\]

\[
= \tilde{\Delta} - \hat{O}_{x_>} \frac{d}{dx_<} (x_< + 1)^{-2} \sum_{j=0}^{\infty} \frac{j + \frac{3}{2}}{j(j + 1)^3(j + 2)^3} \times \]

\[
\times \int_1^{x_<} (\zeta^2 - 1) D_j^{(3/2)}(x_>) C_j^{(3/2)}(\zeta) C_j^{(3/2)}(\cos \theta) C_j^{(3/2)}(\cos \theta') d\zeta =
\]

\[
= \tilde{\Delta} - \hat{O}_{x_>} \frac{d}{dx_<} \int_1^{x_<} \frac{\zeta^2 - 1}{2(x_< + 1)^2} \times
\]

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\[
\times \sum_{j=0}^{\infty} \frac{2j+3}{(j+1)^3(j+2)^3} D_j^{(3/2)}(x_>) C_j^{(3/2)}(\zeta) C_j^{(3/2)}(\cos \theta) C_j^{(3/2)}(\cos \theta') d\zeta =
\]
\[
= \tilde{\Delta} - \hat{O}_{x>} \frac{d}{dx_<} \int_{1}^{x_<} \frac{\zeta^2 - 1}{2\pi(x_< + 1)^2(x_< - 1) \sin^2 \theta \sin^2 \theta'} \times
\]
\[
\times \int_{\cos(\theta - \theta')}^{\cos(\theta + \theta')} \sqrt{\cos(\theta - \theta') - \xi}[\xi - \cos(\theta + \theta')] d\xi d\zeta,
\]
(2.91)

where we have used the relations \(\text{(C.54)}\) and \(\text{(C.54)}\),

\[
\Delta \equiv -\frac{3}{4} F_0(x_<) G_0(x_<) C_0^{(3/2)}(\cos \theta) C_0^{(3/2)}(\cos \theta') = -\frac{1}{4(x_< + 1)^3},
\]
(2.92)

\[
\tilde{\Delta} \equiv \Delta + \hat{O}_{x>} \frac{d}{dx_<} \frac{3}{16(x_< + 1)^2} \int_{1}^{x_<} (\zeta^2 - 1) D_0^{(3/2)}(x_<) \times
\]
\[
\times C_0^{(3/2)}(\zeta) C_0^{(3/2)}(\cos \theta) C_0^{(3/2)}(\cos \theta') d\zeta = -\frac{2}{(x + 1)^3(x' + 1)^3},
\]
(2.93)

and it has been assumed that \(\cos(\theta - \theta') \geq \cos(\theta + \theta')\). If the latter is not true, these two terms has to be interchanged. (Because of the symmetry of the final result \(\text{(2.96)}\), we do not repeat the proof for the second case.) Note also that symmetry with respect to the exchange \(x \leftrightarrow x'\) allowed us to drop the \(x_<\) variables as in the case of \(G_<\).

Above, we deliberately exchanged the order of integration, derivation and summation. When dealing with infinite series, this could be an issue, but luckily, thanks to the fast enough convergence, one can do so. Actually, the convergence is exponential almost everywhere — up to an arbitrarily small neighbourhood of the radius \(r = r'\) and Foubini’s theorem allows us to do so (up to an infinitely small vicinity of the mentioned radius).

Let us remind that an example of the Green function for dragging has been given in figure 2.3.

When treating more than one-dimensional sources, the Green functions have to be integrated over their volume. The first integration of \(\tilde{\Delta}\) brings logarithms or elliptic functions. Both of them represent quite a serious problem for a subsequent quadrature. Applying the operator \(\hat{O}_{x>}\) first and integrating over \(\zeta\) later avoids such difficulties, because

\[
\frac{d}{dx_<} \int_{1}^{x_<} \hat{O}_{x>} \frac{(\zeta^2 - 1)}{2\pi(x_< + 1)^2(x_< - 1) \sin^2 \theta \sin^2 \theta'} \times
\]
\[
\times \frac{d\zeta}{x_< - \zeta \xi + \sqrt{(x_< - \zeta)\xi^2 - (1 - \zeta^2)(1 - \xi^2)}} = \sum_{k=0}^{5} \mathcal{P}_k(x', \theta') \tilde{I}_k(x', \xi),
\]
(2.94)

where the functions \(\mathcal{P}_k\) and \(\tilde{I}_k\) are defined by

\[
\mathcal{P}_0(x', \theta') = \frac{-2}{\pi(x + 1)^3(x' + 1)^3 \sin^2 \theta \sin^2 \theta'},
\]

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Finally, it can be concluded that

\[
\begin{align*}
\mathcal{P}_1(x', \theta') & \equiv \frac{-(x - 1)(x' - 1)}{\pi (x + 1)^3 (x' + 1)^3 \sin^2 \theta \sin^2 \theta'}, \\
\mathcal{P}_2(x', \theta') & \equiv \frac{2xx' + (x - 1)(x' - 1)}{\pi (x + 1)^3 (x' + 1)^3 \sin^2 \theta \sin^2 \theta'}, \\
\mathcal{P}_3(x', \theta') & \equiv \frac{-2}{\pi (x + 1)^3 (x' + 1)^3 \sin^2 \theta \sin^2 \theta'}, \\
\mathcal{P}_4(x', \theta') & \equiv \frac{(x' - 1)(x - 1)(x' + x)}{\pi (x + 1)^3 (x' + 1)^3 \sin^2 \theta \sin^2 \theta'}, \\
\mathcal{P}_5(x', \theta') & \equiv \frac{2\pi (x + 1)(x' + 1) \sin^2 \theta \sin^2 \theta'}{(x - 1)(x' - 1)}.
\end{align*}
\]

\[
I_0(x', \xi) = 1, \\
I_1(x', \xi) = \frac{1}{\xi + 1}, \\
I_2(x', \xi) = \frac{1}{\sqrt{x^2 + (x')^2 + \xi^2 - 1 - 2xx'\xi}}, \\
I_3(x', \xi) = \frac{\xi}{\sqrt{x^2 + (x')^2 + \xi^2 - 1 - 2xx'\xi}}, \\
I_4(x', \xi) = \frac{1}{(\xi + 1)\sqrt{x^2 + (x')^2 + \xi^2 - 1 - 2xx'\xi}}, \\
I_5(x', \xi) = \frac{1}{(x^2 + (x')^2 + \xi^2 - 1 - 2xx'\xi)^{3/2}}.
\]

Finally, it can be concluded that

\[
\mathcal{G}^\omega(x, \theta, x', \theta') = \tilde{\Delta}(x, x') - \int_{\cos(\theta + \theta')}^{\cos(\theta - \theta')} \sqrt{[\cos(\theta - \theta') - \xi][\xi - \cos(\theta + \theta')] \times (2.96)} \\
\times \sum_{k=0}^{5} \mathcal{P}_k(x', \theta') I_k(x', \xi) d\xi = \Delta(x, x') - \sum_{k=0}^{5} \mathcal{P}_k(x', \theta') I_k(x', \theta'),
\]

where

\[
\begin{align*}
I_0(x', \theta') & = \frac{\pi}{2} \sin^2 \theta \sin^2 \theta', \\
I_1(x', \theta') & = \pi (1 - |\cos \theta|)(1 - |\cos \theta'|), \\
I_2(x', \theta') & = \frac{a_{31}}{a_{42}} \left[ -a_{44} K (k) - a_{42} E (k) + a_{41} \left( 1 + \frac{a_{42}}{a_{31}} \right) \Pi \left( -\frac{a_{43}}{a_{31}}, k \right) \right], \\
I_3(x', \theta') & = \frac{a_{41}}{4} \sqrt{\frac{a_{31}}{a_{42}}} \left[ -s_{21} - s_{43} + 2a_{21} K (k) + \frac{a_{42}}{a_{41}} (s_{43} - 3s_{21}) E (k) + a_{21} \frac{a_{42}}{a_{41}} (s_{43} - 3s_{21}) \Pi \left( -\frac{a_{43}}{a_{31}}, k \right) \right], \\
I_4(x', \theta') & = \frac{2a_{41}}{\sqrt{a_{31}a_{42}}} \left[ -\frac{a_{31}}{a_{1} + 1} K (k) + \Pi \left( \frac{a_{43}}{a_{31}}, k \right) - \frac{a_{3} + 1}{a_{1} + 1} \Pi \left( \frac{a_{4} + 1}{a_{1} + 1}, a_{31} \right) \right],
\end{align*}
\]
\[ I_5(x', \theta') = \frac{\sqrt{a_{31}a_{42}}}{a_{21}^2} \left[ (4 - 2k^2)K(k) - 4E(k) \right], \tag{2.97} \]

with
\[
\begin{align*}
  a_1 & \equiv xx' + \sqrt{(x^2 - 1)((x')^2 - 1)}, \\
  a_2 & \equiv xx' - \sqrt{(x^2 - 1)((x')^2 - 1)}, \\
  a_3 & \equiv \cos(\theta - \theta'), \\
  a_4 & \equiv \cos(\theta + \theta') \tag{2.98}
\end{align*}
\]

and
\[
\begin{align*}
  a_{rs} & \equiv a_s - a_r, \\
  s_{rs} & \equiv a_s + a_r, \\
  k & \equiv \sqrt{\frac{a_{21}a_{43}}{a_{31}a_{42}}}. \tag{2.99}
\end{align*}
\]

Note that there is no ambiguity in the notation: the moduli defined by the above equation and by \((2.82)\) coincide.

**Normal derivatives of metric function on the horizon**

Let us return to already proposed vanishing normal derivative of the dragging at the horizon. Given knowledge of Green function we can, ex post, justify such statement.

Assuming \(f(x)\) being general function of dimensionless radius \((2.27)\) its normal derivative at the horizon reads
\[
\left. f_x, r \right|_{r=M^2} = \left. f_x \left( \frac{x}{r^2} - \frac{M}{4r^2} \right) \right|_{r=M^2} = 0. \tag{2.100}
\]

Therefore long as \(f_x, r\) does not diverge at the horizon definition of \(x\) ensures vanishing normal derivative there. GF near the horizon are described by polynomial for each multipole moment. Their derivatives are clearly finite — they are polynomials of \(x\) as well. So GF has to have finite derivative (with respect to \(x\)) at the horizon and subsequently the \(x\) derivatives of perturbations of \(\nu\) and \(\omega\) must be finite there.

In the case of dragging additional the term proportional to \(J_k\) has not been examined yet, see \((2.70)\). Luckily is does not spoil anything (its \(r\)-derivative vanishes at the horizon as well).

### 2.5 Disc solution

The Green functions can be used to calculate a general solution of \((2.9)\) and \((2.10)\). The integrals contain non-trivial combination of elliptic functions, so they are in general not easily calculated, but there exists a special case when the metric functions can be expressed using complete elliptic integrals and elementary functions — the case of a thin disc with constant Newtonian density.

#### 2.5.1 Gravitational potential

There is no evident way how to construct gravitational potential of a thin disc using Green function \((2.81)\). However, one can proceed in analogy with the Newtonian case (see \((2.86)\)). The Newtonian gravitational potential of a disc with constant surface mass density was found by Lass and Blitzer [1983]. Using
their notation (i.e. $V(a)$ denoting gravitational potential of a disc lying in the equatorial plane and extending between origin and the radius $x = a$), it reads

$$V(a) = \pi R \|z_0 H(\tilde{r}_a - \tilde{\rho}) - \frac{\mathcal{R}}{\sqrt{\tilde{z}^2 + (\tilde{\rho} + \tilde{r}_a)^2}} \left\{ \left[ \tilde{z}^2 + (\tilde{\rho} + \tilde{r}_a)^2 \right] E(k(a)) + \right.$$  

$$+ \left( \tilde{r}_a - \tilde{\rho} \right) K(k(a)) + \tilde{z}^2 \frac{\tilde{r}_a - \tilde{\rho}}{\tilde{r}_a + \tilde{\rho}} \Pi \left( \frac{4\tilde{r}_a \tilde{\rho}}{(\tilde{r}_a + \tilde{\rho})^2}, k(a) \right) \right\} = \pi M R x | \cos \theta | H(a^2 - \cos^2 \theta - x^2 \sin^2 \theta) -$$  

$$- \frac{M \mathcal{R}}{\sqrt{a_3(a) a_4(a)}} \left[ a_3(a) a_4(a) E(k(a)) + (a^2 - \cos^2 \theta - x^2 \sin^2 \theta) K(k(a)) + \right.$$  

$$+ \frac{(a_{41} a_{32} - x^2 \cos^2 \theta) x^2 \sin^2 \theta}{a^2 - \cos^2 \theta - x^2 \sin^2 \theta} \Pi \left( \frac{4\sqrt{(a^2 - 1)(x^2 - 1) \sin \theta}}{a_{42} a_{31} - x^2 \cos^2 \theta}, k(a) \right) \right] , \quad (2.101)$$

where $\tilde{r}_a = M \sqrt{a^2 - 1}$ denotes the radius of the ring in coordinates $\tilde{\rho}$ and $\tilde{z}$ which correspond to $B = 1$ (see (2.83) for transformation), $\mathcal{R}$ is an arbitrary parameter scaling the mass density (it is proportional to the surface mass density in the Newtonian case) and $H(x)$ stands for the Heaviside function. Further, the symbols $a_r(x')$, $a_{rs}(x')$ and $k(x')$ are defined analogously to (2.98) and (2.99) (with $\theta' = \pi/2$),

$$a_1(x') = xx' + \sqrt{(x^2 - 1)((x')^2 - 1)}, \quad a_2(x') = xx' - \sqrt{(x^2 - 1)((x')^2 - 1)},$$  

$$a_3(x') = \sin \theta, \quad a_4(x') = -\sin \theta, \quad a_{rs}(x') = a_s(x') - a_r(x'),$$  

$$k^2(x') = \frac{a_{31}(x')}{a_{41}(x')} = \frac{4 \sin \theta \sqrt{(x^2 - 1)((x')^2 - 1)}}{x^2 + (x')^2 - 1 - \sin^2 \theta + 2 \sin \theta \sqrt{(x^2 - 1)((x')^2 - 1)}} =$$  

$$= \frac{4\tilde{r}_a \tilde{\rho}}{(\tilde{r}_a + \tilde{\rho})^2 + \tilde{z}^2}. \quad (2.102)$$

Note that due to the existence of the inner and outer rims of the disc, it is necessary to distinguish them. Because of that and similarity of the expressions, it is useful to indicate the dependence on the disc radius $x'$ explicitly and to substitute there the specific values later accordingly.

Constancy of the surface mass density (in the Newtonian case) makes it easy to get the potential of a disc with inner rim as well — it is sufficient to subtract the potential of two discs extending from the origin with the same surface mass density ($\mathcal{R}$ parameter),

$$\nu_1(x, \theta) = V(x_{R^+}) - V(x_{R^-}), \quad (2.103)$$

where $x_{R^\pm}$ once again denote the radii of the outer/inner rims of the disc.

### 2.5.2 Dragging

No such simple Newtonian analogy exists for dragging (of course), so one has to directly integrate the appropriate Green function over the source. Assuming $S_1 = S = const$ (this assumption will be clarified later), it reads

$$\omega_1 = \int_{x_{R^-}}^{x_{R^+}} S G(x, \theta, x', \pi/2) \, dx' \quad (2.104)$$
The main point leading to evaluation of the integral above is to closely examine the integral form of the Green function (2.91), to take into account that the Green function is symmetric with respect to exchange $x \leftrightarrow x'$ (see (2.96) and related equations), and to realize that the last expression in (2.91) does even make sense if $x_+ < x_-$ (as long as $x_{<-} > 1$, i.e. we are not interested in metric below the black-hole horizon). Altogether, this allows to exchange $x_{<}$ by $x, x'$ without changing the result (in other words, the symmetry ensures that both substitutions $(x_-, x_+) \to (x, x')$ and $(x_-, x_+) \to (x', x)$ lead to the same result), which leads to the following:

$$
\omega_1 = S \int_{x_{R-}}^{x_{R+}} \left\{ \Delta - \frac{d}{dx'} \hat{O}_x \right\} \frac{\zeta^2 - 1}{2\pi(x' + 1)^2(x^2 - 1) \sin^2 \theta} \times \\
\times \int_{-\sin \theta}^{\sin \theta} \sqrt{\sin^2 \theta - \xi^2} \left[ x - \zeta \xi + \sqrt{(x - \zeta \xi)^2 - (1 - \xi^2)(1 - \zeta^2)(1 - \xi^2)} \right] d\xi d\zeta \\
= \int_{x_{R-}}^{x_{R+}} S \hat{\Delta} dx' - S \hat{O}_x \int_{x_{R-}}^{x_{R+}} \frac{\zeta^2 - 1}{2\pi(x' + 1)^2(x^2 - 1) \sin^2 \theta} \times \\
\times \int_{-\sin \theta}^{\sin \theta} \sqrt{\sin^2 \theta - \xi^2} \sum_{j=0}^{7} \left[ Q_j(x_{R+}) I_j(x_{R+}, \xi) - Q_j(x_{R-}) I_j(x_{R-}, \xi) \right] d\xi = \\
= S \frac{(x_{R-} - x_{R+})(x_{R-} + x_{R+} + 2)}{(x + 1)^3(x_{R-} + 1)^2(x_{R+} + 1)^2} \\
- S \sum_{j=0}^{7} \left[ Q_j(x_{R+}) I_j(x_{R+}) - Q_j(x_{R-}) I_j(x_{R-}) \right],
$$

(2.105)

where

$$
Q_0(x') = \frac{1}{\pi \sin^2 \theta (x + 1)^3(x' + 1)^2}, \quad Q_1(x') = \frac{(1 - x)(x' - 1)^2}{4\pi \sin^2 \theta (x + 1)^3(x' + 1)^2},
$$

$$
Q_2(x') = \frac{x'(1 - 2x)}{\pi \sin^2 \theta (x + 1)^3(x' + 1)^2}, \quad Q_3(x') = \frac{1}{\pi \sin^2 \theta (x + 1)^3(x' + 1)^2},
$$

$$
Q_4(x') = \frac{(x + x')(x - 1)(x' - 1)^2}{4\pi \sin^2 \theta (x + 1)^3(x' + 1)^2}, \quad Q_5(x') = 0,
$$

$$
Q_6(x') = \frac{(1 - x)}{4\pi \sin^2 \theta (x + 1)^3}, \quad Q_7(x') = \frac{(1 - x)(x' - x)}{4\pi \sin^2 \theta (x + 1)^3},
$$

(2.106)

$I_0, \ldots, I_5$ are defined by (2.95),

$$
I_6(x', \xi) = \frac{1}{\xi - 1},
$$

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\[
\dot{I}_7(x', \xi) = \frac{1}{(\xi - 1)\sqrt{x^2 + (x')^2 + \xi^2 - 1 - 2xx'\xi}},
\]

and \( I_k(x') \) correspond to (2.97) with \( \theta' = \pi/2 \), i.e.

\[
\begin{align*}
I_0(x') &= \frac{\pi}{2} \sin^2 \theta, \\
I_1(x') &= \pi(1 - |\cos \theta|), \\
I_2(x') &= \sqrt{\frac{a_{31}(x')}{a_{42}(x')}} \left\{ -a_{41}(x')K(k(x')) - a_{42}(x')E(k(x')) + 2x' \frac{a_{41}(x')}{a_{31}(x')} \Pi \left( -\frac{2 \sin \theta}{a_{31}(x')}, k(x') \right) \right\}, \\
I_3(x') &= \frac{a_{41}(x')}{4} \sqrt{\frac{a_{31}(x')}{a_{42}(x')}} \left\{ [a_2(x') - 3a_1(x')] K(k(x')) - 6x' \frac{a_{42}(x')}{{a_{41}(x')}} E(k(x')) + \frac{a_2}{a_{31}(x')} - a_2^2(x') + 8x^2(x')^2 \Pi \left( -\frac{2 \sin \theta}{a_{31}(x')}, k(x') \right) \right\}, \\
I_4(x') &= \frac{2a_{41}(x')}{\sqrt{a_{31}(x') a_{42}(x')}} \left\{ \frac{-a_{31}(x')}{a_{1}(x') + 1} K(k(x')) + \Pi \left( -\frac{a_{43}(x')}{a_{31}(x')}, k(x') \right) - \frac{a_3(x') + 1}{a_1(x') + 1} \Pi \left( \frac{(a_1(x') + 1)a_{43}(x')}{(a_4(x') + 1)a_{31}(x')}, k(x') \right) \right\}, \\
I_5(x') &= \frac{\sqrt{a_{31}(x') a_{42}(x')}}{4(x^2 - 1)((x')^2 - 1)} \left\{ [4 - 2k^2(x')] K(k(x')) - 4E(k(x')) \right\}, \\
I_6(x') &= \pi \left( |\cos \theta| - 1 \right), \\
I_7(x') &= \frac{2a_{41}(x')}{\sqrt{a_{31}(x') a_{42}(x')}} \left\{ \frac{-a_{31}(x')}{a_{1}(x') - 1} K(k(x')) + \Pi \left( -\frac{2 \sin \theta}{a_{31}(x')}, k(x') \right) + \frac{1 - \sin \theta}{a_1(x') - 1} \Pi \left( \frac{2 \sin \theta [a_1(x') - 1]}{a_{31}(x') (\sin \theta + 1)}, k(x') \right) \right\}.
\end{align*}
\]

2.5.3 Physical properties

Physical properties of the obtained disc are discussed in more detail in paper [Čížek and Semerák 2017]. Using the normal one-sided derivatives of the perturbation (2.103) and (2.105) at the equatorial plane, one has

\[
\begin{align*}
\nu_{1,z=0^+} &= 2\pi R \left( 1 + \frac{M^2}{4r^2} \right) = \frac{2\pi RM}{r} \left( 1 + \frac{v_0^2}{v_0^2} \right), \\
\omega_{1,z=0^+} &= -\frac{S}{2r} \frac{x - 1}{(x + 1)^3} = -8SM^2r \left( 2r - M \right)^2 \left( 2r + M \right)^6
\end{align*}
\]

on the surface of the disc (elsewhere normal derivatives vanish due to the reflection symmetry). Unperturbed orbital velocity \( v_0 \) is given by (1.55). Let us stress that all the relations given below are only valid on the surface of the disc, otherwise mass densities are zero.

Considering a two-component dust disc, i.e. energy-momentum tensor in the form (1.15), normal derivatives in the equatorial plane are linked to the dust
densities by relations (1.40)

\[ \nu_{1,z}|_{z=0^+} = 2\pi(\sigma_+ + \sigma_-) \frac{1 + v_0^2}{1 - v_0^2} = (\sigma_+ + \sigma_-) \frac{4r^2 + M^2}{4r^2 - 8rM + M^2}, \]  
(2.111)

\[ \omega_{1,z}|_{z=0^+} = -8\pi(\sigma_+ - \sigma_-) \frac{\Omega_0}{1 - v_0^2} = -8\pi(\sigma_+ - \sigma_-) \frac{8r\sqrt{Mr}}{4r^2 - 8Mr + M^2} \frac{(2r - M)^2}{(2r + M)^3}, \]  
(2.112)

where \( \Omega_0 \) denotes the angular velocity corresponding to \( v_0 \). Combining these two relations yields the surface mass densities

\[ \sigma_{\pm} = \frac{4r^2 - 8Mr + M^2}{8r^2} \left[ R \pm S \frac{(Mr)^{\frac{3}{2}}}{2\pi(2r + M)^{\frac{3}{2}}} \right]. \]  
(2.113)

For the single-component interpretation, the parameters are obtained using (1.17),

\[ \sigma = + \frac{\sigma_+ + \sigma_-}{2} + \sqrt{\left( \frac{\sigma_+ + \sigma_-}{2} \right)^2 + 4\sigma_+\sigma_- v_0^2 \frac{1}{(1 - v_0^2)^2}}, \]  
(2.114)

\[ P = - \frac{\sigma_+ + \sigma_-}{2} + \sqrt{\left( \frac{\sigma_+ + \sigma_-}{2} \right)^2 + 4\sigma_+\sigma_- v_0^2 \frac{1}{(1 - v_0^2)^2}}, \]  
(2.115)

\[ v^2 = \frac{\sigma v_0^2 - P}{\sigma - P v_0^2}. \]  
(2.116)

The sign of \( v \) depends on ratio of the surface mass-densities \( \sigma_{\pm} \): for \( \sigma_+ > \sigma_- \) it is positive, otherwise negative.

An example of the disc parameters is given in figure 2.4. The rim(s) are not present there explicitly, because introducing them would only restrict the domain of the functions accordingly. Also, only the part where the circular orbital velocity makes sense \( (0 \leq v \leq 1) \) is plotted. Note that for the lower-radius-case the two-component interpretation has no sense since the obtained mass density is negative for one of the streams (whereas the single-component interpretation is still possible).

Total mass and angular momentum can be obtained using the asymptotic behaviour of the metric or by Komar’s integrals. Both results coincide so only the latter option is discussed here. The total mass and angular momentum are given by

\[ M = \frac{1}{8\pi} \lim_{\Sigma \to \infty} \int_{\Sigma} \eta^{\alpha\beta} dS_{\alpha\beta} = \]  
(2.117)

\[ = \frac{1}{8\pi} \int_{\text{horizon}} \eta^{\alpha\beta} dS_{\alpha\beta} + \int_{\text{disc}} (2T^{\alpha\beta} \eta^\beta - T\eta^\alpha) d\Sigma = M_H + M_D, \]

\[ J = -\frac{1}{16} \lim_{\Sigma \to \infty} \int_{\Sigma} \xi^{\alpha\beta} dS_{\alpha\beta} = \]  
(2.118)

\[ = -\frac{1}{16} \int_{\text{horizon}} \xi^{\alpha\beta} d\Sigma_{\alpha\beta} + \frac{1}{2} \int_{\text{disc}} (2T^{\alpha\beta} \xi^\beta - T\xi^\alpha) d\Sigma = J_H + J_D, \]
Figure 2.4: Example of parameters of the disc obtained using the closed-form formulas. The configuration is described by $R = 0.1$, $S = 60$ and $M = 1$. Left charts describe parameters of the single-component interpretation, right ones correspond to two-component interpretation (solid lines indicate the positive-direction stream). *Top Left:* Two-stream surface mass densities. *Bottom Left:* Two-stream orbital velocities. *Top Right:* One-stream surface mass density (solid line) and azimuthal pressure (dashed line). *Bottom Right:* Bulk velocity of the matter.
where $S$ is a sphere tending to spatial infinity and $\mathcal{M}_H$, $\mathcal{M}_D$, $\mathcal{J}_H$ and $\mathcal{J}_D$ denote contributions of the black hole and of the disc. Generally, each of them is a function of the expansion parameter $\lambda$ as well. Let us also remind that $\eta^\alpha$ and $\xi^\alpha$ stand for the Killing vector fields.

First, since we are interested in a thin disc, its contribution is better expressed using the surface energy-momentum tensor (1.9) instead of $T^\alpha_{\beta}$,

$$
\mathcal{M}_D = 2\pi \int_{\text{disc}} (S^\phi_\phi - S^t_t) \rho B \rho d\rho = 2\pi \int_{\text{disc}} (\sigma + P + 2\Omega S^t_\phi) \rho B \rho d\rho, \quad (2.119)
$$

$$
\mathcal{J}_D = 2\pi \int_{\text{disc}} S^t_\phi \rho B \rho d\rho = 2\pi \int_{\text{disc}} (\sigma + P) \frac{v}{1 - v^2} \rho B \rho d\rho. \quad (2.120)
$$

Considering the relation between the surface mass density and normal derivative across the disc (1.40), we have

$$
\mathcal{M}_D = \int_{\text{disc}} \nu_z|_{z=0+} \rho B d\rho + O \left( \frac{1}{\lambda^2} \right), \quad (2.121)
$$

$$
\mathcal{J}_D = \int_{\text{disc}} \rho^3 B^3 e^{-4\nu} \omega_z|_{z=0+} d\rho + O \left( \frac{1}{\lambda^2} \right), \quad (2.122)
$$

where staticity of the background solution has been used. Finally, employing the knowledge of the linear perturbation of metric, it can be concluded that

$$
\mathcal{M}_D = \int_{\text{disc}} \nu_z|_{z=0+} \rho B d\rho + O \left( \lambda^2 \right), \quad (2.123)
$$

$$
\mathcal{J}_D = \int_{\text{disc}} \rho^3 B^3 e^{-4\nu} \omega_z|_{z=0+} d\rho + O \left( \lambda^2 \right). \quad (2.124)
$$

Hence, the parameters of the disc, assuming linear order of precision, read

$$
\mathcal{M}_{D1} = 2\pi R \int_{r_{R-}}^{r_{R+}} \left( 1 + \frac{M^2}{4r^2} \right) \left( 1 - \frac{M^2}{4r^2} \right) r dr = \pi R \lambda \left[ r_{R+}^2 - r_{R-}^2 - \frac{M^4}{16} \left( \frac{1}{r_{R-}^2} - \frac{1}{r_{R+}^2} \right) \right], \quad (2.125)
$$

$$
\mathcal{J}_{D1} = \frac{SM^2}{8} \int_{r_{R-}}^{r_{R+}} \left( 1 - \frac{M^2}{4r^2} \right) dr = \frac{S M^2}{8} \left[ r_{R+} - r_{R-} - \frac{M^2}{4} \left( \frac{1}{r_{R-}} - \frac{1}{r_{R+}} \right) \right], \quad (2.126)
$$

where $r_{R\pm}$ denotes inner/outer radius of the disc.

It is easy to show that the black-hole contributions are described by

$$
\mathcal{M}_H = M + O \left( \lambda^2 \right), \quad (2.128)
$$

$$
\mathcal{J}_H = O \left( \lambda^2 \right). \quad (2.129)
$$

(see the paper Čížek and Semerák [2017] for details).
3. Belinskii–Zakharov method

3.1 Unknown generating function

In order to use BZM technique to generate solution of Einstein field equations one has to solve corresponding system of linear differential equations (1.74). Paper by de Castro and Letelier [2011] is going to be followed in this chapter. Assuming staticity of seed metric (1.8) the 2-metric $g_{0}$ is diagonal. The same behaviour can be expected by generating matrix $\Psi_0$. Its non-zero components are expressed by

$$(\Psi_0)_{\phi\phi} = (\rho^2 - \lambda^2 - 2\lambda z)e^{-2F},$$  
$$(\Psi_0)_{tt} = -e^{2F},$$

where $F(\rho, z, \lambda)$ denotes function satisfying differential equations

$$\rho \frac{\partial F}{\partial \rho} - \lambda \frac{\partial F}{\partial z} + 2\lambda \frac{\partial F}{\partial \lambda} = \rho \frac{\partial \nu}{\partial \rho},$$

$$\rho \frac{\partial F}{\partial z} + \lambda \frac{\partial F}{\partial \rho} = \rho \frac{\partial \nu}{\partial z}.$$  

(3.3)

(3.4)

Reader can check that introduction of term $\rho^2 - \lambda^2 - 2\lambda z$ instead of general function is in accordance with (1.74).

The seed metric is usually chosen to be Minkowski one. In this case BH at the axis of symmetry are generated. This section employs Bach–Weyl ring (see section 1.4.3) instead. Its gravitational potential $\nu$ can be expressed in the terms of Legendre functions as

$$\nu = -m \sum_{l=0}^{\infty} b^l P_l(0) \nu_l$$

where

$$\nu_l = (\rho^2 + z^2)^{-\frac{l+1}{2}} P_l \left( \frac{z}{\sqrt{\rho^2 + z^2}} \right).$$

(3.5)

(3.6)

Due to the linearity of (3.3) and (3.4) one can search their solution in form

$$F = -m \sum_{l=0}^{\infty} b^l P_l(0) F_l$$

where $F_l$ satisfies (3.3) and (3.4) with $\nu_l$ on the r.h.s. and $b$ denotes radius of the ring.

As the original paper pointed operators in equations (3.3) and (3.4) commutes with partial derivative with respect to $z$. Moreover the functions $\nu_l$ can be expressed by

$$\nu_l = \frac{(-1)^l}{l!} \frac{\partial^l \nu_0}{\partial z^l},$$

(3.7)

(3.8)
therefore analogous expression for general $F_l$ can expected

$$F_l = \frac{(-1)^l}{l!} \frac{\partial^l F_0}{\partial z^l},$$  \hspace{1cm} (3.9)

where

$$F_0 = \frac{z + r}{(z + r + \lambda)r}.$$

At the end we can express $F$ as infinite sum

$$F = -m \sum_{l=0}^{\infty} b^l P_l \frac{(-1)^l}{l!} \frac{\partial^l}{\partial z^l} \left( \frac{z + r}{(z + r + \lambda)r} \right).$$  \hspace{1cm} (3.11)

Castro & Letelier ended simplifying $F$ there and begun calculating solution numerically using truncated series (3.11). However there is a way to calculate $F$ with Bach–Weyl ring as seed solution in the closed form. At first let us define generating functions of components of (3.7) as

$$F(\xi) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \frac{\rho^l}{\partial z^l} F_0(\rho, z, \lambda) = \frac{z - \xi + \sqrt{\rho^2 + (z - \xi)^2}}{\left[ z - \xi + \sqrt{\rho^2 + (z - \xi)^2} + \lambda \right] \sqrt{\rho^2 + (z - \xi)^2}} = \frac{\rho^2 - z\lambda + \xi\lambda - \lambda \sqrt{\rho^2 + (z - \xi)^2}}{(\rho^2 - \lambda^2 - 2z\lambda + 2\xi\lambda) \sqrt{\rho^2 + (z - \xi)^2}},$$  \hspace{1cm} (3.12)

$$G(\xi) = -m \sum_{l=0}^{\infty} b^l P_l(0) \xi^l = -\frac{m}{\sqrt{1 + (b\xi)^2}}.$$  \hspace{1cm} (3.13)

In the first equation the fact that the operator is just a shift operator have been used. The second case is well known a generating function of Legendre polynomials (f.e. see [DLMF (18.12.4)]). Moreover orthogonality relation

$$\int_0^{2\pi} e^{ik\phi} e^{-il\phi} d\phi = -i \oint_C t^k t^{-l} \frac{dt}{t} = 2\pi \delta_{kl},$$  \hspace{1cm} (3.14)

where $t = e^{i\phi}$ and $C$ is a complex integration path — circle in positive direction around origin (see figure 3.1).

All together lead to

$$F = -m \sum_{l=0}^{\infty} b^l P_l \frac{(-1)^l}{l!} \frac{\partial^l}{\partial z^l} \left( \frac{z + r}{(z + r + \lambda)r} \right) =$$

$$= \sum_{l=0}^{\infty} b^l P_l e^{-il\phi} N^{-l} e^{i\phi} N^l \frac{(-1)^l}{l!} \frac{\partial^l}{\partial z^l} \left( \frac{z + r}{(z + r + \lambda)r} \right) =$$

$$= -\frac{m}{8\pi} \int_0^{2\pi} \left\{ \sum_{l=0}^{\infty} b^l P_l e^{-il\phi} N^{-l} \right\} \left\{ \sum_{k=0}^{\infty} e^{ik\phi} N^k \frac{(-1)^k}{k!} \frac{\partial^k}{\partial z^k} \left( \frac{z + r}{(z + r + \lambda)r} \right) \right\} d\phi =$$

Keep in mind that $r = \sqrt{\rho^2 + z^2}$, therefore it must be taken into account when calculating derivatives.

In the original article it was created in such way.
Figure 3.1: Integration path and singularities employed during calculation of $F$. Singularities and cuts are given by bold lines (circles). Exact position of $t_{sg}$ with respect to integration path $C$ is a priori not known (whether it is inside/outside of the contour).

$$\frac{1}{2\pi} \int_{0}^{2\pi} G \left( \frac{e^{-i\phi}}{N} \right) F(N e^{i\phi}) d\phi = -\frac{i}{2\pi} \oint_{C} G \left( \frac{1}{t} \right) F(t) \frac{dt}{t}, \quad (3.15)$$

where $N$ is suitable normalization constant and $t = Ne^{i\phi}$. In these steps we have deliberately changed order of summation and integration. It could be proven such steps are justified from mathematical point of view, however it is easier to omit such technicalities and check that the final result (3.27).

Before continuing further let us spend some time on singularities of generating functions $F$ and $G$. In the latter case the situation is simple. There is a complex cut between $t = ib$ and $t = -ib$. It is convenient to choose it in such way, that it lies on the imaginary axis.

The former case is a bit more complicated. $\sqrt{\rho^2 + (z - l)^2}$ introduces complex cut between $t = z + i\rho$ and infinity and another between $t = z - i\rho$ and infinity.

4It is necessary to have $N > b$ for (3.13) to converge.

5The idea of the formal proof is based on Fubini’s theorem and uniqueness of function of complex variables. Radius of convergence (centered around $\xi = 0$) of (3.13) is $b$ and in the case of (3.12) it reads min $\left( \sqrt{\rho^2 + z^2}, |t_{sg}| \right)$ (based on analysis of singularities given below).

Assuming domain where $b < \sqrt{\rho^2 + z^2}$ and $b < |t_{sg}|$ (i.e. the point $(\rho, z)$ is far enough) power series converges exponentially there and Fubini’s theorem can be used to exchange summation and integration order (as well as splitting of infinite series into two distinct ones in (3.15)).

Uniqueness of functions of complex variable can be used to extend solution from domain given above. The restrictions on convergence are, in fact, given by introducing expansions. Therefore they pose (up to the cut discussed later in this chapter) only technical difficulties coming from chosen procedure.
Assuming that $z$ and $\rho$ are large enough\footnote{The expansion \((3.5)\) converges when \(r > b\), i.e. it is well behaved at infinity.} \((N^2 < \rho^2 + z^2)\) the integration path avoids them. Another singularity is given by \(z - t + \lambda + \sqrt{\rho^2 + (z - t)^2} = 0\). When \(\lambda = 0\) then \(\rho = 0\) and \(t = z\) and so if \(|z| > N\) it lies outside integration path. When \(\lambda \neq 0\) there is \((\text{possible} — \text{the condition is necessary but not sufficient})\) divergence at the point

\[
t_{sg} = -\frac{\rho^2 - 2\lambda z - \lambda^2}{2\lambda}.
\]

It describes “technical” singularity introduced artificially during calculation. \(\lambda\) is going to be such that this singularity does not contribute, see \((3.19)\). Finally \(t\) exist in denominator alone so \(t = 0\) is singularity of the integrand of \((3.15)\) as well. Illustration of their positions is given by figure 3.1

Integrand in \((3.15)\) is meromorphic (up to the cuts), therefore integration path \(\mathcal{C}\) can be deformed to \(\mathcal{C}' = \mathcal{C}_+ \cup \mathcal{C}_j \cup \mathcal{C}_- \cup \mathcal{C}_\uparrow \cup \mathcal{C}_{sg}\) (\(\mathcal{C}_{sg}\) is present in the integration path when \(t_{sg}\) is inside circle \(\mathcal{C}\)) without changing the result. By defining

\[
I_X = -\frac{i}{2\pi} \int_{C_X} \frac{G(t^{-1})F(t)}{t} dt,
\]

where \(X \in \{+, -, \uparrow, \downarrow, \text{sg}\}\) we can write

\[
F = I_+ + I_- + I_\uparrow + I_\downarrow + I_{\text{sg}} [+I_{\text{sg}}].
\]

First one

\[
I_{\text{sg}} = \lim_{\epsilon \to 0^+} -\frac{i}{2\pi} \int_0^{2\pi} G\left(\frac{1}{t_{sg} + \epsilon e^{i\phi}}\right) \frac{F(t_{sg} + \epsilon e^{i\phi})}{t_{sg} + \epsilon e^{i\phi}} i\epsilon e^{i\phi} d\phi = \\
= \frac{G(t_{sg}^{-1})}{2\pi t_{sg}} \lim_{\epsilon \to 0^+} \int_0^{2\pi} F(t_{sg} + \epsilon e^{i\phi}) e^{i\phi} d\phi = \\
= \frac{m \lambda \text{sgn} (\rho^2 - 2\lambda z - \lambda^2)}{\pi \sqrt{(\rho^2 - 2\lambda z - \lambda^2)^2 + 4b^2 \lambda^2}} \lim_{\epsilon \to 0^+} \epsilon (\lambda - |\lambda|) \int_0^{2\pi} \frac{-1}{2\epsilon \lambda e^{i\phi}} + O(1) \right) e^{i\phi} d\phi = \\
= \frac{m \text{sgn} (\rho^2 - 2\lambda z - \lambda^2)}{\sqrt{(\rho^2 - 2\lambda z - \lambda^2)^2 + 4b^2 \lambda^2}} (|\lambda| - \lambda).
\]

where it is assumed \(\lambda \in \mathbb{R}\). Given \(\lambda = \mu_{(k)}\) defined by \((1.80)\) stronger restriction holds — \(\lambda \geq 0 —\) and \(I_{\text{sg}}\) must vanish. Analogously
\[ I_\pm = \lim_{\epsilon \to 0} \frac{-i}{2\pi} \int_{\mp ib \pm \epsilon}^{1} \frac{F(t)G(t^{-1})}{t} dt = \lim_{\epsilon \to 0} \frac{b}{2\pi} \int_{\mp 1}^{1} \frac{F(ib\nu \pm \epsilon)}{ib\nu + \epsilon} G \left( \frac{1}{ib\nu \pm \epsilon} \right) d\nu = \]

where \( + \) corresponds to \( \uparrow \) and \(- \) to \( \downarrow \). In both cases integrals do not contribute. The last one read

\[ F = \nu(\rho, z) - \frac{m}{\pi} \left\{ \sqrt{\rho b(1 - k^2)} C_1 C_3 \left[ C_2 + 2\rho \left( b^2 + t_{sg}^2 \right) \right] \Pi \left( -\frac{C_2^2}{C_3^2}, k \right) \right\} \]

where \( t = ib \). \( \sqrt{\nu^2(-1 \pm \epsilon)} = \pm i\nu + O(\epsilon) \) have been used to simplify result. Lengthy calculation reveals

\[ \int_{-1}^{1} \frac{F(ib\nu)}{\sqrt{1 - \nu^2}} d\nu = \left\{ \frac{k}{2\sqrt{\rho b}} \left[ (t_{sg}(b - \rho) - zb) C_1 + 1 \right] K(k) + X'(t_{sg}) \right\} + \frac{\sqrt{\rho b(1 - k^2)}}{k} \frac{C_1}{C_3} \left[ C_2 + 2\rho \left( b^2 + t_{sg}^2 \right) \right] \Pi \left( -\frac{C_2^2}{C_3^2}, k \right) \]

where \( K(\theta) \), \( E(\theta) \) and \( \Pi(n, k) \) denotes complete elliptic integrals of the first, second and third kind,

\[ k = \sqrt{\frac{4\rho b}{(\rho + b)^2 + z^2}}, \]  

\[ C_1 = -\frac{\text{sgn}(\lambda) \sqrt{(z - t_{sg})^2 + \rho^2}}{C_2}, \]  

\[ C_2 = t_{sg}^2(b - \rho) - 2zb\nu + b(z^2 - b\rho + \rho^2), \]  

\[ C_3 = -t_{sg}^2z + t_{sg}(\rho^2 + z^2 - b^2) + b^2z, \]  

and \( X'(t_{sg}) \) is suitable function of \( t_{sg} \) alone. Finally

\[ F = \nu(\rho, z) - \frac{m}{\pi} \left\{ \sqrt{\rho b(1 - k^2)} C_1 C_3 \left[ C_2 + 2\rho \left( b^2 + t_{sg}^2 \right) \right] \Pi \left( -\frac{C_2^2}{C_3^2}, k \right) \right\} + \]
Figure 3.2: \( F(\rho, z) \). This particular example was calculated using parameters \( m = \pi, b = 1 \) and \( t_{sg} = 1 \). Choice of \( X(t_{sg}) \) is given by (3.28).

\[
\frac{k}{2\sqrt{\rho b}} \left\{ \left[ (t_{sg}(b - \rho) - zb) C_1 - 1 \right] K(k) + X(t_{sg}) \right\},
\]

(3.27)

where \( X(t_{sg}) \) stands arbitrary piecewise continuous function of \( t_{sg} \). It is convenient to choose

\[
X(t_{sg}) = -\frac{\pi}{\sqrt{t^2_{sg} + b^2}} \left\{ 2 - \text{sgn}(\lambda z) \left[ 1 + \right. \right.
\]

\[
\left. + \text{sgn} \left( \rho^2 + z^2 - b^2 - 2zt_{sg} + \sqrt{(\rho^2 + z^2 - b^2)^2 + 4b^2z^2} \right) + \right.
\]

\[
\left. + \text{sgn} \left( \rho^2 + z^2 - b^2 - 2zt_{sg} - \sqrt{(\rho^2 + z^2 - b^2)^2 + 4b^2z^2} \right) \right\},
\]

(3.28)

in accordance with (3.22). It fixes position of the cut (it extends in the \( z = 0 \) plane from ring to infinity) and ensures there is only one (see figure 3.2).

Check that solution (3.27) satisfies (3.3) and (3.4) can be performed by straightforward calculation. Also \( F \to \nu \) when \( \lambda \to 0 \) (i.e. \( t_{sg} \to \infty \)) as required by boundary condition (1.75). Because \( C_1 \approx \frac{1}{t_{sg}(b - \rho)}, C_2 \approx t^2_{sg}(b - \rho), C_3 \approx -zt^2_{sg} \) the coefficients in (3.27) behaves like

\[
\frac{C_2}{C_3} \to \frac{\rho - b}{z},
\]

\[
[t_{sg}(b - \rho) - zb] C_1 - 1 \to 0,
\]

(3.29)

On the axis \((\rho = 0)\) the situation is a bit different. Limits of mentioned expressions behaves differently —

\[
\frac{C_2}{C_3} \to 0,
\]

\[
\frac{\sqrt{\rho b}(1 - k^2)}{k} C_1 \to \frac{\pi}{2\sqrt{\rho^2 + z^2}},
\]

\[
\frac{C_2}{C_3} \to \frac{\pi}{4\sqrt{\rho^2 + z^2}},
\]

\[
\frac{1}{\sqrt{t^2_{sg} + b^2}} \to \frac{1}{\sqrt{\rho^2 + z^2}}.
\]

This ensure validity on one half-axis. However such “wrong” limit affects only infinitesimally small part of the domain and it will not spoil the solution as whole.
\[
\frac{\sqrt{\rho_b (1 - k^2)}}{k} C_1 \frac{C_2 + 2 \rho \left( b^2 + i_{sg}^2 \right)}{C_3} \rightarrow 0, \\
\frac{1}{\sqrt{i_{sg}^2 + b^2}} \rightarrow 0 \tag{3.30}
\]

proving above mentioned limit of \( F \).

**General Weyl metric**

It should be noted that procedure used earlier to find spectral function \( F \) can be used more generally. Given Weyl metric its gravitational potential can be expanded into multipole moments analogously to \( \nu_l \) (\( \nu_l \) being defined by \( \nu_0 \))

\[
\nu = \sum_{l=0}^{\infty} \beta_l \nu_l \tag{3.31}
\]

using suitable coefficients \( \beta_l \). With their knowledge generating function can be, possibly, found\(^8\)

\[
\mathcal{G}(\xi) = \sum_{l=0}^{\infty} \beta_l \xi^l. \tag{3.32}
\]

At the end, if we are lucky enough, spectral function \( F \) is given by complex integral \( \nu_0 \).

### 3.2 Two–soliton solutions

#### 3.2.1 Generated metric

The focus is given to two–soliton solutions due to their simplicity. General formulas for new metric have been already introduced in section \( \nu_0 \) — unphysical metric is described by \( \nu_0 \) and \( \nu_1 \), physical one is given by \( \nu_0 \) and \( \nu_1 \). (Semi-)Explicit expressions for unphysical metric components. \( \nu_0 \)–\( \nu_1 \) with adopted assumptions given by this chapter reads

\[
m_t^{(k)}(\rho, z) = -C_t^{(k)} e^{-2F(\rho, \rho, z)} \tag{3.33},
\]

\[
m_\phi^{(k)}(\rho, z) = \frac{C_\phi^{(k)}}{\lambda_k(\rho, z)} e^{2F(\rho, \rho, z)} \tag{3.34},
\]

\[
\Gamma_{kl}(\rho, z) = \left\{ C_\phi^{(l)} \frac{\rho^2}{\lambda_k(\rho, z) \lambda_l(\rho, z)} e^{2[F(\rho, \rho, z) + F(\rho, \rho, z) - \nu(\rho, z)]} - C_t^{(l)} e^{2[F(\rho, \rho, z) - F(\rho, \rho, z) + \nu(\rho, z)]} \right\} / \left[ \rho^2 + \lambda_k(\rho, z) \lambda_l(\rho, z) \right] \tag{3.35},
\]

\[
\Pi_{kl}(\rho, z) = [\Gamma_{kl}(\rho, z)]^{-1} \quad \text{and} \quad \nu_1 \quad \text{takes form}
\]

\[
g_{tt} = -e^{2\nu(\rho, z)} - e^{4\nu(\rho, z)} \sum_{k, l=1}^{2} C_t^{(k)} C_t^{(l)} \Pi_{kl}(\rho, z) e^{-2[F(\rho, \rho, z) + F(\rho, \rho, z)]}, \tag{3.36}
\]

\(^8\)The problem of finding \( G \) is much easier since it is not function of \( \rho \) and \( z \).
\[
g_{t\phi} = -\rho^2 \sum_{k,l=1}^{2} \frac{\Pi_{kl}(\rho, z)}{\lambda_k(\rho, z) \lambda_l(\rho, z)} C^{(k)}_t C^{(l)}_\phi e^{2[F(\lambda_k, \rho, z) - F(\lambda_l, \rho, z)]},
\]

(3.37)

\[
g_{\phi\phi} = \rho^2 e^{-2\nu(\rho, z)} - \rho^4 e^{-4\nu(\rho, z)} \sum_{k,l=1}^{2} \frac{\Pi_{kl}(\rho, z)}{\lambda_k(\rho, z) \lambda_l(\rho, z)}
\times \frac{C^{(k)}_\phi C^{(l)}_\phi}{\lambda_k(\rho, z) \lambda_l(\rho, z)} e^{2[F(\lambda_k, \rho, z) + F(\lambda_l, \rho, z)]},
\]

(3.38)

The last unknown function of the solution, \( f \), formula (1.79) reads

\[
f = C_2 e^{2\zeta - 2\nu} \rho^2 \left( 2 \prod_{k=1}^{2} \frac{\lambda_k^2(\rho, z)}{\rho^2 + \lambda_k^2(\rho, z)} \right) \det \Gamma_{kl}(\rho, z),
\]

(3.39)

where \( \zeta \) and \( \nu \) denote metric function of seed solution (see (1.58)).

Also physical solution can be written using algebraic expression of \( \lambda \). Normalization needed to get physical 2–metric can be obtained from (1.84)

\[
\det(g) = -\rho^6 \frac{1}{[\lambda_1(\rho, z) \lambda_2(\rho, z)]^2}.
\]

(3.40)

Moreover (1.86)

\[
f^{ph} = C_2^{ph} e^{2\zeta - 2\nu} \frac{(\lambda_1(\rho, z) \lambda_2(\rho, z))^3}{[\lambda_2(\rho, z) - \lambda_1(\rho, z)]^2} \frac{\det \Gamma_{kl}(\rho, z)}{\rho^2}
\]

(3.41)

gives the remaining physical metric function.

### 3.2.2 Asymptotic behaviour at spatial infinity

General analysis of asymptotic behaviour is already given by Belinskii and Verdaguer [2004]. For detailed analysis refer to this book.

Let us begin with asymptotic behaviour (as \( r \to \infty \)) of spectral function \( F \) because the other functions are linked to it. Fixing piecewise continuous part by (3.28) leads to

\[
F(\rho, z, t_{\text{sg}}) \to -m [1 - \text{sgn}(\lambda) \text{sgn}(\cos \theta)] - m [1 + \text{sgn}(\lambda) \cos \theta] + O \left( \frac{1}{r^2} \right).
\]

(3.42)

Other functions follows. From (1.84) we obtain

\[
\det(g) = -r^2 \sin^2 \theta \tan^{-4} \left( \frac{\theta}{2} \right) \left[ 1 - 2 r^2 \sum_{k=1}^{2} \alpha_k + O \left( \frac{1}{r^2} \right) \right]
\]

(3.43)

and assuming\(^9\) \( \alpha = \alpha_1 = -\alpha_2 \)

\[
g^{ph}_{tt} = -1 + 2 \left[ m + \frac{\alpha (C^{(1)}_t C^{(2)}_\phi + C^{(1)}_\phi C^{(2)}_t)}{C^{(1)}_t C^{(2)}_\phi - C^{(1)}_\phi C^{(2)}_t} \right] \frac{1}{r} + O \left( \frac{1}{r^2} \right),
\]

(3.44)

\(^9\)Such assumption is physical corresponding to position of solitons aligned with the ring.
\[
\mathbf{g}_{t\phi}^{ph} = \frac{\alpha}{C_t^{(1)} C_{\phi}^{(2)} - C_{\phi}^{(1)} C_t^{(2)}} \times \times \frac{C_t^{(1)} C_t^{(2)} \Delta_+^4 + C_{\phi}^{(1)} C_{\phi}^{(2)}}{\Delta_+^2} + \frac{C_{\phi}^{(1)} C_{\phi}^{(2)}}{\Delta_+^2} \Delta_+^4 \cos \theta \right] + O \left( \frac{1}{r} \right), \tag{3.45}
\]

\[
\mathbf{g}_{\phi\phi}^{ph} = \sin^2 \theta r^2 + \sin^2 \theta \left[ 2m + 2 \frac{\alpha \left( C_t^{(1)} C_{\phi}^{(2)} + C_{\phi}^{(1)} C_t^{(2)} \right)}{C_t^{(1)} C_{\phi}^{(2)} - C_{\phi}^{(1)} C_t^{(2)}} \right] r + O \left( 1 \right), \tag{3.46}
\]

where \( \Delta_\pm \) describes discontinuity of \( F(\rho, z, \tau_{\text{gg}}) \). Exact choice of \( \Delta_\pm \) corresponds to the sign of \( z \), \( \Delta_- \) to \( \theta < \pi/2 \) and \( \Delta_+ \) to \( \theta > \pi/2 \), where

\[
\Delta_+ = 1, \quad \Delta_- = \begin{cases} 
1 & \text{when } \rho < b \\
e \sqrt{m} \exp \left( \sqrt{\frac{-\alpha}{m}} \right) & \text{when } \rho > b.
\end{cases} \tag{3.47}
\]

In the limit \( r \to \infty \) the branch \( \rho < b \) of \( \Delta_- \) can be ignored.

Rescaling of parameters \( C_t^{(k)} \) and \( C_{\phi}^{(k)} \) does not change the result, therefore we can choose their scaling arbitrarily — see (3.36), (3.37) and (3.38). It is convenient to put \( \alpha = C_t^{(1)} C_{\phi}^{(2)} - C_{\phi}^{(1)} C_t^{(2)} \). Generated solution asymptotically approaches Kerr–NUT solution. Comparing first non-trivial order of expansion in \( 1/r \) one can see mass, angular momentum and NUT parameter are given by

\[
M_{\text{Kerr}} = m - C_t^{(1)} C_{\phi}^{(2)} - C_{\phi}^{(1)} C_t^{(2)}, \tag{3.48}
\]

\[
a_{\text{Kerr}} = \frac{C_t^{(1)} C_t^{(2)} \Delta_+^4 + C_{\phi}^{(1)} C_{\phi}^{(2)}}{\Delta_+^2}, \tag{3.49}
\]

\[
b_{\text{Kerr}} = \frac{C_t^{(1)} C_t^{(2)} \Delta_+^4 - C_{\phi}^{(1)} C_{\phi}^{(2)}}{\Delta_+^2}. \tag{3.50}
\]

The expressions for \( a_{\text{Kerr}} \) and \( b_{\text{Kerr}} \) are flawed. They (generally) change value when crossing equatorial plane. There are two ways to solve this. One is to choose \( F \) in such way that there is no discontinuity. It requires no, or at least two seed ring solutions compensating each other. This option is not going to be discussed here.

Also special choice of constants that ensuring absence of discontinuity in the equatorial plane can be attempted. And, again, there are two possibilities — with vanishing \( C^{(k)} \) and without. Former one (as it will be shown later) leads to static metric. \( C_t^{(1)} C_t^{(2)} = C_{\phi}^{(1)} C_{\phi}^{(2)} = 0 \) and so \( a_{\text{Kerr}} = b_{\text{Kerr}} = 0 \). Detailed

\[\text{[10]}\]

There are several reasons why it is so. Mentioning most important ones (some topics are going to be discussed later in this thesis):

- The nature of discontinuities in the first derivatives gives no physically reasonable solution. Also it does not change with introduction of second ring.
- Compensation requires the second ring to have negative mass. Preliminary calculations showed this behaviour does not vanish after application of BZM.
- Aim of this section is completing some missing points of paper [de Castro and Letelier [2011]. They are interested in single ring seed metric only.
discussion is given by section 3.3.1. The latter approach focuses more closely on (3.45). There is no way (up to the static case) to ensure that $a_{\text{Kerr}}$ and $b_{\text{Kerr}}$ does not simultaneously change value in the equatorial plane. However the term proportional to $b_{\text{Kerr}}$ is multiplied by $\cos \theta$ therefore it does vanish in equatorial plane (at least in the leading order of expansion). $b_{\text{Kerr}}$ can be discontinuous there and “only” discontinuity of normal derivatives appears.

Focusing solely on $a_{\text{Kerr}}$ from

$$C_t^{(1)} C_t^{(2)} \Delta_+^4 + C_\phi^{(1)} C_\phi^{(2)} \Delta_-^4 = \frac{C_t^{(1)} C_t^{(2)} \Delta_+^4 + C_\phi^{(1)} C_\phi^{(2)} \Delta_+^4}{\Delta_+^4}$$

(3.51)

introduces condition

$$C_\phi^{(1)} C_\phi^{(2)} - C_t^{(2)} C_t^{(1)} e^{\frac{4m}{\sqrt{\alpha^2 + b^2}}} = 0.$$  (3.52)

It also ensures continuity of metric in the whole equatorial plane (see chapter 3.3.2 and equation (3.98)). Inserting it into (3.50) gives $b_{\text{Kerr}}|_+ = -b_{\text{Kerr}}|_-$, where $b_{\text{Kerr}}|_+$ denotes NUT parameter of solution with $z > 0$ and $z < 0$. So something like NUT parameter is present. However it is symmetric with respect to the equatorial plane, i.e., it contains $|\cos \theta|$ term instead of ordinary cosine.

We can conclude that asymptotic structure of generated metric closely related to the Kerr–NUT solution (up to the possible discontinuity). There are two possible approaches to make the discontinuity disappear. First one leads to static solution and the second one generates solution symmetric with respect to the equatorial plane (at least asymptotically). In the latter case there is no way to avoid NUT-like term.

### 3.2.3 Axis

On the axis

$$F(0, z, t_{\text{sg}}) = -\frac{m}{\sqrt{\text{max}(z, t_{\text{sg}})^2 + b^2}}$$

(3.53)

where $\lambda \geq 0$ has been assumed. Moreover

$$\nu(0, z) = -\frac{m}{\sqrt{z^2 + b^2}}.$$  (3.54)

and

$$\lambda = t_{\text{sg}} - z + \sqrt{(t_{\text{sg}} - z)^2 + \rho^2} = \begin{cases} 2(t_{\text{sg}} - z) + O(\rho^4) & \text{when } z < t_{\text{sg}} \\ \frac{\rho^2}{2(z - t_{\text{sg}})} + O(\rho^4) & \text{when } z > t_{\text{sg}} \end{cases}$$

(3.55)

Careful in the vicinity of axis has to be used as $\lambda$ can vanish there and limit approach must be employed. Due to the the technical aspects of the discussion and appearance physically severe problems in the equatorial plane it is present here.

### 3.2.4 Equatorial plane

In the equatorial plane there is, generally, a discontinuity of generated metric due to the discontinuity of spectral function $F(\rho, z, \lambda)$. In some special cases it can be avoided. Complexity of this analysis requires standalone section 3.3.2 dedicated to it.
3.2.5 Castro & Letelier’s solution revisited

At the first glance solution (3.27) and the one given by de Castro and Letelier [2011] cannot be the same. However two different expansions are used in that paper, one in the area \( r^2 \equiv \rho^2 + z^2 < b^2 \) (denoted “in”, representing Taylor expansion of \( F \) from \( r = 0 \)) and another one when \( r^2 > b^2 \) (denoted “out”, representing Taylor expansion of \( F \) with origin at \( r = \infty \)). Such expansions requires \( r = 0 \) and \( r = \infty \) being regular points of \( \nu \) and \( F \). The latter is problematic.

As seen in figure 3.2 there is cut in equatorial plane extending from ring towards infinity so \( r = \infty \) is not a regular point. No Taylor expansion of (3.27) exist there. Keeping in mind, that solution (3.27) is given up to the function of \( t_{sg} \) and \( b \) it is possible to choose different function \( X(t_{sg}) \) to obtain \( F \) with cut extending between \( r = 0 \) and ring. In fact arbitrary curve connecting edge of domain of \( F \) (i.e. \( (\rho, z) \in (0; +\infty) \times (-\infty; +\infty) \)) and ring \((\rho, z) = (b, 0)\) can be chosen and obtain function with cut extending along this curve. In the original paper there has to be non-trivial behaviour along the surface \( r = b \).

Apart from such complication, one can simply follow the procedure there in order to obtain new metric (as in this chapter). Only thing not seen there is discontinuity of the metric (or, in case of lucky choice of constants, jump in the derivatives) itself.

3.3 Avoiding discontinuity

3.3.1 Diagonal matrix \( \Gamma_{kl}(\rho, z) \)

It is possible to get rid of discontinuity in the \( t-\phi \) part of the matrix by suitable choice of \( C^{(k)} \) ensuring \( \Gamma_{kl}(\rho, z) \) is diagonal. Such assumption requires \( C_t^{(2)} = C_\phi^{(1)} = 0 \). Following (3.35) \( \Gamma_{kl}(\rho, z) \) can be expressed as

\[
\begin{align*}
\Gamma_{11}(\rho, z) &= -\frac{e^{2\nu(\rho,z)} - 4F(\lambda_1, \rho,z)}{\rho^2 + \lambda_1^2(\rho,z)} \left[C_t^{(1)}\right]^2, \\
\Gamma_{12}(\rho, z) &= \Gamma_{21}(\rho, z) = 0, \\
\Gamma_{22}(\rho, z) &= \frac{\rho^2 e^{4F(\lambda_2, \rho,z) - 2\nu(\rho,z)}}{\lambda_2^2(\rho,z) [\rho^2 + \lambda_2^2(\rho,z)]} \left[C_\phi^{(2)}\right]^2.
\end{align*}
\]

Corresponding non–normalized metric takes form

\[
\begin{align*}
g_{tt} &= -e^{2\nu(\rho,z)} + e^{2\nu(\rho,z)} \frac{\rho^2 + \lambda_1(\rho,z)^2}{\lambda_1(\rho,z)^2}, \\
g_{t\phi} &= 0, \\
g_{\phi\phi} &= -\rho^2 e^{-2\nu(\rho,z)} - \rho^2 e^{-2\nu(\rho,z)} \frac{\rho^2 + \lambda_2(\rho,z)^2}{\lambda_2(\rho,z)^2} = -\rho^2 e^{-2\nu(\rho,z)} \frac{\rho^2}{\lambda_2(\rho,z)^2}.
\end{align*}
\]

\[\text{“In” and “out” solutions are well adopted in their domains. Considering their convergence and behaviour they does not exhibit cut there. Therefore the cut has to be present at the radius of the ring. Exact analysis is not given in the original paper nor here (as previous argument makes it partly obsolete).}\]

\[\text{Up to the possible exchange of indices 1 and 2.}\]
and using (3.40) physical metric is calculated

\[ g^{ph}_{tt} = -e^{2\nu(\rho, z)} \frac{\lambda_2(\rho, z)}{\lambda_1(\rho, z)} = e^{2\nu'(\rho, z)}, \quad (3.62) \]

\[ g^{ph}_{t\phi} = 0, \quad (3.63) \]

\[ g^{ph}_{\phi\phi} = \rho^2 e^{-2\nu(\rho, z)} \frac{\lambda_1(\rho, z)}{\lambda_2(\rho, z)} = \rho^2 e^{-2\nu'(\rho, z)}, \quad (3.64) \]

where

\[ \nu' = \nu(\rho, z) + \frac{1}{2} \ln \left[ \frac{\alpha_2 - z + \sqrt{(\alpha_2 - z)^2 + \rho^2}}{\alpha_1 - z + \sqrt{(\alpha_1 - z)^2 + \rho^2}} \right]. \quad (3.65) \]

denotes generated gravitational potential. This is a simple superposition of Bach–Weyl ring solution with Schwarzschild black-hole. Such metric is well known and does not require further discussion here.

The unknown part (at least based on my knowledge) is expression of the remaining metric function \( f \) or \( \zeta \) in closed form. (3.41) gives

\[
\ln f^{ph} = 2\zeta' - 2\nu' = \ln C_2^{ph} + \ln f + \ln \left[ \frac{\lambda_1^3 \lambda_2}{\lambda_2 - \lambda_1} \right] \left( \frac{\rho^2 + \lambda_1^2}{\rho^2 + \lambda_2^2} \right)^2 + 2 \left[ F(\lambda_2, \rho, z) - F(\lambda_1, \rho, z) \right].
\]

(3.66)

Up to the symmetric case (where \( \alpha_1 = -\alpha_2 \)) the metric function \( f^{ph} \) obtained by BZM exhibit discontinuity. It has physical reason — situation, where ring is not aligned with BH must contain some kind of singularity giving additional \( z \)-force keeping the ring and BH at their locations.

Assuming \( \alpha_1 = -\alpha_2 = \alpha \) and introducing \( d_+ = \sqrt{(\alpha + z)^2 + \rho^2} \) metric functions have form

\[ \nu' = -\frac{2mK(k)}{\pi \sqrt{(\rho + b)^2 + z^2}} + \frac{1}{2} \ln \left( \frac{d_+ + d_- - 2\alpha}{d_+ + d_- + 2\alpha} \right), \quad (3.67) \]

\[ \zeta' = \frac{m}{4\pi^2 b^2} \left[ \frac{(b^2 - r^2)(2 - k^2)}{(r^2 + b^2)(1 - k^2)} E(k) - 4K(k) E(k) - \frac{(3b^2 + r^2)(2 - k^2)}{(r^2 + b^2)} K(k)^2 \right] + \frac{1}{2} \ln \left( \frac{(d_+ + d_-)^2 - 4\alpha^2}{4d_+ d_-} \right) + 2 \left[ F(\lambda_2, \rho, z) - F(\lambda_1, \rho, z) \right], \quad (3.68) \]

where we have chosen constant \( C_2^{ph} \) in such way, that no conical singularity on the axis appears, i.e. that

\[ \zeta'|_{\rho=0} = 0. \]

(3.69)

Figure 3.3 gives an example of such generated metric.

### 3.3.2 Continuity of physical metric

After special case leading to the static metric, let us focus on the singularities in the equatorial plane much more rigorously. It is clear (from physical reasons already given) there has to be singularity when the BH is not aligned with ring, i.e. \( \alpha_1 = -\alpha_2 = \alpha \). For convenience in this chapter one-sided limits at the equatorial plane are denoted by \( \tilde{f} \equiv f|_{z=0^+} \). Also usage of \( \Delta_{\pm} \) defined by (3.47) simplifies discussion.
Continuity of $f^{th}$ across equatorial plane

Formulas from chapter 3.2.1 contains only continuous terms up spectral function $F(\rho, z, t_{sg})$ that is also fully responsible for discontinuity of metric. $F(\rho, z, t_{sg})$ defined by (3.27) and (3.28) is continuous everywhere up to the equatorial plane where $\rho > b$. There it jumps by constant (with respect to $\rho$ and $z$) value, i.e.

$$F(\lambda, \rho, z)_{|z=0^+} - F(\lambda, \rho, z)_{|z=0^-} = \begin{cases} 0 & \text{when } \rho < b \\ \frac{m_{sgn(\lambda)}}{\sqrt{t_{sg}^2 + b^2}} & \text{when } \rho > b \end{cases}.$$  

(3.70)

Given $\alpha_1 = -\alpha_2 = \alpha$ the dependence of difference of one-sided limits only on squares of $t_{sg}$ can be used to simplify previous expression

$$F(\mu_{(1,2)}, \rho, z)_{|z=0^+} - F(\mu_{(1,2)}, \rho, z)_{|z=0^-} = \begin{cases} 0 & \text{when } \rho < b \\ \frac{m}{\sqrt{\alpha^2 + b^2}} & \text{when } \rho > b \end{cases} = \frac{1}{2} \ln \Delta_- - \frac{1}{2} \ln \Delta_+.$$  

(3.71)

or, equivalently,

$$F(\mu_{(1,2)}, \rho, z)_{|z=0^+} - F(\mu_{(1,2)}, \rho, z)_{|z=0^-} = \frac{1}{2} \ln \Delta_+ - \frac{1}{2} \ln \Delta_-.$$  

(3.72)

Expressions for generated metric does not contain spectral function $F$ itself, but rather $F_k(\rho, \phi, z) \equiv e^{2F(\lambda_k, \rho, z)}$. Its one-sided limits reads

$$F_k_{|z=0^\pm} = \frac{F_k}{\Delta_\pm}.$$  

(3.73)

Using fact, that $\mu_{(k)}$ is continuous across $z = 0$ plane, we can express behaviour of (3.33) and (3.34) near the equatorial plane by

$$m_{(k)}^\phi_{|z=0^\pm} = -\frac{C_{(k)}^\phi}{F_k} \Delta_\pm, \quad m_{(k)}^\phi_{|z=0^\pm} = \frac{C_{(k)}^\phi}{F_k} \Delta_\pm^{-1}. $$  

(3.74)

Subsequently (from (3.35)) limits of matrix $\Gamma_{kl}$ have form

$$\Gamma_{kl}(\rho, z)_{|z=0^\pm} = X_{kl} \Delta_\pm^2 + Y_{kl} \Delta_\pm^{-2},$$  

(3.75)
where $X_{kl}$ and $Y_{kl}$ does not depend on side of limit. They are defined by

$$X_{kl} = -\frac{e^{2\mu}}{F_k F_l} \frac{C^{(k)}_l C^{(l)}_k}{\rho^2 + \mu (k) \mu (l)}, \quad Y_{kl} = \frac{\rho^2 F_k F_l}{e^{2\mu} \rho + \mu (k) \mu (l)} \frac{C^{(k)}_\phi C^{(l)}_\phi}{\alpha^2}. \quad (3.76)$$

Using relation $\mu (1) \mu (2) = \rho^2$ several identities for $X_{kl}$ and $Y_{kl}$ can be found, notably

$$\frac{(C^{(1)}_l C^{(2)}_k)(C^{(2)}_l C^{(1)}_k)}{F_1 \mu (1)} F_2 \mu (2) = 1 = \frac{(\alpha^2 + \rho^2) X_{1212}}{\rho^2 X_{12}^2}, \quad (3.77)$$

$$\frac{(C^{(1)}_\phi C^{(2)}_\phi)(C^{(2)}_\phi C^{(1)}_\phi)}{F_1 \mu (1)} F_2 \mu (2) = 1 = \frac{(\alpha^2 + \rho^2) Y_{1222}}{\rho^2 Y_{12}^2}. \quad (3.78)$$

First let us focus on analysis of $f^{ph}$, because it depends only on $\Gamma_{kl}$ itself and not its inversion. $\textbf{(3.41)}$ simplifies to

$$f^{ph} |_{z=0^\pm} = e^{2\mu - 2\mu} \frac{C^{\rho^2} \rho^4}{4\alpha^2} \det \Gamma_{kl}(\rho, z)|_{z=0^\pm}.$$

Ignoring singular case$^{13}$ $\alpha = 0$ the metric function $f^{ph}$ is continuous if (and only if) $\det \Gamma_{kl}(\rho, z)$ does not jump in the equatorial plane. $\textbf{(3.75)}$ is used to express determinant

$$\det \Gamma_{kl}(\rho, z)|_{z=0^\pm} = (\Gamma_{11} \Gamma_{22} - \Gamma_{12}^2)|_{z=0^\pm} = D_4 \Delta_4^+ + D_0 + D_{-4} \Delta_4^-, \quad (3.80)$$

where

$$D_4 = X_{11} X_{22} - X_{12}^2 = -\frac{\alpha^2}{\rho^2} X_{11} X_{22},$$

$$D_0 = X_{11} Y_{22} + Y_{11} X_{22} - 2 X_{12} Y_{12},$$

$$D_{-4} = Y_{11} Y_{22} - Y_{12}^2 = -\frac{\alpha^2}{\rho^2} Y_{11} Y_{22}. \quad (3.81)$$

There is no jump in determinant of $\Gamma_{kl}$ as long as

$$\det \Gamma_{kl}(\rho, z)|_{z=0^+} = \det \Gamma_{kl}(\rho, z)|_{z=0^-} \Leftrightarrow (\Delta_4^+ - \Delta_4^-) \left( D_4 - \frac{D_{-4}}{\Delta_4^+ \Delta_4^-} \right) = 0. \quad (3.82)$$

To ensure that either $\Delta_4^+ = \Delta_4^-$ (therefore no jump in spectral function $F$ is present — this is the case when $\rho < b$) or $\textbf{14}$ (assuming $\rho > b$)

$$\frac{e^{2\mu}}{e^{\sqrt{\alpha^2 + \eta^2}}} \Delta_4^+ \Delta_4^- = \frac{D_{-4}}{D_4} = \frac{Y_{11} Y_{22}}{X_{11} X_{22}} = \left[ \frac{F_1 F_2 C^{(1)}_{\phi} C^{(2)}_{\phi}}{e^{4\mu} C^{(1)}_t C^{(2)}_t} \right]^2. \quad (3.83)$$

$^{13}$Physically it correspond to extreme BH.

$^{14}$Take note that $C^{(1)}_t C^{(2)}_{\phi} \neq 0$. $C^{(1)}_t C^{(2)}_t = 0$ implies $C^{(1)}_{\phi} C^{(2)}_{\phi} = 0$ and it leads to diagonal case, which was analyzed in previous section.
Straightforward calculation reveals

\[ F(\lambda_1, \rho, z) + F(\lambda_2, \rho, z) = \nu \quad \Rightarrow \quad F_1 F_2 = e^{2\nu}. \]  

(3.84)

So (3.83) simplifies to

\[ e^{4m} e^{\sqrt{\alpha^2 + \rho^2}} = \left| \begin{array}{cc} C^{(1)}_t C^{(2)}_t \\ C^{(1)}_t C^{(2)}_t \end{array} \right|. \]  

(3.85)

This is a bit surprising result, because \( D_4 \) and \( D_{-4} \) both depend on \( \rho \) and \( z \) in non-trivial way. However at the end spatial dependence of the functions disappears and only constants (with respect to \( \rho \) and \( z \)) remains.

**Continuity of \( g_{tt}^{ph} \) across equatorial plane**

Combining (1.85) and (1.84) gives

\[ g_{tt}^{ph} \bigg|_{z=0^\pm} = \pm \rho \ g_{ab} \bigg|_{z=0^\pm}, \]  

(3.86)

where \( a, b \in (t, \phi) \) and the \( \pm \) in front of r.h.s. is arbitrary fixed (and does not depend on direction of limit). The physical metric is continuous as long as its non–normalized component is.

(3.36) reads

\[ g_{tt} \bigg|_{z=0^\pm} = -e^{2\nu} - e^{4\nu} \left[ \sum_{k,l=1}^{2} \frac{\Pi_{kl}(\rho, z)}{\mu_{(k)} \mu_{(l)}} F_k(\rho, z) F_l(\rho, z) \right] \bigg|_{z=0^\pm}. \]  

(3.87)

Due to the continuity of seed metric focus will be given to the bracket. To analyze \( f^{ph} \) matrix \( \Pi_{kl} \) (defined as inverse of \( \Gamma_{kl} \)) is required. General formula for inversion of \( 2 \times 2 \) matrix is quite simple

\[ \Pi_{kl} = \frac{1}{\det \Gamma} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12} & \Gamma_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \Gamma_{22} & -\Gamma_{12} \\ -\Gamma_{12} & \Gamma_{11} \end{pmatrix} \]  

(3.88)

so

\[ \left[ \sum_{k,l=1}^{2} \frac{\Pi_{kl}(\rho, z)}{\mu_{(k)} \mu_{(l)}} C^{(k)}_t C^{(l)}_t \bigg|_{z=0^\pm} \right] = \frac{A_{tt} \Delta^4_+ + B_{tt} \det \Gamma}{\det \Gamma \bigg|_{z=0^\pm}} \]  

(3.89)

where (employing relations from previous section)

\[ A_{tt} = \frac{X_{22} C^{(1)}_t^2}{\mu_{(1)}^2 F_1^2 - 2 \mu_{(2)} F_1 F_2} + \frac{X_{11} C^{(2)}_t^2}{\mu_{(2)}^2 F_1^2} = \]  

\[ = 4 \frac{X_{12}^2}{e^{2\nu}} - 4 \frac{X_{22} X_{11}}{e^{2\nu}} \frac{\alpha^2 + \rho^2}{\rho^2} = 0, \]  

(3.90)

\[ B_{tt} = \frac{Y_{22} C^{(1)}_t^2}{\mu_{(1)}^2 F_1^2} - \frac{2 Y_{12} C^{(1)}_t C^{(2)}_t}{\rho^2 F_1 F_2} + \frac{Y_{11} C^{(2)}_t^2}{\mu_{(2)}^2 F_1^2} = \]  

\[ = 4 \frac{Y_{12} X_{12}}{e^{2\nu}} - \frac{2 \sqrt{\alpha^2 + \rho^2}}{e^{2\nu}} \left( \frac{Y_{22} X_{11}}{\mu_{(1)}} + \frac{Y_{11} X_{22}}{\mu_{(2)}} \right). \]  

(3.91)

\( A_{tt} = 0 \) implies continuity of the numerator of (3.89) across \( z = 0 \) plane. Moreover assuming (3.85) holds also denominator does not jump there and so continuity of \( g_{tt} \) across equatorial plane does not introduce any additional conditions.
Continuity of $g_{\phi\phi}^{ph}$ across equatorial plane

Due to (3.86) it is sufficient to prove continuity of non–normalized metric function $g_{t\phi}$. Given (3.37) $g_{\phi\phi}$ reads

$$g_{t\phi} \bigg|_{z=0\pm} = -\rho^2 \left[ \sum_{k,l=1}^{2} \Pi_{k\ell}(\rho, z) \frac{C_{\ell}^{(k)} C_{\phi}^{(\ell)} F_1}{\mu(\rho) F_k} \right] \bigg|_{z=0\pm} = -\rho^2 \left( \frac{A_{t\phi} \Delta^2_+ + B_{t\phi} \Delta^2_-}{\det \Gamma} \right),$$

(3.92)

where

$$A_{t\phi} = \frac{X_{22} C_{t}^{(1)} C_{\phi}^{(1)}}{\mu(1)} - \frac{X_{12}}{\rho^2} \left( \frac{C_{t}^{(1)} C_{\phi}^{(2)} F_2}{\mu(2) F_1} + \frac{C_{t}^{(2)} C_{\phi}^{(1)} F_1}{\mu(2) F_2} \right) + \frac{X_{11} C_{t}^{(2)} C_{\phi}^{(2)}}{\mu(3)} =$$

$$= \frac{\alpha e^{2\rho} C_{t}^{(2)} C_{\phi}^{(1)}}{2\rho^4 \alpha^2 + \rho^2} \left[ \frac{C_{t}^{(2)} C_{\phi}^{(1)}}{\mu(1) F_2} - \frac{C_{t}^{(1)} C_{\phi}^{(2)}}{\mu(2) F_1} \right],$$

(3.93)

$$B_{t\phi} = \frac{Y_{22} C_{t}^{(1)} C_{\phi}^{(1)}}{\mu(1)} - \frac{Y_{12}}{\rho^2} \left( \frac{C_{t}^{(1)} C_{\phi}^{(2)} F_2}{\mu(2) F_1} + \frac{C_{t}^{(2)} C_{\phi}^{(1)} F_1}{\mu(2) F_2} \right) + \frac{Y_{11} C_{t}^{(2)} C_{\phi}^{(2)}}{\mu(3)} =$$

$$= \frac{C_{\phi}^{(1)} C_{\phi}^{(2)}}{2\rho^4 e^{2\rho} \sqrt{\alpha^2 + \rho^2}} \left[ \frac{C_{t}^{(2)} C_{\phi}^{(1)}}{\mu(1) F_2} - \frac{C_{t}^{(1)} C_{\phi}^{(2)}}{\mu(2) F_1} \right].$$

(3.94)

Assuming (3.85), i.e. continuity of $\det \Gamma$ across equatorial plane $g_{t\phi}$ does not jump there as long as

$$A_{t\phi} \Delta^2_+ + B_{t\phi} \Delta^2_- = A_{t\phi} \Delta^2_+ + B_{t\phi} \Delta^2_-$$

$$\Leftrightarrow (\Delta^2_+ - \Delta^2_-) \left( A_{t\phi} - \frac{B_{t\phi}}{\Delta^2_+ - \Delta^2_-} \right) = 0.$$  

(3.95)

Therefore either $F(\lambda, \rho, z)$ is continuous there ($\rho < b$) or $^{15}$ (assuming $\rho > b$)

$$e^{\frac{4m}{\sqrt{\alpha^2 + \rho^2}}} = \Delta^2_+ \Delta^2_- = \frac{B_{t\phi}}{A_{t\phi}} = -\frac{C_{\phi}^{(1)} C_{\phi}^{(2)} F_1^2 F_2^2}{C_{t}^{(1)} C_{t}^{(2)} e^{4\rho}} = \frac{C_{\phi}^{(1)} C_{\phi}^{(2)}}{C_{t}^{(1)} C_{t}^{(2)}}.$$  

(3.96)

This condition is compatible with (3.85) — it only fixes sign for the term inside absolute value.

Continuity of $g_{\phi\phi}^{ph}$ across equatorial plane

$g_{\phi\phi}^{ph}$ has to be continuous across equatorial plane as simple consequence of previous calculations. We know that determinant of $t\phi$ part of metric is $g_{t\phi}^{ph} g_{\phi\phi}^{ph} = (g_{t\phi}^{ph})^2 = -\rho^2$. Therefore

$$g_{\phi\phi}^{ph} = \frac{(g_{\phi\phi}^{ph})^2 - \rho^2}{g_{tt}^{ph}}.$$  

(3.97)

And because r.h.s. contains only continuous expressions (given (3.96) holds) $g_{\phi\phi}^{ph}$ must have the same property.

$^{15}$The special case when $A_{t\phi} = B_{t\phi} = 0$ is omitted here because it leads to the already discussed static case.
Conclusion of continuity conditions

The sufficient (and necessary) condition to generate continuous metric is given by (3.96). It can be rewritten to contain static case as well

\[ C_\phi^{(1)} C_\phi^{(2)} - C_t^{(2)} C_t^{(1)} e^{\frac{4m}{\sqrt{\alpha^2 + \rho^2}}} = 0. \]  

(3.98)

Clearly it coincides with asymptotic condition (3.52). It should not be surprising because the asymptotic condition is just a special case of general one (3.98).

Discussion performed in this chapter showed that (3.52) is not only necessary, but also sufficient to prevent metric discontinuity.

3.3.3 Normal derivatives across equatorial plane

\[ \partial f_{\phi h} / \partial z \bigg|_{z=0^\pm} \]

Starting with spectral function reveals that, surprisingly enough\(^{16}\), its first derivatives are continuous across \( z = 0 \) plane and reads

\[ F_{z} \equiv \frac{dF(\lambda_{1,2}, \rho, z)}{dz} \bigg|_{z=0^\pm} = \frac{m \text{sgn} (\lambda)}{2\pi \sqrt{\alpha^2 + \rho^2}} \left[ K(k(\rho,0)) - \frac{\rho + b}{\rho - b} \right]. \]  

(3.99)

It follows that

\[ \frac{\partial F_k}{\partial z} \bigg|_{z=0^\pm} = \frac{d\phi^{(2)}(\mu_{(k)}(\rho, z), \rho, z)}{dz} \bigg|_{z=0^\pm} = 2 \frac{F_k|_{z=0^\pm}}{\Delta^\pm} \]  

(3.100)

and

\[ \frac{\partial m^{(k)}_\phi}{\partial z} \bigg|_{z=0^\pm} = -2c^{(k)}_\phi \frac{F_k}{\mu^{(k)}} \frac{F_z}{\Delta^\pm} = -2m^{(k)}_\phi \frac{F_z}{\Delta^\pm}. \]  

(3.101)

\[ \frac{\partial m^{(k)}_t}{\partial z} \bigg|_{z=0^\pm} = c^{(k)}_\phi \frac{F_k}{\mu^{(k)}} \left( 2F_z + \frac{1}{\sqrt{\alpha^2 + \rho^2}} \right) \Delta_{\pm}^{-1} = 2m^{(k)}_\phi \frac{F_z}{\sqrt{\alpha^2 + \rho^2}} + \frac{m^{(k)}_\phi}{\sqrt{\alpha^2 + \rho^2}}. \]  

(3.102)

Derivatives of \( \Gamma_{kl} \) can be obtained by straightforward calculation

\[ \frac{\partial \Gamma_{kl}}{\partial z} \bigg|_{z=0^\pm} = X_{kl, z} \Delta_{\pm}^2 + Y_{kl, z} \Delta_{\pm}^{-2}, \]  

(3.103)

where

\[ X_{kl, z} = X_{kl} \left[ -4F_z + \frac{2}{\sqrt{\alpha^2 + \rho^2}} \frac{\mu^{(k)} \mu^{(l)}}{\mu^{(k)} + \mu^{(l)}} \right], \]  

(3.104)

\[ Y_{kl, z} = Y_{kl} \left[ 4F_z + \frac{2}{\sqrt{\alpha^2 + \rho^2}} \frac{\rho^2 + 2\mu^{(k)} \mu^{(l)}}{\rho^2 + \mu^{(k)} + \mu^{(l)}} \right]. \]  

(3.105)

\(^{16}\) It is not as surprising considering that (3.3) and (3.4) impairs two conditions on first derivatives of \( F \) that are continuous. Supplemented with with boundary condition (1.75) it is sufficient to ensure continuity of first derivatives.
and (dis)continuity of the determinant reads
\[
\frac{\partial \det \Gamma}{\partial z} \bigg|_{z=0^\pm} = D_{4z} \Delta_\pm^4 + D_{0z} + D_{-4z} \Delta_\pm^{-4},
\] (3.106)

where
\[
D_{4z} = X_{11,z}X_{22} + X_{11}X_{22,z} - 2X_{12}X_{12,z} = D_4 \left[ -8F_{,z} + \frac{2}{\sqrt{\alpha^2 + \rho^2}} \right],
\] (3.107)
\[
D_{0z} = X_{11,z}Y_{22} + X_{11}Y_{22,z} + Y_{11,z}X_{22} + Y_{11}X_{22,z} - 2X_{12}Y_{12} - 2X_{12}Y_{12,z} = \frac{4D_0}{\sqrt{\alpha^2 + \rho^2}},
\] (3.108)
\[
D_{-4z} = Y_{11,z}Y_{22} + Y_{11}Y_{22,z} - 2Y_{12}Y_{12,z} = D_{-4} \left[ 8F_{,z} + \frac{3}{\sqrt{\alpha^2 + \rho^2}} \right].
\] (3.109)

At the end using (3.41), derivative of the metric function is obtained
\[
\frac{\partial f^{ph}}{\partial z} \bigg|_{z=0^\pm} = \frac{\partial e^{2-2\nu}}{\partial z} \bigg|_{z=0^\pm} + \frac{\partial \det \Gamma}{\partial z} \bigg|_{z=0^\pm} = -\frac{\rho^6}{\alpha^2 \sqrt{\alpha^2 + \rho^2}} + \Delta_\pm^4 + D_{0z} + D_{-4z} \Delta_\pm^{-4}.
\] (3.110)

It does not jump at the equatorial plane if
\[
D_{4z} \Delta_+^4 + D_{0z} + D_{-4z} \Delta_+^{-4} = D_{4z} \Delta_-^4 + D_{0z} + D_{-4z} \Delta_-^{-4} \Rightarrow (\Delta_+^4 - \Delta_-^4) \left( D_{4z} - \frac{D_{-4z}}{\Delta_+^4 \Delta_-^4} \right) = 0.
\] (3.111)

There are, once again, two possibilities how to satisfy this condition — either \( \Delta_+^2 = \Delta_-^2 \) or
\[
\Delta_+^4 \Delta_-^4 = \frac{D_{-4z}}{D_{4z}} = \frac{D_{-4} \frac{8F_{,z}}{\sqrt{\alpha^2 + \rho^2}} + \frac{3}{\sqrt{\alpha^2 + \rho^2}}}{D_4 \frac{-8F_{,z}}{\sqrt{\alpha^2 + \rho^2}}} = \Delta_+^4 \Delta_-^4 \left[ 1 + \frac{16F_{,z} \sqrt{\alpha^2 + \rho^2} + 1}{-8F_{,z} \sqrt{\alpha^2 + \rho^2} + 2} \right],
\] (3.112)

Where (3.96) is assumed to hold.\(^{17}\) Therefore generated metric must have discontinuity in first derivatives in the equatorial plane (assuming we have got rid of discontinuity of metric itself).

\(^{17}\)It does not make sense to ensure continuity of the first derivatives when the metric itself is discontinuous.
Figure 3.4: Metric functions of generated metric using ring at radius $b = 3$ with mass $m = 2$ and BH determined by $\alpha = 1$. Constants $C_t^{(k)}$ and $C_\phi^{(l)}$ are chose in such way that metric is continuous and total angular momentum is $a_{\text{Kerr}} = 0.1$. Left: Gravitational potential $\nu'$. Right: Dragging $\omega'$.

Continuity of derivatives

Omitting static case it was shown, that it is impossible to obtain metric that has continuous derivatives across equatorial plane. Because of the behaviour of $f^{ph}$ function and modified NUT-like term at the infinity it is clear that the generated solution has severe physical problems. Therefore there is no need to express (and analyze) derivatives of 2-metric $g$ in great detail.

Generally speaking the gravitational potential will have continuous derivatives across equatorial plane. However dragging $\omega$ and subsequently $f^{ph}$ (as already proven) does not. This leads to conclusion that even potential interpretation as BH surrounded by infinite disc (with given inner radius) fails. It would require really exotic matter that will only create dragging without gravitational potential. Similar result of discussion is given by Semerák [2002].

3.4 Example

Example of generated metric is present at figure 3.4. Seed metric contains ring of mass $m = 2$ at radius $b = 3$. Position of the black-hole is determined by $\alpha = 1$ and the constants takes value

$$
C_t^{(1)} = 0.00738, \quad C_\phi^{(1)} = -1.00068, \quad (3.113)
$$

$$
C_t^{(2)} = 1.00000, \quad C_\phi^{(2)} = -0.92557. \quad (3.114)
$$

Such choice is consistent with continuity condition $(3.98)$ (and the degree of freedom involving scaling of all $C$’s, which does not affect generated metric, is fixed by choice $C_t^{(2)} = 1$). The only left degree of freedom is fixed by choice of total angular momentum of generated spacetime.

Asymptotic physical quantities describing generated spacetime, calculated using $(3.48)$, $(3.49)$ and $(3.50)$, are

$$
M_{\text{Kerr}} = 3.00137, \quad (3.115)
$$

$$
a_{\text{Kerr}} = 0.10000, \quad (3.116)
$$
Figure 3.5: Detailed behaviour of metric function near axis. Solid line represents metric function at \( z = 0 \), doted denotes \( z = 0.5 \) and \( z = 2 \). Scaling is chosen to exhibit problematic behaviour of the metric. \textit{Left:} Gravitational potential (up to the exponent) \( e^{2\nu} \). \textit{Right:} Dragging \( \omega \).

\[ b_{\text{Kerr}} = -0.85241. \]  

(3.117)

What can be seen is that metric potential \( \nu' \) is continuous even with its first derivatives across equatorial plane. However the discontinuity of first derivatives of dragging \( \omega' \) at radii \( r > b \).

Moreover the behaviour at the axis (and its vicinity) is less regular then required. One of the metric functions \( (g_{\phi\phi}^{ph}) \) vanishes near axis \( (\rho = 0) \) and another one \( (g_{t\phi}^{ph}) \) diverges at the same place\(^{18}\) (see figure 3.5).

\(^{18}\)Divergence of both functions at the same place could be expected because the determinant of \( t - \phi \) part of metric is \(-\rho^2\), therefore it is regular.
Conclusion

Inspired by accretion structures likely to exist around astrophysical black-holes, we explored some possibilities of finding stationary and axisymmetric solutions of the Einstein field equations generated by a black-hole encircled, symmetrically, by a thin disc or ring (spatially a two- or one-dimensional source). Two main approaches have been followed — perturbation of a background solution (a Schwarzschild black-hole) with appropriate boundary conditions, and generating a black-hole in some suitable non-vacuum “seed” space-time (e.g. that of a disc or a ring) using the Belinskii–Zakharov inverse-scattering transformation.

In the first approach (perturbations), we followed closely the procedure due to Will [1974] who studied a perturbation by a slowly rotating light singular ring. Our contribution consists in putting the Green functions of this perturbation problem into a closed form (involving elliptic integrals), which is more suitable for further work than the original Will’s multipole expansion. In particular, we illustrated on an example that one can, in case of some simple mass distributions, integrate the Green functions and obtain a linear perturbation of the metric for a black-hole encircled by a thin disc (which astrophysically is more realistic than a ring). The results are just being published in Čížek and Semerák [2017] and we have summarized them in chapter 2 of this thesis.

To mention just one specific point, we managed to also abandon — besides the multipole expansion of the Green functions — the “rotational” expansion, namely what in the Will’s version is represented as expansion with respect to the angular velocity of the horizon. This has been possible due to a freedom in the black-hole angular momentum. We fixed the latter in such a way that the angular momentum stays zero even after the perturbation. This does not mean that the black-hole does not rotate (with respect to the asymptotic inertial frame) — its angular velocity is non-zero as it is dragged along by rotation of the surrounding disc.

The second part of the thesis has been devoted to the Belinskii–Zakharov generating method. Inspired by the paper de Castro and Letelier [2011], we specifically checked whether the black-hole cannot be “implemented” in a suitable seed metric (namely that of the envisaged external source, e.g. a ring) by a two-soliton version of the inverse scattering procedure. Similarly as already experienced in the literature, we found that one can, in this way, only generate physically reasonable static space-times; in the more desired, stationary case, there is always present an unphysical discontinuity (corresponding to a supporting surface) in the plane of the source. Physically, one can imagine that the inverse-scattering procedure does introduce the black-hole, but “does not care” about adjusting the seed source to such a configuration that it could — without any struts — stay stationary in the new situation.

To again mention one slight “achievement”, we have at least found in this way, in a closed form, the second metric function of the corresponding Weyl metric for a Schwarzschild black-hole encircled by a Bach–Weyl ring. Previously this has been only known for the Schwarzschild metric and for the Bach–Weyl ring alone, but, due to the non-linearity in this second function, for the “superposed” space-time it has always been obtained only numerically.
Our main future plan is to study the black-hole–disc solution found in the second chapter in more detail and compare it with the results of numerical treatment of similar configurations which have appeared in the literature.
A. Metric function transformations

In this appendix transformations between used metric forms are presented. For this purpose let us denote quantities related to Weyl–Lewis–Papapetrou form of metric (1.5) by subscript $\text{WLP}^*$ and quantities related to Carter–Thorne–Bardeen form of metric (1.6) $\text{CTB}$. The coordinates $(t, \phi, \rho$ and $z$) are in both cases the same. They are introduced before split of the metric forms is done. The difference is in the interpretation of $t$–$\phi$ part of metric implying only metric functions $\nu$, $\omega$, $B$ and $\zeta$ change.

Comparing expressions of line element $ds^2$ in (1.5) and (1.6) lead to

$$-e^{2\nu_{\text{WLP}}} = -e^{2\nu_{\text{CTB}}} + \rho^2 B_{\text{CTB}}^2 e^{-2\nu_{\text{CTB}}} \omega_{\text{CTB}}^2,$$

(A.1)

$$-e^{2\nu_{\text{WLP}}} A_{\text{WLP}} = -\rho^2 B_{\text{CTB}}^2 e^{-2\nu_{\text{CTB}}} \omega_{\text{CTB}},$$

(A.2)

$$-e^{2\nu_{\text{WLP}}} A_{\text{WLP}}^2 + \rho^2 B_{\text{CTB}}^2 e^{-2\nu_{\text{WLP}}} = \rho^2 B_{\text{CTB}}^2 e^{-2\nu_{\text{CTB}}},$$

(A.3)

$$e^{2\nu_{\text{WLP}}} = e^{2\nu_{\text{CTB}} - 2\nu_{\text{CTB}}}.$$  

(A.4)

Transformation of metric functions follow. To get Carter–Thorne–Bardeen form (1.6) relations

$$\nu_{\text{CTB}} = \nu_{\text{WLP}} - \frac{1}{2} \ln \left( 1 - \frac{e^{4\nu_{\text{WLP}}} A_{\text{WLP}}^2}{\rho^2 B_{\text{WLP}}^2} \right),$$

(A.5)

$$\omega_{\text{CTB}} = \frac{\rho^2 B_{\text{WLP}}^2 e^{-2\nu_{\text{CTB}}} e^{2\nu_{\text{WLP}}} A_{\text{WLP}}^2}{e^{2\nu_{\text{WLP}}} A_{\text{WLP}}},$$

(A.6)

$$B_{\text{CTB}} = B_{\text{WLP}},$$

(A.7)

$$\zeta_{\text{CTB}} = \zeta_{\text{WLP}} - \frac{1}{2} \ln \left( 1 - \frac{e^{4\nu_{\text{WLP}}} A_{\text{WLP}}^2}{\rho^2 B_{\text{WLP}}^2} \right)$$

(A.8)

are to be applied. Analogously formulas

$$\nu_{\text{WLP}} = \nu_{\text{CTB}} + \frac{1}{2} \ln \left( 1 - \rho^2 B_{\text{CTB}}^2 e^{-4\nu_{\text{CTB}}} \omega_{\text{CTB}}^2 \right),$$

(A.9)

$$A_{\text{WLP}} = \frac{\rho^2 B_{\text{CTB}}^2 e^{-4\nu_{\text{CTB}}} \omega_{\text{CTB}}^2}{1 - \rho^2 B_{\text{CTB}}^2 e^{-4\nu_{\text{CTB}}} \omega_{\text{CTB}}^2},$$

(A.10)

$$B_{\text{WLP}} = B_{\text{CTB}},$$

(A.11)

$$\zeta_{\text{WLP}} = \zeta_{\text{CTB}} + \frac{1}{2} \ln \left( 1 - \rho^2 B_{\text{CTB}}^2 e^{-4\nu_{\text{CTB}}} \omega_{\text{CTB}}^2 \right)$$

(A.12)

give metric functions of Weyl–Lewis–Papapetrou form (1.5).
B. Christoffel symbols

B.1 Weyl–Lewis–Papapetrou form of metric

Given metric in the form (1.5) non-zero Christoffel symbols read

\[
\Gamma^t_{t\rho} = \nu_{,\rho} + \frac{AA_{,\rho}}{2\rho^2 B^2} e^{4\nu}, \quad (B.1)
\]

\[
\Gamma^t_{t\phi} = \frac{1}{2} A_{,\phi} - \frac{A}{\rho} - \frac{A B_{,\rho}}{B} + \frac{e^{4\nu} A^2 A_{,\rho}}{2\rho^2 B^2}, \quad (B.2)
\]

\[
\Gamma^t_{t\rho} = 2 A\nu_{,\rho} + A\frac{AA_{,\rho}}{\rho} - A\frac{B_{,\rho}}{B} + e^{4\nu} A^2 A_{,\rho}, \quad (B.3)
\]

\[
\Gamma^t_{t\phi} = 2 A\nu_{,\phi} + A\frac{AA_{,\phi}}{\rho} - A\frac{B_{,\phi}}{B} + e^{4\nu} A^2 A_{,\phi}, \quad (B.4)
\]

\[
\Gamma^\phi_{t\rho} = -\frac{A_{,\rho}}{2\rho^2 B^2} e^{4\nu}, \quad (B.5)
\]

\[
\Gamma^\phi_{t\phi} = -\frac{A_{,\phi}}{2\rho^2 B^2} e^{4\nu}, \quad (B.6)
\]

\[
\Gamma^\phi_{\phi\rho} = \frac{B_{,\rho}}{B} + \frac{1}{\rho} - \nu_{,\rho} - \frac{AA_{,\rho}}{2\rho^2 B^2} e^{4\nu}, \quad (B.7)
\]

\[
\Gamma^\phi_{\phi\phi} = \frac{B_{,\phi}}{B} - \nu_{,\phi} - \frac{AA_{,\phi}}{2\rho^2 B^2} e^{4\nu}, \quad (B.8)
\]

\[
\Gamma^\rho_{t\phi} = \nu_{,\phi} e^{4\nu-2\lambda}, \quad (B.9)
\]

\[
\Gamma^\rho_{t\rho} = [A_{,\rho} + A_{,\phi}] e^{4\nu-2\lambda} \quad (B.10)
\]

\[
\Gamma^\rho_{\phi\rho} = [A^2 \nu_{,\rho} + AA_{,\rho}] e^{4\nu-2\lambda} + \left[ \nu_{,\rho} - \frac{B_{,\rho}}{B} - \frac{1}{\rho} \right] \rho^2 B e^{-2\lambda}, \quad (B.11)
\]

\[
\Gamma^\rho_{\phi\phi} = \lambda_{,\rho} - \nu_{,\rho}, \quad (B.12)
\]

\[
\Gamma^\rho_{\rho\rho} = \nu_{,\rho} - \lambda_{,\rho}, \quad (B.13)
\]

\[
\Gamma^\rho_{\rho\phi} = \nu_{,\rho} - \lambda_{,\phi}, \quad (B.14)
\]

\[
\Gamma^\rho_{\phi\phi} = \nu_{,\phi} e^{4\nu-2\lambda}, \quad (B.15)
\]

\[
\Gamma^\phi_{t\phi} = [A_{,\phi} + A_{,\phi}] e^{4\nu-2\lambda} \quad (B.16)
\]

\[
\Gamma^\phi_{\phi\phi} = [A^2 \nu_{,\phi} + AA_{,\phi}] e^{4\nu-2\lambda} + \left[ \nu_{,\phi} - \frac{B_{,\phi}}{B} \right] \rho^2 B e^{-2\lambda}, \quad (B.17)
\]

\[
\Gamma^\rho_{\rho\rho} = \nu_{,\rho} - \lambda_{,\rho}, \quad (B.18)
\]

\[
\Gamma^\rho_{\rho\phi} = \nu_{,\rho} - \lambda_{,\phi}, \quad (B.19)
\]

\[
\Gamma^\rho_{\phi\phi} = \nu_{,\phi} e^{4\nu-2\lambda}, \quad (B.20)
\]
B.2 Carter–Thorne–Bardeen form of metric

Assuming the metric takes form [1.6] non-vanishing connection coefficients are expressed by

\[
\Gamma^t_{\rho \rho} = \nu_{\rho} - \frac{1}{2} \rho^2 B^2 e^{-4\nu} \omega_{\rho}, \quad (B.21)
\]

\[
\Gamma^t_{\tau z} = \nu_{\tau} - \frac{1}{2} \rho^2 B^2 e^{-4\nu} \omega_{\tau}, \quad (B.22)
\]

\[
\Gamma^t_{\phi \rho} = \frac{1}{3} \rho^2 B^2 e^{-4\nu} \omega_{\rho}, \quad (B.23)
\]

\[
\Gamma^t_{\phi z} = \frac{1}{2} \rho^2 B^2 e^{-4\nu} \omega_{\phi}, \quad (B.24)
\]

\[
\Gamma^\phi_{\tau \rho} = 2 \omega_{\nu, \rho} - \frac{1}{2} \omega_{,\rho} - \omega B_{,\rho} B - \omega \frac{\omega_{,\rho}}{\rho} - \frac{1}{2} \rho^2 B^2 e^{-4\nu} \omega^2 \omega_{,\rho}, \quad (B.25)
\]

\[
\Gamma^\phi_{\tau z} = 2 \omega_{\nu, z} - \frac{1}{2} \omega_{,z} - \omega B_{,z} B - \frac{1}{2} \rho^2 B^2 e^{-4\nu} \omega^2 \omega_{,z}, \quad (B.26)
\]

\[
\Gamma^\phi_{\phi \rho} = \frac{B_{,\rho}}{B} + \frac{1}{\rho} - \nu_{\rho} + \frac{1}{2} \rho^2 B^2 e^{-4\nu} \omega_{,\rho}, \quad (B.27)
\]

\[
\Gamma^\phi_{\phi z} = \frac{B_{,z}}{B} - \nu_{z} + \frac{1}{2} \rho^2 B^2 e^{-4\nu} \omega_{,z}, \quad (B.28)
\]

\[
\Gamma^\rho_{tt} = \left[ \nu_{,\rho} - \frac{B_{,\rho}}{B} \frac{1}{\rho} - \frac{\omega_{,\rho}}{\omega} \right] \rho^2 B^2 e^{-2\lambda} \omega^2 \omega_{,\rho} + \nu_{,\rho} e^{4\nu - 2\lambda}, \quad (B.29)
\]

\[
\Gamma^\rho_{\tau \rho} = \left[ -\nu_{,\rho} + \frac{B_{,\rho}}{B} + \frac{1}{\rho} + \frac{\omega_{,\rho}}{2\omega} \right] \rho^2 B^2 e^{-2\lambda} \omega, \quad (B.30)
\]

\[
\Gamma^\rho_{\phi \phi} = \left[ \nu_{,\rho} - \frac{B_{,\rho}}{B} - \frac{1}{\rho} \right] \rho^2 B^2 e^{-2\lambda}, \quad (B.31)
\]

\[
\Gamma^\rho_{\rho \rho} = \lambda_{,\rho} - \nu_{,\rho}, \quad (B.32)
\]

\[
\Gamma^\rho_{\rho z} = \lambda_{,z} - \nu_{,z}, \quad (B.33)
\]

\[
\Gamma^\rho_{zz} = -\lambda_{,\rho} + \nu_{,\rho}, \quad (B.34)
\]

\[
\Gamma^z_{tt} = \left[ \nu_{,z} - \frac{B_{,z}}{B} - \frac{\omega_{,z}}{\omega} \right] \rho^2 B^2 e^{-2\lambda} \omega^2 \omega_{,z} + \nu_{,z} e^{4\nu - 2\lambda}, \quad (B.35)
\]

\[
\Gamma^z_{\tau \rho} = \left[ -\nu_{,z} + \frac{B_{,z}}{B} + \frac{\omega_{,z}}{2\omega} \right] \rho^2 B^2 e^{-2\lambda} \omega, \quad (B.36)
\]

\[
\Gamma^z_{\phi \phi} = \left[ \nu_{,z} - \frac{B_{,z}}{B} \right] \rho^2 B^2 e^{-2\lambda}, \quad (B.37)
\]

\[
\Gamma^z_{\rho \rho} = \lambda_{,z} - \nu_{,z}, \quad (B.38)
\]

\[
\Gamma^z_{\rho z} = \lambda_{,\rho} - \nu_{,\rho}, \quad (B.39)
\]

\[
\Gamma^z_{zz} = \lambda_{,z} - \nu_{,z}, \quad (B.40)
\]
C. Properties of Legendre / Gegenbauer functions

Legendre functions $P_n(x)$, $Q_n(x)$ and Gegenbauer functions $C_n^{(3/2)}(x)$, $D_n^{(3/2)}(x)$ are special cases of general Gegenbauer functions $C_n^{(\lambda)}(x)$ and $D_n^{(\lambda)}(x)$. Due to this generality of their definition (using associated Legendre functions) and their properties are focus of this appendix.

Unless cited the results are based on so far unpublished paper about series of Gegenbauer functions. This appendix summarizes relevant content of this unfinished article and sketches corresponding proofs.

C.1 Definition of Gegenbauer functions

Gegenbauer polynomials $C_n^{(\lambda)}(x)$ are polynomial solutions of ultraspheric differential equation [DLMF, section 18.8]

$$\frac{d}{dx} \left[ (1-x^2)^{\lambda+\frac{1}{2}} \frac{dy(x)}{dx} \right] + n(n+2\lambda)(1-x^2)^{\lambda-\frac{1}{2}}y(x) = 0, \quad (C.1)$$

where $\lambda > 0$ and $n \in \mathbb{Z}^+$. Special case when $\lambda = \frac{1}{2}$ reduces it to Legendre differential equation (see [DLMF, (14.2.1)]). The second solution (diverging at $x = 1$) is going to be denoted Gegenbauer function of the second kind $D_n^{(\lambda)}(x)$ in analogy with Legendre function of second kind $Q_n(x)$.

Formally they can be defined using associated Legendre functions

$$C_n^{(\lambda)}(x) \equiv \left| 1-x^2 \right|^{\lambda+\frac{1}{2}} \frac{\sqrt{2\pi} \Gamma(n+2\lambda)}{2^{\lambda} \Gamma(\lambda) \Gamma(n+1)} \begin{cases} P_{\frac{1}{2}+\lambda+n}^{\frac{1}{2}-\lambda}(x) & \text{when } |x| \leq 1 \\ P_{\frac{1}{2}+\lambda+n}^{-\frac{1}{2}+\lambda+n}(x) & \text{when } x \geq 1 \end{cases}, \quad (C.2)$$

$$D_n^{(\lambda)}(x) \equiv \left| 1-x^2 \right|^{\lambda+\frac{1}{2}} \frac{\sqrt{2\pi} \Gamma(n+2\lambda)}{2^{\lambda} \Gamma(\lambda) \Gamma(n+1)} \begin{cases} Q_{\frac{1}{2}+\lambda+n}^{\frac{1}{2}-\lambda}(x) & \text{when } |x| < 1 \\ e^{i\pi(\lambda-\frac{1}{2})}Q_{\frac{1}{2}+\lambda+n}^{\frac{1}{2}-\lambda}(x) & \text{when } x > 1 \end{cases}, \quad (C.3)$$

where $P_{\mu}^{\nu}(x)$, $P_{\nu}^{\mu}(x)$, $Q_{\mu}^{\nu}(x)$ and $Q_{\nu}^{\mu}(x)$ denotes associated Legendre functions of first (second) kind. The pairs differs in position of its cut in complex plane. First and third one jumps when $x \in (+1; \infty)$ and the remaining two are discontinuous at $x \in (-1; +1)$. The notation of associated Legendre polynomials is in accordance with [DLMF, chapter 14].

The split of definitions on real axis is in order to have real valued function there (with possible discontinuity at $x = 1$). Generally assuming $x \in \mathbb{C}$ and removing absolute value (in accordance with value of $x$) they can be regarded as definition of two functions of complex variable differing by complex phase (and

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1In this appendix $x$ denotes arbitrary variable. It is not linked to coordinate $x$ defined by (2.27). Analogously all other variables (notable example $\lambda$) does not have its physical meaning.

2Some technical aspects can be omitted here. However main points are given and reader can finish formal proofs.
position of cut). It should be remarked that in case of polynomials $C_{n}(\lambda)(x)$ there is no singularity at $x = 1$ and both definitions coincide.

Main interest of this thesis are special cases $\lambda = \frac{1}{2}$ and $\lambda = \frac{3}{2}$. The former one leads to well known Legendre functions and in the latter case (and assuming $n \in \mathbb{Z}^+$) definitions (C.2) and (C.3) reads

\[ C_{n}(\lambda)(x) = \frac{(n + 2)(n + 1)}{\sqrt{|1 - x^2|}} \begin{cases} P_{n+1}^{-1}(x) & \text{when } |x| \leq 1 \\ P_{n+1}^{-1}(x) & \text{when } x \geq 1 \end{cases} \]  
\[ D_{n}(\lambda)(x) = \frac{(n + 2)(n + 1)}{\sqrt{|1 - x^2|}} \begin{cases} Q_{n+1}^{-1}(x) & \text{when } |x| < 1 \\ -Q_{n+1}^{-1}(x) & \text{when } x > 1 \end{cases}. \]

\[ C.2 \quad \text{Basic properties} \]

\subsection{C.2.1 \quad \text{Values at special points}}

Values of Gegenbauer polynomials take simple form at several special points. Based on \[\text{DLMF, Table 18.6.1}\] and assuming $n \in \mathbb{Z}^+$

\[ C_{n}(\lambda)(1) = \frac{\Gamma(n + 2\lambda)}{n!\Gamma(2\lambda)}, \]  
\[ C_{2n+1}(0) = 0, \quad C_{2n}(0) = (-1)^n \frac{\Gamma(n + \lambda)}{n!\Gamma(\lambda)}. \]

In case of Legendre polynomials (C.6) and (C.7) reads

\[ P_{n}(1) = 1, \]  
\[ P_{2n+1}(0) = 0, \quad P_{2n}(0) = (-1)^n \frac{(2n)!}{4^n(n!)^2}. \]

and if $\lambda = \frac{3}{2}$

\[ C_{n}(\frac{3}{2})(1) = \frac{(n + 2)(n + 1)}{2}, \]  
\[ C_{2n+1}(0) = 0, \quad C_{2n}(0) = (-1)^n \frac{(2n + 1)!}{4^n(n!)^2}. \]

To get values at $x = -1$ one can use parity of $C_{n}(\lambda)(x)$. It is determined by parity of $n$, i.e.

\[ C_{n}(-x) = (-1)^n C_{n}(\lambda)(x). \]

See \[\text{DLMF, Table 18.6.1}\].

\subsection{C.2.2 \quad \text{Orthogonality}}

Gegenbauer polynomials $C_{n}(\lambda)(x)$ forms basis of space of analytic functions of variable $x$. Also they follows relation of orthogonality \[\text{DLMF, section 18.2}\]

\[ \int_{-1}^{1} C_{n}(\lambda)(x)C_{m}(\lambda)(x)(1 - x^2)^{\frac{\lambda - 1}{2}}dx = \delta_{mn} \frac{2^{1-2\lambda}\pi\Gamma(n + 2\lambda)}{(n + \lambda)n!\Gamma(\lambda)^2}. \]  

\[ \text{This fact is exploited in chapter 2.} \]

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Two special cases we are interested in read
\begin{align*}
\int_{-1}^{1} P_m(x)P_n(x) \, dx &= \delta_{mn} \frac{2}{2n+1}, \quad (C.14) \\
\int_{-1}^{1} C_n^{(3/2)}(x)C_n^{(3/2)}(x)(1-x^2) \, dx &= \delta_{mn} \frac{2(n+1)(n+2)}{2n+3}. \quad (C.15)
\end{align*}

C.2.3 Asymptotic behaviour

Infinite order

As \( n \to +\infty \) the functions behaves like
\begin{align*}
C_{n}^{(\lambda)}(\cos \theta) &= (\sin \theta)^{-\lambda} \frac{2n^{\lambda-1}}{2\Gamma(\lambda)} \left[ \cos ((\lambda + n)\theta) + O \left( \frac{1}{n} \right) \right], \quad (C.16) \\
D_{n}^{(\lambda)}(\cos \theta) &= (\sin \theta)^{-\lambda} \frac{\pi n^{\lambda-1}}{2\Gamma(\lambda)} \left[ -\sin ((\lambda + n)\theta - \frac{\pi}{2}) + O \left( \frac{1}{n} \right) \right], \quad (C.17)
\end{align*}

where \( \theta \in (0; \pi) \) and
\begin{align*}
C_{n}^{(\lambda)}(x) &= (x^2 - 1)^{-\frac{1}{2}} \frac{n^{\lambda-1}}{2\Gamma(\lambda)} \left( x + \sqrt{x^2 - 1} \right)^{\lambda+n} \left[ 1 + O \left( \frac{1}{n} \right) \right], \quad (C.18) \\
D_{n}^{(\lambda)}(x) &= (x^2 - 1)^{-\frac{1}{2}} \frac{\pi n^{\lambda-1}}{2\Gamma(\lambda)} \left( x + \sqrt{x^2 - 1} \right)^{-\lambda-n} \left[ 1 + O \left( \frac{1}{n} \right) \right], \quad (C.19)
\end{align*}

assuming \( x > 1 \).

Also when \( C_{n}^{(\lambda)}(x) \) does not diverge at \( x = 1 \) the asymptotic behaviour there is given by \( (C.6) \)
\begin{equation}
C_{n}^{(\lambda)}(1) = \frac{n^{2\lambda-1}}{\Gamma(2\lambda)} \left[ 1 + O \left( \frac{1}{n} \right) \right]. \quad (C.20)
\end{equation}

Infinite argument

As \( x \to \infty \) the functions behaves like
\begin{align*}
C_{n}^{(\lambda)}(x) &\approx \frac{2^n \Gamma(n+\lambda)}{\Gamma(\lambda) \Gamma(n+1)} x^n, \quad (C.21) \\
D_{n}^{(\lambda)}(x) &\approx \frac{\pi \Gamma(n+2\lambda)}{2^{2\lambda+n} \Gamma(\lambda) \Gamma(n+\lambda+1)} x^{-n-2\lambda}, \quad (C.22)
\end{align*}

assuming \( n \geq 0 \) and \( \lambda > 0 \). Special cases we are primary interested in read
\begin{align*}
P_{n}(x) &\approx \frac{\Gamma(2n+1)}{[\Gamma(n+1)]^2 2^n} x^n, \quad (C.23) \\
Q_{n}(x) &\approx \frac{2^n [\Gamma(n+1)]^2}{\Gamma(2n+2)} x^{-n-1} \rightarrow 0, \quad (C.24) \\
C_{n}^{(3/2)}(x) &\approx \frac{\Gamma(2n+2)}{[\Gamma(n+1)]^2 2^n} x^n, \quad (C.25) \\
D_{n}^{(3/2)}(x) &\approx \frac{2^{n+1} \Gamma(n+2) \Gamma(n+3)}{\Gamma(2n+4)} x^{-n-3} \rightarrow 0. \quad (C.26)
\end{align*}

\footnote{Also see DLMF (18.15.10) for Gegenbauer polynomials.}
\footnote{See DLMF sec. 14.8(iii)] and definitions of Gegenbauer functions.}
Behaviour at 1

In the case when \( x \to 1^+ \) it can be shown \[ \lim_{x \to 1^+} C_n^{(\lambda)}(x) = \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)\Gamma(2\lambda)}, \] (C.27)

\[ \lim_{x \to 1^+} D_n^{(\lambda)}(x) \approx (2x-2)^{\frac{3}{2}-\lambda} \frac{\sqrt{\pi} \Gamma(\lambda - \frac{1}{2})}{2\Gamma(\lambda)}, \] (C.28)

where it is assumed \( n \geq 0 \) (in both cases) and \( \lambda > 0 \) (in the case of \( C_n^{(\lambda)}(x) \)) or \( \lambda > 1/2 \) (in the case of \( D_n^{(\lambda)}(x) \)). Legendre functions (\( \lambda = 1/2 \)) goes to \[ \lim_{x \to 1^+} P_n(x) = 1, \] (C.29)

\[ \lim_{x \to 1^+} Q_n(x) \approx \frac{1}{2} \ln(x-1) + O(1) \] (C.30)

and Gegenbauer polynomials with \( \lambda = 3/2 \)

\[ \lim_{x \to 1^+} C_n^{(3/2)}(x) = \frac{(n+1)(n+2)}{2}, \] (C.31)

\[ \lim_{x \to 1^+} D_n^{(3/2)}(x) \approx \frac{1}{2(x-1)}. \] (C.32)

C.2.4 Recurrence relation

Recurrence relation of Gegenbauer polynomials is well known. Analogous relation for the Gegenbauer functions of the second kind hold as well. Generally

\[(n+1)A_{n+1}^{(\lambda)}(x) + (n+2\lambda-1)A_{n}^{(\lambda)}(x) = 2(n+\lambda)xA_n^{(\lambda)}(x), \] (C.33)

where \( A_n^{(\lambda)}(x) \) stands for an arbitrary linear combination of Gegenbauer functions of the first and second kind.\[^8\]

C.2.5 Sum of double product

One consequence of recurrence relation given earlier is

\[(x-t)\sum_{l=0}^{n} 2(l+\lambda) \frac{\Gamma(l+1)}{\Gamma(l+2\lambda)} A_l^{(\lambda)}(x)B_l^{(\lambda)}(t) = \]

\[= \frac{\Gamma(n+2)}{\Gamma(n+2\lambda)} \left[ A_{n+1}^{(\lambda)}(x)B_{n}^{(\lambda)}(t) - A_{n}^{(\lambda)}(x)B_{n+1}^{(\lambda)}(t) \right] + C^{(\lambda)}(x,t), \] (C.34)

where \( n \in \mathbb{Z}_0^+ \), \( x, t \in \mathbb{R} \) (or \( \mathbb{C} \)). Series of functions \( A_n^{(\lambda)}(x) \) and \( B_n^{(\lambda)}(x) \) are assumed to obey relation (C.33) and \( C^{(\lambda)}(x,t) \) denotes suitable function\[^7\] independent on \( n \).

\[^6\]Combine [DLMF] sec. 14.8(ii) with definitions (C.2) and (C.3).

\[^7\]In the latter case it is based on [DLMF] (14.8.9)] because (C.28) fails when \( \lambda = 1/2 \).

\[^8\]It can be proven using recurrence relations of (associated) Legendre functions and linearity of (C.33).

This function can be obtained, for example, putting \( n = 0 \)

\[ C^{(\lambda)}(x,t) = \frac{1}{\Gamma(2\lambda)} \left[ 2\lambda(x-t)A_0^{(\lambda)}(x)B_0^{(\lambda)}(t) - A_1^{(\lambda)}(x)B_0^{(\lambda)}(t) + A_0^{(\lambda)}(x)B_1^{(\lambda)}(t) \right]. \] (C.35)
Proof. Due to the existence of $C^{(\lambda)}(x, t)$ in formula (C.34) it is sufficient to prove that the the difference of the r.h.s of (C.34) (let us denote it $R_n$) with $n$ and $n-1$ equals the difference of the l.h.s. (i.e. the additional term in the sum). (C.33) ensures that

\[
\frac{\Gamma(n+2)}{\Gamma(n+2\lambda)} A^{(\lambda)}_{n+1}(x)B^{(\lambda)}_n(t) - \frac{\Gamma(n+1)}{\Gamma(n+2\lambda-1)} A^{(\lambda)}_n(x)B^{(\lambda)}_{n-1}(t) = \quad (C.36)
\]

\[
= \frac{\Gamma(n+1)}{\Gamma(n+2\lambda)} \left[ (n+1)A^{(\lambda)}_{n+1}(x)B^{(\lambda)}_n(t) - (n+2\lambda-1)A^{(\lambda)}_n(x)B^{(\lambda)}_{n-1}(t) \right]
\]

And the difference of the r.h.s of (C.34) with $n$ and $n-1$ takes form

\[
R_n - R_{n-1} = \left[ \frac{\Gamma(n+2)}{\Gamma(n+2\lambda)} A^{(\lambda)}_{n+1}(x)B^{(\lambda)}_n(t) - \frac{\Gamma(n+1)}{\Gamma(n+2\lambda-1)} A^{(\lambda)}_n(x)B^{(\lambda)}_{n-1}(t) \right] -
\]

\[
- \left[ \frac{\Gamma(n+2)}{\Gamma(n+2\lambda)} A^{(\lambda)}_n(x)B^{(\lambda)}_{n+1}(t) - \frac{\Gamma(n+1)}{\Gamma(n+2\lambda-1)} A^{(\lambda)}_{n-1}(x)B^{(\lambda)}_n(t) \right] =
\]

\[
= \frac{\Gamma(n+1)}{\Gamma(n+2\lambda)} \left[ 2x(n+\lambda)A^{(\lambda)}_n(x)B^{(\lambda)}_n(t) -
\right.
\]

\[
- (n+2\lambda-1) \left( A^{(\lambda)}_{n-1}(x)B^{(\lambda)}_{n-1}(t) + A^{(\lambda)}_n(x)B^{(\lambda)}_{n-1}(t) \right) -
\]

\[
- \frac{\Gamma(n+1)}{\Gamma(n+2\lambda)} \left[ 2t(n+\lambda)A^{(\lambda)}_n(x)B^{(\lambda)}_n(t) -
\right.
\]

\[
- (n+2\lambda-1) \left( A^{(\lambda)}_n(x)B^{(\lambda)}_{n-1}(t) + A^{(\lambda)}_{n-1}(x)B^{(\lambda)}_{n-1}(t) \right) \]

\[
= \frac{\Gamma(n+1)}{\Gamma(n+2\lambda)} 2(x-t)(n+\lambda)A^{(\lambda)}_n(x)B^{(\lambda)}_n(t). \quad (C.37)
\]

Comparing this with the difference of the l.h.s. of (C.34) completes the proof. \qed

There are two notable special (extreme) cases of relation (C.34). Sum involving only Gegenbauer polynomials

\[
(x-t) \sum_{l=0}^{n} 2(l+\lambda) \frac{\Gamma(l+1)}{\Gamma(l+2\lambda)} C^{(\lambda)}_l(x)C^{(\lambda)}_{l+1}(t) =
\]

\[
= \frac{\Gamma(n+2)}{\Gamma(n+2\lambda)} \left[ C^{(\lambda)}_{n+1}(x)C^{(\lambda)}_n(t) - C^{(\lambda)}_n(x)C^{(\lambda)}_{n+1}(t) \right] \quad (C.38)
\]

and “mixed” product of Gegenbauer functions of the first and second kind

\[
(x-t) \sum_{l=0}^{n} 2(l+\lambda) \frac{\Gamma(l+1)}{\Gamma(l+2\lambda)} C^{(\lambda)}_l(x)D^{(\lambda)}_{l+1}(t) =
\]

\[
= \frac{\Gamma(n+2)}{\Gamma(n+2\lambda)} \left[ C^{(\lambda)}_{n+1}(x)D^{(\lambda)}_n(t) - C^{(\lambda)}_n(x)D^{(\lambda)}_{n+1}(t) \right] - \frac{2\pi |t^2-1|^{\frac{1}{2}-\lambda}}{4\lambda^2(\lambda)}.
\]

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Both series are a generalization of well-known equalities of Legendre polynomials [DLMF (14.18.6-7)]

\[(x - t) \sum_{l=0}^{n} (2l + 1) P_l(t) P_l(x) = (n + 1) [P_{n+1}(x) P_n(t) - P_n(x) P_{n+1}(t)], \quad \text{(C.40)}\]

\[(x - t) \sum_{l=0}^{n} (2l + 1) P_l(t) Q_l(t) = (n + 1) [P_{n+1}(x) Q_n(t) - P_n(x) Q_{n+1}(t)] - 1. \quad \text{(C.41)}\]

And when \( \lambda = \frac{3}{2} \) (C.38) and (C.39) becomes

\[\begin{align*}
(x - t) \sum_{l=0}^{n} \frac{(2l+3)C^{(3/2)}_l(x)C^{(3/2)}_l(t)}{(l + 1)(l + 2)} &= \frac{C^{(3/2)}_{n+1}(x)C^{(3/2)}_n(t) - C^{(3/2)}_n(x)C^{(3/2)}_{n+1}(t)}{n + 2}, \quad \text{(C.42)} \\
(x - t) \sum_{l=0}^{n} \frac{(2l+3)C^{(3/2)}_l(x)D^{(3/2)}_l(t)}{(l + 1)(l + 2)} &= \frac{C^{(3/2)}_{n+1}(x)D^{(3/2)}_n(t) - C^{(3/2)}_n(x)D^{(3/2)}_{n+1}(t)}{n + 2} - \frac{1}{|t^2 - 1|}. \quad \text{(C.43)}
\end{align*}\]

**Proof of (C.38).** Exact knowledge of low-order Gegenbauer polynomials (see for example [Erdelyi, 1981, page 175, (12)])

\[C^{(\lambda)}_0(x) = 1, \quad C^{(\lambda)}_1(x) = 2\lambda x, \quad \text{(C.44, C.45)}\]

can be used to prove \( C(x) = 0 \) using (C.38) with \( n = 0 \), QED. \( \Box \)

**Proof of (C.39).** (C.44) and (C.45) lead to

\[C^{(\lambda)}_1(x)D^{(\lambda)}_0(x) - C^{(\lambda)}_0(x)D^{(\lambda)}_1(x) = 2\lambda x D^{(\lambda)}_0(x) - D^{(\lambda)}_1(x). \quad \text{(C.46)}\]

On the other hand the cross-products of associated Legendre functions are given by literature, for example [DLMF (14.2.5),(14.2.11)],

\[\begin{align*}
P^{\mu}_{\nu+1}(x)Q^\mu_\nu(x) - P^\mu_\nu(x)Q^\mu_{\nu+1}(x) &= \frac{\Gamma(\nu + \mu - 1)}{\Gamma(\nu - \mu + 2)}, \quad \text{(C.47)} \\
P^{\mu}_{\nu+1}(x)Q^\mu_\nu(x) - P^\mu_\nu(x)Q^\mu_{\nu+1}(x) &= e^{i\pi\mu} \frac{\Gamma(\nu + \mu - 1)}{\Gamma(\nu - \mu + 2)}. \quad \text{(C.48)}
\end{align*}\]

So

\[\begin{align*}
C^{(\lambda)}_1(x)D^{(\lambda)}_0(x) - C^{(\lambda)}_0(x)D^{(\lambda)}_1(x) &= \frac{2\pi \Gamma(2\lambda)}{2^{2\lambda} \Gamma^2(\lambda)} \left\{ \begin{array}{ll}
(1 - x^2)^{\frac{1}{2} - \lambda} & \text{when } |x| < 1 \\
(x^2 - 1)^{\frac{1}{2} - \lambda} & \text{when } x > 1
\end{array} \right. \\
&= \frac{2\pi \Gamma(2\lambda)}{2^{2\lambda} \Gamma^2(\lambda)} |1 - x^2|^{\frac{1}{2} - \lambda}. \quad \text{(C.49)}
\end{align*}\]

where the absolute value makes sense on the real axis\(^\text{10}\). Combining (C.46) and (C.49) leads to

\[D^{(\lambda)}_1(x) = 2\lambda x D^{(\lambda)}_0(x) - \frac{2\pi \Gamma(2\lambda)}{4^{\lambda} \Gamma^2(\lambda)} |x^2 - 1|^{\frac{1}{2} - \lambda}. \quad \text{(C.50)}\]

\(^\text{10}\)If complex \( x \) is considered the absolute value is to be understood in the sense of corresponding branch.
and, surprisingly, function $C$ can be calculated as given by (C.39):

$$C^{(\lambda)}(x, t) = -\frac{2\pi}{4^\lambda \Gamma^2(\lambda)} |t^2 - 1|^{-\frac{1}{2}}$$  \hspace{1cm} (C.51)

\[\square\]

### C.3 Miscellaneous relations

#### C.3.1 Relation between integration and differentiation

One of the consequences of ultraspherical differential equation (C.1) can be used to exchange differentiation and integration in special case. Assuming $\lambda > 0$

$$\int_{1}^{x} (z^2 - 1)^{\lambda - \frac{1}{2}} C^{(\lambda)}(z) dz = \frac{(x^2 - 1)^{\lambda + \frac{1}{2}}}{n(n + 2\lambda)} \frac{dC^{(\lambda)}(x)}{dx}$$  \hspace{1cm} (C.52)

When $\lambda = \frac{1}{2}$ it takes form

$$\int_{1}^{x} P_n(z) dz = \frac{(x^2 - 1)}{n(n + 1)} \frac{dP_n(x)}{dx}$$  \hspace{1cm} (C.53)

and if $\lambda = \frac{3}{2}$

$$\int_{1}^{x} (z^2 - 1) C^{(3/2)}(z) dz = \frac{(x^2 - 1)^{2}}{n(n + 3)} \frac{dC^{(3/2)}(x)}{dx}.$$  \hspace{1cm} (C.54)

#### C.3.2 Product simplification

Product of two Gegenbauer polynomials can be expressed using integral of single polynomial [DLMF (18.17.5)].

$$C^{(\lambda)}_n(\cos \theta_1) C^{(\lambda)}_n(\cos \theta_2) = \frac{2}{2^{2\lambda} \Gamma^2(\lambda)} \frac{\Gamma(n + 2\lambda)}{\Gamma(n + 1)} \int_{0}^{\pi} C^{(\lambda)}_n(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi)(\sin \phi)^{2\lambda - 1} d\phi,$$

where $\theta_1, \theta_2 \in (0; \pi)$. This formula is crucial expressing sums of triple/quadruple products

#### C.3.3 Generalized Neumann integral

The Gegenbauer function of the second kind can be obtained using integral relation

$$D^{(\lambda)}_n(t) = \frac{1}{2}(t^2 - 1)^{\frac{1}{2} - \lambda} \int_{-1}^{1} \frac{C^{(\lambda)}_n(x)(1 - x^2)^{\lambda - \frac{1}{2}}}{t - x} dx.$$  \hspace{1cm} (C.56)
where \( t > 1 \) and \( \lambda > 0 \). When \( \lambda = \frac{1}{2} \) equation (C.56) reduces to the Neumann’s integral \([DLMF, (14.12.13)]\)

\[
Q_\alpha(t) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(x)}{t - x} \, dx. \tag{C.57}
\]

and assuming \( \lambda = \frac{3}{2} \)

\[
D^{(3/2)}_n(t) = \frac{1}{2(t^2 - 1)} \int_{-1}^{1} \frac{c^{(3/2)}_n(x)(1 - x^2)}{t - x} \, dx. \tag{C.58}
\]

**Proof.** Using orthogonality of Gegenbauer polynomials (C.13) l.h.s. of (C.39) (divided by \((x - t)\)) leads to

\[
\frac{(n+\lambda)\Gamma^2(\lambda)\Gamma(n+1)}{2^{1-2\lambda}\pi\Gamma(n+2\lambda)} \int_{-1}^{1} C_n^{(\lambda)}(x)(1-x^2)^{\lambda-\frac{1}{2}} \sum_{l=0}^{m} 2(l+\lambda) \frac{\Gamma(l+1)}{\Gamma(l+2\lambda)} C_l^{(\lambda)}(x)D^{(\lambda)}_l(t) \, dx =
\]

\[
= \begin{cases} 0 & \text{when } m < n \\ \frac{2(n+\lambda)\Gamma(n+1)}{\Gamma(n+2\lambda)} D^{(\lambda)}_n(t) & \text{when } m \geq n. \end{cases} \tag{C.59}
\]

As \( m \to +\infty \) it goes (up to some normalization) to Gegenbauer function of the second kind. So

\[
\lim_{m \to +\infty} \frac{\Gamma^2(\lambda)}{2^{2-2\lambda}\pi} \int_{-1}^{1} C_n^{(\lambda)}(x)(1-x^2)^{\lambda-\frac{1}{2}} \sum_{l=0}^{m} 2(l+\lambda) \frac{\Gamma(l+1)}{\Gamma(l+2\lambda)} C_l^{(\lambda)}(x)D^{(\lambda)}_l(t) \, dx = D^{(\lambda)}(t). \tag{C.60}
\]

The r.h.s. of (C.39) (divided by \((x - t)\)) will be split into two parts. The one, containing Gegenbauer functions, vanishes as \( m \to \infty \). The other forms r.h.s. of (C.56). Let us begin with the vanishing part. Using integral analogous to (C.60) gives\(^\text{11} \)

\[
\begin{align*}
& \left| \frac{\Gamma^2(\lambda)}{2^{2-2\lambda}\pi} \int_{-1}^{1} \frac{C_n^{(\lambda)}(x)D^{(\lambda)}_m(t)}{\Gamma(m+2\lambda)} \frac{C_{m+1}^{(\lambda)}(x)D^{(\lambda)}_m(t)}{x-t} (1-x^2)^{\lambda-\frac{1}{2}} \, dx \right| \approx \\
& \approx \left| \frac{\Gamma^2(\lambda)m^{2-2\lambda}}{2^{2-2\lambda}\pi} \int_{-1}^{1} \frac{2m^{\lambda-1}}{2\lambda\Gamma(\lambda)} \cos [(\lambda + m) \arccos x] (t^2 - 1)^{-\frac{3}{2}} \times \\
& \times \frac{\pi m^{\lambda-1}}{2\lambda\Gamma(\lambda)} (t + \sqrt{t^2 - 1})^{-\lambda-m} \left( \frac{1-x^2} {x-t} \right)^{\frac{\lambda-1}{2}} \, dx \right| \leq \\
& \leq \left( (t^2 - 1)^{-\frac{3}{2}} \frac{t + \sqrt{t^2 - 1}}{2} \right)^{\lambda-m} \int_{-1}^{1} \left| \cos [(\lambda + m) \arccos x] \left( \frac{1-x^2} {x-t} \right)^{\frac{\lambda-1}{2}} \right| \, dx \leq
\end{align*}
\]

\(^{11} f(m) \approx g(m) \) is to be understood as \( f(m) \) behaves like \( g(m) \) as \( m \to +\infty \), i.e.

\[
\lim_{m \to +\infty} \frac{f(m)}{g(m)} = 1. \tag{C.61}
\]

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\[
(t^2 - 1)^{-\frac{3}{2}} \left( t + \sqrt{t^2 - 1} \right) - \lambda \frac{1}{2} \left( t + \sqrt{t^2 - 1} \right) - \lambda - m
\int_{-1}^{1} (1 - x^2)^{-\frac{3}{2}} \frac{1}{t - x} \, dx =\]

\[
= (t^2 - 1)^{-\frac{3}{2}} \left( t + \sqrt{t^2 - 1} \right) - \lambda \frac{1}{2} \frac{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{1}{4} \right)}{2 \Gamma \left( \frac{5}{4} \right) (t - 1)} \rightarrow 0.
\]

All limits are considered \( m \to \infty \) (with the assumption \( t > 1 \) and usage of \( \text{(C.17)} \) and \( \text{(C.16)} \)). Analogous relation can be found for the part of \( \text{(C.39)} \) involving \( C_n^{(\lambda)}(x)D_m^{(\lambda)}(t) \).

The rest

\[
\frac{\Gamma^2(\lambda)}{2^{2-2\lambda} \pi} \int_{-1}^{1} \frac{-2 \pi (t^2 - 1)^{\frac{1}{2}} - \lambda \lambda (1 - x^2)^{\lambda - \frac{1}{2}}}{4 \lambda \Gamma^2(\lambda)} \frac{1}{x - t} \, dx =\]

\[
= \frac{1}{2} (t^2 - 1)^{\frac{1}{2}} - \lambda \lambda \int_{-1}^{1} \frac{C_n^{(\lambda)}(x) (1 - x^2)^{\lambda - \frac{1}{2}}}{t - x} \, dx,
\]

forms the r.h.s. of \( \text{(C.56)} \).

\section*{C.4 Triple product}

\subsection*{C.4.1 CCC}

Let us begin with (infinite) generalization of \( \text{(C.38)} \). Assuming \( x, y, z \in (-1; +1), (1 - x^2 - y^2 - z^2)^2 - 4x^2y^2z^2 \neq 0 \) and \( \lambda > 0 \)

\[
\sum_{l=0}^{\infty} 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} \right]^2 C_l^{(\lambda)}(x)C_l^{(\lambda)}(y)C_l^{(\lambda)}(z) =\]

\[
= \frac{\pi [1 - x^2 - y^2 - z^2 + 2xyz]^{\lambda - 1}}{\Gamma(\lambda) 4^{\lambda - 1} [1 - x^2 - y^2 - z^2]^\lambda} H \left( 1 - x^2 - y^2 - z^2 + 2xyz \right),
\]

holds\footnote{\( H(x) \) stands for Heaviside step function as stated earlier.}. When \( 1 - x^2 - y^2 - z^2 + 2xyz < 0 \) the sum will vanish (as indicated by Heaviside function) regardless on possible ill–definiteness of power term. However the convergence is only conditional when \( \lambda \leq 1 \). The proof is carried out later (see page \text{94}) because there are several lemmas needed for that.

In the case of Legendre polynomials \( (\lambda = 1/2) \) \( \text{(C.64)} \) reduces to result \( \text{Baranov, 2006, (2.21)} \)

\[
\sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right) P_n(x)P_n(y)P_n(z) = \frac{H \left( 1 - x^2 - y^2 - z^2 + 2xyz \right)}{\pi \sqrt{1 - x^2 - y^2 - z^2 + 2xyz}}
\]

and when \( \lambda = 3/2 \) it simplifies to

\[
\sum_{l=0}^{\infty} \frac{2l + 3}{(l + 1)^2(l + 2)^2} C_l^{(3/2)}(x)C_l^{(3/2)}(y)C_l^{(3/2)}(z) =\]

\[
\text{(C.66)}
\]
\[ \frac{8\sqrt{1-x^2-y^2-z^2+2xyz}}{\pi(1-x^2)(1-y^2)(1-z^2)}H \left(1-x^2-y^2-z^2+2xyz\right). \]

Starting with asymptotic behaviour of l.h.s. of (C.64). By direct usage of (C.16) it can be shown

\[ 2(l + \lambda) \left\{ \frac{\Gamma(l+1)}{\Gamma(l+2\lambda)} \right\}^2 C_{i}^{(\lambda)}(\cos \theta_{1})C_{i}^{(\lambda)}(\cos \theta_{2})C_{i}^{(\lambda)}(\cos \theta_{3}) = \]

\[ \frac{16}{8^{\lambda} \Gamma^{3}(\lambda)n^{\lambda}} \prod_{n=1}^{3} \frac{\sin((\lambda+n)\theta_{i})}{\sin \theta_{i}} + O \left( \frac{1}{n} \right), \]  
(C.67)

where \( \lambda > 0 \) and \( \theta_{1}, \theta_{2}, \theta_{3} \in (0; \pi) \). Moreover asymptotic behaviour of convolution of general function with (rapidly oscillating) sine is required. Let \( q(\theta) \) denote a function with bounded first (and second when \( \epsilon \neq 0 \)) derivatives in the interval \( \langle \rho; \sigma \rangle , \epsilon \in \langle 0; 1 \rangle, \theta_{0} \in \langle \rho; \sigma \rangle \) and \( \psi \in \mathbb{R} \). Then

\[ \int_{\rho}^{\sigma} q(\theta) \frac{\sin \left( \Lambda(\theta - \psi) \right)}{\left( \theta - \theta_{0} \right)^{\epsilon}} d\theta = O \left( \frac{1}{\Lambda^{1-\epsilon}} \right) \]  
(C.68)

as \( \Lambda \to \infty \).

Proof. Let us begin with \( \epsilon = 0 \) (i.e. without the singular part). Defining

\[ Q(\theta) = \begin{cases} q(\theta) & \text{when } \theta \in \langle \rho; \sigma \rangle \\ 0 & \text{otherwise} \end{cases} \]  
(C.69)

(C.68) takes form

\[ \int_{\rho}^{\sigma} q(\theta) \sin \left( \Lambda(\theta - \psi) \right) d\theta = \int_{-\infty}^{+\infty} Q(\theta) \sin \left( \Lambda(\theta - \psi) \right) d\theta = \]

\[ = \frac{1}{2} \int_{-\infty}^{+\infty} \left[ Q(\theta) - Q \left( \theta - \frac{\pi}{\Lambda} \right) \right] \sin \left( \Lambda(\theta - \psi) \right) d\theta, \]  
(C.70)

where the fact that \( \sin(x - \pi) = -\sin x \) has been used. This integral can be split into three parts. In one of them both \( Q(\theta) \) and \( Q(\theta - \pi/\Lambda) \) vanishes. Such part clearly does not contribute.

In the second one (spanning over two intervals) only one of them is non-zero. However length of the integration intervals is \( 2\pi/\Lambda \). Because the function \( q(\theta) \) is bounded (consequence of its continuity over the interval \( \langle \rho; \sigma \rangle \)) this part of the integral vanishes like \( O \left( \Lambda^{-1} \right) \).

The last part have both \( Q(\theta) \) and \( Q(\theta - \pi/\Lambda) \) are non-zero. Using mean value theorem it can be shown

\[ \frac{1}{2} \int_{x + \pi \Lambda}^{y} \left[ Q(\theta) - Q \left( \theta - \frac{\pi}{\Lambda} \right) \right] \sin \left( \Lambda(\theta - \psi) \right) d\theta = \]
\[ \pi \int_{x}^{y} \frac{dQ}{d\theta} (\xi(\theta)) \sin (\Lambda(\theta - \psi)) \, d\theta, \] (C.71)

where \( \xi(\theta) \) is suitable function. Because the first derivative of \( q(\theta) \) is bounded and the integration interval is finite the integral itself is bounded as well. On the other hand factor in front of it vanishes like \( O(\Lambda^{-1}) \).

Each part disappears at least as fast as \( O(\Lambda^{-1}) \) so the sum vanishes similarly, QED.

When \( \epsilon \in (0; 1) \) the situation is a bit more tricky. However we can split the ratio

\[ \frac{q(\theta)}{(\theta - \theta_0)^\epsilon} = \frac{q(\theta_0)}{(\theta - \theta_0)^\epsilon} + \frac{q(\theta) - q(\theta_0)}{(\theta - \theta_0)^\epsilon} \]

where \( \xi(\theta) \) is suitable function given by mean value theorem. The term containing derivative of \( q \) is regular in the integration interval and has bounded derivative.

Using result for \( \epsilon = 0 \) proven earlier

\[ \int_{x}^{y} (\theta - \theta_0)^{1-\epsilon} \frac{dq}{d\theta} (\xi(\theta)) \sin (\Lambda(\theta - \psi)) \, d\theta = O \left( \frac{1}{\Lambda} \right) \] (C.73)

as \( \Lambda \to \infty \) takes care of second part of the split (C.72). The rest is described by

\[ \int_{x}^{y} \frac{q(\theta_0)}{(\theta - \theta_0)^\epsilon} \sin (\Lambda(\theta - \psi)) \, d\theta = q(\theta_0) \Im \left\{ e^{i\Lambda(\theta_0 - \psi)} \int_{x}^{y} \frac{y-\theta_0}{\theta-\theta_0} e^{i\Lambda \theta} \, d\theta \right\} = \]

\[ = q(\theta_0) \Im \left\{ e^{i\Lambda(\theta_0 - \psi)} \left( \frac{i}{\Lambda} \right)^{1-\epsilon} [\gamma(1 - \epsilon; -i\Lambda y) - \gamma(1 - \epsilon; -i\Lambda x)] \right\} = \]

\[ = O \left( \frac{1}{\Lambda^{1-\epsilon}} \right), \] (C.74)

where \( \gamma(a, z) = \Gamma(a) - \Gamma(a, z) \) stands for the (in)complete gamma functions. Asymptotic behaviour of the last one, according to [Olver 1974, p. 109-110]

\[ \Gamma(a, iz) = (iz)^a e^{(iz)} \left[ 1 + O \left( \frac{1}{z} \right) \right] = O \left( z^a \right), \] (C.75)

has been used. This justifies the last equality in (C.74) and so it completes the proof.

A bit more complicated relation holds for convolution of function with pole behaving like \( x^{-1} \). Once again let \( q(\theta) \) denote a function with bounded first and second derivative in the interval \( (x; y) \). Then

\[ \int_{x}^{y} \frac{q(\theta)}{\theta} \sin (\Lambda \theta) \, d\theta = \begin{cases} q(0)\pi + O \left( \frac{1}{\Lambda} \right) & \text{when } 0 \in (x; y) \\ O \left( \frac{1}{\Lambda} \right) & \text{when } 0 \not\in (x; y) \end{cases} \] (C.76)

as \( \Lambda \to \infty \).

\[ ^{13} \text{This is why bounded } \text{second \ derivative is required.} \]
Proof. If $0 \notin (x; y)$ relation (C.68) can be applied ($q(\theta)/\theta$ has bounded derivatives up to second in the integration interval, ...). So it has to vanish like $O(\Lambda^{-1})$.

In the other case, i.e. $x < 0 < y$, it is useful to separate “singular part” of the integrand

$$\int_x^y \frac{q(\theta)}{\theta} \sin(\Lambda \theta) d\theta = \int_x^y \frac{q(\theta) - q(0)}{\theta} \sin(\Lambda \theta) d\theta + \int_x^y \frac{q(0)}{\theta} \sin(\Lambda \theta) d\theta. \quad (C.77)$$

As we can see $(q(\theta) - q(0))/\theta$ is regular in the vicinity of $\theta = 0$ and it has bounded first derivative (assuming the value at zero is defined suitably). Therefore we can again use (C.68) to prove that it vanishes like $O(\Lambda^{-1})$ as $\Lambda \to \infty$. The rest is given by

$$\int_x^y \frac{q(0)}{\theta} \sin(\Lambda \theta) d\theta = \frac{\Lambda y}{\Lambda x} \int_x^y \frac{\sin(\theta)}{\theta} d\theta = q(0) [\text{Si}(\Lambda y) - \text{Si}(\Lambda x)] = q(0) \left[ \pi + O\left(\frac{1}{\Lambda}\right) \right]. \quad (C.78)$$

where $\text{Si}(a)$ denotes sine integral and its asymptotic behaviour has been used — it tends to $\pm \pi/2$ as $a \to \pm \infty$ (up to the part vanishing like $O(\Lambda^{-1})$, see [DLMF, (6.2.14)]).

Proof of (C.64). Let us begin with the question of convergence. As it is seen in (C.67) terms of the series decreases (grows) like $n - \lambda$. Therefore when $\lambda \leq 0$ the series diverges.\footnote{This forms the requirement of positive $\lambda$.} On the other hand if $\lambda > 1$ the series converges absolutely.

The situation is less clear when $\lambda \in (0; 1)$. The proof of conditional convergence will be carried out using Dirichlet’s criterion. Let

$$a_n = \frac{16}{8^\lambda \Gamma^3(\lambda) n^\lambda}, \quad (C.79)$$

$$b_n = \prod_{i=1}^3 \frac{\sin((\lambda + n)\theta_i)}{\sin \theta_i}.$$

Then $a_n$ vanishes monotonously as $n^{-\lambda}$ when $n \to \infty$. Moreover $\sum_{n=1}^N b_n$ is bounded.\footnote{Because

$$\prod_{i=1}^3 \frac{\sin((\lambda + n)\theta_i)}{\sin \theta_i} = \frac{\sin((\lambda + n)(-\theta_1 + \theta_2 + \theta_3)) + \sin((\lambda + n)(\theta_1 - \theta_2 + \theta_3))}{4 \sin \theta_1 \sin \theta_2 \sin \theta_3}$$

$$+ \frac{\sin((\lambda + n)(\theta_1 + \theta_2 - \theta_3)) + \sin((\lambda + n)(\theta_1 + \theta_2 + \theta_3))}{4 \sin \theta_1 \sin \theta_2 \sin \theta_3} \quad (C.80)$$

and

$$\left| \sum_{n=1}^N \sin((\lambda + n)\Theta) \right| = \left| \sum_{n=1}^N e^{i(\lambda+n)\Theta} \right| =$$

$\int_{\infty}^{\infty}$.
The r.h.s. of (C.67) contains also remainder disappearing at least as fast as $n^{-1-\lambda}$. This remainder is clearly absolutely convergent and so it does not change behaviour of the series as $n \to \infty$.

Denoting $y = \cos \theta_1$ and $z = \cos \theta_2$ and using (C.55) it can be shown that

$$
\sum_{l=0}^{\infty} 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} \right]^2 C^{(\lambda)}_l(x) C^{(\lambda)}_l(\cos \theta_1) C^{(\lambda)}_l(\cos \theta_2) = \quad (C.82)
$$

$$
= \lim_{N \to \infty} \sum_{l=0}^{N} 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} \right]^2 C^{(\lambda)}_l(x) C^{(\lambda)}_l(\cos \theta_1) C^{(\lambda)}_l(\cos \theta_2) =
$$

$$
= \lim_{N \to \infty} \frac{2}{2^{2\lambda}\Gamma^2(\lambda)} \int_0^\pi (\sin \phi)^{2\lambda-1} \left\{ \sum_{l=0}^{N} 2(l + \lambda) \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} C^{(\lambda)}_l(x) C^{(\lambda)}_l(\xi) \right\} \, d\phi =
$$

$$
= \lim_{N \to \infty} \frac{2}{2^{2\lambda}\Gamma^2(\lambda)} \frac{\Gamma(N + 2)}{\Gamma(N + 2\lambda)} \int_0^\pi \frac{\cos(\theta_1 - \theta_2)}{[\sin(\theta_1 \sin \theta_2)]^{2\lambda-1}} \frac{[(\cos(\theta_1 - \theta_2) - \xi)(\xi - \cos(\theta_1 + \theta_2))]^{\lambda-1}}{x - \xi} \times \frac{C^{(\lambda)}_{n+1}(x) C^{(\lambda)}_N(\xi) - C^{(\lambda)}_N(\xi) C^{(\lambda)}_{N+1}(\xi)}{x - \xi} \, d\xi,
$$

where $\xi = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi$.

Let us make a few notes concerning regularity of the integrated term. The denominator of the last fraction vanishes when $\xi = x$. However there is no singularity. The nominator $C^{(\lambda)}_{n+1}(x) C^{(\lambda)}_N(\xi) - C^{(\lambda)}_N(\xi) C^{(\lambda)}_{N+1}(\xi)$ vanishes there as well. Moreover because $N \in \mathbb{Z}_0^+$ the nominator is polynomial it can be written in the form $(x - \xi) Poly(x, \xi)$, where $Poly(x, \xi)$ stands for some other (suitable) polynomial of $x$ and $\xi$.

Another potential threat arises from $[[(\cos(\theta_1 - \theta_2) - \xi)(\xi - \cos(\theta_1 + \theta_2))]^{\lambda-1}$. With $\lambda < 1$ the term is singular at both ends of the integration interval\cite{Footnote}. However such singularities are of the type $\int_0^x x^{-\epsilon} dx$ with $\epsilon \in (0; 1)$ and so their integral is finite.

To proceed further examination of asymptotic behaviour of $C^{(\lambda)}_{n+1}(x) C^{(\lambda)}_N(\xi)$ -

$$
= \begin{cases} 0 & \text{when } \Theta = 2k\pi \land \lambda = k' + \frac{1}{2} \\
\pm \infty & \text{when } \Theta = 2k\pi \land \lambda \neq k' + \frac{1}{2} \\
\left\| \begin{pmatrix} \cos(\Theta) \\
\sin(\Theta) \\
\end{pmatrix} \right\| \leq \frac{2}{1 - \cos \Theta} & \text{otherwise}
\end{cases}
$$

The condition of convergence read $\theta_1 \pm \theta_2 \pm \theta_3 \neq k\pi$ for any choice of signs. Equivalently it can be given by

$$
0 \neq \sin(\theta_1 + \theta_2 + \theta_3) \sin(-\theta_1 + \theta_2 + \theta_3) \sin(\theta_1 - \theta_2 + \theta_3) \sin(\theta_1 + \theta_2 - \theta_3) = (1 - \cos^2 \theta_1 - \cos^2 \theta_2 - \cos^2 \theta_3)^2 - 4 \cos^2 \theta_1 \cos^2 \theta_2 \cos^2 \theta_3. \quad (C.81)
$$

\footnote{16} It should be noted the power is well defined because the bracket is not negative inside the integration interval.
\[ C_N^{(\lambda)}(\xi)C_{N+1}^{(\lambda)}(\xi) \] as \( N \to \infty \) is necessary. Using (C.10) yields

\[
C_n^{(\lambda)}(x)C_N^{(\lambda)}(\xi) - C_N^{(\lambda)}(\xi)C_{N+1}^{(\lambda)}(x) = R_N(\psi, \vartheta) + \hspace{1cm} (C.83)
\]

\[
\frac{N^{2\lambda - 2}}{2^{2\lambda - 2} \Gamma^2(\lambda) \sin^2 \psi \sin^2 \vartheta} \sin \left( \left( \lambda + N + \frac{1}{2} \right) (\psi + \vartheta) \right) \sin \left( \frac{\vartheta - \psi}{2} \right) + \hspace{1cm}
\]

\[
\sin \left( \left( \lambda + N + \frac{1}{2} \right) (\vartheta - \psi) \right) \sin \left( \frac{\psi + \vartheta}{2} \right),
\]

where \( R_N(\psi, \vartheta) = O\left(N^{2\lambda - 3}\right) \) and \( \psi, \vartheta \in (0; \pi) \) are defined by \( x = \cos \psi, \xi = \cos \vartheta \). The l.h.s. of equation (C.83) vanishes at least as fast as \( x - \xi \) when \( x \to \xi \). The r.h.s., without the remainder \( R_N(\psi, \vartheta) \), behaves similarly. Therefore remainder \( R_N(\psi, \vartheta) \) must vanish there as well.

Inserting this into (C.82) leads to (omitting the limit and WLOG assuming \( \theta_1 \geq \theta_2 \))

\[
\sum_{l=0}^{N} 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} \right]^2 C_l^{(\lambda)}(x)C_l^{(\lambda)}(\cos \theta_2)C_l^{(\lambda)}(\cos \theta_2) = \hspace{1cm} (C.84)
\]

\[
= \frac{2\Gamma(N + 2)}{2^{2\lambda} \Gamma^2(\lambda) \Gamma(N + 2\lambda)} \int \frac{[(\cos(\theta_1 - \theta_2) - \xi)(\xi - \cos(\theta_1 + \theta_2))]^{\lambda - 1}}{\sin \theta_1 \sin \theta_2} \frac{R_N(\psi, \vartheta)}{x - \xi} d\xi + \hspace{1cm}
\]

\[
+ \frac{8N^{2\lambda - 2}}{2^{2\lambda} \Gamma^2(\lambda) \Gamma(N + 2\lambda)} \int_{\theta_1 - \theta_2}^{\theta_1 + \theta_2} \frac{[(\cos(\theta_1 - \theta_2) - \cos \vartheta)(\cos \vartheta - \cos(\theta_1 + \theta_2))]^{\lambda - 1}}{\sin \theta_1 \sin \theta_2} \hspace{1cm}
\]

\[
\times \sin \left( \left( \lambda + N + \frac{1}{2} \right) (\psi + \vartheta) \right) \sin \left( \frac{\vartheta - \psi}{2} \right) + \sin \left( \left( \lambda + N + \frac{1}{2} \right) (\vartheta - \psi) \right) \sin \left( \frac{\psi + \vartheta}{2} \right) d\vartheta = \hspace{1cm}
\]

\[
= \frac{4^{1 - 2\lambda}}{\Gamma^4(\lambda) \sin \theta_1 \sin \theta_2} \frac{\sin^2 \lambda \psi}{\sin \theta_1 \sin \theta_2} \times \hspace{1cm}
\]

\[
\times \int_{\theta_1 - \theta_2}^{\theta_1 + \theta_2} T(\vartheta) \left\{ \frac{\sin [\Lambda(\psi + \vartheta)]}{\sin \left( \frac{\psi + \vartheta}{2} \right)} + \frac{\sin [\Lambda(\vartheta - \psi)]}{\sin \left( \frac{\vartheta - \psi}{2} \right)} \right\} d\vartheta + O \left( \frac{1}{N} \right),
\]

where \( \Lambda = n + \lambda + \frac{1}{2} \),

\[
T(\vartheta) = \left[ \frac{\sin \left( \left( \lambda + N + \frac{1}{2} \right) (\vartheta - \psi) \right)}{\sin \vartheta} \right]^{\lambda - 1}
\]

and the upper limit of the integral assumes\(^{17}\) \( \theta_1 + \theta_2 \leq \pi \). Moreover we have used the fact, that the integral containing remainder \( R_N(\psi, \vartheta) \) is regular (see previous paragraph) and it vanishes like \( O\left(N^{-1}\right) \) when considered with rest of the integrand.

The expression \( T(\vartheta) \) is potentially singular. When \( \lambda > 1 \) the problem causes \( \sin \vartheta \) in the denominator. This term vanishes only when \( \vartheta = 0 \) or \( \vartheta = \pi \). Based on bounds of the integration \( \vartheta \) can have such value only if \( \theta_1 = \theta_2 \) or \( \theta_1 + \theta_2 = \pi \)

\(^{17}\)If \( \theta_1 + \theta_2 > \pi \) then it should be replaced by \( 2\pi - \theta_1 - \theta_2 \) in the upper integration limit and the rest of the proof has to be modified accordingly. Because nothing new is necessary for this modification explicit discussion is not given here.
ensuring regularity of \( T(\vartheta) \). In the other case, \( \lambda \in (0; 1) \), the divergence is of the order \( x^{-t} \) and its discussion was performed above — its integral is finite.

Because \( T(\vartheta) \) diverges at most like \( x^{1-\lambda} \) and \( \sin \left( \frac{\vartheta + \psi}{2} \right) \) is regular\(^{18}\), \( (C.68) \) can be used to prove

\[
\int_{\vartheta_1 - \vartheta_2}^{\vartheta_1 + \vartheta_2} T(\vartheta) \frac{\sin [\Lambda(\psi + \vartheta)]}{\sin \left( \frac{\vartheta + \psi}{2} \right)} d\vartheta = O \left( \Lambda^{-\min(1, \lambda)} \right) \tag{C.86}
\]

The second part of integral (containing \( \vartheta - \psi \)) is a bit more complicated due to the (possible) existence of singularity when \( \psi = \vartheta \). First let us isolate that singularity

\[
\frac{1}{\sin \left( \frac{\vartheta - \psi}{2} \right)} = \frac{2}{\vartheta - \psi} + f(\vartheta, \psi), \tag{C.87}
\]

where \( f(\vartheta, \psi) \) is suitable function that does not tend to infinity when \( \vartheta \to \psi \). Therefore

\[
\int_{\vartheta_1 - \vartheta_2}^{\vartheta_1 + \vartheta_2} T(\vartheta) \frac{\sin [\Lambda(\vartheta - \psi)]}{\sin \left( \frac{\vartheta - \psi}{2} \right)} d\vartheta = \int_{\vartheta_1 - \vartheta_2}^{\vartheta_1 + \vartheta_2} 2T(\vartheta) \frac{\sin [\Lambda(\vartheta - \psi)]}{\vartheta - \psi} d\vartheta + \int_{\vartheta_1 - \vartheta_2}^{\vartheta_1 + \vartheta_2} T(\vartheta) f(\vartheta, \psi) \sin [\Lambda(\vartheta - \psi)] d\vartheta =
\]

\[
= \begin{cases} 
2\pi T(\psi) + O \left( \Lambda^{-\min(1, \lambda)} \right) & \text{when } \psi \in (\theta_1 - \theta_2; \theta_1 + \theta_2) \\
O \left( \Lambda^{-\min(1, \lambda)} \right) & \text{when } \psi \notin (\theta_1 - \theta_2; \theta_1 + \theta_2) 
\end{cases}, \tag{C.88}
\]

where we have used \( (C.68) \) to prove that the second integral vanishes and \( (C.76) \) to find result of the first integral (in the limit \( N, \Lambda \to +\infty \)).

All together forms relation

\[
\sum_{l=0}^{n} 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} \right]^2 C_n^{(\lambda)} \cos \psi C_n^{(\lambda)} \cos \theta_1 C_n^{(\lambda)} \cos \theta_2 = \tag{C.89}
\]

\[
= \pi \frac{\sin \left( \frac{\theta_1 + \theta_2 + \psi}{2} \right) \sin \left( \frac{\theta_1 - \theta_2 + \psi}{2} \right) \sin \left( \frac{\theta_1 + \theta_2 - \psi}{2} \right)}{\Gamma^4(\lambda) [2 \sin \theta_1 \sin \theta_2 \sin \psi]^{2\lambda - 1}} \times
\]

\[
\times H \left[ \theta_2^2 - (\psi - \theta_1)^2 \right] + O \left( \frac{1}{\eta^{\min(1, \lambda)}} \right).
\]

In case of Legendre functions (\( \lambda = \frac{1}{2} \)) the result correspond exactly to \cite{Baranov2006} (2.21). However at the first glance the result is not symmetric with respect to the \( x, \cos \theta_1, \cos \theta_2 \) as it should be. Also we have assumed \( \theta_1 + \theta_2 \leq \pi \). Careful analysis reveals the argument of the Heaviside function can be replaced with the content of the square bracket with sines and it whichs symmetry is evident. Moreover this symmetric form need not to assume \( \theta_1 + \theta_2 \leq \pi \) (the proof is analogous) and therefore the final result, after using limit \( n \to \infty \), states

\[
\sum_{l=0}^{\infty} 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} \right]^2 C_n^{(\lambda)}(x) C_n^{(\lambda)}(y) C_n^{(\lambda)}(z) = \tag{C.90}
\]

\(^{18}\psi \in (0; \pi) \) and \( \vartheta \in (0; \pi) \).
\[
\frac{\pi[1 - x^2 - y^2 - z^2 + 2xyz]^\lambda}{\Gamma^4(4\lambda - 1) \left[(1 - x^2)(1 - y^2)(1 - z^2)\right]^\lambda} H \left(1 - x^2 - y^2 - z^2 + 2xyz\right),
\]
where, as mentioned above, \(x, y, z \in (-1; 1)\). The convergence is as fast as \(O\left(N^{-\min(1, \lambda)}\right)\).

\[\Box\]

**C.4.2 DCC**

Triple product of Gegenbauer functions (DCC) reads

\[
\sum_{l=0}^{\infty} 2(l + \lambda) \left[\frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)}\right]^2 D_l(\lambda)(x)C_l(\lambda)(y)C_l(\lambda)(z) = \frac{\pi}{2^{2\lambda}} \binom{\lambda}{\lambda} \binom{\lambda}{\lambda} \frac{2F_1\left(\frac{1}{2}, 1; \frac{1}{2}; \frac{1}{2}; (x - yz)^2\right)}{(x - yz)^2},
\]
where \(2F_1\left(a, b; c; x\right)\) stands for the Gauss hypergeometric function, \(x > 1 + \epsilon\), \(y, z \in (-1; 1)\) and \(\lambda \in (0; +\infty)\) and the convergence is absolute and uniform.

One can also check the formula with \(\lambda = \frac{1}{2}\) corresponds to the result obtained by Baranov [2006]. Specific cases of (C.91) that are main focus are given by

\[
\sum_{l=0}^{\infty} (2l + 1)Q_l(x)P_l(y)P_l(z) = \frac{1}{\sqrt{x^2 + y^2 + z^2 - 2xyz - 1}},
\]

and

\[
\sum_{l=0}^{\infty} \frac{2l + 3}{(l + 1)^2(l + 2)^2} D_l^{(3/2)}(x)C_l^{(3/2)}(y)C_l^{(3/2)}(z) = \frac{x - yz - \sqrt{x^2 + y^2 + z^2 - 2xyz - 1}}{(x^2 - 1)(y^2 - 1)(z^2 - 1)},
\]

During proof of (C.91) integral

\[
\int_0^\pi \frac{(\sin \phi)^{2\lambda - 1}}{a - b \cos \phi} d\phi = B \left(\frac{\lambda}{2}, a^2\right) 2F_1\left(\frac{1}{2}, 1; \frac{1}{2}; \frac{b^2}{a^2}\right),
\]

where \(B(x, y)\) denotes beta function is encountered. Such relation holds when \(|b| > |a| > 0\).

**Proof.**

\[
\int_0^\pi \frac{(\sin \phi)^{2\lambda - 1}}{a - b \cos \phi} d\phi = \frac{1}{a} \int_0^\pi \sum_{l=0}^{\infty} \left(\frac{b}{a}\right)^l (\sin \phi)^{2\lambda - 1}(\cos \phi)^l d\phi =
\]

\[
= \frac{1}{a} \sum_{l=0}^{\infty} \left(\frac{b}{a}\right)^l \int_0^\pi (\sin \phi)^{2\lambda - 1}(\cos \phi)^l d\phi =
\]

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\[ \sum_{l=0}^{\infty} \left( \frac{b}{a} \right)^{2l} \left( \sin \phi \right)^{2l-1} \left( \cos \phi \right)^{2l} d\phi = \]
\[ \sum_{l=0}^{\infty} \left( \frac{b}{a} \right)^{2l} B\left( \lambda, l + \frac{1}{2} \right) = \frac{B\left( \lambda, \frac{1}{2} \right)}{a} \, _2F_1\left( \frac{1}{2}, 1; \frac{b^2}{a^2} \right), \]
where the fact that series decreases exponentially fast allows to use Fubini’s theorem.

**Proof of (C.91).** Starting with convergence \cite{DLMF} (18.10.4) can be used to prove \(|C_n^{(\lambda)}(\xi)| < X_\lambda|C_n^{(\lambda)}(1)|\) for suitable constants \(X_\lambda\). Assuming \(l\) large enough
\[ 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} \right]^2 D_i^{(\lambda)}(x)C_i^{(\lambda)}(y)C_i^{(\lambda)}(z) \]
\[ < \left( x^2 - 1 \right)^{-\frac{\lambda}{2}} \frac{\pi X_\lambda^2}{2^{\lambda-3} \Gamma(\lambda) \Gamma(2\lambda)} t^\lambda \left( x + \sqrt{x^2 - 1} \right)^{-\lambda - l}. \]

The last (exponential) part dominate behaviour of the series and so it the original sum \((C.91)\) is absolutely convergent. Moreover \(x^2 - 1 > 2\epsilon\) and \(x + \sqrt{x^2 - 1} > 1 + \epsilon\). Therefore (assuming \(1 < p \leq q\) and large enough
\[ \sum_{l=p}^{q} 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} \right]^2 D_i^{(\lambda)}(x)C_i^{(\lambda)}(y)C_i^{(\lambda)}(z) \]
\[ < \frac{16 \pi X_\lambda^2}{2^{\frac{\lambda}{2}} \Gamma(\lambda) \Gamma(2\lambda) e^{\frac{\lambda}{2}} (1 + \epsilon)^{\lambda - 1}} p^\lambda \left( 1 - \frac{\epsilon}{1(1 + \epsilon)} \right)^p, \]
which clearly vanishes as \(p \to \infty\). Also it does not depend on \(x, y\) and \(z\). Cauchy criterion proves uniform convergence of the series.

Again relation \((C.55)\) was employed to convert single Gegenbauer polynomial in \((C.39)\) into product of two of them.
\[ \sum_{l=0}^{n} 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} \right]^2 D_i^{(\lambda)}(x)C_i^{(\lambda)}(\cos \theta_1)C_i^{(\lambda)}(\cos \theta_2) = \]
\[ = \frac{2}{2^{2\lambda} \Gamma^2(\lambda)} \int_0^\pi \sin^{2\lambda - 1} \sum_{l=0}^{n} 2(l + \lambda) \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} D_i^{(\lambda)}(x)C_i^{(\lambda)}(\xi) d\phi = \]
\[ = \frac{2}{2^{2\lambda} \Gamma^2(\lambda)} \int_0^\pi \sin^{2\lambda - 1} \frac{\Gamma(n + 2)}{\Gamma(n + 2\lambda)} \left[ C_n^{(\lambda)}(\xi) D_i^{(\lambda)}(x) - \right. \]
\[ - \left. C_n^{(\lambda)}(\xi) D_n^{(\lambda)}(x) \right] d\phi = \]
\[ = \frac{2}{2^{2\lambda} \Gamma^2(\lambda)} \cos(\theta_1 + \theta_2) \int_0^{\cos(\theta_1 + \theta_2)} \frac{\Gamma(n + 2)}{\sin \theta_1 \sin \theta_2} \left[ (\cos(\theta_1 - \theta_2) - \xi)(\xi - \cos(\theta_1 + \theta_2))^\lambda - 1 \right] \times \]
\[ \left[ (\cos(\theta_1 - \theta_2) - \xi)(\xi - \cos(\theta_1 + \theta_2))^\lambda - 1 \right] d\phi = \]
\[ \frac{4}{2^{2\lambda} \Gamma^2(\lambda)} \Gamma(n + 2) \times \]
\[ \frac{\Gamma(n + 2)}{\sin \theta_1 \sin \theta_2} 2^{2\lambda - 1} (\xi - x) \]

\(^{19}l > \lambda\) and the remainder in \((C.17)\) \(|O\left( \frac{1}{\lambda} \right)| < 1.\)
\(^{20}\)That is that assumptions of \((C.96)\) holds and \(p > \frac{4}{(1 + \epsilon/2)^{\lambda - 1}}, \) i.e. \(\left( \frac{1 + \epsilon}{2} \right)^\lambda < 1 + \frac{\epsilon}{2}.\)
\[ \left[ c_{n+1}^{(\lambda)}(\xi)D_{n}^{(\lambda)}(x) - c_{n}^{(\lambda)}(\xi)D_{n+1}^{(\lambda)}(x) \right] - \frac{2\pi |x^2 - 1|^{1-2\lambda}}{4\lambda \Gamma^2(\lambda)} \right\} d\xi, \]

where \( \xi = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi \) (as in the previous proof). The integral can be separated into two parts — one containing cross product of Gegenbauer functions of the first and second kind\textsuperscript{21} and the one independent on \( n \).

Now for the evaluation of the limit itself. Because we are interested in the limit \( n \to +\infty \) the first part of the integral vanish (exponential decrease \( D_{n}^{(\lambda)}(x) \) dominate its behaviour). Therefore

\[ \sum_{l=0}^{\infty} 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} \right]^2 D_{l}^{(\lambda)}(x)C_{l}^{(\lambda)}(\cos \theta_1)C_{l}^{(\lambda)}(\cos \theta_2) = \]

\[ = \frac{\pi |x^2 - 1|^{1-2\lambda}}{2^{2\lambda-1}2\Gamma^2(\lambda)} \int_{0}^{\pi} \frac{(\sin \phi)^{2\lambda-1}}{x - \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos \phi} d\phi. = \]

\[ = \frac{\pi (x^2 - 1)^{1-2\lambda}}{2^{2\lambda-1}2\Gamma^2(\lambda)\Gamma(2\lambda)(x - yz)^2} F_1 \left( \frac{1}{2} \frac{1}{2} \frac{1 - y^2 - z^2 + y^2z^2}{(x - yz)^2} \right), \]

where \([C.94]\) has been used.

\[ \square \]

It could be useful to note that the restriction posed on arguments of \([C.91]\) can be loosened — it holds as long as \( y, z \in (-1; +\infty) \) and \( a > \max(y, 1) \max(z, 1) + \sqrt{\max(y, 1)^2 - 1} (\max(z, 1)^2 - 1) \). As \( n \to \infty \) the function summed in \([C.91]\) vanishes exponentially. Due to that is can be easily seen that partial sums \( (\sum_{n=1}^{l} \text{with arbitrary } l) \) are uniformly convergent (as a function of \( y \) and \( z \)). This holds even when \( y \) and \( z \) are complex numbers where above criterion holds for their real part and their imaginary part is sufficiently small (i.e. in the vicinity of real axis). Also partial sums are analytic functions of \( y \) and \( z \) (partial sums are just polynomials). Therefore the series is convergent and the result is analytic function of (complex) variables \( y \) and \( z \). Uniqueness theorem for functions of complex variables allows us to loosen the requirement \( y, z \in (-1; 1) \).

**C.4.3 DDC, DDD**

The other two kinds of series containing triple product of Gegenbauer functions, i.e.

\[ \sum_{l=0}^{\infty} 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} \right]^2 D_{l}^{(\lambda)}(x)D_{l}^{(\lambda)}(y)C_{l}^{(\lambda)}(z) \quad ([C.99]) \]

and

\[ \sum_{l=0}^{\infty} 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2\lambda)} \right]^2 D_{l}^{(\lambda)}(x)D_{l}^{(\lambda)}(y)D_{l}^{(\lambda)}(z) \quad ([C.100]) \]

can be obtained using \([C.56]\) on \([C.91]\) once (twice). No additional problem arises from the exchange of order of sum and integral, because the series convergence is uniform\textsuperscript{22} Because of straightforwardness of the procedure and not using them

\[ 21(\xi - x) \text{ in the denominator does not introduce singularity. It can be shown analogously to the proof of regularity of the integrand in proof of } ([C.64]). \]

\[ 22 \text{This is in contrast with } ([C.64]). \]
in the thesis explicit relations is not given. Reader can finish them (including proof) if necessary.

C.5 Quadruple product

C.5.1 CCCC
Assume \( x, y, z, t \in (-1; +1) \). Then
\[
\sum_{l=0}^{\infty} 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2 \lambda)} \right]^3 C_i^{(\lambda)}(x)C_i^{(\lambda)}(y)C_i^{(\lambda)}(z)C_i^{(\lambda)}(t) = \quad \text{(C.101)}
\]

\[
= \frac{16 \pi}{2^{6\lambda} \Gamma(4(\lambda)) \Gamma(2\lambda)} \left[ (1 - x^2)(1 - y^2)(1 - z^2)(1 - t^2) \right]^{\lambda - \frac{1}{2}} \times
\]

\[
\times \text{F}_D^{(4)} \left( \lambda; 1 - \lambda, 1 - \lambda, 1 - \lambda, 1 - \lambda; \lambda - \frac{1}{2}, \lambda - \frac{1}{2}; 2\lambda; \\
M_+ - M_-, M_+ - M_-, M_+ - M_-, M_+ - M_-; 1 + M_+, 1 + M_+ \right),
\]

where \( \text{F}_D^{(4)} (a; b_1, b_2, b_3, b_4; c; x_1, x_2, x_3, x_4) \) denotes Lauricella function of four variables. \( M_+ \), \( M_- \) stands for
\[
M_+ = \max \left\{ x y \pm \sqrt{(1 - x^2)(1 - y^2)}, t \pm \sqrt{(1 - z^2)(1 - t^2)} \right\}, \quad \text{(C.102)}
\]
\[
M_- = \min \left\{ x y \pm \sqrt{(1 - x^2)(1 - y^2)}, t \pm \sqrt{(1 - z^2)(1 - t^2)} \right\} \quad \text{(C.103)}
\]

and its convergence is absolute when \( \lambda > 1/2 \). If \( \lambda \in (0; 1/2) \) the convergence is conditional (Dirichlet’s criterion is employed) and additional condition(s) \( \arccos x \pm \arccos y \pm \arccos z \pm \arccos t \neq k\pi \) (for all combinations of signs) must hold.

Proof of this theorem is not given here. It follows closely the proof of \( \text{(C.64)} \), only using \( \text{(C.55)} \) twice.

C.5.2 DCCC
Let \( a > 1, x, y, z \in (-1; 1) \) and
\[
\sum_{l=0}^{\infty} 2(l + \lambda) \left[ \frac{\Gamma(l + 1)}{\Gamma(l + 2 \lambda)} \right]^3 D_i^{(\lambda)}(a)C_i^{(\lambda)}(x)C_i^{(\lambda)}(y)C_i^{(\lambda)}(z) = \frac{\pi(a^2 - 1)^{\frac{1}{2} - \lambda}}{2^{4\lambda - 2}\Gamma(4(\lambda)) \Gamma(2\lambda)}.
\]

\[
\text{F}_D^{(4)} (a; b_1, b_2, b_3, b_4; c; x_1, x_2, x_3, x_4) = \sum_{i_1, i_2, i_3, i_4=0}^{\infty} \frac{(a)_{i_1+i_2+i_3+i_4}(b_1)_{i_1}(b_2)_{i_2}(b_3)_{i_3}(b_4)_{i_4}(c)_{i_1+i_2+i_3+i_4}}{(c_1)_{i_1}(c_2)_{i_2}(c_3)_{i_3}(c_4)_{i_4}} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4},
\]

where \( (a)_n = \Gamma(a + n)/\Gamma(a) \) denotes Pochhammer Symbol. In this appendix integral relation
\[
\text{F}_D^{(4)} (a; b_1, b_2, b_3, b_4; c; x_1, x_2, x_3, x_4) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}dt}{(1-x_1t)^{b_1}(1-x_2t)^{b_2}(1-x_3t)^{b_3}(1-x_4t)^{b_4}}
\]

have significant role.
\[
\frac{1}{[(1 - y^2)(1 - z^2)]^{\lambda - \frac{1}{2}}} \int_{yz - \sqrt{(1 - y^2)(1 - z^2)}}^{yz + \sqrt{(1 - y^2)(1 - z^2)}} \times \\
\times \, _2F_1 \left( \frac{1}{2}, 1; \frac{1}{\lambda} + \frac{1}{2}; \frac{(1 - x^2)(1 - \xi^2)}{(a - x \xi)^2} \right) \left( 1 - y^2 - z^2 - \xi^2 + 2yz\xi \right)^{\lambda - 1} \frac{a - x\xi}{a - x^2} \, d\xi,
\]

(C.104)

and the convergence is absolute (once again exponential decrease of \( D_I^{(\lambda)}(a) \) dominates behaviour).

To prove this relation it is sufficient to apply relation (C.55) on (C.91) and uniform convergence of triple product.\(^{24}\)

Special case of Legendre functions (\( \lambda = \frac{1}{2} \)) was already considered in Baranov\(^{2006} \) (including result in closed from instead of integral (C.104)). It reads

\[
\sum_{l=0}^{\infty} (2l + 1)Q_l(a)P_l(x)P_l(y)P_l(z) =
\]

\[
= \int_{yz - \sqrt{(1 - y^2)(1 - z^2)}}^{yz + \sqrt{(1 - y^2)(1 - z^2)}} \frac{\pi \sqrt{a^2 + x^2 + \xi^2 - 1 - 2ax\xi \sqrt{1 - y^2 - z^2 - \xi^2 + 2yz\xi}}}{\sqrt{1 - y^2 - z^2 - \xi^2 + 2yz\xi}} \, d\xi.
\]

(C.105)

and for \( \lambda = \frac{3}{2} \) (C.104) reduces to

\[
\sum_{l=0}^{\infty} \frac{2l + 3}{(l + 1)^3(l + 2)} \frac{D_I^{(3/2)}(a)C_I^{(3/2)}(x)C_I^{(3/2)}(y)C_I^{(3/2)}(z)}{\sqrt{1 - y^2 - z^2 - \xi^2 + 2yz\xi}} \times
\]

\[
= \int_{yz - \sqrt{(1 - y^2)(1 - z^2)}}^{yz + \sqrt{(1 - y^2)(1 - z^2)}} \sqrt{1 - y^2 - z^2 - \xi^2 + 2yz\xi} \times
\]

\[
\times \frac{[a - x\xi + \sqrt{a^2 + x^2 + \xi^2 - 1 - 2ax\xi}]}{\pi(a^2 - 1)(1 - x^2)(1 - y^2)(1 - z^2)(1 - \xi^2)} \, d\xi.
\]

(C.106)

The requirements for (C.104) can be loosened. Analogously to the discussion performed for (C.91) the series converges as long as

\[
a + \sqrt{a^2 - 1} > \max(x + \sqrt{x^2 - 1}, 1) \max(y + \sqrt{y^2 - 1}, 1) \max(z + \sqrt{z^2 - 1}, 1).
\]

(C.107)

Similarly (C.104) holds as long as the criterion above holds.\(^{25}\)

C.5.3 DDCC, DDDC, DDDD

Other series including four Gegenbauer functions can be solved using the same approach given by section C.4.3. Applying (multiple times) (C.56) on (C.104) and considering exponential convergence (to use Foubini’s theorem) lead to the desired result. Once again due to the straightforwardness of the procedure and that these sums are not used in this thesis these integrals are nor given here explicitly.

\(^{24}\)In the case of quadruple product uniform convergence is present as well.

\(^{25}\)The argumentation to prove that is analogous as well.
Bibliography


Attachments

Papers created during my doctoral studies relevant to this thesis are attached below. [Čížek and Semerák 2012] is given as it was submitted (and accepted) only due to technical problems with obtaining published version.
Thin-disc Perturbation of a Schwarzschild Black Hole

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Abstract. The perturbation scheme used by Will (1974) is revisited in order to find the metric for an (originally) Schwarzschild black hole surrounded by a light and slowly rotating ring. Focusing on linear order, we show that the method might also be adapted to the case of a thin disc. However, the expansions included are not much suitable for numerical processing due to a rather bad convergence.

Introduction

Disc-like structures around very compact bodies play a key role in such highly active astrophysical sources like active galactic nuclei, X-ray binaries, supernovas and gamma-ray bursts. Very compact objects generate very strong and inhomogeneous gravitational fields, so general relativity has to be used in their description. The matter (ring or disc) around should also be included in this description, because its gravitational effect may not always be negligible, and because the properties of the accretion system can be very sensitive to the precise shape of the field. However, general relativity is non-linear and the fields of multiple sources cannot be obtained by a simple superposition. Recently, such more complicated fields are being successfully treated numerically (this even applies to strongly time-dependent cases including collisions and gravitational collapse), but, for the present, the compass of explicit “analytical” solution terminates at systems with very high degree of symmetry, factually at static and axially symmetric cases. It would be most desirable to extend this compass to stationary cases, namely those admitting rotation. The book by Klein & Richter (2005) summarises the exact-solution prospects in this direction.

If the gravity of external matter is weak, the problem may be tackled as a small (stationary and axisymmetric) perturbation of the central-source metric, determined by linearised Einstein equations. The method can be iterated; in a limit case (many iterations), it goes over to a solution in terms of series. The result then need not any longer represent a tiny variation of some “almost right” metric — it may even be put together on Minkowski, with the “strong” part (e.g. a black hole) “dissolved” within the fundamental systems. The main problem here is convergence and meaning of the series.

It is for 35 years now that the paper by Will (1974) remains the major reference in the subject. It provided the field of a slowly rotating and light thin equatorial ring around a black hole by mass- and rotational perturbation of the Schwarzschild metric. Unfortunately, the perturbation strategy is strongly dependent on how “nice” the background metric is — and the Schwarzschild metric is exceptional in this respect: the Will’s procedure can neither be simply followed for the Kerr background, nor for a more complicated static one. Thus only partial questions have been answered explicitly in this direction, in particular the one of deformation of the black-hole horizon (Demianski 1976; Chrzanowski 1976). (Interestingly, these results do not agree on certain points, mainly in the limit of an extreme black hole.)

In the present note, we check whether the Will’s scheme can be adapted to the case of a (finite, annular) thin disc. Although suggesting a positive answer, we also indicate that the expansions that come out in this approach converge rather badly and their numerical processing is problematic.

Einstein equations and boundary conditions for the hole–disc system

We consider asymptotically flat space-times (without the cosmological term) that are stationary, axially symmetric and circular (orthogonally transitive), i.e. where a time-like and a space-like Killing vector fields \( \eta^\mu \equiv \frac{\partial}{\partial t} \), \( \xi^\mu \equiv \frac{\partial}{\partial \phi} \) exist and commute, and where the tangent planes to meridional directions (orthogonal to both Killing vectors) are integrable. In the isotropic-type spheroidal coordinates the metric with these properties can e.g. be written in the “Bardeen-Carter” form\(^1\) (e.g. Bardeen 1973)

\[
ds^2 = -e^{2\nu}dt^2 + r^2B^2e^{-2\nu}\sin^2\theta(d\phi - \omega dt)^2 + e^{2\zeta-2\nu}(dr^2 + r^2d\theta^2),
\]

where the unknown functions \( \nu, B, \omega, \zeta \) only depend on meridional coordinates \( r \) and \( \theta \).

\(^1\)Metric signature is \((-++++)\) and geometrised units are used in which \( c = G = 1 \), Greek indices run 0-3 and partial derivative is denoted by a comma.
Apart from the asymptotic flatness, the boundary conditions include those on the symmetry axis, on the black-hole horizon and on the external-source surface. Regularity of the axis requires that \( e^\xi \to B \) there. Should the circumferential radius \( \sqrt{g_{\phi\phi}} \) grow linearly with proper cylindrical radius \( r \) (thus with \( \rho = r \sin \theta \)), there must be \( g_{\phi\phi} \propto O(\rho^2) \), and from finiteness of \( \omega \) one also has \( -g_{\phi\phi} = \frac{8}{3} \rho^2 \omega = O(\rho^2) \). The stationary horizon is characterised by \( e^{2\nu} = 0 = \omega = \text{const} \equiv \Omega \) (thus \( \omega = 0 \)). Regularity of \( g_{\phi\phi} \) and \( g_{r\phi} \) then requires \( rB = 0 \) and \( e^{2\xi} = 0 \) on the horizon. More specifically, in order that the azimuthal and latitudinal circumferences of the horizon be positive and finite, the latter has to be valid for \( Be^{-2\nu} \) and \( e^{2\xi} - 2\nu \). (Let us add that one of the field equations implies that \( \omega,\nu,B \) and \( \nu,\nu,\omega \) have to be finite on a regular horizon, so \( \omega,\nu \) has to vanish there, too.)

Let the external matter have the form of an infinitesimally thin disc in the equatorial plane, stretching over some interval of radii. We will assume that it bears neither charge nor current (no EM fields) and that the space-time is reflection symmetric with respect to its plane \((z = r \cos \theta = 0)\). The metric is then continuous everywhere, but has finite jumps in first normal derivatives \( g_{\mu\nu,z} \) across the disc. The functions \( B, \nu, \omega, \xi \) are even in \( z \), their \( z \)-derivatives are odd in \( z \) and even powers and multiples (for example, \( B_z \nu_z \)) are even in \( z \) (therefore, the latter have no jumps on \( z = 0 \)).

In order that the space-time be stationary, axially symmetric and orthogonally transitive, the disc elements must only move along surfaces spanned by Killing fields, namely they must follow spatially circular orbits with steady angular velocity \( \Omega \equiv \frac{d}{dt} \). This corresponds to four-velocity \( u^\mu = u^t(1,0,0,\Omega), \ u^\phi = \frac{1}{r}(\dot{\theta} + g_{\phi\phi} \Omega,0,0,\dot{\theta} + g_{\phi\phi} \Omega) \), where \( \Omega^2 = e^{2\nu} (1 - \nu^2) \equiv \rho B e^{-2\nu}(\Omega - \omega) \) the linear velocity with respect to the local zero-angular-momentum observer (ZAMO). For the thin discs without radial pressure \( T_{\rho\phi} = 0 \), the surface energy-momentum tensor

\[
\int_{z=0}^\infty T^\mu_{\rho\phi} g_{zz} \ dz = \int_{z=0}^{0^+} T^\mu_{\rho\phi} e^{2\xi-2\nu} dz \equiv S^\mu_{\rho}(\rho) \quad \Rightarrow \quad T^\mu_{\rho} e^{2\xi-2\nu} \equiv S^\mu_{\rho}(\rho) \delta(z)
\]

has only three non-zero components \( S^t_{\rho}, S^\phi_{\rho}, S^\rho_{\rho} \). If \( (S^\rho_{\rho} - S^t_{t})^2 + 4S^\phi_{\rho}S^\rho_{\rho} \geq 0 \), it can be diagonalised to \( S_{\mu\nu} = \sigma w^\mu w^\nu + Pu^\mu w^\nu \), where \( \sigma \) and \( P \) (more precisely, \( \sigma e^{-\xi} - \xi \) and \( Pe^{-\nu} - \nu \)) stand for the surface density and continuous everywhere, but has finite jumps in first normal derivatives \( g_{\mu\nu,z} \) across the disc. The functions \( B, \nu, \omega, \xi \) are even in \( z \), their \( z \)-derivatives are odd in \( z \) and even powers and multiples (for example, \( B_z \nu_z \)) are even in \( z \) (therefore, the latter have no jumps on \( z = 0 \)).

Circular space-times are described by 5 independent Einstein equations. For our thin disc they read

\[
\nabla \cdot (\rho \nabla B) = 0,
\]

\[
\nabla \cdot (B \nabla \nu) - \frac{\rho^2 B^3}{2e^{2\nu}} (\nabla \omega)^2 = 4\pi Be^{2\xi - 2\nu}(T^\phi - 2\omega T^t - T^t_{\phi}) = 4\pi B \frac{(\sigma + P)(1 + \nu^2)}{1 - \nu^2} \delta(z),
\]

\[
\nabla \cdot (\rho^2 B^3 e^{-2\nu} \nabla \omega) = -10\pi Be^{2\xi - 2\nu} T^t_{\phi} = -10\pi \rho B^2 \frac{(\sigma + P)\nu}{e^{2\nu}(1 - \nu^2)} \delta(z),
\]

plus two equations for \( \zeta,\rho \) and \( \zeta,\phi \) (or, \( \zeta,\nu \) and \( \zeta,\omega \)) which are integrable provided that the vacuum equations for \( \nu \) and \( \omega \) are satisfied; \( \nabla \) and \( \nabla \cdot \) denote gradient and divergence in fictive Euclidean three-space. The axis boundary condition \( e^\xi = B \) implies that the \( \zeta \) function can elsewhere be obtained according to \( \zeta(r,\theta) = \int_0^\theta \zeta,\phi \ d\theta + \ln B \), where \( \zeta,\phi \) follows from the last two field equations.

Relation between the \( g_{\mu\nu,z} \) jumps across the disc and \( S^\mu_{\rho} \) can now be obtained by integrating the field equations over the infinitesimal interval \([z = 0^-], [z = 0^+]\). Only the terms proportional to \( \delta(z) \) (i.e. the source terms on the r.h. side and the terms linear in \( B_{zz}, \nu_{zz}, \omega_{zz} \) and \( \zeta_{zz} \) on the l.h. side) contribute according to \( \int_{z=0^-}^{0^+} \nu_{zz} \ dz = 2\nu_{z}(z = 0^+) \) (etc.), so we have (at \( z = 0^+ \))

\[
\nu_{z}(0^+) = 2\pi \frac{(\sigma + P)(1 + \nu^2)}{1 - \nu^2}, \quad \omega_{z}(0^+) = -8\pi \frac{(\sigma + P)(\Omega - \omega)}{1 - \nu^2}, \quad \zeta_{z}(0^+) = 4\pi \frac{\sigma \nu^2 + P}{1 - \nu^2}.
\]

Thin discs in a counter-rotating interpretation

In interpreting the counterweight of its (and the black hole’s) gravity, the disc may either be considered as a solid structure (a set of circular hoops), or, to the contrary, as a non-coherent mix of azimuthal streams. In the astrophysical context, one usually adheres to the latter extreme possibility: that the disc is composed of two non-interacting streams of particles following circular orbits in opposite azimuthal
Substituting (11), (12) into equations (4), (5), one obtains

\[ S^{\mu \nu} = \sigma_+ u_+^\mu u_+^\nu + \sigma_- u_-^\mu u_-^\nu , \]

where the +/- signs indicate the stream orbiting in a positive/negative sense of \( \phi \) (with respect to \( u^\nu \)). The four-velocities are of the form \( u_+^\mu = u_+^1 (1, 0, 0, \Omega_\pm) \) form again, now with \( \Omega_\pm \) corresponding to free motion, i.e. to the roots of \( g_{tt, \pm} + 2 g_{\phi, \pm} \Omega + \gamma g_{\phi, \pm} \Omega^2 = 0 \). Explicitly,

\[ \Omega_\pm = \sqrt{\frac{r^2 \omega_{\rho} + 2 \rho \omega (1 - \rho v_{\rho}) \pm \sqrt{r^4 (\omega_{\rho})^2 + 4 \rho v_{\rho} \rho^2 (1 - \rho v_{\rho})}}{2 \rho (1 - \rho v_{\rho})}} \quad (8) \]

Comparing the two forms of the energy-momentum tensor, the parameters of the counter-rotating picture \((\sigma_+, \sigma_-, \Omega_\pm)\) are found to be related to the “total”, one-stream parameters \((\sigma, P, \Omega)\) by

\[ \sigma_+ + \sigma_- = \sigma - P, \quad P = - \frac{\sigma}{\sigma_+} \frac{u_+^\mu u_+^\nu}{u_-^\mu u_-^\nu} \left( \equiv s^2 \right) \quad \Rightarrow \quad \frac{\sigma + P}{\sigma_+ + \sigma_-} = 1 + \frac{s^2}{1 - s^2} \quad (9) \]

\[ \sigma_\pm (u_\pm^1)^2 = \pm \frac{s^2}{\Omega_+ - \Omega_-} \quad (10) \]

**Perturbation scheme**

It is convenient to choose \( B = 1 - \frac{k^2}{\rho^2} \) as a solution of equation (3). With such a choice, the horizon lies where \( B = 0 \), hence on \( r = k/2 \). The main task remains: to solve the coupled equations (4) and (5).

We will expand the metric functions \( \nu, \omega \) and \( \zeta \) as

\[ \nu = \sum_{j=0}^{\infty} \nu_j \lambda^j, \quad \omega = \sum_{j=0}^{\infty} \omega_j \lambda^j, \quad \zeta = \sum_{j=0}^{\infty} \zeta_j \lambda^j, \quad (11) \]

where the coefficients \( \nu_j, \omega_j \) and \( \zeta_j \) depend on \( r \) and \( \theta \) and the parameter \( \lambda \) is proportional to the disc mass. More precisely, let it be related by

\[ (\sigma + P) \delta(z) \equiv \lambda \Sigma(r) \delta(z) = \lambda \Sigma(r) \frac{1}{r} \delta(\cos \theta) = - \lambda \Sigma(r) \frac{1}{r} \delta \left( \theta - \frac{\pi}{2} \right), \quad (12) \]

where \( \delta \) denotes \( \delta \)-distribution. The functions \( \nu_0, \omega_0 \) and \( \zeta_0 \) represent the black-hole background, i.e. they are given by the Schwarzschild metric (in isotropic coordinates)

\[ ds^2 = - \left( \frac{2r - M}{2r + M} \right)^2 dt^2 + \left( 1 + \frac{M}{2r} \right)^4 \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \quad (13) \]

thus

\[ \nu_0 = \ln \frac{2r - M}{2r + M}, \quad \omega_0 = 0, \quad \zeta_0 = \ln B. \quad (14) \]

Substituting (11), (12) into equations (4), (5), one obtains

\[ \sum_{k=0}^{\infty} \lambda^k \nabla \cdot (B \nabla \nu_k) = \sum_{k=2}^{\infty} \lambda^k \left[ \frac{1}{2} r^2 \sin^2 \theta B^3 e^{-4\nu_0} \sum_{l=0}^{k-2} \exp \left( -4 \sum_{j=1}^{\infty} \nu_j \lambda^j \right) \sum_{m=1}^{k-l-1} \nabla \omega_m \cdot \nabla \omega_{k-l-m} \right] + \]

\[ 4 \pi B \Sigma(r) \frac{1}{r} \delta(\cos \theta) \sum_{k=0}^{\infty} \lambda^{k+1} \frac{1 + v^2}{1 - v^2} \delta_k, \quad (15) \]

\[ \sum_{k=0}^{\infty} \lambda^k \nabla \cdot (r^2 \sin^2 \theta B^3 e^{-4\nu_0} \nabla \omega_k) = - \sum_{k=2}^{\infty} \lambda^k \sum_{l=1}^{k-1} \nabla \cdot \left( r^2 \sin^2 \theta B^3 e^{-4\nu_0} \exp \left( -4 \sum_{j=1}^{\infty} \nu_j \lambda^j \right) \nabla \omega_l \right) - \]

\[ - 16 \pi B \Sigma(r) \delta(\cos \theta) \sum_{k=0}^{\infty} \lambda^{k+1} \frac{e^{-2v}}{1 - v^2} \delta_k, \quad (16) \]

---

2The interpretation is only possible where the expression under the square root is non-negative; physically, this is not satisfied for discs with “too much matter on larger radii”.  

3Different solutions are possible, but changing \( B \) generally does not make a real physical difference anyway — it in fact corresponds to a certain re-definition of coordinates. Only the \( B = \frac{1}{r \sin \theta} \) choice leads to different, plane-wave solutions.
where \([f_k]\) means the coefficient standing at \(\lambda^k\) in \(f\).

If the background is static, the first-order equations only contain mass-energy terms multiplied by the background metric on the r.h. sides, because the first-line sums do not contribute. In higher orders, the r.h. sides only contain lower-order terms then. (For non-static backgrounds the equations do not decouple so easily.) Now, the eigenfunctions with respect to \(\theta\) of the operator on the l.h. side of (15) and (16) are Legendre polynomials \(P_l(\cos \theta)\) and Gegenbauer polynomials \(T_l^{3/2}(\cos \theta)\), respectively. Introducing a dimensionless radius \(x = \frac{r}{\Sigma(r)}\left(1 + \frac{M}{r}\right)\), we may thus write

\[
\nu_1 = \sum_{l=0}^{\infty} \nu_{1l}(x) P_l(\cos \theta), \quad \omega_1 = \sum_{l=0}^{\infty} \omega_{1l}(x) T_l^{3/2}(\cos \theta).
\]

(17)

In the first \(\lambda\)-order, equations (15) and (16) then appear as

\[
\sum_{l=0}^{\infty} \left\{ \frac{d}{dx} \left[ (x^2 - 1) \frac{d}{dx} \nu_{1l} \right] - l(l+1) \right\} P_l(\cos \theta) = 4\pi r \Sigma(r) \frac{1 + v^2}{1 - v^2} \delta(\cos \theta),
\]

(18)

\[
\sum_{l=0}^{\infty} \left\{ \frac{d}{dx} \left[ (x+1)^4 \frac{d}{dx} \omega_{1l} \right] - \frac{(x+1)^4}{x-1} l(l+3) \omega_{1l} \right\} T_l^{3/2}(\cos \theta) = -\frac{\pi(2r + M)^3 \Sigma(r)}{4M^2(2r - M)} \frac{v}{1 - v^2} \sin \theta.
\]

(19)

The source terms on the r.h. sides of (15), (16) (let us denote them \(R(r, \theta), S(r, \theta)\)) can also be decomposed thanks to the orthogonality relations valid for the polynomials; the \(l\)-th order terms read

\[
R_l = \frac{2l + 1}{2} \int_{-1}^{1} R(r, \theta) P_l(\cos \theta) d(\cos \theta) = 2(2l + 1) \pi P_l(0) r \Sigma(r) \frac{1 + v^2}{1 - v^2},
\]

(20)

\[
S_l = \frac{2l + 3}{2(l + 1)(l + 2)} \int_{-1}^{1} S(r, \theta) T_l^{3/2}(\cos \theta) \sin^2 \theta d(\cos \theta) = -\frac{\pi T_l^{3/2}(0)(2l + 3)}{8M^2(l + 1)(l + 2)} \frac{(2r + M)^3 \Sigma(r)v}{(2r - M)} \frac{1}{1 - v^2}.
\]

(21)

Demanding the equality for each and every order, one thus arrives at equations (for every integer \(l \geq 0\))

\[
\frac{d}{dx} \left[ (x^2 - 1) \frac{d}{dx} \nu_{1l} \right] - l(l+1) \nu_{1l} = R_l,
\]

(22)

\[
\frac{d}{dx} \left[ (x+1)^4 \frac{d}{dx} \omega_{1l} \right] - \frac{(x+1)^4}{x-1} l(l+3) \omega_{1l} = S_l.
\]

(23)

Now, a general solution of a linear differential equation can be written as a sum of a particular solution and some function from its fundamental system. In our case the fundamental systems are vector spaces with dimension 2. Let us find their bases. It is easy to verify that

\[
P_l(x) \quad \text{and} \quad Q_l(x) = P_l(x) \int_{x}^{\infty} \frac{d\xi}{[P_l(\xi)]^2(\xi^2 - 1)}
\]

(24)

are two independent solutions of (22) without the r.h. side. Similarly (though not so obviously),

\[
F_l(x) = 2F_1 \left( -l, l + 3; 4; \frac{x + 1}{2} \right) \quad \text{and} \quad G_l(x) = F_l(x) \int_{x}^{\infty} \frac{d\xi}{[F_l(\xi)]^2(\xi + 1)^4}
\]

(25)

represent the desired couple for (23); here \(2F_1(\alpha, \beta; \gamma; z)\) stands for the hypergeometric function. Asymptotically, for \(r \to \infty\), the basis polynomials behave as \(P_l \to r^l, Q_l \to r^{-l-1}, F_l \to r^l\) and \(G_l \to r^{-l-3}\). Near the horizon (\(r = M/2\), i.e. \(x = 1\)), \(P_l\) and \(F_l\) are regular, but \(Q_l\) and \(G_l\) diverge (except for \(G_0\)).

With the above fundamental system and the requirement of asymptotic flatness and regularity on the horizon, the Green functions \((\mathcal{G})\) of the equations (22) and (23) have to look like

\[
\mathcal{G}_l^r(x, x') = \begin{cases} -Q_l(x) P_l(x') & \text{for } x > x' \\ -P_l(x) Q_l(x') & \text{for } x < x' \end{cases},
\]

(26)
\[
G_l^\omega(x, x') = \begin{cases} 
-G_l(x)F_l(x') & \text{for } x > x' \\
-F_l(x)G_l(x') & \text{for } x < x' 
\end{cases}
\]

(27)

For a thin ring, the surface density is \( \Sigma(r) = \tau(1 + 2M/r)^4(r - r_{\text{ring}}) \), where \( \tau \) is the linear density (constant due to the axial symmetry). Substituting this into (20) and (21) and subsequently into (22) and (23), the first-order perturbation due to the ring finally comes out as

\[

\nu_1 = \sum_{j=0}^{\infty} \left( \frac{2l + 1}{v^2} \right) P_l(0)(2r_{\text{ring}} + M)^4 \frac{G_l^\omega(x, x_{\text{ring}})}{8r_{\text{ring}}^3} \rho B \frac{1 + v^2}{1 - v^2} P_l(0)(2r_{\text{ring}} + M)^4 \frac{G_l^\omega(x, x_{\text{ring}})}{8r_{\text{ring}}^3} \cos \theta, \\

\omega_1 = -\sum_{j=0}^{\infty} \frac{\pi T_{l}^{3/2}(0)(2l + 3)}{8M^3(l + 1)(l + 2)} v \frac{(2r_{\text{ring}} + M)^7}{v^2 (2r_{\text{ring}} - M)} \frac{G_l^\omega(x, x_{\text{ring}})}{8r_{\text{ring}}^3} \cos \theta.
\]

Two important properties can immediately be noticed. First, the perturbation of dragging is constant over the horizon \( (x = 1) \). Actually, \( F_l(1) = 0 \) for \( l > 0 \), so \( G_l^\omega(1, x') = 0 \) for any \( x' > 1 \) and, for \( l = 0 \), \( T_l^{3/2}(0) = 1 \) independently of \( \theta \). Second, unlike \( G_l^\omega \), the \( G_l^\omega \) is only uniquely defined (boundary conditions) for \( l > 0 \). Namely, it is possible to modify the perturbation to \( \omega'_l = \omega_l + C \delta_0(x)T_l^{3/2}(0) \cos \theta \) (with \( C \) constant), assuming that \( G_l(0) = \frac{1}{2} \delta_{l,0} \tau \) which is regular on the horizon and vanishes at infinity. This modification satisfies the vacuum Einstein equation everywhere outside the black hole and is independent of \( \theta \) as well. The constant \( C \) can be chosen in such a way, for instance, that \( \omega'_l = 0 \) at the horizon (so the hole is kept non-rotating with respect to infinity).

The mass and angular momentum of the system can be determined from Komar integrals which also allow to separate the contributions of the hole and of the disc. For the disc part, their original expression in terms of Killing vector fields may be rewritten in various ways using the field equations. If \( T_{\text{r}} = 0 \) and \( T_{\phi} = 0 \), one has

\[
\mathcal{M} = 2\pi \int_{\text{disc}} (S_\phi^\omega - S_\phi) \rho B \, \mathrm{d} \rho = 2\pi \int_{\text{disc}} \Sigma(r) \frac{1 + v^2}{1 - v^2} \left( 1 - \frac{M^2}{4r^2} \right) r \, \mathrm{d} r,
\]

\[
\mathcal{J} = 2\pi \int_{\text{disc}} S_\phi^\omega \rho B \, \mathrm{d} \rho = 2\pi \int_{\text{disc}} \Sigma(r) \frac{v}{1 - v^2} \frac{(2r + M)^4}{16r^2} \, \mathrm{d} r.
\]

Perturbation by an annular thin disc

Knowing the Green functions, one might attempt to find the thin-disc perturbation by “integrating over the rings”. However, this leads to complicated integrals, so we will rather try a different way.

If we know some particular solution \( \nu_0^\text{part} \), \( \omega_0^\text{part} \) which satisfies equations (22), (23) between certain radii \( r_{\text{in}} < r < r_{\text{out}} \) above the horizon, we may try to modify it in such a way that it could be extended to the whole space-time, that it will satisfy the boundary conditions on the horizon and at infinity and that the respective energy-momentum tensor be interpretable as a thin finite (annular) equatorial disc with surface density \( \Sigma(r) \) between \( r_{\text{in}} \) and \( r_{\text{out}} \). A natural guess is \( \nu_l = A_l^0 \tilde{P}_l(x) \) at \( r < r_{\text{in}} \), \( \nu_l = B_l^0 \tilde{Q}_l(x) \) at \( r > r_{\text{out}} \) and \( \nu_l = A_l^0 \tilde{P}_l(x) + B_l^0 \tilde{Q}_l(x) + \nu_l^\text{part} \) in between. The constants \( A_l^0, B_l^0 \) should be fixed so that the function \( \nu_l \) and its first derivative be continuous on the shells \( r_{\text{in}} \) and \( r_{\text{out}} \). A suitable solution for \( \omega_l \) could be searched for in a similar manner.

With the \( l \)-dependences adopted from (20), (21), we start from the r.h. sides

\[
R_l = C_R \frac{2l + 1}{2} P_l(0), \quad S_l = C_S \frac{2l + 3}{2(l + 1)(l + 2)} T_l^{3/2}(0) \frac{(x + 1)^3}{(x - 1)},
\]

(30)

where \( C_R \) and \( C_S \) are some constants.\(^4\) The respective particular solutions of (22), (23) are

\[

\nu_l^\text{part} = \begin{cases} 
0 & \text{for } l \text{ odd} \\
-C_R \frac{2l + 1}{2(l + 1)} P_l(0) & \text{for } l > 0 \text{ even} \\
\frac{4}{C_R} \ln(x^2 - 1) & \text{for } l = 0
\end{cases},

\omega_l^\text{part} = \begin{cases} 
0 & \text{for } l \text{ odd} \\
-C_S \frac{2l + 3}{2(l + 1)(l + 2)} T_l^{3/2}(0) & \text{for } l > 0 \text{ even} \\
-C_S \left[ \frac{2l + 1}{2(l + 1)(l + 2)} + \left( \frac{2l + 1}{2(l + 1)} \right) T_l^{3/2}(0) \right] + \ln(x - 1) & \text{for } l = 0
\end{cases}
\]

\(^4\)Ordinary dimensionless, while \( C_S \) has the dimension of \( 1/\text{length} \). They should be chosen so that the physical requirements on the corresponding energy-momentum tensor (energy conditions, in particular) be satisfied in the disc region.
Now we must ensure, in the above manner, that the solutions composed of these particular solutions and the respective fundamental systems match smoothly enough \((C')\) at the disc rims. Also, we demand the r.h. sides \((30)\) to be interpretable as generated by a thin disc in the chosen radial range, i.e. we put them equal to \((20), (21)\) and check how the radial functions \(\Sigma(r), v(r)\) thus determined look. Alternatively, in a counter-rotating picture of the disc, the functions to determine are surface densities of the geodesic streams \(\sigma_+ , \sigma_-\), their velocities being fixed to the Schwarzschild circular-geodesic value.

The explicit relations between \((\sigma_+, \sigma_-, v_\pm)\) and \((\sigma, P, v)\) are rather complicated in general, but in the Schwarzschild field, where the two geodesic orbital velocities \(\Omega_\pm\) are just opposite and the corresponding ZAMO-measured linear velocities are \(v_\pm = \pm \sqrt{M^2 - \ell^2}\), they reduce to

\[
\sigma = \frac{\Sigma + \sigma_+ + \sigma_-}{2}, \quad P = \frac{\Sigma - \sigma_+ - \sigma_-}{2}, \quad v = \frac{K - \sqrt{K^2 - 4L^2}}{2L},
\]

where

\[
\Sigma = \frac{\sqrt{K^2 - 4L^2}}{1 - v_\pm^2}, \quad K \equiv (\sigma_+ + \sigma_-)(1 + v_\pm^2), \quad L \equiv (\sigma_+ - \sigma_-) v_+.
\]

Due to them, the r.h. sides of \((20), (21)\) can be rewritten, respectively,

\[
-2(2l + 1)\pi P_1(0)r(\sigma_+ + \sigma_-)\frac{1 + v_\pm^2}{1 - v_\pm^2}, \quad \frac{\pi T_1^3/2(0)(2l + 3)(2r + M)^3 (\sigma_+ - \sigma_-)v_+}{8M^2(l + 1)(l + 2)(2r - M) - 1 - v_\pm^2},
\]

which is then to be equated with the r.h. sides of \((30)\) in order to find \(\sigma_+\) and \(\sigma_-\) corresponding to some specific choice of \(C_R\) and \(C_S\).

Substituting into equations \((28)\) and \((29)\), we find the mass and angular momentum of the disc,

\[
\mathcal{M} = \frac{C_R}{2} \int_{r_{in}}^{r_{out}} \left(1 - \frac{M^2}{4r^2}\right) dr = \frac{C_R}{2} (r_{out} - r_{in}) \left(1 - \frac{M^2}{4r_{in}r_{out}}\right),
\]

\[
\mathcal{J} = \frac{C_S}{32} \int_{r_{in}}^{r_{out}} \frac{(2r + M)^7}{r^4(2r - M)} dr.
\]

**Numerical example**

Now we give a specific example of the first-order annular-disc perturbation. We place the disc between \(r_{in} = 5M\) and \(r_{out} = 6M\) and choose the constants in \((30)\) as \(C_R = 0.03, C_S = -0.001M^{-1}\). The corresponding disc mass and angular momentum are \(\mathcal{M} = 0.015M\) and \(\mathcal{J} = 0.12M^2\). Should the disc be interpreted as made of two counter-rotating geodesic streams, the necessary surface densities come out as shown in figure 1. The corresponding parameters of the “total” one-stream interpretation are drawn in figure 2. The energy conditions are clearly satisfied.

Finally we will compute the linear metric perturbations \(\nu_1\) and \(\omega_1\) as described above. Restricting the infinite sums in orthogonal polynomials to their 14-th order, the total gravitational potential and the dragging angular velocity are obtained as shown in figure 3.

Unfortunately, the terms of these sums (the magnitude of which is mainly given by the functions \(Q_l\) and \(G_l\), see \((24)\) and \((25)\)) decrease rather slowly and oscillate while almost compensating each other. Even with very high precision numerics (the figures spent about a day on a handsome PC\(^5\)) the resulting curves do not look satisfactory in detail (notice the anomaly near the \(z\)-axis in the \(\omega\)-plot, in particular). This mainly applies to the radii where the disc exists. In addition, the latter’s profile in the \(\theta\)-direction being \(\delta\)-function, it would require very high frequency terms (i.e. high orders of the Legendre and Gegenbauer polynomials) in order to be well rendered by the decompositions. The problems are not so serious in the case of gravitational potential, because this is anyway dominated by the black hole. But exactly the opposite is true with dragging which is purely generated by the perturbing agent. We learned, by increasing the number of series terms, that the contours smooth out only very reluctantly.

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\(^5\)In order to make this statement observer independent, let us specify the PC has a 3GHz processor.
Figure 1. Left: Velocity magnitude \( v_+ = -v_- \) of the counter-rotating components, as measured by ZAMO. (The gray line indicates Keplerian orbital velocity in a pure Schwarzschild field.) Right: Respective “real” surface densities \( \sigma_+ e^{v_+ - \zeta} \) (upper curve) and \( \sigma_- e^{v_- - \zeta} \) (lower one). The radius is in the units of \( M \), the surface densities in \( M^{-1} \).

Figure 2. Left: “Total” speed \( v \). Middle: “Real” surface density \( \sigma e^{v - \zeta} \). Right: “Real” pressure \( P e^{v - \zeta} \). The radius is in the units of \( M \) and the surface density and pressure in \( M^{-1} \).

Figure 3. Left: Contours of the gravitational potential \( \nu = \nu_{\text{Schwarzschild}} + \nu_1 \) in the meridional plane \( (\rho = r \sin \theta, z = r \cos \theta) \). The lines correspond to the potential levels \(-1.0, -0.9, \ldots, -0.1\) (as going outwards from the centre). Right: Contours of the dragging angular velocity \( \omega = \omega_1 \) (remember that \( \omega_0 \equiv \omega_{\text{Schwarzschild}} = 0 \)) in the same plane. The lines correspond to the values \( 9 \cdot 10^{-5} M^{-1} \ldots 1 \cdot 10^{-5} M^{-1} \) (as going away from the disc at \( z = 0, 5M < \rho < 6M \)). The axes are in the units of \( M \).

Concluding remarks

We have shown that the method used by Will (1974) in order to perturb the Schwarzschild metric by a light and slowly rotating ring can be adapted to the case of a thin annular disc. Focusing on the
first-order perturbation, we found that the formulas useful for analytic/symbolic calculations (angular decompositions, in particular) are less suitable for numerical evaluation. Actually, a very slow convergence of the series and instabilities encountered in counting up “the second-kind functions” may explain why no explicit results of this type have yet appeared in the literature (as far as we are aware).

There remains a number of points that should be clarified in future, for instance the dependence of solutions on the parameters $C_R, C_S, r_{\text{in}}$ and $r_{\text{out}}$ (as illustrated by more numerical examples) or the effect of the perturbation on various properties of the field (like deformation of the horizon). Also, stability of the above obtained disc sources should be analysed in order to judge whether the solutions might have some (astro)physical relevance. A special question is the validity of the linear approximation. It cannot be answered in few general inequalities, it will be necessary to check, for each particular solution, whether the $\nu_1(r, \theta)$- and $\omega_1(r, \theta)$- terms are really negligible as compared with the $\nu_0(r, \theta)$-terms. For the moment we can only add to our example that both the values encountered in Fig. 3 and the total mass of the disc seem to indicate that these conditions are satisfied in this case.

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Linear perturbations of a Schwarzschild black hole by thin disc

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Abstract. The article written by C.M.Will (1974) describes a method how to obtain a perturbation of an (originally) Schwarzschild metric by a slowly rotating light ring. This scheme is revisited in order to adapt it to the case of a thin disc. It turns out for some class of mass distribution on the disc bounded between some radii the solution (or at least its decomposition into spherical harmonics) can be found explicitly.

1. Introduction
Many of the very compact active astrophysical objects (for example active galactic nuclei and X-ray binaries) are believed to contain disc-like structures. However these very compact objects generates strong and inhomogenous gravitational field so Newton’s theory is not sufficient to describe them. On the other hand general relativity is non-linear and the simple superposition cannot be used. Recently these and even more complicated (for example time dependant) fields are successfully treated numerically. However so far the only known “analytical” solutions of this problem contains a very high degree of symmetry, usually they are axisymmetric and stationary. Even in this “symmetric” model the equations are quite complicated so it is a bit surprising in the static case the superposition can be used and the equations solved explicitly (Weyl class of solutions).

The fundamental treatment of stationary axisymmetric thin discs can be found in the paper by Bardeen [1]. It was succeeded by the article by Will [2], which remains a major reference in this subject even now.

2. Basic concepts
Considering axisymmetric stationary case we can write metric in form

$$ds^2 = -e^{2\nu}dt^2 + r^2B^2e^{-2\nu}\sin^2\theta(d\varphi - \omega dt)^2 + e^{2\zeta-2\nu}\left(dr^2 + r^2d\theta^2\right),$$

where $\nu$, $\omega$, $B$ and $\zeta$ stands for an unknown metric functions of coordinates $r$ and $\theta$.

The energy-momentum tensor of thin dust disc can expressed

$$T^\alpha_\beta = \sigma e^{2\nu-2\zeta}u^\alpha u_\beta r^{-1}\delta}\cos\theta, $$

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where $\sigma$ describes surface mass density and $u^\alpha$ is four-velocity of the fluid. It is convenient to write four-velocity in the form $u^\alpha = e^{\nu} \frac{\partial}{\partial \tau} (1, 0, 0, \Omega)$, where $\Omega = \frac{d\phi}{dt}$ is coordinate angular velocity and $v = r \sin \theta e^{-2\nu}(\Omega - \omega)$ linear velocity\(^1\) of the fluid.

It can be shown function $B$ does not contain physically relevant information. As long as it satisfies the equation $\nabla \cdot (r \sin \theta \nabla B) = 0$ (where the divergence $\nabla \cdot$ and gradient $\nabla$) operators are the same as in the virtual Euclidean space-time with spherical coordinates $r, \theta$ and $\phi$) it can be chosen arbitrarily\(^2\) and the solutions for different $B$ will only differ in the choice of coordinates.

The important pair Einstein equations is

$$\nabla \cdot (B \nabla \nu) - \frac{1}{2} v^2 \sin^2 \theta e^{-4\nu} \nabla \cdot \nabla \omega = 4\pi B \sigma \frac{1 + v^2}{1 - v^2} \frac{1}{r} \delta(\cos \theta),$$

$$\nabla \cdot \left(r^2 \sin^2 \theta B^3 e^{-4\nu} \nabla \omega \right) = -16\pi B^2 \sigma e^{-2\nu} \frac{v}{1 - v^2} \delta(\cos \theta)$$

and the last two independent equations can be used to express $\zeta$ using known $\nu$ and $\omega$.

Now let us assume the background metric is perturbed by a thin dust disc. The metric function can be written in form $\nu = \nu_0 + \nu_1 + \cdots$ where the lower index corresponds with the power of the disc mass it contains (and analogously for the functions $\omega$ and $\zeta$). In the subsequent text we will work only with the linear perturbations.

The Schwarzschild black hole in the isotropic coordinates takes form

$$ds^2 = -\left(\frac{2r - M}{2r + M}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\right),$$

so the background metric functions are

$$B = 1 - \frac{M^2}{4r^2}, \quad \nu_0 = \ln \left(2r - M \over 2r + M\right), \quad \omega_0 = 0, \quad \zeta_0 = \ln \left(1 - \frac{M^2}{4r^2}\right).$$

The linear perturbation of this black hole by disc will lead us to the equations

$$\frac{\partial}{\partial x} \left[ \frac{(x^2 - 1)}{x} \frac{\partial \nu_1}{\partial x} \right] + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta \frac{\partial \nu_1}{\partial \theta} \right] = 4\pi \sigma \frac{1 + v^2}{1 - v^2} \delta(\cos \theta),$$

$$\frac{\partial}{\partial x} \left[ \frac{(x+1)^3}{x-1} \frac{\partial \omega_1}{\partial x} \right] + \frac{(x+1)^2}{x^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left[ \sin^2 \theta \frac{\partial \omega_1}{\partial \theta} \right] = -\pi \left(2r + M\right)^3 \frac{v}{4M^2} \frac{1}{2r - M} \sigma \frac{1}{1 - v^2} \delta(\cos \theta)$$

where $x = \frac{r}{M} \left[1 + M^2/(4r^2)\right]$ is new radial coordinate. Considering the particles are moving on geodesics their velocity can be written\(^3\) as $v = (x - 1)^{-1/2}$.

To solve these equations we will use expansion into the Legendre (for the function $\nu_1$) and Gegenbauer ($\omega_1$) polynomials\(^4\). So let us define

$$\nu_1 = \sum_{n=0}^{\infty} \nu_1 (x) P_n (\cos \theta), \quad \omega_1 = \sum_{n=0}^{\infty} \omega_1 (x) T_n^{3/2} (\cos \theta).$$

\(^1\) With respect to the zero angular momentum observer.

\(^2\) With the exception $B = \frac{1}{r \sin \theta}$, which represents plane waves solutions instead of stationary axisymmetric ones.

\(^3\) In the zeroth order of expansion. We need not to consider first order since the velocity is used in the source term of the equations, which is clearly proportional to the disc mass solely because of $\sigma$.

\(^4\) It can be shown the series converges absolutely anywhere besides the axis. On the axis $\nu_1$ converges absolutely, however in the case of $\omega_1$ it is still not clear. The conjecture is $\omega_1$ converges, but not absolutely.
Moreover we will write the r.h.s. of Einstein equations using these polynomials

\[
4\pi r^2 \frac{1 + \nu^2}{1 - \nu^2} \delta(\cos \theta) = \sum_{n=0}^{\infty} R_n P_n(\cos \theta), \quad \text{where } R_n = (4n + 2)\pi P_n(0)x\chi(x) \quad \text{and (10)}
\]

\[
-\frac{\pi}{4M^2} \frac{2\nu}{2r - M} \delta(\cos \theta) = \sum_{n=0}^{\infty} S_n T_n^{3/2}(\cos \theta), \quad \text{where } S_n = \frac{\pi T_n^{3/2}(0)(2n+3)(x+1)^3}{-2M(n+1)(n+2)} \quad \text{(11)}
\]

In both expressions \(\chi(x) = r\sigma/(x - 2)\) stands for the new function used to describe mass density. Also we can see \(R_n = S_n = 0\) when \(n\) is odd.

After substitution into linearized Einstein equations and their separation we will obtain

\[
\frac{d}{dx} \left[ (x^2 - 1) \frac{d\nu_1}{dx} \right] - n(n+1)\nu_1 = R_n, \quad \text{(12)}
\]

\[
\frac{d}{dx} \left[ (x+1)^4 \frac{d\omega_1}{dx} \right] - n(n+3)(x+1)^3 \frac{\omega_1}{x-1} = S_n. \quad \text{(13)}
\]

The base of fundamental system of the first equation is the \(n\)-th Legendre polynomial \(P_n(x)\) and the Legendre function of the second kind \(Q_n(x)\). Analogously we can obtain fundamental system of the second equation. It consists of the polynomial \(F_n(x)\) and the “function of the second kind” \(G_n(x)\) defined as

\[
F_n(x) = 2F_1(-n, n + 3; (x+1)/2), \quad G_n(x) = F_n(x) \int_{x}^{\infty} \frac{d\xi}{[F_n(\xi)]^2(\xi + 1)^4}, \quad \text{(14)}
\]

where \(2F_1(a, b; c; z)\) denotes hypergeometric function. Assuming asymptotic flatness and regularity of the perturbation one can construct corresponding Green functions

\[
G_n^\nu(x, x') = -P_n[\min(x, x')]Q_n[\max(x, x')] \quad \text{and } G_n^\nu(x, x') = -F_n[\min(x, x')]G_n[\max(x, x')]. \quad \text{(15)}
\]

These can be used to calculate first perturbation, however the integrals are quite complicated.

In special case, where \(\chi(x)\) is an even polynomial, particular solution can be found explicitly. \(\nu_1^{\text{part}}\) is exactly

\[
x\chi(x) = \sum_{l=0}^{m} c_{2l+1} P_{2l+1}(x) \quad \Rightarrow \quad \nu_1^{\text{part}} = \sum_{l=0}^{m} \frac{c_{2l+1}}{2(l+1)(2l+1) - n(n+1)} P_{2l+1}(x), \quad \text{(16)}
\]

where \(m\) and \(c_j\) are some constants. The case of \(\omega_1^{\text{part}}\) is slightly more difficult, but it takes form

\[
\omega_1^{\text{part}} = \frac{A_n(x)}{\sqrt{x+1}} + C_n F_n(x) \ln \left( \frac{\sqrt{x+1} + \sqrt{2}}{\sqrt{x+1} - \sqrt{2}} \right), \quad \text{(17)}
\]

where \(A_n(x)\) stands for a suitable polynomial and \(C_n\) is some constant.

However we would like to obtain solution with disc spanned between some radii \(r_{\min}\) and \(r_{\max}\) and we want the solution to be regular on the horizon and asymptotically flat. Its uniqueness is clear from the uniqueness of the Green functions. It can be also described by

\[
\nu_1 = \begin{cases} 
A_\leq P_n(x) + B_\leq Q_n(x) + \nu_1^{\text{part}}(x) & \text{for } x \leq x_{\min} \\
A_\geq P_n(x) + B_\geq Q_n(x) & \text{for } x_{\min} \leq x \leq x_{\max} \\
B_\geq Q_n(x) & \text{for } x \geq x_{\max} 
\end{cases}, \quad \text{(18)}
\]

where \(A_\leq, A_\geq, B_\leq\) and \(B_\geq\) are constants obtained by the the requirement of continuity of metric functions and their first derivatives at the radii of the rims of the disc (and analogously we can express \(\omega_1\)).

3. Numerical examples

Let us present some numerical examples of the procedure described above. In both cases we take \(M = 1\), \(r_{\min} = 5\) and \(r_{\max} = 6\) and we consider only first 30 orders in the “harmonic” expansion (i.e. \(P_n(\cos \theta)\) and \(T_n^{3/2}(\cos \theta)\)).
The first one (Fig. 1 and 2) deals with $\chi = (40 - x^2) \cdot 10^5$ and the second one (Fig. 3 and 4) with $\chi = 10^{-7}$.

**Figure 1.** Gravitational potential $\nu = \nu_0 + \nu_1$. The lines corresponds with the potential values $-1.5, -1.4, \ldots, -0.1$.

**Figure 2.** Dragging $\omega = \omega_0 + \omega_1$. The lines corresponds with the values $8 \cdot 10^{-6}, 7 \cdot 10^{-6}, \ldots, 1 \cdot 10^{-6}$.

**Figure 3.** Gravitational potential $\nu = \nu_0 + \nu_1$. The lines corresponds with the potential values $-0.20, -0.18, \ldots, -0.04$.

**Figure 4.** Dragging $\omega = \omega_0 + \omega_1$. The lines corresponds with the values $10 \cdot 10^{-9}, 9 \cdot 10^{-9}, \ldots, 1 \cdot 10^{-9}$.

4. Conclusion
Analogously to the paper by Will [2] we were able to obtain procedure calculating the first order perturbations of Schwarzschild black hole by a thin dust disc for some class of matter distributions. However mentioned procedure is not suited well for second or higher order expansion in the mass because of the problems associated with multiplication of spherical harmonics. Moreover it is not well suited for the numerical calculations (as we can see in the numerical examples).

5. Acknowledgements
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References
Linear perturbations of a Schwarzschild black hole by thin disc - convergence

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Abstract. In order to find the perturbation of a Schwarzschild space-time due to a rotating thin disc, we try to adjust the method used by [4] in the case of perturbation by a one-dimensional ring. This involves solution of stationary axisymmetric Einstein’s equations in terms of spherical-harmonic expansions whose convergence however turned out questionable in numerical examples. Here we show, analytically, that the series are almost everywhere convergent, but in some regions the convergence is not absolute.

Keywords: general relativity, black holes, perturbation techniques, accretion discs

PACS: 04.70.Bw, 04.25.-g, 02.30.Lt, 04.20.-q

LINEAR PERTURBATION OF BLACK HOLE

The orthogonally transitive, stationary and axisymmetric space-time can be described by the metric [1]

\[ ds^2 = -e^{2\nu} dt^2 + r^2 B^2 e^{-2\nu} \sin^2 \theta (d\phi - \omega dt)^2 + e^{2\xi - 2\nu} (dr^2 + r^2 d\theta^2), \]

where \( t, \phi \) are Killing coordinates and \( r, \theta \) are isotropic coordinates covering the meridional planes; \( B, \nu, \omega \) and \( \zeta \) denote functions of \( r \) and \( \theta \) which are determined by Einstein’s equations. In the thin-disc case, the energy-momentum tensor reads

To calculate them we must specify the energy-momentum tensor. In the case of thin dust disc it can be expressed as

\[ T^\alpha_\beta = \sigma e^{2\xi - 2\nu} u^\alpha_\beta \frac{1}{r} \delta(\cos \theta), \]

where \( \sigma \) is a surface density and \( u^\alpha \) is the disc-matter four-velocity which can be expressed as

\[ u^\alpha = \frac{e^{-\nu}}{1 - \nu^2} (1, 0, 0, \Omega) \]

in terms of linear velocity with respect to the zero-angular-momentum observer \( v = r \sin \theta Be^{-2\nu} (\Omega - \omega) \) (\( \Omega = d\phi/dt \) is the corresponding angular velocity at infinity).

The field equation for \( B \) reads

\[ \nabla \cdot (r \sin \theta \nabla B) = 0, \]
where $\nabla$ and $\nabla \cdot$ denote gradient and divergence in a Euclidean 3D space (represented in spherical coordinates $r, \theta, \phi$). Otherwise $B$ can be chosen arbitrarily\(^1\).

The most important Einstein equations are those for the dragging and gravitational potentials, $\nu$ and $\omega$,

\[
\nabla \cdot (B \nabla \nu) - \frac{1}{2} r^2 \sin^2 \theta e^{-4\nu} \nabla \omega \cdot \nabla \omega = 4 \pi B \sigma \frac{1 + \nu^2}{1 - \nu^2} \frac{1}{r} \delta (\cos \theta),
\]

\[
\nabla \cdot (r^2 \sin^2 \theta B e^{-4\nu} \nabla \omega) = -16 \pi B^2 \sigma e^{-2\nu} \frac{\nu}{1 - \nu^2} \delta (\cos \theta).
\]

Knowing $B$, $\nu$ and $\omega$, the last function $\zeta$ can be obtained by line integration of the remaining two relevant field equations.

We are interested in a perturbation of the Schwarzschild metric which in isotropic coordinates reads

\[
ds^2 = -\left(\frac{2r-M}{2r+M}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)
\]

also it corresponds to $\nu = \ln \frac{2r-M}{2r+M}$ and $\omega = 0$. We start from choosing $B = 1 - \frac{M^2}{4r^2}$. Then one can rewrite the equations (5), (6) using the perturbation expansions

\[
\nu = \ln \left(\frac{2r-M}{2r+M}\right) + \sum_{n=0}^{\infty} \delta \nu_n(x) P_n(\cos \theta) + O(\epsilon^2),
\]

\[
\omega = \sum_{n=0}^{\infty} \delta \omega_n(x) C_n^{3/2}(\cos \theta) + O(\epsilon^2),
\]

(with the $O$-remainders omitted), where $\epsilon = \text{disc mass} / M$ is mass expansion parameter, $x \equiv \frac{r}{M} \left(1 + \frac{M^2}{4r^2}\right)$ new "radial" coordinate and $P_n$ and $C_n^{3/2}$ are Legendre and Gegenbauer polynomials, respectively, while decomposing the source terms on the r.h. sides in the same manner. Demanding equality in each $n$-order, the equations (5), (6) leads to

\[
\frac{d}{dx} \left[ (x^2 - 1) \frac{d}{dx} \delta \nu_n \right] - n(n+1) \delta \nu_n = 2(2n+1)\pi P_n(0) r \sigma(r) \frac{1 + \nu^2}{1 - \nu^2}
\]

\[
\frac{d}{dx} \left[ (x+1)^4 \frac{d}{dx} \delta \omega_n \right] - \frac{(x+1)^3}{x-1} n(n+3) \delta \omega_n = -\frac{\pi C_n^{3/2}(0)(2n+3)(2r+M)^3 \sigma(r) \nu}{8M^2(n+1)(n+2)(2r-M)} \frac{1 + \nu^2}{1 - \nu^2}.
\]

This set can be solved to obtain expansions of the linear perturbation of functions $\nu$ and $\omega$ which are induced by the chosen source (thin disc) and which can be made regular both at the horizon and at radial infinity\(^2\); see e.g. [2].

---

\(^1\) With the exception of the $B = 1/(r \sin \theta)$ case which leads to plane-wave space-times.

\(^2\) One can choose freely one constant which represents angular velocity of the horizon. We will omit it since it is not important for convergence. Besides that the system is determined.
TABLE 1. Coefficients of Green functions of the problem

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>$N_0(x,x')$</th>
<th>$L_n(x,x')$</th>
<th>$\Sigma(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G^\nu_n$</td>
<td>0</td>
<td>0</td>
<td>$1/2$</td>
<td>0</td>
<td>$-2\pi(2n+1)C^\nu_0(0)$</td>
<td>$r\sigma(r)\frac{1}{1+\nu}$</td>
</tr>
<tr>
<td>$G^\omega_n$</td>
<td>1</td>
<td>3</td>
<td>$3/2$</td>
<td>1</td>
<td>$(x-1)(x'-1)^{\frac{n+3}{n}}$</td>
<td>$\frac{\pi(2n+3)(n+3)C^\omega_0(0)(x-1)(x'-1)^{1/2}}{2M(n+1)(n+2)}$</td>
</tr>
</tbody>
</table>

However, numerical illustrations of the results revealed unsatisfactory behaviour in the region close to the axis, mainly of the $\omega$ function. Suspecting bad convergence of the employed perturbation series, we have tried to check this issue analytically. As briefly summarized below, we found that the $\omega$-expansion does not converge absolutely in certain regions and that its convergence may become problematic at certain parts of the axis.

CONVERGENCE OF THE SERIES

The general disc solution can be obtained convolving rings of matter. This ring solution (see [4]) corresponds (up to the numerical factor) to the Green function of equation,

$$ f_n(x,x') = N_n(x,x') \left\{ \begin{array}{ll} P_{n-\delta}^{(\alpha,\beta)}(x)Q_{n-\delta}^{(\alpha,\beta)}(x') & x < x' \\ P_{n-\delta}^{(\alpha,\beta)}(x')Q_{n-\delta}^{(\alpha,\beta)}(x) & x' < x \end{array} \right., $$

(12)

where $P_{n-\delta}^{(\alpha,\beta)}$, $Q_{n-\delta}^{(\alpha,\beta)}$ are Jacobi functions of the first and second kind and the coefficients $\alpha$, $\beta$, $\delta$ and the function $N(x,x')$ are written explicitly in the table 1.

The whole disc solution takes form

$$ f(r, \theta) = \sum_{n=0}^{\infty} \int_{\text{disc}} C_n^\nu(\cos \theta)L_n(x,x')\Sigma(x')P_{n-\delta}^{(\alpha,\beta)}(\min(x,x'))Q_{n-\delta}^{(\alpha,\beta)}(\max(x,x'))dx', $$

(13)

where $f(r, \theta)$ is $\nu$ or $\omega$ and functions and constants are chosen in accordance to the table 1. It should be noted $C_n^\nu(0) = 0$ when $n$ is odd so in the rest of this paper we will consider only "even" contributions to the Green functions.

To analyze asymptotic behaviour (with respect to $n \to \infty$) it is convenient to express Jacobi function in terms of Legendre functions:

$$ P_n^{(1,3)}(x)Q_n^{(1,3)}(x') = H(x,x') \sum_{k=0}^{4} \sum_{l=0}^{4} X_{k,l} \left[ 1 + O\left( \frac{1}{n} \right) \right] P_{n+k}(x)Q_{n+l}(x'), $$

(14)

where $H(x,x')$ is rational function of $x$ and $x'$ and $X_{k,l}$ are constants.

Using relations for modified Bessel functions $I_0$ and $K_0$ (see [3]) we can write

$$ P_{n+k}(\cosh \xi)Q_n(\cosh \xi) = \sqrt{\frac{\xi}{\sinh \xi}} \sinh \xi I_0\left( \frac{2n+2k+1}{2} \xi \right) K_0\left( \frac{2n+1}{2} \xi \right) \left[ 1 + O\left( \frac{1}{n} \right) \right] = (15) $$

$$ = \frac{1}{2n\sqrt{\sinh \xi \sinh \xi}} e^{\xi k} e^{(\xi - \xi)(n+1/2)} \left[ 1 + O\left( \frac{1}{n} \right) \right]. $$
We will also need to know asymptotic behaviour of the "spherical harmonics"

\[ C_n^\gamma(\cos \theta) = \begin{cases} 2n^{\gamma-1}\cos((n+\gamma)\theta-\frac{\pi}{2}\gamma)+O(\frac{1}{n}) & \text{when } 0 < \theta < \pi \smallskip \vspace{2mm} \\ n^{2\gamma-1}\frac{1+O(\frac{1}{n})}{\Gamma(2\gamma)} & \text{when } \cos \theta = 1 \end{cases} \]  

(16)

This expression can be also used to find properties of \( L_n(x,x') \) when \( n \to \infty \).

Taking all together we can conclude that

- At radii, where \( \sigma(r) = 0 \) (i.e. without matter) the exponential in (15) will dominate remaining terms and so there is exponential convergence.
- At the radii with \( \sigma(r) \neq 0 \) the behaviour depends on the actual position.
  - Between axis and equatorial plane there will be conditional convergence (as \( n^{-1} \)) in the case of ring and absolute convergence (as \( n^{-2} \)) in the case of disc.
  - At the equatorial plane there will be logarithmic divergence in the case of ring and absolute convergence (as \( n^{-2} \)) in the case of disc.
  - On the axis the situation is much more complicated. When the source is ring, there is conditional convergence (as \( n^{-1/2} \)) when considering gravitational potential and divergence (as \( n^{1/2} \)) for dragging. In the disc case convergence is one order faster, i.e. absolute for \( \nu \) (as \( n^{-3/2} \)) and conditional for \( \omega \) (as \( n^{-1/2} \)).

CONCLUSIONS

The method used by Will [4] when considering perturbation of a Schwarzschild black hole by a light slowly rotating ring can be extended to also enable disc perturbation. On the other hand, it involves expansions which do not behave well numerically. We have shown here that the series used in the first perturbation order are convergent almost everywhere, but the convergence is indeed slow at radii where the source is present (in the equatorial plane).

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PERTURBATION OF A SCHWARZSCHILD BLACK HOLE DUE TO A ROTATING THIN DISC

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ABSTRACT

Will (1974) treated the perturbation of a Schwarzschild black hole due to a slowly rotating light concentric thin ring by solving the perturbation equations in terms of a multipole expansion of the mass-and-rotation perturbation series. In the Schwarzschild background, his approach can be generalized to the perturbation by a thin disc (which is more relevant astrophysically), but, due to a rather bad convergence properties, the resulting expansions are not suitable for specific (numerical) computations. However, we show that Green’s functions represented by the Will’s result can be expressed in a closed form (without multipole expansion) which is more useful. In particular, they can be integrated out over the source (thin disc in our case), to yield well converging series both for the gravitational potential and for the dragging angular velocity. The procedure is demonstrated, in the first perturbation order, on the simplest case of a constant-density disc, including physical interpretation of the results in terms of a one-component perfect fluid or a two-component dust on circular orbits about the central black hole. Free parameters are chosen in such a way that the resulting black hole has zero angular momentum but non-zero angular velocity, being just carried along by the dragging effect of the disc.

Keywords: gravitation, black hole physics, accretion discs

1. INTRODUCTION

Disc-like structures around very compact bodies are likely to play a key role in the most energetic astrophysical sources like active galactic nuclei, X-ray binaries, supernovas and gamma-ray bursts. Analytical modeling of such structures relies on various simplifying assumptions, the basic ones being their stationarity, axial symmetry and test (non-gravitating) nature (see e.g. Kato et. al 2008). The last assumption is justified by two arguments: (i) in the above astrophysical systems, the disc mass is typically much smaller than that of the black hole (or neutron star) in their center, so the latter surely dominate the gravitational potential as well as the “radial” field; (ii) black holes are the strongest possible (extended) gravitational sources (and neutron stars are just slightly less compact), so they would - at least in a certain region - even dominate the field if their mass was less than that of the matter around. However, such arguments need not hold for a latitudinal component of the field (namely perpendicular to the disc),1 and, most importantly, the additional matter may in fact dominate the second and higher derivatives of the metric (curvature). These higher derivatives are in turn crucial for stability of the matter’s motion, and thus the tricky issue of self-gravity enters the problem. Actually, a real, massive matter may thus assume quite a different configuration than a test matter (Abramowicz et. al 1984). One should also add that even if the accreting matter really had only a tiny effect on the geometry, it could still change the observational record of the source significantly, in particular, it may change the long-term dynamics of bodies orbiting in the system (e.g. Suková & Semerák 2013 and references therein).

Hence, the properties of accretion systems may be sensitive to the precise shape of the field (Semerák 2003, 2004). Unfortunately, general relativity is non-linear and the fields of multiple sources mostly cannot be obtained by a simple superposition. Such more complicated fields are being successfully treated numerically (this even applies to strongly time-dependent cases including gravitational collapse, collisions and waves), but, for the present, the compass of explicit analytical solution terminates at systems with very high degree of symmetry, practically at static and axially symmetric cases. It would be most desirable to extend this to stationary cases, namely those admitting rotation. Stationary axisymmetric problem is usually represented in the form of the Ernst equation, but actually being tackled is the corresponding linear problem (Lax pair of equations whose integrability condition is just the Ernst equation). Exact solutions of this problem have been searched for in several ways. Klein & Richter (2005) and Meinel et. al. (2008) summarized “straightforward” but rather involved treatment of the respective boundary-value problem, providing both the black-hole and the thin-disc solutions, plus prospects of how to also obtain their non-linear superpositions. Other attempts employed the “solution-generating” techniques - mathematical procedures which transform one stationary axisymmetric metric into another and can in principle provide any solution of this type. The practical power of these methods strongly depends on how simple is the “seed” metric, so usually a static one is started from. Using the soliton (inverse-scattering) method of Belinsky & Zakharov, Tomimatsu (1984), Krori & Bhattacharjee (1990), Chandhuri & Das (1997) and Zellerin & Semerák (2000) generated black holes immersed in external fields, but at least the case corresponding to a hole surrounded by a thin disc (Zellerin & Semerák 2000) turned out to be unphysical (Semerák 2002). More successful seem to have been Bretón et. al (1997) who started from a different representation

1 For a Schwarzschild black hole which is spherically symmetric, there is no latitudinal field of course, so any additional source would automatically dominate this component.
of the static axisymmetric seed and managed to "install" a rotating black hole in it (see also Bretón et. al (1998) for a charged generalization).

If the external-matter gravity is weak, the problem may be treated as a small perturbation of the central-source metric, determined by linearized Einstein equations. In doing so, one can restrict to a special type of perturbations, for example, to stationary and axisymmetric ones. The method can be iterated; in a limit case (many iterations), it goes over to a solution in terms of series. The result then need not any longer represent a tiny variation of any "almost right" metric: it may even be put together on Minkowski background, with the "strong" part (e.g. a black hole) "dissolved" within the fundamental systems of the equations. The main problem of this scheme is convergence and meaning of the series.

No less than 43 years ago, the paper by Will (1974), published in this journal, became a seminal reference in the subject. It provided the gravitational field of a light and slowly rotating thin equatorial ring around an (originally Schwarzschild) black hole by mass-and-rotational perturbation of the Schwarzschild metric. (See also the following paper Will (1975) where basic properties of the obtained solution were discussed.) Unfortunately, the perturbation-scheme success depends strongly on how simple the background metric is – and the Schwarzschild metric is exceptionally simple: the Will’s procedure cannot be simply extended to a Kerr background. Consequently, only partial questions have been answered explicitly in these directions, in particular the one of deformation of the Kerr black-hole horizon (Demianski 1976; Chrzanowski 1976) (interestingly, these two results do not agree on certain points, mainly in the limit of an extreme black hole).2

Recently the Will's black-hole–ring problem has been revisited by Sano & Tagoshi (2014), but using the perturbation approach of Chrzanowski, Cohen and Kegeles in which the metric is found on the basis of solution of the Teukolsky equation for the Weyl scalars. The Will’s results have also been followed by Hod who analyzed the behaviour of the innermost stable circular orbit in the black-hole–ring field (Hod 2014) and the relation between the angular velocity of the horizon and the black-hole and ring angular momenta (Hod 2015).

In the present note, we check whether the Will’s scheme can be adapted to the case of a thin equatorial stationary and axisymmetric disc. [Preliminary results were presented in Čižek & Semerák (2009) and Čižek (2011).] In converging to a positive answer, we first observed that the expansions in spherical harmonics that typically come out in this approach (even in computing just the linear terms) converge rather badly and their numerical processing is problematic. Much more effective is the usage of Green functions of the problem, namely the perturbations generated by an infinitesimal linearized metric, determined by a positive answer, we first observed that the expansions in spherical harmonics that typically come out in this approach (even in computing just the linear terms) converge rather badly and their numerical processing is problematic. Much more effective is the usage of Green functions of the problem, namely the perturbations generated by an infinitesimal linearized metric, determined by which

The paper is organized as follows. In section 2 we introduce equations describing the gravitational field of a thin disc. Section 3 summarizes Will’s approach and section 4 discusses its (un)suitability for a numerical treatment. In section 5 we compute Green functions of the problem in a closed form and in section 6 we show, on linear perturbation by a thin annular concentric disc, that they can be integrated in order to obtain a perturbation generated by a given (stationary and axisymmetric) distribution of mass. The resulting series converge much better and allow to compute specific configurations explicitly.

Notation and conventions: our metric signature is (−+++). Greek indices run 0-3 and partial derivative is denoted by a comma. Complete elliptic integrals are given in terms of

\[ K(k) := \int_0^\frac{\pi}{2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}, \quad E(k) := \int_0^\frac{\pi}{2} \sqrt{1 - k^2 \sin^2 \alpha} \, d\alpha, \quad \Pi(n,k) := \int_0^\frac{\pi}{2} \frac{d\alpha}{(1 - n \sin^2 \alpha)\sqrt{1 - k^2 \sin^2 \alpha}}. \]

2. BLACK-HOLE & THIN-DISC SYSTEM: EINSTEIN EQUATIONS AND BOUNDARY CONDITIONS

We will search for the black-hole–disc field by perturbation of the Schwarzschild metric, while restricting to the simplest space-times which can host rotating sources, namely to those which are stationary and axially symmetric. In addition, we will consider asymptotically flat space-times, without cosmological term, and will require their orthogonal transitivity (i.e., the motion of sources will be limited to stationary circular orbits). In such space-times, the time and axial Killing vector fields \( n^\mu = \frac{\partial}{\partial r} \) and \( \xi^\mu = \frac{\partial}{\partial \phi} \) exist and commute, and the tangent planes to meridional directions (locally orthogonal to both Killing vectors) are integrable. Needless to say, it is assumed that there exists an axis of the space-like symmetry, namely a connected 2D (time-like) set of fixed points of the space-like isometry. In isotropic-type spheroidal coordinates \((t, r, \theta, \phi)\) (of which \( t \) and \( \phi \) have been chosen as parameters of the Killing symmetries), the metric with these properties can – for instance – be written in the "Carter-Thorne-Bardeen" form (e.g. Bardeen 1973)

\[ ds^2 = -e^{2\nu}dt^2 + e^{2\nu}e^{2\nu}d\varphi^2 + e^{2\nu}e^{-2\nu}d\theta^2 + e^{2\nu}e^{-2\nu}(d\varphi - \omega dt)^2 + e^{2\nu}e^{-2\nu}(d\varphi + \omega dt)^2, \]

where the unknown functions \( \nu, B, \omega \) and \( \zeta \) depend only on \( r \) and \( \theta \) covering the meridional surfaces. Besides the above coordinates, we will also occasionally use the Weyl-type cylindrical coordinates \( \rho = r \sin \theta \) and \( z = r \cos \theta \).

Apart from the asymptotic flatness, the boundary conditions have to be fixed on the symmetry axis, on the black-hole horizon and on the external-source surface. Regularity of the \( \text{axis} \) (local flatness of the orthogonal surfaces \( z = \text{const at } \rho = 0 \)) requires that \( \zeta \mapsto B \) at \( \rho \to 0^+ \). The invariants \( g_{tt} = g_{\theta \theta} \eta^2, g_{\phi \phi} = g_{\phi \phi} \xi^2, g_{\phi \phi} = g_{\phi \phi} \xi^3, g_{\phi \phi} = g_{\phi \phi} \xi^3 \xi^2 \) have to be

\(^2\) Note that another approximation possibility is the post-Newtonian expansion. The composition of a rotating gravitational center with a massive ring in Keplerian rotation was tackled, using the gravito-electromagnetic analogy, by Ruggiero (2016).
even functions of \( \rho \) (in order not to induce a conical singularity on the axis). Should the circumpolar radius \( \sqrt{g_{\phi\phi}} \) grow linearly with proper cylindrical radius \( \rho [e^{\nu}\rho]^{0} \), thus with \( \rho \), there must be \( g_{\phi\phi} \approx O(\rho^{2}) \), and, demanding the finiteness of \( \omega \), also \( -g_{\rho\phi} = g_{\phi\phi} \omega \approx O(\rho^{2}) \).

The stationary horizon is characterized by \( e^{2\nu} = 0 \) and \( \omega = \text{const} = \omega_{H} \) (in our coordinates it specifically means that \( \omega_{r} = 0 \) there). In order that the azimuthal and latitudinal circumferences of the horizon be positive and finite, the functions \( B e^{-2\nu} \) and \( e^{2\nu} \omega \) have to be such; the latter ensures regularity of \( g_{rr} \) as well. Hence, \( Br = 0 \) and \( e^{2\nu} \) on the horizon. (Let us add in advance that the field equations also imply that \( \omega_{r}, B, \omega \) and \( \zeta, z \) have to be finite on the horizon, so \( \omega_{H} \) has to vanish there as well.)

Now for boundary conditions on the external source. We assume that the latter has the form of an infinitesimally thin disc in the equatorial plane \( z = 0 \), stretching over some interval of radii lying above the central black-hole horizon. We assume that the disc bears neither charge nor current (there are no EM fields) and that the space-time is reflection symmetric with respect to its plane. The metric is then continuous everywhere, but has finite jumps in the first normal derivatives \( g_{\alpha\beta}, \zeta, z \) across the disc. The functions \( \nu, B, \omega \) and \( \zeta, z \) must be even in \( z \), their \( \rho \)-derivatives are odd in \( \zeta \), and even powers and multiples of derivatives (for example, \( B_{,\zeta} \zeta_{,\zeta} \)) are even in \( \zeta \) (therefore they do not jump across \( z = 0 \)).

In order that the space-time be stationary, axially symmetric and orthogonally transitive, the disc elements must only move along surfaces spanned by the Killing fields, namely they must follow spatially circular orbits with steady angular velocity \( \Omega = \frac{\omega}{\rho} \). This corresponds to four-velocity

\[
\begin{align*}
\omega^{\alpha} &= \frac{\eta^{\alpha} + \Omega \xi^{\alpha}}{|\eta^{\alpha} + \Omega \xi^{\alpha}|} = u^{\alpha}(1, 0, 0, \Omega), & u_{\alpha} &= -u^{\nu}e^{2\nu}\delta^{\alpha}_{\nu} + u^{\beta}B\nu (-\omega, 0, 0, 1) \quad (2)
\end{align*}
\]

with

\[
(4) = \frac{e^{-2\nu}}{1 - B^{2}\rho^{2} e^{-4\nu}(\Omega^{2} - \omega^{2})} = \frac{e^{-2\nu}}{1 - \omega^{2}}.
\]

represents linear velocity with respect to the local zero-angular-momentum observer (ZAMO). For the thin discs \( (T_{z}^{\alpha} = 0, T^{\rho} = 0) \) without radial pressure \( (T^{\rho} = 0) \), the surface-energy-momentum tensor

\[
\int_{-\infty}^{\infty} T_{\beta}^{\alpha} g_{zz} dz = \int_{-\infty}^{\infty} T_{\beta}^{\alpha} e^{2\nu} dz =: S_{\beta}^{\alpha}(\rho) \quad \Rightarrow \quad T_{\beta}^{\alpha} e^{2\nu} =: S_{\beta}^{\alpha}(\rho) \delta(z) \quad (3)
\]

has only three non-zero components \( (S_{\lambda}^{\phi}, S_{\phi}^{\rho}, S_{\phi}^{\phi}) \), representing energy density, orbital-momentum density and azimuthal pressure, respectively. If \( (S_{\lambda}^{\phi} - S_{\phi}^{\phi})^{2} + 4S_{\phi}^{\rho} S_{\phi}^{\phi} \geq 0 \), it can be diagonalized to \( S^{\alpha\beta} = \sigma u^{\alpha}u^{\beta} + Pu^{\alpha}w^{\beta} \), where \( \sigma \) and \( P \) (more precisely, \( \sigma e^{\nu-\zeta} \) and \( Pe^{\nu-\zeta} \)) stand for the surface density and azimuthal pressure in a co-moving frame and \( w^{\alpha} \) is the “azimuthal” vector perpendicular to \( u^{\alpha} \), with components \( w^{\alpha} = \frac{1}{\rho^{2}} (u_{\phi}, 0, 0, -u_{z}) \), \( w_{\alpha} = \rho B (-u_{\phi}, 0, 0, u^{\nu}) \). Hence, the surface-tensor components read

\[
S_{\lambda}^{\phi} = -\sigma - (\sigma + P) u^{\phi}u_{\rho}, \quad S_{\phi}^{\rho} = (\sigma + P) u^{\nu}u_{\phi}, \quad S_{\phi}^{\phi} = P + (\sigma + P) u^{\phi}u_{\phi}. \quad (4)
\]

Orthogonally transitory stationary and axisymmetric space-times are described by 5 independent Einstein equations. In our case of a thin disc, the energy-momentum tensor has only \( T_{\phi}^{\phi} \) and \( T_{\phi}^{\rho} \) components and the equations read\(^3\)

\[
\nabla \cdot (\rho \nabla \zeta) = 0, \quad \nabla \cdot (B \nabla \nu) - \frac{B^{2}B_{\nu}^{2}}{2e^{4\nu}} (\nabla \omega)^{2} = 4\pi B e^{2\nu} - 2\Omega T_{\phi}^{\phi} - T_{\phi}^{\phi} = 4\pi B (\sigma + P) \frac{1 + u^{2}}{1 - \omega^{2}} \delta(z), \quad (5)
\]

\[
\nabla \cdot (B^{2} \rho^{2} e^{-4\nu} \nabla \omega) = -16\pi B e^{2\nu} T_{\phi}^{\phi} = -16\pi B^{2} e^{-2\nu} (\sigma + P) \frac{u^{\nu}}{1 - \omega^{2}} \delta(z), \quad (6)
\]

\[
\zeta_{\phi} + \zeta_{\phi} + (\nu_{\phi})^{2} + (\nu_{\phi})^{2} - 3B^{2} \rho^{2} (\omega_{,\phi}^{2} + (\omega_{,\rho})^{2}) = 8\pi e^{2\nu} - 2\Omega T_{\phi}^{\phi} - \omega T_{\phi}^{\phi} = \sigma \frac{u^{\nu}}{1 - \omega^{2}} \delta(z), \quad (7)
\]

\[
\zeta_{\rho} + \zeta_{\rho} - \zeta_{\phi} (B_{\phi})_{,\phi} = -B (\nu_{\rho}^{2} - (\nu_{\phi})^{2}) - \frac{1}{4} (\nu_{\rho}^{2} - (\nu_{\phi})^{2})^{2} + \frac{1}{4} B^{2} \rho^{2} e^{-4\nu} (\omega_{,\phi}^{2} + (\omega_{,\phi})^{2}), \quad (8)
\]

\[
\zeta_{\rho} (B_{\phi})_{,\rho} + \zeta_{\phi} (B_{\phi})_{,\phi} = -2B (\nu_{\rho}^{2} - (\nu_{\phi})^{2}) + \frac{1}{2} B^{2} \rho^{2} e^{-4\nu} \omega_{,\phi}^{2}, \quad (9)
\]

\[
\zeta_{\rho} (B_{\phi})_{,\rho} + \zeta_{\phi} (B_{\phi})_{,\phi} = -2B (\nu_{\rho}^{2} - (\nu_{\phi})^{2}) + \frac{1}{2} B^{2} \rho^{2} e^{-4\nu} \omega_{,\phi}^{2}, \quad (10)
\]

where \( \nabla \) and \( \nabla \cdot \) denote gradient and divergence in an (auxiliary) Euclidean three-space. The last two equations (for \( \zeta \)) are integrable, provided the first three vacuum equations hold. The axis boundary condition \( e^{\nu} = B \) implies that the \( \zeta \) function can elsewhere be obtained according to \( \zeta(r, \theta) = \int_{0}^{\theta} \zeta_{\phi} \text{d}z + \ln B \), where \( \zeta_{\phi} \) follows from the last two field equations.

\(^3\) Equation (8) is not independent, but we include it here since it provides the jump of \( \zeta_{\phi} \) across the equatorial plane given later in section 2.2.
The treatment of the stationary axisymmetric problem (5)–(10) usually starts from a suitable solution of the first equation (5). In the parametrization (coordinates) we use here, it is convenient to choose

\[ B = 1 - \frac{k^2}{4a^2}. \]  

With such a choice, the horizon lies where \( B = 0 \), hence on \( r = k/2 \). This reveals the meaning of the constant \( k \) (which is supposed to be positive), in particular, for Schwarzschild one has \( k = M \); for Kerr it would be \( k = M + \sqrt{M^2 - a^2} \), with \( a \) the centre’s specific angular momentum (\( k = 0 \) would correspond to an extreme black hole, or to a Minkowski space-time).

The main task is to solve the coupled equations (6) and (7), and then to integrate equations (9) and (10) for \( \zeta \). With the choice \( B = 1 - \frac{k^2}{4a^2} \) (thus with \( B_{\theta} = 0 \)) and written out explicitly in the \((t, \rho, \nu, \phi)\) coordinates, these equations read

\[ (r^2 \nu_\rho)_\rho + r^2 \nu_\rho (\ln B)_\rho + \nu_{\rho \theta} + \nu_{\theta \rho} \cot \theta = \frac{B^2 r^2}{2e^{4\nu}} \sin^2 \theta \left[ r^2 (\omega_\rho)^2 + (\omega_\theta)^2 \right] + 4\pi r^2 (\sigma + P) \frac{1 + v^2}{1 - v^2} \delta(z), \]

\[ r^2 \omega_{\rho \rho} + 4r \omega_{\rho \nu} (1 - r \nu_\nu) + 3r^2 \omega_\rho (\ln B)_\rho + \omega_{\rho \theta} + 3\omega_{\theta \rho} \cot \theta - 4\omega_{\nu \nu} \nu_{\theta \rho} = -\frac{16\pi r e^{4\nu}}{B \sin \theta} (\sigma + P) \frac{v}{1 - v^2} \delta(z), \]

\[ (2 - B) r \zeta_\rho \cot \theta + B \zeta_\rho = B \left[ (r^2 (\nu_\rho)^2 - (\nu_\theta)^2) + 2B - 2 \frac{1}{4} B^2 e^{2\nu} \sin^2 \theta \left[ r^2 (\omega_\rho)^2 - (\omega_\theta)^2 \right] \right], \]

\[ Br \zeta_\rho \cot \theta + (2 - B) \zeta_\theta = 2Br \nu_{\rho \nu} \nu_{\theta} + 2(1 - B) \cot \theta - \frac{1}{4} B^2 e^{-4\nu} \omega_{\rho \rho} \omega_{\theta \theta} \sin^2 \theta. \]

2.1. Counter-rotating interpretation of thin discs

When asking about counterbalance to its (and black holes) gravity, the disc may either be considered as a solid structure (a set of circular hoops), or, to the contrary, as a non-coherent mix of azimuthal streams. In the astrophysical context, one usually adheres to the latter extreme possibility: that the disc is composed of two non-interacting streams of particles which follow stationary circular orbits in opposite azimuthal directions (Morgan & Morgan 1969; Lynden-Bell & Pineault 1978; Lamberti & Hamity 1981; Bičák et al. 1993; Bičák & Ledvinka 1993; Klein & Richter 1999; González & Espitia 2003; García-Reyes & González 2004). These orbits are geodesic if there is no radial stress acting within the disc \((T^n_\rho = 0)\). The surface energy-momentum tensor is thus decomposed as

\[ S^{\alpha \beta} = \sigma_{\alpha} u_{\rho \alpha} u_{\beta} + \sigma_{\beta} u_{\rho \beta} u_{\alpha}, \]

where the \(+/-\) signs indicate the stream orbiting in a positive/negative sense of \( \phi \), as taken with respect to the “average” fluid four-velocity \( u^\alpha \). The four-velocities are of the \( u^\alpha = u^\alpha (1, 0, 0, \Omega_{\pm}) \) form again, so

\[ S^\phi_\rho = B \rho e^{-2\nu} \left( \frac{\sigma + v_\rho}{1 - v^2} + \frac{\sigma - v_\rho}{1 - v^2} \right), \]

\[ S^\rho_\rho = -\omega S^\rho_\rho = \frac{\sigma + v_\rho^2}{1 - v^2} + \frac{\sigma - v_\rho^2}{1 - v^2}, \]

\[ S^t_\rho = -S^\rho_t = \frac{\sigma + v_\rho}{1 - v^2} + \frac{\sigma - v_\rho}{1 - v^2}, \]

now with \( \Omega_\pm \) (and the corresponding \( v_\rho \)) given by free circular motion, i.e., by the roots of equation \( g_{tt, \alpha} + 2g_{t \phi, \alpha} \Omega + g_{\phi \phi, \alpha} \Omega^2 = 0 \). Such a motion is only possible in the equatorial plane \((z = 0)\) in general, where just the radial component of acceleration remains relevant, vanishing if

\[ \Omega = \Omega_{\pm} = \pm \frac{g_{\phi \phi} \omega_\rho}{g_{\phi \phi, \rho}} \pm \sqrt{\left( \frac{\omega + g_{\phi \phi} \omega_\rho}{g_{\phi \phi, \rho}} \right)^2 - \frac{g_{tt, \rho}}{g_{\phi \phi, \rho}}}; \]

in particular, with the \( B = 1 \) choice, this expression reduces to

\[ \Omega_{\pm} = \pm \frac{\rho^2 \omega_\rho}{2 \rho (1 - v^2)} \pm \frac{4e^{4\nu} \rho \nu_\rho (1 - v^2)}{2 \rho (1 - v^2)}. \]

The interpretation is only possible where the expression under the square root is non-negative; physically, this is not satisfied for discs with “too much matter on larger radii”: in such a case, the total gravitational pull at a given location points outwards, so “no angular velocity is low enough” to admit Keplerian orbiting there. Parameters of the counter-rotating picture \((\sigma_{\pm}, \Omega_{\pm})\) and the “total”, one-stream parameters \((\sigma, \Omega, P)\) are related by comparing the respective two forms of the energy-momentum tensor, \( \sigma_{\alpha} u_{\rho \alpha} + Pu_{\rho \beta} u_{\beta} = \sigma_{\pm} u_{\rho \alpha} u_{\beta} + \sigma_{\beta} u_{\rho \beta} u_{\alpha} \). From the trace of this equation and from its projections onto \( u_{\alpha} u_{\beta} \) and \( u_\alpha u_\beta \), while using a suitable expression of the scalar products

\[ u_{\pm \alpha}^\rho u_{\alpha} = \frac{\rho vv_\rho - 1}{\sqrt{1 - v^2}} \frac{1}{\sqrt{1 - v^2}}, \]

\[ u_{\alpha}^\rho u_{\beta} = \frac{v_\alpha - v}{\sqrt{1 - v^2}} \frac{1}{\sqrt{1 - v^2}} \]

\[ g_{\alpha \beta} u_{\alpha} u_{\beta} = \frac{v_\alpha - v}{\sqrt{1 - v^2}} \frac{1}{\sqrt{1 - v^2}} \]

When working in the Weyl-type coordinates \((t, \rho, \nu, \phi)\) and with the corresponding, Weyl-Lewis-Papapetrou form of the metric, the \( B \)-equation is usually being satisfied by \( B = 1 \). This choice is advantageous in most respects, but it makes the horizon “degenerate” to a central segment of the symmetry axis \( \rho = 0 \), which is not suitable for discussion of its properties. Different solutions for \( B \) are also possible for the “Carter-Thorne-Bardeen” form of the metric we use here; however, changing \( B \) generally does not imply a real physical difference, it effectively corresponds to a certain re-definition of coordinates, cf. section 6.1 (only the \( B = 1/\rho \) choice leads to different, plane-wave solutions).
Disc perturbation of a Schwarzschild black hole

one finds
\[ \sigma - P = \sigma_+ + \sigma_- , \]
\[ \sigma = \frac{\sigma_+ (vv_+ - 1)^2}{(1 - v^2)(1 - v^2_+)} + \frac{\sigma_- (vv_- - 1)^2}{(1 - v^2)(1 - v^2_-)} , \]
\[ P = \frac{\sigma_+ (v_+ - v)^2}{(1 - v^2)(1 - v^2_+)} + \frac{\sigma_- (v_- - v)^2}{(1 - v^2)(1 - v^2_-)} . \]
(21)

\[ \frac{P}{\sigma} = \frac{u_+^2 w_+ u_+ w_+}{u_-^2 u_- u_-^2 u_-} = \frac{(v_+ - v)(v_- - v)}{(1 - vv_+)(1 - vv_-)} \implies \frac{\sigma + P}{\sigma_+ + \sigma_-} = 1 + \frac{P}{\sigma} = \ldots , \]
(22)

\[ \sigma_\pm(u_\pm^2)^2 = \pm \frac{S_\phi^2 - \Omega_+ S_\phi^b}{\Omega_+ - \Omega_-} \implies \sigma_\pm = \pm \frac{\sigma_\pm (v_+ - v - P v_+ - v_-)}{(1 - v^2_+)(1 - v^2_-)} . \]
(23)

We may also, for example, compare expressions for \( S_{\alpha \beta}S^{\alpha \beta} \), finding the relation
\[ \sigma^2 + P^2 = \sigma_+^2 + \sigma_-^2 + 2\sigma_+ \sigma_- (g_{\alpha \beta}u_\alpha u_\beta)^2 = \sigma^2 + \sigma_-^2 + 2\sigma_+ \sigma_- \frac{(1 - v_+ - v_-)^2}{(1 - v^2_+)(1 - v^2_-)} ; \]
(24)

which in combination with \( \sigma - P = \sigma_+ + \sigma_- \) also leads to
\[ \sigma P = \sigma_+ \sigma_- \frac{(v_+ - v_-)^2}{(1 - v^2_+)(1 - v^2_-)} \quad \text{and} \quad \left( \sigma_+ - \sigma_- \right)^2 = (\sigma + P)^2 - 4\sigma P \frac{(1 - v_+ v_-)^2}{(v_+ - v_-)^2} . \]
(25)

2.2. One-stream and two-stream interpretations: integrating jumps in the field equations

Relation between the jumps of \( g_{\alpha \beta, \mu} \) across the disc and \( S_\beta^2 \) are obtained by integrating the field equations (5)–(8) over the infinitesimal interval \( (z = 0^+ , z = 0^-) \). Only the terms proportional to \( \delta(z) \) (i.e. the source terms on the right-hand sides and the terms linear in \( B_{zz} , \nu_{zz} , \omega_{zz} \) and \( \zeta_{zz} \) on the left-hand side) contribute according to \( \int_{z=0^-}^{z=0^+} \nu_{zz} dz = 2\nu_z(z=0^+) \) (etc.), so we have \( B_z(z=0^+) = 0 \) and
\[ \nu_z(z=0^+) = 2\pi \left( S_\phi^2 - 2\omega S_\phi^b - S_\phi^b \right) = 2\pi (\sigma + P) \frac{1 + v^2}{1 - v^2} = 2\pi \left( \sigma_+ \frac{1 + v^2_+}{1 - v^2_+} + \sigma_- \frac{1 + v^2_-}{1 - v^2_-} \right) , \]
(26)
\[ \omega_z(z=0^+) = -8\pi \frac{S_\phi^2}{B^2 \rho^2 e^{-\omega}} = -8\pi (\sigma + P) \frac{\Omega - \omega}{1 - v^2} = -8\pi \left( \sigma_+ \frac{\Omega_+ - \omega}{1 - v^2_+} + \sigma_- \frac{\Omega_- - \omega}{1 - v^2_-} \right) , \]
(27)
\[ \zeta_z(z=0^+) = 4\pi \left( S_\phi^2 - \omega S_\phi^b \right) = 4\pi \frac{\sigma v^2 + P}{1 - v^2} = 4\pi \left( \sigma_+ \frac{v^2_+ + P}{1 - v^2_+} + \sigma_- \frac{v^2_- + P}{1 - v^2_-} \right) , \]
(28)

where we have expressed the results in terms of the one-stream as well as two-stream form of \( S_\beta^2 \).

3. PERTURBATION SCHEME

We will look for a solution of equations (6) and (7) in the form of series, expanding
\[ \nu = \sum_{j=0}^{\infty} \nu_j \lambda^j , \quad \omega = \sum_{j=0}^{\infty} \omega_j \lambda^j , \quad \zeta = \sum_{j=0}^{\infty} \zeta_j \lambda^j , \]
(29)
where the coefficients \( \nu_j , \omega_j \) and \( \zeta_j \) depend on \( r \) and \( \theta \) (or \( r \) and \( z \)) and the dimensionless parameter \( \lambda \) is proportional to the ratio of the disc mass to the black-hole mass \( M \). More specifically, let it be related by
\[ (\sigma + P)\delta(z) \equiv \lambda \Sigma(r)\delta(z) = \lambda \Sigma(r) \frac{1}{r} \delta(\cos \theta) = -\lambda \Sigma(r) \frac{1}{r} \delta(\theta - \pi/2) , \]
(30)
where \( \delta \) denotes the \( \delta \)-distribution of \( \Sigma \equiv \sigma + P \) is an “effective” surface density. The functions \( \nu_0 , \omega_0 \) and \( \zeta_0 \) represent the black-hole background, i.e. the Schwarzschild metric which in isotropic coordinates (recall that \( \rho = r \sin \theta \) and \( z = r \cos \theta \) reads
\[ ds^2 = -\left( \frac{2r - M}{2r + M} \right)^2 dt^2 + \left( 1 + \frac{M}{2r} \right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) , \]
(31)
hence
\[ \nu_0 = \ln \frac{2r - M}{2r + M} , \quad \omega_0 = 0 , \quad B = e^{\nu_0} = 1 - \frac{M^2}{4r^2} , \]
(32)
and the corresponding orbital velocity is
\[ \Omega_0 = \frac{8 \sqrt{Mr^3}}{(2r + M)^3} , \quad v_0 = \frac{2 \sqrt{Mr}}{2r - M} , \]
(33)
Substituting (29) and (30) into (6) and (7) and subtracting the pure-Schwarzschild terms, one obtains
\[
\sum_{k=1}^{\infty} \lambda^k \nabla \cdot (B \nabla \nu_k) = \sum_{k=2}^{\infty} \lambda^k \left\{ \frac{(2r + M)^2}{2r^3(2r - M)} \sum_{l=0}^{k-2} \exp \left( -4 \sum_{j=1}^{\infty} \lambda^j \nu_j \right) \sum_{m=1}^{k-l-1} \nabla \omega_m \cdot \nabla \omega_{k-l-m} \right\} + 4\pi \frac{B}{r} \sum_{k=0}^{\infty} \lambda^{k+1} \left[ \frac{1 + v^2}{1 - v^2} \right]_k,
\]
(34)
\[
\sum_{k=1}^{\infty} \lambda^k \nabla \cdot \left( \frac{(2r + M)^2}{2r^3(2r - M)} \right) \nabla \omega_k = -\sum_{k=2}^{\infty} \lambda^k \sum_{l=1}^{k-1} \nabla \cdot \left( \frac{(2r + M)^2}{2r^3(2r - M)} \right) \left[ \exp \left( -4 \sum_{j=1}^{\infty} \lambda^j \nu_j \right) \right]_{k-l} - 16\pi \left( 1 + \frac{M}{2r} \right)^4 \sum_{k=0}^{\infty} \lambda^{k+1} \left[ \frac{1 + v^2}{1 - v^2} \right]_k,
\]
(35)
where \([f]_k\) means the coefficient standing at \(\lambda^k\) in Taylor expansion of \(f\). Since the background is static, the first-order equations only contain mass-energy terms multiplied by the background metric on the right-hand sides, because the first-line sums do not contribute. In higher orders, the right-hand sides contain only lower-order terms. (For non-static backgrounds the equations do not decouple so easily.)

Now, the eigen-functions with respect to \(\theta\) of the operator on the left-hand sides of (34) and (35) are the Legendre polynomials \(P_j(\cos \theta)\) and the Gegenbauer polynomials \(C_j^{(3/2)}(\cos \theta)\), respectively.\(^5\) Introducing a dimensionless radius \(x := \frac{r}{M} \left( 1 + \frac{M^2}{r^2} \right)\), we may thus write
\[
\nu_l = \sum_{j=0}^{\infty} \nu_{lj}(x) P_j(\cos \theta), \quad \omega_l = \sum_{j=0}^{\infty} \omega_{lj}(x) C_j^{(3/2)}(\cos \theta).
\]
(36)
Substituting this into (34) and (35) leads to
\[
\sum_{j=0}^{\infty} \left\{ \frac{d}{dx} \left[ (x^2 - 1) \frac{d \nu_{lj}}{dx} \right] - j(j+1) \nu_{lj} \right\} P_j(\cos \theta) = R_l(x, \theta),
\]
(37)
\[
\sum_{j=0}^{\infty} \left\{ (x^2 - 1) \frac{d}{dx} \left[ (x + 1)^4 \frac{d \omega_{lj}}{dx} \right] - (x + 1)^4 j(j+1) \omega_{lj} \right\} C_j^{(3/2)}(\cos \theta) = S_l(x, \theta),
\]
(38)
where \(R_l(x, \theta)\) and \(S_l(x, \theta)\) stand, up to an \(l\)-independent multiplication factor, for the coefficients of expansion (with respect to \(\lambda\)) of the right-hand sides of equations (34) and (35); specifically,
\[
\sum_{l=0}^{\infty} R_l(x, \theta) \lambda^l = \frac{r^2}{B} \text{[r.h.s. of (34)]}, \quad \sum_{l=0}^{\infty} S_l(x, \theta) \lambda^l = \frac{B r^4}{M^4 \sin^2 \theta} \text{[r.h.s. of (35)].}
\]
(39)
Provided that \(R_l\) and \(S_l\) do not diverge on the axis \(\theta=0, \pi\), these coefficients can be also decomposed as
\[
R_l(x, \theta) = \sum_{j=0}^{\infty} R_{lj}(x) P_j(\cos \theta), \quad S_l(x, \theta) = \sum_{j=0}^{\infty} S_{lj}(x) C_j^{(3/2)}(\cos \theta),
\]
(40)
where
\[
R_{lj}(x) = \frac{2j + 1}{2} \int_{-1}^{1} R_l(x, \theta) P_j(\cos \theta) d(\cos \theta),
\]
(41)
\[
S_{lj}(x) = \frac{2j + 3}{2(j + 1)(j + 2)} \int_{-1}^{1} S_l(x, \theta) C_j^{(3/2)}(\cos \theta) \sin^2 \theta d(\cos \theta).
\]
(42)
Demanding that the equations (37) and (38) hold for each order (multipole moment) separately, one obtains a system
\(^5\) In Will's article the latter are denoted by \(T_l^{3/2}(\cos \theta)\).
of independent ordinary differential equations

\[ \frac{d}{dx} \left[ (x^2 - 1) \frac{d\nu_j}{dx} \right] - j(j + 1) \nu_j = R_{ij}, \]  

\[ (x^2 - 1) \frac{d}{dx} \left[ (x + 1)^4 \frac{d\omega_j}{dx} \right] - (x + 1)^4 j(j + 3) \omega_j = S_{ij}, \]  

where \( l \in \mathbb{N}, j \in \mathbb{Z}_0^+ \). They only contain lower-order source terms (assumed to be given) for every \( l \) and are to be supplemented by boundary conditions on the horizon \( x = 1 \) (\( \nu_j \) and \( \omega_j \) are supposed to be regular there) and at spatial infinity (\( \nu_j \) and \( \omega_j \) should vanish there). Of many techniques available for such equations, we shall focus on finding their Green functions.

Let us start from fundamental systems of equations (43) (44). The first one is the Legendre differential equation whose fundamental system can be expressed as linear combination of Legendre functions of the first and second kinds

\[ P_j(x) \quad \text{and} \quad Q_j(x) = P_j(x) \int_1^\infty \frac{d\xi}{(\xi^2 - 1)\left[ P_j(\xi)\right]^2}. \]  

For the second equation, Will (1974) used the substitution \( t = (x + 1)/2 \) which transforms it to the hypergeometric differential equation, having two generators of the fundamental system,

\[ F_j(x) = 2F_1 \left( -j, j + 3; \frac{x + 1}{2} \right) \quad \text{and} \quad G_j(x) = F_j(x) \int_1^\infty \frac{d\xi}{(\xi + 1)^4 \left[ F_j(\xi)\right]^2}, \]  

where \( 2F_1(a, b; c; \xi) \) denotes the Gauss hypergeometric function. Note that since \( j \in \mathbb{Z}_0^+ \), \( F(x) \) is in fact a polynomial of degree \( j \). Asymptotically (as \( x \to \infty \)), \( P_j(x) \sim x^j \), \( Q_j(x) \sim x^{-l-1} \), \( F(x) \sim x^j \) and \( G(x) \sim x^{-j-3} \). At the horizon \( x = 1 \), \( Q_j(x) \) and \( G_j(x) \) diverge (except \( G_0(x) \) which will be discussed later).

Given the above boundary conditions, the Green functions of equations (37), (38) can be found in the form

\[ G_j^j(x, x') = \begin{cases} -Q_j(x) P_j(x') & \text{for } x \geq x' \\ -P_j(x) Q_j(x') & \text{for } x \leq x' \end{cases}, \]  

\[ G_j^\omega(x, x') = \begin{cases} -G_j(x) F_j(x') & \text{for } x \geq x' \\ -F_j(x) G_j(x') & \text{for } x \leq x' \end{cases}, \]  

and their inhomogeneous solutions as

\[ \nu_j(x) = \int_1^\infty R_{ij}(x') G_j^\nu(x, x') \, dx', \]  

\[ \omega_j(x) = \int_1^\infty S_{ij}(x') G_j^\omega(x, x') \, dx' + \frac{J_i \delta_j^0}{(x + 1)^3}, \]  

where \( J_i \) are arbitrary constants, representing a choice of the black-hole spin (see section 3.1). Such a solution is unique up to a coordinate transformation. Note that one could add to the Green functions terms proportional to \( P_0(x) = F_0(x) = 1 \) which is everywhere finite, but such an addition only corresponds to a rescaling of time and (thus) of the coordinate angular velocity, so it has no invariant physical effect.

Using (36), we may write

\[ \nu_l(x, \theta) = \sum_{j=0}^{\infty} \nu_j(x) P_j(\cos \theta) = \sum_{j=0}^{\infty} R_{ij}(x') G_j^\nu(x, x') P_j(\cos \theta) \, dx' = \int_1^1 R_l(x', \theta') G_l^\nu(x, \theta, x', \theta') \, dx' \, d(\cos \theta'), \]  

\[ \omega_l(x, \theta) = \sum_{j=0}^{\infty} \omega_j(x) C_j^{(3/2)}(\cos \theta) = \sum_{j=0}^{\infty} S_{ij}(x') G_j^\nu(x, x') C_j^{(3/2)}(\cos \theta) \, dx' = \int_1^1 S_l(x', \theta') G_l^\omega(x, \theta, x', \theta') \, dx' \, d(\cos \theta'), \]  

(51)
where
\[ G^{\nu}(x, \theta, x', \theta') := -\sum_{j=0}^{\infty} \frac{2j + 1}{2} P_j(\min(x, x')) Q_j(\max(x, x')) P_j(\cos \theta') P_j(\cos \theta), \]
\[ G^{\nu}(x, \theta, x', \theta') := -\sum_{j=0}^{\infty} \frac{2j + 3}{2(j+1)(j+2)} F_j(\min(x, x')) G_j(\max(x, x')) C_j^{(3/2)}(\cos \theta) C_j^{(3/2)}(\cos \theta') \left[ + \frac{J_l}{(x+1)^{l}} \right] \]
are Green's functions of homogeneous parts of equations (34) and (35). These represent a perturbation by an infinitesimal (2D) circular ring placed at \( x', \theta' \) in the first order in \( \lambda \). (We omit the \( J_l \) term in the following, see below.)

3.1. Differences from the Will’s article

Will (1974) found the Green functions of equations (47) and (48) or, more precisely, he proposed the inhomogeneous solutions (49) and (50). He employed three expansions of the solutions: with respect to the linear mass density of the ring, with respect to the angular velocity of the horizon, and the multipole expansion performed to convert partial differential equations into the ordinary ones. In contrast, we use only two expansions: with respect to the disc density and the multipole expansion. Below (see section 5), we will even be able to drop the multipole expansion and work solely with expansion with respect to the disc (ring) density. Namely, the rotational expansion can be “reconstructed” using the constants \( J_l \) (not employed in the Will’s paper). This is possible due to \( F_l(1) = 0 \) for \( j > 0 \) (see, for example, (67)), which implies that higher multipole moments do not contribute to the black-hole angular velocity at all, the only important entering terms (in a given order \( \lambda' \) of the mass perturbation) being proportional to \( F_0(x) = 1 \), \( G_0(x) = (x+1)^{-3} \) or \( \omega_0(x) \), where \( \omega_0 \) is the inhomogeneous solution of (44) given by (50) (with \( J_l = 0 \)). The first term \( F_0(x) = 1 \) only adds constant coordinate angular velocity, so it is not a physical degree of freedom (it can be cancelled out by a coordinate transformation). The meaning of the remaining two terms is revealed by their behaviour in a vicinity of the disc (ring): \( G_0 \) is smooth everywhere, but \( \omega_0 \) jumps in its first derivative (in the disc case it thus contributes to the density of the disc energy-momentum). Hence, the term proportional to \( G_0 \) can be interpreted as the black hole’s “own” angular velocity (perturbation towards the Kerr solution), while the terms proportional to \( \omega_0 \) represent rotational perturbations due to the presence of the rotating disc (ring). Needless to say, such an interpretation is not unique, because \( \omega_0 + C_l G_0 \) (\( C_l \) are arbitrary constants) is also a solution of (44) contributing to the source. Since there is no clear way how to say which choice of \( C_l \) is the “correct” one, we will adhere to \( C_l = 0 \) for simplicity.

Having prescribed the black-hole angular velocity
\[ \omega_H = \sum_{l=1}^{\infty} \beta_l \lambda^l, \]
one can find the \( J_l \) constants according to
\[ J_l = 8 \left[ \beta_l - \omega_0(1) \right]_{J_l = 0} = 8 \beta_l + \int_1^\infty \frac{8 S_{00}(x')}{(x'+1)^3} dx', \]
where \( \omega_0 |_{J_l = 0} \) stands for the right-hand side of (50) with \( J_l = 0 \), and \( S_{00} \) contains lower-order terms of the expansion in \( \lambda \) (i.e., it also contains \( J_k \) with \( k < l \)).

4. CONVERGENCE AND RELATED ISSUES

The main attribute of the above procedure is its speed of convergence to the desired solution. The problem is familiar from spectral methods (multipole expansion can actually be viewed as a spectral method): the numerics works well when the desired result is smooth, otherwise convergence problems arise. More specifically, one can expect exponential convergence for analytical function, whereas at most a power-law one for functions having some derivatives discontinuous (see, for example, Grandclément & Novak 2008).

Let us begin with the potential for the ring case. Focusing on the region outside the black hole, \( 1 < x < x_0 \), and regarding the asymptotic behaviour of special functions (see, for example, Olver et. al. 2010), one can, after lengthy calculations, find that
\[ 2j + 1 \left( x_0 + \sqrt{x_0^2 - 1} \right)^{1+j} \frac{P_j(x_0) Q_j(x_0) P_j(0) + 1}{x_0 + \sqrt{x_0^2 - 1}} \frac{f(x, x') + O(1/j)}{x_0 + \sqrt{x_0^2 - 1}}, \]
where \( f(x, x') \) is a suitable function independent of \( j \), \( x_0 := \min(x, x') \) and \( x_0 := \max(x, x') \). This implies that the sum (51) exponentially converges outside the ring radius and conditionally converges (like \( 1/j \)) just on the radius of the ring. Practically, this means that near the ring radius one has to include quite many terms (in \( j \)) in order to get reasonably small oscillations. (Convergence could be expected to be slow near a singular source.)
In the disc case, the situation is better and worse at the same time. After a lengthy calculation again, one finds that
\[
\left| \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{2j+1}{2} R_l(x_R) P_j(\cos \theta) P_j(x_<) Q_j(x_>) P_j(0) \, dx_R \right| < \frac{1}{j^2} \left( \frac{x_< + \sqrt{x_<^2 - 1}}{x^>_2 + \sqrt{x^>_2 - 1}} \right)^{\frac{j+1}{2}} \left[ |f(x, x')| + O(1/j) \right],
\] (58)
where \(x_<\) and \(x_>\) have the same meaning as above, i.e. \(x_< := \min(x, x_R)\) and \(x_> := \max(x, x_R)\), and \(x'_<, x'_>\) stand for analogous quantities taken with respect to the nearest radius lying inside the disc \((x'_R)\), i.e.,
\[
x'_< = \begin{cases} x & \text{when } x \leq x_{\text{out}} \\ x_{\text{out}} & \text{when } x > x_{\text{out}} \end{cases}, \quad x'_> = \begin{cases} x_{\text{in}} & \text{when } x \leq x_{\text{in}} \\ x & \text{when } x \geq x_{\text{in}} \end{cases},
\] (59)
where \(x_{\text{in}}\) and \(x_{\text{out}}\) represent inner and outer rim of the disc.

Equation (58) factually yields a lower bound of the convergence speed. (One can already obtain a speed estimate by checking the simplest example, i.e. constant \(S_{ij}\).) We can thus conclude that in the disc case the convergence is exponentially fast outside the radii of the disc and polynomial (like \(j^{-2}\)) at the radii “within” the disc. Therefore, it is generally impossible to improve the polynomial convergence using interpolation in some infinitesimally small area.

Calculation of dragging \((\omega_{ij})\) shows similar behaviour, i.e. exponential convergence outside the radii of the ring (disc) and polynomial one like \(j^{-1}\) \((j^{-2})\) at the radii of the source. It is somewhat longer to prove this, so let us only say that one can proceed as follows: (i) prove that \(dD_0^{(j)}(x)/dx = -2\gamma D_0^{(j+1)}(x)\), where \(D_0^{(j)}(x)\) is the second independent solution of the ultraspherical differential equation (the “second Gegenbauer function”, see section 5 below), (ii) express \(F_1(x)\) and \(G_j(x)\) using (67), and (iii) consider the asymptotic behaviour of the Gegenbauer functions.

There is one more challenge for numerical precision: integrals (45) and (46), when expressed using elementary functions, have a form of two terms almost cancelling each other. This problem can be managed by recalling that the Legendre function of the second kind \(Q_j(x)\) can be expressed in terms of elementary functions and then evaluated efficiently. Analogously, \(G_j(x)\) can be expressed as
\[
G_j(x) = \frac{(-1)^j(j+2)!}{48(2j+3)!} \left( \frac{2}{x+1} \right)^{j+3} {}_2F_1 \left( j, j+3; 2j+4; \frac{2}{x+1} \right),
\] (60)
which converges very well after the Gauss hypergeometric function is expanded suitably. Let us add that we do not know how to prove the equivalence of (60) and (46) directly. However, it is possible to check that both expressions of \(G_j(x)\) solve the homogeneous part of (44) and that both vanish at \(x \to \infty\). Equation (44) is a second-order ordinary linear differential equation, so its fundamental space is two-dimensional. And since the other solution \(F_1(x)\) does not vanish at infinity, both expressions of \(G_j(x)\) have to be proportional to each other. Comparing their leading terms (of expansion in \(1/x\)), one finally concludes that the functions exactly coincide.

5. GREEN FUNCTIONS

To overcome the convergence problems, one can try to avoid angular expansion. A straightforward way to do so is to express the Green functions \(G^\theta(x, \theta, \varphi, \theta', \varphi')\) and \(G^\varphi(x, \theta, \varphi, \theta', \varphi')\) in a closed form. This section is devoted to this task and uses relations which we will justify in detail elsewhere (Cáček in preparation). As an advertisement for the usage of the closed-form Green functions, see figure 1 where, on an example of the ring perturbation, results are compared computed i) by the usual expansions and ii) using the closed-form Green functions. The first 30 terms of multipole expansion are summed there, but the closed-form Green functions provide better results. See also Tables 1 and 2 where the convergence of the multipole expansion towards exact result (given by the closed-form Green functions) is illustrated. One should admit that the multipole series of this type can usually be re-summed in such a way that their convergence is much better even close to the source radius. Anyway, the main advantage of the closed-form Green functions is that one can better integrate them over the source, as exemplified in section 6 below.

5.1. Green function for the gravitational potential

In order to find the closed form of the Green function (53), one can start from a more general relation which will be proven in an accompanying paper by Cížek (in preparation):\(^6\)
\[
\sum_{j=0}^{\infty} 2(j+\gamma) \frac{\Gamma^j(j+1)}{\Gamma^j(j+2\gamma)} D_j^{(\gamma)}(a) C_j^{(\gamma)}(u) C_j^{(\gamma)}(v) C_j^{(\gamma)}(w) = \frac{\pi^{-\gamma+2}}{\sqrt{2} \Gamma(\gamma) \Gamma(2\gamma)} \int_0^{\infty} \frac{\xi^{\gamma-1} d\xi}{(\alpha^2 - \xi^2)(\beta^2 - \xi^2)(\gamma^2 - \xi^2)} \frac{2 F_1 \left( \frac{1}{2}, 1; \gamma + \frac{1}{2}; \frac{(1-u^2)(1-v^2)(1-w^2)}{(\alpha^2 - u^2)(\beta^2 - v^2)(\gamma^2 - w^2)} \right)}{2^{\gamma-2} \Gamma^4(\gamma) \Gamma(2\gamma) \left( |a^2 - 1| |1 - u^2| |1 - w^2| \right)^{\gamma-2}},
\] (61)

\(^6\) It is also shown there that the presented sum converges absolutely when \(\gamma=3/2\).
where $u, v, w \in (-1; 1)$, $a$ should be greater than 1, and $C_ν^{(γ)}(x)$ and $D_ν^{(γ)}(x)$ denote, respectively, Gegenbauer functions of the first and of the second kind. Namely, $C_ν^{(γ)}(x)$ and $D_ν^{(γ)}(x)$ are two independent solutions of the ultraspherical differential equation

$$(1 - x^2) \frac{d^2 y(x)}{dx^2} - (2γ + 1) x \frac{dy(x)}{dx} + j(j + 2γ) y(x) = 0.$$  \hspace{1cm} (62)

The first solution can be written as

$$C_ν^{(γ)}(x) = |1 - x^2|^{-\frac{1}{2}γ} \frac{\sqrt{2π} Γ(n + 2γ)}{2Γ(γ)Γ(n + 1)} \times \begin{cases} P_{\frac{1}{2}γ}^{|z|} (x) & \text{when } |x| < 1, \\ P_{\frac{1}{2}γ}^{-\frac{1}{2}γ} (x) & \text{when } x \geq 1; \end{cases} \hspace{1cm} (63)$$

it reduces to Gegenbauer polynomials when $n$ is a non-negative integer and to Legendre functions of the first kind (actually Legendre polynomials) for $γ = 1/2$. The second solution can be written analogously,

$$D_ν^{(γ)}(x) = |1 - x^2|^{-\frac{1}{2}γ} \frac{\sqrt{2π} Γ(n + 2γ)}{2Γ(γ)Γ(n + 1)} \times \begin{cases} Q_{\frac{1}{2}γ}^{|z|} (x) & \text{when } |x| < 1, \\ Q_{\frac{1}{2}γ}^{-\frac{1}{2}γ} (x) & \text{when } x \geq 1; \end{cases} \hspace{1cm} (64)$$

it reduces to Legendre functions of the second kind for $γ = 1/2$ (hence how we call it). Above, we have employed the notation $P_n^γ(x)$, $P_n^0(x)$, $Q_n^γ(x)$ and $Q_n^0(x)$, used in Olver et. al. (2010), chapter 14, in order to distinguish the Ferrers functions of the first and of the second kinds (i.e., Legendre functions defined on the cut $|x| < 1$; written in roman) from the associated Legendre functions of the first and of the second kinds (defined on $x > 1$; written in italic).

One more remark to the formula (61) is at place, namely that it requires $u$ to lie within $(-1; 1)$, while for a ring outside of the horizon actually $u > 1$. However, a function of complex variable has a unique extension (up to a Riemann folding), so the solution valid for $u \in (-1; 1)$, when extended into the complex plane, should also yield a (unique) solution of the respective differential equation with different boundary conditions.
The relation (61) provides $G^\nu(x, \theta, x', \theta')$ (up to a multiplication factor) if putting $\gamma = 1/2, a = \max(x, x'), u = \min(x, x'), v = \cos \theta$ and \( w = \cos \theta' \). Namely, as will be shown in Čížek (in preparation) (the result follows from Baranov 2006),

$$\sum_{j=0}^{\infty} (2j+1) Q_j(a) P_j(u) P_j(v) P_j(w) = \frac{2K}{\pi \left((au-vw)^2 - \left((a^2-1)(u^2-1) - (1-v^2)(1-w^2)\right)\right)^{1/2}},$$

(65)

hence, comparing this with (53), we get

$$G^\nu(x, \theta, x', \theta') = - \frac{K}{\pi \left(x x' - \cos \theta \cos \theta' \sin^2 \theta - \left((x^2-1)(x'^2-1) - \sin^2 \theta \sin \theta' \sin^2 \theta' \right)\right)^{1/2}}.$$  

(66)

This is actually an expected result: the Green function corresponds to the potential due to a thin-ring source (situated at $x', \theta'$), which is familiar to be given by the complete elliptic integral $K(k)$ (within general relativity, such a source is known as the Bach-Weyl ring, see e.g. Semerák 2016). Note in passing that equation (6) is quadratic in dragging (ω) and the background metric is static (Schwarzschild solution), so dragging has to be proportional to the perturbation (λ), i.e. the correction from the (\nabla \phi) term is at least of order $\lambda^2$ and does not contribute in the linear order (in higher orders, it behaves like a source term). Without dragging, equation (6) is the same as in the static case (or even a Newtonian one) and one reaches the above result (up to a coordinate transformation, see (80)).
F_j(x) = \frac{12(-1)^j}{j(j+1)^2(j+2)^3(j+3)}\hat{O}_xC_j^{(3/2)}(x), \quad G_j(x) = \frac{(-1)^j}{12}\hat{O}_xD_j^{(3/2)}(x),

where \hat{O}_x := (x-1)\frac{d^2}{dx^2}(x-1) = \frac{d}{dx}(x-1)^2\frac{d}{dx}, \quad (67)

5.2. Green function for dragging

Now we proceed to the second Green function (54). Considering relations
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we can rewrite (54) as

\[ G(\omega) = \Delta - \sum_{j=1}^{\infty} \frac{j + 3}{j(j+1)^3(j+2)^3} \hat{O}_\omega \hat{C}^{(3/2)}(x) \hat{O}_{x>} D_j^{(3/2)}(x>)(\cos \theta) C_j^{(3/2)}(\cos \theta') = \]

\[ = \Delta - \hat{O}_{x>} \frac{d}{dx<} (x< - 1)^3 \sum_{j=1}^{\infty} \frac{j + 3}{j(j+1)^3(j+2)^3} D_j^{(3/2)}(x>) \frac{d}{dx<} C_j^{(3/2)}(x<) C_j^{(3/2)}(\cos \theta) C_j^{(3/2)}(\cos \theta') = \]

\[ = \tilde{\Delta} - \hat{O}_{x>} \frac{d}{dx<} (x< + 1)^3 \sum_{j=0}^{\infty} \frac{j + 3}{j(j+1)^3(j+2)^3} \int_1^{x<} (\xi^2 - 1) D_j^{(3/2)}(x>) C_j^{(3/2)}(\cos \theta) C_j^{(3/2)}(\cos \theta') d\xi = \]

\[ = \tilde{\Delta} - \hat{O}_{x>} \frac{d}{dx<} \int_1^{x<} \frac{\xi^2 - 1}{2(x< + 1)^2} \sum_{j=0}^{\infty} \frac{2j + 3}{j(j+1)^3(j+2)^3} D_j^{(3/2)}(x>) C_j^{(3/2)}(\cos \theta) C_j^{(3/2)}(\cos \theta') d\xi = \]

where

\[ \Delta := - \frac{3}{4} F_0(x<) G_0(x>) C_0^{(3/2)}(\cos \theta) C_0^{(3/2)}(\cos \theta') = - \frac{1}{4(x> + 1)^3}, \]

\[ \tilde{\Delta} := \Delta + \hat{O}_{x>} \frac{d}{dx<} \frac{3}{16 (x< + 1)^2} \int_1^{x<} (\xi^2 - 1) D_0^{(3/2)}(x>) C_0^{(3/2)}(\xi) C_0^{(3/2)}(\cos \theta) C_0^{(3/2)}(\cos \theta') d\xi = \]

\[ = - \frac{2}{(x+1)^3(x^2+1)^3}. \]

We have deliberately switched the order of summation, integration and differentiation, which might in fact be an issue, but we will later check that the result is really a Green function of the original equation (35) with given boundary conditions. Note that the above result is simple thanks to its symmetry with respect to \(x< \leftrightarrow x>\), and also due to the relation

\[ \int_1^{x<} (z^2 - 1) C_n^{(3/2)}(z) \, dz = \frac{(x^2 - 1)^2}{n(n+3)} \frac{dC_n^{(3/2)}(x)}{dx} \]

which is a direct consequence of the ultraspherical differential equation.

Calculation of the integral (69) leads to logarithms or elliptic functions, but, “miraculously”, when one applies the operator \(O_{x>}\) first and only then integrates by \(\xi\), such difficulties do not occur, because

\[ \frac{d}{dx<} \int_1^{x<} \hat{O}_{x>} \frac{\xi^2 - 1}{2(x< + 1)^2(x^2 - 1) \sin^2 \theta \sin^2 \theta'} [x_\theta - \xi_\theta + \sqrt{(x_\theta - \xi_\theta)^2 - (1 - \xi^2)(1 - \xi^2)}] \, d\xi = \sum_{k=0}^{5} P_k(x', \theta') I_k(x', \xi), \]
where we have denoted
\[ P_0(x', \theta') := \frac{-2}{\pi(x+1)^3(x'+1)^3 \sin^2 \theta \sin^2 \theta'}, \]
\[ P_1(x', \theta') := \frac{-2}{\pi(x+1)^4(x'+1)^3 \sin^2 \theta \sin^2 \theta'}, \]
\[ P_2(x', \theta') := \frac{2xx' + (x-1)(x'-1)}{(x+1)^3(x'+1)^3 \sin^2 \theta \sin^2 \theta'}, \]
\[ P_3(x', \theta') := \frac{-2}{(x'-1)(x-1)(x'+1)} \]
\[ P_4(x', \theta') := \frac{2x x' + (x-1)(x'-1)}{(x+1)^3 \sin^2 \theta \sin^2 \theta'}, \]
\[ P_5(x', \theta') := \frac{2}{2\pi(x+1)(x+1) \sin^2 \theta \sin^2 \theta'}, \]
\[ I_0(x', \xi) := 1, \]
\[ I_1(x', \xi) := \frac{1}{\xi + 1}, \]
\[ I_2(x', \xi) := \frac{1}{\sqrt{x^2 + x'^2 + \xi^2 - 1 - 2xx'}}, \]
\[ I_3(x', \xi) := \frac{\xi}{\sqrt{x^2 + x'^2 + \xi^2 - 1 - 2xx'}}, \]
\[ I_4(x', \xi) := \frac{1}{(x^2 + x'^2 + \xi^2 - 1 - 2xx')^{3/2}}, \]
\[ I_5(x', \xi) := \frac{1}{(x^2 + x'^2 + \xi^2 - 1 - 2xx')^{3/2}}. \] (74)

Therefore, we can finally conclude that
\[ G^\omega(x, \theta, x', \theta') = \Delta(x, x') - \int_{\cos(\theta-\theta')}^{\cos(\theta+\theta')} \sqrt{\cos(\theta-\theta') - \xi} \sqrt{\cos(\theta+\theta') - \xi} \sum_{k=0}^{5} P_k(x', \theta') I_k(x', \xi) \, d\xi = \Delta(x, x') - \sum_{k=0}^{5} P_k(x', \theta') I_k(x', \theta'), \]
(75)

where
\[ I_0(x', \theta') := \frac{\pi}{2} \sin^2 \theta \sin^2 \theta', \]
\[ I_1(x', \theta') := \pi(1 - |\cos \theta|)(1 - |\cos \theta'|), \]
\[ I_2(x', \theta') := \sqrt{\frac{a_{31}}{a_{42}}} \left[-a_{41} K(k) - a_{42} E(k) + a_{41} \left(1 + \frac{a_{42}}{a_{31}}\right) \Pi\left(\frac{a_{43}}{a_{31}}, k\right)\right], \]
\[ I_3(x', \theta') := \frac{a_{41}}{4} \sqrt{\frac{a_{31}}{a_{42}}} \left[-(a_{31} - 3a_{43} + 2a_{42}) K(k) + a_{42} \left(2a_{31} - a_{43} + 2\right) E(k) - \frac{a_{21}^2 - a_{21}^3 + 2a_{21}^2 - 2a_{21}a_{43}}{a_{31}} \Pi\left(\frac{a_{43}}{a_{31}}, k\right)\right], \]
\[ I_4(x', \theta') := \frac{2a_{41}}{\sqrt{a_{31}a_{42}}} \left[\frac{K(k)}{a_1 + 1} + \Pi\left(\frac{a_{43}}{a_{31}}, k\right) - a_1 + 1 \Pi\left(\frac{a_1 + 1}{a_1 + 1}, k\right)\right], \]
\[ I_5(x', \theta') := \frac{2a_{41}a_{42}}{a_{21}} \left[(2k^2) K(k) - 2E(k)\right]. \]
(76)

with
\[ a_1 := xx' + \sqrt{(x^2 - 1)(x'^2 - 1)}, \quad a_2 := xx' - \sqrt{(x^2 - 1)(x'^2 - 1)}, \quad a_3 := \cos(\theta - \theta'), \quad a_4 := \cos(\theta + \theta'), \]
(77)
and
\[ a_{r,s} := a_s - a_r, \quad s_{r,s} := a_s + a_r, \quad k := \sqrt{\frac{a_{31}a_{42}}{a_{31}a_{31}}}. \]
(78)

On the horizon the above Green function simplifies considerably,
\[ G^\omega(x=1, \theta, x', \theta') = -\frac{1}{4(x' + 1)^3}. \]
(79)

Note that this is independent of the position on the horizon (i.e. of the angle \( \theta \)), consistently with the well known “rigid rotation” of stationary horizons.

6. DISC SOLUTION

Perturbation equations can be solved by integrating the Green functions over the source mass distribution. Due to the presence of (complete) elliptic integrals in the above concise expression of the Green functions, a numerical integration has to be employed in general. Integration is of course simpler for thin sources (surface distribution) than for extended ones. Below we consider the case of a stationary and axially symmetric thin disc, encircling the central black hole between some two finite radii \( x_{in} \leq x \equiv r/M \leq x_{out} \) in a concentric manner (thus lying in the equatorial
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plane). We will show how to obtain explicit results for the linear perturbation of the metric functions and for the corresponding basic characteristics (density, pressure and velocity distribution) of the source.

6.1. Gravitational potential

First, let us find the linear perturbation of potential, \( \nu_1 \). It is advantageous to choose \( B=1 \) here, since this makes equation (34) the Laplace/Poisson equation, so one can use its solutions known from the classical field theory. Let us denote by \( t, \phi, \tilde{\rho} \) and \( \tilde{z} \) the coordinates corresponding to this choice (time \( t \) and azimuth \( \phi \) do not depend on the choice of \( B \)). Their counterparts \( (\rho, z) \) corresponding to \( B=1-\frac{M^2}{4\pi^2} \) are related by

\[
\tilde{\rho} = \rho \left( M^2 - 4\rho^2 - 4z^2 \right) \left( 4\rho^2 + z^2 \right) = \frac{(4\rho^2 - M^2) \sin \theta}{4\rho}, \quad \tilde{\rho} M \sqrt{x^2 - 1} \sin \theta, \\
\tilde{z} = \frac{z \left( M^2 + 4\rho^2 + 4z^2 \right)}{4\rho} = \frac{(4\rho^2 + M^2) \cos \theta}{4\rho}, \quad \tilde{z} M \cos \theta, \\
\tilde{\zeta} = \frac{1}{2} \ln \left[ \frac{(4\rho^2 + M^2)^2 - 16M^2z^2}{16(\rho^2 + z^2)^2} \right] = \frac{1}{2} \ln \left[ \frac{(4\rho^2 - M^2)^2 - 16\rho^2 M^2 \cos^2 \theta}{16\rho^2} \right] = \frac{1}{2} \ln \left[ \frac{(x^2 - \cos^2 \theta)}{(x + \sqrt{x^2 - 1})^2} \right].
\]

(to recall the function \( \zeta \), let us remind that \( g_{\tilde{\rho}\tilde{\rho}} = g_{rr} = e^{2\zeta - 2\nu} \).

Of the known disc solutions of the Laplace equation, we will choose the one describing a disc which extends from \( \rho = \tilde{\rho} = \tilde{z} = 0 \) to some finite equatorial radius \( x' \) (or \( \tilde{\rho}' \) when expressed in the Weyl radius corresponding to the \( B=1 \) choice) and whose Newtonian surface density \( S \) is constant in the unperturbed \( B=1 \) coordinate system. Such a solution was obtained by Lass & Blitzer (1983),

\[
V(x'; x, \theta) =
\frac{2S|x|}{\sqrt{2x'} + (x')^2} \frac{H(\tilde{\rho}' - \tilde{\rho})}{2S|x| \cos \theta} \left[ \frac{\tilde{\rho}^2 - \rho^2}{\sqrt{2x'} + (x')^2} \right] = \frac{2S|x| \cos \theta}{2S|x| \cos \theta} \frac{H(x^2 - \cos^2 \theta - x^2 \sin^2 \theta)}{2S|x| \cos \theta} \frac{E(k) + (x')^2 - \cos^2 \theta - x^2 \sin^2 \theta) K(k) + \left( a_1 \tilde{\rho}^2 - \rho^2 \right) \Pi \left( \frac{2a_{12} \sin \theta}{a_{12} a_{12} - a_{12} \cos^2 \theta} \right)}{\sqrt{a_{12} a_{12}}},
\]

where \( H(x) \) stands for the Heaviside function, and the \( a_{rs}(x') \) and \( k(x') \) symbols are given by (77) and (78) evaluated at \( \theta' = \pi/2 \).

\[
a_1(x') = xx' + \sqrt{(x^2 - 1)(x'^2 - 1)}, \quad a_2(x') = xx' - \sqrt{(x^2 - 1)(x'^2 - 1)}, \quad a_3(x') = \sin \theta, \quad a_4(x') = -\sin \theta, \\
a_{rs}(x') = a_s(x') - a_r(x'), \quad k^2(x') = \frac{2a_{21} \sin \theta}{a_{31} a_{42}} = \frac{4 \sqrt{(x^2 - 1)(x'^2 - 1)}}{\sqrt{(x^2 - 1)(x'^2 - 1) - \sin^2 \theta}} = \frac{4 \tilde{\rho}' \rho^2}{(\tilde{\rho}' + \rho)^2 + \tilde{z}^2}.
\]

The potential of a disc extending between two finite radii \( x_{in} \leq x \leq x_{out} \) (which we are interested in) is obtained by subtraction of two potentials (81) (with the same density \( S \)) with outer radii at \( x_{out} \) and \( x_{in} \),

\[
\nu_1(x, \theta) = V(x' = x_{out}; x, \theta) - V(x' = x_{in}; x, \theta).
\]

6.2. Dragging

In order to find the perturbation of dragging angular velocity, one has to integrate (69) over the “density” of the disc \( W(x) \) (here assumed to be constant, \( W \)). Since, as already noticed, (69) is symmetric with respect to the exchange
\( x_\leftrightarrow x \), we can substitute \( x_\leftrightarrow \) by \( x' \) and \( x \) by \( x \) without changing the result.

\[
\omega_1(x, \theta) = - \int_{x_{in}}^{x_{out}} W \mathcal{G}^\omega(x, \omega, x', \pi/2) \, dx' =
\]

\[
= - \int_{x_{in}}^{x_{out}} W \Delta(x, x') \, dx' + \int_{x_{in}}^{x_{out}} W \bar{\Omega}_x \frac{d}{dx'} \int_1^{x'} \frac{\zeta^2 - 1}{\pi(x')^2 \sin^2 \theta (x' - 1)} \, dx' \frac{\sqrt{\sin^2 \theta - \xi^2}}{x - \zeta + \sqrt{(x - \zeta)^2 - (1 - \zeta^2)(1 - \xi^2)} - \xi \, d\zeta} dx' =
\]

\[
= \frac{(x_{out} - x_{in})(x_{out} + x_{in} + 2) W}{(x + 1)^3(x_{out} + 1)^2(x_{out} + 1)^2} + W \int_{-\sin \theta}^{\sin \theta} \frac{\hat{\Omega}_x}{2\pi} \frac{\sqrt{\sin^2 \theta - \xi^2}}{x - \zeta + \sqrt{(x - \zeta)^2 - (1 - \zeta^2)(1 - \xi^2)} - \xi \, d\zeta} \frac{(\zeta^2 - 1)}{x_{out}} \]

\[
= \frac{(x_{out} - x_{in})(x_{out} + x_{in} + 2) W}{(x + 1)^3(x_{out} + 1)^2(x_{out} + 1)^2} + W \int_{-\sin \theta}^{\sin \theta} \sqrt{\sin^2 \theta - \xi^2} \sum_{j=0}^{\infty} \left[ Q_j(x_{out}) \bar{T}_j(x_{out}, \xi) - Q_j(x_{in}) \bar{T}_j(x_{in}, \xi) \right] d\xi =
\]

\[
= \frac{(x_{out} - x_{in})(x_{out} + x_{in} + 2) W}{(x + 1)^3(x_{out} + 1)^2(x_{out} + 1)^2} + W \sum_{j=0}^{\infty} \left[ Q_j(x_{out}, \theta) \bar{T}_j(x_{out}, \theta) - Q_j(x_{in}, \theta) \bar{T}_j(x_{in}, \theta) \right] , \tag{84}
\]

where \( x_{in} \) and \( x_{out} \) mark, again, the inner and outer rims of the disc (in the coordinate \( x \)), \( \bar{T}_0, \ldots, \bar{T}_5 \) are defined by (74),

\[
\bar{T}_0(x', \xi) = \frac{1}{\xi - 1} , \quad \bar{T}_1(x', \xi) = \frac{1}{\xi - 1} , \quad \bar{T}_2(x', \xi) = \frac{1}{\xi - 1} , \quad \bar{T}_3(x', \xi) = \frac{1}{\xi - 1} , \quad \bar{T}_4(x', \xi) = \frac{1}{\xi - 1} , \quad \bar{T}_5(x', \xi) = \frac{1}{\xi - 1} , \tag{85}
\]

and \( I_i(x') \) correspond to (76) taken at \( \theta' = \pi/2, \)

\[
I_0(x') = \frac{\pi}{2} \sin^2 \theta , \quad I_1(x') = \pi (1 - |\cos \theta|) , \quad I_2(x') = \sqrt{\frac{a_{31}^2}{a_{41}}} \left[ a_{41} K(k) - a_{42} E(k) + 2 x' a_{41} \right] \left[ 2 \sin \theta \right] , \quad I_3(x') = \frac{\sqrt{a_{31}^2 a_{41}^2}}{a_{41}} \left[ a_{241}^2 K(k) - a_{242}^2 E(k) + 2 x'^2 + \cos^2 \theta - (x' - x)^2 \right] \left[ 2 \sin \theta \right] , \quad I_4(x') = \frac{\sqrt{a_{31}^2 a_{41}^2}}{a_{41}^2} \left[ a_{31}^2 K(k) + 2 \sin \theta + \frac{1}{a_{41}} \right] \left[ 2 \sin \theta \right] , \quad I_5(x') = \frac{\sqrt{a_{31}^2 a_{41}^2}}{a_{41}^2} \left[ a_{31}^2 K(k) + 2 \sin \theta + \frac{1}{a_{41}} \right] , \quad I_6(x') = \pi (|\cos \theta| - 1) , \quad I_7(x') = \frac{\sqrt{a_{31}^2 a_{41}^2}}{a_{41}^2} \left[ a_{31}^2 K(k) + 2 \sin \theta + \frac{1}{a_{41}} \right] , \tag{86}
\]

where the argument \( (x') \) of \( a_1, a_{4x} \) and \( k \) (given by (82)) has been omitted.

7. Behaviour of the Solution at Significant Locations

Regarding the axial and reflective symmetry, one naturally asks whether and how the solution simplifies on the axis and in the equatorial plane. On the axis, \( \sin \theta = 0 \) and \( \cos^2 \theta = 1 \), so, in the potential (81), \( a_3 = 0 \), \( a_4 = 0 \), \( k = 0 \), so
all the elliptic integrals there reduce to $\pi/2$ and

$$a_{31}a_{42} = a_{41}a_{32} = a_1a_2 = x'^2 + x^2 - 1.$$ \hfill (87)

Finally, considering that the disc has to lie above the horizon, i.e., at $x' > 1$, we have $H(x'^2 - 1) = 1$, so the first term of (81) is the same for both $x' = x_{in}$ and $x' = x_{out}$ (and thus cancels out), and hence

$$\nu_1(x, \theta = 0) = -2\pi MS \left( \sqrt{x_{out}^2 + x^2 - 1} - \sqrt{x_{in}^2 + x^2 - 1} \right).$$ \hfill (88)

Also the equatorial form of the potential (81) is somewhat simpler, namely the first term and the one with the $\Pi$ integral vanish, so one is left with

$$\nu_1(x, \theta = \pi/2) = -2MS \left[ \frac{(a_1 - 1)(a_2 + 1) E(k) + (x'^2 - x^2) K(k)}{\sqrt{(a_1 - 1)(a_2 + 1)}} \right]_{x' = x_{out}}.\hfill (89)$$

At radial infinity ($r \to \infty$) the disc potential falls off as

$$\nu_1(r \to \infty) \propto -\frac{\pi S}{r} \left[ r_{out}^2 - r_{in}^2 - \frac{M^4}{16} \left( \frac{1}{r_{in}^2} - \frac{1}{r_{out}^2} \right) \right],\hfill (90)$$

and at the horizon ($x = 1$) it assumes the value

$$\nu_1(x = 1, \theta) = -2\pi MS \left( \sqrt{x_{out}^2 - \sin^2 \theta} - \sqrt{x_{in}^2 - \sin^2 \theta} \right).\hfill (91)$$

The formula for the dragging perturbation $\omega_1(x, \theta)$ is more tricky and does not reduce so much on the axis and in the equatorial plane (however, figure 3 shows that it behaves reasonably). Very simple limits can still be obtained at radial infinity where it falls off as

$$\omega_1(x \to \infty) \propto \frac{W}{4} \frac{x_{out} - x_{in}}{x^2} \quad \implies \quad \omega_1(r \to \infty) \propto \frac{WM^2}{4r^3} \left[ r_{out} - r_{in} + \frac{M^2}{4} \left( \frac{1}{r_{out}} - \frac{1}{r_{in}} \right) \right],$$ \hfill (92)

and on the horizon where it is everywhere the same (independent of $\theta$) and having its sign given by $W$,

$$\omega_1(x = 1) \equiv \omega_H = \frac{W}{8} \frac{(x_{out} - x_{in})(x_{out} + x_{in} + 2)}{(x_{out} + 1)^2(x_{in} + 1)^2}.$$ \hfill (93)

8. PARAMETERS OF THE DISC SOURCE

In order to calculate the physical characteristics of the disc, let us realize that all the densities and one-stream pressure ($\sigma, P, \sigma_+, \sigma_-$) are themselves small, namely of linear perturbation order (see equation (30)) and that the geodesic orbital velocities (18) are given by their unperturbed Schwarzschild values ($\pm \Omega_0$, to which corresponds $v_o = -v_- = v_0 = 2\sqrt{M'M}/(2r - M)$) plus terms $O(\omega)$, where $\omega$ is (of course) linearly small. Consequently, up to the linear order, relations (26)-(28) between normal jumps of the metric gradients across the disc and its physical parameters reduce to

$$\nu_{1,z}(z = 0^+) = 2\pi \left( S_0^+ - S_z^+ \right) = 2\pi (\sigma + P) \frac{1 + v_o^2}{1 - v_o^2} = 2\pi (\sigma_+ + \sigma_-) \frac{1 + v_o^2}{1 - v_o^2},$$ \hfill (94)

$$\omega_{1,z}(z = 0^+) = -8\pi S_0^+ \frac{1}{B^2} \frac{4v_o}{e^{-4v_o}} = -8\pi (\sigma + P) \frac{\Omega_0}{1 - v_o^2} = -8\pi (\sigma_+ + \sigma_-) \frac{\Omega_0}{1 - v_o^2},$$ \hfill (95)

$$\zeta_{1,z}(z = 0^+) = 4\pi S_0^+ \frac{1}{e^{-4v_o}} = 4\pi (\sigma_+ + \sigma_-) \frac{v_o^2}{1 - v_o^2}.$$ \hfill (96)

Differentiating the potential (83), (81) across the disc plane and regarding that $\nu_1(\theta = \pi/2^-) = -\nu_1(\theta = \pi/2^+)$, we find

$$\frac{\partial V}{\partial \theta} \bigg|_{\theta = \frac{\pi}{2}^-} = -2\pi MS x H(x' - x) \implies \nu_{1,z}(z = 0^+) = 2\pi S \left( 1 + \frac{M^2}{4r^2} \right) = \frac{2\pi SM}{r} \frac{1 + v_o^2}{v_o^2},$$ \hfill (97)

(the result only applies where the disc actually lies, i.e. at $x_{in} \leq x \leq x_{out}$, elsewhere it is zero of course), hence, according to (93),

$$\sigma_+ + \sigma_- = \frac{\nu_{1,z}(z = 0^+)}{2\pi} \frac{1 - v_o^2}{1 + v_o^2} = \frac{S}{4r^2} \left( 4r^2 - 8Mr + M^2 \right).$$ \hfill (98)

More involved is to find the equatorial limit of the normal gradient of $\omega_1$ (84). One finds that solely the term $Q_1I_7$ contributes eventually, because $Q_5I_5$ is zero from the beginning, the first separate term of (84) as well as $Q_0I_0$ are
Figure 2. Meridional-plane contours of the gravitational potential $\nu$, given by sum of the Schwarzschild expression and the contribution from the disc. The four examples shown represent an equatorial disc stretching from $\rho = 5M$ to $\rho = 7M$ (it is indicated by a thick black line), with potential (81) scaled by $S = 0.01$ (top left), $S = 0.026$ (top right), $S = 0.1$ (bottom left) and $S = 1.0$ (bottom right) (such a series corresponds to a more and more massive disc). The potential is everywhere negative, with light/dark shading indicating shallow/deeper values (the potential diverges to $-\infty$ at the horizon, while the “weakest” levels reached at top right corners of the plots amount to $-0.20$ at top left, to $-0.34$ at top right, to $-1.00$ at bottom left and to $-8.96$ at bottom right); the black-hole horizon (at $\rho^2 + z^2 = M^2/4$) is represented by the white quarter-circle. Both axes are given in the units of $M$.

Figure 3. Meridional-plane contours of the dragging angular velocity $\omega$, as entirely given by the first-order perturbation due to the disc ($\omega_1$). The disc again stretches from $\rho = 5M$ to $\rho = 7M$ (as indicated by the thick black line). The angular velocity is everywhere positive, with light/dark shading indicating smaller/larger values (they reach about $0.0268/M$ at the disc and fall off to some $0.0047/M$ at top right of the plot); the black quarter-disc at $\rho^2 + z^2 \leq M^2/4$ represents the black hole. Both axes are given in the units of $M$. Since $\omega = \omega_1$ is proportional to $W$, the isolines have the same shape for any $W$, only their values scale with this rotational parameter.

Independent of $\theta$, $Q_5I_6$ is independent of $x'$ (and thus its “out” and “in” values subtract to zero), the $Q_3I_2$ and $Q_3I_3$ terms vanish in the $\theta \to (\pi/2)^-$ limit, and contributions of the terms $Q_1I_1$ and $Q_4I_4$ exactly cancel out each other in
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From \((Q_{7} I_{7})_\theta\) one specifically obtains, in the above limit, non-zero terms

\[
\frac{2Q_{7}(a_{1}+1)}{\sqrt{(a_{1}-1)^{3}(a_{2}+1)}} \left[ -\Pi(n, k) \cos \theta + (1-\sin \theta) \frac{\partial \Pi(n, k)}{\partial \theta} \right], \quad \text{where} \quad n = n(x, x', \theta) := \frac{2\sin \theta (a_{1}+1)}{(1+\sin \theta)(a_{1}-\sin \theta)}
\]

and where the derivative of \(\Pi(n, k)\) contributes solely through the term \(\frac{\Pi(n, k)}{2(1-n)} \frac{\partial \Pi(x, x', \theta)}{\partial \theta}\). Using the asymptotics

\[
\Pi(n, k) \propto \frac{\pi \sqrt{n}}{2\sqrt{n-K^{2}\sqrt{1-n}} + O(1)} \quad \text{valid for} \quad n \to 1^-
\]

in both terms and making the equatorial limit, one finally finds a simple result

\[
\lim_{\theta \to \frac{\pi}{2}} \frac{\partial \omega_{1}}{\partial \theta} \left( = -r \lim_{z \to 0^{+}} \frac{\partial \omega_{1}}{\partial z} \right) = \frac{W}{2} \frac{x-1}{(x+1)^{3}} = 8W M^{2} r^{2} \frac{(2r-M)^{2}}{(2r+M)^{3}},
\]

which, when compared against

\[
\lim_{z \to 0^{+}} \frac{\partial \omega_{1}}{\partial z} = -8\pi(\sigma_{+} - \sigma_{-}) \frac{\Omega_{0}}{1 - \frac{r}{\rho_{0}}} = -8\pi(\sigma_{+} - \sigma_{-}) \frac{8r \sqrt{Mr}}{(2r+M)^{3}} \frac{(2r-M)^{2}}{4r^{2} - 8Mr + M^{2}}
\]

given by (94), implies that

\[
\sigma_{+} - \sigma_{-} = \frac{W M^{2}}{8\pi \sqrt{Mr}} \frac{4r^{2} - 8Mr + M^{2}}{(2r+M)^{3}}.
\]

By adding (97) and (99), we have then

\[
\sigma_{\pm} = \frac{4r^{2} - 8Mr + M^{2}}{8r^{2}} \left[ S \pm \frac{W (Mr)^{3/2}}{2\pi (2r+M)^{3/2}} \right].
\]

To find the one-stream disc characteristics, one first combines (93) and (95), which yields

\[
v^{2} = \frac{\sigma_{\pm}^{2} - P}{\sigma - P_{0}} = \frac{4Mr \sigma - (2r-M)^{2}P}{(2r-M)^{2}\sigma - 4MrP}.
\]

Substituting this for \(v^{2}\) in (94) leads to

\[
(\sigma_{\pm}^{2} - P)(\sigma - P_{0}) = (\sigma_{+} - \sigma_{-})^{2} v_{0}^{2}
\]

and from there, using \(\sigma - P = \sigma_{+} + \sigma_{-}\), we find

\[
\sigma = \frac{\sigma_{+} + \sigma_{-}}{2} + \sqrt{\left( \frac{\sigma_{+} + \sigma_{-}}{2} \right)^{2} + \frac{4\sigma_{+} \sigma_{-} v_{0}^{2}}{(1-v_{0}^{2})^{2}}} = \frac{\sigma_{+} + \sigma_{-}}{2} + \sqrt{\left( \frac{\sigma_{+} + \sigma_{-}}{2} \right)^{2} + \frac{16Mr (2r-M)^{2}\sigma_{+} \sigma_{-}}{(4r^{2} - 8Mr + M^{2})^{2}}},
\]

\[
P = \frac{\sigma_{+} + \sigma_{-}}{2} + \sqrt{\left( \frac{\sigma_{+} + \sigma_{-}}{2} \right)^{2} + \frac{4\sigma_{+} \sigma_{-} v_{0}^{2}}{(1-v_{0}^{2})^{2}}} = \frac{\sigma_{+} + \sigma_{-}}{2} + \sqrt{\left( \frac{\sigma_{+} + \sigma_{-}}{2} \right)^{2} + \frac{16Mr (2r-M)^{2}\sigma_{+} \sigma_{-}}{(4r^{2} - 8Mr + M^{2})^{2}}}.
\]

Note that when substituting these \(\sigma\) and \(P\) back to (101), the resulting \(v\) given by square root of the latter is to be taken with \(+/-\) sign in case that \(\sigma_{+} > \sigma_{-}\) or \(\sigma_{+} < \sigma_{-}\).

In order to demonstrate that the procedure really works and, in particular, to illustrate the role of the parameters \(S\) and \(W\), let us consider a disc spanning between the radii \(\rho_{-} = 5M\) and \(\rho_{+} = 7M\), surrounding a black hole of mass \(M\), and let us plot the results for several different values of \(S\) and/or \(W\). Note that both \(S\) and \(W\) have the dimension of 1/length, practically 1/M. Note also that in order to emphasize their effect, we ignore here the assumption that the perturbation should be very small. More precisely, the first-order perturbation of the gravitational potential is in fact not restricted by this assumption, because rotation/dragging only enters in the second order, so the change of \(\nu\) can be understood as an exact superposition within the non-rotating, static class. The potential \(\nu\) (the sum of the Schwarzschild background \(v_{0}\) and the perturbation \(v_{1}\)) is shown in figure 2; it behaves in an expected manner, namely the disc effect grows with increasing \(S\) (and it is independent of \(W\)). The dragging angular velocity amounts to \(\omega = \omega_{1}\) (since \(v_{0} = 0\)), so the level-contour shape is in fact fixed and does not change with parameters (only the level values do change, in particular they scale with \(W\)); this is shown in figure 3.

Now for parameters of the double-stream and single-stream interpretations, i.e., \(\sigma_{\pm}\) and \(v_{\pm}\), and \(\sigma\), \(P\) and \(v\), respectively. In fact the orbital velocities \(v_{\pm}\) of the double-stream picture need not be computed, namely, in the first perturbation order they are the same as in a pure Schwarzschild field, because the field equations (6) and (7) have right-hand sides proportional to the surface densities which are themselves of linear order. The other quantities are plotted in figure 4 for several choices of the \(S\) and \(W\) parameters. One sees there that an increasing value of \(W\) makes \(\sigma_{+}\) and \(\sigma_{-}\) more and more different, with \(\sigma_{-}\) finally becoming negative, which marks limits of the (counter-)rotating
interpretation. Physically, such a situation means that, for a given mass, the disc has too much angular momentum. Naturally, this is also accompanied by a need for too high orbital velocity; such a circumstance can be “remedied” by increasing the mass, i.e. $S$.

\[
\begin{array}{cccc}
\text{1st column: } W = 25/M & \text{2nd column: } W = 50/M & \text{3rd column: } W = 90/M & \text{4th column: } W = 150/M \\
\end{array}
\]

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Figure 4. Parameters of the disc’s double-stream and single-stream interpretations, plotted as functions of the Weyl radius for a disc stretching between $\rho = 5M$ and $\rho = 7M$ and for several combinations of the parameters $S$ (Newtonian density) and $W$ (scaling the disc rotation). The first/second/third/fourth columns represent the cases given by $W = 25/50/90/150$, with the first row showing densities $\sigma_+$ (solid line), $\sigma_-$ (dotted line) of the single-stream, counter-rotating dust interpretation, the second row showing density $\sigma$ (solid line) and azimuthal pressure $P$ (dotted line) of the single-stream fluid interpretation, and the third row showing the corresponding “bulk” velocity of the fluid $v$ (solid). For each of the values of $W$, three different values of $S$ have been chosen, corresponding to discs of different mass – $S = 0.04, 0.12$ and 0.2. The density and pressure curves obtained for higher $S$ are higher (given by larger values), whereas the corresponding bulk velocities decrease with growing $S$. The ranges of the chosen $W$ and $S$ are largely out of the scope of the linear perturbation, but this is in order to illustrate clearly what they represent. Since all the parameters (densities, pressure, velocity) should be real and positive, and the velocity $v$ has to be $<1$ in addition, one sees immediately that i) the discs with a given mass cannot bear however large angular momentum (with growing $W$, the density $\sigma_+$ of the double-stream interpretation – and consequently also the single-stream parameters – tend to be negative); and, ii) since large $W$ implies the need for high orbital velocities, for too large $W$ the orbital interpretation would have to involve superluminal motion (in the bottom row, from left to right, one sees that all three / two / two / one of the shown cases are “physical” in this respect). The densities $S$, $W$, $\sigma_+$, $\sigma_-$, $\sigma$ as well as the pressure $P$ have the dimension of $1/\text{length}$ and their values are in the units of $1/M$, the speed $v$ is dimensionless.

8.1. Mass and angular momentum of the disc

The total mass and angular momentum of a stationary and axially symmetric space-time can be found from Komar integrals, given by the Killing vector fields $\eta^\mu = \partial^\mu$ and $\xi^\mu = \partial^\mu$. In our case (involving two sources) the integrals over spatial infinity can be split into contributions from the black hole $M$ and $J$ (given by integration over the horizon $H$) and from the disc (integrated over some space-like hypersurface $\Sigma$ covering the black-hole exterior),

\[
\text{mass} = \frac{1}{8\pi} \lim_{S \to \infty} S \int \eta_{\mu\nu} dS_{\mu\nu} = \frac{1}{8\pi} \int \eta_{\mu\nu} dS_{\mu\nu} - \int_{\Sigma \subset H} (2T^i_\nu \eta^\nu - T \eta^\nu) d\Sigma_i = M + \int_{\Sigma \subset H} (T^i_\nu \eta^\nu - T \eta^\nu) d\Sigma_i ,
\]

\[
\text{ang.m.} = -\frac{1}{16\pi} \lim_{S \to \infty} S \int \xi_{\mu\nu} dS_{\mu\nu} = -\frac{1}{16\pi} \int \xi_{\mu\nu} dS_{\mu\nu} + \int_{\Sigma \subset H} (2T^i_\nu \xi^\nu - T \xi^\nu) d\Sigma_i = J + \int_{\Sigma \subset H} (2T^i_\nu \xi^\nu - T \xi^\nu) d\Sigma_i ,
\]

where we have finally employed the “Killing” coordinates $t$ and $\phi$ in which $\eta^\mu = \delta^\mu_t$, $\xi^\mu = \delta^\mu_\phi$, and a natural space-like section $\Sigma = \{ t = \text{const} \}$ (which corresponds to $d\Sigma_i = \delta^\mu_i \sqrt{-g} d^4x$). Clearly $(T^i_\nu - T^i)_{\nu}$ – or actually $\kappa^{2\lambda - 2\nu}(T^i_\nu - T^i)$ – plays the same role as mass density in the Newtonian theory. Note also that the second part of the mass integral represents what is sometimes called Tolman’s formula.

Choosing $B = 1$, the metric determinant reads $-g = \rho^2 e^{\lambda - 4\nu}$, so, if considering the thin $\{ z = 0 \}$-disc as source and thus having $T^t_i \sqrt{-g} = S^t_i \rho \delta(z)$, the disc mass and angular momentum (denoted by $M$ and $J$) come out as

\[
M = 2\pi \int_{\rho_{\text{disc}}} (S^\phi_\text{disc} - S^r_\text{disc}) \rho d\rho , \quad J = 2\pi \int_{\rho_{\text{disc}}} S^\phi_\text{disc} \rho d\rho .
\]
Substituting here for (4) with \( u^\phi = u^t \Omega \) and \( u_\phi = g_{\phi\phi} u^t (\Omega - \omega) \), we have

\[
S_\phi' = (\sigma + P)(u^t)^2 g_{\phi\phi}(\Omega - \omega), \quad S_\phi^0 - S_1^t = (\sigma + P)(u^t)^2 \left[ e^{2\nu} + g_{\phi\phi}(\Omega^2 - \omega^2) \right] = \sigma + 2\Omega S_\phi',
\]

hence

\[
\mathcal{M} = 2\pi \int_{\text{disc}} (\sigma + P + 2\Omega S_\phi') \rho \, d\rho, \tag{107}
\]

\[
\mathcal{J} = 2\pi \int_{\text{disc}} (\sigma + P) \frac{\rho e^{-2\nu}(\Omega - \omega)}{1 - \rho^2 e^{-4\nu}(\Omega - \omega)^2} g_{\phi\phi} \, d\rho; \tag{108}
\]
in the special case of \( \Omega = \text{const} \) the first integral amounts to

\[
\mathcal{M} = 2\pi \int_{\text{disc}} (\sigma + P) \rho \, d\rho + 2\Omega \mathcal{J}.
\]

One can alternatively express, in the integrals, \( S_\phi' \) in terms of jumps of the normal fields, according to (26)–(27). Setting \( B = 1 \) again, we thus have

\[
S_\phi' = \frac{1}{8\pi} \rho^2 e^{-4\nu} \omega_z, \quad S_\phi^0 - S_1^t = \frac{\nu_z}{2\pi} + 2\omega S_\phi',
\]

hence

\[
\mathcal{M} = \int_{\text{disc}} \left( \nu_z - \frac{1}{2} \rho^2 e^{-4\nu} \omega_z \right) \rho \, d\rho, \quad \mathcal{J} = -\frac{1}{4} \int_{\text{disc}} \rho^3 e^{-4\nu} \omega_1, z \, d\rho \tag{109}
\]

(where \( \omega_z \) and \( \nu_z \) are evaluated at \( z \to 0^+ \)). Since the second term in \( \mathcal{M} \) is \( O(\lambda^2) \), in the linear order one is left with just

\[
\mathcal{M}_1 = \int_{\text{disc}} \nu_1, z (z \to 0^+) \rho \, d\rho, \quad \mathcal{J}_1 = -\frac{1}{4} \int_{\text{disc}} \rho^3 e^{-4\nu} \omega_1, z (z \to 0^+) \, d\rho. \tag{110}
\]

Finally, one has to realize that the above formulas hold for \( B = 1 \), whereas our results for \( \nu_1 \) and \( \omega_1 \) have been derived with \( B = 1 - \frac{M^2}{R^2} \). However, adapting the integrals to the latter choice practically means just to write \( B\rho \) instead of \( \rho \) in both integrands, reaching

\[
\mathcal{M}_1 = \int_{\text{disc}} B\rho \nu_1, z (z \to 0^+) \, d\rho, \quad \mathcal{J}_1 = -\frac{1}{4} \int_{\text{disc}} B^2\rho^3 e^{-4\nu} \omega_1, z (z \to 0^+) \, d\rho. \tag{111}
\]

Considering specifically the above constant-density disc, it is clear from (81) and (84) that \( \mathcal{M}_1 \) is scaled by the free parameter \( S \) while \( \mathcal{J}_1 \) is scaled by the second parameter \( W \). We have from (96) and (98), at \( z \to 0^+ \) (i.e., \( \theta \to \pi/2^{-} \)),

\[
\nu_1, \rho = -\nu_1, \theta = \pi S \left( 2r + \frac{M^2}{2r} \right), \quad \omega_1, \rho = -\omega_1, \theta = -8WM^2r^2 \left( \frac{2r - M^2}{2r + M} \right)^2,
\]

which yields, after using \( \rho(\theta = \pi/2) = r \) and \( e^{-4\nu_0} = \frac{(2r + M)^4}{(2r - M)^4} \),

\[
\mathcal{M}_1 = \pi S \int_{\text{in}}^{\text{out}} \left( 2r + \frac{M^2}{2r} \right) \left( 1 - \frac{M^2}{4r^2} \right) \, dr = \pi S \left[ \frac{M^4}{16} \left( \frac{1}{r_{\text{in}}^2} - \frac{1}{r_{\text{out}}^2} \right) \right], \tag{113}
\]

\[
\mathcal{J}_1 = \frac{WM^2}{8} \int_{\text{in}}^{\text{out}} \left( 1 - \frac{M^2}{4r^2} \right) \, dr = \frac{WM^2}{8} \left[ \frac{M^4}{4} \left( \frac{1}{r_{\text{in}}^2} - \frac{1}{r_{\text{out}}^2} \right) \right]. \tag{114}
\]

The same results follow from the asymptotics (89) and (91), if regarding the general behaviour (in an asymptotically flat space-time)

\[
\nu(r \to \infty) \propto -\frac{M_1 + \mathcal{M}_1}{r}, \quad \omega(r \to \infty) \propto 2 \frac{J_1 + \mathcal{J}_1}{r^3}.
\]

The last statement implies that i) the black-hole mass remains \( M \) (which should be so in the linear perturbation order), and that ii) the whole angular momentum of the system (inferred from the asymptotic behaviour of \( \omega \)) is being carried by the disc, hence that the angular momentum of the hole remains zero, \( J = 0 \). Actually, we can verify this
directly by computing the respective Komar integral over the horizon. It is known that the latter can be rewritten (see e.g. Will 1974, equation (21)) as
\[ J = -\frac{k^4}{2^7} \int_0^\pi [B^3 \omega_r e^{-4\nu}]_{r=k/2} \sin^3 \theta \, d\theta, \]
where in our first-order case one takes \( \nu = \nu_0, \omega = \omega_1 \) and \( k = M \). Inspecting the radial gradient \( \omega_{1,r} \), of (84), one finds that all the terms of this expression individually vanish at the horizon ( irrespectively of \( \theta \)), so the above formula really yields \( J = 0 \). Hence, the perturbed black hole is rotating with respect to the asymptotic inertial frame with the non-zero (positive) angular velocity (92), but has a zero angular momentum, which means that it is just being “carried along” by dragging primarily induced by the disc. We stress that this feature is not a necessary outcome of the perturbation procedure, namely it is a consequence of our choice of the constants \( J_i \) (namely \( J_i = 0 \)) which can be employed to fix the black-hole spin in the solution (50) of the inhomogeneous perturbation equation (44); see the discussion in section 3.1.

9. CONCLUDING REMARKS

We have shown that the procedure suggested by Will (1974), originally employed to determine the gravitational perturbation of a Schwarzschild black hole by a slowly rotating and light thin ring, can be also applied to the perturbation due to a thin disc. However, concerning the bad numerical properties of the series involved, we have dropped the angular expansion and rather expressed the Green functions (perturbations due to a thin ring) in a closed form. Such expressions bring more complex special functions, but these can be evaluated effectively with rapidly converging algorithms, so the resulting numerical convergence is much better.

Using the proposed closed-form Green functions, one can in principle obtain an arbitrary order of the perturbation. However, a numerical treatment is necessary in order to analyse specific results, as illustrated on a simple example of the “uniform-density” disc in section 6.

An important point has been to show that the series involved in computation of the Green functions converge (section 4 and the accompanying paper by Čížek (in preparation)). However, what still remains to be answered is whether the perturbation scheme is effective to any order, namely whether the perturbation expansion (29) with parameter proportional to the external-source mass converges. Regarding the structure of the Green functions and the speed of their convergence to zero at infinity, as well as the structure of source terms of higher perturbation orders, one conjectures that the expansion converges at least for some positive disc masses.

Various properties of the obtained solution could be studied now, for instance, deformation of the geometry (as represented by curvature invariants, in particular), deformation of the (originally spherical) horizon, perturbation of the properties of stationary circular motion or influence on geodesic structure. In particular, it will be interesting to see how the perturbation influences the location of important circular equatorial geodesics (mainly of the innermost stable one, usually abbreviated as ISCO), because this should indicate how the actual quasi-stationary accretion disc may differ from its test-matter model. Also, à propos, one could consider a different type of disc (a different density distribution) – preferably close to what follows from models of accretion onto astrophysical black holes – and compare the results with what has been found here for the simplest case of constant density.

9.1. Comparison with black-hole–disc configurations found numerically

Another obvious option is to compare the perturbative solution with the results of numerical treatment of similar source configurations. The most similar of these – a hole with a thin finite annular equatorial disc – was studied by Lanza (1992), while Nishida & Eriguchi (1994) considered a hole with a thick toroid. More recently, Ansorg & Petroff (2005) used a different numerical method to compute stationary and axisymmetric configurations of uniformly rotating constant-density toroids around black holes. They specifically used these solutions to demonstrate that both the central hole and the surrounding toroid may in some cases have negative Komar masses (Ansorg & Petroff 2006). Yet another codes for studying self-gravitating matter around black holes have been developed, and specifically used to find stationary thick-toroid configurations, by Shibata (2007), Montero et al. (2008) and Stergioulas (2011). Finally, Karkowski et al. (2016) analyzed such rotating black-hole–toroid systems in the first post-Newtonian approximation.

The possibility to compare our results with the above “exact” numerical configurations (in particular the thin discs by Lanza) is very limited, mainly because our first-order perturbation only represents gravitation of the disc, not its self-gravitation (no back effect of the source on itself through its field); more generally speaking, the solution does not incorporate any non-linearity of the Einstein equations. There are also other, more definite differences. The numerical results of Lanza (1992) were obtained by numerical “relaxation” of initial configurations provided by “squeezing” the well known (analytical) test thick-disc models in a given Kerr background. More specifically, assuming constant ratio of angular momentum to energy (with respect to infinity) throughout the disc, in our notation
\[ \text{const} = \frac{u_\phi}{-u_t} = \frac{g_{\theta\phi} + g_{\phi\phi} \Omega}{g_{tt} + g_{\phi\phi} \Omega} = \left( \omega + \frac{e^{2\nu}}{B_\rho v} \right)^{-1}, \]
and choosing the inner disc radius and the proportionality constant appearing in the polytropic equation of the disc-gas state, the initial surface density and outer radius of the disc are obtained. Generally, with the increase of that constant,
the surface density (as well as pressure) and mass of the resulting disc rise quickly, while the outer disc radius grows with the specific angular momentum of the disc matter. In contrast to Lanza’s constraint of constant specific angular momentum of the disc’s surface density and outer radius derived accordingly, we have exemplified our perturbation procedure on a disc with prescribed and constant Newtonian surface density (while non-constant angular momentum) and with both inner and outer radii prescribed as well.

Lanza illustrated his numerical scheme on two groups of configurations sequences, one containing a rapidly rotating hole (specified by its horizon area and angular momentum) and one with a slowly rotating hole (in this case specified by its horizon angular velocity \( \omega_H \) and isotropic radius \( k/2 \)). Focusing naturally only to the latter, one finds solution sequences for three different rotation choices in the cited paper: \( \omega_H = 0.0025/M, \omega_H = 0.025/M \) and \( \omega_H = 0.25/M \), in all cases with \( k = M \) and \( r_A = 8M \); each sequence is characterized by a certain fixed value of the angular momentum to energy ratio (constant throughout the disc) and was generated by gradual increase of the disc mass. With the increasing disc mass, also the total mass of the system were found to increase (almost linearly), while the total angular momentum and the horizon area \( A_H \) were increasing slightly faster, with the horizon surface gravity decreasing consequently according to the generic relation \( \pi H = 4\pi k/A_H \). One special point Lanza examined on slowly rotating configurations was that the black-hole rotational angular momentum may decrease to negative values when the disc angular momentum is being increased (while \( \omega_H \) is kept fixed). This happens when the disc is “overtaking” the black hole in the sense that the combined rotational-dragging effect is stronger than the effect due to the hole alone (such a circumstance was already pointed out by Will 1974). Let us remind that in our case the black hole was effectively set to keep zero angular momentum (while acquiring non-zero angular velocity) in the perturbation, as confirmed at the end of section 8.1.

Regarding the differences between assumptions of the above numerical treatment and our perturbative one, the comparison of results obtained by these two methods is going to be rather problematic. Anyway, we now plan to compute various more specific parameters of the disc considered as an example in section 5 (and the following ones), in order to possibly return to this point. Let us thus conclude with one particular surprising observation made by Lanza (1992) which should be simple to compare: for slowly rotating black holes, Lanza found that the presence of the disc can make the horizon’s polar proper circumference larger than the equatorial one, which would suggest that the horizon becomes prolate (along the symmetry axis). This goes against common experience that the black holes stretch towards external sources of gravity (cf. Nishida & Eriguchi 1994 and Ansorg & Petroff 2005 who always got oblate horizons).

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