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On the interplay of combinatorics, geometry, topology and computational complexity

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Contents

Preface

This habilitation thesis consists of reprints of nine papers coauthored by Martin Tancer, together with an introductory commentary.[1] The common topic of the papers is that they focus on the interplay among combinatorics, combinatorial geometry, topology and computational complexity. However, the motivation for the results in these papers also partially comes from other areas of mathematics such as algebra or Riemannian geometry.

The main aim of the introductory commentary is to quickly explain main results of each of the individual papers and to provide the links among the papers. We remark that many details regarding the motivation or the ideas for the proofs are left out from the introductory commentary and we refer to the introductions of each of the individual papers. The aim of the introductory commentary is not to repeat these introductions (although some bit of repetition is perhaps unavoidable in order to explain the results).

The nine papers forming the main body of the thesis are listed at the end of preface. Namely, papers [3, 4, 6] relate to the combinatorial and algorithmic properties of embeddability of simplicial complexes (and its applications) and their contents is explained in Section 1.2. Papers [8, 9] focus on collapsibility and shellability, two combinatorial ways how to simplify a topological space (given as a simplicial complex or a poset). Their contents is explained in Section 1.3. Papers [2, 5, 7] study combinatorial and topological properties of graphs and curves drawn on surfaces and they are explained in Section 1.4. Finally, [1] is a result on the growth of homology of certain complexes associated to graphs and it is explained in Section 1.5

[1] Karim Adiprasito, Eran Nevo, and Martin Tancer. On Betti numbers of flag complexes with forbidden induced subgraphs. *arXiv preprint*, https://arxiv.org/abs/1602.01761, 2016.

[2] Éric Colin de Verdière, Vojtěch Kaluža, Pavel Paták, Zuzana Patáková, and Martin Tancer. A direct proof of the strong Hanani-Tutte theorem on the projective plane. *arXiv preprint*, http://arxiv.org/abs/1608.07855, 2016. To appear in Proceedings of the 24th International Symposium on Graph Drawing and Network Visualization (GD 2016).

[3] Xavier Goaoc, Isaac Mabillard, Pavel Paták, Zuzana Patáková, Martin Tancer, and Uli Wagner. On generalized Heawood inequalities for manifolds: a van Kampen–Flores-type nonembeddability result. In *31st International Symposium on Computational Geometry*, volume 34 of *LIPIcs. Leibniz Int. Proc.*

[1]These papers are in general very recent. For papers that do not currently have a publicly available final journal version yet, we use a preprint from arXiv.

Inform., pages 476–490. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2015. Full version: https://arxiv.org/abs/1610.09063, full version to appear in Israel Journal of Mathematics.

[4] Xavier Goaoc, Pavel Paták, Zuzana Patáková, Martin Tancer, and Uli Wagner. Bounding Helly numbers via Betti numbers. In *31st International Symposium on Computational Geometry*, volume 34 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages 507–521. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2015. Full version: https://arxiv.org/abs/1310.4613, full version to appear in Journey Through Discrete Mathematics. A Tribute to Jiří Matoušek.

[5] Alfredo Hubard, Vojtech Kaluža, Arnaud de Mesmay, and Martin Tancer. Shortest Path Embeddings of Graphs on Surfaces. In Sándor Fekete and Anna Lubiw, editors, *32nd International Symposium on Computational Geometry (SoCG 2016)*, volume 51 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 43:1–43:16, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. Full version: https://arxiv.org/abs/1602.06778.

[6] Jiří Matoušek, Eric Sedgwick, Martin Tancer, and Uli Wagner. Embeddability in the 3-sphere is decidable. In *Computational geometry (SoCG'14)*, pages 78–84. ACM, New York, 2014. Full version: https://arxiv.org/abs/1310.4613, full version to appear in Journal of the ACM.

[7] Jiří Matoušek, Eric Sedgwick, Martin Tancer, and Uli Wagner. Untangling two systems of noncrossing curves. *Israel J. Math.*, 212(1):37–79, 2016.

[8] Martin Tancer. Shellability of the higher pinched Veronese posets. *J. Algebraic Combin.*, 40(3):711–742, 2014.

[9] Martin Tancer. Recognition of collapsible complexes is NP-complete. *Discrete Comput. Geom.*, 55(1):21–38, 2016.

On the interplay of combinatorics, geometry, topology and computational complexity

1.1 Introduction

The fundamental task of combinatorics is to study properties of discrete objects such as their enumeration, extremal properties, interactions or structural properties. Although it is often possible to solve combinatorial problems by intrinsic combinatorial means, in general, combinatorics strongly benefits from interactions with other areas of mathematics (and vice versa).

Combinatorics is strongly linked to the theoretical computer science. Understanding efficient algorithms for recognition of combinatorial objects with certain property or for enumeration of objects is an integral part of combinatorics. For instance, the Kuratowski's planarity [Kur30] criterion is as central result in graph theory as the existence of a linear time algorithm for recognition of planar graphs by Hopcroft and Tarjan [HT74].

A rich mathematical world appears when we further combine these two subjects with questions in combinatorial geometry and topology. From the point of view of the way how these subjects interact, we may distinguish several areas. They include *algorithmic topology*, *combinatorial topology* and *graph drawings* (on surfaces).[1]

Algorithmic topology. The task of algorithmic topology is to design efficient algorithms for topological problems. This usually comes together with some combinatorial model for the topological spaces, maps, etc. in the question so that we may have a finite input for the algorithmic question.

A prominent example in this line of research is the unknot recognition problem which is currently known to belong to NP $\cap$ co-NP [HLP99, Lac16]; but no polynomial time algorithm is known for this problem.

To this area of research we contribute with an algorithm for the 3-embeddability problem [6] (see Section 1.2.1) and with a result on NP-hardness of recognition of collapsible complexes [9] (see Section 1.3.1).

Combinatorial topology. By combinatorial topology we mean the area studying the direct interactions of combinatorial and topological objects. This

[1]This list is neither complete nor pairwise disjoint. As other areas we could name, for example, topological methods in combinatorics or study of metric embeddings. However, we focus only on the areas covered in this thesis. Regarding disjointness, for example [7] partially belongs to all three mentioned areas.

is usually done via properties of simplicial complexes which can be seen both as purely combinatorial objects (hereditary set systems) as well as topological objects (triangulations of topological spaces).

Such interactions include, for example, combinatorial questions on embeddings of simplicial complexes into $\mathbb{R}^d$ or other topological spaces initiated by van Kampen and Flores [vK32, Flo34], study of clique complexes of graphs with forbidden certain subgraphs as a discrete analogue of bounded curvature [JŚ06] or study of Morse spectra of triangulated spaces as in [ABL14], for example.

To this area, we contribute by results on the homology growth of clique complexes of graphs with forbidden certain subgraphs [1] (see Section 1.5) and a significant progress towards Kühnel's conjecture on embeddings of skeleta of simplices into manifolds [3] (see Section 1.2.2). We also provide a very general topological Helly-type theorem [3] (see Section 1.2.2), which can be seen as a result partially belonging to this area and partially to 'topological methods' in combinatorics. Similarly, the result [8] on shellability of higher pinched Veronese posets (see Section 1.3.2) can be seen as result partially belonging to topological combinatorics and partially belonging to combinatorial commutative algebra.

Drawings of graphs on surfaces. It could be easily argued that studying various aspects of drawings of graphs on surfaces is just a part of combinatorial topology described above. However, the lower-dimensional nature of drawing of graphs on surfaces causes that there are very different interesting questions in this area and it brings the area even closer to combinatorics. That is why we consider this area separately.

Classical questions in this area include to determine which graphs can be drawn on which surface without crossings or what are the other combinatorial properties of graphs drawn on surfaces (such as the chromatic number); see [MT01]. Regarding drawings where we allow crossings, it is very interesting to study various aspects of the crossing number of such graphs [Sch13a].

To this area, we contribute with an alternative proof of the strong Hanani-Tutte theorem on the projective plane [2] (see Section 1.4.3) and with results on drawings of graphs on surfaces with shortest paths [5] (see Section 1.4.2).

Finally, contribution [7] interacts with all three areas. Here we show that two systems of pairwise disjoint curves can be untangled with a self-homeomorphism of a surface applied to one of the systems so that there are not too many intersections among the two systems (see Section 1.4.1). Systems of curves are closely related to drawings of graphs. In addition a part of the main result in [7] serves as a verification of the algorithm in [6].

On preliminaries. We assume that the reader is familiar with basic notions from combinatorics, computational complexity and topology. In particular, we frequently use basic properties of simplicial complexes in the text below and we also use the basics of the homology theory. For further reading, we refer, for example, to [Hat01, Mat03].

Organization of the remaining sections. In the forthcoming sections, we briefly explain the contents of the individual papers that are part of this thesis. We group the papers together according to the similarity of the topics, as sketched in Preface. It turns out that this is not the same as what we described in this introductory section according to the way how the fields interact.

1.2 Embeddability of simplicial complexes

Let X and Y be two topological spaces, does X embed into Y? This is a classical important general question in topology. We cannot expect that there would be a simple criterion that would answer this question. Indeed, it includes, for example, the *homeomorphism problem* for manifolds which is known to be algorithmically undecidable. Nevertheless there are important classes of spaces X and Y for which the question can be either fully answered or there are important sufficient and/or necessary conditions to be understood.

We will mostly focus on the case where X and Y are topological spaces triangulated as finite simplicial complexes. In this setting, it is possible to represent X and Y in computer and thus we may ask algorithmic questions on embeddability.[2] From theoretical point of view, the structure of simplicial complex allows linking topological and combinatorial questions on embeddings which has fruitful consequences as we will argue in 1.2.2.

1.2.1 Algorithmic aspects

From algorithmic point of view one of the most natural settings is the following algorithmic question $\textsc{Embed}_{k\to d}$, which depends on two positive integers k and d, $k \leq d$: Given a simplicial complex K of dimension at most k, does K (piecewise linearly) embed into $\mathbb{R}^d$?

The question $\textsc{Embed}_{k\to d}$ was introduced by Matoušek, the author and Wagner in [MTW11] and based on this paper it was one of the central topics of the author's PhD thesis. It was previously known that the cases $\textsc{Embed}_{1\to 2}$ (graph planarity [HT74]) and $\textsc{Embed}_{2\to 2}$ [GR79] are solvable in linear time and that for every $k \geq 3$ fixed, $\textsc{Embed}_{k\to 2k}$ can be decided in polynomial time (this is based on the work of Van Kampen, Wu, and Shapiro; see [MTW11] for a detailed explanation).

For dimension $d \geq 4$, the currently known understanding of the computational complexity of $\textsc{Embed}_{k\to d}$ is the following: for all k with $(2d-2)/3 \leq k \leq d$ it is NP-hard (and even undecidable if $k \geq d-1 \geq 4$) [MTW11], while for $k < (2d-2)/3$ it is polynomial-time solvable, assuming d fixed, as was shown in a series of papers on computational homotopy theory [ČKM+14a, ČKM+14b, KMS13, ČKV13].

Dimension 3. The new contribution (when compared with [MTW11]), presented as a part of this thesis, is the joint work with J. Matoušek, E. Sedgwick and U. Wagner [6] where we show the following.

Theorem 1 (Thm 1.1 & Cor. 1.2 in [6])**.** *The problems* $\textsc{Embed}_{2\to 3}$ *and* $\textsc{Embed}_{3\to 3}$ *are algorithmically decidable.*

Here we only very briefly sketch the main steps; for a more detailed overview of the idea of the proof we refer to Sections 1, 2 and 3 of [6].

In fact, it is sufficient to show algorithmic decidability of $\textsc{Embed}_{2\to 3}$; solution for $\textsc{Embed}_{3\to 3}$ then follows by a combinatorial reduction. The first step is to

[2] However, this is not the only possible representation. For example, a representation as a simplicial set might be a more efficient representation of the same space.

show that it is actually sufficient to establish the following variant of the problem for 3-manifolds.

Theorem 2 (Thm. 1.3 in [6])**.** *There is an algorithm which decides whether a given triangulated* 3*-manifold* X *with boundary embeds into the* 3*-sphere* S^3*.*

Indeed, given a 2-complex K (that is, an instance of $\text{EMBED}_{2\to3}$), we can test all possible *thickenings* of K to a 3-manifold with boundary (up to a homeomorphism). That is, to a manifold which contains K and collapses to K. Then K embeds into $\mathbb{R}^3$ if and only if at least one thickening of K embeds into S^3. By an algorithm of Neuwirth, it is possible to generate all possible thickenings [Neu68] (see also [Sko95]).

The bulk of our contribution is to prove the following result.

Theorem 3 (Thm. 1.4 in [6])**.** *Let* X *be an irreducible* 3*-manifold, neither a ball nor an* S^3*, with incompressible boundary and with a* 0*-efficient triangulation* $\mathcal{T}$*. If* X *embeds in* S^3*, then there is also an embedding for which* X *has a* short meridian γ*, i.e., an essential*[3] *normal curve* $\gamma \subset \partial X$ *bounding a disk in* $S^3 \setminus X$ *such that the length of* γ*, measured as the number of intersections of* γ *with the edges of* $\mathcal{T}$*, is bounded by a computable function of the number of tetrahedra in* $\mathcal{T}$*.*

Here an *irreducible manifold* is such that every embedded 2-sphere in X bounds a ball in X; it has incompressible boundary if any curve in ∂X that bounds a disc in X also bounds a disc in ∂X. A 0-efficient triangulation is a technical term that we do not define here (and we refer to the reprint of [6]). A *normal curve* on ∂X is a closed curve which avoids vertices of the triangulation; it crosses each edge transversally; and it meets each triangle in a finite number of arcs with endpoints on different edges of the triangle. Finally, the *length* of a normal curve is the number of edges of the triangulation it crosses.

Theorem 3 allows to prove Theorem 2 recursively. After standard transformations, we may assume that X satisfies the assumptions of Theorem 3. Then we may enumerate all normal curves up to the length provided by Theorem 3 as candidate meridians, fill them with a thickened disc and recurse.

For the proof of Theorem 3, we already refer to [6].

1.2.2 Combinatorial aspects

From a combinatorial point of view, we plan to present two results regarding embeddability of simplicial complexes.

Almost embeddings. First, we need to introduce a certain important notion. Given a simplicial complex K and a topological space Y, an *almost embedding* is a map $f\colon |K| \to Y$ such that $f(|\sigma|) \cap f(|\tau|) = \emptyset$ whenever σ and τ are disjoint simplices of K.[4] Every embedding is an almost embedding but the converse is not true.

The classical results of van Kampen and Flores [vK32, Flo34] state that the following k-dimensional complexes do not embed into $\mathbb{R}^{2k}$:

[3] Meaning that γ does not bound a disk in ∂X.

[4] Unless stated otherwise, we work with abstract simplicial complexes. For a face (simplex) $\sigma \in K$, the symbol $|\sigma|$ denotes the geometric simplex corresponding to σ in some fixed geometric realization of K. Finally, $|K|$ denotes the *underlying space* of K, that is $\bigcup_{\sigma\in K} |\sigma|$.

- $\Delta^{(k)}_{2k+2}$, that is, the k-skeleton of the $(2k+2)$-simplex, and
- $D_3^{*(k+1)}$, that is, the $(k+1)$-tuple join of the three-element discrete set.

However, the standard proofs provide a stronger conclusion: these complexes do not even almost embed into $\mathbb{R}^{2k}$. In general, almost embeddings are useful tool for understanding embeddings as they are often easier to handle.

(Almost) embeddings on the level of chain maps. For further applications it turned out that it is important to study the (almost) embeddings on the level of chain maps (in $\mathbb{Z}_2$-homology). In a joint work with X. Goaoc, P. Paták, Z. Patáková and U. Wagner [4] we have developed an inductive Ramsey-based approach how to build a certain combinatorially well behaved chain map $C_*(K) \to C_*(\mathbb{R}^d)$ where K is a simplicial complex. As an application of this approach, we have obtained a Helly-type theorem with very weak topological assumptions (see Theorem 4 below). Subsequently, in a joint work with the same group of coauthors and in addition with I. Mabillard [3], we have utilized a modification of this technique to a different problem regarding embeddability of simplicial complexes into manifolds. (In fact, a simplification of this technique is sufficient in [3] which allows to remove the use of the Ramsey theorem and yields improved quantitative bounds.)

A Helly theorem for collections of convex sets with very weak topological assumptions. Now we explain the statement of the main result of [4].

Helly's classical theorem [Hel23] states that a finite family of convex subsets of $\mathbb{R}^d$ must have a point in common if any $d+1$ of the sets have a point in common. In the contrapositive, Helly's theorem asserts that any finite family of convex subsets of $\mathbb{R}^d$ with empty intersection contains a sub-family of size at most $d+1$ that already has empty intersection. This inspired the definition of the *Helly number* of a family $\mathcal{F}$ of arbitrary sets. If $\mathcal{F}$ has empty intersection then its Helly number is defined as the size of the largest sub-family $\mathcal{G} \subseteq \mathcal{F}$ with the following properties: $\mathcal{G}$ has empty intersection and any proper sub-family of $\mathcal{G}$ has nonempty intersection; if $\mathcal{F}$ has nonempty intersection then its Helly number is, by convention, 1. With this terminology, Helly's theorem simply states that any finite family of convex sets in $\mathbb{R}^d$ has Helly number at most $d+1$.

Helly already realized that bounds on Helly numbers independent of the cardinality of the family are not a privilege of convexity: his *topological* theorem [Hel30] asserts that a finite family of open subsets of $\mathbb{R}^d$ has Helly number at most $d+1$ if the intersection of any sub-family of at most d members of the family is either empty or a *homology cell.*[5] Subsequently, several other topological generalizations of the Helly theorem were found. However, as far as we know, all these generalizations require vanishing homology in certain dimension.

Here we offer a generalization that requires only a bounded homology (but possibly non-zero). We consider homology with coefficients in $\mathbb{Z}_2$, and denote by $\tilde{\beta}_i(X)$ the ith reduced Betti number (over $\mathbb{Z}_2$) of a space X. Furthermore, we use the notation $\bigcap \mathcal{F} := \bigcap_{U \in \mathcal{F}} U$ as a shorthand for the intersection of a family of sets.

[5] By definition, a homology cell is a topological space X all of whose (reduced, singular, integer coefficient) homology groups are trivial, as is the case if $X = \mathbb{R}^d$ or X is a single point.

Theorem 4 (Thm 1. in [4]). *For any non-negative integers b and d there exists an integer $h(b,d)$ such that the following holds. If $\mathcal{F}$ is a finite family of subsets of $\mathbb{R}^d$ such that $\tilde{\beta}_i\left(\bigcap \mathcal{G}\right) \leq b$ for any $\mathcal{G} \subsetneq \mathcal{F}$ and every $0 \leq i \leq \lceil d/2 \rceil - 1$ then $\mathcal{F}$ has Helly number at most $h(b,d)$.*

Theorem 4 subsumes many other existential Helly-type theorems as well as it helps to identify new Helly-type theorems for concrete collections of sets. For a detailed overview of consequences and additional background, we refer to Sections 1.1, 1.2 and 1.3 of [4]. For a sketch of a proof, which also explains the relation to (almost) embeddings, we refer the reader to Section 1.4 of [4].

On a conjecture of Kühnel. Now we explain the contents of [3].

The fact that the complete graph K_5 does not embed in the plane has been generalized in two independent directions. One generalization is coming from the solution of the classical *Heawood problem* for graphs on surfaces which implies that the complete graph K_n embeds in a closed surface M (other than the Klein bottle) if and only if $(n-3)(n-4) \leq 6b_1(M)$, where $b_1(M)$ is the first $\mathbb{Z}_2$-Betti number of M. Second generalization is the aforementioned van Kampen–Flores theorem saying that the k-skeleton of the n-dimensional simplex embeds in $\mathbb{R}^{2k}$ if and only if $n \leq 2k+1$.

In [Küh94], Kühnel conjectured the following common generalization.

Conjecture 5 (Kühnel). *Let $n, k \geq 1$ be integers. If $\Delta_n^{(k)}$ embeds in a compact, $(k-1)$-connected $2k$-manifold M with kth $\mathbb{Z}_2$-Betti number $b_k(M)$, then*

$$\binom{n-k-1}{k+1} \leq \binom{2k+1}{k+1} b_k(M). \tag{1.1}$$

In [3], using the aforementioned technique, we obtained the following bound towards the Kühnel conjecture.

Theorem 6 (Thm. 2 in [3]). *If $\Delta_n^{(k)}$ almost embeds into a $2k$-manifold M then*

$$n \leq 2\binom{2k+2}{k} b_k(M) + 2k + 4,$$

where $b_k(M)$ is the kth $\mathbb{Z}_2$-Betti number of M.

The quantitative bound on n from Theorem 6 is much weaker that the bound conjectured by Kühnel. On the other hand Theorem 6 does not require that the manifold is $(k-1)$-connected and also the assumption that $\Delta_n^{(k)}$ *almost* embeds is weaker. As far as we know, the bound from Theorem 6 is a first finite bound on n of this type. In addition, Theorem 6 further generalizes to the case of mappings not covering a same point q-times (where q is a parameter, power of a prime number); see Theorem 3 in [3].

As usual, we refer to the introduction of [3] for more detailed background.

1.3 Collapsibility and shellability

There are various ways how to simplify a simplicial complex step by step while keeping certain topological or combinatorial properties of interest. Two of the most important notions in this respect are *collapsibility* and *shellability* of a simplicial complex which we introduce below.

1.3.1 Collapsibility

Let K be a simplicial complex and let σ be a nonempty non-maximal face of K. We say that σ is *free* if it is contained in only one maximal face τ of K. Let K' be the simplicial complex obtained from K by removing σ and all faces above σ, that is,

$$K' := K \setminus \{\vartheta \in K : \sigma \subseteq \vartheta\}.$$

We say that K' arises from K by an *elementary collapse* (induced by σ and τ). We say that a complex K *collapses* to a complex L if there exist a sequence of complexes $(K_1 = K, K_2, \dots, K_{m-1}, K_m = L)$, called a *sequence of elementary collapses* (from K to L), such that K_{i+1} arises from K_i by an elementary collapse for any $i \in \{1, \dots, m-1\}$. A simplicial complex K is *collapsible* if it collapses to a point.

An important property of elementary collapses is that they preserve homotopy type. Thus, for example, collapsibility of some complex serves as a certificate that the complex is contractible (homotopically trivial). However, even if we start with a complex that is not contractible, it may be very useful to simplify it with collapses to a smaller complex for which we can determine the homotopy type more easily.

From purely theoretical point of view, collapsibility plays an essential role, for example, in PL-topology where it helps to determine properties of regular neighborhoods [RS72] or it is strongly related to the discrete Morse theory [For98] where the Morse functions (roughly) correspond to sequences of collapses. From more practical point of view, an application of collapsibility can be found, for example, in shape reconstruction [AL15].

In [9], we prove the following algorithmic result on collapsibility.

Theorem 7 (Thm. 1 of [9])**.** *It is NP-complete to decide whether a given* 3*-dimensional simplicial complex is collapsible.*

In the statement above '3' can be replaced with any $d \geq 3$. On the other hand, recognition of collapsible 2-dimensional complexes is polynomial time solvable.

It is easy to see that the problem in Theorem 7 belongs to NP by guessing a right sequence of collapses. Thus, the core is to show that the problem is NP-hard. The proof of the NP-hardness builds on a previous work of Malgouyres and Francés [MF08] showing that it is NP-hard to decide whether a given 3-dimensional complex collapses to 1-complex. The reduction of Malgouyres and Francés uses complexes that are (typically) homotopically non-trivial and therefore the resulting 1-complexes (for positive instances) do not further collapse to a point. The key new step in [9] is to overcome this difficulty by gluing suitable fillings to the 'holes' (despite the fact that the exact position of the holes is unknown prior to collapses). This requires introducing several auxiliary triangulated topological spaces including a Bing's house with three rooms, a modification of famous Bing's house with two rooms.

For more detailed background on Theorem 7, we refer to Section 1 of [9] and for a sketch of a proof, we refer to Section 3 of [9].

1.3.2 Shellability

Shellability of a simplicial complex is traditionally considered from a dual perspective when compared with collapsibility: this time, we start with an empty complex and we gradually add faces following certain rules until we reach the target complex.

More concretely, we say that a d-dimensional simplicial complex is *pure* if all its maximal faces (called *facets*) have dimension d. For simplicity we restrict ourselves to finite pure complexes when it is easier to grasp the definition. For such a complex K, a *shelling* of K is an ordering $\sigma_1, \ldots, \sigma_t$ of all facets of K such that for all $k \in \{2, \ldots, t\}$ the complex $B_k := \sigma_k \cap \left(\bigcup_{i=1}^{k-1} \sigma_i\right)$ is pure and $(\dim \sigma_k - 1)$-dimensional; here we regard σ_i as geometric simplices. A complex is *shellable* if it admits a shelling.

When compared with collapsibility, shellings do not necessarily preserve the homotopy type of the complex. However, they still may affect homology (or the homotopy type) only in certain ways. In particular, every shellable complex is homotopy equivalent to a wedge of spheres.

Shellability of posets. An important class of simplicial complexes is obtained as order complexes of posets. That is, given a poset $P = (P, \leq)$, the vertices of the order complex $\Delta(P)$ are elements of P and the simplices are the chains in P.

We will be interested especially in the following restricted case. Let $P = (P, \leq)$ be a graded poset with rank function rk. By $\hat{0}$ we mean the minimum element of P (if it exists) and similarly by $\hat{1}$ we mean the maximum element (if it exists). For $a, b \in P$ we say that *a covers b*, $a \rhd b$, if $a > b$ and there is no c with $a > c > b$. Equivalently, $a > b$ and $\mathrm{rk}(a) = \mathrm{rk}(b) + 1$. Pairs of elements a, b with $a \rhd b$ are also known as *edges* in the Hasse diagram of P. *Atoms* are elements that cover $\hat{0}$. That is, atoms are elements of rank 1 in a poset that contains the minimum element.

From now on, let us assume that P contains the minimum element. Let A be a set of some atoms in P. By $P\langle A\rangle = (P\langle A\rangle, \leq)$ we mean the induced subposet of P with the ground set

$$P\langle A\rangle = \{\hat{0}\} \cup \{b \in P \colon b \geq a \text{ for some } a \in A\}.$$

Now we assume that P contains both the minimum and the maximum element. Let $C(P)$ be the set of maximal chains of P. A *shelling order* is an order of chains from $C(P)$ satisfying the following condition.

(Sh) If c' and c are two chains from $C(P)$ such that c' appears before c, then there is a chain c^* from $C(P)$ appearing before c such that $c \cap c^* \supseteq c \cap c'$ and also c and c^* differ in one level only (that is, $|c \Delta c^*| = 2$ where Δ denotes the symmetric difference).

A poset P is *shellable* if it admits a shelling order. This is equivalent with saying that the order complex of P (which is pure) is shellable as a simplicial complex.

Shellability of a poset serves as a tool how to show that a poset is Cohen-Macaulay. This has further consequences on intrinsic properties of the poset;

see [BGS82]. On algebraic side, the fact that a certain polynomial ring is Koszul can be verified by checking that all intervals of a certain poset associated to the ring are Cohen-Macaulay; see the results of Peeva, Reiner and Sturmfels [PRS98] (this result is explicitly stated as Proposition 1.2 in [8]).

There are various sufficient criteria how to establish shellability of poset. Such criteria were pioneered by Björner [Bjö80] who proved that a certain edge-lexicographic labelling of the poset implies shellability. This criterion was later on extended by Björner and Wachs [BW82] to chain-lexicographic labellings. In the next paragraph we describe a new criterion to prove shellability obtained in [8], that we call A-shellability. This criterion has been successfully applied to show shellability of so-called pinched Veronese posets where the direct application of the other previously known criteria seems to fail.

A-shellability. Now let us assume that $A = (A, \leq^\circ)$ is a partially ordered set of some atoms in P. We say that $P\langle A\rangle$ is *A-shellable* if $P\langle A\rangle$ is shellable with a shelling order respecting the order on A. That is, if c and c' are two maximal chains on $P\langle A\rangle$ and the unique atom of c' appears before the unique atom of c in the $\leq^\circ$ order, then c' appears before c in the shelling.

The strength of this notion is that there are three inductive criteria allowing to prove A-shellability inductively for a well behaved class of posets; see Theorems 2.1, 2.2 and 2.3 in [8]. The choice of the partial order on A allows enough freedom not to overlook some important candidate shelling orders. However, on the other hand, if the order on A is non-trivial, it still preserves certain structure that can be useful in induction.

For more details on A-shellability we refer to Sections 1 and 4 in [8]

Shellability of pinched Veronese posets. By the m-th *Veronese poset with spacing* on n generators, denoted as $(\mathcal{V}_{m,n}, \leq)$ we mean the following poset. Its ground set consists of non-negative integer vectors of length n such the sum of their coordinates is divisible by m. The partial order on $\mathcal{V}_{m,n}$ is given so that $\mathbf{a} \leq \mathbf{b}$ if and only if $\mathbf{a}$ is less or equal to $\mathbf{b}$ in each coordinate. It is not hard to see that every interval in $\mathcal{V}_{m,n}$ is shellable and therefore Cohen-Macaulay.

If we set $m = n$, we just speak of the n-th *Veronese poset* $\mathcal{V}_n := \mathcal{V}_{n,n}$. We can pinch this poset in the following way. We remove the distinguished vector $\mathbf{j}$ which contains 1 in each coordinate. We also remove order relations between vectors that differ exactly by $\mathbf{j}$ (making them incomparable). In this way we thus obtain the n-th *pinched Veronese poset* $(\mathcal{V}_n^\bullet, \preceq)$. It is very interesting that removing this single element $\mathbf{j}$ (and corresponding order relation) strongly influences understanding the properties of the poset.

By using the properties of A-shellability, in [8] we prove the following.

Theorem 8 (Thm 1.1 in [8])**.** *Let $n \geq 4$. For any $\mathbf{z} \in \mathcal{V}_n^\bullet$ the interval $[\mathbf{0}, \mathbf{z}]$ in $\mathcal{V}_n^\bullet$ is a shellable poset, where $\mathbf{0}$ is the zero vector of length n.*

Together with the aforementioned result of Peeva, Reiner and Sturmfels [PRS98], Theorem 8 provides a combinatorial proof of the result of Conca, Herzog, Trung and Valla [CHTV97] that the n-th pinched Veronese ring is Koszul for $n \geq 4$.

For additional background on the pinched Veronese poset and ring, we refer to Sections 1 and 4 in [8].

1.4 Curves and Graphs on surfaces

In this section we describe the contents of [2], [5] and [7]. The unifying topic of these three results is that they deal with curves and/or graphs on surfaces.

Drawing graphs on surfaces can be seen as a lower-dimensional analogue of the embeddability question into the manifolds. However, in the lower dimension we often encounter different phenomena which often yield different answers or at least different approaches how to reach the goal.

1.4.1 Untangling curves on surfaces

The earliest among the three results [7], obtained in a joint work with J. Matoušek, E. Sedgwick and U. Wagner, considers the following problem: we are given two collections $A = (\alpha_1, \dots, \alpha_n)$ and $B = (\beta_1, \dots, \beta_m)$ of simple curves on a surface $\mathcal{M}$ with boundary. Each of the curves is either a closed curve avoiding the boundary or an arc meeting the boundary exactly at the two endpoints of the arc. The curves α_i are pairwise disjoint except that they may possibly share endpoints. Similarly β_j are pairwise disjoint that they may possibly share endpoints. However, there might possibly be many crossings of the curves α_i with the curves β_j. Our aim is to untangle the β_j from the α_i by some boundary-preserving homeomorphism of $\mathcal{M}$; that is we look for a boundary preserving self-homeomorphism $\varphi\colon \mathcal{M} \to \mathcal{M}$ such that the total number of crossings between α_i and $\varphi(\beta_j)$ is as small as possible. We call this minimum number of crossings achievable through any choice of φ the *entanglement number* of the two systems A and B.

In the orientable case, let $f_{g,h}(m, n)$ denote the maximum entanglement number of any two systems $A = (\alpha_1, \dots, \alpha_m)$ and $B = (\beta_1, \dots, \beta_n)$ of almost-disjoint curves on an orientable surface of genus g with h holes. Analogously, we define $\hat{f}_{g,h}(m, n)$ as the maximum entanglement number of any two systems A and B of m and n curves, respectively, on a nonorientable surface of genus g with h holes.

The main results of [7] are the following; they provide bounds on $f_{g,h}(m, n)$ and $\hat{f}_{g,h}(m, n)$ independent of g and h.

Theorem 9 (Thm. 1.1 of [7]). *For planar $\mathcal{M}$, we have $f_{0,h}(m, n) = O(mn)$, independent of h.*

Theorem 10 (Thm. 1.2 of [7]). (i) *For the orientable case, $f_{g,h}(m, n) = O((m+n)^4)$.*

(ii) *For the nonorientable case, $\hat{f}_{g,h}(m, n) = O((m + n)^4)$.*

A small modification of a proof of Theorem 10 provides a bound on $f_{g,h}(m, n)$ and $\hat{f}_{g,h}(m, n)$ which depends on g but is linear in m and n (see Corollary 1.6 in [7]). Such a bound is important for verification of the correctness of the algorithm in [6] and the relation with [6] was our main motivation why we considered this problem.

Independently of us, a similar problem was studied by Geelen, Huynh, and Richter [GHR13], with a rather different and very strong motivation stemming from the theory of graph minors, namely the question of obtaining explicit upper bounds for the graph minor algorithms of Robertson and Seymour. Geelen et al. [GHR13, Theorem 2.1] show that $f_{g,h}(m,n)$ and $\hat{f}_{g,h}(m,n)$ are both bounded by $n3^m$, but only under the assumption that $\mathcal{M} \setminus (\beta_1 \cup \cdots \cup \beta_n)$ is connected (which is sufficient for their needs).

The proof of Theorem 9 relies on a result of Erten and Kobourov [EK05] on simultaneous drawings of graphs with bends in the plane.

The proof of Theorem 10 relies on the bound from Theorem 9 via a suitable cut and glue technique. Two other main ingredients are ideas based on the change of the coordinates principle; see [FM11] and a result on searching for a canonical system of loops in an orientable surface by Lazarus, Pocchiola, Vegter and Verroust [LPVV01]. For a more detailed overview of the proof we refer to Table 1 of [7]. As usual, more additional background can be found in the introduction of [7].

1.4.2 Shortest paths on surfaces

A famous theorem of Fáry [Fár48] states that any simple planar graph can be embedded so that edges are represented by straight line segments.

It is natural to ask whether the following generalization of Fáry's theorem is possible: Given a surface S, is there a metric on S such that every graph embeddable into S can be embedded so that edges are represented by shortest paths? We call such an embedding a *shortest path embedding*. If such a metric exists, we call it a *universal shortest paths metric*.

Motivation to study this question comes from various directions. Apart from the relation to Fáry's theorem, it is also related to a conjecture of Negami [Neg01] which states that there exists a universal constant c such that for any pair of graphs G_1 and G_2 embedded in a surface S, there exists a homeomorphism $h : S \to S$ such that $h(G_1)$ and G_2 intersect transversely at their edges and the number of edge crossings satisfies $cr(h(G_1), G_2) \leq c|E(G_1)| \cdot |E(G_2)|$. The connection is that if two graphs are embedded transversally by shortest path embeddings, then indeed no two edges cross more than once, since otherwise one of them could be shortcut.

Similarly, this question is related to untangling curves in [7]. If we had a stronger version of a result of Erten and Kobourov [EK05] on simultaneous drawings of graphs with bends extended to an arbitrary surface, then we could perhaps improve the bounds in Theorem 10. Answering the question above seems as the first necessary step towards such a result.

We do not know a full answer to the question; however, in a joint work with A. Hubard, V. Kaluža and A. de Mesmay [5] we have reached the following results.

Theorem 11 (Thm. 1 in [5])**.** *The sphere S^2, the projective plane $\mathbb{R}P^2$, the torus T^2, and the Klein bottle K can be endowed with a universal shortest path metric.*

For surfaces of higher genus, a natural approach would be to look for a universal shortest path metric among hyperbolic metrics. However, we show that most of them are not universal shortest path metrics. For understanding the statement of the theorem below: if we allow that each edge is drawn as a concatenation of k shortest paths, we call such a metric k-universal shortest paths metric.

Theorem 12 (Thm. 3 in [5])**.** *For any $\varepsilon > 0$, with probability tending to* 1 *as g goes to infinity, a random hyperbolic metric is not a $O(g^{1/3-\varepsilon})$-universal shortest* *paths metric. In particular, with probability tending to* 1 *as* g *goes to infinity, a random hyperbolic metric is not a universal shortest path metric.*

On the other hand, if we allow k-universal shortest paths metric for k linear in g, then there is already such a metric.

Theorem 13 (Thm. 4 in [5])**.** *For every* $g > 1$*, there exists an* $O(g)$*-universal shortest path hyperbolic metric* m *on the orientable surface* S *of genus* g*.*

1.4.3 Hanani-Tutte theorem on the projective plane

Finally, regarding graphs on surfaces, let us briefly explain the contents of [2], obtained in a joint work with É. Colin de Verdière, V. Kaluža, P. Paták and Z. Patáková.

The strong Hanani-Tutte theorem [CH34, Tut70, PSŠ07] states that whenever a graph can be drawn in the plane in such a way that every pair of disjoint edges crosses evenly, then the graph is actually planar. Apart from the intrinsic combinatorial beauty of this theorem, it can be also seen as an analogue[6] of completeness of van Kampen obstruction for embedding k-complexes into $\mathbb{R}^{2k}$ for $k \geq 3$. It can be also seen as a basis for results on various notions of planarity [Sch13b].

It is an open question whether the strong Hanani-Tutte theorem is valid on other surfaces; that is, if a graph can be drawn on a surface S in such a way that every pair of disjoint edges crosses evenly, then the graph actually embeds into S. Pelsmajer, Schaefer and Stasi [PSS09] proved that the strong Hanani-Tutte theorem is valid on the projective plane via the inspection of the forbidden minors for the projective plane. Unfortunately, this approach cannot be used on other surfaces.

The main aim of [2] is to provide an alternative constructive proof of the strong Hanani-Tutte on the projective plane not relying on forbidden minors. The cost that we pay is that the proof is more complicated. On the other hand, there is a hope that this approach could be extended to other surfaces. (Or it could yield a desired structure for a counterexample if some essential step fails.)

1.5 Homology growth of flag complexes

Last but not least, in a joint work with K. Adiprasito and E. Nevo [1] we study the maximal possible growth of homology of clique complexes over graphs with a fixed forbidden induced minor.

More precisely, let $\mathbb{K}$ be any field, H be any simple finite graph, and

$$b_H(n) = b_H(n, \mathbb{K}) = \max_G \left\{ \sum_{i \geq -1} \dim_{\mathbb{K}} \widetilde{H}_i(\mathrm{cl}(G); \mathbb{K}) \right\}$$

where G runs over all simple graphs on at most n vertices without an induced copy of H, $\mathrm{cl}(G)$ denotes the clique complex of G and $\widetilde{H}_i(\cdot; \mathbb{K})$ denotes the ith reduced homology with coefficients over $\mathbb{K}$. We are interested in the growth of $b_H(n)$ as n tends to infinity.

[6] actually slightly stronger

A similar question was previously considered by Adamaszek [Ada14]. He showed that $b(n) \leq 4^{n/5}$, for

$$b(n) = \max_G \{\sum_{i \geq -1} \dim_{\mathbb{K}} \widetilde{H}_i(\mathrm{cl}(G); \mathbb{K})\}$$

where G runs over *all* graphs on at most n vertices. Adamaszek further showed that for $H = I_3$ (independent set with 3 vertices), the growth is exponential but with a smaller base, at most $\approx 1.2499 < 4^{1/5} \approx 1.3195$. It is also obvious that, if $H = K_d$ is a complete graph on d vertices, then $\mathrm{cl}(G)$ is at most $(d-2)$-dimensional, and thus $b_{K_d}(n) = O(n^{d-1})$.

Our aim is to provide a systematic approach to this question for a general forbidden graph H. A strong part of our motivation also comes from the case $H = C_4$ which can be seen as a discrete analogue of non-positive curvature (for a suitable metric on simplices). It is perhaps a bit surprising that a clique complexes with forbidden induced C_4 exist with arbitrary high homology [JŚ03].

We show that the limit $\lim_{n\to\infty} \sqrt[n]{b_H(n)}$ always exists and that it may attain exactly 5 possible values (four of which we can determine precisely).

Theorem 14 (Thm. 1.2 in [1]). *Let H be any graph. The limit $c_H = \lim_{n\to\infty} \sqrt[n]{b_H(n)}$ exists. In addition:*

(i) If $H \not\subseteq K_{5,5,\ldots}$, then $c_H = 4^{1/5} \approx 1.3195$.

(ii) For every $i \in \{1, \ldots, 5\}$ there is a value c'_i with the following property. If $H = K_{i_1,\ldots,i_m}$ with $5 \geq i_1 \geq \cdots \geq i_m \geq 1$, then $c_H = c'_{i_1}$. Moreover, $c'_5 = 3^{1/4} \approx 1.3161$, $c'_4 = 2^{1/3} \approx 1.2599$, $c'_3 \in [8^{1/14}, \Gamma_4] \approx [1.1601, 1.2434]$, and $c'_2 = c'_1 = 1$.

Here $\Gamma_4 \approx 1.2434$ is a certain fixed constant.

For the interesting case when $H = C_4$, we get the following improved bounds.

Theorem 15 (Thm. 1.4 in [1]). *If $H = K_{2,2} = C_4$ is the 4-cycle, then there are constants $c, C > 0$ such that for any n*

$$cn^{3/2} < b_{C_4}(n) < n^{C\sqrt{\log n}}.$$

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