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KOMBINATORIKA FILTRŮ NA PŘIROZENÝCH ČÍSLECH
COMBINATORICS OF FILTERS ON THE NATURAL NUMBERS

Bakalářská práce

Vedoucí práce: Jonathan Verner

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Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl všechny použité prameny a literaturu. Děkuji svému vedoucímu práce panu Jonathanu Vernerovi za vedení bakálářské práce.

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Abstract

Práce se věnuje kombinatorickým vlastnostem filtrů na přirozených číslech. Obsahuje úvod a motivaci do problematiky mezi definovatelností filtrů a jejich kombinatorikou, definice základních typů filrů: P-filtr, Q-filtr, Rapid filtr; upořádání: Rudin-Kiesler, Rudin-Blass, Katětov and Tukey; konstrukce filtrů; základní definice z kombinatoriky na ω; úvod do deskriptivní teorie množin, topologie a základní výsledky.

Abstract

The work is intended to combinatorial properties of filters on natural numbers as an introduction and motivation to the problematics between definability of the filters and its combinatorics. Basic filter types: P-filter, Q-filter, Rapid filter; orders: Rudin-Kiesler, Rudin-Blass, Katětov and Tukey; filter konstructions; basic definitions related to combinatorics on ω; introduction to basic descriptive set theory and topology and some specific results.

Contents

Introduction

The work is intended to combinatorial properties of filters on natural numbers as an introduction to problematics between definability of the filters and its combinatorics. There are basic filter types: P-filter, Q-filter, Rapid filter; orders: Rudin-Kiesler, Rudin-Blass, Katětov and Tukey; filter products. In Chapter I we show basic definitions related to combinatorics on ω. Chapter II contains introduction to basic descriptive set theory and topology. Chapter III contains more specific results. This work is based on Jonathan Verner's and Radek Honzík's lectures.

Chapter I

In this chapter we want to introduce the basic definitios and facts related to the concept of Ultrafilter. An ultrafilter is a truth-value assignment to the family of subsets of a set, and a method of the convergence to infinity. From the logical view it arises a connection to two-valued logic. From the convergence property it has a connection with topology. An ultrafilter could be regarded as subspace of Cantor set ${}^{\omega}2$. The study of ultrafilters concentrates on certain extremal properties.

2.1 Filters

Definition 2.1 (Filter over a set)**.** A *filter* over a set X is a collection F of subsets of X such that:

1. $X \in F$;
2. if $A \in F$ and $B \in F$, then $A \cap B \in F$;
3. if $A, B \subseteq X$, $A \in F$, and $A \subseteq B$, then $B \in F$.

if $A \in F$ or $X \setminus A \in F$, then F is called *ultrafilter.*[1]

A filter F is *proper* if $\emptyset \notin F$. We always consider only proper filters. A filter F is *principal* if $F = \{A \subseteq X; A \supseteq B \subseteq X\}$. Principal filter is generated by the set B.

Observation 2.1. Principal filter is ultrafilter if B is a singleton.

Principal ultrafilter is always generated by one element. That's why the principal[2] ultrafiler is called trivial. An ultrafilter is principal if and only if it contains a finite set, and so an ultrafilter is non-principal (tree) if and only if it contains all cofinite sets.

Definition 2.2. A filter F is *Fréchet filter* if $F = \{A \subseteq X; \neg Fin^{3}(X) \wedge Fin(X \setminus A)\}$

[1] *In the lecture at the fourth international congress held in Rome in 1908, Frigyes Riesz (1880 - 1956) introduced the concept of ultrafilter. Henri Cartan (1904 - 2008) pointed out the usefulness of this concept nearly thirty years after in the articles of Théorie des filters and Filtres et ultrafilters published in Compt. rend. Acad, Sci. Paris (1937). Since then, the concept of ultrafilter has become one of the most remarkable and most delicate concepts of the set theory.*

[2] *The terminology "principal" is imported from ring theory.*

[3] *X is Fin if every nonempty $y \subseteq P(X)$ has maximal element*

It is called *Free filter*. We know that ω is has $\in$ order. Bounded set of $\langle\omega, \in\rangle$ are finite sets of natural numbers. Frechet filter on ω is nonempty linear ordered set without maximal element. From Fréchet filter[4] are finite sets small and co-finite sets are big.
Recall what is limit of sequence of real numbers. Sequence $\langle a_n \mid n \in \omega\rangle$ has limit a if $\forall V\ \forall^\infty n$ (V is neighbourhood of $A \Rightarrow x_n \in V$)

Observation 2.2. Every filter over a finite set is principal. If filter F is on finite set then F is finite set and $A = \bigcap F \in F$ so filter F is generated by the set A.

Observation 2.3. If A is a nonempty family of filters over X, then $\bigcap A$ is a filter over X.

Observation 2.4. If A is a $\subset$-chain of filters over X, then $\bigcup A$ is a filter over X.

Observation 2.5. If F is a filter and $X \in F$, then $P(X) \cap F$ is a filter over X.

Observation 2.6. Let κ be an infinite cardinal, $|S| \leq \kappa$. The set $\{X \subset S \mid |X| > \kappa\}$ is a nonprincipal filter over S.

Definition 2.3 (Finite intersection property FIP)**.** A system of sets has the finite intersection property, FIP if for every $n \in \omega$ and every family $e_0, ..., e_n \subseteq E$ is true:

$$e_0 \cap ... \cap e_n \neq \emptyset.$$

Observation 2.7. Every $E \subseteq P(X)$ with FIP can be extended into a proper filter. F is defined: $F = \{A \subseteq X \mid \exists n \in \omega \exists e_0, ..., \exists e_n e_0 \cap ... \cap e_n \subseteq A\}$.
F is closed under intersection, i.e. that for $A, B \in F$ we have that $X \cap Y \in F$ because if

$$e_0 \cap ... \cap e_n \subseteq A \text{ and } f_0 \cap ... \cap f_m \subseteq B$$

then

$$e_0 \cap ... \cap e_n \cap f_0 \cap ... \cap f_m \subseteq A \cap B$$

Lemma 2.1. A filter F over X is an ultrafilter if and only if it is maximal

[4] *Other Views of filters. Filters on X are the sets $h^{-1}(\{1\})$ for homomorphism h from P(X) to Boolean algebras. All sets which are mapped onto 1.*

Proof. An ultrafilter U is clearly a maximal filter: Assume that $U \subset F$ and $A \in F \backslash U$. Then $X \backslash A \in U$, and so both $X \backslash A \in F$ and $A \in F$, contradiction. Let F be a filter that is not an ultrafilter. We find $F' \supset F$: Let $B \subseteq X$ be such that neither B nor $X \setminus B$ is in F. Consider the family $G = F \cup \{B\}$; we claim that G has the finite intersection property. If $A \in F$, then $A \cap B \neq \emptyset$, for otherwise we would have $A \subseteq X \setminus B$ and $X \setminus B \in F$. If $A_1, ..., A_n \in F$, we have $A_1 \cap .. \cap A_n \in F$ and so

$$B \cap A_1 \cap .. \cap A_n \neq \emptyset$$

G has finite intersection property, and there is a filter $G \subseteq F'$.
Since $B \in F' \setminus F$, we have $F \subset F'$ □

Theorem 2.1 (Tarski's Ultrafilter Theorem)**.** Every filter can be extended to an ultafilter

Proof. Let F_0 be a filter over X. Let P be the set of all filters F over X such that $F_0 \subseteq F$ an consider the partially ordered set $\langle P, \subset \rangle$. If C is a chain in P, then $\bigcup C$ is a filter and an upper bound of C in P. By the Kuratowski-Zorn lemma there exists a maximal element U in P. This U is an ultrafilter by pevious lemma. □

Note that the use of axiom of choice means that the ultrafilter theorem is non-constructive.
Let F be an ultrafilter containing cofinite filter. Then F cannot be generated by {a}, as $X \setminus \{a\} \in F$. Such ultrafilter is called free. If F is not generated by a singleton then it contains the cofinite filter and is free.

A filter F over S is countably complete (σ-complete) if it is closed on countable intersections. Every principal filter is closed under arbitrary intersections.

Definition 2.4 (Filter Base)**.** A filter *Base* over a set X is a collection B of subsets of X such that:

1. if $A \in B$ and $A' \in B$, then $A \cap A' \in B$;
2. $B \neq \emptyset$ and $\emptyset \notin B$.

Given a filter base B, the filter generated by B is defined as the minimum filter containing B. Every filter is also filter base, so the process of passing from filter base can be viewed as a sort of comletion.

Let X be a non-empty set and C be a non-empty subset of X. Then $\{C\}$ is a filter base. The filter generated by C (i.e., the collection of all subsets containing C) is called the principal filter generated by C.

A filter is a free filter if the intersection of all its memebers is empty. A principal filter is not free. No filter of finite set is free, and indeed is the principal filter generated by the common intersection of all its members. A non-principal filter o an infinite set is not necessarily free. A filter is free if and only if contains the Fréchet filter.

Definition 2.5. An ultrafilter F is an *uniform* ultraflter in X if $|A| = |X|$ for every $A \in F$.

A free ultrafilter on an uncountable set X need not be uniform because every proper filter can be extended to an ultrafilter, which follows from the axiom of choice.

Definition 2.6 (Filter Generators)**.** The set is called Generators of the filter F, if all finite intersections of S gives a filter base.

Theorem 2.2 (Pospíšil[5])**.** The number of uniform ultrafilters on an infinite set X is $2^{2^{|X|}}$

If X is finite then each ultrafilter on X is principal, and so there are exactly $|X|$ ultrafilters. In the folowing proof we will assume that $|X| = \kappa$. Firstly we prove the following lemma. Let us call a family C of subsets of κ uniformly independent if for any distinc sets $X_1, ..., X_n, Y_1, ..., Y_m$ in C, the intersection

$$X_1 \cap ... \cap X_n \cap (\kappa \setminus Y_1) \cap ... \cap (\kappa \setminus Y_m)$$

has cardinality κ.

Lemma 2.2. There exists a uniformly independent family of subsets of κ such that $|C| = 2^\kappa$.

Proof. Let consider the set P of all pairs (F, F') where F is a finite subset of κ and F' is a finite set of finite subsets of κ. Since $|P| = \kappa$, it suffices to find auniformly independent family C' of subsets of P, of size 2^κ. For each $u \subseteq \kappa$, let

$$X_u = \{(F, F') \in P \ |F \cap u \in F'\}$$

and let

$$C = \{X_u \ |u \subseteq \kappa\}$$

[5] *Bedřich Pospíšil (1912-1944) proved the theorem in a work On bicompact spaces published in 1939 at Masaryk Univerzity periodicals in Brno. On the request of the most significant contemporary set thery magazine Fundamenta Mathematicae published Pospíšil had sent revised paper of this work to this magazine. In 1941 he was arrested by Gestapo and sentenced to three years in a concentration camp, from where he returned May 17, 1944 but soon succumbed to the consequences of long imprisonment.*

If u and v are distinct subsets of κ, then $X_u \neq X_v$. For example, if $\alpha \in u$ but $\alpha \notin v$, then let $F = \{\alpha\}$, $F' = \{F\}$, and $(F, F') \in X_u$ while $(F, F') \notin X_v$. Hence $|C| = 2^\kappa$.
To show that C is uniformly independent, let $u_1, ..., u_n, v_1, ..., v_m$ be distinct subsets of κ. For each $i \leq n$ and each $j \leq m$, let α_{ij} be some element of κ such that either $\alpha_{ij} \in u_i \setminus v_j$ or $\alpha_{ij} \in v_j \setminus u_i$. Now let F be any finite subset of κ such that $F \supseteq \{\alpha_{ij} \; |i \leq n, j \leq m\}$ (note that there are κ such finite sets). We have

$$\text{for any } i \leq n, j \leq m,\ F \cap u_i \neq F \cap v_j$$

So if we let $F' = \{F \cap u_i \; |i \leq n\}$, we have

$$(F, F') \in X_{u_i}, i \leq n$$
$$(F, F') \notin X_{v_j}, j \leq m$$

Consequently, the intersection

$$X_{u_1} \cap ... \cap X_{u_n} \cap (\kappa \setminus X_{v_1}) \cap ... \cap (\kappa \setminus X_{v_m})$$

has cardinality κ. □

Proof of Pospíšil Theorem. Let C be an uniformly independent family of subsets of κ. For every function $f : C \to \{0, 1\}$, consider this family of subsets of κ:

$$G_f = \{X \; |\kappa \setminus X \; | \leq \kappa\} \cup \{X \; |f(X) = 1\} \cup \{\kappa \setminus X \; |f(X) = 0\}$$

The family G_f has the finite intersection property, and so there exists an ultrafilter D_f such that $D_f \supseteq G_f$. D_f is uniform. If $f \neq g$, then for some $X \in C, f(X) \neq g(X)$; e.g. F(X) = 1 and g(X) = 0 and then $X \in D_f$ while $\kappa \setminus X \in D_g$. Thus we obtain 2^{2^κ} distinct uniform ultrafiters over κ.

□

Definition 2.7 (σ-complete Filter)**.** A filter F over X is countably complete (σ-complete) if $\bigcap\limits_{n=0}^{\infty} X_n \in F$.

Note, every principal filter is closed under arbitrary intersections.

2.2 Orders on filters on ω

There is a ordering of the ultrafilters which says that U is less than V if it is a quotient of V under some mapping of the natural numbers. This idea commonly appears in work on ultrafilters - and on filters and measures as well. The analysis of this various quasi-orders on the class of all ultrafilters on ω provides an information about the global structure of this class. Intuitively, all non-principal ultrafilters on the set ω look pretty much alike (correspond to each other under a suitable permutation of ω), but such a conjecture is denied by a cardinality argument: There are too many ultrafilters and not enough permutations so that there are non-isomorphic non-principal ultrafilters on ω. It is important problem to find the properties that distinguish them.

Folowing pre-orders on filters defined by pre-image of function f. Let $f : \omega \to \omega$ be any function.

Definition 2.8 (image/pre-image of X via function f)**.** For image and pre-image of the set on function f is used notation:

$$f[A] = \{b \mid \exists a \in A(b = f(a))\}$$
$$f^{-1}[B] = \{b \mid \exists a \in B(\langle b, a\rangle \in f)\}$$

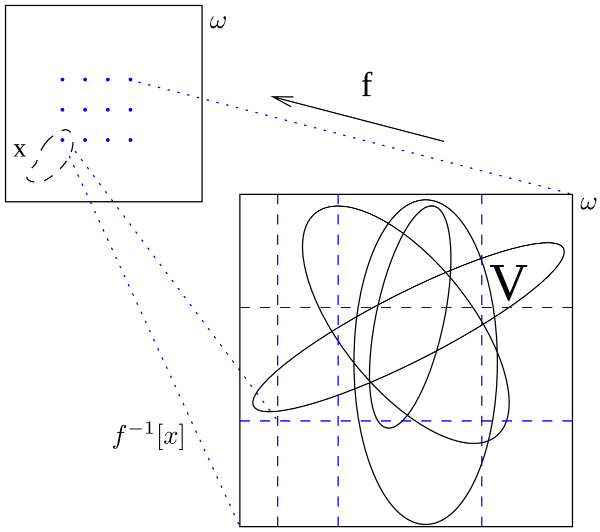
or simply:

$$f[A] = \{f(b) \mid b \in B\}$$
$$f^{-1}[B] = \{b \mid f(a) \in B\}$$

f^{-1} means just inverse relation and $f^{-1}[] : P(\omega) \to P(\omega)$ is again function. Pre-image function pretends intersection: $f^{-1}[A] \cap f^{-1}[B] = f^{-1}[A \cap B]$ and $f[f^{-1}[X]] \subseteq X$.

Definition 2.9 (image of an filter under a function $f : \omega \to \omega$)**.** For $f \in {}^{\omega}\omega$ and filter $V \subseteq P(\omega)$ let

$$f(V) = \{x \subseteq \omega \mid \exists y \in V f[y] \subseteq x\}$$



It is equivalent with:

$$f(V) = \{x \subseteq \omega \mid f^{-1}[x] \in V\}$$

Observation 2.8. If $V \subseteq P(\omega)$ is an ultrafilter over ω, then $U = f(V)$ is also ultrafilter over ω.

Proof. Since $f^{-1}[\omega] = \omega$, so $\omega \in U$, and since $f^{-1}[\emptyset] = \emptyset$ implies $\emptyset \notin U$.
If $x \subseteq x'$ and $x \in f(V)$, then $f[y] \subseteq x$ for some $y \in V$, and therefore $f[y] \subseteq x'$, which shows that $x' \in f(V)$.
If $x, x' \in f(V)$, then $f^{-1}[x], f^{-1}[x'] \in V$, and since V is an ultrafilter,
$f^{-1}[x] \cap f^{-1}[x'] \in V$. Since $f^{-1}[x \cap x'] \in V$ we get $x \cap x' \in f(V)$. □

The next theorem has been prooved by many people including K. Kunen, H.J. Keisler, and M.E. Rudin.

Theorem 2.3. If $f(V) = V$, then $\{n \in \omega \mid f(n) = n\} \in V$

Proof. Let $S = \{n \mid f(n) = n\}$,$R = \{n \mid f(n)r > n\}$, and $T = \{n \mid f(n) < n\}$.
We show that $R \cup T \notin V$.
Suppose that $T \in V$, let $T_n = \{m \mid n$ is least integer such that $f^n(m) \notin T\}$.
Here f^n means the n-fold iterate of f, so that $\bigcup\{T_n \mid n > 0\} = T$. Each of the disjoint sets $\bigcup\{T_{2n} \mid n \in \omega\}$ and $\bigcup\{T_{2n+1} \mid n \in \omega\}$ can be in p only if the other is as well,this gives a conradiction.

Suppose $R_n = \{m \mid n$ is the least integer such that $f^n(m) \notin R\}$. Just as in the previous case one see that neither $\bigcup\{R_{2n} \mid n \in \omega\}$ nor $\bigcup\{R_{2n+1} \mid n \in \omega\}$ can be in V. The set $\omega \setminus \bigcup\{R_n \mid n \in\}$ can be partitioned into two pieces, in the soe manner, such that when one is in V the other is in $f(V) = V$, thus $R \notin V$. □

There are several partial orders on isomorphism types of ultrafilters in the folowing definitions.

Definition 2.10 (Rudin-Keisler order)**.** Let F, G are filters. If there is a function $f : \omega \to \omega$ such that $A \in F$ if and only if $f^{-1}[A] \in G$, then $F \leq_{RK} G$.

$F \equiv_{RK} G$ if and only if there is a permutation $f : \omega \to \omega$ such that $F = \{A \subset \omega \mid f^{-1}(A) \in G\}$. Ultrafilters that are RK equivalent are said to be isomorphic. If $f : \omega \to \omega$ is a function such that $\forall A \in G[f''A \in F]$, then in the case when F and G are ultrafilters on ω, f already witnesses that $F \leq_{RK} G$. The relation $\leq_{RK}$[6] is a quasiorder since the relation is not antisymmetric.

Lemma 2.3. if $f(U) = U$, then $\{n \mid f(n) = n\} \in U$, i.e. f is identity.

Proof. Let $b_1 = \{n \mid f(n) = n\}, b_2 = \{n \mid f(n) < n\}$, and $b_3 = \{n \mid f(n) > n\}$. We show that $b_1 \in U$

If $b_2 \in U$, let $a_n = \{m \mid n$ is the first number such that $f^n(m) \notin b_2\}$. where f^n is n^{th} iteration of f. $\bigcup\limits_{n\leq 1} a_n = b_2 \in U$.

One of $\bigcup\limits_{n\leq 1} a_{2n}$ and $\bigcup\limits_{n\leq 1} a_{2n+1}$ is in U. But $\bigcup\limits_{n\leq 1} a_{2n} \in U$ iff $f[\bigcup\limits_{n\leq 1} a_{2n}] \in U$ iff $\bigcup\limits_{n\leq 1} a_{2n+1} \in U$. This is impossible.

If $b_3 \in U$, again let $c_n = \{m \mid$ is the first number such that $f_n(m) \notin b_3\}$

Similarly, $\bigcup\limits_{n\leq 1} c_n \notin U$. Let d=b $\setminus \bigcup\limits_{n\leq 1} c_n \in U$.

Let $d_0 = \{n \in d \mid n \notin f[d]\}$,

let $d_n = \{m \in d \mid$ n is the least number $m \in f^n[d_0]\}$,

then either $\bigcup\limits_{n\leq 0} d_{2n}$ or $\bigcup\limits_{n\leq 0} d_{2n+1}$ is in U.

But $\bigcup\limits_{n\leq 0} d_{2n} \in U$ iff $f[\bigcup\limits_{n\leq 0} d_{2n}] \in U$ iff $\bigcup\limits_{n\leq 0} d_{2n+1} \in U$.

This is impossible, so b_1 is in U. □

[6] *Kunen was the first who contructed two ultrafilters U and V on ω such that $V \not\leq_{RK} U$ and $U \not\leq_{RK} V$ using only the axioms of ZFC. His techniques actually showed in ZFC alone that the class of ultrafilters on ω has a complicated structure with respect to the ordering $\leq_{RK}$.*

Observation 2.9. $\leq_{RK}$ is a partial order.

Proof. If $U \leq_{RK} V$ and $V \leq_{RK} U$, then $f(U) = V$ and g(V) = U for some $f, g \in {}^{\omega}\omega$. So $f \circ g(U) = U$. $f \circ g(U)$ is the identity on some set $a \in U$, and so g is one-to-one on a. We can split a into two infinite halves b and b', $n \in b$ and $b \in U$ implies that g(n) = g'(n). So $V \cong U$. □

Definition 2.11 (Katětov order)**.** Let F, G are filters. If there is a function $f : \omega \to \omega$ such that $f^{-1}[A] \in G$, for all $A \in F$ then $F \leq_K G$.

Katětov order $\leq_K$ is an extension (the order is more general) of the Rudin-Keisler order to arbitrary filters.

Definition 2.12 (Katětov-Blass order)**.** Let F, G are filters. If there is a finite-to-one function $f : \omega \to \omega$ such that $f^{-1}[A] \in G$, for all $A \in F$ then $F \leq_{KB} G$.

Definition 2.13 (Rudin-Blass)**.** Let F, G are filters. If there is a finite-to-one function $f : \omega \to \omega$ such that $A \in F$ if and only if $f^{-1}[A] \in G$, then $F \leq_{RB} G$.

Definition 2.14 (Tukey order)**.** Let F, G are filters. If there is a function $f : \omega \to \omega$ such that for every $\subseteq$-bounded set $B \in G, f^{-1}[B]$ is $\subseteq$-bounded in F.

The Tukey equivalence class of an ultrafilter is called its Tukey type. $U \equiv_T V$ iff $U \leq_T V$ and $V \leq_T U$ If $V \leq_{RK} U$, then $V \leq_T U$. Thus, every Tukey type is partitioned by some isomorphism types. [Tukey 1940] The directed partial order $([c]^{<\omega}, \subseteq)$ is the maximum Tukey type forall directed partial orders of cardinality c. The to Tukey type has cardinality 2^c, whereas each Rudin-Keisler type has cardinality c. There is no maximal Rudin-Keisler type.

2.3 Ultrafilter constructions

We showed construction of filter via function f. Folowing constructions operate with the set of filters.

Definition 2.15 (Tensor product)**.** Let F, G are filters on ω.
The $F \times G = \{A \subseteq \omega \times \omega \mid \{n \mid A(n) \in G\} \in F\}$ where $A \subseteq \omega \times \omega$ where $A(n)$ is vertical section at n; $A^x(n) = \{m \mid (n, m) \in A\}$
The product $F \times G$ is induced by the base $\{a \times b \mid a \in F \text{ and } b \in G\}$.

This filter can be viewed as a filter on ω if we desired by fixing a bijection between $\omega \times \omega$.

Observation 2.10. Semigroup of filters
Fixing bijection between $\omega \times \omega$ and ω we have:

1. $F \times (G \times H) \equiv (F \times G) \times H$
2. $F \equiv H$, then $F \times G \equiv R \times G$
3. $F \equiv H$, then $G \times F \equiv G \times R$

Definition 2.16 (F-sum)**.** If $\{F_s \mid s \in S\}$ is a set of filters and F is a filter on ω. Then the F-sum of the filters is:
$F - \sum\limits_{s\in\omega} F_s = \{A \subseteq \bigcup\limits_{s\in\omega} \{s\} \times S_s \mid \{s \mid A_x(s) \in F_s\} \in F\}\}$

Definition 2.17 (Free-product filter)**.** Let F, G are filters on ω. $F \otimes G = \{(A, B) \mid A \in F \wedge B \in G\}$

Note. If U, V are ultrafilters then so is $F \times G$. However $U \otimes G$ is never an ultrafilter (e.g. because of the set $\sum\limits_{n<\omega} \{n\} \otimes \omega \setminus n)$

Definition 2.18. If F is a filter, le $[F]^n$ be the filter on $[\omega]^n$ generated by sets of the form $[A]^n$ where $A \in F$. Moreover let $[F]^{<\omega}$ sometimes also denoted $F^{<\omega}$ is the filter on $[\omega]^{<\omega}$ generated by sets of the form $[A]^{<\omega}$ where $A \in F$.

If A is a collection of sets A^+ is closure on finite intersections of A on P(X). We are interested essentially in filters on countable sets.

2.4 Standard combinatorial properties

We define some special sorts of ultrafilters. The first combinational propery of filters, a generalization of the standard P-point property of ultrafilters.

Definition 2.19 (P-filter)**.** A filter F is *P-filter* if for every (descending: $A_0 \supseteq A_1 \supseteq A_2 ...$) countable sequence $\langle A_n \in F \mid n < \omega\rangle$ of elements of F there exists $X \in F$ such that $X \subseteq^* A_n$ for all $n < \omega$. $X \setminus F_n$ is finite.

P-ultrafilters are called P-points (weakly selective). A P-point is a non-principal ultrafilter (A point of topological space is a P-point if its filter of neighbourhoods is closed under countable intersections.)

Observation 2.11. If U is a P-filter, U is an ultrafilter.

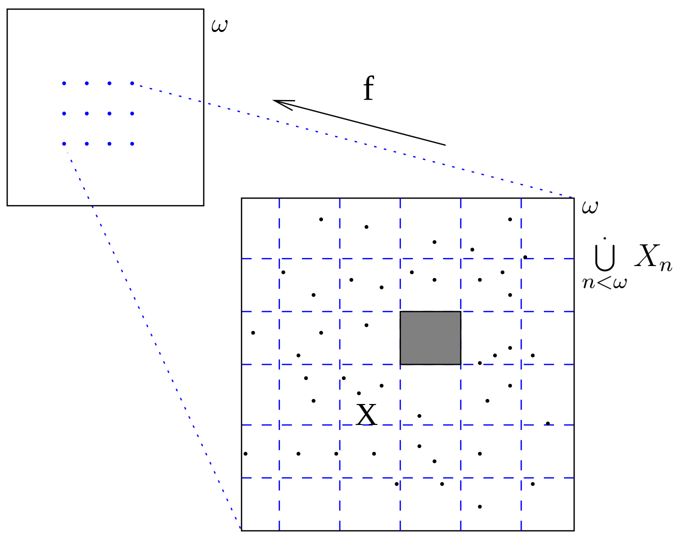
Proof. If $b \notin U$, Complement C(b) is infinite, as U is non-principal, so let $\langle a_n \mid n \in \omega\rangle$ be a partition of C(b). Either $a_n \in U$ for some n, or else there is $a \in U$, $|a \cap a_n| < \omega$ for all n and $|a \cap b| < \omega$. As U is non-principal, in either case $C(b) \in U$. □

Definition 2.20 (P-ultrafilter 1)**.** A ultrafilter U is *P-ultrafilter* (weakly selective): Let there is a ω factoring: $\omega = \dot{\bigcup}_{n<\omega} X_n$ and for U there are satisfied one of folowing items:

1. $\exists n < \omega(X_n \in U)$
2. $\exists (X \in U)(\forall n)(|X \cap X_n| < \omega)$

Definition 2.21 (P-ultrafilter 2)**.** A ultrafilter U is *P-ultrafilter* (weakly selective)
if $(\forall f : \omega \to \omega)(\exists X \in U)(\forall n \in \omega)(|(f \restriction X)^{-1}(n)| < \omega)$,
it means $f \restriction X$ is constant or finite-to-one.

Every function on ω becomes finite-to-one or constant when restricted to some set in U.

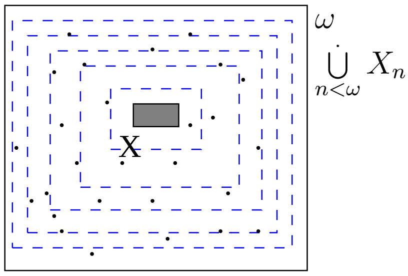


The both previous P-ultrafilter definitions are equivalent using
$f(x) = n \Leftrightarrow x \in X_n$. $\{X_n \; |n < \omega\}$ is ω factorinig. If some $X_n \in U$, we are done. If not, there is a $X \in U$ such that $|f \restriction X| < \omega$. This means $X \cap X_n < \omega$.

Observation 2.12. The definitions of P-filter and P-ultrafilter are equivalent.

Proof. Let there is a factoring $\omega = \dot{\bigcup}_{n<\omega} X_n$. If some set $X_n \in U$ we are finished. If no partition is in ultrafilter we can enumerate theirs complements: $\langle X'_n \,|X'_n = \omega \setminus X_n$ for $n \in \omega\rangle$. For this set exists $X \in U$, and for every $n \in \omega$, $|X \cap X'_n| < \omega$.

The other direction, let $\langle A_n \in U \,|n < \omega\rangle$ is a sequence in U. We may assume without loss of generality that the sequence is strictly decreasing, and $A_0 = \omega$. If U contains the intersection we are finished. If not, let consider the factoring defined $X_n = A_n \setminus A_{n+1}$.



No part this factoring of ω is in U since if $X_n \in U$ then $X_n \cap A_{n+1} = \emptyset \in U$. There is some $X \in U$ where $|X \cap A_n| < \omega$. By induction, $X \subseteq A_0$. Suppose $X \subseteq^* A_n$. $X \cap A_{n+1} = (X \cap A_n) \setminus X_n$, since $X_n \cap X$ is finite we have that $X \cap A_n =^* X \cap A_{n+1}$, so $X \subseteq^* A_{n+1}$.

□

Definition 2.22 (Q-filter)**.** A filter F is *Q-filter* if for every partition P of ω into finite sets there is a selector $A \in F$, set $\forall p \in P(A \cap p \neq \emptyset)$.

Question 2.1. If U is a Q-filter and $1 < k < \omega$ then $[U]^k$ is an ultrafilter.

Definition 2.23 (Rapid-filter)**.** A filter F is *Rapid-filter* if for each function $h \in^\omega \omega$, there is $A \in F$ with $|A \cap h(n)| \leq n$ for every $n < \omega$.

The family of increasing enumerations of elements of Rapid-filter is dominating.

Under CH, can be construct ultrafilters with any combination of the properties: selective, P-point, rapid or negations, except those excluded by the facts that selective implies both P-point and rapid.
In ZFC alone, none of this can be proved. It is consistent that there are no P-points, and it is consistent that there are no rapid ultrafilters.

2.5 Cardinal invariants

There are many cardinal characteristic of the continuum. Following have some connections with the theory of ultrafilters. The continuum could mean $\mathbb{R}$, Cantor space 2^ω, $[\omega]^\omega$ or Baire space ω^ω. These spaces are essentially the same after removal of at most a countable set from each space, there exists a homeomorphism between the modified spaces.
The idea of cardinal characteristics is that for some combinatorial property, $\aleph_0$ and $\mathfrak{c}$ behave differentely. We can look at the least cardinal which behaves like $\mathfrak{c}$, and this is called cardinal characteristic associated with this property. This is, self-evidentelly, uninteresting if CH holds, then all such cardinal characteristics are equal to $\mathfrak{c}$.

Definition 2.24 (Partial pre-orders)**.** $f \leq_F g$ iff $\{n \mid F(n) \leq g(n)\} \in F$.
$f \leq^* g$ iff $\forall^\infty n f(n) \leq g(n)$.

We say f dominates g if $g \leq^* f$.

Definition 2.25. Cardinal invariants
The dominanting number $\mathfrak{d}$ is the minimum cardinality of a dominanting family - a subset of ω^ω such that every g is dominated by an f in the family.
The unbounding number $\mathfrak{b}$ is the minimum cardinality of an unbounded family - a subset of ω^ω not dominated by a single function.
Let $f, g : \omega \to \omega$, and let $F \subseteq P(\omega)$.

$$\mathfrak{d} = min(\{|F| \mid F \subseteq {}^\omega\omega \wedge \forall f : \omega \to \omega\ \exists g \in F(f \leq^* g)\})$$
$$\mathfrak{b} = min(\{|F| \mid F \subseteq {}^\omega\omega \wedge \forall f : \omega \to \omega\ \exists g \in F(g \not\leq^* f)\})$$

A set $S \subseteq \omega$ splits an infinite $A \subseteq \omega$ if both $A \cap S$ and $A \setminus S$ are infinite.
The splitting number $\mathfrak{s}$ is the minimum size of a splitting family - a family $S \subseteq [\omega]^\omega$ such that for all $A \in [\omega]^\omega$, there is $s \in S$ that splits A.
The unsplitting (or reaping) number $\mathfrak{r}$ is the minimum cardinality of an unsplit family - a family $R \subseteq [\omega]^\omega$ such that no single set splits all the sets in R.

$$\mathfrak{s} = min(\{|S| \mid S \subseteq [\omega]^\omega \wedge \forall b \in [\omega]^\omega \exists a \in S(|b \cap a| = \aleph_0 \wedge |b \setminus a| = \aleph_0)\})$$
$$\mathfrak{r} = min(\{|R| \mid R \subseteq [\omega]^\omega \wedge \forall a \in [\omega]^\omega \exists b \in R(|b \cap a| < \aleph_0 \vee |b \setminus a| < \aleph_0)\})$$

We consider nonprincipal ultrafilters on ω. The cardinality of U is $\mathfrak{c}$. We can ask how many sets does it take to generate U (closing under intersections and supersets). This number is called the character of U and is denoted

$\chi(U)$. $\chi(U)$ is always between $\aleph_0$ and $\mathfrak{c}$. The minimum cardinality $\chi(U)$ over all non-principal ultrafilters U on ω is $\mathfrak{r}$. Ultrafilter base has to be unsplit family.

Chapter II

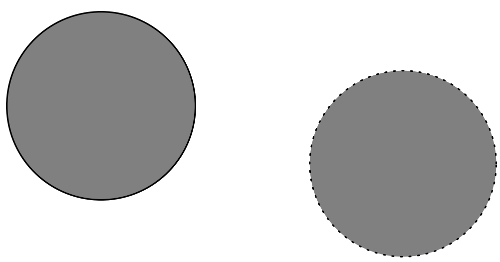
Descriptive set theory deals with sets of reals that are described in some simple way: sets that have simple topological structure (e.g., cotinuous images of closed sets) or are definable in a simple way. The goal of this chapter presents filters on ω in context of their topological and measure-theoretical properties. It means to identify filters on ω with subsets od 2^ω via characteristic functions of their elements. In this way a measurability makes sense. We consider the collection of subsets of natural numbers as a topological space. ${}^2\omega$ carries the ususal topology, and there is the usual measure. Then we may speak about meager filters, measurable filters, filter with the Baire property.

3.1 Topology

Definition 3.1 (Topological space)**.** A *Topological space* is an ordered pair $\langle X, \tau \rangle$, where X is a set and $\tau \subseteq P(X)$ such that:

1. $\emptyset, X \in \tau$;
2. if $A \subseteq \tau$, then $\bigcup A \in \tau$;
3. if $A, B \in \tau$, then $A \cap B \in \tau$.

The collection τ is called topology. Members of the topology are referred to as open sets. The set is called *closed* if its complement is open. The idea behind this definition, at least for the standard spaces, is that an open set is one which contains no point of its boundary. For instance, in 2-dimensional euclidean space, an open disc, meaning the set of points having distance strictly less than some fixed number from a fixed point, forms an open set. Another way of explain this is that wherever in the set is possible to move a little in any direction, and stay in the set. For the closed disc moving any distance may possible leave the set.



Though the definition of closed as the complement of open, it is possible for a set to be both closed and open. In this case the set is called clopen. Obvious examples of clopen sets in all spaces are $\emptyset$ and X, but there may be many more clopen sets than that. The more clopen sets are in the more disconnected spaces.

Definition 3.2 (Neighbourhood)**.** N_x is *neighbourhood* of $x \in X$ if there is an open set O containing x such that $O \subseteq N_x$. If N_x is open, we call it open neigbourhood O_x.

Observation 3.1. Directly from definition, the system of closed sets contains X and $\emptyset$ and is closed under arbitrary intersections and finite unions (De Morgan's laws).

Definition 3.3 (Interior)**.** If Y is a subset of X, let int(Y) be the union of open sets contained in Y.

$$\text{int(Y)} = \bigcup\{O \in \tau \ |O \subseteq Y\}$$

Definition 3.4 (Closure)**.** Let $\overline{Y}$ be the intersection of all closed sets containing Y.

$$\overline{Y} = \bigcap\{C \ |C \text{ is closed and } Y \subseteq C\}$$

Observation 3.2. int(Y) is the greatest open set contained in Y and $\overline{Y}$ is the smallest closed set containing Y in the ordering under inclusion.

Definition 3.5. Set $D \subseteq X$ is *dense* in (X, τ) if $\overline{D}$ = X.

Definition 3.6. Set $B \subseteq P(X)$ is topology *base* if is satisfied

1. for $U, V \in B$ and $x \in U \cap V$ then $\exists W \in B(x \in W \subseteq U \cap W)$.
2. $\forall x \in X \ \exists U \in B(x \in U)$

Definition 3.7 (Compactness)**.** $\langle X, \tau\rangle$ is *compact* if every open cover of X has a finite subcover, where C is an open cover if $C \subseteq \tau$ and $\bigcup C = X$. Conversely if F is a system of closed sets and has FIP then $\bigcup F$ is non-empty.

Definition 3.8 (Filter converges to x)**.** Let F be a filter and $x \in X$. We say that filter converges to x, or that x is a limit of F if all $N_x \subseteq F$.

Example 3.1. Frechet filter F in discrete topology on ω is non-convergent filter. Singleton set {n} cannot belog to F.

Definition 3.9 (Hausdorff space)**.** A *Hausdorff space*[7] is a topological space with a separation property: any two distinct points can be separated by disjoint open sets.

Lemma 3.1. X is hausdorf space if every filter has at most one limit.

Proof. Suppose X is hausdorf and let $x \neq y$. Then there are neighbourhoods U and V of x and y respectively with $U \cap V = \emptyset$. No filter contains both U and V, and so no filter can converge to both x and y. Hence all filters have at most one limit.
Conversely, suppose that x and y do not have disjont neighbourhoods. Then $N_x \cup N_y$ forms a subbase for a filter with converges to both x and y. So if every filter has at most one limit the X is hausdorf. □

So requiring X to be hausdorf is equivalent to requiring unique limits. In hausdorf space we write $\lim_F = x$ to mean x is unique limit of F. Note that not all filters have a limit.

Definition 3.10 (Normal space)**.** A *normal space* is a topological space with a separation property: Any two distinct closed sets can be separated by disjoint open sets.

Definition 3.11 (Continuous function)**.** Let $\langle X, \tau \rangle$, $\langle Y, \sigma \rangle$ are topological spaces and $f : X \to$ Y is function. We say f is *continuous* if for every open set U in Y, $f^{-1}[U]$ is open in X.

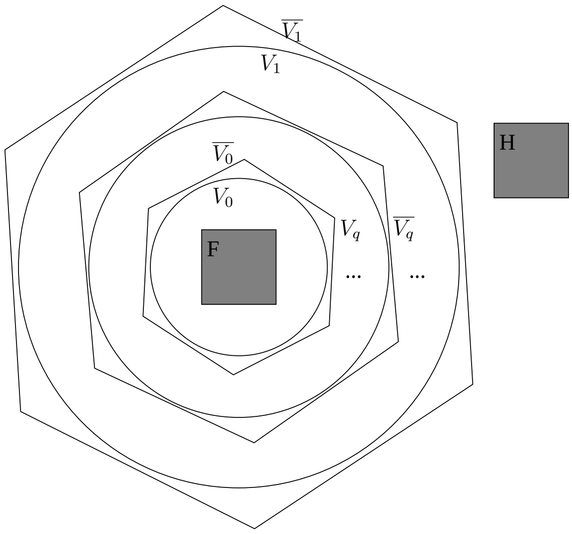
Observation 3.3. A topological space is normal iff for every open set U and every closed $C \subseteq U$, there is an open set V such $C \subseteq V \subseteq \overline{V} \subseteq U$.

Theorem 3.1 (Urysohn's lemma[8])**.** Let $\langle X, \tau \rangle$ is normal space and F, H are closed sets such that $F \cap H = \emptyset$.

Proof. Firstly we construct a system of open sets $\{V_q \mid q \in \mathbb{Q} \cap [0,1]\}$, where is satisfied $V_q \subseteq \overline{V_q} \subseteq V_p \iff q < p \in \mathbb{Q} \cap [0,1]\}$ and $\forall q \in \mathbb{Q} \cap [0,1](F \subseteq V_q \wedge H \cap V_q = \emptyset)$.

[7] *Hausdorff included the separation property in his axiomatic description of general spaces in Grundzüge der Mengenlehre (1914; "Elements of Set Theory"). Although later it was not accepted as a basic axiom for topological spaces, the Hausdorff property is often assumed in certain areas of topological research. It is one of a long list of properties that have become known as "separation axioms" for topological spaces.*

[8] *Urysohn's lemma has the usefull applications. For example Urysohn Metrization Theorem. If X is a normal space with a countable basis, then we can use the continuous function from X to [0, 1] to assign numerical coordinates to the points of X and obtain an embedding of X into $\mathbb{R}^\omega$. From this, every countable normal space is a metric space.*



The construction uses the induction. Firstly we enumerate $\mathbb{Q} \cap [0,1]$ as $\langle q_n \mid n < \omega \rangle$ where $g_0 = 0$ and $q_1 = 1$. Using the previous observation we setup V_0 and V_1 as the first step.
The induction step:
For V_{q_i} we use $k > i$ maximal where $q_i < q_{k+1}$. For $\overline{V_{q_i}} \subseteq V_{q_{i+}}$ we have $V_{q_{k+1}}$ where

$$V_{q_i} \subseteq \overline{V_{q_i}} \subseteq V_{q_{k+1}} \subseteq \overline{V_{q_{k+1}}} \subseteq V_{q_{i+1}}$$

The function f is defined:

$$f(x) = \begin{cases} \inf(\{q \mid x \in \overline{V_q}\}) \text{ if } x \in \bigcup_{q \in \mathbb{Q} \cap [0,1]} \overline{V_q} \\ 1 \text{ otherwise} \end{cases}$$

We show the function f is continuous. Let (q_1, q_2) is open interval in $\mathbb{Q} \cap [0,1]$. Firstly let $q_2 < 1$ then

$$f^{-1}[(q_1, q_2)] = U_{(q_1,q_2)} = \bigcup_{q \in (q_1,q_2)} v_q \setminus \overline{V_{q_1}}$$

is open set. Now let $q_2 = 1$ then

$$f^{-1}[(q_1, q_2)] = U_{(q_1,q_2)} \cup X \setminus \overline{V_1}$$

is open set. □

Theorem 3.2. A topological space X is compact iff every ultrafilter o X converges to at least one point.

Proof. Suppose that X is compact, and let F be an ultrafilter on X. Then F has FIP, since it is closed under finite intersections, and $\emptyset \notin F$. Compactness causes that there is some point $x \in \bigcup_{B \in F} \overline{B}$. This means that every open neighbourhood of x meets every $B \in F$. Let U be an open neigbourhood of x. Since no member of F is disjoint from U, we see that in particular $U^c \notin F$. Since F is an ultrafilter, it must be that $U \in F$. This proves that F converges to x.
For the converse, suppose that every ultrafilter converges and let F be a family of subsets of X that has FIP. Then F generates a filter, which can be extended to an ultrafilter G. By assumption, G converges to some point x. Consider $B \in F$. Since G converges to x, every neigbourhood of x meets B. This says exactly that $x \in \overline{B}$, so, since this is true of every $B \in F$, we have $x \in \bigcup_{B \in F} \overline{B}$. This proves that X is compact. □

Definition 3.12 (P-point)**.** A point p in topological space X is called a *P-point* if the intersection of countably many neighbourhoods of p is again a neighbourhood of p.

Definition 3.13 (Weak P-point)**.** A point in a topological space that is not an acumulation point of any countable subset of the space. Every P-point is a weak P-point.

Let $2^{<\omega}$ denotes the set of all finite sequences of 0,1. The ordering by inclusion of these sequences turns $\langle 2^{<\omega}, \subseteq \rangle$ into a tree. $\langle 2^{<\omega}, \subseteq \rangle$ is the full binary tree of height ω.

Definition 3.14 (Cantor space)**.** The pair $\langle 2^{\omega}, \tau \rangle$ is called *Cantor space* with topology generated by base set: $B = \{o | o \supset s \in 2^{<\omega}\}$ (set of all cofinal branches).

Observation 3.4. Cantor space has a countable base.
(A set of all finite sequences is countable.)

Observation 3.5. Cantor space has a base composed of clopen sets.[9]
(A complement of any base set is union of base sets which differ from the initial sequence.)

3.2 Borel sets

Borel hierarchy is used to describe a collection of subsets of $\mathbb{R}$. Level one consists of all open and closed sets, and levels 2, 3, 4, ... are obtained by taking countable unions ad itersections of previous level.

[9] *The space is totally disconnected.*

Definition 3.15 (F_σ)**.** A subset $F \subset \mathbb{R}$ is F_σ[10] if it is countable union of closed sets.

Example 3.2. *Fin*
Finite subsets of ω, is a countable union of singletons of the Cantor space, a F_σ-set.

Definition 3.16 (G_δ)**.** A subset $G \subset \mathbb{R}$ is G_δ[11] if it is countable intersection of open sets.

3.3 Meager sets

Meagre set of first category is a set that, considered as a subset of a topological space, is in a precise sense small or negligible.

Definition 3.17 (Nowhere dense set)**.** Given a topological space X, a subset A of X is *nowhere dense* if for every non-empty open set O there is a non-empty open set $O' \subseteq O$ such that $O' \cap A = \emptyset$.

A subset B of X is nowhere dense if there is no neighbourhood on which B is dense: for any nonempty open set U in X, there is a nonempty open set V contained in U such that V and B are disjoint.

Definition 3.18 (Meagre set)**.** Given a topological space X, a subset A of X is meagre if it can be expressed as the union of countably many nowhere dense subsets of X.

The rational numbers are meagre as a subset of the reals. The Cantor set is meagre as a subset of the reals, but not as a space, since it is complete metric space.

3.4 Filters and convergence

The notion of the filter converence is a generalization of the classical notion of the convergence of a sequence. The use of filter is way how to talk about convergence in arbitrary topological space. Let N_x be a set of all neighbourhoods of x. The neighbourhood is open set which contains x. N_x has following properties:

[10] *F_σ comes from French: The F stands for fermé, meaning "closed,"while the sigma stands for somme, meaning "sum."*
[11] *G_δ comes from German: The G stands for Gebiet, meaning "area," while the delta stands for Durchschnitt, meaning "intersection."*

1. $X \in N_x$;
2. if $A \in N_x$ and $B \in N_x$, then $A \cap B \in N_x$;
3. if $A, B \subseteq N_x$, $A \in N_x$, and $A \subseteq B$, then $B \in N_x$.
4. $\emptyset \notin N_x$.

The neighbourhood satisfies filter properties. It is called neighbourhood filter.

Recall that a sequence $\langle x_n \mid n \in \omega \rangle$ from some space P is mapping $f : \omega \to P$. The concept of limit is extended. As large sets are regarded the elements of F.

Definition 3.19. F-lim [12] $x_n = a$ if for all neighbourhoods V $\{n \mid x_n \in V\} \in F$

Standard limit is defined: $\lim\limits_{n \to \infty} x_n = a$ iff $\forall \epsilon \exists n_0 \forall n > n_0 (|a_n - a| < \epsilon)$ In other words for all neighbourhoods V of the point a almost all sequence members are in V. This definition of limit is equivalent F-limit with F is Frécher filter.

Observation 3.6. Let S is sequence $\langle x_n \mid n < \omega \wedge x_n \in \mathbb{R} \rangle$ and a is a limit point. $a \in \overline{\{x_n \mid n < \omega\} \setminus \{a\}}$ and $A = \{X \subseteq \omega \mid \lim\limits_{n \in X} x_n = a\}$
If A is non-empty, A is closed under union and subsets. It leads as to the folowing chapter.

[12] *Filter convergence was formulated by Henri Cartan around 1937 and explored by Bourbaki in the 1940s.*

Chapter III

In this chapter we show the basic particular result and its connections between combinatorics and descriptive set theory. We explain the relation between submeasures and ideals on ω.

4.1 Ideals and filters

Definition 4.1 (Ideal over a set)**.** A *ideal* over a set X is a collection I of subsets of X such that:

1. $\emptyset \in I$;
2. if $A \in F$ and $B \in F$, then $A \cup B \in I$;
3. if $A, B \subseteq X$, $A \in F$, and $A \subseteq B$, then $A \in I$.

Given an ideal I we denote by I^* the dual filter, consisting of complements of the sets in I. Similarly, if F is a filter on X, F^* denotes the dual ideal.
$I^* = \{A \subseteq x \mid X \setminus A \in I\}$

Duality between ideals and filters serves to examinate only one of this concepts which is i particular situation better. The sentences could be transformed using De Morgan's laws.

Definition 4.2 (P-ideal)**.** A ideal I is *P-ideal* if for every (descending: $A_0 \supseteq A_1 \supseteq A_2 ...$) countable sequence $\langle A_n \in I \mid n < \omega \rangle$ of elements of I there exists $B \in I$ such that $B \supseteq^* A_n$ for all $n < \omega$. $A \setminus I_n$ is finite.

The ideal convergence is dual to the filter convergence. Let I be an ideal on ω. Let $x_n \in \mathbb{R} (n \in \omega)$ and $x \in \mathbb{R}$. We say that the sequence (x_n) is I-convergent to x if $\{n \in \omega \mid \epsilon \leq |x_n - x|\} \in I$ for every $\epsilon > 0$. We write I-lim $x_n = x$. If I = Fin, then I-convergence is equivalent to the classical convergence.

4.2 Submeasure

There is a close connection between G_σ filters an analytic P-filters, and lower semicontinuous submeasures.

Definition 4.3. A submeasure on a set X is a function $\varphi : P(X) \to [0, \infty]$ satisfying:

1. $\varphi(\emptyset) = 0$,
2. if $A \subseteq B$ then $\varphi(A) \leq \varphi(B)$,
3. $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$,

To avoid trivialities, we require $\varphi(X) < \infty$ for all finite sets X.
If φ is a submeasure on ω and satisfies $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$
then φ is called a *lower semicontinuous submeasure* (lscsm). To each lscsm φ on ω correspond the following ideals:

$$Fin(\varphi) = \{A \subseteq \omega \,|\varphi < \infty\}$$
$$Exh(\varphi) = \{A \subseteq \omega \mid \lim_{n\to\infty} \varphi(\tfrac{A}{n}) = 0\}$$

The first is called the exhaustive ideal of φ an second one the finite ideal of φ.

From definition $Exh(\varphi) \subseteq Fin(\varphi), Fin(\varphi)$ is an F_σ ideal and $Exh(\varphi)$ is an $F_{\sigma\delta}$ P-ideal.

Definition 4.4. A set $A \subseteq P(\omega)$ is *hereditary* if it is closed under subsets.

We note the functions $\cup$ an $\cap$ are continuous as functions in the Cantor space topology.

Lemma 4.1. For any hereditary F_σ set H there exists a family $\{F_n \,|n \in \omega\}$ of hereditary closed sets such that $H = \bigcup_{n\in\omega} F_n$ and $F_n \subseteq F_{n+1}$ for $n \in \omega$.

Proof. Let $H = \bigcup_{n\in\omega} D_n$ where D_n is closed for $n \in \omega$.
$F_n = \{\bigcup_{k\leq n} D_k \cap A \,|A \subseteq \omega\}$ satisfies the condition. □

Theorem 4.1 (Mazur)**.** Let I be an ideal on ω. Then I is an F_σ iff there is a lscsm φ such that I = $Fin(\varphi)$.

The idea of the proof is to define such sets with the indexes which satisfies submeasure conditions.

Proof. By the Lemma we can assume that $I = \bigcup_{n\leq\omega} F'_n$ where each F'_n is hereditary closed and $F'_n \subseteq F'_{n+1}$ for each n. Now we define inductively:
$F_0 = F'_0$
$F_{n+1} = \{x \cup y \,|x, y \in F_n\} \cup F'_{n+1}$
For every $x \in Fin$ we have $f(x) = min(\{n + 1|x \in F_n\})$ which satisfies:

1. $\forall x, y \in Fin$ ($x \subseteq y$ implies $f(x) \leq f(y)$)
2. $\forall x, y \in Fin$ ($f(x \cup y) \leq f(x) \cup f(y)$)
3. $\lim_{n \to \infty} f(n) = +\infty$

If we have $f : Fin \to \mathbb{R}_+ \cup \{\emptyset\}$ then for every n we define

$$F_n = \{x \subseteq \omega | \; \forall k (f(x \cap k) \leq n)\}.$$

For fixed k the set is a finite sum of basic clopen sets, so F_n is closed and $I = \bigcup_{n \leq \omega} F_n$. I is hereditary, closed under finite unions and $\omega \notin I$. □

Example 4.1. $I_{\frac{1}{n}} = \{A \subseteq \omega \mid \sum_{n \in A} \frac{1}{n} < \infty\}$ is F_σ P-ideal where submeasure φ is defined: $\varphi(A) = \sum_{n \in A} \frac{1}{n}$

Analytic P-ideals are characterized by the folowing theorem.

Theorem 4.2 (Solecki)**.** I is an analytic P-ideal iff there is a lscsm φ such that I = $Exh(\varphi)$.
The proof can be found in [Sole97].

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