Charles University
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Doctoral Thesis

Asymptotical Structure of Combinatorial Objects

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Declaration

This is to certify that I have written this thesis on my own and that the references include all the sources of information I have exploited. I authorize Charles University to lend this document to other institutions or individuals for the purpose of scholarly research.

Prague, May 9, 2007

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Zdeněk Dvořák

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Contents

Introduction

This thesis presents and studies the concept of *bounded expansion* and related graph parameters introduced recently by Jaroslav Nešetřil and Patrice Ossona de Mendez. Let us first present a motivation for this new graph property. In this brief introduction, we are quite informal and use several standard graph theory notions without defining them first. We refer the reader to Chapter 1 or any textbook on graph theory for precise definitions.

One of the oldest topics in the graph theory is the graph coloring. Already in 1852, Guthrie and De Morgan proposed what became the famous Four Color Conjecture. This conjecture turned out to be surprisingly difficult, and it was only proved in 1976 by Appel and Haken [7, 8] (a simpler and easier to verify proof was later provided by Robertson et al. [83]).

The Four Color Theorem claims that every planar graph can be colored using at most four colors. However, if we are only interested in whether there exists a constant c such that every planar graph can be colored by at most c colors, the problem becomes much simpler – by Euler's formula, every planar graph is 5-degenerate, hence it can be colored by at most 6 colors. Note that similarly, a graph with maximum degree d can be colored by at most $d+1$ colors.

The graph coloring became one of the most studied problems in graph theory, and with that, many variants of the basic problem have arisen. One of them is the *acyclic coloring* – coloring a graph in such a way that no two adjacent vertices have the same color and no cycle is colored by only two colors. A quite complex proof of Borodin [20] shows that there exists an acyclic coloring of every planar graph by at most five colors. Similarly to the case of Four Color Theorem, one might ask whether there is a simple way how to see that the acyclic chromatic number of planar graphs is bounded by a constant? Note that the degeneracy is no longer sufficient to ensure this property, since for example the graph obtained from K_n by subdividing each edge by one vertex is 2-degenerate, but its acyclic chromatic number is $\Omega(\sqrt{n})$.

The result of Nešetřil and Ossona de Mendez [66] can be viewed as such a way. They have proved that every graph G has a minor H such that $\chi_a(G) \leq O\left(\chi(H)^2\right)$. Together with the fact that all graphs in a proper minor-closed class are degenerate, this implies that the acyclic chromatic number is bounded by a constant for any such class of graphs, in particular for planar graphs as well.

It is also easy to see that the acyclic chromatic number of a graph is bounded by the chromatic number of the square of the graph, which implies that the acyclic chromatic number of a graph with maximum degree d is at most $d^2 + 1$.

Unlike the case of the ordinary coloring, we needed separate arguments to show that the acyclic chromatic number is bounded by a constant both for planar graphs and for graphs with bounded maximum degree – the chromatic number of the square of a planar graph may be arbitrarily high, and graphs with bounded maximum degree may contain arbitrary minors. Naturally, the question is whether it is possible to present an argument that would work for both of these graph classes. One might hope that such an argument could be easy to generalize for a wide range of other graph classes and graph properties.

The cornerstone for such a result is finding a property that both proper minor-closed graph classes and graphs with bounded maximum degree share. It turns out that both of these classes have bounded expansion, in the sense defined in Section 1.4, and Nešetřil and Ossona de Mendez [70] have proved that graphs with bounded expansion have a bounded acyclic chromatic number. Therefore, we indeed have a natural way to see for many graph classes that they have bounded acyclic chromatic number.

The acyclic coloring is just one of many graph coloring notions that can be defined by requiring some prescribed type of subgraphs to have many colors. An interesting question is how many colors we can request on a particular type of subgraphs, and still be able to color any graph in an arbitrary proper minor-closed class by a bounded number of colors. For example, it is possible to require that each cycle has at least three colors (the acyclic chromatic number is bounded in any proper minor-closed class of graphs), but it is not possible to require this for a path with three vertices: Such coloring corresponds to the coloring of the square of the graph, and the number of colors of this coloring is bounded from below by the maximum degree of the graph. However, planar graphs and graphs in many other proper minor-closed classes have unbounded maximum degree. A surprising result of Nešetřil and Ossona de Mendez [74] shows how to determine exactly the maximum number of colors that we may request for any graph. Later, Nešetřil and Ossona de Mendez [70] have showed that this property in fact does not rely on the classes being minor-closed – the essential property again turns out to be their bounded expansion.

One of the reasons for the importance of the study of proper minor-closed classes is that many algorithmic problems that are hard in general can be solved or approximated quickly for graphs in such a class. Classes with bounded expansion are more general, and they appear in many common applications (intuitively, many graphs arising from the geometric considerations have the property that there are few vertices that are near to any chosen vertex, which implies that their expansion is bounded), thus one might ask whether such algorithms generalize for these graph classes. Indeed, Nešetřil and Ossona de Mendez [71] describe many such algorithmic applications, even more emphasizing the importance of the concept of the bounded expansion.

Motivated by these striking results and unexpected connections, we further investigate the properties of the classes of graphs with bounded expansion, as well as the related concepts. The thesis is structured as follows:

- We start by introducing the definitions and notation.
- In Chapter 2, we describe some of the known properties of graphs with bounded expansion and results regarding them.

In the following chapters we present our own contributions:

- In Chapter 3, we study the existence of subdivisions of certain graphs as subgraphs. We present a characterization of graphs with bounded expansion in the terms of forbidden subdivisions that cannot appear in such graphs. We also provide similar characterizations for the acyclic chromatic number and the arrangeability of the graph, and in this way expose more clearly the relationship of the bounded expansion to these graph parameters. We apply these characterizations on problems regarding the game chromatic number.

  Additionally, we study the existence of clique subgraphs with edges subdivided by a constant number of vertices, use this to characterize graphs with exponential expansion and relate this property to the existence of big expander-like subgraphs.

- In Chapter 4, we consider several algorithmic questions regarding the bounded expansion – we show that determining the expansion precisely is NP-complete even for graphs with degree bounded by four, discuss several classes of graphs for that the expansion can be determined in polynomial time, and show an approximation algorithm with polynomial approximation factor.

- In Chapter 5, we provide several results regarding the concepts arising from the study of the properties of graph classes with bounded expansion – the tree-depth of a graph and the subgraph coloring.

- We finish by some concluding remarks and open problems, in Chapter 6.

Chapter 1

Definitions

Let us introduce definitions and notation regarding the concepts we use.

1.1 Graphs

Most of the thesis is devoted to the study of graphs and their properties. We expect the reader to be familiar with the elementary results and notions of the graph theory, to the extent covered for example by the textbook by Matoušek and Nešetřil [64]. Unless specified otherwise, we deal with simple undirected graphs, without loops and parallel edges. If G is a graph, let $V(G)$ denote the set of its vertices and $E(G)$ the set of its edges. For a vertex v of a graph G, let $N(v)$ denote the open neighborhood of v, i.e., the set of the vertices adjacent to v, and $d(v) = |N(v)|$ the degree of v.

Let $\Delta(G)$ denote the maximum degree and $\delta(G)$ the minimum degree of the graph G. If $\Delta(G) = \delta(G) = d$, then the graph G is called *d-regular*. The 3-regular graphs are also called *cubic*. If $\Delta(G) \leq 3$, we say that G is *subcubic*.

Let P_n denote a path with n vertices. The *length* of a path is the number of its edges, i.e., the length of P_n is $n-1$. The *distance* of two vertices u and v in a graph is the length of the shortest path between u and v.

In a rooted tree, a *level* of a vertex is its distance from the root. The *depth* of a rooted tree is the maximum of the levels of its vertices.

1.2 Vertex Orderings

Several of our definitions and proofs involve constructing a linear ordering of vertices of a graph that satisfies some properties. Given a linear ordering L of the vertices of a graph G, let $L^+(v)$ be the set of vertices of G that are after v

in this ordering (not including v), and $L^-(v)$ the set of vertices that are before it (again, we do not include v in this set). More precisely, if the vertices are ordered by L in the sequence v_1, v_2, …, v_n, then $L^-(v_i) = \{v_1, v_2, \ldots, v_{i-1}\}$ and $L^+(v_i) = \{v_{i+1}, v_{i+2}, \ldots, v_n\}$ for each $i = 1, \ldots, n$.

Given a fixed ordering L of the vertices of a graph G, we let the *back-degree* $d^-(v)$ of a vertex v be the number of neighbors of v before it in L, $d^-(v) = |N(v) \cap L^-(v)|$.

1.3 Minors and Subdivisions

The definition of the measure of the expansion of the graph that we investigate is based on average degrees of minors of a graph, and it is closely related to average degrees of subdivisions inside the graph.

To *contract* an edge $e = \{u, v\}$ of a graph G means to identify the vertices u and v into a single vertex, remove the edge e and suppress the possibly arising parallel edges. The graph obtained from G by contracting the edge e is denoted by G/e. To *suppress* a vertex v of degree two means to contract one of the edges incident with v.

A graph H is a *minor* of a graph G (denoted by $H \prec G$) if it can be obtained from G by contracting edges and removing vertices and edges. In other words, the vertices of H correspond to vertex disjoint connected subgraphs of G and if two vertices form an edge of H, then the corresponding subgraphs contain adjacent vertices. For $w \in V(H)$, let $\mathrm{sgofv}(G, H, w)$ be the subgraph of G corresponding to the vertex w. For $v \in V(G)$, let $\mathrm{repr}(G, H, v)$ be the vertex w of H such that the subgraph $\mathrm{sgofv}(G, H, w)$ contains v (if such a vertex exists), and let $\mathrm{sgofv}(G, H, v) = \mathrm{sgofv}(G, H, \mathrm{repr}(G, H, v))$ be the subgraph that contains v. Sometimes, we use the following observation: We may assume that all the subgraphs $\mathrm{sgofv}(G, H, w)$ are trees.

A class of graphs $\mathcal{G}$ is called *minor-closed* if for each $G \in \mathcal{G}$, every minor of G belongs to $\mathcal{G}$ as well. A minor-closed class $\mathcal{G}$ is *proper* if it is not the class of all graphs. Equivalently, a minor-closed class $\mathcal{G}$ is proper if does not contain all complete graphs. Each minor-closed class $\mathcal{G}$ corresponds to a set of *forbidden minors* $\mathcal{F}$ – the set of graphs that do not belong to $\mathcal{G}$, but all their minors do. Note that a graph belongs to $\mathcal{G}$ if and only if it does not contain a minor belonging to $\mathcal{F}$, in particular, no graph in $\mathcal{F}$ is a minor of another graph in $\mathcal{F}$. The famous result of Robertson and Seymour [87] states that the set of forbidden minors for any minor-closed class of graphs is finite.

Given a graph G, the *eccentricity* of a vertex $v \in V(G)$ is the maximum distance from v to any other vertex of a graph G. The *radius* of a graph G is the minimum of the eccentricities of its vertices. Given an integer $r \geq 0$ and a graph G with radius at most r, a *center* of G is a vertex with eccentricity at most r. Note that there may be several centers in a graph; usually, we select one of them arbitrarily. Note also that this definition of center does not in general require the center to be a vertex with the minimum eccentricity.

The *depth* of a minor $H \prec G$ is the maximum of the radii of the subgraphs $\mathrm{sgofv}(G, H, w)$ for $w \in V(H)$. Note that we may require the subgraphs to be trees without changing the depth of the minor. It is also often useful to consider the trees to be rooted in a center.

A graph $G' = \mathrm{sd}_t(G)$ is the *t-subdivision* of a graph G, if G' is obtained from G by replacing each edge by a path with exactly t inner vertices. Similarly, the graph G' is a *$\leq t$-subdivision* of G if the graph G' can be obtained from G by subdividing each edge by at most t vertices (the number of vertices may be different for each edge). By a small misuse of the notation (a $\leq t$-subdivision of G is not determined uniquely), we write $G' = \mathrm{sd}_{\leq t}(G)$. We call a graph G *subdivided* if no two vertices of G of degree greater than two are adjacent.

If H' is a subdivision of a graph H and H' is a subgraph of a graph G, we say that G *contains a subdivision* of the graph H. Note that if G contains a $\leq t$-subdivision of H, then H is a minor of G of depth at most $\lceil \frac{t}{2} \rceil$, but the reverse claim does not hold – there exist graphs such that $H \prec G$, but G does not contain any subdivision of H. One notable exception is the case that H is subcubic. Then, if H is a minor of G of depth d, then a $\leq 4d$-subdivision of H is a subgraph of G.

If a subdivision of H is a subgraph of G, a phrase that H is a *topological subgraph* or a *topological minor* of G is sometimes used in literature. We do not use this terminology, since we usually need to specify the properties of the subdivision more precisely.

Stars and their subdivisions are used intensively in our characterizations and proofs, warranting a need for a special terminology: For an integer $t \geq 0$, a *$\leq t$-star* S is a $\leq t$-subdivision of a star and a t-star is the t-subdivision of a star, i.e., the ordinary stars are 0-stars. The subdivision of a star consists of *ray vertices* (the leaves), *middle vertices* (the vertices of degree two created by subdividing the edges) and the *center vertex*. A ray vertex and the middle vertices on the path from the ray vertex to the center form a *ray* of the $\leq t$-star. For a $\leq t$-star S that is a subgraph of a graph G and a set $T \subseteq V(G)$, let $\mathrm{rays}_T(S)$ be the set of the ray vertices of S that belong to T. The (T,t)-degree $d_t^T(v)$ of a vertex v of G is the maximum of $|\,\mathrm{rays}_T(S)|$ for all $\leq t$-stars S with the center v that are subgraphs of G.

We also use the term *double-star* for 1-star. The *middle edges* of a double star are the edges that are incident with the center, while the remaining edges are the *ray edges*. The *double back-degree* of v is $d_2^-(v) = d_1^{L^-(v)}(v)$, i.e., the maximum number of ray vertices of a double-star with the center v that are before v in the ordering L.

1.4 Greatest Reduced Average Density and Classes With Bounded Expansion

The notion of the graph expansion is defined in th terms of the *greatest reduced average density*.

The *average degree* of a graph G is equal to $\frac{2|E(G)|}{|V(G)|}$. The *average density* of a graph G is half of its average degree, i.e., $\frac{|E(G)|}{|V(G)|}$. The *maximum average density* $\nabla_0(G)$ is the maximum of the average densities of the subgraphs of G.

A related parameter is the degeneracy of a graph. A graph G is *t-degenerate* if the minimum degree of every subgraph of G is at most t. Equivalently, G is t-degenerate if there exists an ordering L of vertices of G such that $d^-(v) \leq t$ for each vertex v. The *degeneracy* $\nabla_0^d(G)$ of the graph G is the minimum t such that G is t-degenerate. If a graph G with n vertices is t-degenerate, then it has at most tn edges, hence $\nabla_0(G) \leq \nabla_0^d(G)$. Note that this also implies that a graph with average degree at least t contains a subgraph whose minimum degree is at least $\frac{t}{2}$. On the other hand, if G is not $t-1$-degenerate, then it contains a subgraph with minimum (and thus also average) degree at least t, hence $\nabla_0^d(G) \leq 2\nabla_0(G)$.

For an integer $r \geq 0$, the *greatest reduced average density of rank r* $\nabla_r(G)$ of a graph G is the maximum of the average densities of all minors of G of depth at most r. For example, $\nabla_1(G)$ is the maximum average density of all graphs that can be obtained from G by contracting the edges of a star forest. The greatest reduced average density may be an arbitrary rational number. In some (especially computational) contexts, the variant in that we replace the average density by the minimum degree is easier to work with. We define $\nabla_r^d(G)$ as the maximum degeneracy of a minor of G of depth at most r. Obviously, $\nabla_r(G) \leq \nabla_r^d(G) \leq 2\nabla_r(G)$, and unlike the greatest reduced average density, $\nabla_r^d(G)$ is always an integer.

A class of graphs $\mathcal{G}$ has *bounded expansion* with the bounding function f, if for each $G \in \mathcal{G}$, $\nabla_r(G) \leq f(r)$. Let us present several examples of such classes:

- Every proper minor-closed class $\mathcal{G}$ has the expansion bounded by a constant function: Every minor of a graph $G \in \mathcal{G}$ belongs to $\mathcal{G}$, and all graphs in a proper minor-closed class are t-degenerate for some constant t. On the other hand, the class $\mathcal{G}_c$ of graphs for that $\nabla_r(G) \le c$ for each r is proper minor-closed, hence each class of graphs whose expansion is bounded by a constant function is a subclass of a proper minor closed class.

- A class of graphs $\mathcal{G}$ that do not contain a subdivision of any of graphs in some set $\mathcal{S}$ (possibly infinite) as a subgraph has the expansion bounded by the function $f(r) = 2^{r-1}(\min_{H\in\mathcal{S}} |V(H)|)^{2^{r+1}}$, by Nešetřil and Ossona de Mendez [73].

- For a constant $c > 1$, the class of graphs with maximum degree at most c has expansion bounded by the function $f(r) = \frac{1}{2}c(c-1)^r$.

- The class consisting of the graphs $\mathrm{sd}_{|V(G)|}(G)$ for all graphs G has expansion bounded by the function $f(r) = \max(2, r)$, since for a graph G on n vertices, $\nabla_r(\mathrm{sd}_n(G)) \le 2$ if $2r < n$ and $\nabla_r(\mathrm{sd}_n(G)) \le \frac{n-1}{2}$ if $2r \ge n$.

1.5 Arrangeability

The *arrangeability* of a graph is a graph parameter with a bit artificially looking definition, that nevertheless appears in many different contexts – it bounds the acyclic chromatic number, the game chromatic number (Kierstead and Trotter [55]), and the Ramsey number (Chen and Schelp [23]) of a graph in a natural way (some of the authors rather consider *admissibility* of the graph, which differs from the arrangeability only by a polynomial factor). As we will see later, the arrangeability of a graph G can in some sense be viewed as $\nabla_{1/2}(G)$.

A graph G is *p-arrangeable* if there exists a linear ordering L of vertices of G such that every vertex v of G satisfies

$$\left| L^-(v) \cap \bigcup_{u\in N(v)\cap L^+(v)} N(u) \right| \le p,$$

i.e., the neighbors of v after it have only few different neighbors before v. Note that in particular every vertex v has back-degree at most $p+1$ – if w is the last of the vertices in $L^-(v) \cap N(v)$, then $(L^-(v) \cap N(v)) \setminus \{w\} \subseteq L^-(w) \cap \bigcup_{u\in N(w)\cap L^+(w)} N(u)$. It follows that a p-arrangeable graph is $(p+1)$-degenerate.

1.6 Graph Colorings

A *coloring* of a graph G by k colors is a function from the vertices of G to a k-element set. Graph colorings are one of the most intensively studied subjects in the graph theory, and various generalizations and variants of the graph coloring have been investigated. In this thesis, we deal with several of these types of coloring.

A coloring of vertices of a graph G is *proper* if no two adjacent vertices have the same color. The minimum k such that the graph G has a proper coloring by k colors is called the *chromatic number* of G and denoted by $\chi(G)$.

A proper coloring of a graph G is *acyclic* if the union of each two color classes induces a forest, i.e., there is no cycle colored by two colors. The minimum k such that the graph G has an acyclic coloring by k colors is called the *acyclic chromatic number* of G and denoted by $\chi_a(G)$. Note that the fact that G has low acyclic chromatic number implies that G is degenerate, as it is a union of a small number of trees.

The *star chromatic number* $\chi_s(G)$ of G is the minimum k such that G can be properly colored by k colors in such a way that union of each two colors induces a star forest. It is a well-known fact that the acyclic and the star chromatic number of a graph are almost equal – by Albertson et al. [2], $\chi_a(G) \leq \chi_s(G) \leq \chi_a(G)(2\chi_a(G) - 1)$. The study of the star chromatic number and its generalizations was one of the motivations for the definition of the bounded expansion.

Another interesting variant of graph coloring is game coloring. The *graph coloring game* with k colors and a graph G has the following rules: There are two players, Alice and Bob, who take turns. Each move of Alice or Bob consists of coloring a so far uncolored vertex of G by one of the k colors in such a way that the obtained partial coloring of G is proper. Alice wins if the whole graph G is colored, while Bob wins if he prevents this, i.e., manages to ensure that there is an uncolored vertex such that all k colors are used in its neighborhood. The *game chromatic number* $\chi_g(G)$ of a graph G is defined as the minimum k such that Alice has a winning strategy. The game chromatic number is quite difficult to work with, and it has some rather surprising properties, e.g, a subgraph $G' \subseteq G$ may have greater game chromatic number than G: The game chromatic number of the complete bipartite graph $K_{n,n}$ is 3, while the game chromatic number of $K_{n,n}$ without a single perfect matching is n. It is not even known whether there exists a graph G such that Alice wins with k colors, but loses with $k + 1$ colors.

An easier to work parameter related to the game chromatic number is the game coloring number. The *game coloring number* $\mathrm{col}(G)$ of a graph G is the minimum number k for that Alice wins the *marking game*: Alice and Bob are marking vertices of a graph in such a way that in the moment when a vertex is marked, it has at most $k-1$ marked neighbors. Alice wins if all the vertices of the graph are marked, while Bob wins if this becomes impossible. It is easy to see that the game coloring number is the upper bound on the game chromatic number, and that it is monotone (game coloring number of a subgraph of G is at most $\mathrm{col}(G)$, and if Alice wins the marking game for k, she also wins it for $k+1$). Note that a graph with game coloring number c is $(c-1)$-degenerate, however there exist 2-degenerate graphs with arbitrarily large game coloring number.

Another way how to make the game chromatic number more tractable is to consider the *hereditary game chromatic number* $\chi_{hg}(G)$ – the maximum of the game chromatic numbers of all subgraphs of G. The hereditary game chromatic number at least obviously is monotone with respect to taking subgraphs.

Finally, let us define two almost equivalent notions of coloring that have a direct connection to the tree-depth of a graph (see the next section). A *rank coloring* of a graph by colors $1, \ldots, t$ is a coloring in such a way that each path between two vertices with the same color contains a vertex with greater color. In the *centered coloring*, we require that in each connected subgraph of G, some color appears exactly once. Trivially, a rank coloring is also centered. Note also that given a centered coloring by t colors, we can construct a rank coloring by the same number of colors.

1.7 Tree-width and Tree-depth

The *tree-width* $\mathrm{tw}(G)$ of a graph G is an important and intensively studied parameter, especially because of its connections to the theory of graph minors and because of its algorithmic applications. There are many equivalent definitions of tree-width, let us state the two of them that we need.

A graph is *chordal* if it does not contain a cycle of length greater than three as an induced subgraph. A *clique number* of a graph is the size of the largest clique. The tree-width of a graph G is by one lower than the minimum clique number of all chordal supergraphs of G. This definition is elegant and exposes the connection of the tree-width and the tree-depth (one of the graph parameters we study, see the definition later in this section), but it says very little about the structure of the graphs with bounded tree-width.

The second definition of tree-width is often used in the algorithmic applications. This definition brings out the tree-like structure of the graphs with bounded tree-width, which is useful for recursive dynamic programming-type algorithms. A connected graph G has tree-width at most k if there exists a rooted tree T together with induced subgraphs $G_u \subseteq G$ and sets of at most $k+1$ *border vertices* $S_u \subseteq V(G_u)$ associated with each node u of T, satisfying the following properties: The set S_u separates $V(G_u) \setminus S_u$ from $V(G) \setminus V(G_u)$, the root r of the tree is associated with the graph $G_r = G$ and $S_r = \emptyset$, and each node u of T is of one of the following types:

- The node u is a leaf of T, the graph G_u consists of a single vertex v, and $S_u = \{v\}$, or
- the node u has a single child w, $G_w = G_u - v$ and $S_u = S_w \cup \{v\}$, or
- the node u has a single child w, $G_u = G_w$ and $S_u = S_w \setminus \{v\}$, or
- the node u has exactly two children w_1 and w_2 such that $S_u = S_{w_1} = S_{w_2} = V(G_{w_1}) \cap V(G_{w_2})$, and $G_u = G_{w_1} \cup G_{w_2}$.

The tree T together with the associated subgraphs and sets of border vertices is called the *tree of the construction* of G. The tree T describes the construction of G starting with single vertices, and proceeding by adding new vertices together with the edges that join them to some of the border vertices, and taking unions of graphs that intersect only in the small sets of border vertices.

Several important types of graphs have small tree-width, e.g., trees have tree-width one and series-parallel and outerplanar graphs have tree-width at most two. The tree-width of a graph appears in many contexts – VLSI layouts, Cholesky factorization, expert systems, evolution theory, natural language processing, etc. (Bodlaender [13]). A theorem of Robertson and Seymour [84] states that for any planar graph H, the graphs without H as a minor have bounded tree-width. Also, many problems that are NP-complete become solvable in polynomial or even linear time on graphs with tree-width bounded by a constant. In particular, the series of results started by Courcelle [27, 28] shows that a wide class of problems (verifying properties that can be formulated in Monadic Second Order Logic) can be solved in linear time for graphs with constant tree-width. See the survey of Bodlaender [13] for more results regarding the graphs with bounded tree-width.

Given the abundance of the algorithms for graphs with small tree-width, one might be interested in whether it is possible to recognize such graphs, and to construct their trees of construction. In general, determining the tree-width of a graph is NP-complete (Arnborg et al. [9]), even for graphs with bounded maximum degree (Bodlaender [13]). However, for fixed k, it can be determined in polynomial time whether the tree-width of a graph is at most

k, and to construct the tree of the construction that witnesses this tree-width if this is the case (Reed [82], Bodlaender and Kloks [16], Bodlaender [12]).

A parameter related to the tree-width with a natural connection to the expansion in graphs is the *tree-depth*. The *closure* of a rooted tree T is the graph obtained from T by joining each vertex v by edges with all its ancestors (the vertices on the path from v to the root of T). The *depth* of a forest F of rooted trees is the maximum of the depths of the trees in the forest, and the *closure* of F is the disjoint union of the closures of the trees of F. The *tree-depth* $\mathrm{td}(G)$ of a graph G is the minimum d such that G is a subgraph of the closure of a forest of depth $d-1$. Graphs with small tree-depth naturally generalize stars – star forests are exactly the graphs with tree-depth two.

We define the tree-depth this way for consistency with the paper of Nešetřil and Ossona de Mendez [74]. It might appear more natural to set the tree-depth of star forests to one, corresponding with our definition of the depth or the radius of a tree. To avoid confusion, we decided against this. Also, the definition we use has the advantage that $\mathrm{td}(G)$ is exactly equal to the minimum number of colors in a rank coloring of G.

1.8 Clique-width

Another parameter related to tree-width is clique-width. A *clique-width* of a graph G is the minimum number k for that there exists a coloring of G by k colors such that G together with this coloring can be obtained from single-vertex colored graphs by a finite sequence of the following operations:

- Given two k-colored graphs G_1 and G_2, take their disjoint union.
- Given a k-colored graph G_1 and two colors $c_1 \neq c_2$, change the color of all vertices of G_1 colored by c_1 to c_2.
- Given a k-colored graph G_1 and two colors $c_1 \neq c_2$, join by an edge each pair of vertices v_1 and v_2 such that the color of v_1 is c_1 and the color of v_2 is c_2.

The clique-width of a graph G is denoted by $\mathrm{cw}(G)$. The notion of clique-width has been introduced by Courcelle et al. [29], and found many applications in the design of algorithms (Courcelle, Makovsky and Rotics [30, 31], Gerber and Kobler [43]). Courcelle and Olariu [32] have showed that clique-width has several desirable properties that other graph width parameters lack, e.g., robustness on some graph operations (the complement and the square of a graph G have the clique-width bounded by a function of the clique-width of G).

The clique-width of a graph is bounded by a function of its tree-width, more precisely $\mathrm{cw}(G) \leq 3(2^{\mathrm{tw}(G)-1})$ by Corneil and Rotics [26]. They also found examples of graphs with tree-width k but clique-width $2^{\Omega(k)}$ for any k. The reverse inequality cannot be true, e.g., cliques have clique-width two, but arbitrarily large tree-width. However, Gurski and Wanke [46] have at least showed the following statement:

Theorem 1.1 (Gurski and Wanke [46]) *Let $t > 1$ be an arbitrary integer. If a graph G does not contain $K_{t,t}$ as a subgraph, then $\mathrm{tw}(G) \leq 3(t-1)\,\mathrm{cw}(G) - 1$.*

As Fellows et al. [41] showed, determining the clique-width of a graph is NP-complete. Regarding the fixed-parameter cases, it is possible to decide in polynomial time whether $\mathrm{cw}(G) \leq 3$ (Corneil et al. [25]). The complexity of the problems of determining whether $\mathrm{cw}(G) \leq k$ is open for all constants $k \geq 4$. On the other hand, there exists a polynomial-time algorithm that given a graph G with $\mathrm{cw}(G) = k$, finds a construction that shows that $\mathrm{cw}(G) \leq 2^{3k+2} - 1$ (Oum and Seymour [78]).

1.9 Separators and Expanders

For a graph G, an *α-vertex separator* is a set $S \subseteq V(G)$ such that each component of $G - S$ has at most $\alpha|V(G)|$ vertices. Let $\mathrm{sep}_\alpha(G)$ be the size of the smallest α-vertex separator of G. Usually, we consider the case $\alpha = \frac{2}{3}$. We say just *separator* for the 2/3-vertex separator. A well-known theorem of Lipton and Tarjan [60] states that any planar graph on n vertices has a separator of size at most $O(\sqrt{n})$. This was generalized to all proper minor-closed graph classes by Alon, Seymour and Thomas [6].

The *vertex expansion* $\mathrm{vexp}(G)$ of a graph G on n vertices is defined as

$$\mathrm{vexp}(G) = \min_{U \subseteq V(G), |U| \leq \frac{n}{2}} \frac{\left|\left(\bigcup_{u \in U} N(u)\right) \setminus U\right|}{|U|}.$$

A graph is an *expander* if it has bounded degree and large vertex expansion. More precisely, a class $\mathcal{G}$ of d-regular graphs is a family of expanders with expansion $c > 0$ if for each $G \in \mathcal{G}$, $\mathrm{vexp}(G) \geq c$. Note that expanders do not have separators of sublinear size.

1.10 Probability Theory

In some of our proofs, we use probabilistic arguments. Let $\mathrm{Prob}\,[K]$ denote the probability of an event K and $\mathrm{E}\,[X]$ the expected value of a random variable X. We use the following variants of the well-known estimates, see e.g. [4] for reference.

Lemma 1.2 (Markov Inequality) *If X is a nonnegative random variable and $a > 0$, then*

$$\mathrm{Prob}\left[X \geq a\right] \leq \frac{\mathrm{E}\left[X\right]}{a}.$$

Lemma 1.3 (Chernoff Inequality) *Let $X_1, \ldots, X_n$ be independent random variables, each attaining values 1 with probability p and 0 with probability $1-p$. Let $X = \sum_{i=1}^{n} X_i$. For any $t \geq 0$,*

$$\mathrm{Prob}\left[X \geq np + t\right] < e^{-\frac{t^2}{2(np+t/3)}},$$

and

$$\mathrm{Prob}\left[X \leq np - t\right] < e^{-\frac{t^2}{2(np+t/3)}}.$$

In particular, we use the following special cases of Chernoff Inequality: $\mathrm{Prob}\left[X \geq 2np\right] < e^{-\frac{3}{8}np}$, and $\mathrm{Prob}\left[X \leq \frac{np}{2}\right] < e^{-\frac{3}{28}np}$.

Given a class of random graphs, we say that the graphs in this class have some property *asymptotically almost surely* (a.a.s.), if the probability that a graph in this class with n vertices has this property is at least $1 - f(n)$, where $\lim_{n\to\infty} f(n) = 0$.

Chapter 2

Overview of Known Results

Let us now provide a brief overview of the known properties of graphs with bounded expansion and graphs whose $\nabla_r(\cdot)$ is bounded at least for all $r \leq r_0$. Most of the results in this chapter are by Nešetřil and Ossona de Mendez [70, 71, 72]. We start by introducing one of the original motivations for the bounded expansion property – relationship to star colorings and their generalizations.

2.1 Star Coloring and Low Tree-depth Coloring

As we mentioned in the introduction, the acyclic chromatic number of planar graphs is bounded by a constant (by 5 due to Borodin [20]). Since the star chromatic number of a graph G is bounded by $\chi_a(G)(2\chi_a(G)-1)$ (Albertson et al. [2]), the star chromatic number of planar graphs is also bounded (the best known bound by Albertson et al. [2] is that the star chromatic number of a planar graph is at most 20). Recently, several researchers asked the question for what classes of graphs this result can be generalized. Nešetřil and Ossona de Mendez [66] have proved that the star chromatic number of graphs in any proper minor-closed class is bounded by some constant (depending on the class). More generally, DeVos et al. [34] have proved the following statement concerning the *low tree-width coloring*:

Theorem 2.1 (DeVos et al. [34]) *For any fixed integer $p > 1$ and any proper minor-closed class $\mathcal{G}$, there exists a constant c such that every graph in $\mathcal{G}$ can be colored by at most c colors in such a way that the union of any $i \leq p$ colors induces a graph of tree-width at most $i-1$.*

A more precise generalization of the star coloring is the *low tree-depth coloring*. For a fixed integer $p \geq 1$, the *p-tree-depth coloring* is defined analogically to the low tree-width one – the coloring such that the union of any $i \leq p$ colors induces a graph of tree-depth at most i. The minimum number of colors of such a coloring of a graph G is denoted by $\chi_p(G)$. In particular, $\chi_1(G) = \chi(G)$, $\chi_2(G) = \chi_s(G)$ and $\chi_{|V(G)|}(G) = \mathrm{td}(G)$. The definition of the low tree-depth coloring can also be reformulated in the following way: Every nonempty subgraph $G' \subseteq G$ is colored by at least $\min(p+1, \mathrm{td}(G'))$ colors. In [74], Nešetřil and Ossona de Mendez have proved the following theorem:

Theorem 2.2 (Nešetřil and Ossona de Mendez [74]) *For any integer $p \geq 0$ and any proper minor-closed class of graphs $\mathcal{G}$, there exists a constant c such that $\chi_p(G) \leq c$ for every $G \in \mathcal{G}$.*

Since a low tree-depth coloring is also a low tree-width coloring, this result strengthens Theorem 2.1.

There is a close relationship between the graphs with low tree-depth colorings by a few colors and graphs with bounded expansion. It is well-known that $\chi(G) = \chi_1(G)$ is bounded for graphs with bounded maximum average density $\nabla_0(G)$. Similarly, Nešetřil and Ossona de Mendez [66] in fact shows that $\chi_s(G) = \chi_2(G)$ is bounded for graphs G such that all minors of G obtained by contracting edges of a star forest have bounded maximum average degree, i.e., $\nabla_1(G)$ is bounded. On the other hand, if G has small $\chi_2(G)$, its edge set is a union of edges of a small number of trees, hence G has small maximum average degree. The following hypothesis is natural: There exist functions f_1, f_1', f_2 and f_2' such that for any integer $p \geq 0$ and any graph G,

$$\chi_p(G) \leq f_1'\left(\nabla_{f_1(p)}(G)\right),$$

and

$$\nabla_p(G) \leq f_2'\left(\chi_{f_2(p)}(G)\right).$$

In [70], Nešetřil and Ossona de Mendez proved that this hypothesis is true. More precisely, they proved the following inequalities:

Theorem 2.3 ([70]) *For any graph G and any integer $r \geq 0$,*

$$\nabla_r(G) \leq (2r+1)\binom{\chi_{2r+2}(G)}{2r+2},$$

and

Theorem 2.4 ([70]) *There exist polynomials P_i such that for any graph G and any integer $p \geq 0$, if $t = 1 + (p-1)(2 + \lceil \log_2 p \rceil)$, then*

$$\chi_p(G) \leq P_t\left(\nabla_{2^{t+1}-1}(G)\right).$$

The polynomials P_i can be deduced from the proof, however their degree is quite large (in the order of i^{i^2}). Unlike the previous (more special) results regarding the minor-closed classes, the proofs of these inequalities do not use Structural Theorem of Roberson and Seymour [86], and yield a linear-time algorithm for finding such a coloring, see the paper of Nešetřil and Ossona de Mendez [71] and the discussion in Section 2.6 for details. As we note in Section 5.2, a stronger version of Theorem 2.4 (Theorem 5.6) follows from our results.

2.2 Subgraph Coloring

In this section, we present another motivation for introducing the notion of the tree-depth and the low tree-depth coloring. A *graph function* is a function φ that assigns each nonempty graph G an integer between one and $|V(G)|$ and assigns the same number to the isomorphic graphs. For a graph G, let $\chi_\varphi(G)$ be the minimum number k such that there exists a coloring of the vertices of G by k colors with the property that the number of colors used in every nonempty subgraph $H \subseteq G$ is at least $\varphi(H)$. Note that we do not restrict the subgraph H to be induced. The number $\chi_\varphi(G)$ is always defined, since coloring the vertices of G by $|V(G)|$ different colors obviously satisfies the condition of the subgraph coloring.

This notion generalizes most of the locally constrained variants of the graph coloring. For example,

- If $\varphi(K_2) = 2$ and $\varphi(G) = 1$ for all other graphs G, then $\chi_\varphi(G) = \chi(G)$ is the ordinary graph coloring.
- If $\varphi(K_2) = 2$, $\varphi(C) = 3$ for each cycle C, and $\varphi(G) = 1$ for all other graphs, then $\chi_\varphi(G) = \chi_a(G)$ is the acyclic coloring.
- If $\varphi(K_2) = 2$, $\varphi(P_4) = 3$ and $\varphi(G) = 1$ for all other graphs, then $\chi_\varphi(G) = \chi_s(G)$ is the star coloring.
- If $\varphi(G) = \min(p+1, \mathrm{td}(G))$, then $\chi_\varphi(G)$ is the minimum number of colors of a p-tree-depth coloring.

With this notion, Theorem 2.1 of DeVos et al. [34] can be restated in the following way: For any p, if $\varphi(G) = 1 + \min(p, \mathrm{tw}(G))$ and $\mathcal{G}$ is any proper minor-closed class of graphs, then there exists a constant c such that $\chi_\varphi(G) \leq c$ for any $G \in \mathcal{G}$. We call a graph function φ with this property *minor-closed class coloring bounded*, i.e., the function is minor-closed class coloring bounded if for any proper minor-closed class $\mathcal{G}$ there exists a constant c such that $\chi_\varphi(G) \leq c$ for any $G \in \mathcal{G}$. For a graph H, let $\varphi_{H,k}$ be the function defined by $\varphi_{H,k}(H) = k$ and $\varphi_{H,k}(H') = 1$ for any $H' \neq H$. The *upper chromatic number* $\bar{\chi}(H)$ of a graph H is the maximum k such that the function $\varphi_{H,k}$ is minor-closed class coloring bounded.

The main question considered by Nešetřil and Ossona de Mendez in [74] is: How large can the values of a minor-closed class coloring bounded function φ be? For example, $\varphi(P_3)$ may be at most two, since if $\varphi(P_3) = 3$, then $\chi_\varphi(G) \geq \Delta(G) + 1$, and planar graphs (or graphs in almost any other infinite minor-closed class of graphs) have unbounded maximum degree. The following theorem of Nešetřil and Ossona de Mendez [74] shows that the tree-depth of the graph is the correct bound:

Theorem 2.5 (Nešetřil and Ossona de Mendez [74]) *If φ is a minor-closed class coloring bounded function, then $\varphi(H) \leq \mathrm{td}(H)$ for any graph H.*

Obviously, it is not possible to require $\varphi(H) = \mathrm{td}(H)$ for every H, since then $\chi_\varphi(G) = \mathrm{td}(G)$, and the tree-depth is not bounded on many proper minor-closed classes of graphs (e.g., on paths). However, Theorem 2.2 shows that there exists an infinite sequence of minor-closed class coloring bounded functions whose limit is the function $\varphi(H) = \mathrm{td}(H)$. This means that $\bar{\chi}(H) = \mathrm{td}(H)$.

Interestingly, it turns out that this characterization extends to a much wider set of of graph classes – the classes with bounded expansion. We call a function φ *bounded expansion class coloring bounded*, if the subgraph chromatic number $\chi_\varphi(\cdot)$ is bounded on any class of graphs with bounded expansion. Since bounded expansion class coloring bounded function is also minor-closed class coloring bounded, Theorem 2.5 implies that $\varphi(H) \leq \mathrm{td}(H)$ for any bounded expansion class coloring bounded function. On the other hand, Theorem 2.4 shows that for each $p \geq 0$, the function $\varphi(H) = \min(p + 1, \mathrm{td}(H))$ is bounded expansion class coloring bounded. We consider the minor-closed class coloring bounded and bounded expansion class coloring bounded functions in Section 5.4.

We may also define $\chi^i_\varphi(G)$ as the minimum number k such that there exists a coloring of the vertices of G by k colors with the property that the number of colors used in every nonempty **induced** subgraph $H \subseteq G$ is at least $\varphi(H)$. The behavior of the chromatic numbers $\chi^i_\varphi(\cdot)$ was studied much less than $\chi_\varphi(\cdot)$, see Section 5.3 for some results.

2.3 Homomorphisms

The bounded expansion also has interesting consequences regarding the properties of the ordering of graphs by the homomorphism relation. A *homomorphism* from a graph G to a graph H is a function $f : V(G) \to V(H)$ such that whenever $e = \{u, v\}$ is an edge in G, then $\{f(u), f(v)\}$ is an edge of H. If there exists a homomorphism from G to H, we write $G \to H$, otherwise we write $G \not\to H$. For a set of graphs $\mathcal{F}$, let $\mathrm{Forb}_h(\mathcal{F})$ be the set of all graphs G that satisfy $F \not\to G$ for every $F \in \mathcal{F}$. If $G \to H$ and $H \to G$, the graphs G and H are *homomorphism equivalent*. The relation $\to$ is transitive and reflexive, hence it forms a quasiorder on the class of all finite graphs. See for example Hell and Nešetřil [49] for an overview of the properties and results regarding the graph homomorphisms.

One of the reasons for the importance of the study of graph homomorphisms is that many types of graph coloring can be expressed using this notion. For example, $\chi(G) \leq t$ if and only if $G \to K_t$. A well-known theorem of Grötzsch [45] states that every triangle-free planar graph is 3-colorable, i.e., $G \to K_3$ for every graph G in the class $\mathcal{P}_3$ of triangle-free planar graphs. Using the partial order terminology, K_3 is an upper bound for $\mathcal{P}_3$. One might ask whether $\mathcal{P}_3$ has a smaller bound. This indeed turns out to be the case – there exists a triangle-free 3-colorable graph H such that $G \to H$ for any $G \in \mathcal{P}_3$ (this has been showed by Nešetřil and Ossona de Mendez [68, 74] in the setting of proper minor-closed classes). In fact, there exists a smaller bound for $\mathcal{P}_3$ below any given bound, i.e., the class $\mathcal{P}_3$ has no supremum (Nešetřil and Ossona de Mendez [67]).

A similar result holds for the class of the subcubic graphs. By Brooks Theorem (Brooks [21], Lovász [62]), all connected subcubic graphs (except for K_4) are 3-colorable. Häggkvist and Hell [47] and Dreyer et al. [36] have showed that the class of subcubic triangle-free graphs also has a 3-colorable triangle-free bound. In fact, for every finite set $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ of connected graphs there exists a graph $H \in \mathrm{Forb}_h(\mathcal{F})$ with the following properties:

- $H \to K_3$, and
- $G \to H$ for every subcubic graph $G \in \mathrm{Forb}_h(\mathcal{F})$ (except for K_4).

This property can be interpreted as the existence of "restricted dualities", in the sense described below.

A pair F and D of graphs is called a *dual pair* if for every graph G, $F \not\to G$ if and only if $G \to D$. Equivalently, D is the maximum of $\mathrm{Forb}_h(\{F\})$. One of the interesting properties of the dual pairs is that it can be determined in polynomial time whether $G \to D$ (by testing whether $F \to G$) – this is a very rare property, since determining whether $G \to D$ is NP-complete for most graphs D (Hell and Nešetřil [48], Bang-Jensen et al. [11]).

Dual pairs of graphs and even of relational structures were characterized by Nešetřil and Tardif [76]. All the dualities turn out to be of the form (T, D_T), where T is a finite (relational) tree and D_T is uniquely determined by T (however, its structure is far more complex). These results imply that while in most classes of structures there are infinitely many dualities, they are relatively quite rare. However, a much richer spectrum of dualities arises when we restrict the notion to a particular class of graphs.

A *restricted duality* for a class of graphs $\mathcal{G}$ is a pair consisting of a finite set of connected graphs $\mathcal{F}$ and a finite graph $D \in \mathrm{Forb}_h(\mathcal{F})$ such that for any $G \in \mathcal{G}$, $G \in \mathrm{Forb}_h(\mathcal{F})$ if and only if $G \to D$. In other words, the class $\mathcal{G} \cap \mathrm{Forb}_h(\mathcal{F})$ has an upper bound D belonging to $\mathrm{Forb}_h(\mathcal{F})$.

A class $\mathcal{G}$ is said to have *all restricted dualities* if for any finite set of connected graphs $\mathcal{F}$, the class $\mathcal{G} \cap \mathrm{Forb}_h(\mathcal{F})$ has an upper bound in the class $\mathrm{Forb}_h(\mathcal{F})$. Obviously, not all graph classes have all restricted dualities. The main result of Nešetřil and Ossona de Mendez [72] is the following sufficient condition:

Theorem 2.6 ([72]) *Any class of graphs with bounded expansion has all restricted dualities.*

Since proper minor-closed classes and bounded degree graphs form classes of bounded expansion, this theorem generalizes both of the mentioned results. It also has many interesting consequences and connections:

The well-known Hadwiger conjecture states that every graph without a K_t minor has chromatic number at most $t-1$. In other words, if $\mathcal{G}_t$ is the class of all graphs without K_t minor, the conjecture claims that K_{t-1} is a maximum (in the homomorphism ordering) of $\mathcal{G}_t$. In fact, as Naserasr and Nigussie [65] and independently Nešetřil and Ossona de Mendez [67] showed, the Hadwiger conjecture is equivalent to the claim that every proper minor-closed class has a maximum. Theorem 2.6 shows at least that $\mathcal{G}_t$ has a K_t-free upper bound.

The *exact p-power* of a graph G is the graph $G^{\sharp p}$ on the vertex set $V(G)$, in that two vertices u and v are joined by an edge if and only if there exists a path u from to v in G of length exactly p. If p is even, then the graph $G^{\sharp p}$ may have arbitrarily large chromatic number even if G is a tree. Surprisingly, Theorem 2.6 shows that if p is odd, this is not the case: Let $\mathcal{G}$ be any class with bounded expansion, and let $\mathcal{F} = \{C_p\}$ consist of the cycle on p vertices. By Theorem 2.6, there exists a graph D of odd girth greater than p such that for each $G \in \mathcal{G}$, $C_p \not\to G$ if and only if $G \to D$. This implies the if $G \in \mathcal{G}$ has odd-girth greater than p, then $G^{\sharp p}$ has a proper coloring by $|V(D)|$ colors. Note that $|V(D)|$ is a constant dependent only on $\mathcal{G}$ and p.

The proof of Theorem 2.6 uses the following lemma, that claims that there are only finitely many cores with bounded tree-depth (a graph G is a *core* if G is not homomorphic to any of its proper subgraphs):

Lemma 2.7 ([74]) *For any positive integer p there exists a number N such that if G is a connected graph with $\mathrm{td}(G) = p$, then G contains a subgraph H with at most N vertices such that $G \to H$.*

Let D_p be the finite set of cores with tree-depth at most p. Theorem 2.4 thus implies the following claim: For any class of graphs with bounded expansion $\mathcal{G}$ and any $p > 0$, there exists N such that any graph $G \in \mathcal{G}$ has a partition to at most N parts with the property that each connected component of the subgraph induced by the union of any $j \le p$ parts is homomorphism equivalent to a graph in D_j. This interprets Theorem 2.4 as a kind of a "sparse regularity lemma", claiming that each graph in a bounded expansion class has a partition such that that the graphs induced by a few of the parts are of quite precisely described types.

2.4 Tree-depth

The previous sections indicate the importance of the tree-depth. Let us mention some of the results regarding this graph property. The tree-depth was introduced by Nešetřil and Ossona de Mendez [74], but the equivalent or similar notions appeared in several contexts, e.g. as the *minimum height of an elimination tree* (Deogun et al. [33]), a *rank function* of a graph (Nešetřil and Švejdarová [77]), or the rank coloring [14, 50, 81, 90, 91].

The tree-depth is minor-monotone ($\mathrm{td}(H) \le \mathrm{td}(G)$ if $H \prec G$), and bounds the tree-width of the graph – $\mathrm{tw}(G) < \mathrm{td}(G) \le (\mathrm{tw}(G) + 1)\log_2 n$ holds for any graph G with n vertices. The bound $\mathrm{tw}(G) < \mathrm{td}(G)$ is an easy consequence of the fact that the closure of any rooted forest is chordal, and for a chordal graph, the tree-width is equal to the size of the maximum clique minus one. The upper bound is by Bodlaender et al. [15] and Nešetřil and Ossona de Mendez [74]. However, a graph with low tree-width may have an arbitrarily large tree-depth, since a path on 2^k vertices has tree-depth $k+1$. In fact, the tree-depth of a graph G is high if and only if G contains a long path:

Lemma 2.8 (Nešetřil and Ossona de Mendez [71]) *If k is the number of the vertices of the longest path in a graph G, then $\lceil \log_2(k+1) \rceil \leq \mathrm{td}(G) \leq \binom{k+2}{2} - 1$.*

The tree-depth of a graph G also can be defined using the following inductive scheme (that gives rise to the *elimination tree* of the graph, whose depth corresponds to the tree-depth of G):

- If $|V(G)| = 1$ then $\mathrm{td}(G) = 1$.
- If G is not connected and $G_1, \ldots, G_k$ are its components, then $\mathrm{td}(G) = \max\{\mathrm{td}(G_i) | i = 1, \ldots, k\}$.
- If G is connected and $|V(G)| > 1$, then $\mathrm{td}(G) = 1 + \min\{\mathrm{td}(G - v) | v \in V(G)\}$.

Determining the tree-depth of a graph is NP-complete in general; in fact, unless $P = NP$, there is no polynomial time algorithm that would approximate the tree-depth with an error bounded by n^ε, where ε is a constant $0 < \varepsilon < 1$ and n is the order of the graph, by Bodlaender [15]. On the other way, there are many ways how to see that it is possible to decide whether $\mathrm{td}(G) \leq k$ for any fixed integer k:

- The inductive definition of the tree-depth provides an algorithm with time complexity $O(n^k)$.
- Since the graphs with $\mathrm{td}(G) \leq k$ form a proper minor-closed class, there is a finite number of forbidden minors for this class, and an algorithm to recognize the members of this class in time $O(n^3)$ exists by Robertson and Seymour [85, 87].
- As Nešetřil and Ossona de Mendez [74] showed, every graph with $\mathrm{td}(G) > k$ contains a subgraph H of bounded size with $\mathrm{td}(H) = k+1$ (we discuss this in greater detail in Section 5.1). This implies that there is a finite number of forbidden subgraphs for the class of graphs with $\mathrm{td}(G) \leq k$.
- Nešetřil and Ossona de Mendez [71] provide a linear-time algorithm to verify whether $\mathrm{td}(G) \leq k$ for any fixed integer k.

Also, efficient algorithms are known for some special classes of graphs, e.g., trees, cographs, permutation graphs, circular-arc graphs and cocomparability graphs of bounded dimension (Deogun et al. [33], Schaffe [90], Scheffler [91]), graphs with bounded tree-width (Bodlaender et al. [14]), star-like graphs and split graphs (Hsieh [50]). On the other hand, determining tree-depth is NP-complete when restricted to cobipartite graphs (Pothen [81]).

We can also bound the tree-depth of graphs in many graph classes. One such bound (by Nešetřil and Ossona de Mendez [74]) relates the tree-depth of a graph to the size of its separators. For a graph G, let $s_G(k)$ be the maximum of $\mathrm{sep}_{1/2}(H)$ of a subgraph $H \subseteq G$ with at most k vertices.

Lemma 2.9 ([74]) *For any graph G with n vertices,*

$$\mathrm{td}(G) \leq \sum_{i=1}^{\lfloor \log_2 n \rfloor} s_G\left(\frac{n}{2^i}\right).$$

This lemma implies the already mentioned bound $\mathrm{td}(G) \leq (\mathrm{tw}(G) + 1)\log_2 n$, and bounds the tree-depth of graphs in minor-closed classes:

Lemma 2.10 ([74]) *If a graph G does not contain K_h (for some integer $h > 0$) as a minor, then $\mathrm{td}(G) \leq (2+\sqrt{2})\sqrt{h^3 n}$.*

2.5 Separators and Expanders

The term "bounded expansion" suggests that it might be interesting to consider the relationship with the previously studied notion of the graph expansion and expander graphs. Both expanders and separators are studied intensively for their applications in design of algorithms (Lipton and Tarjan [61], Alber et al. [1], Baker [10]), derandomization (Gilman [44], Kabanets [53]) and graph theory (Wigderson and Zuckerman [94]). To make the distinction clear, in this section we use the term *bounded-depth minor expansion* for the definition of the expansion of a graph G based on the values $\nabla_r(G)$.

The notion of the vertex expansion is in some sense dual to the notion of the bounded-depth minor expansion: The bounded-depth minor expansion is small unless there exists an obstruction (a bounded-depth minor with many edges), whereas the vertex expansion is small if there exists an obstruction for it to be large (a small separator). As we describe below, a large vertex expansion implies a large bounded-depth minor expansion.

Alon, Seymour and Thomas [6] have showed that graphs in any proper minor-closed class of graphs have separators whose size is sublinear in the number of vertices, hence the graphs in such a class are not expanders. Nešetřil and Ossona de Mendez [71] extended this result for classes with subexponential bounded-depth minor expansion:

Theorem 2.11 ([71]) *Let f be a function such that $\log f(n) = o(n)$. If $\mathcal{G}$ is a class of graphs with expansion bounded by f, then the graphs in $\mathcal{G}$ have sublinear vertex separators.*

This theorem cannot be improved significantly, since a random cubic graph on n vertices almost surely has no separator of size $\frac{n}{20}$ (Bollobás [18], Kostochka and Melnikov [58]), hence if $\log f(n) = (\log 2)n$, the graphs in the class do not necessarily have sublinear separators.

The result is based on the following theorem by Plotkin et al. [80]:

Theorem 2.12 (Plotkin et al. [80]) *Given a graph G with m edges and n vertices, and integers l and h, there is an $O(mn/l)$ time algorithm that will either produce a K_h-minor of G of depth at most $l \log_2 n$, or will find a separator of size at most $O(n/l + 4lh^2 \log n)$.*

Can Theorem 2.11 be reversed, claiming that a graph with exponential expansion does not have small separators? Such a claim would be obviously false, since the large bounded-depth minor expansion can be caused by a relatively small subgraph of the graph, but the whole graph still can contain a small separator. Nevertheless, there is a more interesting counterexample to this hypothesis: Consider the graph $G = \mathrm{sd}_{\log n}(K_n)$. This graph has $\Omega(n^2 \log n)$ vertices, exponential bounded-depth minor expansion (since $\nabla_{\log n}(G) = n$), however the vertices of degree greater than two form a separator whose size is n, i.e., sublinear in the number of vertices.

Formulating the question more carefully, we want to ask whether a graph with a large bounded-depth minor expansion contains a (reasonably large) subgraph without small separators. This obviously handles the case of the expansion being caused by only a part of the graph. The other counterexample for the naive version of the question (the graph $\mathrm{sd}_{\log n}(K_n)$) contains a $\log n$-subdivision of a random 3-regular graph, that almost surely does not have a separator whose size would be sublinear in n. Since such a graph has $N = \theta(n \log n)$ vertices, all of its separators have size $\Omega\left(\frac{N}{\log N}\right)$. Therefore, even in this setting it is impossible to match Theorem 2.11 exactly, and claim that such a graph does not have a sublinear separator. However, as we show in Section 3.2.2, the lower bound of $\Omega\left(\frac{N}{\log N}\right)$ is correct.

2.6 Algorithmic Considerations

The proofs of most of the results mentioned in the previous sections are constructive and can be modified to provide efficient algorithms. In particular, there exists a linear time algorithm that given a graph G from a class with bounded expansion, finds the p-tree-depth coloring of G whose existence is guaranteed by Theorem 2.4 (Nešetřil and Ossona de Mendez [71]). The multiplicative constant of the algorithm depends on the expansion of the class and on p. The same paper also shows how such a coloring can be used to solve several other problems efficiently.

Consider the problem of deciding whether a graph G with n vertices contains a subgraph isomorphic to a fixed graph H with k vertices. In general, the best known algorithm (by Nešetřil and Poljak [75]) for this problem has time complexity $O(n^{\frac{\alpha k}{3}})$, where α is the exponent of the square matrix multiplication. Using the best known multiplication algorithm of Coppersmith and Winograd [24], this gives time complexity $O(n^{0.792k})$. The special case of planar graphs was studied by Plehn and Voigt [79], Alon et al. [5] and finally Eppstein ([38, 39]) who gave a linear-time algorithm. Eppstein [39] also shows that counting the number of subgraphs isomorphic to a fixed graph H in a graph with bounded tree-width can be done in linear time. Together with the existence of the low tree-depth coloring, this implies the following result:

Theorem 2.13 ([71]) *Let $\mathcal{G}$ be a class with bounded expansion and H a fixed graph. Then, there exists a linear time algorithm that for any $G \in \mathcal{G}$ determines the number of subgraphs of G isomorphic to H.*

Monadic Second-Order Logic (MSOL) is the extension of First-Order Logic (FOL) that includes vertex and edge sets and the relation of belonging to these sets. A well-known result of Courcelle [27, 28] states that a property expressible in MSOL can be verified for graphs with bounded tree-width in linear time. A similar result for graphs with bounded expansion follows:

Theorem 2.14 ([71]) *Let $\mathcal{G}$ be a class of graphs with bounded expansion and let p be a fixed integer. Let ϕ be a FOL sentence. Then, there exists a linear time algorithm to check whether the following sentence is true: $(\exists X)(|X| \leq p) \wedge (G[X] \models \phi)$.*

For instance:

Theorem 2.15 ([71]) *Let $\mathcal{G}$ be a class of graphs with bounded expansion and H a fixed graph. Then, there exist linear time algorithms that for a graph $G \in \mathcal{G}$ decide whether*

- *$H \to G$,*
- *H is a subgraph of G,*
- *H is an induced subgraph of G.*

We consider some further algorithmic questions (regarding complexity of determining or approximating the values $\nabla_r(\cdot)$ that determine the expansion of the graph) in Chapter 4, and apply these results to improve the bounds on the number of colors in the low tree-depth coloring of a graph with bounded expansion in Section 5.2.

2.7 Relationships of Graph Classes

Figure 2.1 summarizes the known and conjectured relationships between various graph classes and graph parameters we consider with respect to the bounded expansion. The arrows in the figure should be interpreted in the following sense: if P_1 and P_2 are quantitative graph properties and $P_1 \to P_2$, then there exists a function f such that $P_2(G) \leq f(P_1(G))$ for any graph G. For qualitative properties, the arrows can be read as implications (e.g., planar graphs are a proper minor-closed class of graphs), and an arrow from a qualitative property P_1 to a quantitative parameter P_2 reads as that the graphs with property P_1 have the parameter P_2 bounded by a constant, or vice versa (e.g., graphs with arrangeability bounded by a constant have linear Ramsey numbers). The dotted arrows correspond to the conjectured relationships.

Most of the claims depicted in the figure are discussed in detail elsewhere in this thesis, however let us briefly summarize the nontrivial ones here:

- Nešetřil and Ossona de Mendez [69] have proved that the arrangeability of a graph G is bounded by a function of $\nabla_1(G)$. It also follows from our Theorem 3.7.
- It is easy to see that a p-arrangeable graph has acyclic chromatic number at most $2p+2$: Given the ordering L that witnesses the arrangeability, we assign each vertex v a color different from the colors of vertices in
$$L^-(v) \cap \left(N(v) \cup \bigcup_{u \in N(v) \cap L^+(v)} N(u) \right),$$
and observe that this coloring is acyclic. This fact also follows from Theorem 3.3 (with a much worse bound).

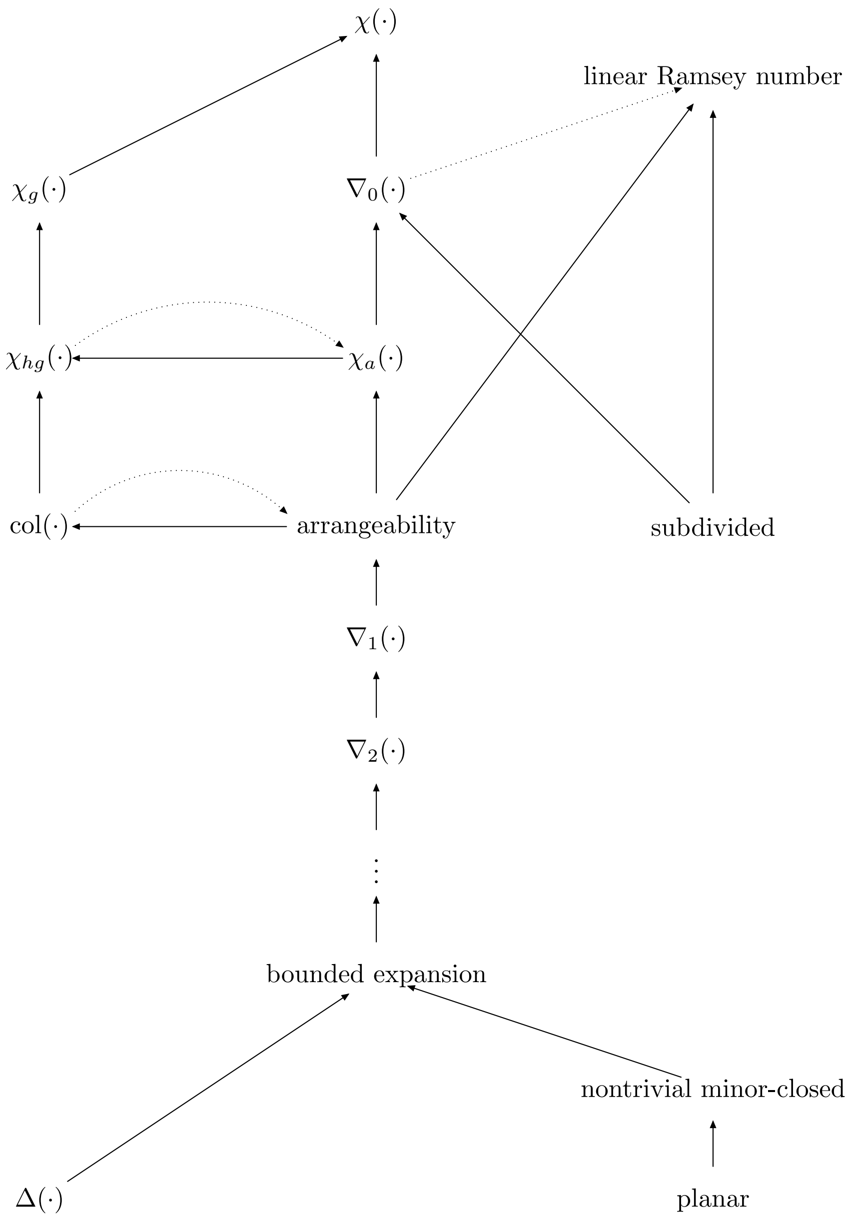

Figure 2.1: The relationships of various graph classes.

- Zhu and Dinsky [35] have proved that a function of the acyclic chromatic number bounds the hereditary game chromatic number, and conjectured that the reverse is true as well.

- The game coloring number that bounds the (hereditary) game chromatic number was defined in order to capture the concept common in many proofs regarding the game chromatic number. Kierstead and Trotter [55] have proved that a function of the arrangeability bounds the game chromatic number, and their proof in fact works for the game coloring number as well. Inspired by the conjecture of Zhu and Dinsky and by the relationship between the acyclic chromatic number and the arrangeability that follows from Theorems 3.3 and 3.7, we pose Conjecture 3.3 stating that the reverse is true as well, i.e., that the arrangeability of the graph is bounded by a function of its coloring number.

- A class of graphs $\mathcal{G}$ has *linear Ramsey number* if there exists a constant c such that a monochromatic copy of any graph $G \in \mathcal{G}$ appears in any coloring of the edges of the graph $K_{c|V(G)|}$ by two colors. The famous Erdős-Burr conjecture [22] states that every degenerate graph has linear Ramsey number. The best known results regarding the conjecture are by Chen and Schelp [23] that states that the classes of graphs with bounded arrangeability have a linear Ramsey number, and by Alon [3] that shows that the subdivided graphs have a linear Ramsey number.

Chapter 3

Subdivisions

In this chapter, we study the relationship between the graph expansion and other similar graph parameters and the existence of certain subdivisions in such graphs. We provide a characterization of several of the graph parameters discussed in the previous chapter in terms of forbidden subdivisions. In addition to precisely describing graph classes in that these parameters are bounded by a constant, these characterizations help to clarify the relationship among the parameters.

Also, we consider the existence of a large cliques with edges subdivided by a number of vertices bounded by a constant in graphs with large minimum degree and use this result to characterize graphs with exponential expansion. This characterization implies that such graphs contain large expander-like subgraphs.

3.1 Forbidden Subdivisions

One of the parameters we consider is the acyclic chromatic number. Let us first mention several past results. Borodin [20] has proved that the acyclic chromatic number of every planar graph is at most 5. Nešetřil and Ossona de Mendez [66] have proved that every graph G has a minor H such that $\chi_a(G) = O\left(\chi(H)^2\right)$. This implies that the acyclic chromatic number is bounded by a constant for every proper minor-closed class of graphs. However, this result does not describe all graph classes with bounded acyclic chromatic number, e.g., $\mathrm{sd}_1(K_{n,n})$ has acyclic chromatic number 3, but it contains K_n as a minor.

On the other hand, Wood [95] has proved that the acyclic chromatic number of $\mathrm{sd}_1(G)$ is bounded by a function of the chromatic number of the graph G and vice versa:

Theorem 3.1 (Wood [95], Corollary 3) *For each graph G,*

$$\sqrt{\frac{\chi(G)}{2}} < \chi_a(\mathrm{sd}_1(G)) \le \max(3, \chi(G)).$$

A simple corollary of this theorem is the following characterization:

Corollary 3.2 *Let G be a graph with $\chi_a(G) = c$. If H is a graph such that $\chi(H) \ge 2c^2$, then G does not contain $\mathrm{sd}_1(H)$ as a subgraph.*

In Section 3.1.1, we prove that this statement essentially describes all graphs with bounded acyclic chromatic number, i.e., that if G has high acyclic chromatic number, it contains as a subgraph a $\le$ 1-subdivision of a graph with high chromatic number:

Theorem 3.3 *Let $c \ge 4$ be an integer and let $d = 56(c-1)^2 \frac{\log(c-1)}{\log c - \log(c-1)}$. Let G be a graph with acyclic chromatic number greater than $c(c-1)^{c\binom{d}{2}}$, i.e., $\chi_a(G) \approx \exp(c^7 \log^3 c)$. If $\chi(G) \le c$, then G contains a subgraph $G' = \mathrm{sd}_1(G'')$ such that the chromatic number of the graph G'' is c.*

This result generalizes the result of Nešetřil and Ossona de Mendez [66], although our bound is much weaker than the quadratic one. The consequence of Theorem 3.3 and Corollary 3.2 is the following characterization of graph classes with the bounded acyclic chromatic number:

Corollary 3.4 *Let $\mathcal{G}$ be any class of graphs such that the chromatic number of every graph in $\mathcal{G}$ is bounded by a constant. The acyclic chromatic number of graphs in $\mathcal{G}$ is bounded by a constant if and only if there exists a constant c such that every graph G such that $\mathrm{sd}_1(G)$ is a subgraph of a graph in $\mathcal{G}$ satisfies $\chi(G) \le c$.*

Arrangeability turns out to be closely related to the acyclic chromatic number. In Section 3.1.2, we show a precise characterization of graphs G with bounded arrangeability in terms of average degrees of graphs whose $\le$ 1-subdivisions are subgraphs of G, analogical to Theorem 3.3 and Corollary 3.4. Rödl and Thomas [89] have shown that every graph with arrangeability p^8 contains a subdivision of the clique K_p as a subgraph; a result similar to ours is implicit in their proof. However, the result we obtained is slightly stronger. Komlós and Szemerédi ([56, 57, 92]) have proved that every simple graph with average degree at least d^2 contains a subdivision of K_d as a subgraph, hence Theorem 3.7 implies that every graph with arrangeability $\Omega(p^6)$ contains a subdivision of K_p as a subgraph. Nešetřil and Ossona de

Mendez [69] have proved that the arrangeability of a graph G is bounded by a function of $\nabla_1(G)$. This fact is also an easy consequence of Theorem 3.7.

Similarly, the greatest reduced average density of rank r of a graph G can be characterized by the average degrees of graphs whose $\leq 2r$-subdivisions are contained in G. More precisely, if G is a $\leq 2r$-subdivision of a graph with minimum degree d, then $\nabla_r(G) \geq \frac{d}{2}$. In Section 3.1.3, we prove that on the other hand, graphs with large $\nabla_r(G)$ contain $\leq 2r$-subdivisions of graphs with high minimum degree. In the view of this result, one can consider the arrangeability to be $\nabla_{1/2}(G)$.

Also, in Section 4.3, we show an approximation algorithm that given a graph G with $\nabla_r(G) = d$, produces a witness that proves that $\nabla_r(G) \leq f(d)$, for some function f. This witness is a linear ordering of the vertices of G that satisfies certain properties, thus allowing us to interpret the greatest reduced average densities as stronger versions of the arrangeability.

3.1.1 Acyclic Chromatic Number

The goal of this section is to prove Theorem 3.3. We first prove a lemma regarding the graphs with high density. Note that a graph G with acyclic chromatic number at most c cannot have high density, as G is a union of $\binom{c}{2}$ forests.

Lemma 3.5 *Let $c \geq 4$ be an integer and let G be a graph with the minimum degree $d > 56(c-1)^2\frac{\log(c-1)}{\log c - \log(c-1)}$, (i.e., $d = \Omega(c^3 \log c)$). Then the graph G contains a subgraph G' that is the 1-subdivision of a graph with chromatic number c.*

Proof: Every graph contains a bipartite subgraph with at least half of the edges of the original graph, i.e., G contains a bipartite subgraph G_1 with average degree more than $\frac{d}{2}$. The graph G_1 cannot be $\frac{d}{4}$-degenerate, since otherwise the average degree of G_1 would be at most $\frac{d}{2}$. Let G_2 be a subgraph of G_1 with minimum degree at least $d_2 = \frac{d}{4}$. The graph G_2 is bipartite, let $V(G_2) = A \cup B$ be a partition of its vertices to two independent sets such that $|A| \leq |B|$. Let $a = |A|$ and $b = |B|$. Since the minimum degree of G_2 is at least d_2, it follows that $d_2 \leq a \leq b$.

Let $q = 7\frac{\log(c-1)}{\log c - \log(c-1)}$. Note that $\frac{d_2}{q} \geq 10$. We construct a subgraph G_3 in the following way: if $b \geq qa$, then let $G_3 = G_2$, $A' = A$ and $B' = B$. Otherwise, we choose sets $A' \subseteq A$ and $B' \subseteq B$ as described in the next paragraph, and let G_3 be the subgraph of G_2 induced by A' and B':

Let A' be a subset of A obtained by taking each element of A randomly independently with probability $p = \frac{b}{qa}$. The expected size of A' is $ap = \frac{b}{q}$, and by Chernoff Inequality, the size of A' is more than $\frac{2b}{q}$ with probability less than $e^{-\frac{3b}{8q}} \le e^{-\frac{3d_2}{8q}} \le e^{-\frac{15}{4}} < 0.5$. Consider a vertex v of B with degree $s \ge d_2$ in G_2, and let s' be the number of neighbors of v in A' and $r(v) = \frac{s'}{s}$. The expected number of neighbors of v in A' is ps. By Chernoff Inequality, the probability that $s' < \frac{p}{2}s$ is less than $e^{-\frac{3ps}{28}} \le e^{\frac{3}{28}\cdot\frac{b}{a}\cdot\frac{d_2}{q}} \le e^{-\frac{15}{14}} < 0.35$. Let B' be the set of vertices v of B such that $r(v) \ge \frac{p}{2}$. The expected value of $|B \setminus B'|$ is less than $0.35b$, and by Markov Inequality, $\text{Prob}\left[|B \setminus B'| \ge 0.7b\right] \le 0.5$. Therefore, the probability that the set A' has size at most $\frac{2b}{q}$ while the set B' has size at least $0.3b$ is greater than zero. We let A' and B' be a pair of sets that satisfies these properties.

Let $a' = |A'|$ and $b' = |B'|$. Observe that the degree of every vertex of B' in G_3 is at least $\frac{b}{2qa}d_2 \ge \frac{1}{2q}d_2 = (c-1)^2 = d_3$, and that $b' \ge 0.3b \ge 0.15qa'$. Let $D_1, \ldots, D_{b'} \ge d_3$ be the degrees of vertices of B'.

We show that the graph G_3 contains as a subgraph the 1-subdivision of a graph with chromatic number c. Suppose for contradiction that each graph whose 1-subdivision is a subgraph of G_3 has chromatic number at most $c-1$. Let us consider only the subgraphs whose vertices of degree 2 created by subdividing edges belong to B'. There are exactly $N_G = \prod_{i=1}^{b'} \binom{D_i}{2}$ such subgraphs and $N_C = (c-1)^{a'}$ colorings of A' by $c-1$ colors.

Let φ be a coloring of A' by $c-1$ colors. We determine the number of subgraphs $H \subseteq G_3$ such that all vertices of B' have degree 2 in H, and φ is a proper coloring of the graph obtained from H by suppressing the vertices in B'. Let us consider a vertex v in B' of degree D. Since φ is proper, the two edges incident with v in H lead to vertices with different colors. Let M be the neighborhood of v, $|M| = D$. Let m_i be the number of vertices of M colored by φ with the color i. The number s of the pairs of neighbors of v that have different colors satisfies

$$s = \sum_{1 \le i < j \le c-1} m_i m_j = \frac{1}{2} \sum_{1 \le i,j \le c-1, i \ne j} m_i m_j = \frac{1}{2} \sum_{i=1}^{c-1} m_i (D - m_i)$$

$$s = \frac{1}{2}\left(D^2 - \sum_{i=1}^{c-1} m_i^2\right) \le \frac{1}{2}\left(D^2 - \frac{D^2}{c-1}\right).$$

Therefore, the number of the subgraphs of G_3 for that φ is proper is at most $N_P = \left(\frac{1}{2}\left(1 - \frac{1}{c-1}\right)\right)^{b'} \prod_{i=1}^{b'} D_i^2$. For each subgraph of G_3 there exists at least one proper coloring, hence $N_G \le N_C N_P$, and we obtain

$$(c-1)^{a'} \left(\frac{1}{2}\left(1 - \frac{1}{c-1}\right)\right)^{b'} \prod_{i=1}^{b'} D_i^2 \ge \prod_{i=1}^{b'} \binom{D_i}{2}$$

$$(c-1)^{a'}\left(1-\frac{1}{c-1}\right)^{b'} \geq \prod_{i=1}^{b'}\left(1-\frac{1}{D_i}\right) \geq \left(1-\frac{1}{d_3}\right)^{b'}$$

Since $b' \geq 0.15qa'$ and $\left(1-\frac{1}{d_3}\right)\left(1-\frac{1}{c-1}\right)^{-1} = \frac{c}{c-1} > 1$, it follows that

$$(c-1)^{a'} \geq \left(\left(1-\frac{1}{d_3}\right)\left(1-\frac{1}{c-1}\right)^{-1}\right)^{0.15qa'}$$

$$(c-1) \geq \left(\frac{c}{c-1}\right)^{0.15q}$$

This is a contradiction, since

$$\left(\frac{c}{c-1}\right)^{0.15q} > \left(\frac{c}{c-1}\right)^{\frac{\log(c-1)}{\log c-\log(c-1)}} = c-1.$$

∎

Let us now prove the main theorem of this section.

Proof of Theorem 3.3: We prove the contravariant implication: "Let G be a graph with $\chi(G) \leq c$. If all graphs whose 1-subdivision is a subgraph of G have chromatic number at most $c-1$, then G has acyclic chromatic number at most $c_1 = c(c-1)^{c\binom{d}{2}}$."

Let us assume that G is a graph with chromatic number at most c, such that all graphs whose 1-subdivision is a subgraph of G have chromatic number at most $c-1$. By Lemma 3.5, the graph G is d-degenerate. Let $L = v_1, \ldots, v_n$ be an ordering of the vertices of G in such a way that each vertex has at most d neighbors after it; let $N_t = L^+(v_t) \cap N(v)$ be the set of the neighbors of v_t that are after v_t in the ordering L, and let $v_{t,j}$ be the j-th of these neighbors, for $j = 1, 2, \ldots, |N_t|$.

Suppose that there exists a proper coloring φ of G such that each set N_t is rainbow (i.e., no two vertices in N_t have the same color). Let us consider an arbitrary cycle C in G. Let v be the vertex of C that appears first in the ordering L, and let u and w be the neighbors of v in C. The colors $\varphi(u)$, $\varphi(v)$ and $\varphi(w)$ are mutually distinct, hence C is not colored by two colors. Therefore, the coloring φ is acyclic.

Let us now construct a coloring φ that satisfies this property. Let φ_0 be a fixed proper coloring of G by c colors. For $i = 1, \ldots, c$ and $1 \leq j_1 < j_2 \leq d$, we define the graph G_{i,j_1,j_2} in the following way: the vertices of G_{i,j_1,j_2} are the vertices of G, and for each t such that $\varphi_0(v_t) = i$, we join by an edge the vertices v_{t,j_1} and v_{t,j_2} (if both of them exist). Note that the 1-subdivision of

G_{i,j_1,j_2} is a subgraph of G, hence G_{i,j_1,j_2} can be colored by $c-1$ colors. Let φ_{i,j_1,j_2} be such a coloring.

We color each vertex v of G with the $c\binom{d}{2}+1$-tuple $\varphi(v)$ consisting of $\varphi_0(v)$ and $\varphi_{i,j_1,j_2}(v)$ for $i=1,\ldots,c$ and $1 \le j_1 < j_2 \le d$. Each N_t is rainbow in this coloring, as the vertices $v_{t,a}$ and $v_{t,b}$ get distinct colors in the coloring $\varphi_{\varphi_0(v_t),a,b}$. Also, the coloring φ is proper since the coloring φ_0 is proper. Therefore, we found the acyclic coloring φ of G by c_1 colors, hence the claim of the theorem holds. ∎

3.1.2 Arrangeability

Let us start with a simple observation regarding the arrangeability. Let G be the 1-subdivision of a graph with minimum degree $d > 2$, and consider the ordering L that shows that its arrangeability is at most p. Let v be the last vertex of degree greater than two in this ordering. Note that $|N(v)\cap L^-(v)| \le p+1$ and $|N(v)\cap L^+(v)| \le p$. Since the degree of v is at least d, the graph G is not p-arrangeable for $p < \frac{d-1}{2}$. The goal of this section is to prove that on the other hand, every graph with large arrangeability contains a ≤ 1-subdivision of a graph with large minimum degree. First, we show the following characterization of graphs with small arrangeability:

Lemma 3.6 *If G is a p-arrangeable graph, then there exists ordering L of vertices of G such that each vertex has the back-degree at most $p+1$ and the double back-degree at most $2p+1$. On the other hand, if there exists an ordering L of vertices of a graph G such that the back-degree of each vertex is at most d_1 and the double back-degree is at most d_2, then the graph G has arrangeability at most d_1d_2.*

Proof: Suppose first that the graph G is p-arrangeable, and let L be an ordering of the vertices of G that witnesses its arrangeability. Let S be a double-star in G with center v. As we observed before, the back-degree of v is at most $p+1$, hence S has at most $p+1$ middle vertices in $L^-(v)$. By the p-arrangeability of G, the double-star S also has at most p middle vertices in $L^+(v)$ whose ray vertex belongs to $L^-(v)$. Therefore, $d_2^-(v) \le 2p+1$.

Let us now assume that L is an ordering of vertices of the graph G such that for each v, $d^-(v) \le d_1$ and $d_2^-(v) \le d_2$. Consider an arbitrary vertex v, and let $X = L^-(v)\cap\bigcup_{u\in N(v)\cap L^+(v)} N(u)$. For each vertex $x \in X$, let us choose one of its neighbors $u_x \in N(v)\cap L^+(v)$ arbitrarily, and let $U=\{u_x : x \in X\}$. By the definition of the double back-degree, $|U| \le d_2$. On the other hand, each vertex in U has at most d_1 neighbors before it, hence $|X| \le d_1d_2$. Therefore, the ordering L witnesses that G is d_1d_2-arrangeable. ∎

Let us formulate the main theorem of this section:

Theorem 3.7 *Let $d \geq 1$ be an arbitrary integer and $p = 4d^2(4d+5)$. Let G be a d-degenerate graph. If G is not p-arrangeable, then G contains a subgraph $G' = \mathrm{sd}_1(G'')$ such that the minimum degree of G'' is at least d.*

Proof: Let G be a d-degenerate graph that is not p-arrangeable and let n be the number of vertices of G. Let $d_1 = 4d$ and $d_2 = d(4d+5)$ and consider the following algorithm that attempts to construct an ordering of vertices of G such that for each vertex v, $d^-(v) \leq d_1$ and $d_2^-(v) \leq d_2$: We set $G_0 = G$. In the step $i > 0$, if there exists a vertex v of G_{i-1} such that the degree of v in G_{i-1} is at most d_1 and each double-star S in G with center v has at most d_2 ray vertices in $V(G_{i-1})$, then let $v_{n-i+1} = v$ and $G_i = G_{i-1} - v$. Note that we consider also the double-stars that are not subgraphs of G_{i-1}. Obviously, if this algorithm succeeds in each step, the ordering $L = v_1, v_2, \ldots, v_n$ satisfies the required properties.

By Lemma 3.6, since $p = d_1 d_2$ and G is not p-arrangeable, this algorithm fails on G. This means that there exists i such that each vertex v of G_{i-1} has more than d_1 neighbors in G_{i-1}, or is a center of a double-star in G with more than d_2 ray vertices in $V(G_{i-1})$. Let V_1 be the set of vertices of degree greater than d_1 in G_{i-1}, and let $V_2 = V(G_{i-1}) \setminus V_1$. Let $n_1 = |V_1|$ and $n_2 = |V_2|$. Each vertex v in V_2 has degree at most d_1 in G_{i-1}, hence v is a center vertex of a double-star with more than $d_2 - d_1$ ray vertices in $V(G_{i-1})$ and all middle vertices in $V(G) \setminus V(G_{i-1})$; let us choose such a double-star S_v for each vertex $v \in V_2$ arbitrarily. Let X be the set of the chosen double-stars, $X = \{S_v | v \in V_2\}$.

Let M be the set of middle vertices of double-stars in X, let $m = |M|$, and let G_X be the bipartite graph on $V_2 \cup M$ (note that V_2 and M are disjoint) whose set of edges consists of all the middle edges of the double-stars in X. Since the graph G is d-degenerate, the average degree of G_X is at most $2d$. On the other hand, each vertex in V_2 has degree at least $d_2 - d_1$ in G_X, hence $d(m + n_2) \geq |E(G_X)| \geq n_2(d_2 - d_1)$. It follows that $m \geq \frac{n_2(d_2-d_1-d)}{d} = 4dn_2 = d_1 n_2$. For each vertex $u \in M$, let us choose an arbitrary double-star $T_u \in X$ such that u is a middle vertex of T_u. Let us remove u together with the corresponding ray vertex from each double-star in X except for T_u. Let X' be the set of double-stars obtained this way. The double-stars in X' have disjoint middle vertices, and in total m rays (the ray vertices do not have to be disjoint).

Let us consider the graph H with the vertex set $V_1 \cup V_2$ in that the vertices u and v are adjacent if:

1. $u \in V_1$ and $\{u, v\}$ is an edge of G_{i-1}, or

2. u is a center of a star $T \in X'$, and v is a ray vertex of T.

A ≤ 1-subdivision of H is a subgraph of G. Observe that the graph H has $n_1 + n_2$ vertices, and at least $\frac{1}{2}(n_1 d_1 + m)$ edges. Using the lower bound $m \geq d_1 n_2$, we conclude that $|E(H)| \geq \frac{d_1}{2}(n_1 + n_2)$.

Let H' be the subgraph of H, whose edges are only the edges connecting the ray vertices of the double-stars in X' with their centers. The graph G is d-degenerate, hence H' has at least $(\frac{d_1}{2} - d)(n_1 + n_2) = d(n_1 + n_2)$ edges. Therefore, the average degree of H' is at least $2d$. It follows that H' has a subgraph G'' with the minimum degree at least d. The graph $G' = \mathrm{sd}_1(G'')$ is a subgraph of G that satisfies the claim of the theorem. ∎

3.1.3 Greatest Reduced Average Density

We now focus on the characterization of graphs with bounded expansion. A non-empty graph H is *an average (resp. minimum) degree* (r, d)*-witness* if there exists a (r, d)*-witness decomposition* $D = \{(S_1, s_1), \ldots, (S_k, s_k)\}$ of H, i.e., vertex-disjoint nonempty induced subgraphs S_1, S_2, $\ldots$, S_k of H such that $V(S_1)$, $V(S_2)$, $\ldots$ partition $V(H)$, and vertices $s_i \in V(S_i)$ such that

- the subgraphs S_i are trees, and
- for each vertex $v \in S_i$, the distance of v from s_i in S_i is at most r, and
- for each $i \neq j$, there is at most one edge between S_i and S_j in H, and
- the average (resp. minimum) degree of the minor obtained from H by identifying all the vertices of each tree S_i with s_i is at least d.

Observe that $\nabla_r(G) \geq d$ if and only if G contains an average degree $(r, 2d)$-witness as a subgraph. Therefore, if $\nabla_r(G) \geq d$ then G contains a minimum degree (r, d)-witness as a subgraph.

The *size* of the decomposition is the number of its trees. The vertices s_i of a witness decomposition are called *centers*. We consider the trees to be rooted in the centers. The edges that belong to the trees of the decomposition are called *internal* and the remaining edges are *external*. For a non-center vertex $v \in V(S_i)$, the unique internal edge from v on the shortest path to s_i is called the *parent edge* and its vertex different from v is called the *parent vertex*.

We always assume that each leaf of a tree S_i is incident to at least one external edge, unless the leaf is the center of the tree; otherwise, the leaf vertex may be removed from the witness. When an external edge is removed from the decomposition, we also repeatedly remove leaves that are incident to no external edges. Similarly, the operation of *removal* of a tree from the decomposition D of a graph H is defined in the following way: The decomposition $D' = \{(S'_1, s'_1), \ldots\}$ of a graph $H' = \bigcup_i V(S'_i)$ is obtained from the decomposition $\{(S_1, s_1) \ldots, (S_{t-1}, s_{t-1}), (S_{t+1}, s_{t+1}), \ldots\}$ by repeatedly removing the leaves that are incident with no external edges. Given an internal edge $e \in S_t$, the decomposition D' of the graph H is obtained from D by *splitting on* e if D' consists of trees $S_1, \ldots, S_{t-1}, S_{t+1}, \ldots$, and the two trees S'_t and S''_t obtained from S_t by removing e. If $s_t \in S'_t$ then the center of S'_t is s_t and the center of S''_t is the common vertex of e and S''_t. The edge e becomes external by this operation. *Expunging* of a center vertex v is performed by first splitting on all the internal edges incident to v, and then removing the tree consisting of v.

Lemma 3.8 *Let H be a minimum degree (r,d)-witness with a decomposition $D = \{(S_1, s_1), (S_2, s_2), \ldots\}$ and let $d_1 = \sqrt[r+1]{\frac{d}{4}}$. There exists a minimum degree (r, d_1)-witness $H' \subseteq H$ with a decomposition $D' = \{(S'_1, s'_1), (S'_2, s'_2), \ldots\}$ such that the degree of each center is at least d_1.*

Proof: We construct the new decomposition D' by repeatedly expunging the vertices v such that v is a center and its degree is less than d_1, as long as any such vertices exist. Let us show that the decomposition D' obtained by this construction is non-empty.

Let k be the size of D and let e be the number of external edges of D. Note that $e \geq \frac{d}{2}k$. Let us count the number of external edges that get removed by expunging the vertices. If an edge e is removed by expunging a vertex v, let us assign e to the tree in D that contains the vertex v. When a vertex is expunged, its degree is less than d_1. The depth of each tree in the decomposition D is at most r, thus there are at most d_1^{r+1} edges assigned to each tree. Therefore, at most $d_1^{r+1}k$ external edges are removed. Since $e \geq \frac{d}{2}k > d_1^{r+1}k$, the decomposition D' is non-empty, and it obviously satisfies the claim of the lemma. ∎

Consider a minimum degree (r,d)-witness H with the decomposition D such that the degree of each center is at least d. Given a non-center vertex $v \in S_i$, let $B(v)$ be the component of $S_i - s_i$ that contains v. The vertex vertex $v \in S_i$ is called *lonely* v is a center of S_i, or if there is only one external edge incident to the vertices in $B(v)$ and this edge is incident to v. Note that in this case, $B(v)$ is a path with the end vertex v. An external edge $e = \{u, v\}$ is called *critical* if u or v is lonely, and *bicritical* if both u and v are lonely. Observe that there exists an (r,d)-witness $H' \subseteq H$ with a decomposition D' such that the degree of each center is at least d and each external edge is critical.

Theorem 3.9 *Let $r, d \geq 1$ be arbitrary integers and let $p = 4\,(4d)^{(r+1)^2}$. If $\nabla_r(G) \geq p$, then G contains a subgraph G' that is a $\leq 2r$-subdivision of a graph with minimum degree d.*

Proof: Let G be a graph with $\nabla_r(G) \geq p$. As we noted before, there exists a minimum degree (r, p)-witness $H \subseteq G$. Let $d_1 = \sqrt[r+1]{\frac{p}{4}} = (4d)^{r+1}$. By Lemma 3.8, there exists a minimum degree (r, d_1)-witness $H' \subseteq H$ with a decomposition $D' = \{(S'_1, s'_1), (S'_2, s'_2), \ldots\}$ such that the degree of each center is at least d_1. Furthermore, we may assume that each external edge in the decomposition D' is critical.

Let $b = \sqrt[r+1]{d_1} = 4d$. We create a new decomposition D'' of a graph $H'' \subseteq H'$ by splitting on the parent edges of all non-center vertices whose degree is greater than b. After splitting on edge uv, where u is the parent vertex of v, if u is not lonely, then we also remove the edge e. Let us call the center vertices of D' the *old centers*, and the center vertices of D'' that are non-center in D' the *new centers*. The decomposition D'' satisfies the following properties:

1) All old centers have degree at least d_1 and all new centers have degree at least b, and

2) all non-center vertices have degree at most b, and

3) all external edges are critical, and

4) all external edges between trees with the new centers are bicritical (all such edges were internal in D''), and

5) the lonely vertex of each external edge that is not bicritical belongs to a tree whose center is old.

We construct a graph G'' in the following way: For each tree S with the center s in the decomposition D'' and for each component C of $S - s$ that is incident with more than one external edge, we select one external edge incident to a vertex in C arbitrarily. Let W be the set consisting of all vertices incident with the selected edges or with the bicritical edges of D''. The graph G'' is the induced subgraph of H'' with the vertex set that consists of the centers of D'', the vertices in W and the vertices on the paths that join the vertices of W with the centers of their trees. Observe that all the non-center vertices of G'' have degree exactly 2, i.e., the graph G'' is a $\leq 2r$-subdivision of some graph F''.

Let us compute average degree of F''. Let n_{old} be the number of old centers, n_{new} the number of new centers, $n = n_{\text{old}} + n_{\text{new}}$ the number of vertices of F'' and m the number of edges of F''. The properties 1), 3), 4) and 5) of D'' imply that the degree of each new vertex in F'' is at least b, hence $m \geq \frac{b}{2} n_{\text{new}}$. On the other hand, the total number of external edges in D'' is at least $\frac{d_1}{2} n_{\text{old}}$, and by the properties 2) and 3) of D'', the number of external edges is decreased at most b^r times during the construction of G'', i.e., $m \geq \frac{d_1}{2b^r} n_{\text{old}} = \frac{b}{2} n_{\text{old}}$. Hence $m \geq \frac{b}{4} n$, and the average degree of F'' is at least $\frac{b}{2}$.

Therefore, there exists a subgraph $F' \subseteq F''$ such that the minimum degree of F' is at least $\frac{b}{4} = d$. The corresponding subgraph $G' \subseteq G''$ is a $\leq 2r$-subdivision of F', hence the claim of the theorem holds. ∎

On the other hand, if G is a graph with $\delta(G) = d$, then $\nabla_r(\mathrm{sd}_{2r}(G)) \geq \frac{d}{2}$, hence a graph has large rank r greatest reduced average density if and only if it contains a $\leq 2r$-subdivision of a graph with large minimum degree. More precisely:

Corollary 3.10 *For any integer $r \geq 0$, there exist functions f_1 and f_2 such that for any graph G,*

- *if G satisfies $\nabla_r(G) \geq f_1(c)$, then it contains a $\leq 2r$-subdivision of a graph with minimum degree c, and*
- *if G contains a $\leq 2r$-subdivision of a graph with minimum degree $f_2(d)$, then $\nabla_r(G) \geq d$.*

3.1.4 Lower Bounds

Can the theorems derived in the previous sections can be strengthened, to ensure existence of subdivisions with larger minimum degree or chromatic number? In this section, we describe several constructions that constrain such improvements.

Theorem 3.11 *For each $d \geq 3$ and $r > 0$, there exists a graph G with maximum degree d and $\nabla_r(G) = \frac{1}{2} d(d-1)^r$.*

Proof: The graph G is constructed from the complete graph G' on $d(d-1)^r+1$ vertices in the following way: We replace each vertex v by a rooted tree T_v of depth r, whose inner vertices have degree d (such a tree has $d(d-1)^{r-1}$ leaves). For each pair of vertices $u, v \in V(G')$, we join by an edge arbitrary leaves of T_u and T_v in such a way that each leaf has degree d in G. Since G' is a minor of G of depth r, it follows that $\nabla_r(G) \geq \frac{1}{2} d(d-1)^r$. On the other hand, the maximum degree of G is d, thus each minor of depth at most r of G has maximum degree at most $d(d-1)^r$, hence $\nabla_r(G) = \frac{1}{2} d(d-1)^r$. ∎

Theorem 3.12 *For each $d \geq 3$, there exists a graph G with maximum degree d and arrangeability at least $\frac{d(d-2)}{8}$.*

Proof: Let G be an arbitrary d-regular graph of girth at least five, and L be an ordering that witnesses that G is p-arrangeable. Let

$$d_2^L(v) = \left| L^-(v) \cap \bigcup_{u \in N(v) \cap L^+(v)} N(u) \right|.$$

We count the sum $S = \sum_{v \in V(G)} d_2^L(v)$ in two ways. Since $d_2^L(v) \leq p$ for each vertex v, if n is the number of vertices of G, then $S \leq pn$. On the other hand, if $x \neq y$ is a pair of vertices in $N(v) \cap L^-(v)$ and x is before y in the ordering L, then x is one of the vertices counted by $d_2^L(y)$. Since the girth of G is greater than four, for each vertex v, any pair of vertices $x, y \in N(v) \cap L^-(v)$ contributes exactly one to S, hence

$$S = \sum_{v \in V(G)} \binom{d^-(v)}{2} = -\frac{1}{4}dn + \frac{1}{2} \sum_{v \in V(G)} \left(d^-(v)\right)^2.$$

Using the inequality between arithmetic and quadratic mean, we get

$$\sum_{v \in V(G)} \left(d^-(v)\right)^2 \geq \frac{\left(\sum_{v \in V(G)} d^-(v)\right)^2}{n} = \frac{1}{4}d^2 n,$$

hence $pn \geq S \geq \frac{d(d-2)}{8}n$, and the inequality $p \geq \frac{d(d-2)}{8}$ follows. ∎

The graphs constructed in Theorems 3.11 and 3.12 cannot contain subdivisions of graphs with minimum degree greater than d. It follows that in Theorem 3.9, it is not possible to improve the bound for $\nabla_r(G)$ to $o(d^{r+1})$, and in Theorem 3.7, we must require the arrangeability to be at least $\Omega(d^2)$.

The gap between the bounds in the case of the acyclic chromatic number is much wider. The best lower bound we know is the following: The graph $K_{n,n}$ has acyclic chromatic number $n+1$, but each graph whose 1-subdivision is a subgraph of $K_{n,n}$ has chromatic number $O(\sqrt{n})$. This sharply contrasts with the exponential bound of Theorem 3.3. It would be interesting to decrease the upper bound or to find an example showing that an exponential bound is necessary.

3.1.5 Game Chromatic Number

Zhu and Dinsky [35] have proved that the game chromatic number of a graph is bounded by a function of the acyclic chromatic number, and conjectured that each graph with high acyclic chromatic number contains a subgraph with high game chromatic number, i.e., that the hereditary game chromatic number and the acyclic chromatic number are bounded by a function of each other. The consequence of Corollary 3.4 is that this conjecture is implied by the following statement:

Conjecture 3.1 *There exists a function f such that for each graph G, if $\chi(G) \geq f(c)$ then the game chromatic number of $\mathrm{sd}_1(G)$ is at least c.*

It is easy to show that the game chromatic number of $\mathrm{sd}_1(K_n)$ is at least $\log_4 n$. Rödl [88] has proved that a graph with large chromatic number contains a large clique or a triangle-free subgraph with large chromatic number. Therefore, it would suffice to prove the following equivalent claim:

Conjecture 3.2 *There exists a function f such that for each triangle-free graph G, if $\chi(G) \geq f(c)$ then the game chromatic number of $\mathrm{sd}_1(G)$ is at least c.*

There are known examples of graphs with acyclic chromatic number 3 and arbitrarily large game coloring number – Kierstead and Trotter [54] have proved that the game coloring number of $\mathrm{sd}_1(K_{n,n})$ is $\theta(\log n)$. On the other hand, the game coloring number is bounded by the arrangeability of a graph. It is natural to conjecture the following:

Conjecture 3.3 *There exists a function f such that for each graph G, if G is not $f(c)$-arrangeable then the game coloring number of G is at least c.*

By Theorem 3.7, the equivalent statement is that there exists a function f' such that $\delta(G) \geq f'(c)$ implies that the game coloring number of $\mathrm{sd}_1(G)$ is at least c.

3.2 Clique Subdivisions and Separators

We consider the problem outlined in Section 2.5: Do the graphs with large bounded-depth minor expansion contain large subgraphs that do not have small separators (i.e., have properties similar to expanders)? One notable example that we have mentioned are the graphs $\mathrm{sd}_{\log n}(K_n)$, that demonstrate that an exponential expansion in general cannot imply a subgraph without a separator of size $\Omega\left(\frac{N}{\log N}\right)$, where N is the number of the vertices of the subgraph. We show that this is the worst-case: We argue that every graph with exponential expansion contains a subgraph "similar" to $\mathrm{sd}_{\log n}(K_n)$, and hence also a subgraph without a separator of size $o\left(\frac{N}{\log N}\right)$.

3.2.1 Clique Subdivisions

We consider the existence of clique subdivisions in graphs with large minimum degree. A random graph on n vertices asymptotically almost surely does not contain a clique of size greater than $\Omega(\log n)$. On the other hand, it is easy to see that a.a.s., it contains $\mathrm{sd}_1(K_{\sqrt{n}})$ as a subgraph. Mader [63] and Erdős and Hajnal [40] have conjectured that there exists a constant c such that any graph with average degree cp^2 contains a subdivision of K_p. This conjecture was finally proved by Komlós and Szemerédi [56, 57] and Bollobás and Thomason [19]. A similar result holds for minors – a graph with average degree $\Omega(p\sqrt{\log p})$ contains K_p as a minor, by Kostochka [59] and Thomason [93].

Consider a graph G on n vertices with minimum degree n^ε, for some constant $0 < \varepsilon < 1$. If G is random, the expected diameter of G would be constant (dependent on ε), and it is easy to prove that it contains a subdivision of a large clique such that each edge is subdivided by a constant number of vertices. The question is, whether this claim holds in general. The proof of Bollobás and Thomason [19] uses an argument from that the lengths of the paths that replace the edges of the clique are not easy to derive. The proof of Komlós and Szemerédi [57] is more straightforward – it finds a subgraph of G that behaves as an expander, shows that each expander contains a subdivision of a graph with $O(d)$ vertices and $\Omega(d^2)$ edges (where d is the average degree of G), and in this dense graph finds a subdivision of a clique using Regularity Lemma. The edges of the clique are subdivided polylogarithmic number of times (this comes from the use of the expander to boost the degree).

Below, we derive a result that shows that a graph with minimum degree n^ε contains the c-subdivision of a clique on n^μ vertices as a subgraph, for some constant c and μ (depending only on ε). The constant μ is much smaller than $\frac{\varepsilon}{2}$ of the previous results, though. We use the following lemma to boost the exponent in the minimum degree:

Lemma 3.13 *For any ε $(0 < \varepsilon < 1)$, there exists n_0 such that every graph G on $n \geq n_0$ vertices with minimum degree at least n^ε contains as a subgraph $\mathrm{sd}_1(K_{n^{\varepsilon^3}})$ or the 1-subdivision of a graph with $n_1 = \Omega(n^{\varepsilon-\varepsilon^3})$ vertices and* *minimum degree at least* $d = \Omega\left(n_1^{\varepsilon+\varepsilon^2\frac{1-\varepsilon-\varepsilon^2}{1-\varepsilon+\varepsilon^3}}\right)$.

Proof: Let

$$n_0 = \max\left(40^{\frac{1}{\varepsilon-\varepsilon^3}}, \left(\frac{4}{3}\right)^{\frac{1}{1-\varepsilon+\varepsilon^3}}, \left(\frac{28}{3}\right)^{\frac{1}{\varepsilon^3}}\right),$$

and $p = 2n^{-\varepsilon+\varepsilon^3}$ (note that $p \le 1$). Let A be a subset of $V(G)$ obtained by taking each vertex randomly independently with probability p. The expected size of A is $pn = 2n^{1-\varepsilon+\varepsilon^3}$, and by Chernoff Inequality,

$$\text{Prob}\left[|A| \ge 2pn\right] < e^{-\frac{3}{8}pn} \le e^{-1} < \frac{1}{2},$$

since $n \ge \left(\frac{4}{3}\right)^{\frac{1}{1-\varepsilon+\varepsilon^3}}$.

Consider an arbitrary vertex $v \in V(G)$ of degree $d \ge n^\varepsilon$. The expected number of neighbors of v in A is $dp \ge 2n^{\varepsilon^3}$, and by Chernoff Inequality, the probability that the number of neighbors is at most n^{ε^3} is less than $e^{-\frac{3}{14}n^{\varepsilon^3}} \le \frac{1}{5}$, since $n \ge \left(\frac{28}{3}\right)^{\frac{1}{\varepsilon^3}}$. Let B' be the set of vertices in $V(G)$ that have at least n^{ε^3} neighbors in A. The expected value of the size $|V(G) \setminus B'|$ is at most $\frac{1}{5}n$, hence by Markov Inequality, $\text{Prob}\left[|V(G) \setminus B'| \ge \frac{2}{5}n\right] \le \frac{1}{2}$. Therefore, with nonzero probability, $|A| \le 4n^{1-\varepsilon+\varepsilon^3}$ and $|B'| \ge \frac{3}{5}n$; let us fix such a pair of sets, and set $B = B' \setminus A$. Note that $|B| \ge \frac{n}{2}$, since $n \ge 40^{\frac{1}{\varepsilon-\varepsilon^3}}$.

Let us form a graph G' with the vertex set A, whose edges correspond to the vertices of B: We order the vertices of B arbitrarily, and for each vertex $v \in B$, if $N(v) \cap A$ is not yet a clique in G', we join two nonadjacent vertices in $N(v) \cap A$. In the end, if there exists a vertex $v \in B$ such that $N(v) \cap A$ is a clique in G', then G contains $\text{sd}_1(K_{n^{\varepsilon^3}})$ as a subgraph. Otherwise, G' (whose 1-subdivision is a subgraph of G) has $|B| \ge \frac{n}{2}$ edges and $|A| \le 4n^{1-\varepsilon+\varepsilon^3}$ vertices, hence its average degree is at least $\frac{1}{4}n^{\varepsilon-\varepsilon^3}$. The graph G' contains a subgraph G'' whose minimum degree is at least $d_1 = \frac{1}{8}n^{\varepsilon-\varepsilon^3}$.

Let $n_1 = |V(G'')|$. Note that $\frac{1}{8}n^{\varepsilon-\varepsilon^3} \le n_1 \le 4n^{1-\varepsilon+\varepsilon^3}$, in particular, $n = \Omega\left(n_1^{\frac{1}{1-\varepsilon+\varepsilon^3}}\right)$. Expressing d_1 relatively to n_1, we obtain

$$d_1 = \Omega\left(n_1^{\frac{\varepsilon-\varepsilon^3}{1-\varepsilon+\varepsilon^3}}\right) = \Omega\left(n_1^{\varepsilon+\varepsilon^2\frac{1-\varepsilon-\varepsilon^2}{1-\varepsilon+\varepsilon^3}}\right),$$

hence G'' satisfies the conditions of the lemma. ∎

Note that if $\varepsilon \leq \frac{\sqrt{5}-1}{2}$, then the application of Lemma 3.13 increases the exponent in the expression for the minimum degree. The same technique can be used to show that for $\varepsilon > 0.5$, the graph with minimum degree n^ε contains the 1-subdivision of a large clique:

Lemma 3.14 *There exists n_0 such that every graph G with $n \geq n_0$ vertices and minimum degree at least $4n^{0.6}$ contains $\mathrm{sd}_1(K_{n^{0.1}})$ as a subgraph.*

Proof: The proof proceeds analogically to the proof of Lemma 3.13. We set $p = \frac{1}{2}n^{-\frac{1}{2}}$, and choose elements of A randomly independently with probability p. With high probability, the set A has at most $n^{\frac{1}{2}}$ elements. We let B' be the set of vertices that have at least $n^{0.1}$ neighbors in A, and show that with high probability, $|B'| \geq \frac{3}{5}n$. Then, we choose the set A of size at most $n^{\frac{1}{2}}$ and a set $B \subseteq B'$ disjoint with A such that $|B| \geq \frac{n}{2}$.

We form a graph G' with the vertex set A, whose edges correspond to the vertices of B: For each vertex $v \in B$ such that $N(v) \cap A$ is not yet a clique in G', we add an edge between two nonadjacent vertices in $N(v) \cap A$. If there exists a vertex $v \in B$ such that $N(v) \cap A$ is a clique in G', then G contains $\mathrm{sd}_1(K_{n^{0.1}})$ as claimed. However, if this were not the case, then G' would be a simple graph with at most $n^{\frac{1}{2}}$ vertices and at least $\frac{n}{2}$ edges, which is a contradiction. We conclude that G must contain the 1-subdivision of a clique with $n^{0.1}$ vertices as a subgraph. ∎

The main result is a simple consequence of Lemmas 3.13 and 3.14:

Theorem 3.15 *For each ε $(0 < \varepsilon \leq 1)$ there exist integers n_0 and c_0 and a real number $\mu > 0$ such that every graph G with $n \geq n_0$ vertices and minimum degree at least n^ε contains $\mathrm{sd}_c(K_{n^\mu})$ as a subgraph, for some $c \leq c_0$.*

Proof: If $\varepsilon > 0.6$, then the statement follows from Lemma 3.14. Consider the case $\varepsilon \leq 0.6$. We apply Lemma 3.13 repeatedly to increase the exponent – note that if $\mathrm{sd}_a(G_1)$ is a subgraph of G and $\mathrm{sd}_b(G_2)$ is a subgraph of G_1, then $\mathrm{sd}_{(a+1)(b+1)-1}(G_2)$ is a subgraph of G. On the interval $\langle 0, 0.6\rangle$, the function $\frac{1-\varepsilon-\varepsilon^2}{1-\varepsilon+\varepsilon^3}$ is greater than 0.06, hence each application of Lemma 3.13 increases the exponent by at least $0.06\varepsilon^2$. After at most $\frac{10}{\varepsilon^2}$ applications of Lemma 3.13, the exponent exceeds 0.6, and we apply Lemma 3.14. Each application of Lemma 3.13 decreases the size of the graph only by a polynomial factor and we need to repeat the lemma only a constant number of times (dependent on ε), hence we can set n_0 big enough so that the assumptions of Lemmas 3.13 and 3.14 are satisfied. It also follows that the size of the clique we find is $\Omega(n^\mu)$ for some constant μ dependent on ε. The edges of this clique are subdivided at most $c_0 = 2^{\frac{10}{\varepsilon^2}+1} - 1$ times. ∎

3.2.2 Expander-like Subgraphs

Let us now apply Theorem 3.15 on the problems regarding graphs with the exponential expansion.

Theorem 3.16 *For any real number ε ($0 < \varepsilon \leq 1$) and an integer $k > 0$, there exist constants n_0 and $\mu > 0$ such that if a graph G with $n \geq n_0$ vertices satisfies $\nabla_{k \log n}(G) = \Omega(n^\varepsilon)$, then G contains K_{n^μ} as a minor of depth $O(\log n)$.*

Proof: Since $\nabla_{k \log n}(G) = \Omega(n^\varepsilon)$, G contains a minor G' of depth $k \log n$ with minimum degree $\Omega(n^\varepsilon)$. The graph G' has at most n vertices, hence by Theorem 3.15, there exist constants $\mu > 0$ and c_0 (depending only on ε) such that G' contains $\mathrm{sd}_c(K_{n^\mu})$ as a subgraph, for some $c \leq c_0$. It follows that G' contains K_{n^μ} as a minor of depth at most $(c+1)k \log n \leq (c_0+1)k \log n = O(\log n)$. ∎

If G has K_{n^μ} as a minor of depth $c \log n$, then $\nabla_{c \log n}(G) \geq n^\mu$, hence G has exponential expansion if and only if it contains a minor of K_{n^μ} of logarithmic depth. We use this fact to provide a counterpart to Theorem 2.11.

For a graph G and a set $U \subseteq V(G)$, let ∂U be the number of edges between U and $V(G) \setminus U$. We use the following (special case of) theorem proved by Bollobás [18]:

Theorem 3.17 (Bollobás [18]) *A random 3-regular graph G on n vertices a.a.s. satisfies $\partial U \geq \frac{3}{20}|U|$ for every $U \subseteq V(G)$ such that $|U| \leq \frac{n}{2}$.*

Let $w : E(G) \to R^+$ be any function that assigns a positive number to each edge of G. For $H \subseteq G$, let

$$w(H) = |V(H)| + \sum_{\{u,v\} \in E(G), \{u,v\} \cap V(H) \neq \emptyset} w(\{u,v\}).$$

Lemma 3.18 *Let G be a 3-regular graph G on n vertices such that $\partial U \geq \frac{3}{20}|U|$ for every $U \subseteq V(G)$ with $|U| \leq \frac{n}{2}$. Let $w : E(G) \to R^+$ be any function that assigns a positive number to each edge of G. Let $W = w(G)$ and $M = \max_{e \in E(G)} w(e)$. If $S \subseteq V(G)$ is a set such that each component H of $G - S$ satisfies $w(H) \leq \frac{2}{3}W$, then $|S| \geq \frac{W}{63(3M+1)}$.*

Proof: Let H_1, H_2, …, H_k be the components of $G - S$. By the assumptions, $w(H_i) \leq \frac{2}{3}W$ for each $i = 1, \ldots, k$. Let $s = |S|$. The subgraph of G induced by S has at most $\frac{3}{2}s \leq 3s$ edges, hence $\sum_{i=1}^{k} w(H_i) \geq W - (3M+1)s$.

Observe that the graphs H_i can be partitioned into two subgraphs G_1 and G_2 such that $\min(w(G_1), w(G_2)) \geq \frac{W}{3} - (3M+1)s$. At least one of the subgraphs (say G_1) contains at most $\frac{n}{2}$ vertices. Let $n_1 = |V(G_1)|$. Since $n_1 \leq \frac{n}{2}$, there are at least $\frac{3}{20}n_1$ edges between G_1 and S. The graph G is 3-regular, thus $s \geq \frac{n_1}{20}$. On the other hand, the number of the edges incident with the vertices of G_1 is at most $3n_1$, hence $w(G_1) \leq (3M+1)n_1$. It follows that $s \geq \frac{n_1}{20} \geq \frac{w(G_1)}{20(3M+1)} \geq \frac{W/3-(3M+1)s}{20(3M+1)}$. Therefore, $s \geq \frac{W}{63(3M+1)}$. ∎

We can now show that the graph with exponential expansion contains a subgraph without a small separator:

Theorem 3.19 *For each ε ($0 < \varepsilon < 1$) and each integer $k > 0$, there exist constants $\mu > 0$ and n_0 such that if G is a graph on $n \geq n_0$ vertices with $\nabla_{k \log n}(G) \geq n^\varepsilon$, then G contains a subgraph H on $N \geq n^\mu$ vertices such that each separator in H has size $\Omega\left(\frac{N}{\log N}\right)$.*

Proof: By Theorem 3.16, there exists a constant μ such that if G is sufficiently large, it contains K_{n^μ} as a minor of depth $O(\log n)$. By Theorem 3.17, there exists a 3-regular graph H' on $n'' = n^\mu$ vertices such that $\partial U \geq \frac{3}{20}|U|$ for every $U \subseteq V(H')$ with $|U| \leq \frac{n''}{2}$. The graph G contains H' as a minor of depth $O(\log n)$. Since H' is 3-regular, a graph H that is a $\leq O(\log n)$-subdivision of H' is a subgraph of G. Let $N = |V(H)|$.

Let us define a function $w : E(H') \to R^+$ by setting $w(e)$ to be the number of vertices of the path that replaces e in H, for each $e \in E(H')$. Note that $w(H') = N$ and $M = \max_{e \in E(H')} w(e) = O(\log n) = O(\log N)$. Let S be a minimum separator in H. Suppose that a vertex $v \in S$ has degree two, and let v_1 and v_2 be the vertices of degree three such that v belongs to the path that replaces the edge $\{v_1, v_2\}$ of H'. Then, assuming that n is large enough, $S \setminus \{v\} \cup \{v_1\}$ or $S \setminus \{v\} \cup \{v_2\}$ is a separator in H, hence without loss on generality, only vertices of degree three belong to S. Let F be a component of $H - S$ and let F' be the corresponding component of $H' - S$. Since S is a separator in H, $w(F') = |V(F)| \leq \frac{2}{3}N$, hence we can apply Lemma 3.18 and conclude that $|S| = \Omega\left(\frac{N}{M}\right) = \Omega\left(\frac{N}{\log N}\right)$. Therefore, H does not have small separators. ∎

We get the following corollary:

Corollary 3.20 *If $\mathcal{G}$ is a class of graphs closed on taking subgraphs such that each graph in $\mathcal{G}$ on n vertices contains a separator of size $o\left(\frac{n}{\log n}\right)$, then $\mathcal{G}$ has subexponential expansion.*

Chapter 4

Algorithmic Aspects

Given the algorithmic applications of graphs with bounded expansion, the question whether it is possible to determine the greatest reduced average densities of a graph G quickly deserves a close attention.

In the algorithmic setting, it is more natural to study the complexity of the problem of determining $\nabla_r^d(G)$ for various graph classes. As mentioned before, $\nabla_r^d(G)$ is a good approximation of $\nabla_r(G)$, and it appears more tractable – not only because $\nabla_0^d(G)$ is much easier to determine than $\nabla_0(G)$, but also because the values of $\nabla_r^d(G)$ are integers, whereas $\nabla_r(G)$ may be an arbitrary nonnegative rational number. Also, unlike the average degree, the minimum degree can be verified locally. Nevertheless, we derive some results regarding determining $\nabla_r(G)$ as well.

It is a well-known fact that the greedy algorithm based on the following formula can be used to determine $\nabla_0^d(G)$: Let v be a vertex of minimum degree d in G. Then $\nabla_0^d(G) = \max(d, \nabla_0^d(G-v))$. Therefore, it is possible to determine $\nabla_0^d(G)$ in linear time. It is also possible (but much more complicated) to determine $\nabla_0(G)$ in polynomial time using the matroid partition algorithm of Edmonds [37].

Let us also note that problems of determining whether $\nabla_r(G) \geq x$ and $\nabla_r^d(G) \geq x$ are in NP – after guessing the rank r contraction, we verify that the maximum average degree or the degeneracy of the contraction is at least x. We could in fact avoid using the algorithms for maximum average degree and degeneracy, by guessing the subgraph on that the maximum is achieved as well.

In the rest of this chapter, we first show that for any $r \geq 1$, the problems of determining $\nabla_r^d(G)$ and $\nabla_r(G)$ are NP-complete, even if restricted to a class of graphs with bounded maximum degree. Then, we show several classes of graphs for that the problems can be solved in polynomial time, and present an approximation algorithm for general graphs. The existence of the approximation algorithm also has theoretical consequences, discussed later in Section 5.2.

4.1 NP-completeness

Bodlaender et al. [17] have studied the problem of determining the maximum degeneracy over all contractions of a graph, i.e., determining $\nabla^d_{|V(G)|}(G)$. They proved that this problem is NP-complete for general graphs and noted that $\nabla^d_{|V(G)|}(G) \le \mathrm{tw}(G)$. They also showed that for a fixed constant x, it is possible to determine whether $\nabla^d_{|V(G)|}(G) \ge x$ in cubic time: The set of graphs such that $\nabla^d_{|V(G)|}(G) < x$ forms a proper minor-closed class, hence we can use the results of Robertson and Seymour [85, 87] and verify this property by checking a finite number of forbidden minors. Also, since $\nabla^d_{|V(G)|}(G)$ is bounded by a constant for any proper minor-closed class, it follows that $\nabla^d_{|V(G)|}(G)$ can be determined in a polynomial time for graphs in any such class.

Note that this is not necessarily the case for fixed r – the graphs with $\nabla^d_r(G) < x$ or $\nabla_r(G) < x$ do not form a minor-closed class, and we show that determining whether $\nabla^d_r(G) \ge 4$ is NP-complete. Nevertheless, we do not know whether the problem is NP-complete for graphs in some proper minor-closed class.

The following theorems are simple variations of the construction of Bodlaender et al. [17]:

Theorem 4.1 *For any $r \ge 1$, the problem of determining whether $\nabla^d_r(G) \ge x$ (where the input consists of both G and x) is NP-complete.*

Proof: It suffices to show that the problem is NP-hard. We proceed by a transformation from Vertex Cover Problem. The instance of Vertex Cover Problem consists of a graph G and an integer $k \le |V(G)|$, and the question is whether there exists a set $W \subseteq V(G)$ such that $|W| = k$ and every edge of G is incident with at least one vertex in $|W|$. This problem is NP-complete, see e.g. Garey and Johnson [42].

Let G be an arbitrary graph on n vertices and k an integer such that $k \le n$. For an integer t, the graph G^k_t is obtained by the following construction: We take the complement $\overline{G}$ of G, a set U of k vertices $u_1, \ldots, u_k$, and a set of t vertices $w_1, \ldots, w_t$. We add all edges of the clique on the vertices $w_1, \ldots, w_t$, and join each vertex $u_1, \ldots, u_k$ and $w_1, \ldots, w_t$ with each vertex of $\overline{G}$ by an edge, see Figure 4.1.

Let us show that $\nabla^d_r(G^k_2) \ge n + 1$ if and only if G has a vertex cover of size at most k.

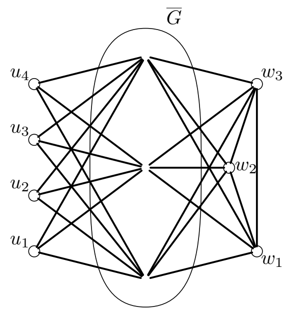

Figure 4.1: The construction of the graph G_t^k (for $k = 4$ and $t = 3$).

Suppose first that $\nabla_r^d(G_2^k) \geq n+1$, and let H be the corresponding minor of G_2^k with minimum degree at least $n+1$. Let X be the set of edges that are contracted and R the set of vertices that are removed from G_2^k to obtain H. The degree of each vertex of U in G_2^k is n, hence each such vertex either belongs to R or is incident to an edge in X. It follows that H has at most $n+2$ vertices, hence H must be a complete graph on $n+2$ vertices. In particular, only the edges between vertices of U and the vertices of $\overline{G}$ may belong to X. Note that we may assume that $R = \emptyset$ and that each vertex of U is incident with exactly one edge of X. For each $i = 1, \ldots, k$, let s_i be the vertex of $\overline{G}$ such that the edge $\{s_i, u_i\}$ belongs to X. Let $S = \{s_i | i = 1, \ldots, k\}$. Let $e = \{v_1, v_2\}$ be an edge of G. The pair $\{v_1, v_2\}$ is a non-edge of $\overline{G}$, but the corresponding vertices must be joined by an edge in H. Therefore, at least one of the vertices is incident with an edge of X and belongs to S. This shows that S is a vertex cover of G of size at most k.

On the other hand, suppose that G has a vertex cover S such that $|S| \leq k$. We may add vertices to the vertex cover, hence we may assume that $|S| = k$. Let s_1, s_2, …, s_k be the vertices of G_2^k corresponding to the vertices of S. We contract all the edges $\{u_i, s_i\}$ ($i = 1, \ldots, k$). Note that the graph obtained in this way is a complete graph on $n+2$ vertices – a non-edge in $\overline{G}$ corresponds to an edge in G, hence each non-edge in $\overline{G}$ contains at least one vertex s_i of S, and the vertex u_i identified with s_i is adjacent to all vertices of $\overline{G}$. Therefore, $\nabla_r^d(G_2^k) \geq n+1$.

We showed that $\nabla_r^d(G_2^k) \geq n+1$ if and only if G has a vertex cover of size at most k. Since the graph G_2^k can be constructed in polynomial time, it follows that the problem of determining whether $\nabla_r^d(\cdot) \geq x$ is NP-complete. ∎

A similar result can be showed for $\nabla_r(\cdot)$:

Theorem 4.2 *For any $r \geq 1$, the problem of determining whether $\nabla_r(G) \geq x$ is NP-complete.*

Proof: It suffices to show that the problem is NP-hard. We again exhibit a transformation from Vertex Cover Problem. We show that $\nabla_r(G^k_{n+1}) \geq n$ if and only if G has a vertex cover of size at most k. If G has a vertex cover of size k, we find K_{2n+1} (whose average density is n) as a rank r contraction of G^k_{n+1} analogically to Theorem 4.1.

On the other hand, let us assume that $\nabla_r(G^k_{n+1}) \geq n$, and let H be the corresponding minor of G^k_{n+1} with average degree at least $2n$. Let X be the set of edges that are contracted and R the set of vertices that are removed from G^k_{n+1} to obtain H. We may assume that the minimum degree of H is greater than n, since removing the vertex of degree at most n does not decrease the average degree of H. Therefore, each vertex of U either belongs to R or is incident with an edge in X. It follows that H has at most $2n+1$ vertices, hence $H = K_{2n+1}$. The rest of the proof is identical to the proof of Theorem 4.1 – we may assume that exactly one edge incident to each vertex in U is contracted, and the matched vertices of $\overline{G}$ form a vertex cover in G.

Since the graph G^k_{n+1} can be constructed in polynomial time, it follows that the problem of determining whether $\nabla_r(G) \geq x$ is NP-complete. ∎

Since the problem is NP-complete for general graphs, it makes sense to consider its restrictions to smaller classes of graphs. The following lemma presents a transformation for the problem of determining $\nabla^d_r(\cdot)$ from the problem of determining $\nabla^d_1(\cdot)$ that shows that for graph classes closed on subdivision of an edge, determining $\nabla^d_r(\cdot)$ is at least as hard as determining $\nabla^d_1(\cdot)$, for any $r \equiv 1 \pmod 3$.

Lemma 4.3 *Let G be a graph with minimum degree at least three and $t \geq 0$ an integer, and let $G_1 = \mathrm{sd}_{2t}(G)$. Then, $\nabla^d_{3t+1}(G_1) = \nabla^d_1(G)$.*

Proof: Let S be a star forest in G and X the set of edges of G that are incident with two different trees of S, and let G' be the minor of G whose vertices correspond to the trees of S and the edges to the edges of X. For each edge $e \in E(G)$, let P_e be the corresponding path with $2t$ inner vertices of degree two in G_1.

We construct a forest S_1 in G_1 in the following way: We subdivide the edges of each star of S by $2t$ vertices, and for each edge $e = \{x, y\} \in X$, attach half of the path P_e to each of the trees that contain vertices x and y. The minor of G_1 obtained by contracting the trees in S_1 is equal to G', hence $\nabla^d_1(G) \leq \nabla^d_{3t+1}(G_1)$.

Let us now show the reverse inequality. Let S_1 be a forest with trees of depth at most $3t+1$ in G_1. Note that $\nabla^d_{3t+1}(G_1) \geq 3$, hence we may assume that the minor H obtained by contracting S_1 has minimum degree at least three. For each tree $T \in S_1$, let T' be the corresponding tree in G obtained by suppressing all the vertices of T that have degree two in G_1. Observe that T' is a star: Suppose that v is a center of T, and let v' be the vertex of G_1 with degree greater than two that is nearest to v. Then, all the vertices of T of degree greater than two in G_1 are neighbors of v' in G.

Let S be the set of all the stars T' for $T \in S_1$. Contracting the stars in S induces a minor that is a supergraph of H, hence $\nabla_1^d(G) \geq \nabla_{3t+1}^d(G_1)$. Consequently, $\nabla_1^d(G) = \nabla_{3t+1}^d(G_1)$ ∎

Let us now consider the problem of determining whether $\nabla_1^d(G) \geq x$, for x fixed. Unlike the case studied by Bodlaender et al. [17], the class of graphs such that $\nabla_1^d(G) < x$ is not minor-closed, hence we do not obtain an algorithm easily. In fact, it turns out that the problem is NP-complete for $x \geq 4$, and Lemma 4.3 implies that determining whether $\nabla_{3t+1}^d(G) \geq x$ is also NP-complete, for any integer $t > 0$. For $x = 2$, the problem is trivial – $\nabla_r^d(G) \geq 2$ if and only if G contains a cycle. We do not know what is the complexity of the question whether $\nabla_r^d(G) \geq 3$.

To prove the NP-hardness of determining whether $\nabla_1^d(G) \geq 4$, we first consider the following (a bit artificial looking) problem that we call Double Matching Cover: The instance of the problem consists of a cubic graph G together with a partial matching M (called the *input matching*) of G. The question is whether there exists a perfect matching X of G such that for every edge $e = \{u, v\} \in E(G) \setminus (X \cup M)$, at least one of the vertices u or v is incident with an edge belonging to $X \cap M$. We call X the *solution matching*. If M were a perfect matching, $X = M$ would obviously satisfy the conditions of the problem specification, however we show that in general, the problem is hard. The proof uses a reduction from Exact 3-Set Cover Problem: the instance of the problem consists of a set Q and a set B of 3-element subsets of Q such that every element of Q belongs to exactly three sets in B. The question is whether there exists $Y \subseteq B$ such that every element of Q belongs to exactly one set in Y. This problem is NP-complete (see Garey and Johnson [42]).

In the proof, we need to construct several gadgets. A *gadget* is a graph G' with a partial matching M' and several half-edges that are used to connect it with other gadgets (in our constructions, the connecting half-edges never belong to the input matching). When we construct a gadget, we also describe the possible solution matchings restricted to G', and especially concentrate on their restrictions to the connecting half-edges. There are three possible states a half-edge e may have with respect to the solution matching X: it may be a *solution half-edge*, i.e., $e \in X$. Or, it may be a *covered half-edge*, i.e., $e \notin X$, but the vertex of e that belongs to G' is incident with an edge

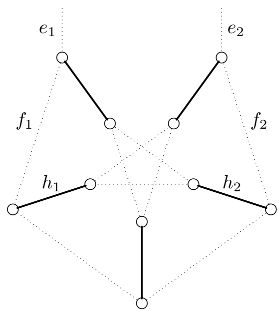

Figure 4.2: The gadget COVERED.

in $X \cap M'$. Finally, the half-edge may be *isolated*, i.e., it does not belong to $X \cup M'$, and it is not incident with an edge in $X \cap M'$.

If (G'_1, M'_1) and (G'_2, M'_2) are two gadgets connected by an edge e, and X_1 and X_2 are the restrictions of a solution matching X to G_1 and G_2, then the half-edges corresponding to e may have one of the following states in the restricted solutions X_1 and X_2: Both of them are solution half-edges, or both of them are covered, or one of them is covered and the other one is isolated. In particular, it is not possible for both of them to be isolated.

Let us now construct several useful gadgets. In the figures of the gadgets, we draw the edges belonging to the input matching M' by solid lines and the remaining edges by dotted lines. Note that the girth of all the gadgets is at least five.

- The gadget COVERED(e_1, e_2) with two half-edges e_1 and e_2 (Figure 4.2) is obtained from the Petersen graph by splitting one edge and putting all the edges between the inner and the outer cycle to the input matching M. Since this gadget has even number of vertices, either both e_1 and e_2 belong to the solution matching or neither of them does. However, it is not possible for both of them to be solution edges – if that were the case, f_1 and f_2 would have to be covered, hence h_1 and h_2 would have to belong to the solution matching, and the matching cannot be extended to the remaining four vertices. On the other hand, we may set $X = M$ and make both half-edges e_1 and e_2 covered.

- The gadget COPY(e_1, e_2) with two half-edges e_1 and e_2 (Figure 4.3) is obtained from COVERED(f_1, f_2) by adding the end vertices v_1 and v_2 of f_1 and f_2, joining them by an edge f belonging to the input matching, and adding the half-edges e_1 and e_2 incident with v_1 and v_2. The solution matching restricted to COPY(e_1, e_2) has either both e_1 and e_2 as solution half-edges, or f as a solution edge and both e_1 and e_2 as covered half-edges.

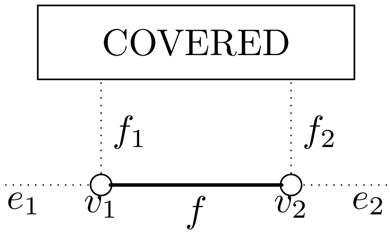

Figure 4.3: The gadget COPY.

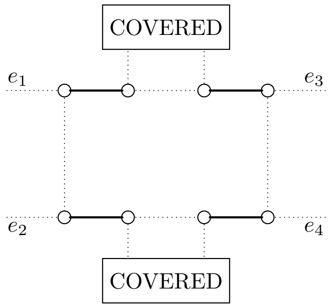

Figure 4.4: The gadget PXOR.

- The gadget PXOR(e_1, e_2, e_3, e_4) is depicted in Figure 4.4. In the restriction of the solution to the gadget, there are the following possible states of the half-edges:

  - e_1 and e_3 are solution half-edges, e_2 and e_4 are covered.
  - e_2 and e_4 are solution half-edges, e_1 and e_3 are covered.
  - All the half-edges are covered.
  - All the half-edges are isolated.

- The gadget XOR(e_1, e_2, e_3, e_4) is depicted in Figure 4.5. We claim that in any solution matching X, either e_1 and e_4 are solution edges and e_2 and e_3 are covered, or vice versa. For both of these choices, it is easy to construct the appropriate restriction of the solution matching to XOR(e_1, e_2, e_3, e_4), hence it suffices to show that there are no other possibilities. By the properties of PXOR, at most one of e_1 and e_2 belongs to the solution, and similarly at most one of e_3 and e_4 does.

  Consider first the case that e_1 belongs to the solution matching X. Then, f_1, f_4 and e_4 belong to X as well, and e_2, f_2, f_3 and e_3 are covered.

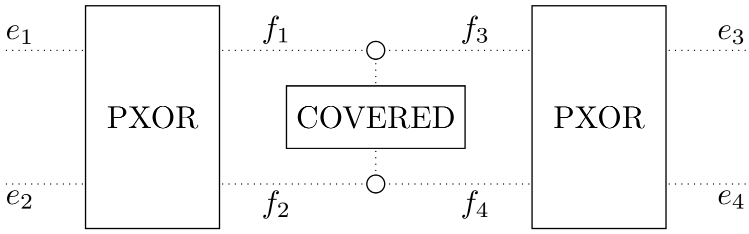

Figure 4.5: The gadget XOR.

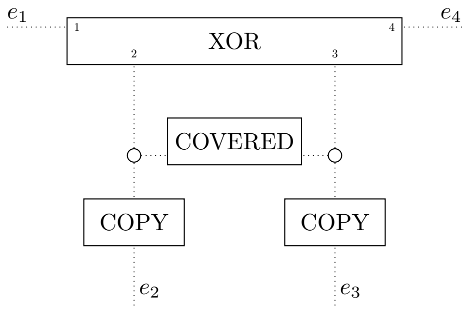

Figure 4.6: The gadget SPLIT.

By symmetry, it remains to consider the case that neither e_1 nor e_2 belong to the solution matching. By a parity argument and the properties of PXOR, neither e_3 nor e_4 belong to the solution matching in this case. However, that means that none of f_1, f_2, f_3 and f_4 belong to X. This is a contradiction, since X is a perfect matching.

- The gadget SPLIT(e_1, e_2, e_3, e_4) is depicted in Figure 4.6 (note the ordering of the connecting edges of the XOR gadget). In any solution matching, either all the half-edges e_1, e_2, e_3 and e_4 belong to the solution, or all of them are covered.

- The gadget ONE(e_1, e_2, e_3) (Figure 4.7) consisting of exactly one vertex incident with three half-edges has exactly one of the half-edges in the solution and all the remaining ones isolated.

Let us prove the auxiliary hardness result regarding Double Matching Cover Problem:

Lemma 4.4 *Double Matching Cover Problem is NP-hard, even when restricted to graphs of girth five.*

Proof: We show a reduction from Exact 3-Set Cover Problem, let (Q, B) be its instance. Using the gadgets we constructed, the transformation proceeds

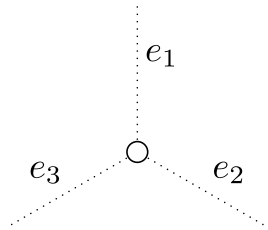

Figure 4.7: The gadget ONE.

in the obvious manner – we use a copy C_b of the COPY gadget for each element $b \in B$ to choose whether it is included in the cover (the half-edges of the gadget are solution edges) or not (the half-edges are covered), a copy O_q of the ONE gadget for each element $q \in Q$ to ensure that exactly one of the sets it belongs to is chosen to the cover, and the SPLIT gadgets to join the half-edges of O_q with the gadgets C_b for the sets b such that $q \in b$, thus obtaining an instance (G, M) of Double Matching Cover Problem. The minor complication arises from the parity constraints – the gadgets C_b used to represent the sets $b \in B$ cannot have only one output edge – hence we need two copies of the graphs O_q and the connecting SPLIT gadgets. Nevertheless, the size of the construction is linear in $|Q| + |B|$, and the properties of the gadgets imply that the exact set covers of (Q, B) correspond to the solution matchings in (G, M) and vice versa. The construction of the gadgets also ensures that G does not contain any cycles shorter than five, hence the claim of the lemma follows.

For the sake of completeness, let us describe the construction of (G, M) and the bijection between the solutions of the two problems more precisely: The graph G (and the input matching M) consists of

- the copies C_b of the COPY gadget with half-edges e_b and e'_b for each set $b \in B$, and
- the copies O_q and O'_q of the ONE gadget with half edges $f_{q,1}, f_{q,2}, f_{q,3}$ and $f'_{q,1}, f'_{q,2}, f'_{q,3}$, for each $q \in Q$, and
- the copies S_b and S'_b of the gadgets SPLIT with the half-edges g_b, $h_{b,1}$, $h_{b,2}$, $h_{b,3}$ and g'_b, $h'_{b,1}$, $h'_{b,2}$, $h'_{b,3}$, for each $b \in B$.

For each $q \in Q$, let b_1^q, b_2^q and b_3^q be the three sets in B that contain Q. For each $b \in B$, let us assign numbers 1, 2 and 3 to its elements in an arbitrary order, and for $q \in b$, let $\mathrm{idx}(b, q)$ be the number of the element q in the set b. The half-edges are connected in the following way: We connect e_b with g_b and e'_b with g'_b for each $b \in B$. For each $q \in Q$ and $b \in B$ such that

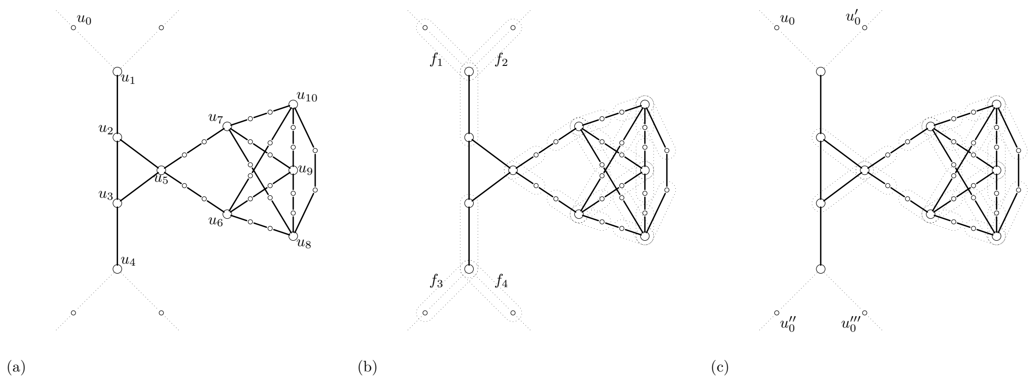

Figure 4.8: The replacement of an edge in the input matching.

$q \in b$, if i is the integer such that $b_i^q = b$ and $j = \mathrm{idx}(b, q)$, then we connect $h_{b,j}$ with $f_{q,i}$ and $h'_{b,j}$ with $f'_{q,i}$.

Let $Y \subseteq B$ be the solution of the exact set cover instance (Q, B). We construct the solution matching in (G, M) by including all the edges incident to S_b and S'_b for $b \in Y$, and extending this matching to all the gadgets. Properties of the gadgets and of the exact set cover ensure that this is possible.

On the other hand, if X is a solution matching in (G, M), we let Y consist of the sets $b \in B$ such that the edges incident to S_b are in X (the properties of the SPLIT gadget ensure that either all of them belong to X or none does). The properties of the ONE gadget imply that Y is an exact set cover of (Q, B).

We showed that (G, M) has a solution matching if and only if (Q, B) has an exact set cover, hence Double Matching Cover Problem is NP-hard. ∎

We are now ready to prove the result regarding the hardness of determining $\nabla_1(\cdot)$ in the fixed parameter case:

Theorem 4.5 *The problems of determining whether $\nabla_1^d(\cdot) \geq 4$ and $\nabla_1(\cdot) \geq 2$ are NP-complete.*

Proof: We show a reduction from Double Matching Cover Problem. Let G be a cubic graph of girth five and M a partial matching in G. We construct a graph G' by subdividing each edge of $E(G)\backslash M$ by one vertex and replacing all edges $e = \{x, y\}$ of M by copies U_e of the graph U depicted in Figure 4.8(a), where x is identified with u_1 and y with u_4. We need to show that $\nabla_1^d(G') \geq 4$ if and only if there exists a solution matching X in (G, M).

Suppose first that X is a solution matching in (G, M). We construct the minor of G' of depth one and minimum degree 4 by:

- For all edges $e = \{v_1, v_2\} \in X \setminus M$, identifying the vertex that subdivides e in G' with the vertices v_1 and v_2, and
- for all edges $e \in X \cap M$, contracting the edges of U_e that are emphasized in Figure 4.8(b), except possibly for the edges f_1, f_2, f_3 and f_4, in case the vertices of degree two incident with them were already identified with vertices in another copy of U, and
- for all edges $e \in M \setminus X$, contracting the edges of U_e emphasized in Figure 4.8(c). Note that in this case, one of u_0 and u_0', and one of u_0'' and u_0''' is a center of a star.

These rules ensure that every vertex of degree two is suppressed and each vertex of degree three is identified with one other vertex of degree three. Since the girth of G is five, the vertices created this way have degree four. An inspection of Figures 4.8(b) and (c) shows that the minimum degree of all vertices inside the copies of U is four as well. Therefore, if (G, M) has a solution matching, then $\nabla_1^d(G') \geq 4$.

Let us prove the reverse implication. Suppose that H is a minor of G' of depth one and minimum degree at least 4. We let X be the set of edges $e = \{u, v\}$ of G such that

- $e \notin M$, and $\mathrm{repr}(G', H, u) = \mathrm{repr}(G', H, v)$, or
- $e \in M$, and the edges of U_e were contracted in the way depicted in Figure 4.8(b).

We need to show that X is a solution matching. We first show that no vertex of G' is removed during the construction of H, i.e., that every vertex $v \in V(G')$ belongs to the subgraph $\mathrm{sgofv}(G', H, v')$ for some $v' \in V(H)$.

Let us for contradiction assume that this is not the case – let $v \in V(G')$ be a removed vertex. Since G' is connected, we may assume that v is adjacent to a vertex $u \in V(G')$ such that $u \in \mathrm{sgofv}(G', H, u')$ for a vertex $u' \in V(H)$, and that the degree of u in G' is at least three (otherwise we could remove u as well, without affecting the degree of any vertex of H). Let $W = V(\mathrm{sgofv}(G', H, u'))$. Since $d(u') \geq 4$ and $\Delta(G') = 4$, the set W must contain at least three vertices of degree ≥ 3, or one vertex of degree ≥ 3 and one vertex of degree four. Also, as the depth of H is at most one, the vertices of W form a star in G'. These two conditions can only be satisfied if all the vertices of W of degree ≥ 3 belong to one of the copies of U. We label the vertices of this copy in the way depicted in Figure 4.8(a). Note that at least one of u_1 and u_4 is not removed, as $U - \{u_1, u_4\}$ does not have a minor of depth 1 and minimum degree 4. Let us assume that u_1 is not removed.

If u_5 were removed, then the degree of u_2 and u_3 is at most two, hence both of them must be suppressed or removed. Since the degree of $\operatorname{repr}(G', H, u_1)$ is at least four, u_2, u_3 and u_4 are not removed. But it is impossible to suppress the vertices u_2 and u_3 and at the same time make the degree of both $\operatorname{repr}(G', H, u_1)$ and $\operatorname{repr}(G', H, u_4)$ at least four, thus u_5 cannot be removed. Similarly, u_5 must be the center of $\operatorname{sgofv}(G', H, u_5)$, otherwise the vertices u_6 and u_7 would have to be removed, which leads to a contradiction. Also, an analogical argument shows that the vertices $u_6, \ldots, u_{10}$ are not removed.

Neither u_2 nor u_3 can be removed, since the degree of $\operatorname{repr}(G', H, u_5)$ is four. If u_4 were removed, u_3 would have to be suppressed, but then the degree of $\operatorname{repr}(G', H, u_5)$ would be at most three. We conclude that no vertex of U is removed, and we may assume that the removed vertex is u_0. Let us consider the set W. Obviously, $\{u_1, u_2\} \subseteq W$, and W must have at least three elements. Since u_5 is a center of $\operatorname{sgofv}(G', H, u_5)$, it cannot belong to W, hence $W = \{u_1, u_2, u_3\}$. This is a contradiction, since in that case the degree of $\operatorname{repr}(G', H, u_5)$ would be at most three.

Next, we show that the edges of each copy of U in G' are contracted in the ways depicted in Figure 4.8(b) or (c). Let $W_1 = V(\operatorname{sgofv}(G', H, u_1))$ and $W_4 = V(\operatorname{sgofv}(G', H, u_4))$. There are three cases:

- $u_2, u_3 \in W_1$. As we argued before, u_5 is a center of $\operatorname{sgofv}(G', H, u_5)$, but then the degree of $\operatorname{repr}(G', H, u_5)$ would be at most three, hence this case cannot occur.

- $u_2 \in W_1$, and $u_3 \notin W_1$. Since the degree of $\operatorname{repr}(G', H, u_1)$ is four, $\operatorname{repr}(G', H, u_3) \neq \operatorname{repr}(G', H, u_5)$, and since the degree of $\operatorname{repr}(G', H, u_3)$ is four, $u_3 \in W_4$. This is the situation depicted in Figure 4.8(b).

- $u_2 \notin W_1$. By symmetry, it follows that $u_3 \notin W_4$. The only way how the degree of all the vertices $\operatorname{repr}(G', H, u_2)$, $\operatorname{repr}(G', H, u_3)$ and $\operatorname{repr}(G', H, u_5)$ can be four in this case is the one depicted in Figure 4.8(c).

Let us now show that X is a perfect matching. Let v be any vertex of G. Consider first the case that v is incident with an edge e of M, and let e_1 and e_2 be the other edges incident with v. Consider the copy U_e of U that replaces e in G', and assume that $u_1 = v$. If $e \notin X$, then the edges of U_e are contracted as in Figure 4.8(c). Since the degree of $\operatorname{repr}(G', H, u_1)$ is four, one of the vertices that subdivide e_1 or e_2 must be a center of a star, hence either e_1 or e_2 belongs to X. Similarly, in the case that v is not incident with an edge of M, to make the degree of $\operatorname{repr}(G', H, v)$ four, the vertex v

must be identified with a vertex v' of degree three corresponding to one of the neighbors of v in G, and the edge $\{v, v'\}$ belongs to X.

Finally, we prove that X is a solution matching. Consider an arbitrary edge $e \in E(G) \setminus (X \cup M)$, and let x be the 2-vertex that splits the edge e in G'. Since x is not removed, it must be identified with one of the vertices v incident with e (it cannot be identified with both, since in that case e would belong to X). However, this is only possible if an edge $f \in M$ is incident with v, and the corresponding copy of U is contracted as in Figure 4.8(b). Therefore, $f \in X$, and X indeed satisfies the properties of the solution matching.

We proved that $\nabla_1^d(G') \geq 4$ if and only if there exists a solution matching X in (G, M). Since G' can be constructed from G and M in polynomial time, the problem of determining whether $\nabla_1^d(\cdot) \geq 4$ is NP-hard.

Regarding the problem of determining whether $\nabla_1(\cdot) \geq 2$, observe that any minor of G' of depth at most one that does not contain a vertex of degree ≤ 2 has maximum degree at most four. Therefore, $\nabla_1(G') \geq 2$ if and only if all the vertices of the minor have degree four, i.e., $\nabla_1^d(G') \geq 4$. It follows that the problem of determining whether $\nabla_1(\cdot) \geq 2$ is NP-complete as well. ∎

Note that this theorem also implies that determining $\nabla_1^d(\cdot)$ or $\nabla_1(\cdot)$ is NP-complete even if restricted to graphs with maximum degree four. The result of Theorem 4.5 can be easily generalized for any fixed constant ≥ 4:

Theorem 4.6 *For any constant $x \geq 4$, the problem of determining whether $\nabla_1^d(\cdot) \geq x$ is NP-complete.*

Proof: We proceed by induction. The basic case is Theorem 4.5. Let $x > 4$ be an arbitrary constant, and let us assume that it is NP-hard to determine whether $\nabla_1^d(\cdot) \geq x - 1$. Let G be a graph, and let G' be the graph obtained from G by adding a universal vertex v joined with all the vertices of G. We prove that $\nabla_1^d(G) \geq x - 1$ if and only if $\nabla_1^d(G') \geq x$, thus showing that the problem of determining whether $\nabla_1^d(\cdot) \geq x$ is NP-complete.

Suppose first that $\nabla_1^d(G) \geq x - 1$, and let H be a minor of G of depth one with minimum degree $x - 1$. The graph H has least x vertices, hence the graph H' obtained from H by adding a universal vertex joined to all the vertices of H has minimum degree x. The graph H' is a minor of depth one of G', thus $\nabla_1^d(G') \geq x$.

Let us show the reverse implication. Suppose that $\nabla_1^d(G') \geq x$ and let H' be a minor of G' of depth one with minimum degree x. If $\mathrm{repr}(G', H', v)$ does not exist, then H' is a minor of G and $\nabla_1^d(G) \geq x$, thus assume that

$v' = \mathrm{repr}(G', H', v)$ exists. The graph $H' - v'$ is a minor of G' and its minimum degree is at least $x - 1$, hence $\nabla_1^d(G') \geq x - 1$. ∎

4.2 Polynomial Cases

The problems of determining $\nabla_r(G)$ and $\nabla_r^d(G)$ can be solved in polynomial time for graphs with bounded tree-width (note that $\nabla_r^d(G) \leq \mathrm{tw}(G)$). Let us formulate the algorithm for determining $\nabla_1(G)$ more precisely, the algorithms for determining the other parameters are similar.

Theorem 4.7 *If G is a graph on n vertices with tree-width at most k and a tree U of the construction that witnesses that its tree-width is at most k is given, then $\nabla_1(G)$ can be determined in time $O(4^{k^2} n^3)$.*

Proof: Let G be a graph of tree-width at most k. We may assume that G is connected, otherwise we can determine $\nabla_1(\bullet)$ for each component of G separately, and return the maximum. A graph G_u together with a set $S_u \subseteq V(G_u)$ of at most $k+1$ border vertices is associated with each node u of the tree U. A *description* $D = (R \cup T_1 \cup T_2 \cup \ldots \cup T_t, C, O, N)$ for G_u and S_u consists of

- a partition of S_u into several sets, $S_u = R \cup T_1 \cup T_2 \cup \ldots \cup T_t$ (the set R may be empty, all the sets T_i are nonempty), and
- a set $C \subseteq S_u$ such that $C \cap R = \emptyset$, each set T_t contains at most one vertex of C, and if $c \in C \cap T_i$ for some i, then all other elements of T_i are adjacent to c, and
- a set O whose elements are some of the sets T_i such that $C \cap T_i = \emptyset$, and
- a graph N whose vertices are the sets T_i, such that if there are adjacent vertices belonging to the sets T_i and T_j, then T_i and T_j form an edge in N (there may be also other edges in N).

The description characterizes a minor of G of depth one with respect to the vertices in S_u. The set R corresponds to the vertices that are removed, the vertices in each set T_i are contracted into a single vertex. The set C contains the centers of the contracted stars that belong to S_u. The sets T_i that do not contain any element of C are rays of a star whose center does not belong to S_u – the center for T_i belongs to $V(G) \setminus V(G_u)$ if $T_i \in O$, and to $V(G_u) \setminus S_u$ otherwise. The graph N describes which vertices of the minor are adjacent inside G_u. Note that there are less then $d = O(2^{k^2})$ different descriptions.

Given a minor H of G of depth one, we may construct a description D in an obvious way, according to the rules described in the previous paragraph. A description D' is *consistent* with H if D' is equal to D up to a permutation of the sets T_i. For a minor H, let $H[G_u]$ be the graph obtained from G_u by removing all vertices that are removed and identifying all the vertices of G_u that are identified during the construction of H. The edges of $H[G_u]$ correspond to the edges of G_u, however $H[G_u]$ does not have to be a minor of G_u, in case the centers of the stars T_i are outside of G_u. Let $\mathcal{G}$ be the class of all connected graphs G' such that G_u is an induced subgraph of G' and S_u is a cut in G'. For a description D, let $\mathcal{H}_u(D)$ be the set of all graphs $H[G_u]$, where H is a minor of G' of depth at most one for a graph $G' \in \mathcal{G}$, and H is consistent with D.

For a graph G_u and S_u, a description D, and for every $m \leq V(G_u)$, let $e_u(D, m)$ be the maximum number of edges of a graph $H' \in \mathcal{H}_u(D)$ with m vertices. We set $e_u(D, m) = -\infty$ if no such minor exists. The root r of the construction tree contains the graph G with the empty set of border vertices. There exists only one description D for the empty set, and $\mathcal{G} = \{G\}$, hence $\nabla_1(G) = \max\{\frac{e_r(D,m)}{m} | 1 \leq m \leq |V(G)|\}$. It suffices to show how to determine the numbers $e_u(D, m)$. The algorithm processes the $O(n)$ nodes of the tree U recursively, starting from the leaves. Let us consider the possible types of the nodes of U separately:

- *u is a leaf, G_u is a graph with a single vertex v, and $S_u = \{v\}$.* There are the following descriptions in this case: The description D_1 in that $R = \{v\}$ and all the other sets are empty, the description D_2 such that $T_1 = \{v\}$, $C = \emptyset$ and $O = \{T_1\}$, and the description D_3 such that $T_1 = \{v\}$, $C = \{v\}$ and $O = \emptyset$. The values of e_u that are not $-\infty$ are $e_u(D_1, 0) = 0$, $e_u(D_2, 1) = 0$ and $e_u(D_3, 1) = 0$.

- *u is a node with a single child w such that $G_u = G_w$ and $S_u = S_w \setminus \{v\}$.* Suppose D is a description for G_w and S_w. In this case, we call D *valid* if $v \in R$, or $v \in C$, or $v \in T_i$ for some i such that $T_i \notin O$. A description that is not valid cannot be consistent with any minor of G of depth one, as if $v \in T_i$ such that $T_i \in O$, v could not be joined with the center of T_i that is outside G_u (since S_u is a cut). Given a valid description D, the *restricted* description D' is obtained from D by removing v from the set of the partition $R \cup T_1 \cup \ldots$ it belongs to, from the set C, and in case $T_i = \{v\}$ for some i, removing T_i from the graph N. Observe that for a description D' for G_u and S_u, $e_u(D', m)$

is the maximum of $e_w(D,m)$ over all valid descriptions of G_w and S_w whose restricted description is D'.

- *u is a node with a single child w such that $G_w = G_u - v$ and $S_u = S_w \cup \{v\}$.* In this case, we call a description D for G_u and S_u *valid* if the following cases are avoided:

  – $T_i = \{v\}$ for some i, and T_i is in N joined with some T_j such that v is not adjacent to any vertex in T_j, or

  – $v \in T_i$ such that $T_i \cap C = \emptyset$ and $T_i \notin O$, or

  – $v \in T_i$ such that $T_i \cap C = \{c\}$ and $v \neq c$ is not adjacent to c, or

  – $v \in T_i \cap C$ and there exists a vertex in T_i that is not adjacent to v.

  Since $N(v) \cap V(G_u) \subseteq S_w$, $e_u(D,m) = -\infty$ for the descriptions D that are not valid. The description D' of G_w and S_w is *extended with adding x edges* to a valid description $D = (R \cup T_1 \cup \ldots T_t, C, O, N)$ if $D' = (R' \cup T'_1 \cup \ldots T'_{t'}, C', O', N')$ satisfies the following properties:

  – The partition $R' \cup T'_1 \cup \ldots T'_{t'}$ is obtained from the partition $R \cup T_1 \cup \ldots T_t$ by removing the vertex v, and

  – $C' = C \setminus \{v\}$, and

  – if $v \in C \cap T_i$, then $O' = O \cup \{T_i \setminus \{v\}\}$, otherwise $O' = O$, and

  – N' is a subgraph of N, and all the edges in $E(N) \setminus E(N')$ correspond to the edges incident with v, and

  – $x = |E(N)| - |E(N')|$.

  For a description D for G_u and S_u, let $\delta_D = 1$ if $T_i = \{v\}$ for some some i, and $\delta_D = 0$ otherwise. The value $e_u(D,m)$ is the maximum of $x + e_w(D', m - \delta_D)$ over all descriptions D' of G_w and S_w that extend by adding x edges to D, where $x = 0, 1, \ldots, \binom{k+1}{2}$.

- Finally, let us consider the case that *u is a node with children w_1 and w_2 such that $S_u = S_{w_1} = S_{w_2} = V(G_{w_1}) \cap V(G_{w_2})$ and $G_u = G_{w_1} \cup G_{w_2}$.* For each description D for G_u and S_u with t sets T_i and for each m, we consider all pairs m_1 and m_2 such that $m_1 + m_2 = m + t$, and all pairs D_1 and D_2 of descriptions for G_{w_1}, S_{w_1} and G_{w_2}, S_{w_2} that satisfy the following properties:

  – The partition $R \cup T_1 \cup \ldots$ of the vertices of S_u is the same in D, D_1 and D_2, and

- the sets C, C_1 and C_2 of centers of D, D_1 and D_2 are the same, and
- the graph N of the description D is the union of the graphs N_1 and N_2 of the descriptions D_1 and D_2, and
- the set O of D is the intersection of the sets O_1 and O_2 for D_1 and D_2.

Observe that $e_u(D, m)$ is the maximum of $e_{w_1}(D_1, m_1) + e_{w_2}(D_2, m_2) - |E(N_1) \cap E(N_2)|$ over all such pairs m_1, m_2 and D_1, D_2.

It is easy to perform each of these steps in time $O(d^2 n^2)$, where d is the maximum number of descriptions. It follows that the time complexity of the algorithm is $O(4^{k^2} n^3)$. ∎

Using the result of Gurski and Wanke [46], this implies that determining whether $\nabla_1(G) \leq k$ for a fixed k can be done in polynomial time for graphs with bounded clique-width:

Theorem 4.8 *Let k and c be constants. There exists a polynomial time algorithm that for any graph G with $\mathrm{cw}(G) \leq c$ decides whether $\nabla_1(G) \leq k$.*

Proof: If $K_{2k+1,2k+1}$ were a subgraph of G, then $\nabla_1(G) \geq \nabla_0(G) \geq k + \frac{1}{2}$. Therefore, if $\nabla_1(G) \leq k$ then $\mathrm{tw}(G) < 6k\,\mathrm{cw}(G) \leq 6kc$ by Theorem 1.1. Using the algorithm of Bodlaender [12], we can determine whether tree-width of G is at most $6kc$ in linear time, and find the tree of its construction. Therefore, we can apply Theorem 4.7 to determine whether $\nabla_1(G) \leq k$. ∎

4.3 Approximation

Since determining $\nabla_r(\cdot)$ is hard, we are interested in approximating it. We were not able to find a constant factor approximation algorithm, but we describe an algorithm that given a graph G and an integer $d \geq 0$, finds either a minor of depth $\leq r$ and minimum degree at least d, or a witness that $\nabla_r(G) = O\left(d^{(2r+1)(r+1)^2}\right)$. This approximation algorithm is a generalization of the algorithm used in the proof of Theorem 3.7. We also use the fact that since we are only interested in polynomial approximation, we may use Theorem 3.9 and look for bounded depth subdivisions rather than minors.

Let us recall that given a graph G and a set $T \subseteq V(G)$, the (T, t)-degree $d_t^T(v)$ of a vertex $v \in V(G)$ is the maximum number of rays of a $\leq t$-star S in G with center v, such that all the ray vertices of S belong to

T. Consider the following algorithm $\texttt{APPROX\_NABLA}(t, D)$: Given a graph G with n vertices and integers $t, D \geq 0$, we construct an ordering of vertices $v_1, \ldots, v_n$ of G such that the $d_t^{T_i}(v_{i+1}) < D$ for $i = 1, \ldots, n-1$, where $T_i = \{v_1, \ldots, v_i\}$. The algorithm selects the vertices starting from v_n. Suppose we have already determined $v_{i+1}, \ldots, v_n$, i.e., we know the set $T_i = V(G) \setminus \{v_{i+1}, \ldots, v_n\}$. If there exists a vertex $v \in T_i$ such that $d_t^{T_i \setminus \{v\}}(v) < D$, we set $v_i = v$. Otherwise, the algorithm fails. Let us show several properties of this algorithm:

Lemma 4.9 *If the algorithm* $\texttt{APPROX\_NABLA}(t, D)$ *succeeds for a graph* G *and a graph* $H' \subseteq G$ *is a* $\leq t$*-subdivision of a graph* H*, then the average degree of* H *is at most* $2(D-1)$*.*

Proof: Let $L = v_1, \ldots, v_n$ be the ordering of the vertices of G found by the algorithm. Let $H' \subseteq G$ be a $\leq t$-subdivision of a graph H. Let f be the function that maps vertices of H to the corresponding vertices of G, and let $L' = w_1, \ldots, w_k$ be the ordering of vertices of H such that $f(w_i)$ is before $f(w_{i+1})$ in the ordering L for $i = 1, \ldots, k-1$. By the properties of the `APPROX_NABLA` algorithm, the back-degree of each vertex of H is at most $D-1$, hence H has at most $(D-1)k$ edges, and the average degree of H is at most $2(D-1)$. ∎

This lemma together with Theorem 3.9 shows that if the algorithm succeeds for G and $t = 2r$, then $\nabla_r(G) = O\left((4D)^{(r+1)^2}\right)$. We need to show that on the other hand, if the algorithm fails, then $\nabla_r(G)$ is large. We start with the following technical claim:

Lemma 4.10 *Let* G *be a directed graph without loops on* n *vertices in that the outdegree of each vertex is at most one. Then, there exists a set* $U \subseteq V(G)$ *of size at least* $\frac{n}{3}$ *such that the subgraph of* G *induced by* U *contains no edges.*

Proof: We use induction. The claim is obviously true if $n \leq 3$; suppose that $n > 3$, the lemma holds for all graphs with less than n vertices and G is a graph on n vertices. Note that we may assume that G is connected.

Suppose first that G contains a vertex v with indegree zero. Since G is connected, v has outdegree one. Let (v, u) be the edge going from v. We use the induction hypothesis on $G - \{u, v\}$, thus obtaining a set U' of size at least $\frac{n-2}{3}$, and let $U = U' \cup \{v\}$. The size of U is at least $\frac{n-2}{3} + 1 > \frac{n}{3}$, hence the claim holds.

Next, consider the case that each vertex of G has non-zero indegree. Since the outdegree of each vertex is at most one and G is connected, this is only possible if G is a directed cycle. We can put $\left\lfloor \frac{n}{2} \right\rfloor \geq \frac{n}{3}$ of its vertices to U. Therefore, the size of U is at least $\frac{n}{3}$ as required. ∎

Let us now show that failure of the algorithm `APPROX_NABLA` implies the existence of an obstacle for small $\nabla_r(G)$.

Theorem 4.11 *Let $t, D \geq 0$ be integers such that $d = \frac{\sqrt[t+1]{3D}}{3} \geq \frac{5}{3}$. If G is a graph such that the algorithm* `APPROX_NABLA`(t, D) *fails, then there exists $H' \subseteq G$ such that $H' = \mathrm{sd}_{\leq t}(H)$ and H is a graph with the average degree at least d.*

Proof: Assume for contradiction that the algorithm fails on a graph G, but every subgraph whose $\leq t$-subdivision is a subgraph of G has the average degree less than d. Let $T \subseteq V(G)$ be the set of vertices that causes the algorithm to fail, i.e., such that each vertex $v \in T$ satisfies $d_t^{T\setminus\{v\}}(v) \geq D$. Let $k = |T|$. For a vertex $v \in T$, let S_v be an arbitrary $\leq t$-star with center v and at least D ray vertices in $T \setminus \{v\}$, and let $S_0 = \{S_v | v \in T\}$. We may assume that the middle vertices of the stars S_v belong to $V(G) \setminus T$.

We construct sets $M_1, \ldots, M_t \subseteq V(G) \setminus T$ and sets of $\leq t$-stars $S_1, \ldots, S_t$ such that for $i = 1, \ldots, n$,

(P1) the centers of the stars in S_i are mutually distinct and belong to T, and

(P2) each $\leq t$-star $S \in S_i$ is a subgraph of a $\leq t$-star $S' \in S_{i-1}$ obtained by removing some of the rays, i.e., each ray of S is also a ray of S', and

(P3) the level of each ray vertex of any star $S \in S_i$ is at least $i + 1$, and

(P4) each vertex $v \in M_i$ appears as a vertex of level i in exactly one of the $\leq t$-stars in S_i, and if v belongs to any other $\leq t$-star $S \in S_i$, then the level of v in S is greater than i, and

(P5) $m_i = |M_i| \geq \frac{D}{(3d)^i} k$.

We generate the sets M_i and S_i sequentially, starting with $i = 1$. Suppose we have already constructed the sets M_j and S_j for $j < i$, and let us describe how to obtain the sets M_i and S_i.

Note that the stars in S_{i-1} have together exactly m_{i-1} rays if $i > 1$ and at least $m_0 = Dk$ rays if $i = 1$. For each $\leq t$-star in S_{i-1}, let us remove all the rays whose ray vertex has level i. Let S'_i be the set of the stars obtained this way and let m'_i be the total number of rays of the $\leq t$-stars in S'_i. Let x be the number of the removed rays, and let us consider the subgraph $G_1 \subseteq G$ formed by the removed rays. By Properties (P2) and (P4), the middle vertices of the removed rays are mutually distinct, hence G_1 contains as a subgraph a

$(i-1)$-subdivision of a graph with at least $\frac{x}{2}$ edges (each edge may appear twice, once for each of its end vertices) and with at most k vertices. By the assumption, the average degree of this graph is less then d, hence $x < dk$ and $m'_i > m_{i-1} - dk$. This step ensures that S_i will satisfy Property (P3).

If $i = 1$, we let $S''_i = S'_i$ and $m''_i = m'_i$. If $i > 1$, we find a set of stars S''_i that ensures that S_i will satisfy the second part of Property (P4), i.e., that if a vertex v appears on level i in one of the stars, its level in any other star is at least i. We form a directed graph F whose vertices are the rays of $\leq t$-stars in S'_i such that rays r_1 and r_2 form an edge (r_1, r_2) if the vertex of r_1 of level i belongs to r_2 and its level in r_2 is less than i. Since the vertices of the rays with level less than i are mutually distinct (by Properties (P2) and (P4)), the outdegree of each ray is at most one, hence we may apply Lemma 4.10 – there exists a set of $m''_i \geq \frac{m'_i}{3}$ rays such that their vertices with level i are distinct from all vertices with level less than i. Let S''_i be the set of $\leq t$-stars with these rays.

We set M_i to be the set of the level i vertices of the stars in S''_i, and let G_2 be the subgraph of G formed by the rays of the $\leq t$-stars of S''_i truncated to the level i. Note that $M_i \subseteq V(G) \setminus T$. The graph G_2 is an $(i-1)$-subdivision of a graph with m''_i edges and $k + m_i$ vertices. The average degree of this graph is less than d, hence $2m''_i < d(k + m_i)$. It follows that

$$m_i > \frac{2m''_i}{d} - k \geq 2\frac{m_{i-1} - dk}{3d} - k \geq \left(2\frac{\frac{D}{(3d)^{i-1}} - d}{3d} - 1\right)k$$

$$m_i > \left(2\frac{D}{(3d)^i} - \frac{5}{3}\right)k \geq \frac{D}{(3d)^i}k,$$

hence M_i satisfies Property (P5).

For each vertex $v \in M_i$, let r_v be an arbitrary ray of a $\leq t$-star in S''_i such that v belongs to r_v and its level is i, and let S_i be the set of $\leq t$-stars with rays $\{r_v | v \in M_i\}$. Observe that S_i satisfies Properties (P1), (P2) and (P4).

Let $H'' \subseteq G$ be the graph formed by the rays of the $\leq t$ stars of S_t. By Properties (P1)–(P5), H'' contains a subgraph H' with at least $\frac{m_t}{2}$ edges such that H' is the t-subdivision of a graph H with at most k vertices. This is a contradiction, since the average degree of H is at least $\frac{m_t}{k} = d$. ∎

Therefore, the algorithm `APPROX_NABLA` can indeed be used to approximate $\nabla_r(G)$. A bit problematic part of the algorithm is determining $d^T_{2r}(v)$, as Itai, Perl and Shiloach [51] proved that finding the maximum number of vertex disjoint paths of length at most ℓ between two vertices is NP-complete if $\ell \geq 5$ (and found polynomial-time algorithms in case $\ell \leq 4$). Nevertheless, this problem is easy to approximate within $\ell - 1$ factor, which suffices for our purposes. Given a graph G, a set $T \subseteq V(G)$, a vertex $v \in V(G) \setminus T$ and an integer $t > 0$, let $a_t^T(v)$ be the number of vertex-disjoint paths found by the following algorithm: We choose an arbitrary path of length at most $t+1$ from v to a vertex in T, and remove all the vertices of the path except for v. We repeat this procedure until the distance from v to T exceeds $t+1$. By maintaining the first $t+1$ levels of the breath-first search from v, we can implement this algorithm with time complexity $O(t(n+e))$, where $n = |V(G)|$ and $e = |E(G)|$. Let P_1 be the set of paths found by this algorithm, and P_2 the set of $d_t^T(v)$ rays of a $\leq t$-star with center v and all ray vertices in T. Each path in P_1 intersects at most $t+1$ paths in P_2 in vertices different from v, hence $a_t^T(v) \leq d_t^T(v) \leq (t+1)a_t^T(v)$.

Theorem 4.12 *For each integer $r > 0$, there exists an algorithm that for a connected graph G on n vertices with $\nabla_r(G) = d$ returns a subgraph $H' \subseteq G$ such that $H' = \mathrm{sd}_{\leq 2r}(H)$ and the average degree of H is $\Omega\left(d^{\frac{1}{(2r+1)(r+1)^2}}\right)$. The time complexity of the algorithm is $O(rn^2e)$.*

Proof: We use the variant `APPROX_NABLA*` of the algorithm `APPROX_NABLA` in that we replace determining $d_t^T(v)$ by determining $a_t^T(v)$: We construct an ordering $v_1, \ldots, v_n$ of vertices of G in reverse. Suppose that we already know $v_{i+1}, \ldots, v_n$. When we are selecting v_i, we choose a vertex such that $v_i \in T = V(G) \setminus \{v_{i+1}, \ldots, v_n\}$ and the value $D_i = a_{2r}^{T\setminus\{v_i\}}(v_i)$ is minimal. We let $D = \max\{D_i | i = 1, \ldots, n\}$. Observe that the algorithm `APPROX_NABLA`$(2r, D)$ cannot succeed on G, since $a_{2r}^{T\setminus\{v_i\}}(v_i) \leq d_{2r}^{T\setminus\{v_i\}}(v_i)$, and let T be the set of vertices on that the algorithm fails. We construct the sets S_i and M_i as described in the proof of Theorem 4.11, and finally obtain a subgraph $H' = \mathrm{sd}_{\leq 2r}(H) \subseteq G$ such that the average degree of H is at least $\frac{\sqrt[2r+1]{3D}}{3}$.

Since $d_{2r}^T(v) \leq (2r+1)a_{2r}^T(v)$ for all sets T and vertices v, the algorithm `APPROX_NABLA`$(2r, (2r+1)D+1)$ succeeds on G, hence by Lemma 4.9, the average degree of any graph whose $\leq 2r$-subdivision is a subgraph of G is at most $2(2r+1)D$. By Theorem 3.9, $d = \nabla_r(G) < 4\,(8(2r+1)D+4)^{(r+1)^2}$, i.e., $D = \Omega\left(d^{\frac{1}{(r+1)^2}}\right)$. It follows that the average degree of H is at least $\Omega\left(d^{\frac{1}{(2r+1)(r+1)^2}}\right)$.

Let us now consider the time complexity of the algorithm. Let $e = |E(G)|$. First we run the the algorithm `APPROX_NABLA*`, this requires time $O(rn^2e)$. Then we construct the graph H'. Creating sets M_i and S_i takes linear time, hence this phase requires time $O(re)$, and it is dominated by the first phase.
∎

The algorithm `APPROX_NABLA` can be used to produce a witness that the expansion of a graph is small. This witness is a linear ordering of the vertices of G that satisfies certain properties, thus emphasizing the connection between the arrangeability and the greatest reduced average density. This ordering may also be useful for design of algorithms for graphs with bounded expansion, and provides some information about the structure of such graphs. We present one application of this fact in Section 5.2.

Chapter 5

Tree-depth and Subgraph Coloring

In this chapter, we provide several simple results regarding the tree-depth, especially properties of the tree-depth critical graphs. Also, we study the low tree-depth and subgraph coloring and their variations.

5.1 Forbidden Subgraphs

Nešetřil and Ossona de Mendez [74] have showed that the class of graphs with $\mathrm{td}(G) \leq k$ has a finite number of forbidden subgraphs. Their proof is based on the fact that the class of graphs with $\mathrm{td}(G) \leq k$ is minor-closed and hence it has a finite number of forbidden minors. Since we have no description of what the forbidden minors are, this proof does not provide an explicit bound on the size of the minimal forbidden subgraphs. A tower function bound can be derived from their result stating that for any integer k, there are only finitely many cores with tree-depth $\leq k$. However, a direct argument shows a much better bound – these graphs have at most $2^{2^{k-1}}$ vertices:

Theorem 5.1 *For any integer $k > 0$, if G is a graph with $\mathrm{td}(G) > k$, then G contains a connected subgraph H with $\mathrm{td}(H) > k$ and $|V(H)| \leq 2^{2^{k-1}}$.*

Proof: We may assume that G is connected, otherwise we focus on the component of G that determines its tree-depth. Also, without loss on generality, $\mathrm{td}(G) = k + 1$. We prove the statement by induction:

If $\mathrm{td}(G) = 2$, then G contains at least one edge, and we may set $H = K_2$. If $\mathrm{td}(G) = 3$, then G is not a star forest, i.e., it contains P_4 or K_3 as a subgraph.

Suppose now that $\mathrm{td}(G) = k+1$ for $k \geq 3$, and assume that the statement holds for all smaller values of tree-depth. If G contains P_{2^k} as a subgraph, then we may set $H = P_{2^k}$. Otherwise, each two vertices in G are connected by a path of length at most $2^k - 2$.

Since $\mathrm{td}(G) > k - 1$, by induction hypothesis G contains a subgraph H_0 with $\mathrm{td}(H_0) \geq k$ and $m \leq 2^{2^{k-2}}$ vertices $v_1, \ldots, v_m$. For each $i = 1, \ldots, m$, the graph $G - v_i$ has tree-depth greater than $k - 1$, hence $G - v_i$ contains a subgraph H_i with at most $2^{2^{k-2}}$ vertices and tree-depth at least k.

If there exists i such that $V(H_0) \cap V(H_i) = \emptyset$, then we let H consist of H_0, H_i and the shortest path that connects them. For every vertex v of H, the graph $H - v$ contains H_0 or H_i as a subgraph, hence the tree-depth of $H - v$ at least k and $\mathrm{td}(H) > k$. Also, $|V(H)| \leq 2^{2^{k-2}+1} + 2^k - 3 \leq 2^{2^{k-1}}$ (for $k \geq 3$).

On the other hand, if all the graphs H_i intersect H_0, then we set $H = H_0 \cup H_1 \cup \ldots \cup H_m$. Since all the graphs H_i are connected, the graph H is connected as well, and it has at most $m + m(2^{2^{k-2}} - 1) \leq 2^{2^{k-1}}$ vertices. Similarly to the previous case, the graphs $H - v_i$ contain H_i as a subgraph (for $i = 1, \ldots, m$), and the graph $H - v$ for v different from $v_1, \ldots, v_m$ contains H_0 as a subgraph, hence $\mathrm{td}(H) > k$. ∎

We call a graph G *tree-depth critical* if any proper subgraph of G has strictly smaller tree-depth than G. The double-exponential bound of Theorem 5.1 seems to be far from optimal. Indeed, we do not know any example of a tree-depth critical graph with more than exponential size. We can obtain more precise descriptions of tree-depth critical graphs under some additional assumptions, for example, we can fully characterize the tree-depth critical trees.

We call a tree G *decomposable*, if it is a single vertex, or it has 2^k vertices for some integer $k > 0$ and it contains an edge e such that both components of $G - e$ are decomposable. Note that the edge e splits the tree into two components of the same size. We show that a tree is tree-depth critical if and only if it is decomposable.

Let us note that for each tree G, $\mathrm{td}(G) \leq \lfloor \log_2 |V(G)| \rfloor + 1$ – this follows by induction from the fact that each tree G contains a vertex v such that each component of $G - v$ has at most $\frac{|V(G)|}{2}$ vertices.

Lemma 5.2 *Every decomposable tree G has tree-depth $1 + \log_2 |V(G)|$ and it is tree-depth critical. Additionally, if G is a decomposable tree with $|V(G)| > 1$, then for any vertex $v \in V(G)$ there exists a leaf $u \neq v$ of G such that the tree created from $G - u$ by adding a leaf adjacent to v is decomposable.*

Proof: We proceed by induction. Obviously, the claims hold for the trees with at most two vertices, thus we consider the case G is a decomposable tree on 2^k ($k > 1$) vertices, and we assume that the statement of the lemma is true for all smaller trees. Let e be the edge that splits G into two halves, and let G_1 and G_2 be the components of $G-e$; both G_1 and G_2 are decomposable, and by the induction hypothesis, their tree-depth is k.

As we noted, the tree-depth of G is at most $1 + \log_2 |V(G)| = k + 1$. On the other hand, for every $v \in V(G)$, the graph $G - v$ contains G_1 or G_2 as a subgraph, hence $\mathrm{td}(G) = k+1$. Also, $\mathrm{td}(G-v) \leq 1 + \lfloor \log_2(|V(G)|-1) \rfloor = k$, hence G is tree-depth critical.

Consider an arbitrary vertex $v \in V(G)$, and let us show that there exists a leaf u of G that we can move to v while preserving decomposability. Without loss of generality, we may assume that $v \in V(G_1)$. By the the induction hypothesis, there exists a vertex $u' \in V(G_1)$ such that the tree created from $G_1 - u'$ by adding a leaf adjacent to v is decomposable. If u' is not incident to the edge e, we may set $u = u'$. Otherwise, let v' be the vertex of G_2 incident to e, and let u'' be the leaf of G_2 that can be moved to v'. In this case, we can set $u = u''$: Moving the leaf u'' to v has the same result as moving it to v', moving the leaf u' to v, and replacing the edge e by an edge between u'' and the vertex of G_1 that used to be adjacent to u'. ∎

We are now ready to prove the characterization of the tree-depth critical trees. If G is a tree and u and v two distinct vertices of G, let $G_u(v)$ be the component of $G - u$ that contains v.

Theorem 5.3 *For any integer $k \geq 0$, a tree G is tree-depth critical with $\mathrm{td}(G) = k + 1$ if and only if G is a decomposable tree with 2^k vertices.*

Proof: Lemma 5.2 shows that a decomposable tree on 2^k vertices is tree-depth critical and has tree-depth $k + 1$, hence we only need to show that a tree-depth critical tree is decomposable. We proceed by induction. The statement holds for graphs with $\mathrm{td}(G) = 1$, thus consider some tree-depth critical tree G with $\mathrm{td}(G) = k + 1$ for $k > 0$ and assume that the statement holds for all trees with smaller tree-depth.

For any vertex v of G, the graph $G - v$ contains a component of tree-depth k. We first show that G contains an edge $\{x, y\}$ such that $\mathrm{td}(G_x(y)) = \mathrm{td}(G_y(x)) = k$: We let v_0 be an arbitrary vertex of G, and for each $i > 0$, we let v_i be a vertex adjacent to v_{i-1} that belongs to a component of $G - v_{i-1}$ of tree-depth k. The sequence $v_0, v_1, \ldots$ is a walk in G, and since G is a tree, there exists i such that $v_i = v_{i+2}$. We let $x = v_i$ and $y = v_{i+1}$. The graph $G_x(y)$ contains a tree-depth critical subgraph G_1 of tree-depth k, and $G_y(x)$

contains a tree-depth critical subgraph G_2 of tree-depth k. Additionally, there is the unique path P in G that connects G_1 with G_2. Observe that since G is tree-depth critical, G is exactly the union of G_1, G_2 and P.

By the induction hypothesis, G_1 and G_2 are decomposable. We need to show that P has no inner vertices, thus proving that G is decomposable. Suppose that this is not the case, and let w be the inner vertex of P adjacent to a vertex v of G_1. By Lemma 5.2, the graph G_1 contains a leaf u such that the graph obtained from G_1 by moving the leaf u to v is decomposable, and thus also tree-depth critical. This however implies that we may remove the vertex u from G and consider w to be its replacement. The created graph has tree-depth $k+1$, thus contradicting the criticality of G. Therefore, G is just a union of G_1 and G_2 joined by a single edge, hence it is decomposable.
$\blacksquare$

It is also possible to enumerate the forbidden subgraphs for some small values of k: A graph G has tree-depth greater than one if and only if G contains K_2 as a subgraph, and $\mathrm{td}(G) > 2$ if and only if G is not a star forest, i.e., if G contains K_3 or P_4 as a subgraph. The list of forbidden subgraphs for $\mathrm{td}(G) \le 3$ is a bit harder to obtain:

Theorem 5.4 *For any graph G, $\mathrm{td}(G) > 3$ if and only if G contains one of the graphs depicted in Figure 5.1 as a subgraph.*

Proof: Since each of the graphs in Figure 5.1 has tree-depth four, it suffices to show that any connected graph with tree-depth four contains one of them as a subgraph. Suppose for contradiction that this is not the case, and let G be a connected graph with tree-depth four that contains none of the graphs in Figure 5.1 as a subgraph. We may assume that G is minimal, i.e., that $\mathrm{td}(G-e) = 3$ and $\mathrm{td}(G-v) = 3$ for any edge $e \in E(G)$ and any vertex $v \in V(G)$. The graph G cannot contain any cycles of length greater than four.

Let G' be a 2-connected subgraph of G. The graph G' can be constructed from an arbitrary cycle in G' by repeatedly appending paths joining two distinct vertices. Suppose that $|V(G')| \ge 5$, then G' must contain a 4-cycle C whose vertices (in the order along the cycle) are $v_1v_2v_3v_4$, and we may start the construction of G' from this cycle. Since G' does not contain a cycle of length greater than four, we can only join two of its opposite vertices (say v_1 and v_3) by paths of length two – let v_5 be a vertex of such a path. If the subgraph induced by $V(C) \cup \{v_5\}$ contains any of the edges $\{v_2, v_4\}$, $\{v_2, v_5\}$ or $\{v_4, v_5\}$, then G contains C_5 as a subgraph, hence we may assume

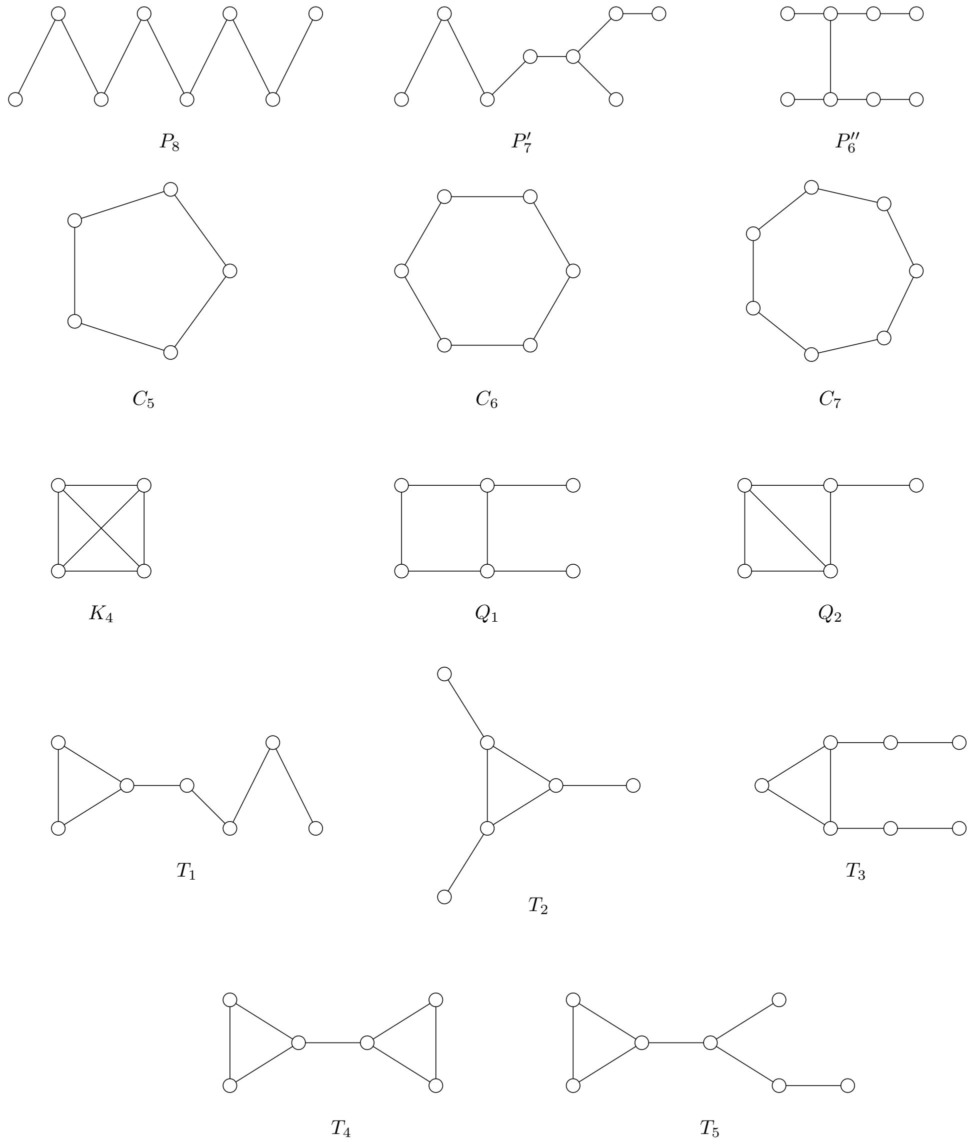

Figure 5.1: Forbidden subgraphs for $\mathrm{td}(G) \leq 3$.

that this is not the case. Also, none of v_2, v_4 and v_5 may be incident with any other vertex of G, otherwise G would contain Q_1. Consider the graph H obtained from G by removing the edge $\{v_1, v_5\}$. By the minimality of G, $\operatorname{td}(H) = 3$. The graph H is connected, hence H contains a vertex v such that $H - v$ is a star forest. If $v = v_1$ or $v = v_3$, then $G - v$ is a star forest, which is contradiction with $\operatorname{td}(G) = 4$. However, $H - v$ for any other vertex v contains P_4 as a subgraph. This is a contradiction, hence we may assume that any 2-connected subgraph of G has at most four vertices.

Let us now consider the case that G contains a 4-cycle $C = v_1v_2v_3v_4$. If both edges $\{v_1, v_3\}$ and $\{v_2, v_4\}$ are in G, then G contains K_4 as a subgraph, thus we may assume this is not the case. Suppose first that $\{v_1, v_3\}$ is an edge (thus $\{v_2, v_4\}$ is not an edge). If v_2 or v_4 is adjacent to a vertex outside of C, then G contains Q_2 as a subgraph. Otherwise, consider the graph H obtained from G by removing the edge $\{v_1, v_3\}$. By the minimality of G, there exists a vertex v such that $H - v$ is a star forest. The vertex v must belong to C. Since $G - v$ is not a star forest, $v \neq v_1$ and $v \neq v_3$, hence we may assume that $v = v_2$. But then $H = C$, and tree-depth of G would be only three, which is a contradiction; therefore, any 4-cycle in G is induced.

Let $C = v_1v_2v_3v_4$ be an induced 4-cycle in G. Since G does not contain Q_1 as a subgraph, the vertices of $V(G) \setminus V(C)$ can only be adjacent to two non-adjacent vertices of C, say v_1 and v_3. We may also assume that at least one such vertex v_5 is adjacent to v_1. Let us consider the graph H obtained from G by removing the edge v_1v_4. By the minimality of G, there exists a vertex v such that $H - v$ is a star forest. Since $v_5v_1v_2v_3v_4$ is a path, v must be v_1, v_2 or v_3. If $v = v_1$ or $v = v_3$, then $G - v$ is a star forest, hence $v = v_2$. However, this means that $G - v_1$ is a star forest, which is a contradiction, thus G does not contain any 4-cycle.

Consider now the case that G contains a triangle $C = v_1v_2v_3$. The graph G cannot contain another triangle disjoint from C, since otherwise it would contain T_1 or T_4 as a subgraph. Together with the fact that each nontrivial 2-connected subgraph of G is a triangle, this implies that all the triangles in G intersect in one vertex. We may assume that there is at least one vertex v_4 not belonging to C adjacent to v_1, and that all triangles in G contain the vertex v_1. The vertex v_1 is a cut-vertex in G. The graph $G - v_1$ is not a star forest, hence one of its components contains a triangle or P_4. All triangles in G contain the vertex v_1, hence one of the components of $G - v_1$ contains a path P of length three.

If P is disjoint with C, then G contains a subgraph T_1 or T_5. It follows that G contains only one triangle C, and that the path P intersects $C - v_1$. If the degree of both v_2 and v_3 is greater than two, then G contains the subgraph T_2, thus we may assume that degree of v_2 is two and that $P = v_2v_3v_5v_6$ for some vertices v_5 and v_6. Similarly, $G - v_3$ contains P_4 as a subgraph, hence we may assume that there is a vertex v_7 adjacent to v_4. However, the graph G then would contain T_3 as a subgraph. Therefore, G does not contain a triangle, and it must be a tree.

It is however easy to verify using Lemma 5.3 that the only tree-depth critical trees with tree-depth four are P_8, P'_7 and P''_6. It follows that any graph with $\mathrm{td}(G) > 3$ contains one of the graphs in Figure 5.1 as a subgraph. ∎

The inspection of the forbidden subgraphs of Theorem 5.4 also gives us the set of forbidden minors for the property $\mathrm{td}(G) \leq 3$: all the graphs in Figure 5.1 except for C_6 and C_7 (that have C_5 as a minor). Note that the forbidden subgraphs have at most 8 vertices, hence Theorem 5.1 is not sharp even in this case (it only claims that the subgraphs have at most 16 vertices).

5.2 Low Tree-depth Coloring

The algorithm `APPROX_NABLA` described in Section 4.3 can be used to provide a proof that the expansion of the graph is small – the ordering L of vertices of the graph G such that Lemma 4.9 and Theorem 3.9 can be applied. By Theorems 2.3 and 2.4 (proved by Nešetřil and Ossona de Mendez [70]), a graph has small expansion if and only if it has a low tree-depth coloring by a small number of colors. Naturally, there should be a way how to use the ordering L to find a low tree-depth coloring of G, thus proving a result similar to Theorem 2.4:

Theorem 5.5 *Let G be a graph with n vertices and e edges, $p, D > 0$ integers, and let $b = 2^p - 2$. Given a linear ordering L of vertices of G such that $d_b^{L^-(v)} < D$ for each $v \in V(G)$, one can find a p-tree-depth coloring of G by at most $D^{(b+1)^2}$ colors in time $O(D^{(b+1)^2} n)$.*

The easy consequence of Theorem 5.5 is the following claim that strengthens Theorem 2.4:

Theorem 5.6 *For each $p > 0$, there exists a polynomial f of degree $O(8^p)$ such that for each graph G,*

$$\chi_p(G) \leq f\left(\nabla_{2^{p-1}-1}(G)\right).$$

Proof: Let $d = \nabla_{2^{p-1}-1}(G)$ and $b = 2^p - 2$. This means that any graph H whose $\leq b$-subdivision is a subgraph of G has average density at most

d. Let $D = \frac{(6d+3)^{b+1}}{3}$. By Theorem 4.11, the algorithm APPROX_NABLA(b, D) succeeds on G, hence there exists a linear ordering L of the vertices of G such that $d_b^{L^-(v)} < D$ for each $v \in V(G)$. By Theorem 5.5, there exists a p-tree-depth coloring of G by at most $D^{(b+1)^2} = O(d^{(b+1)^3})$ colors. The degree of the polynomial that bounds the number of colors is $(b+1)^3 = O(8^p)$. ∎

For a vertex $v \in V(G)$ and a set $T \subseteq V(G)$, let $Q_b^T(v)$ be the set of vertices in T that are reachable from v by a path of length at most $b+1$ whose inner vertices do not belong to T. To show the correctness of the algorithm of Theorem 5.5, we need the following lemmas.

Lemma 5.7 *Let $b \geq 0$ and $D > 0$ be integers and G be a graph together with an ordering L. If $d_b^{L^-(v)}(v) < D$ for each $v \in V(G)$, then $|Q_b^{L^-(v)}(v)| \leq (D-1)^{b+1}$ for each vertex v.*

Proof: For each vertex v there exists a tree R rooted in v whose depth is at most $b+1$, its leaves are the members of $Q_b^{L^-(v)}(v)$ and all the inner vertices belong to $L^+(v)$. Each vertex inner vertex or root $u \in V(R)$ has degree less than D in R, since R contains a $\leq b$-star with center u and $d_R(u)$ rays whose middle vertices belong to $L^+(u)$ and the ray vertices belong to $L^-(u)$. It follows that the number of the leaves of the tree is at most $(D-1)^{b+1}$, i.e., $|Q_b^{L^-(v)}(v)| \leq (D-1)^{b+1}$. ∎

Lemma 5.8 *For any $k > 0$, if G is a connected graph with $\mathrm{td}(G) = k+1$ and v a vertex of G such that $\mathrm{td}(G-v) = k$, then the eccentricity of v is at most $2^k - 1$.*

Proof: Let u be the vertex of G whose distance d from v is maximal (i.e., equal to the eccentricity of v), and let P be the shortest path joining u and v. The graph $G-v$ contains the path $P-v$, whose length is $d-1$. Since $\mathrm{td}(P_{2^k}) = k+1$, it follows that $G-v$ does not contain a path of length 2^k-1, hence $d \leq 2^k - 1$. ∎

For a fixed graph G, its ordering L and an integer b, let $Q(v) = Q_b^{L^-(v)}(v)$, and let $Q'(v) \subseteq \{v\} \cup L^-(v)$ be the set of vertices obtained by iterating the operation Q $b+1$ times, i.e., $Q'(v) = X_{b+1}(v)$, where $X_0(v) = \{v\}$ and $X_{i+1}(v) = X_i(v) \cup \bigcup_{u \in X_i(v)} Q(u)$ for $i \geq 0$. Lemma 5.7 implies that $|Q'(v)| \leq D^{(b+1)^2}$. We use the following algorithm TDCOLOR to color the vertices of the graph G: We process the vertices in the order given by L, and we color each vertex by the smallest color that does not appear on any other vertex in $Q'(v)$.

Proof of Theorem 5.5: Since $v \in Q'(v)$ and $|Q'(v)| \le D^{(b+1)^2}$, the algorithm `TDCOLOR` colors the graph by at most $D^{(b+1)^2}$ colors. We need to argue that the coloring produced by the algorithm is a proper p-tree-depth coloring of G. We show this by induction: let G' be a nonempty connected subgraph of G of tree-depth $p' \le p+1$, and assume that the algorithm `TDCOLOR` assigns at least k colors to any subgraph of G of tree-depth $k < p'$. We may assume that $p' > 1$, since at least one color is used in any nonempty subgraph of G. Furthermore, we may assume that G' is tree-depth critical, i.e., for any vertex u of G', $\mathrm{td}(G'-u) < \mathrm{td}(G')$.

Let u be the first vertex of G' in the ordering L. Since $\mathrm{td}(G'-u) < \mathrm{td}(G')$, the distance from u to any other vertex of G' is at most $2^p - 1 = b+1$ by Lemma 5.8. Consider a vertex $v \in V(G') \setminus \{u\}$ and let P be a shortest path from u to v in G'. We can partition P to at most $b+1$ edge-disjoint paths $P_1, P_2, \ldots$, such that for each path $P_i = v_i^1 v_i^2 \ldots v_i^\ell$, the vertex v_i^1 belongs to $L^-(v_i^\ell)$ and all the inner vertices of the path belong to $L^+(v_i^\ell)$. This implies that $v_i^1 \in Q(v_i^\ell)$, hence $u \in Q'(v)$, and the color assigned to u by the algorithm `TDCOLOR` is different from the colors assigned to all the other vertices of G'. The tree-depth of $G'-u$ is $p'-1$, hence by the induction hypothesis, $G'-u$ is colored by at least $p'-1$ colors. This implies that G' is colored by at least p' colors. Therefore, the algorithm `TDCOLOR` constructs a proper p-tree-depth coloring.

Let us now consider the time complexity of the algorithm. If we already have constructed the sets $Q(v)$ for each vertex v, we can compute the sets $Q'(v)$ in time $O(D^{(b+1)^2}n)$, and select for each vertex v a color unused on $Q'(v)$ with the same time complexity. We construct the sets $Q(v)$ in the reverse ordering to L. Note that

$$Q(v) \subseteq (N(v) \cap L^-(v)) \cup \bigcup_{u \in N(v) \cap L^+(v)} Q(u).$$

Also, if R_u is a tree of the shortest paths from u to the vertices in $Q(u)$ whose inner vertices belong to $L^+(u)$, we may choose the trees so that

$$R_v - (N(v) \cap L^-(v)) \subseteq \bigcup_{u \in N(v) \cap L^+(v)} R(u).$$

Therefore, we may construct R_v and $Q(v)$ by finding the shortest paths in the union of the trees of vertices in $N(v) \cap L^+(v)$. Since the size of each tree R_u is $O(bD^{b+1})$, we can do that in time $O(bD^{b+1}d(v))$. Thus, we can construct all the sets $Q(v)$ in time $O(n + bD^{b+1}e)$, where $e = |E(G)$. Since $e \le Dn$, the time complexity of the whole algorithm is $O(D^{(b+1)^2}n)$. ∎

5.3 Induced Subgraph Coloring

In this section, we consider the variant of the subgraph coloring in that we require that **induced** subgraphs have many colors. Let us recall that for a graph H and an integer k, $\varphi_{H,k}$ is the graph function defined by $\varphi_{H,k}(H) = k$ and $\varphi_{H,k}(H') = 1$ for any $H' \neq H$. Analogically to the upper chromatic number, we define the *induced upper chromatic number* $\bar{\chi}^i(H)$ as the maximum k such that the function $\chi^i_{\varphi_{H,k}}(\cdot)$ is bounded by a constant on any proper minor closed class.

The induced upper chromatic number of a graph H can be expressed in the terms of a structural property of H similar to the tree-depth, although the description is more complicated. We need several definitions:

Given a connected graph G, let us call a rooted tree T whose set of vertices is a superset of $V(G)$ a *spine* of G if for each edge $\{u, v\} \in E(G)$, u is an ancestor of v in T or vice versa. Note that the minimum depth of a spine of G is equal to $\mathrm{td}(G) - 1$. Let $\ell_T(v)$ denote the level of a vertex $v \in V(G)$ in T. Given a rooted tree T of depth t, a graph G is a *partial closure* of T if $V(G) = V(T)$, T is a spine of G, and between each two levels of T, either all the possible edges are present or none is. In other words, there exists a set $M_T \subseteq \binom{\{0,1,\ldots,t\}}{2}$ such that $\{u, v\} \in E(G)$ if and only if u is an ancestor of v in T or vice versa, and $\{\ell_T(u), \ell_T(v)\} \in M_T$.

For a graph H, let t be the the minimum depth of a rooted tree T such that there exists a partial closure of T that contains H as an induced subgraph. We define the *induced tree-depth* of H as $\mathrm{td}^i(H) = t + 1$. Each graph is a partial closure of a path, hence $\mathrm{td}(H) \leq \mathrm{td}^i(H) \leq |V(H)|$. If H' is an induced subgraph of H, then $\mathrm{td}^i(H') \leq \mathrm{td}^i(H)$. Also, the induced tree-depth of a graph is bounded by a function of its tree-depth:

Lemma 5.9 *For any connected graph H, $\mathrm{td}^i(H) < 2^{\mathrm{td}(H)}$. Additionally, if $\Delta(H) = k$, then $\mathrm{td}^i(H) \leq \sum_{j=0}^{k} \binom{\mathrm{td}(H)}{j+1}$.*

Proof: Let T be a spine of H of depth $d = \mathrm{td}(H) - 1$ such that $V(T) = V(H)$. For a vertex $v \in V(H)$ and an integer i $(0 \leq i < \ell_T(v))$, let $a_i(v)$ be the ancestor of v in T whose level is i. Let $\sim_T$ be the equivalence in that $u \sim_T v$ if $\ell_T(u) = \ell_T(v)$ and $\{v, a_i(v)\} \in E(G) \Leftrightarrow \{u, a_i(u)\} \in E(G)$ for each $i = 0, 1, \ldots, \ell_T(v) - 1$. The maximum number of the classes of the equivalence $\sim_T$ on the level i is 2^i, hence the total number D of the classes of $\sim_T$ satisfies $D \leq \sum_{i=0}^{d} 2^i = 2^{d+1} - 1$.

If $\Delta(H) = k$, we can improve this bound – the maximum number of the classes of this equivalence on the level i is $\sum_{j=0}^{k}\binom{i}{j}$, hence

$$D \le \sum_{j=0}^{k}\sum_{i=0}^{d}\binom{i}{j} = \sum_{j=0}^{k}\binom{d+1}{j+1}.$$

Let us assign numbers $c(v) \in \{0, \ldots, D-1\}$ to the vertices of H in such a way that $c(u) = c(v)$ if and only if $u \sim_T v$ and $\ell_T(u) < \ell_T(v) \Rightarrow c(u) < c(v)$. We create a tree T' from T by subdividing each edge $\{u, v\} \in T$ by $|c(u) - c(v)| - 1$ vertices. Note that the level of each vertex $v \in V(H)$ in T' is $c(v)$. Let G be the partial closure of T' in that the vertices at levels $i < j$ are adjacent if and only if there exists an edge $\{u, v\} \in E(H)$ such that $c(u) = i$ and $c(v) = j$. We claim that H is an induced subgraph of G, thus showing that $\mathrm{td}^i(H) \le D$.

If $\{u, v\}$ is an edge in H and u is an ancestor of v in T, then u is also an ancestor of v in T' and by the construction of G, the vertices u and v are adjacent in G. On the other hand, consider the case that u and v are not adjacent in H and $\ell_T(u) \le \ell_T(v)$. Suppose for contradiction that $\{u, v\} \in E(G)$. This means that u is an ancestor of v and there exist vertices $u' \sim_T u$ and $v' \sim_T v$ such that $\{u', v'\} \in E(H)$. This contradicts the definition of the equivalence $\sim_T$, since $u = a_{\ell_T(u)}(v)$ and $u' = a_{\ell_T(u)}(v')$, but $\{u, v\} \in E(H) \not\Leftrightarrow \{u', v'\} \in E(H)$. Therefore, H is an induced subgraph of G. ∎

Let us now apply the same idea as Nešetřil and Ossona de Mendez [74] used to show that $\bar{\chi}(H) \le \mathrm{td}(H)$:

Lemma 5.10 *For any connected graph H, $\bar{\chi}^i(H) \le \mathrm{td}^i(H)$.*

Proof: Let $t = \mathrm{td}^i(H)$ and $n = |V(H)|$. The graph H is an induced subgraph of a partial closure G of a rooted tree S of depth $t-1$. We may assume that each vertex of S has at most n sons. Let $k > 0$ be an integer. Let S' be the rooted tree of depth $t-1$ in that all vertices on level $i < t-1$ have $(n-1)k^{t-i-1} + 1$ sons. Let G' be the partial closure of S' in that the vertices on the levels $i < j$ are adjacent if and only if the vertices on levels i and j are adjacent in G.

Consider an arbitrary coloring of G' by k colors, and the corresponding coloring of S'. By a repeated application of the pigeonhole principle, S' contains a subtree T' of depth $t-1$ in that each inner vertex has n sons, such that all the vertices of T' on the same level have the same color. Let F be the corresponding induced subgraph of G'. The graph H is an induced subgraph of G and G is an induced subgraph of F, hence G' contains H as an induced subgraph colored by at most t colors.

Consider the class $\mathcal{G}$ of all graphs with tree-depth at most $\mathrm{td}^i(H)$. This is a proper minor-closed class, and by the argument we presented, for any integer $k > 0$, there exists a graph $G' \in \mathcal{G}$ such that for any coloring of G' by k colors, G' contains H as an induced subgraph colored by at most $\mathrm{td}^i(H)$ colors. The inequality $\bar{\chi}^i(H) \le \mathrm{td}^i(H)$ follows. ∎

Given a rooted tree T, a tree is obtained from T by *expunging a level i* if all the vertices of T on level i are removed, and we add the edges between the vertices of T on level $i+1$ and their ancestors on level $i-1$. To prove the other inequality between $\bar{\chi}^i(H)$ and $\mathrm{td}^i(H)$, we need the following lemma:

Lemma 5.11 *Let H be a connected graph with $\mathrm{td}^i(H) = t$ and G a graph with tree-depth at most $t-1$. Then, there exists a coloring of G by less than 2^{t-1} colors such that every induced subgraph of G isomorphic to H contains at least t colors.*

Proof: By Lemma 5.9, the graph G is an induced subgraph of a partial closure G' of a tree T with depth at most $2^{t-1}-2$. Let us color each level of the tree T by a separate color, and consider the corresponding coloring φ of G. This coloring uses at most $2^{t-1}-1$ colors.

Let H be an induced subgraph of G. Let T' be the graph obtained from T by removing all the vertices that do not lie on a path connecting two vertices of H. Note that T' is a rooted tree whose root and leaves belong to $V(H)$. Let T'' be the rooted tree obtained from T' by expunging all levels that do not contain a vertex of H. Let d be the depth of T''. Note that the coloring φ assigns $d+1$ colors to H. On the other hand, the subgraph G'' of G' induced by $V(T'')$ is a partial closure of T'', and H is an induced subgraph of G'', hence $d+1 \ge \mathrm{td}^i(H)$. Therefore, the coloring φ assigns at least $\mathrm{td}^i(H)$ colors to any induced subgraph of G isomorphic to H. ∎

Let us now show that $\bar{\chi}^i(H)$ and $\mathrm{td}^i(H)$ are equal.

Theorem 5.12 *For any connected graph H, $\bar{\chi}^i(H) = \mathrm{td}^i(H)$.*

Proof: Let $t = \mathrm{td}^i(H)$. By Lemma 5.10, $\bar{\chi}^i(H) \le t$, hence it suffices to show that $\bar{\chi}^i(H) \ge t$. Let $\mathcal{G}$ be any class with bounded expansion, and G any graph in $\mathcal{G}$. By Theorem 2.4, there exists a constant k (independent on the choice of G) such that G has a $t-1$-tree-depth coloring φ_0 by at most k colors. For a color a, let S_a be the set of vertices of G colored by a. For a set $A \subseteq \{1,\ldots,k\}$, let G_A be the subgraph of G induced by $\bigcup_{a\in A} S_a$. If $|A| < t$, then the tree-depth of G_A is at most $t-1$. By Lemma 5.11, there exists a coloring φ_A of each such graph G_A by at most $D = 2^{t-1} - 1$ colors such that any induced subgraph of G_A isomorphic to H is colored by at least t colors. Let us extend all the colorings φ_A to G by setting the color of each vertex that does not belong to $V(G_A)$ to 1.

Let A_1, A_2, ..., A_m (where $m = \binom{k}{t-1}$) be all the subsets of $\{1, \ldots, k\}$ whose size is $t-1$. We define the coloring φ of G such that the color $\varphi(u)$ of each vertex $u \in V(G)$ is the tupple $(\varphi_0(u), \varphi_{A_1}(u), \ldots, \varphi_{A_m}(u))$. The function φ is a coloring of G by at most kD^m colors, which is a constant independent on the graph G.

Suppose H is an induced subgraph of G. Either H intersects at least t classes of the coloring φ_0, or it is an induced subgraph of G_A for some set $A \subseteq \{1, \ldots, k\}$ of size $t-1$. In that case, the coloring φ_A assigns at least t colors to H. Therefore, the coloring φ assigns at least t colors to H. It follows that $\bar{\chi}^i(H) \geq t$. ∎

We can use Theorem 5.12 to determine $\bar{\chi}^i(H)$ for some graphs.

Theorem 5.13 *The smallest graph H for that $\bar{\chi}^i(H) \neq \bar{\chi}(H)$ is C_5.*

Proof: By Theorems 2.5, 2.2 and 5.12, it suffices to consider the tree-depth and the induced tree-depth of the graphs. Let H be any connected graph with at most four vertices. If H is a single vertex, then $\mathrm{td}(H) = \mathrm{td}^i(H) = 1$. If $\mathrm{td}(H) = 2$, then H is a star, and $\mathrm{td}(H) = \mathrm{td}^i(H) = 2$. If $\mathrm{td}(H) = 4$, then $H = K_4$ and $\mathrm{td}^i(H) = 4$. Therefore, it suffices to consider the case $\mathrm{td}(H) = 3$.

If H is a tree, then we can root it in an arbitrary non-leaf vertex, thus showing that H is its own partial closure. If $H = C_4$ or H is K_4 without one edge, then H is a partial closure of the star $K_{1,3}$ rooted in one of its leaves. Finally, if H is a triangle with one added edge, then it is a partial closure of the $\leq$ 1-star with two rays of length one and two. In all these cases, $\mathrm{td}^i(H) = 3$. The smallest graph H with $\mathrm{td}(H) \neq \mathrm{td}^i(H)$ thus has at least five vertices.

The path P_5 is equal to the 1-star with two rays, hence $\mathrm{td}^i(P_5) = \mathrm{td}(P_5) = 3$, thus the smallest graph with $\mathrm{td}(H) \neq \mathrm{td}^i(H)$ has at least five edges.

Let us now show that $\mathrm{td}^i(C_5) = 5 > \mathrm{td}(C_5)$. Suppose for contradiction that T is a spine with depth three of a 5-cycle H, and that H can be extended to a partial closure of T. Such a spine has the root v_1 and exactly one vertex v_2 of level one, where both v_1 and v_2 belong to $V(H)$, since H is 2-connected and its tree-depth is four. There also are exactly two vertices $v_3, v_4 \in V(H)$ whose level is the same (either two or three). Since the complement of C_5

does not contain $K_{2,2}$ as a subgraph and we may exchange the position of v_1 and v_2 in the tree T, we may assume that $\{v_1, v_3\}$ is an edge. Since T can be extended to a partial closure, $\{v_1, v_4\}$ is an edge as well. Since v_3 and v_4 do not have a common neighbor other than v_1, exactly one of v_3 and v_4 must be adjacent with v_2. However, this contradicts the assumption that H can be extended to a partial closure of T.

Therefore, C_5 is indeed the smallest graph for that the tree-depth and the induced tree-depth differ. ∎

Lemma 5.9 shows that $\mathrm{td}^i(P_n) \leq O(\log^3 n)$. In fact, the induced tree-depth of a path is logarithmic:

Theorem 5.14 *For any integer $k > 1$, $\mathrm{td}^i(P_{2^k-1}) \leq 2k-2$.*

Proof: Let T be the complete binary rooted tree of depth $k-1$, and let us embed the path $P = P_{2^k-1}$ to the closure of T. Note that the embedding is unique. We analyze the number D of classes of the equivalence $\sim_T$. As we showed in Lemma 5.9, the induced tree-depth of P is bounded by D.

Each edge of P is incident with a leaf of T, hence all the vertices on level $i < k-1$ are equivalent. Let us now consider the classes on the level $k-1$. The end vertices of P are equivalent, and they are not equivalent with any other leaf of T. Every other leaf of T is adjacent to a vertex on level $k-2$ and a vertex on level less than $k-2$, hence there are $k-2$ other classes of $\sim_T$ on level $k-1$. Together, $D \leq 2k-2$. ∎

5.4 Coloring Bounded Functions

As described in Section 2.2, the values of the minor-closed class coloring bounded and the bounded expansion class coloring bounded functions are bounded from above by $\mathrm{td}(G)$. However, not all functions that are smaller than $\mathrm{td}(G)+1$ are coloring bounded, e.g. if $\varphi(G) = \mathrm{td}(G)$, then $\chi_\varphi(H) = \mathrm{td}(H)$ is not bounded even for paths. Also, although the results for the minor-closed class coloring bounded and the bounded expansion class coloring bounded functions referenced in Section 2.2 are similar, the two classes of functions are not identical: Consider the function φ such that $\varphi(\mathrm{sd}_n(K_n)) = n$ for each $n > 0$, and $\varphi(G) = 1$ otherwise. This function is minor-closed class coloring bounded, but not bounded expansion class coloring bounded. The precise characterizations of the minor-closed class coloring bounded functions and the bounded expansion class coloring bounded functions are easy to derive, though:

Theorem 5.15 *A graph function φ is minor-closed class coloring bounded if and only if $\varphi(G) \leq \mathrm{td}(G)$ for each graph G, and additionally for each $h > 0$ exists N such that all graphs with $\varphi(H) > N$ contain K_h as a minor.*

Proof: Suppose first that φ is minor-closed class coloring bounded. By Theorem 2.5, $\varphi(G) \leq \mathrm{td}(G)$. For an arbitrary $h > 0$, consider the class $\mathcal{G}_h$ of all graphs that do not contain K_h as a minor. The class $\mathcal{G}_h$ is a proper minor-closed class, hence there exists a constant N such that $\chi_\varphi(G) \leq N$ for each $G \in \mathcal{G}_h$. Since $\chi_\varphi(G) \geq \varphi(G)$, it follows that $\varphi(G) \leq N$ for each graph in $\mathcal{G}_h$, which is equivalent to the statement of the lemma.

On the other hand, suppose that φ is a graph function such that $\varphi(G) \leq \mathrm{td}(G)$ for each graph G and that for each $h > 0$ exists N such that all graphs with $\varphi(H) > N$ contain K_h as a minor. Let $\mathcal{G}$ be any proper minor-closed class of graphs. Since $\mathcal{G}$ is proper, there exists h such that K_h is not a minor of any graph in $\mathcal{G}$. Let N be an integer such that all graphs with $\varphi(H) > N$ contain K_h as a minor. Then, all graphs $G \in \mathcal{G}$ satisfy $\varphi(G) \leq N$. This implies that $\chi_\varphi(G) \leq \chi_{N-1}(G)$ for each $G \in \mathcal{G}$, and by Theorem 2.2, $\chi_\varphi(\cdot)$ is bounded by a constant on $\mathcal{G}$. ∎

Similarly, for the bounded expansion class coloring bounded functions, we have the following statement:

Theorem 5.16 *A function φ is bounded expansion class coloring bounded if and only if $\varphi(G) \leq \mathrm{td}(G)$ for each graph G, and additionally for each nondecreasing function f there exists N such that all graphs H with $\varphi(H) > N$ satisfy $\nabla_r(H) > f(r)$ for some r.*

Proof: If φ is bounded expansion class coloring bounded, then $\varphi(G) \leq \mathrm{td}(G)$ by Theorem 2.5. Let f be an arbitrary function, and let $\mathcal{G}$ be the class of all graphs with expansion bounded by f. There exists a constant N such that $\chi_\varphi(H) \leq N$ for each graph $H \in \mathcal{G}$, hence no graph with $\varphi(H) > N$ belongs to this class.

Let us now prove the reverse implication. Let φ be a graph function such that $\varphi(G) \leq \mathrm{td}(G)$ for each graph G, and let $\mathcal{G}$ be a graph class with expansion bounded by f. By the assumptions of the theorem, the function φ is bounded by some constant N on $\mathcal{G}$, hence $\chi_\varphi(G) \leq \chi_{N-1}(G)$ for each $G \in \mathcal{G}$. By Theorem 2.2, the number $\chi_{N-1}(G)$ is bounded by a constant on the class $\mathcal{G}$. ∎

Chapter 6

Conclusion

We conclude by a quick overview of the contributions of the thesis, and posing several open problems motivated by our results.

- We have provided a characterization of graphs with small acyclic chromatic number (Section 3.1.1), arrangeability (Section 3.1.2) and expansion (Section 3.1.3) in the terms of forbidden subdivision.

- We used these results to simplify some problems regarding game chromatic number (Section 3.1.5).

- We characterized graphs with exponential expansion by the existence of clique minors with small depth. To obtain this result, we showed that graphs with large minimal degree contain a subdivision of a clique in that each edge is subdivided only by a constant number of vertices (Section 3.2.1). We also showed that these graphs contain large expander-like subgraphs (Section 3.2.2).

- We studied the complexity of determining $\nabla_r(\cdot)$ and $\nabla_r^d(\cdot)$, showing their NP-completeness even for graphs with bounded maximum degree (Section 4.1) and an algorithm to compute them in polynomial time for graphs with bounded tree-width and for graphs with bounded clique-width in the fixed parameter case (Section 4.2). We also found an approximation algorithm for general graphs (Section 4.3).

- We applied the witness of the small expansion obtained from the approximation algorithm to construct a low tree-depth coloring of a graph (Section 5.2).

- We considered the set of forbidden subgraphs for small tree-depth, described precisely the tree-depth critical trees and determined the set of forbidden subgraphs for $\mathrm{td}(\cdot) \leq 3$ (Section 5.1).

- We introduced the induced upper chromatic number and showed an equivalent graph property similar to tree-depth.

6.1 Open problems

Many of our results seem to be far from tight, or leave open questions:

- Is it possible to improve the bounds of Theorems 3.3, 3.7 or 3.9 regarding the existence of the subdivisions of graphs with high minimum degree or chromatic number? While the bounds of Theorem 3.7 and Theorem 3.9 have at least the correct order of magnitude, the gap between the bounds for Theorem 3.3 is quite large.

- Conjectures 3.1 and 3.2 for the game chromatic number are wide open. It seems that proving them would require a deep understanding of the properties of the chromatic number of subgraphs of a graph with high chromatic number. Conjecture 3.3 regarding the game coloring number might be easier to prove or disprove.

- The size of the clique guaranteed by Theorem 3.15 is quite small. Is it true that a graph with minimum degree $\Omega(n^{\varepsilon})$ contains a subdivision of $K_{n^{\varepsilon/2}}$ in that each edge is subdivided a constant number of times? Perhaps a more refined analysis similar to the proof of Komlós and Szemerédi [57] could provide such a result.

- Is it possible to approximate $\nabla_r(G)$ within a better factor than the one of Section 4.3? Possibly within a constant factor? Since determining whether $\nabla_1^d(\cdot) \geq 4$ is NP-complete, $\nabla_1^d(\cdot)$ cannot be approximated within factor better than $\frac{4}{3}$. A more involved analysis of the proof of Theorem 4.5 using the inapproximability of set cover also rules out PTAS for $\nabla_1(\cdot)$.

- What is the complexity of the problem of determining whether $\nabla_1^d(G) \leq 3$?

- We showed that determining whether $\nabla_1(G) \leq k$ is solvable in polynomial time for graphs with bounded clique-width if k is fixed. What is the complexity of the problem if k is a part of the input?

- Does there exist a proper minor-closed class for that determining $\nabla_1(G)$ is NP-complete? There exists such a class with bounded expansion (graphs with maximum degree four), by Theorem 4.5.

- By Theorem 5.3, a tree-depth critical tree with tree-depth $k+1$ has 2^k vertices, and among the forbidden subgraphs in Theorem 5.4, the trees have the largest number of vertices. Is it possible that 2^k is the correct upper bound on the size of the tree-depth critical graphs, rather than the double-exponential bound of Theorem 5.1?

- By Theorem 5.6, $\chi_p(G)$ is bounded by a function of $\nabla_{2^{p-1}-1}(G)$. Is the exponential depth of the considered minors necessary? The fact that the tree-depth of a path is logarithmic in its length (thus we may need to have a control of vertices in the exponential distance) indicates that this indeed might be the case, but we were not able to construct an example of graphs with bounded $\nabla_{2^{o(p)}}(G)$ and unbounded $\chi_p(G)$.

- Is computing the induced tree-depth of a graph NP-complete? What about the fixed parameter case? Do there really exist graphs whose induced tree-depth is exponential in their tree-depth?

- An important area of research regarding the planar graphs and graphs on surfaces is the study of light configurations, i.e., existence of subgraphs such that all of their vertices have a small degree (see e.g. the survey paper of Jendrol' and Voss [52]). The graphs with bounded maximum degree (another canonical example of a class with bounded expansion) obviously contain light subgraphs. Can something be told about the existence of light configurations in graphs with bounded expansion in general? There are obstacles for their existence that need to be forbidden in some way (e.g., $K_{c,n}$ for constant c), which however is the case for planar graphs as well (minimum degree or connectivity constraints are used to avoid $K_{2,n}$).

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