

REPORT ON THE PhD THESIS BY Tomáš Jakl:

d-Frames as algebraic duals of bitopological spaces

The present thesis provides an extensive, well rounded, detailed, and thoroughly written study of **d-frames** which are the algebraic duals of **bitopological spaces**.

The thesis consists of one introductory, one survey, and four "main" chapters. The survey chapter start by recalling the well-known correspondence between **topological spaces** and **frames** (in the order-theoretic sense), which extends into adjunction between their respective categories and specializes to a categorical equivalence between the subcategories of spatial frames and sober spaces.

The chapter then continues by presenting the work of Jung, Moshier, and others, which can be summarized under the slogan "we redo it for the **bitopological spaces**" (which are nothing else than spaces with two topologies). They have established that one can mimic the story of frames and topological spaces, replacing frames by **d-frames** and topological spaces by their bi-version. More precisely, it is still true that we have an adjunction between the corresponding categories, which - moreover - restricts to an equivalence between spatial d-frames and sober bitopological spaces. This duality subsumes some known dualities, including Priestley and Stone's ones.

So, what is a d-frame? Informally, it is a pair of frames L^+ and L^- , related by two compatibility relations which govern the interplay of the two natural orders (representing respectively truth and knowledge) in the bilattice over the direct product of the frames L^+ and L^- .

Now that we learned that d-frames are a useful algebraic tool to understand bitopological spaces, it makes sense to investigate the category **d-Frm** of d-frames. This is done in Chapter 3, whose main results states that **d-Frm** is bicomplete. The author has approached this problem by first exploring a bigger category of proto d-frames, which can easily be shown to be bicomplete (Section 3.1) and as **d-Frm** is proved to be its reflective subcategory (Section 3.2), the result follows. The subsequent sections then explore the consequences of this result. The main theorem and its consequences are perhaps to-be-expected generalizations of the classical topological results and their interest is rather technical, but to put it all together in such a neat fashion is no small feat. Here I would like to praise especially Section 3.5 which contains a concrete clever descriptions of free d-frames.

Chapter 4 deals with generalizations of the **Vietoris construction** for topological spaces and frames. In Section 4.1 the student defines its natural generalization for bitopological spaces and studies which bitopological notions it preserves. Section 4.2 does the same for d-frames. The subsequent two sections investigate the categorical properties of the resulting constructions, which turn out to be quite the same as in the usual topological setting, and culminate by a careful proof of the expected fact that the bitopological and d-frame versions of the Vietoris construction are indeed dual. From the last section I would like to mention the student's observation that this duality generalizes the duality theory between (positive) modal algebras and its topological duals.

This section is more interesting and more difficult than the previous one, partly due to the beauty of the Vietoris construction. However, the work has still a certain character of "mimicking the classical topological results". Nevertheless, it should be mentioned that especially Section 4.2 contains a lot of hard work. In addition, the author presents a novel kind of model for positive modal logic in Proposition 4.5.8. The subsequent Remark 4.5.10 implies some relation between frames including worlds and co-worlds (arising from canonical extensions) and bitopologies, but unfortunately this hint is not developed further, which is a pity.

Chapter 5 we gives new bitopological proofs of some old results in the theory of stably compact spaces. Due to the lack of familiarity with its subject matter I cannot offer any meaningful comments here.

The last chapter develops a **logic of d-frames**, which is intended to capture reasoning with observable properties. The first step is to relate d-frames to bilattices whose relation to logic is rather well explored in the literature. Subsequently, standard ways of equipping bilattices with negation, implication and modal operations are investigated and reviewed in the new setting. The author shows that bilattices (which is he considering in a rather restricted distributive setting, it would interesting to know some deeper reasons for this restriction) can be identified with a subcategory of **d-Frm** and similarly for their topological duals. In Section 6.3 the logic of d-frames is finally presented as an adaptation of Vicker's logic of frames. The student provides a calculus and shows its completeness (the proof relies on the one for Vickers' frame logic, and requires a non-standard argument because of the presence of infinitary operations).

Generalization conundrum: the thesis deals with a lot of interesting tools and techniques such as frames and topologies, Vietoris construction, domains, and geometric logic. However, there seem to be a rather tedious underlying pattern, which consists of **generalizing** all these notions and theory relating them to the bitopological setting. It makes one wonder whether there are indeed brand new ideas behind the work or whether much of the results were to be expected. In some cases I am tempted to guess a positive answer to the above questions.

But let me stress here that this not-so-enthusiastic impression comes from someone quite outside the topic of the thesis and is **more than balanced** by the vast variety of mathematical techniques the student has demonstratively mastered and the diversity of possible applications of the achieved results he has explored in his thesis.

The summary: The thesis is clearly and rigorously written and contains many hard-to-prove original results with a wide array of possible applications and so its author has **demonstrated beyond any doubt his ability to successfully undertake an independent research.**

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