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Report on "Cut Elimination and Consistency Proofs" by Anna Horska, submitted for a Ph.D. thesis in Logic to the Faculty of Arts of the Charles University.

This thesis concerns two of the central topics in proof theory: questions about the complexity of eliminating cuts in the first-order logic of arithmetic, and properties of cut-elimination for the propositional sequent calculus.

The first part of this thesis contains a new proof-theoretic analysis of the Gentzen's original consistency proof for first-order logic with the omega-rule. This original proof by Gentzen remained unpublished for decades after Brouwer complained about it lack of constructivity, as Gentzen eventually withdrew it in the face of this criticism. Indeed, it was not published in full until much later in 1974. This original proof is nonetheless the simplest and most transparent of Gentzen's consistency proofs, and still of considerable interest.

The thesis starts with a definition of Gentzen's proof system. It then turns to the ordinal analysis of the cut-elimination for this system. The theory of the Veblen Phi functions, Φ_α, where α is an ordinal, is developed carefully. Horska then establishes that the ordinals implicit in the Gentzen cut-elimination method correspond to $\Phi_\omega(0)$, where ω is the first infinite ordinal. This is a surprising result, since the usual present-day constructions for cut-elimination, due originally to Tait, yield instead the much smaller ordinal ε_0. It should be noted that these ordinal bounds are intrinsic to the cut-elimination process because they represent the ordinals that are associated with the well-founded deduction trees that result from the cut-elimination. In other words, the ordinal bounds, if they are tight, are an essential feature of the cut-elimination process, not just an artifact of the proof method used to establish that cut-elimination holds.

This question of the difference between the Gentzen method ordinal bounds of $\Phi_\omega(0)$ and the Tait method ordinal bounds of ε_0 is taken up in Part II of the thesis. First, however, Part I concludes by formalizing the entire Gentzen cut-elimination in Peano arithmetic. This formalization is carried out meticulously, and it is established that Peano arithmetic, augmented with transfinite induction up to $\Phi_\omega(0)$ can indeed prove the correctness of the construction. A surprising part of the construction is that uses (transfinite) induction not just for primitive recursive predicates, but also for Π_3 formulas. Whether it is necessary to use transfinite induction on Π_3 formulas is left open, but I suspect this could be improved with a more refined proof.

Part II of the thesis considers the difference in the ordinal bounds of $\Phi_\omega(0)$ for the Gentzen cut-elimination method (as were established for the first time in Part I of the thesis), and the ordinal bounds of ε_0 for the Tait method of cut-elimination. The difference between the two methods lies in the fact that the Gentzen cut-elimination method always takes a top-most cut to be eliminated first, whereas the Tait cut-elimination method always takes a top-most cut *of the highest logic complexity* to be eliminated first. The bounds of $\Phi_\omega(0)$ as just established by Horska is only an upper bound at the present state of knowledge. However, the bound of ε_0 for the Tait method is known to be optimal. Horska asks the question of whether the upper bound of $\Phi_\omega(0)$ is also tight. The first question is of course whether the upper bound of ε_0 also applies to the Gentzen method, in which case it would be tight. Horska, however, suggests even stronger scenario: Perhaps the Gentzen cut-elimination method and the Tait cut-elimination method (when made appropriately deterministic to avoid problems with formulas which are solely weakly introduced) yield exactly the same cut-free proofs? Part II answers this latter question affirmatively for sequent calculus proofs for propositional logic by proving a Church-Rosser property for elimination of cuts in the propositional sequent calculus. This is a surprising result, and provides hope that the same holds for first-order cut elimination.

This result (Theorem 4 on page 93) applied a form of "general cut elimination" (Definition 39) that encompasses both the Gentzen method and the Tait method for selecting cuts. This "general cut elimination" includes the restriction that the cut selected for elimination must not be below a cut of higher complexity. I must admit I cannot see why this is important for the proof; indeed, as far as I can see, the only use of this restriction is avoid considering cases where a cut on a non-atomic formula lies below a cut on a non-atomic formula. It seems possible that this restriction can be removed.

Summary: This is a very interesting thesis which makes significant progress on our understanding of cut-elimination, one of the most important tools for classical proof complexity. It establishes a new analysis of the original cut-elimination proof of Gentzen; indeed, it provides the first ordinal analysis of this method. It poses a fundamental question about the importance of the order in which cuts are eliminated. This question is left open for first-order logic, but it is answered in a very strong way for propositional logic by proving a Church-Rosser property for a general cut elimination method.

The thesis concerns questions about long-standing, central constructions in cut-elimination. It is remarkable that Gentzen's cut elimination method has not had a more rigorous analysis in the past, but the only works I know are those of Tait [17] and van Plato [9]. This makes Horska's thesis all the more timely and relevant.

The exposition and mathematical rigor of the thesis are excellent. There are places where the English grammar could use small improvements; however, the clarity is excellent, there are never problems in understanding the arguments, and the arguments are exposited in an intuitive, yet rigorous, fashion.

I encourage Horska to publish parts of the thesis as articles to have a wider visibility and impact. In particular, the first part of Part I and the entire Part II would each make excellent articles, not only because they establish new fundamental results, but also for the very interesting questions they raise.

In conclusion, the thesis meets the standard customarily required for a doctoral dissertation. I recommend the dissertation for a public defense. My assessment is a grade of “Pass”, and I recommend the thesis be accepted for as a doctoral dissertation in Logic.

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