

PhD Thesis of Tomáš Jakl

## **d-Frames as algebraic duals of bitopological spaces**

(Assessment by one of the supervisors)

Theory of d-frames the Thesis is concerned with investigates a structure connecting some topological and algebraic facts (by dualities or more general adjunctions), with a strong motivation (and interesting applications) in theoretical computer science. Let me say right away that the defendant has not only fulfilled the original task (as indicated in the title), but that his research went far beyond that. The results are a very substantial contribution to the theory and its applications (in particular in logic of computing). Important problems are solved, some aspects of the theory are shown in a new perspective and new paths are opened.

The concept of a d-frame was introduced by Achim Jung and Drew Moshier in connection with Stone type dualities concerning bitopological spaces. A d-frame is constituted by a pair  $L_+, L_-$  of frames endowed with relations of *consistency* and *totality*,  $\text{con}$  and  $\text{tot}$ , subsets of  $L_+ \times L_-$ , satisfying certain natural axioms. The terms “consistency” and “totality” suggest a logical connotation, and indeed, d-frames turned out to be, a.o., a very useful tool in the study of the logical aspects of computer science (in which the defendant also substantially contributed).

The Thesis is divided into six chapters followed by a helpful Appendix. The author writes rigorously and concisely, but in a style that should make the text readable to advantage not only for topologist – point-free, classical or categorical – but also for logicians and duality theorists, and I think he should be successful with such readership.

Chapter 1, “Introduction” explains the general aims of the theory. The motivating dualities (and adjunctions) are explained (with particular attention paid to the logical aspects), and the usefulness of d-frames indicated.

In the following chapter, “Bispaces and d-frames” the author starts with explaining the phenomenon of bitopology, that is, the fact that in many useful cases the space structure studied is made more transparent if viewed as two topologies naturally creating the given global one, or associated with it. The immediate point-free counterpart of a bispaces is a biframe (intensively studied in literature). It turns out, however, that the bispaces often appearing in computer science have certain features that suggests a spe-

cific structure of a *d-frame*, more natural for the duality aspects in this area (and not only for them). This structure is analyzed, special d-frames introduced (separation properties etc.), the comparison with the compact pospaces dealt with, and the famous dualities put into the perspective of the natural adjunction of bispaces and d-frames.

The relations of consistence and totality in  $L_+ \times L_-$  constituting the structure of a d-frame have to satisfy very natural axioms. However natural (and necessary for the applications) these conditions are, they do not come automatically in the constructions. The next, third, chapter is devoted to the category of d-frames and shows how one can cope with such difficulties. First one deals with the basic problem of quotients. The standard and very transparent construction of a quotient from general frames cannot be applied without substantial modification, because it produces objects with the essential properties damaged. The problem is solved by introducing the category of pd-frames **pd-Frm** in which the axioms are relaxed so that they are easily seen to be preserved by natural constructions of limits and colimits, and then proving (producing two alternative constructions) that **d-Frm** is reflective in **pd-Frm**. Further in this chapter, a factorization system in **d-Frm** is produced, and the problem of free construction solved and discussed (it is worth noting that as an example of a free construction one can present a coproduct without having to use factorization). Another example, the Vietoris construction is a sort of bridge to the following chapter.

Chapter 4 concerns Vietoris constructions for bispaces and d-frames. This is, by a few pages, the longest chapter of the Thesis, and the topic deserves it. If one follows in a frame the classical Vietoris approach one obtains a naturally defined space (of “compact elements”); this construction lacks the most basic property one would like to have (and which one would need in applications): the original object is not embedded into the result. This problem was solved by Johnstone in 1985 by presenting a construction based on a suitably chosen equation system in a free frame. Having prepared the technique in the previous chapter, the author is able to proceed similarly in the new (more complex) context. But first one has to explain this extended situation. The chapter starts with a very useful discussion of the constructions for posets and spaces, followed by a Vietoris construction(s) for bispaces and its (their) properties. Then the author presents a construction of the Vietoris objects for d-frames. Important properties are proved and discussed, in particular the relations between the upper, lower and composed variants are shown, the comonad structure established, and relations between some associated constructions are proved. The chapter is

completed with an applications to modal and coalgebraic logic.

After sizable chapters 3 and 4 comes a chapter on stably compact spaces. It is much shorter, but also important. The topic of stable compactness played a role in the introduction the notion of a d-frame, and the connections call for discussion. The author uses the previously presented theory to show some facts in a new perspective. This is not to say that this chapter should contain remakes of known facts only; there are also results that are quite new.

Finally, the last chapter “Belnap-Dunn logic of bispaces” is devoted to the relation of d-frames with a version of the logic of observable properties (which admits paraconsistent reasoning, that is, allows reasoning with conflicting or incomplete information). A logical system based on the category of d-frames is produced and its soundness and completeness is proved. Although the last fact constitutes only a small part of the Thesis, it is a very important achievement and I think it may play a considerable role in the future.

It might not be in a good taste to cumulate superlatives when assessing a Thesis in the role of a supervisor. Let me just state that I am very well contented with both the results and the form of the Thesis. I can warmly recommend it to be accepted.

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Aleš Pultr