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Report on the Habilitation Thesis by Tomáš Bárta: "Asymptotic Behavior of Gradient-Like Systems"

In the habilitation thesis of Tomáš Bárta several results on convergence of solutions of abstract evolution equations of the form

$$u'(t) + F(u(t)) = 0, \qquad t > 0, \tag{1}$$

for large times to stationary solutions are considered. Moreover, rates for this convergence in various situations and the optimality of these rates are considered as well. A fundamental tool to show convergence for large times in evolution equations with many stationary solutions is the so-called Lojasiewicz(-Simon) gradient inequality

$$|E(u) - E(u_\infty)|^{1-\theta} \leq C\|DE(u)\|$$

for all u a neighborhood of a critical point u_∞ of E, where $\theta \in (0, \frac{1}{2}]$. This inequality has played an important role throughout the last decades for such problems and was applied to large varity of evolutions equations, including very complex system of partial differential equation, free boundary value problems and geometric evolution equation. Moreover, it was also used to show stability of certain equilibria. The Lojasiewicz-Simon gradient inequality was generalized by Kurdyka to

$$\Theta(|E(u) - E(u_\infty)|) \leq C\|DE(u)\|$$

for all u a neighborhood of a critical point u_∞ of E, where $\frac{1}{\Theta}$ is integrable on $(0, \varepsilon)$ for some $\varepsilon > 0$. The standards applications of the Lojasiewicz-Simon gradient inequality carry over to this generalization, which is also called Kurdyka-Lojasiewicz-Simon inequality.

In order to apply the (Kurdyka-)Lojasiewicz-Simon inequality link between the evolution equation and the energy type functional E is needed, which is connected to the structure of the system. A classical example is the case of a gradient flow, where $F(u) = DE(u)$. More generally, (1) is called gradient-like system if

$$\langle DE(u), F(u)\rangle > 0 \qquad \text{if } F(u) \neq 0,$$

which implies that E is a (strict) Lyapunov function for the evolution equation. In the first paper, which is presented in the Habilitation thesis, the author shows together with Ralph Chill and Eva Fašangová that for ordinary differential equations (on finite dimensional Riemannian manifold) actually every gradient-like system is a gradient system if $DE(u)$ is replaced by the gradient $\nabla_{\tilde{g}} E$ with respect to a suitable metric $\tilde{g}$. Moreover, equivalence of the new metric $\tilde{g}$ with the given metric g of the Riemannian manifold is shown to be equivalent to an angle condition between $\nabla_g E(u)$ and $F(u)$. In the second paper the condition that a Kurdyka-Lojasiewicz-Simon inequality holds close to the equilibria of the system is weakened to the condition that

$$-\frac{d}{dt}E(u(t)) \geq c\|u'(t)\|$$

for all (sufficiently large) t. This is done for first order equations and abstract second order equations. The crucial step in the proof of convergence is usually to show this inequality with the of a (Kurdyka-)Lojasiewicz-Simon inequality. In the third paper Tomáš Bárta and Eva Fašangová consider a general damped wave equation with a (degenerate) damping function. The previous techniques are used to show convergence to stationary solutions under quite general assumptions for the damping function. Finally, in the fourth and fifth paper the Mr. Bárta studies rates of convergence to stationary solutions. First such rates are proved for first order systems under quite general assumptions depending on suitable relations between $E(u(t))$ and $\|F(u(t))\|$. Then the results are applied to the damped wave equations and optimality of the results are discussed.

Altogether the thesis is written very well, the results and their relation to previous results are explained clearly. Only in Section 3.2 the notation is partly inconsistent and the precise assumptions are not clear at every point. The results are interesting, reasonable and most of them appeared already in good to very good journals. But mostly they are refinements of known techniques. For my taste the applications are concentrated too much on damped wave equations although the techniques are applicable to many other systems as well. Tomáš Bárta has shown a good overview of the context of his presented research results and can develop new ideas. In that respect I believe that he is able to supervise PhD theses. Therefore I recommend acceptance of his Habilitation thesis.

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