REPORT ON VÍT TUČEK'S DOCTORAL THESIS 'INVARIANT DIFFERENTIAL OPERATORS FOR 1-GRADED GEOMETRIES'

The thesis presents a rich blend of several areas of Mathematics relaying on Algebra, Geometry, and Analysis, and focusing on computational aspects there. While the motivation comes from understanding the design of differential operators with prescribed symmetries, the infinite dimensional representation theory and cohomological computations are the main tools (and goals) here.

In certain sense, Vít Tuček's work provides rare bridging between seemingly far groups of researchers.

1. The content of the dissertation

The submitted work reflects three papers by Vít Tuček:

(1) **Construction of conformally invariant differential operators**, Hypercomplex analysis and applications, 249-260, Trends Math., Birkhäuser / Springer Basel AG, Basel, 2011.
(2) **Hyperplane section $\mathbb{OP}^2_0$ of the complex Cayley plane as the homogeneous space F_4/P_4** (joint with Pazourek, Karel; Franek, Peter) Comment. Math. Univ. Carolin. 52 (2011), no. 4, 535549.
(3) **Yamabe operator via BGG sequences**, Arch. Math. (Brno) 48 (2012), no. 5, 411422.

All three papers are attached as Appendix C in the Thesis.

Instead of providing the usual short historical survey, and description of the current state of art in the areas related to the attached published results, the main text of the thesis aims at explaining and developing the core of the wider area of mathematics in question. As such, it is rather independent of the three attached papers and it has rather form of a decent short research monograph (more than 80 pages), followed by set of explicit computational results (25 pages) and complete code of the software developed by the author as a part of his work (15 pages).

The aim of the work is impressive. I shall comment more on the four individual parts of the thesis below.

2. The main text

The goal of the three chapters is to lead the reader through the thrilling relations between infinite dimensional modules and the invariant differential operators. While the finite dimensional modules are linked to the cohomological description of the internal structure of the duals to infinite jet modules, the so called generalized Verma modules, in some situations we should start with more general infinite dimensional modules straight from the beginning. More explicitly, the author guides the reader quickly to the infinite dimensional situations were the cohomologies enjoy roughly the same structure as with the finite dimensional modules. This is the content of the first chapter. Starting seemingly from scratch, the reader suddenly stares at the translation principles between the so called O categories, sophisticated equivalencies of Enright and Shelton and the intriguing concept of globalization of the Harish Chandra modules.

All this is necessary to understand the programme coined in the first and third of the attached articles – to discuss the patterns of invariant differential operators for classical parabolic Cartan geometries with the singular infinitesimal character in a similar way as the regular ones.

Clearly this task is too difficult, but still Vít Tuček succeeded to give a fair feeling and overview, including the explanation that actually the programme could work for a very specific class of the geometries only, those related to the so called Hermitian symmetric spaces and the so called unitarizable highest weight modules for them. Those are dealt with in the second chapter, with special focus on the original approach to octonian planes worked out in the second attached paper.

The third chapter is devoted to the unitarizable highest weight modules whose importance was revealed in the previous text. The goal is to compute the nilpotent cohomology of those modules explicitly, and the important cases are described, one by one.

3. The software

As already mentioned, the computational task has been achieved with the help of the original software written by Vít Tuček. The algorithms have been implemented in the open source system Sage. Needles to say that understanding of the algorithms is by itself a highly non-trivial task. Implementing them is a great achievement.

The software is carefully structured and commented, and even quick reading shows that the author met several serious gaps in the procedures of the system, some of them going back to misprints or mistakes in the original papers.

4. Further comments

Perhaps due to the very high demands and challenging goals, the main text would very much need further work and careful polishing in order to become a really useful research monograph.

There are numerous places with chaotic notation, some quite unpleasant misprints, or several undefined symbols (e.g. exchanging Σ and J on several places on p. 12, or the introduction of concepts in the second paragraph on p. 39).

On top of such technicalities, many of the difficult parts would have to be told in much more detail.

5. Conclusions

Summarizing, Vít Tuček has provided a very unconventional and interesting thesis providing survey on relations between infinite dimensional representation theory of Lie groups and the invariant differential operators with certain specific symmetries, including his own essential results and original and potentially very useful general computer software.

Vít Tuček clearly has shown the abilities of an advanced, very broadly educated, and strong researcher. Certainly, the submitted dissertation meets the expectations and I consider this dissertation to be worthy of public defense.

In Warsaw, September 5, 2017

Jan Slovák