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Vít Tuček

Invariant differential operators for 1-graded geometries

Mathematical Institute of Charles University

Supervisor of the doctoral thesis: prof. RNDr. Vladimír Souček, DrSc.
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Author: Vít Tuček

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Abstract: In this thesis we classify singular vectors in scalar parabolic Verma modules for those pairs $(\mathfrak{sl}(n, \mathbb{C}), \mathfrak{p})$ of complex Lie algebras where the homogeneous space $\mathrm{SL}(n, \mathbb{C})/P$ is the Grassmannian of k-planes in $\mathbb{C}^n$. We calculate cohomology of nilpotent radicals with values in certain unitarizable highest weight modules. According to [BH09] these modules have BGG resolutions with weights determined by this cohomology. Such resolutions induce complexes of invariant differential operators on sections of associated bundles over Hermitian symmetric spaces. We describe formal completions of unitarizable highest weight modules that one can use to modify method from [CD01] that constructs sequences of differential operators over any 1-graded (aka almost Hermitian) geometry. We suggest uniform description of octonionic planes that could serve as a basis for better understanding of the exceptional Hermitian symmetric space for group E_6.

Keywords: Hermitian symmetric space, unitarizable highest weight module, nilpotent Lie algebra cohomology, octonionic plane

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I dedicate this work to my parents.

Contents

Introduction

This thesis started with investigation of the higher symmetries of the Laplacian [Eas05; Tuč11]. These are differential operators on $\mathbb{R}^n$ that preserve the space of harmonic functions and one can construct them using the so called ambient construction. The explicit construction of invariant operators is in general difficult problem. For parabolic geometries (of which is the conformal geometry prominent example) this is equivalent to finding homomorphism of parabolic Verma modules. These homoromphisms are in turn completely determined by so called singular vectors. Recently there has been new development which uses isomorphism of parabolic Verma modules with differential operators with constant coefficients where the action of the Lie algebra in question is rather complicated [KT17] but which is rather straightforward to calculate for the so called $|1|$-graded parabolic geometries [Kob+15]. One can then apply algebraic Fourier transform to reinterpret the set of singular vectors as polynomial solutions of system of PDEs with polynomial coefficients. Since these PDEs come from the Lie algebra action one can use representation theory to considerably simplify this problem. This joint work with Libor Křižka is presented in the section 2.4.1.

Not all these invariant differential operators are natural. Meaning that not all invariant operators acting between sections of associated bundles over the homogeneous spaces G/P admit curved analogues which would act on sections of bundles associated to a Cartan geometry modeled on the pair (G, P). The counterexample is a conformally invariant power of the Laplace operator [Gra92; GH04]. There is however a big class of operators that always admits curved analogues and moreover with quite favourable properties. For each finite-dimensional irreducible representation of $\mathfrak{g}$ there is a sequence of such operators which even form a complex in the flat case. This is the famous resolution of Bernstein, Gelfand and Gelfand [BGG75]. The existence of curved analogues was proved in [ČSS01] and this construction was simplified in [CD01]. Among these so called BGG operators are various interesting ‘geometric’ operators whose kernels give e.g. projective Killing tensor fields or conformal Killing spinors. All these operators have the property that the dimension of their solution space is bounded above by the dimension of the so called tractor bundle associated to the representation one has started with. In other words, these operators are all overdetermined as this property holds even locally and thus for any small subset of the manifold the dimension of solution space is finite-dimensional. Thus if one wants to obtain operators such as conformally invariant modification of the Laplace operator, one has to consider infinite-dimensional representations. Moreover, the article [STV06] shows that in all cases the operator is determined by the nilpotent cohomology as in the finite dimensional case. It turns out that the construction of [CD01] with little modification goes through also for a certain infinite-dimensional representation. Basically the same proof works for any unitarizable module of highest weight. For details we refer to [Tuč12] which is contained in the appendix C. These modules occur only for so called Hermitian symmetric spaces which are recalled in chapter 2. There are five classical series of such spaces and two exceptional ones. Thus one is lead to investigate the nilpotent cohomology of this class of modules. This is investigated in chapter 3. In contrast with the finite-dimensional representations where the structure of the cohomology for abelian nilradical is much simpler than for nilradical of a general parabolic subalgebra, the cohomology of unitarizable highest weight modules contains considerable combinatorial obstacle. Namely, one has to consider intersection of the cone(s) of unitarizable weights with the cones given by root hyperplanes. This can get pretty complicated. In the end, a computer program, whose listing is in appendix B, was written and can be used to compute not only cohomologies of unitarizable weights but also the finite-dimensional ones even in the relative case. A table of results for low ranks is compiled in appendix A.

Going back to the starting example of the ambient construction for $\mathbb{R}^n$ we can motivate the last original part of this thesis. The ambient construction in the flat case amounts to working on $\mathbb{R}^{n+1,1}$ or rather it's projectivization where the $\mathbb{R}^n$ is conformally compactified as the sphere S^n. The isometry group $\mathrm{SO}(n+1,1)$ of the ambient space induces conformal transformation of this embedded S^n. The projectivization of the two-sheeted hyperboloid of the defining quadric provides a model for the hyperbolic space. All points of the projectivization of $\mathbb{R}^{n+1,1}$ are lines and as such can be identified with orthogonal projectors of rank 1. There is a well known scalar product on matrices and one can wonder what kind of Riemannian strucutre it induces on subvariety of rank 1 idempotents. This leads to uniform description of octonionic planes which were usually defined as homogeneous spaces of their isometry group. These groups are of type F_4 and as such are a bit tricky to define. The usual way is to define them as automorphism groups of certain Jordan algebras. The article [HSV09] provided elementary definition of these octonionic planes on case by case basis. The caveat being that the identification with the homogeneous spaces uses classification of so called Osserman manifolds. We provide uniform and much simpler way to obtain this elementary description of octonionic planes in section 2.3. Complexification of the projective octonionic plane then gives one of the two exceptional Hermitian symmetric spaces.

1. Lie algebras and their modules

In this chapter we fix some notation and review several structural results about Lie algebras. First we start with some basic definitions concerning complex Lie algebras.

1.1 Complex simple Lie algebras

Let $\mathfrak{g}^1 = \mathfrak{g}$ and define inductively the so called *lower central series* of $\mathfrak{g}$ by $\mathfrak{g}^{k+1} = [\mathfrak{g}, \mathfrak{g}^k]$. A subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ is called *nilpotent* if $\mathfrak{n}^k = 0$ for some $k \in \mathbb{N}$.

Let $\mathfrak{g}^{(1)} = \mathfrak{g}$ and define inductively the so called *derived series* of $\mathfrak{g}$ by $\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$. A subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is called *solvable* if $\mathfrak{b}^{(k)} = 0$ for some $k \in \mathbb{N}$. A *Borel subalgebra* $\mathfrak{b}$ of $\mathfrak{g}$ is any maximal solvable subalgebra of $\mathfrak{g}$.

We denote by $\mathrm{ad}(X)$ the Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{g}$ given by $Y \mapsto [X, Y]$. This is the *adjoint representation* of $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ is said to be *semisimple* if it has no nonzero solvable ideal and it is called *simple* if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and the only ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$. Any semisimple Lie algebra is a direct sum of simple Lie algebras. An *semisimple element* X of $\mathfrak{g}$ is an element of $\mathfrak{g}$ such that $\mathrm{ad}\, X : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable.

A *Cartan subalgebra* of a complex Lie algebra $\mathfrak{g}$ is a maximal commutative subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ consisting of semisimple elements. For a real Lie algebra $\mathfrak{g}_\mathbb{R}$ we define its Cartan subalgebra $\mathfrak{h}_\mathbb{R}$ as such that its complexification $\mathfrak{h}_\mathbb{C}$ is a Cartan subalgebra of the complexification of $\mathfrak{g}_\mathbb{R}$.

A *reductive* Lie algebra $\mathfrak{g}$ is a Lie algebra that decomposes as $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$, where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$ and the algebra $\mathfrak{g}_{ss} = [\mathfrak{g}, \mathfrak{g}]$ is the *semisimple part* of $\mathfrak{g}$. Of course, as its name suggest, the algebra $\mathfrak{g}_{ss}$ is semisimple.

Let $\mathfrak{g}$ be a real or complex Lie algebra let B be a bilinear form $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{k}$. We say that B is invariant if all $\mathrm{ad}\, X$, $X \in \mathfrak{g}$ are skew-symmetric operators relative to B, i.e.

$$B([X, Y], Z) = -B(Y, [X, Z]), \quad \forall X, Y, Z \in \mathfrak{g}.$$

The most important invariant form is the so called *Killing form* which we will denote B and which exists for any Lie algebra $\mathfrak{g}$. It is defined by

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ ad(Y)).$$

If $\phi : \mathfrak{g} \to \mathfrak{g}$ is any automorphism of the Lie algebra $\mathfrak{g}$, then by definition $\mathrm{ad}(X) \circ \phi = \phi \circ \mathrm{ad}(X)$ for any $X \in \mathfrak{g}$. This implies

$$\mathrm{ad}(\phi(X)) \circ \mathrm{ad}(\phi(Y)) = \phi \circ \mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \phi^{-1}$$

and thus $B(\phi(X), \phi(Y)) = B(X, Y)$. The Cartan criterion states that a Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form B is nondegenerate.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and choose a Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$. Any two Cartan subalgebras of $\mathfrak{g}$ are conjugate by an inner automorphism of $\mathfrak{g}$. The roots Φ of $(\mathfrak{g}, \mathfrak{h})$ are linear functionals $\alpha : \mathfrak{h} \to \mathbb{C}$ such that the corresponding *root space* $\mathfrak{g}_\alpha = \{X \in G \,|\, \forall H \in \mathfrak{h} : [H, X] = \alpha(H)X\}$ is nonempty.

In other words, roots are the weights of the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}$. The roots form a finite subset $\Phi \subset \mathfrak{h}^*$ and we have the *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

If $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, then $B(X, Y) = 0$ unless $\alpha = -\beta$. Thus the Killing form induces a nondegenerate pairing on $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$. The restriction of B to $\mathfrak{h}$ is nondegenerate and, in particular, for each linear functional $\lambda \in \mathfrak{h}^*$, there is a unique element $\widetilde{H_\lambda} \in \mathfrak{h}$ such that $\lambda(H) = B(H, \widetilde{H_\lambda})$ for all $H \in \mathfrak{h}$. We can define the bilinear form $(\, , \,)$ on $\mathfrak{h}^*$ by

$$(\lambda, \mu) = B(\widetilde{H_\lambda}, \widetilde{H_\mu}).$$

The restriction of $(\, , \,)$ to the real span of Φ is positive definite (and in particular has real values).

Let us summarize the properties of Φ

1. For any $\alpha \in \Phi$ the only nontrivial complex multiple of α that is also a root is $-\alpha$, i.e.
$$z\alpha \in \Phi, \quad z \in \mathbb{C} \Longleftrightarrow z \in \{1, -1\}.$$

2. The roots spaces $\mathfrak{g}_{-\alpha}$ are one-dimensional and the subspace spanned by $\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha$ and $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha]$ is a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

3. For $\alpha, \beta \in \Phi$, $\beta \neq -\alpha$ we have
$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} \mathfrak{g}_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi \\ 0 & \text{otherwise.} \end{cases}$$

4. Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$ and $z \in \mathbb{C}$. A functional $\alpha + z\beta$ can be a root if and only if $z \in \mathbb{Z}$. The set of such roots form an unbroken chain
$$\beta - p\alpha, \beta - (p-1)\alpha, \ldots, \beta + (q-1)\alpha, \beta + q\alpha,$$
where $p, q \geq 0$ and $p - q = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

For a root $\alpha \in \Phi$ we define the *coroot* $\alpha^\vee$ as

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)} \widetilde{H_\alpha}.$$

Nonzero vectors in $\mathfrak{g}_\alpha$ are called a *root vectors*. From the properties of Φ follows that we can always find root vectors $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ such that the triple $E_\alpha, F_\alpha, H_\alpha$ satisfies the canonical $\mathfrak{sl}(2)$ relations

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F$$

Choose a basis $v_1, \ldots, v_r$ of $\mathfrak{h}$ and define a linear functional $\lambda \in \mathfrak{h}^*$ to be *positive* if there is an index j such that $\lambda(v_i) = 0$ for $i < j$ and $\lambda(v_j) > 0$. The *positive roots* Φ^+ are then the roots which are positive and we obtain a disjoint union $\Phi = \Phi^+ \cup \Phi^-$. The *negative roots* Φ^- are defined as $-\Phi^+$ and we will write

$\alpha > 0$ or $\alpha < 0$ to indicate whether the root is positive or negative. By Δ we will denote the set of *simple roots* associated with Φ and the notion of positivity for $\mathfrak{h}^*$. The simple roots are define as the set of those positive roots, which cannot be written as a sum of positive roots. The set of simple roots Δ forms a basis of $\mathfrak{h}^*$. Alternatively, one can start with a subset Δ of Φ that form a basis of $\mathfrak{h}^*$ and declare it to be the set of simple roots. The positive and negative roots are then obtained as positive or negative linear combinations of elements of Δ.

Let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be a system of simple roots for $(\mathfrak{g}, \mathfrak{h})$. The *Cartan matrix* is defined as

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

A *Dynkin diagram* is defined as a graph with vertex for each simple root and ith and jth vertex are joined by $a_{ij}a_{ji}$ many edges. If a two joined vertices correspond to roots of different lengths, one orients the edges by arrow pointing from the longer root to the shorter one.

For a root $\alpha \in \Phi$ we define *root reflection* s_α as

$$s_\alpha : \phi \mapsto \phi - \frac{2(\phi, \alpha)}{(\alpha, \alpha)}\alpha.$$

It is a reflection with respect to the hyperplane orthogonal to α on the Euclidean space formed by the real span of Φ endowed with the restriction of $(\, , \,)$. The set of roots is preserved under root reflections $s_\alpha(\Phi) = \Phi$ and the group $W(\Phi)$ generated by these reflections is known under the name *Weyl group*. In fact, it is sufficient to take reflections with respect to simple roots to generate the whole Weyl group, i.e. $W(\Delta) = W(\Phi)$. The *length function* l on W is defined via the minimal number of simple reflections needed to express $w \in W$.

1.1.1 Borel and parabolic subalgebras

Given a complex Lie algebra $\mathfrak{g}$ with a chosen Cartan subalgebra $\mathfrak{h}$ and a system of positive roots Φ^+ we get a *Cartan decomposition* (*triangular decomposition*) as

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}, \quad \text{where } \mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \text{ and } \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha.$$

The Lie subalgebras $\mathfrak{n}$, $\mathfrak{n}^-$ are nilpotent and $\mathfrak{n}^-$ is called the *opposite* Lie subalgebra to $\mathfrak{n}$. The subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is the *standard Borel subalgebra* and we will denote by $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$ the opposite Borel subalgebra. It is clear from the triangular decomposition that $\mathfrak{b} \cap \mathfrak{b}^- = \mathfrak{h}$. All Borel subalgebras are conjugated by inner automorphism to the standard Borel subalgebra.

A *parabolic subalgebra* $\mathfrak{p}$ of $\mathfrak{g}$ is a subalgebra that contains a Borel subalgebra, *standard parabolic subalgebra* is then a subalgebra that contains the standard Borel subalgebra. All parabolic subalgebras are conjugated by inner automorphism to a standard parabolic subalgebra.

Any parabolic subalgebra $\mathfrak{p}$ has a decomposition

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$$

into its Levi part $\mathfrak{l}$ and nilpotent part $\mathfrak{u}$. The Levi part is a reductive Lie subalgebra of $\mathfrak{g}$.

Standard parabolic subalgebras are classified by subset of simple roots (Proposition 3.2.1 of [ČS09]). To a standard parabolic subalgebra $\mathfrak{p}$ we assign the subset

$$\Sigma_{\mathfrak{p}} = \{\alpha \in \Delta : \mathfrak{g}_{-\alpha} \nsubseteq \mathfrak{p}\}.$$

Conversely, the standard parabolic subalgebra $\mathfrak{p}_\Sigma$ corresponding to a subset $\Sigma \subseteq \Delta$ is the sum of the standard Borel subalgebra $\mathfrak{b}$ and all negative root spaces corresponding to roots Φ_Σ which can be written as a linear combination of elements of $\Delta \setminus \Sigma$

$$\mathfrak{p}_\Sigma = \mathfrak{b} \oplus \sum_{\alpha \in \Phi_\Sigma} \mathfrak{g}_{-\alpha}.$$

In particular, given $\Sigma \subset \Delta$ we get

$$\mathfrak{l} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi_\Sigma} \mathfrak{g}_\alpha, \quad \mathfrak{u} = \sum_{\alpha \in \Phi^+ \setminus \Phi_\Sigma} \mathfrak{g}_\alpha.$$

For $\Sigma \subseteq \Sigma' \subseteq \Delta$ we have $\mathfrak{p}_{\Sigma'} \leq \mathfrak{p}_\Sigma \leq \mathfrak{g}$ and the two extreme choices $\Sigma = \emptyset$, $\Sigma = \Delta$ lead to $\mathfrak{p} = \mathfrak{g}$ and $\mathfrak{p} = \mathfrak{b}$ respectively.

The *opposite parabolic* subalgebra $\overline{\mathfrak{p}}$ is obtained by switching negative roots to positive and vice versa. In other words, $\mathfrak{g}_\alpha$ is contained in $\overline{\mathfrak{p}}$ if and only if $\mathfrak{g}_{-\alpha}$ is in $\mathfrak{p}$.

It is convenient to denote parabolic subalgebras by Dynkin diagrams with crossed or otherwise marked nodes. Namely, the standard parabolic subalgebra $\mathfrak{p}_\Sigma$ of $(\mathfrak{g}, \Phi^+)$ is denoted by the Dynkin diagram of $\mathfrak{g}$ where the nodes corresponding to Σ are represented by crosses instead of dots. By erasing of these crossed nodes one obtains the Dynkin diagram of the semisimple part of $\mathfrak{l}$ and the crossed nodes correspond precisely to the generators of the center of $\mathfrak{l}$.

Elements of $\mathfrak{h}^*$ are called weights. Weights λ such that (λ, α) is greater (less) or equal to zero for all $\alpha \in \Phi_\Sigma$ are called $\mathfrak{p}$-*dominant* ($\mathfrak{p}$-*antidominant*). In case they satisfy this condition for all roots $\alpha \in \Phi$ we call them $\mathfrak{g}$-(anti)dominant or just (anti)dominant. The weights dual to simple coroots are called fundamental weights. There is a distinguished element of $\mathfrak{h}^*$ sometimes called the lowest form and usually denoted by ρ. It is defined as the sum of fundamental weights or equivalently as half the sum of positive roots.

The following lemma (whose proof can be found in [ČS09]) shows that the parabolic subalgebras are equivalent to gradings of the lie algebra $\mathfrak{g}$.

Lemma 1.1.1. *There is a bijective correspondence between parabolic subalgebras of $\mathfrak{g}$ and gradings $\mathfrak{g} = \oplus_{i=-k}^{k} \mathfrak{g}_i$ of $\mathfrak{g}$.*

Given $\Sigma \subset \Delta$, the set $\mathfrak{g}_i$ $(i \neq 0)$ is defined to be $\oplus_{\phi \in A_i} \mathfrak{g}_\phi$, where A_i contains elements $\phi = \sum_{\alpha_j \in \Delta} c_j \alpha_j$ such that $\sum_{\{j : \alpha_j \in \Sigma\}} c_j = i$, and $\mathfrak{g}_0 = \mathfrak{h} \oplus_{\phi \in A_0} \mathfrak{g}_\phi$.

Given a grading $\oplus_j \mathfrak{g}_j$, the parabolic subalgebra is then $\mathfrak{p} = \oplus_{j \geq 0} \mathfrak{g}_j$.

So we can see that given a parabolic Lie algebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ there is a grading on $\mathfrak{g}$ such that $\mathfrak{g}_0 = \mathfrak{l}$ and $\mathfrak{u} = \bigoplus_{i \geq 1} g_i$. The associated filtration to a grading $\mathfrak{g} = \oplus_{i=-k}^{k} \mathfrak{g}_i$ of $\mathfrak{g}$ is defined by $\mathfrak{g}^i = \bigoplus_{j \geq i} \mathfrak{g}_j$. Sometimes it's convenient to use the following notation as well $\mathfrak{g}_+ = \bigoplus_{i>0} \mathfrak{g}_i$ and $\mathfrak{g}_- = \bigoplus_{i<0} \mathfrak{g}_i$. The next lemma shows structural properties of this grading as well as relationship with the Killing form.

Lemma 1.1.2 (Proposition 3.1.2 of [ČS09])**.** *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra over $\Bbbk = \mathbb{R}$ or $\mathbb{C}$ and let $B : \mathfrak{g} \otimes \mathfrak{g} \to \Bbbk$ be a nondegenerate invariant bilinear form. Then we have:*

1. *There is a unique element* $E \in \mathfrak{g}$, *called the* grading element, *such that* $[E, X] = jX$ *for all* $X \in \mathfrak{g}_j$. *The element* E *lies in the center of the subalgebra* $\mathfrak{g}_0 \leq \mathfrak{g}$.

2. *The* $|k|$*-grading on* $\mathfrak{g}$ *induces a* $|k_i|$*-grading for some* $k_i \leq k$ *on each ideal* $s \subseteq \mathfrak{g}$. *In particular,* $\mathfrak{g}$ *is a direct sum of* $|k_i|$*-graded simple Lie algebras, where* $k_i \leq k$ *for all* i *and* $k_i = k$ *for at least one* i.

3. *The isomorphism* $\mathfrak{g} \to \mathfrak{g}^*$ *provided by* B *is compatible with the filtration and the grading of* $\mathfrak{g}$. *In particular,* B *induces dualities of* $\mathfrak{g}_0$*-modules between* $\mathfrak{g}_i$ *and* $\mathfrak{g}_{-i}$ *and the filtration component* $\mathfrak{g}^i$ *is exactly the annihilator (with respect to* B*) of* $\mathfrak{g}^{-i+1}$. *Hence,* B *induces a duality of* $\mathfrak{p}$*-modules between* $\mathfrak{g}/\mathfrak{g}^{-i+1}$ *and* $\mathfrak{g}^i$, *and in particular between* $\mathfrak{g}/\mathfrak{p}$ *and* $\mathfrak{p}_+$.

4. *For* $i < 0$ *we have* $[\mathfrak{g}_{i+1}, \mathfrak{g}_{-1}] = \mathfrak{g}_i$. *If no simple ideal of* $\mathfrak{g}$ *is contained in* $\mathfrak{g}_)$, *then this also holds for* $i = 0$.

5. *Let* $A \in \mathfrak{g}_i$ *with* $i > 0$ *be an element such that* $[A, X] = 0$ *for all* $X \in \mathfrak{g}_{-1}$. *Then* $A = 0$. *If no simple ideal of* $\mathfrak{g}$ *is contained in* $\mathfrak{g}_)$, *then this also holds for* $i = 0$.

In particular, the Killing form B has the following anti-diagonal block matrix form with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \mathfrak{g}_k$

$$B = \begin{pmatrix} 0 & \cdots & B_{k,-k} \\ \vdots & \ddots & \vdots \\ B_{-k,k} & \cdots & 0 \end{pmatrix},$$

where $B_{i,j}$ denote the restriction of B to $\mathfrak{g}_i \otimes \mathfrak{g}_j$.

Since the grading element E is in the center of $\mathfrak{g}_0 = \mathfrak{l}$ it acts by scalar on any irreducible $\mathfrak{l}$-module $\mathbb{V}$. We will call this scalar *geometric weight* of $\mathbb{V}$.

The Weyl group has natural action on weights by orthogonal transformations and we call a weight singular (regular) if $(\lambda + \rho, \alpha)$ is (not) zero. The reason for this ρ shift is that in applications one has to consider not the defining acting of the Weyl group W but rather it's *affine action*

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

Let w be an element of W. A *reduced expression* for w is $w = s_{i_1} \cdots s_{i_k}$ where k is as small as possible and all s_{i_j} are simple reflections. In particular $l(w) = k$. The *Bruhat order* on W is defined as follows: $w' \leq w$ if and only if a reduced expression for w' is subexpression of some reduced expression of w. For any standard parabolic subalgebra $\mathfrak{p}_\Sigma$ one can form the parabolic Weyl group W_Σ which is generated by simple reflections belonging to roots of $\mathfrak{l}_\Sigma$. It is a theorem of Kostant [Kos61] that for any $w \in W$ there is a unique decomposition $w = w_\Sigma w^\Sigma$ such that $l(w) = l(w_\Sigma) + l(w^\Sigma)$. The element w^Σ is called *minimal length representative* of w and one of it's fundamental properties is that it maps $\mathfrak{g}$-dominant weights to $\mathfrak{p}$-dominant weights. The set of all minimal length representative is denoted by W^Σ and it inherits a partial order from W.

1.2 BGG category $\mathcal{O}$

This chapter contains the description of Bernstein-Bernstein-Gelfand category $\mathcal{O}$ and a small recapitulation of Enright-Shelton equivalences which will be needed later. In this section we will denote the Levi lie group / algebra by K / $\mathfrak{k}$ since later we will consider only the cases when the Levi subalgebra is actually the complexification of maximal compact subalgebra of certain real form of $\mathfrak{g}$.

For any $\lambda \in \mathfrak{h}^*$ we denote by $\mathbb{C}_\lambda$ the one dimensional representation of $\mathfrak{h}$ on with character λ. We extend the action to $\mathfrak{b} := \mathfrak{n} \oplus \mathfrak{h}$ by letting $\mathfrak{n}$ act trivially. The *Verma module* $N(\lambda)$ is then defined as $N(\lambda) := \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} \mathbb{C}_\lambda$. We will denote its highest weight vector by v_λ.

Let λ be a $\mathfrak{k}$-dominant and integral weight and denote by $F(\lambda)$ the finite dimensional irreducible $\mathfrak{k}$-module. We can extend any irreducible representation of $K_\mathbb{C}$ to P and to $\overline{P}$ by letting $\mathfrak{p}_+$ and $\mathfrak{p}_-$ act trivially. The *generalized* or *parabolic Verma module* $M(\lambda)$ is defined as $M(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} F(\lambda)$. In cases where we will need to deal with Verma modules for different algebras we will denote them by M_Σ where Σ is the subset of Δ defining the parabolic subalgebra $\mathfrak{p}$.

It is well known and easy to prove that $M(\lambda)$ contains a maximal nontrivial submodule $J(\lambda)$ and we denote by $L(\lambda)$ the corresponding irreducible quotient of $M(\lambda)$. We can easily see that $M(\lambda) \simeq S(\mathfrak{p}_-) \otimes F(\lambda)$ as $K_\mathbb{C}$ representations, where $S(\mathfrak{p}_-)$ is the symmetric algebra over the Lie algebra $\mathfrak{p}_-$. In the case of $M(\lambda)$, the geometric weight corresponds to the polynomial degree shifted by the weight of $F(\lambda)$.

The *(parabolic) category* $\mathcal{O}_\mathfrak{p}$ is the full subcategory of $\mathfrak{U}(\mathfrak{g})$ modules whose objects M satisfy:

1. M is a finitely generated $\mathfrak{U}(\mathfrak{g})$ module
2. Viewed as a $\mathfrak{U}(\mathfrak{l})$ module, M is a direct sum of finite dimensional simple modules.
3. M is locally $\mathfrak{u}$-finite

1.2.1 Translation principle

The whole category $\mathcal{O}^\mathfrak{p}$ decomposes into so called *infinitesimal blocks* $\mathcal{O}^\mathfrak{p}_\mu$, where μ is an antidominant weight of Φ. These blocks are full subcategories consisting of modules on which the center of the universal enveloping algebra acts by a character induced from μ. One can move between different blocks using so called translation functors which in favorable cases provide equivalence of these subcategories. These functors map Verma modules to Verma modules and simple modules to simple modules. The favorable cases are determined by so called facets which are in turn defined via root hyperplanes. The bottom line is that once two weights μ, μ' have the same signs of scalar products with positive roots the infinitesimal blocks $\mathcal{O}^\mathfrak{p}_\mu$ and $\mathcal{O}^\mathfrak{p}_{\mu'}$ are equivalent. See chapter 7 of [Hum08] for details.

1.2.2 Shapovalov form

Let σ be involutive antiautomorphism on $\mathfrak{U}(\mathfrak{g})$ such that it's restriction on the real form $\mathfrak{g}_0$ is $-\mathrm{Id}$. With our choice of Cartan subalgebra $\mathfrak{h}$ we have that

$\sigma : X_\beta \mapsto X_{-\beta}$ for $X_\beta \in \mathfrak{g}_\beta$ and $\sigma : h_\alpha \mapsto h_\alpha$ for $h_\alpha \in \mathfrak{h}$.

Let $P : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{h})$ be the projection defined by the splitting

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{h}) \oplus (\mathfrak{n}_- \mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{n}_+)$$

Definition 1.2.1. *The* universal Shapovalov form *on* $\mathfrak{U}(\mathfrak{g})$ *is defined as*

$$\langle u_1, u_2 \rangle = P(\sigma(u_1)u_2).$$

It is a bilinear form on $\mathfrak{U}(\mathfrak{g})$ *with values in* $\mathfrak{U}(\mathfrak{h}) = S(\mathfrak{h})$.

For $\lambda \in \mathfrak{h}^*$ *we define the* Shapovalov form on the Verma module $N(\lambda)$ *by*

$$\langle u_1 v_\lambda, u_2 v_\lambda \rangle = P(\sigma(u)v)(\lambda).$$

It is easy to see that $\langle u \cdot v, v' \rangle = \langle v, \sigma(u) \cdot v' \rangle$ for all $v, v' \in M_\lambda$ and for all $u \in \mathfrak{U}(\mathfrak{g})$. Forms with such a property are called *contravariant forms*.

The following proposition and its proof can be found e.g. in [Hum08].

Proposition 1.2.2. *Contravariant forms have the following properties:*

1. *If the* $\mathfrak{U}(\mathfrak{g})$*-module* M *has a contravariant form* $(v, v')_M$*, then the weight spaces are orthogonal. I.e.* $(M_\mu, M_\nu) = 0$ *whenever* $\mu \neq \nu$ *in* $\mathfrak{h}^*$.

2. *Suppose* $M = \mathfrak{U}(\mathfrak{g}) \cdot v$ *is a highest weight module generated by a maximal vector* v *of weight* λ*. If* M *has a nonzero contravariant form, then the form is uniquely determined up to a scalar multiple by the (nonzero!) value* $(v, v)_M$.

3. *If* $\mathfrak{U}(\mathfrak{g})$*-modules* M_1, M_2 *have contravariant forms* $(v, v')_{M_1}$ *and* $(w, w')_{M_2}$ *, then* $M := M_1 \otimes M_2$ *also has a contravariant form, given by*

   $$(v \otimes w, v' \otimes w')_M := (v, v')_{M_1}(w, w')_{M_2}.$$

   In case both of the forms are nondegenerate, so is the product form.

4. *If* M *has a contravariant form and* N *is a submodule, the orthogonal complement* $N^\perp := \{v \in M | (v, v')_M = 0$ *for all* $v' \in N\}$ *is also a submodule.*

Proof. 1. We use the fact that $\sigma(h) = h$ for $h \in \mathfrak{h}$. Let v be a vector of weight μ and let v' be a vector of weight ν. The for any $h \in \mathfrak{h}$ we have

   $$\mu(h)(v, v')_M = (h \cdot v, v')_M = (v, \sigma(h) \cdot v')_M = (v, h \cdot v')_M = \nu(h)(v, v')_M.$$

   Since v, v' were arbitrary we must have $(v, v') = 0$ for $\mu \neq \nu$.

2. In view of the already proven point it suffices to look at values of the form on a weight space M_μ. Typical vectors $v, v' \in M_\mu$ can be written as $u \cdot v, u' \cdot v$ for suitable $u, u' \in \mathfrak{U}(\mathfrak{n})$. Note that since u takes M_λ into M_μ, the element $\sigma(u) \in \mathfrak{U}(\mathfrak{n}^-)$ takes M_μ into M_λ (which is spanned by v). Then $(v, v')_M = (u \cdot v, u' \cdot v)_M = (v, \sigma(u)u' \cdot v)_M$, which is a scalar multiple of $(v, v)_M$ depending just on the action of $\mathfrak{U}(\mathfrak{g})$ and not on the choice of the form.

   The remaining points are elementary.

□

Important consequence of this is the following lemma.

Lemma 1.2.3. *The maximal submodule of $N(\lambda)$ is the radical of the Shapovalov form.*

Proof. Let $v \in J(\lambda)$ be arbitrary. Then $\langle v, v_\lambda \rangle = 0$ since clearly v and v_λ have different weights. Since $J(\lambda)^\perp$ is a submodule of $N(\lambda)$ containing the generating vector v_λ the statement follows. □

The generalized Verma module $M(\lambda)$ is a quotient of the Verma module $N(\lambda)$. Hence the simple quotient $L(\lambda)$ of $M(\lambda)$ is also a quotient of $N(\lambda)$ by its maximal submodule. We get induced contravariant forms on $L(\lambda)$ and $M(\lambda)$ which we also call Shapovalov forms.

1.2.3 Equivalences of Enright and Shelton

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ of Hermitian type. (I.e. giving rise to a $|1|$-grading on $\mathfrak{g}$.) Let Σ be the set of singular simple roots for an antidominant weight μ:

$$J = \{\alpha \in \Delta \,|\, (\mu + \rho, \alpha^\vee) = 0\}.$$

Theorem 1.2.4 (8, Prop 2.3 of [BN05])**.** *Let*

$$W^{\Sigma,J} = \{w \in W^\Sigma \,|\, \forall \alpha \in \Sigma : w < ws_\alpha \text{ and } ws_\alpha \in W^\Sigma\}$$

and define $L(w)$ to be the simple quotient of the parabolic Verma module $M_\Sigma(w) = M(w_\Sigma w \cdot \mu)$ where w_Σ is the longest element of the parabolic Weyl group W_Σ. Then

$$W^{\Sigma,J} \to \{\text{simple modules in } \mathcal{O}^{\mathfrak{p}}_\mu\}$$

given by $w \mapsto L(w)$ is a bijection.

Theorem 1.2.5 ([ES87], [ES89])**.**

1. *Suppose that either Φ has one root length or that it has roots of two lengths and all the roots in J are short. Then there exists an equivalence of categories*

   $$\mathcal{E} : \mathcal{O}^{\mathfrak{p}}_\mu \to \mathcal{O}^{\mathfrak{p}'}_{\text{reg}},$$

   where $\mathfrak{p}'$ is a parabolic subalgebra of Hermitian type of a complex simple Lie algebra $\mathfrak{g}'$ of rank smaller or equal to the rank of $\mathfrak{g}$. Moreover, there is an isomorphism of posets $W^{\Sigma,J} \to W'^{\Sigma'}$ such that the functor $\mathcal{E}$ sends parabolic Verma modules $M_\Sigma(w)$ to parabolic Verma modules $M_{\Sigma'}(w')$ and similarly for simple modules.

2. *Suppose that Φ has two root lengths and J contains a long root. Then there exists an equivalence of categories*

   $$\mathcal{E} : \mathcal{O}^{\mathfrak{p}}_\mu \to \mathcal{O}^{\mathfrak{p}'}_{\text{reg}} \oplus \mathcal{O}^{\mathfrak{p}'}_{\text{reg}},$$

   where $\mathfrak{p}'$ is a parabolic subalgebra of Hermitian type of a complex simple Lie algebra $\mathfrak{g}'$ of rank smaller or equal to the rank of $\mathfrak{g}$. More precisely the poset $W^{\Sigma,J}$ is a disjoint union $W^{\Sigma,J}_{\text{odd}} \cup W^{\Sigma,J}_{\text{even}}$ of two poset and the category

$\mathcal{O}^{\mathfrak{p}}_{\mu}$ has a decomposition into a direct sum $\mathcal{O}^{\mathfrak{p}}_{\mu} = \mathcal{O}^{\mathfrak{p}}_{\mu,\mathrm{even}} \oplus \mathcal{O}^{\mathfrak{p}}_{\mu,\mathrm{odd}}$ such that all extensions between modules in different summands is zero. There exists isomorphisms of posets $W^{\Sigma,J}_{\mathrm{odd}} \to W'^{\Sigma'}$ and $W^{\Sigma,J}_{\mathrm{even}} \to W'^{\Sigma'}$ and corresponding equivalences of categories $\mathcal{E}_{\mathrm{odd}} : \mathcal{O}^{\mathfrak{p}}_{\mu,\mathrm{odd}} \to \mathcal{O}^{\mathfrak{p}'}_{\mathrm{reg}}$ and $\mathcal{E}_{\mathrm{even}} : \mathcal{O}^{\mathfrak{p}}_{\mu,\mathrm{even}} \to \mathcal{O}^{\mathfrak{p}'}_{\mathrm{reg}}$ such that $M_{\Sigma}(w) \to M_{\Sigma'}(w')$ and similarly for their simple quotients.

Let's illustrate this equivalence with an example.

Example 1.2.6 ([BH09]). *Consider the complex simple Lie algebra $\mathfrak{sl}(6)$ and let $\Sigma = \{\alpha_3\}$. Since the Weyl group is a permutation group acting on ϵ-coordinates, the weights which are singular have repeating values. Pick $\mu = (0, 1, 2 \,|\, 3, 3, 4)$. The permutations of $\mu + \rho$ which are highest weights plus ρ of simple modules in $\mathcal{O}^{\Sigma,J}$ have their first three and last three entries strictly decreasing. There are only six possible cases, namely*

$$(4, 3, 2 \,|\, 3, 1, 0) \quad (4, 3, 1 \,|\, 3, 2, 0) \quad (4, 3, 0 \,|\, 3, 2, 1)$$

and their counterparts with first three and last three entries swapped. Now these three weights are equivalent to

$$(4, 2 \,|\, 1, 0) \quad (4, 1 \,|\, 2, 0) \quad (4, 0 \,|\, 2, 1).$$

If we impose that the weight for $\mathfrak{sl}(6)$ has to have entries increasing by 1 this mapping on weights has inverse since we know that we have to put the number 3 into two places and there's only one way to do it in each case so that one obtains $\mathfrak{l}$-dominant weight. This directly generalizes to all other $\mathfrak{sl}$ cases as any weight can be brought by translation functors to a weight that starts with 0, has entries which are nondecreasing and which increase only by 1.

Remark 1.2.7. *The proof of these equivalences is not entirely satisfactory. The functors implementing the Enright-Shelton equivalences are combinations of derived Zuckermann functors, extension functor $M \to \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{q})} (M \otimes F \otimes L)$ where L is the one-dimensional module of weight $\rho(\mathfrak{u}_{\mathfrak{q}})$ of a certain parabolic subalgebra $\mathfrak{q}$. Determining the effect of these functors on modules of singular character is hard. Independent proof of one version of the Enright-Shelton equivalence was given by [Soe88] using Beilinson-Bernstein localization. Another independent proof for the A series was given by [BFK99]. See also [PS16].*

1.3 Cohomology and homology of Lie algebras

In this section we denote by $\mathfrak{m}$ an arbitrary Lie algebra and by $\mathbb{V}$ its representation. We will denote the action of $\mathfrak{m}$ on $\mathbb{V}$ by a dot. We follow [Kos61] and [ČS09].

The chain spaces of *Lie algebra homology* $C_k(m, \mathbb{V})$ of the algebra $\mathfrak{m}$ with values in $\mathbb{V}$ are defined as $\Lambda^k \mathfrak{m} \otimes \mathbb{V}$. The Lie algebra *homology differential*

$$\delta_{\mathfrak{m}} : C_{k+1}(\mathfrak{m}, \mathbb{V}) \to C_k(\mathfrak{m}, \mathbb{V})$$

is defined by the following formula

$$\delta_{\mathfrak{m}}(Z_0 \wedge \cdots Z_k \otimes v) = \sum_{i=0}^{k} (-1)^{i+1} Z_0 \wedge \cdots \wedge \widehat{Z_i} \wedge \cdots \wedge Z_k \otimes Z_i \cdot v + \\ + \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_0 \wedge \cdots \wedge \widehat{Z_i} \wedge \cdots \wedge \widehat{Z_j} \wedge \cdots \wedge Z_k \otimes v,$$

where $Z_i \in \mathfrak{m}$ for $i = 0, \dots, k$. If the algebra $\mathfrak{m}$ is abelian, then the second term in the sum is zero. The cochain spaces of *Lie algebra cohomology* $C^k(\mathfrak{m}, \mathbb{V})$ are defined as $\mathrm{Hom}(\Lambda^k \mathfrak{m}, \mathbb{V})$. The Lie algebra *cohomology differential*

$$\partial_{\mathfrak{m}} : C^k(\mathfrak{m}, \mathbb{V}) \to C^{k+1}(\mathfrak{m}, \mathbb{V})$$

is defined as

$$(\partial_{\mathfrak{m}} \phi)(X_0, \dots, X_n) = \sum_{i=0}^{n} (-1)^i X_i \cdot \phi(X_0, \dots, \widehat{X_i}, \dots, X_n) + \\ + \sum_{i<j} (-1)^{i+j} \phi([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_n),$$

where $X_0, \dots, X_n \in \mathfrak{m}$ and $\phi : \Lambda^k \mathfrak{m} \to \mathbb{V}$. Again, we can forget the second term if the algebra $\mathfrak{m}$ is commutative.

There is a natural identification of $C_k(\mathfrak{m}^*, \mathbb{V})$ and $C^k(\mathfrak{m}, \mathbb{V})$ coming from the natural isomorphism $\Lambda^k \mathfrak{m}^* \otimes \mathbb{V} \simeq \mathrm{Hom}(\mathfrak{m}, \mathbb{V})$.

In the special case when we consider $\mathfrak{m}$ to be a nilradical $\mathfrak{p}_-$ of some opposite parabolic subalgebra $\mathfrak{p}$, we have the Killing form that induces an isomorphism ${\mathfrak{p}_-}^* \simeq \mathfrak{p}_+$. Since we can identify $C^k(\mathfrak{p}_-, \mathbb{V}) = \mathrm{Hom}(\Lambda^k \mathfrak{p}_-, \mathbb{V})$ with $\Lambda^k {\mathfrak{p}_-}^* \otimes \mathbb{V}$, we see we can consider the Lie algebra cohomology differential ∂ as an operator on the chain spaces of Lie algebra homology $\partial : \Lambda^k {\mathfrak{p}_-}^* \otimes V \to \Lambda^{k+1} {\mathfrak{p}_-}^* \otimes V$. After these identifications we get the formula

$$\partial(Z_1 \wedge \dots \wedge Z_k \otimes v) = \sum_{i=1}^{2p} \epsilon^i \wedge Z_1 \wedge \dots \wedge Z_k \otimes e_i \cdot v.$$

The Lie algebra cohomology differential is $\overline{P}$-equivariant. In particular, both ∂ and δ are L-equivariant and consequently they preserve the geometric weight.

The following two propositions are well known.

Proposition 1.3.1. *Suppose M is a $\mathfrak{g}-$module. Let $\mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra with nilradical $\mathfrak{n}$ and Levi factor $\mathfrak{l}$.*
(a) Let M^* denote the $\mathfrak{g}-$module dual to M. Then there is a natural isomorphism

$$H_p(\mathfrak{n}, M^*) \cong H^p(\mathfrak{n}, M)^*$$

where $H^p(\mathfrak{n}, M)^*$ denotes the $\mathfrak{l}-$module dual to $H^p(\mathfrak{n}, M)$.
(b) Let d denote the dimension of $\mathfrak{n}$. Then there is a natural isomorphism

$$H_p(\mathfrak{n}, M) \cong H^{d\text{-}p}(\mathfrak{n}, M) \otimes \Lambda^d \mathfrak{n}$$

Proposition 1.3.2 (Proposition 3.3.6 of [ČS09])**.** *Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra with complexification $\mathfrak{g}^{\mathbb{C}}$ and let $\mathbb{V}$ be a complex representation of $\mathfrak{g}$. Then the real cohomology spaces $H^*(\mathfrak{g}_-, \mathbb{V})$ are naturally complex vector spaces and we have a natural isomorphism of $\mathfrak{g}_0$-modules*

$$H^*_{\mathbb{R}}(\mathfrak{g}_-, \mathbb{V}) \simeq H^*_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}_-, \mathbb{V}).$$

Finally let us recall interpretation of the Lie algebra (co)homology as derived functors which plays a crucial role in the proof of existence of the BGG resolution for regular Kostant modules. See [EHP14; BH09] and references therein.

The right standard resolution of $\mathbb{C}$ is the complex of free right $U(\mathfrak{n})-$modules given by

$$\cdots \to \Lambda^{p+1}\mathfrak{n} \otimes U(\mathfrak{n}) \to \Lambda^{p}\mathfrak{n} \otimes U(\mathfrak{n}) \to \cdots \to \mathfrak{n} \otimes U(\mathfrak{n}) \to U(\mathfrak{n}) \to 0.$$

Applying the functor

$$- \otimes_{U(\mathfrak{n})} M$$

to the standard resolution we obtain a complex

$$\cdots \to \Lambda^{p+1}\mathfrak{n} \otimes M \to \Lambda^{p}\mathfrak{n} \otimes M \to \cdots \to \mathfrak{n} \otimes M \to M \to 0$$

of left $\mathfrak{l}-$modules called *the standard* $\mathfrak{n}-$*homology complex*. Here $\mathfrak{l}$ acts via the tensor product of the adjoint action on $\Lambda^p\mathfrak{n}$ with the given action on M. Since $U(\mathfrak{g})$ is free as $U(\mathfrak{n})-$module, one can show that the pth-homology of the standard complex is in fact the pth $\mathfrak{n}-$homology group $H_{\mathrm{p}}(\mathfrak{n}, M)$.

The *zero* $\mathfrak{n}-$*cohomology* of a $\mathfrak{g}-$module M is the $\mathfrak{l}-$module

$$H^0(\mathfrak{n}, M) = \mathrm{Hom}_{U(\mathfrak{n})}(\mathbb{C}, M).$$

This suggest that applying the left exact functor from the category of $\mathfrak{g}-$modules to the category of $\mathfrak{l}-$modules

$$\mathrm{Hom}_{U(\mathfrak{n})}(-, M)$$

to the standard resolution of $\mathbb{C}$, this time by free left $U(\mathfrak{n})-$modules, will yield the Lie algebra cohomology groups which is indeed the case.

Hence we see that the we can see the Lie algebra (co)homology as the Tor and Ext functors.

1.3.1 Disjoint operators and Hodge decomposition

Let V be a topological vector space and let ∂ and δ be two linear operators on V such that $\partial^2 = \delta^2 = 0$. Such operators are called *disjoint* if

$$\partial\,\delta\,x = 0 \text{ implies } \delta\,x = 0$$

and

$$\delta\,\partial\,x = 0 \text{ implies } \partial\,x = 0$$

for all $x \in V$. In other words, δ and ∂ are disjoint if and only if

$$\ker\partial \cap \mathrm{im}\,\delta = 0 \quad \& \quad \ker\delta \cap \mathrm{im}\,\partial = 0. \tag{1.1}$$

The *Laplacian* is then defined as

$$\Box = \partial\,\delta + \delta\,\partial\,.$$

We note that $\Box = (\delta + \partial)^2$ and hence one can consider $\delta + \partial$ as the Dirac operator for the Laplacian $\Box$.

Proposition 1.3.3. *Let the notation be as above and assume ∂ and δ are disjoint and* $\dim V < \infty$*. Then*

$$\ker \Box = \ker \partial \cap \ker \delta \quad \& \quad \operatorname{im} \Box = \operatorname{im} \delta \oplus \operatorname{im} \partial\,. \tag{1.2}$$

*Also one has a direct sum decomposition (*Hodge decomposition*),*

$$V = \operatorname{im} \partial \oplus \ker \Box \oplus \operatorname{im} \delta\,. \tag{1.3}$$

Proof. To prove the first equality of (1.2) we just use the definition of disjointness (1.1) of δ and ∂. The inclusion $\ker \delta \cap \ker \partial \subseteq \ker \Box$ is trivial. Let $x \in \ker \Box$ and put $y = -\delta\,\partial\, x$. Then $\delta\, y = 0$ and also $y = \partial\,\delta\, x$. Thus $\delta\,\partial\,\delta\, x = 0$ and by disjointness this implies that $\partial\,\delta\, x = 0$ which, for the same reason, implies $\delta\, x = 0$. Similarly $\partial\, x = 0$. The remaining two decompositions of (1.2) are direct consequence of $\ker \Box = \ker \delta \cap \ker \partial$ and of disjointness of δ and ∂.

It is obvious that $\operatorname{im} \Box \subseteq \operatorname{im} \partial + \operatorname{im} \delta$ and by (1.2) and disjointness (1.1) we have $(\operatorname{im} \delta + \operatorname{im} \partial) \cap \ker \Box = 0$. If follows that $\operatorname{im} \Box \cap \ker \Box = 0$. Since V is finite-dimensional, we get $\dim \ker \Box + \dim \operatorname{im} \Box = \dim V$ and (1.3) and the second equality of (1.2) follows. □

Corollary 1.3.4. *Under the assumptions of the previous proposition, there are decompositions*

$$\ker \delta = \operatorname{im} \delta \oplus \ker \Box, \quad \ker \partial = \operatorname{im} \partial \oplus \ker \Box\,. \tag{1.4}$$

Consequently we have isomorphisms

$$\ker \Box \simeq \ker \partial \,/\, \operatorname{im} \partial \quad \& \quad \ker \Box \simeq \ker \delta \,/\, \operatorname{im} \delta \tag{1.5}$$

given by restriction to $\ker \Box$ *of the canonical mappings* $\ker \partial \to \ker \partial \,/\, \operatorname{im} \partial$ *and* $\ker \delta \to \ker \delta \,/\, \operatorname{im} \delta$.

Proof. This is a direct consequence of the Hodge decomposition (1.3). □

Remark 1.3.5. *For any two linear mappings δ and ∂ on a vector space V such that* $\operatorname{im} \delta \cap \operatorname{im} \partial = 0$ *we have that*

$$\ker(\delta + \partial) = \ker \delta \cap \ker \partial\,.$$

Under our assumptions we get that $\ker(\delta + \partial) = \ker \Box$*, since* $\operatorname{im} \delta \subseteq \ker \delta$ *and by disjointness* (1.1) $\operatorname{im} \delta \cap \operatorname{im} \partial = 0$.

To better understand the situation we can represent these operators in a block matrix form with respect to the Hodge decomposition (1.3)

$$\partial = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B & 0 & 0 \end{pmatrix} \quad \Box = \begin{pmatrix} AB & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & BA \end{pmatrix},$$

where A is the restriction of ∂ to $\operatorname{im} \delta$ and B is the restriction of δ to $\operatorname{im} \partial$. Disjointness of ∂ and δ then amounts to nondegeneracy of A and B.

There is a convenient way to construct pairs of disjoint operators.

Proposition 1.3.6. *Let δ be a differential on a vector space V endowed with a non-degenerate bilinear (or sesquilinear) form $\langle\,,\,\rangle$ and let ∂ be the adjoint of δ with respect to $\langle\,,\,\rangle$. If the restrictions of $\langle\,,\,\rangle$ to $\operatorname{im}\delta$ and $\operatorname{im}\partial$ are non-degenerate, then δ and ∂ are disjoint and the Hodge decomposition* (1.3) *is orthogonal with respect to $\langle\,,\,\rangle$.*

Proof. It is trivial to see that ∂ is a differential, for we have

$$0 = \langle \delta^2 x, y\rangle = \langle x, \partial^2 y\rangle$$

for all $x, y \in V$.

To prove (1.1), let $x \in V$ be such that $\delta\,\partial\, x = 0$. We get

$$0 = \langle \delta\,\partial\, x, y\rangle = \langle \partial\, x, \partial\, y\rangle$$

for all $y \in V$. By our assumptions is the bilinear (sesquilinear) map $(x, y) \to \langle \partial\, x, \partial\, y\rangle$ non-degenerate and hence we conclude that $\partial\, x = 0$. Exchanging the roles of δ and ∂ in this computation we get that also $\partial\,\delta\, x = 0$ implies $\delta\, x = 0$.

Orthogonality of the Hodge resolution follows from $\ker\delta = (\operatorname{im}\partial)^{\perp}$ and $\ker\partial = (\operatorname{im}\delta)^{\perp}$. □

Corollary 1.3.7. *Let V be endowed with a scalar product $\langle\,,\,\rangle$ and a differential δ. Let ∂ be the adjoint of δ. Then δ and ∂ are disjoint operators.*

Proof. This is trivial as the restriction of $\langle\,,\,\rangle$ to any subspace is non-degenerate. □

If V is infinite-dimensional and L is any closed subspace of finite codimension, then any algebraic complementary subspace to L in V is also a topological complement ([SW99]). In these cases we demand that δ and ∂ are continuous with closed image[1]

The authors of [HPR06] proved that if $\mathbb{V}$ is a unitarizable $(\mathfrak{g}, K)$-module and $\mathfrak{u}_+$ is abelian nilradical of a θ-stable parabolic subalgebra[2], then the Hodge decomposition of $C_\bullet(\mathfrak{u}_+, \mathbb{V})$ is still valid.

1.3.2 Kostant theorem and Kostant modules

It was proven in [Kos61] that for a finite-dimensional $\mathfrak{g}$-representation $\mathbb{V}$ and a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}_+$ there is a positive definite scalar product on $C_\bullet(\mathfrak{p}_+, \mathbb{V})$ with respect to which are the differentials ∂ and δ adjoint. It follows that there is a direct sum *Hodge decomposition* of $\mathfrak{l}$-modules $C_\bullet(\mathfrak{p}_+, \mathbb{V}) = \operatorname{im}\partial \oplus \ker\Box \oplus \operatorname{im}\delta$ and moreover $\ker\delta = \ker\Box \oplus \operatorname{im}\delta$ and $\ker\partial = \ker\Box \oplus \operatorname{im}\partial$. It follows that $H_\bullet(\mathfrak{p}_+, \mathbb{V}) \simeq \ker\Box \simeq H^\bullet(\mathfrak{p}_-, \mathbb{V})$.

Theorem 1.3.8 ([Kos61])**.** *Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\mathfrak{p}_\Sigma$ be a standard parabolic subalgebra with Levi decomposition $\mathfrak{p}_\Sigma = \mathfrak{l}_\Sigma \oplus \mathfrak{u}_\Sigma$. Let W^Σ be*

[1]We note that for a finite-dimensional vector space V are these topological conditions automatically satisfied for all linear operators.

[2]See the definition 2.1.7.

the poset of minimal coset representatives. For every $\mathfrak{g}$-integral and $\mathfrak{g}$-dominant weight λ there is isomorphism of $\mathfrak{l}_\Sigma$ modules

$$H^i(\mathfrak{u}_\Sigma, L(\lambda)) \simeq \bigoplus_{\substack{w \in W^\Sigma \\ l(w)=i}} F(w \cdot \lambda), \tag{1.6}$$

where $L(\lambda)$ is the finite dimensional $\mathfrak{g}$-module with highest weight λ and $F(w\cdot\lambda)$ are finite dimensional $\mathfrak{l}_\Sigma$ modules with highest weights $w \cdot \lambda$ where $w \cdot \lambda = w(\lambda + \rho) - \rho$ is the affine action of $w \in W$.

This theorem gave rise to a definition of *Kostant modules* which are (not necessarily finite-dimensional) modules for which there is a similar formula for cohomology. It started with [Col85] and was later developed in [BH09]. The definition roughly says that a simple highest weight module L is Kostant module if there is a graded poset P with a length function in W such that the ith cohomology is given by the affine action of elements of P of length i. For simple modules with regular weight this poset P is given by (sub)poset of minimal coset representatives W^Σ. For singular weights the situation is much more complicated, the "working definition" is that the poset P is given by $W^{\Sigma,J}$ from theorem 1.2.4. We refer to [BH09] and [EHP14] for further details and examples. For our applications we only need to know that the unitarizable highest weight modules of chapter 3 are examples of Kostant modules.

The article [BH09] classifies Kostant modules for arbitrary blocks of category $\mathcal{O}^\mathfrak{p}$ for $|1|$-graded parabolic $\mathfrak{p} \leq \mathfrak{g}$. The classification is given by certain subdiagrams of the Dynkin diagram. Given subdiagram of the Dynkin diagram one can use the Enright-Shelton equivalence functor to simple modules of the Lie algebra defined by the smaller diagram and one obtains Kostant modules for the bigger Lie algebra. Moreover all Kostant modules arise in this way. Let J denote the set of singular simple roots, i.e. $J = \{\alpha \in \Delta \,|\, (\lambda + \rho, \alpha^\vee) = 0\}$. The columns α and α' give the crossed roots in the Dynkin diagrams. Each Kostant module corresponds to a connected subdiagram of $\mathcal{D}'$ containing α'. Each such subdiagram gives a parabolic pair and the Kostant modules correspond to finite-dimensional modules for the semisimple part of the Levi factor of $\mathcal{D}'$.

$\mathcal{D}$	α	$\lvert J\rvert$	$\mathcal{D}'$	α'
$(A_n, A_{r-1} \times A_{n-r})$	α_r	t	$(A_{n-2t}, A_{r-t-1} \times A_{n-r-t})$	α_{r-t}
(B_n, B_{n-1})	α_1	1 (short)	$\varnothing$	–
(B_n, B_{n-1})	α_1	1 (long)	$\varnothing$	–
(C_n, A_{n-1})	α_n	t (all short)	(D_{n+1-2t}, A_{n-2t})	α_{n+1-2t}
(C_n, A_{n-1})	α_n	t (1 long)	(D_{n+1-2t}, A_{n-2t})	α_{n+1-2t}
(D_n, D_{n-1})	α_1	1	$(A_1, \varnothing)$	α_1
(D_n, D_{n-1})	α_1	2 ($\{n-1, n\}$)	$\varnothing$	–
(D_n, A_{n-1})	α_n	t	(D_{n-2t}, A_{n-2t-1})	α_{n-2t}
(E_6, D_5)	α_6	1	(A_5, A_4)	α_5
(E_6, D_5)	α_6	2	$\varnothing$	–
(E_7, E_6)	α_7	1	(D_6, D_5)	α_1
(E_7, E_6)	α_7	2	$(A_1, \varnothing)$	α_1
(E_7, E_6)	α_7	3 ($\{2, 5, 7\}$)	$\varnothing$	–

Table 1.1: Data for singular Hermitian symmetric categories

1.4 Harish-Chandra modules and their globalizations

Our ultimate goal is to use unitarizable highest weight modules in geometric applications as in [Tuč12]. For these applications we actually need modules which are naturally $\overline{P}$-modules but not P-modules which poses a serious problem as the structure group of our Cartan geometry is P and thus we need some way to "globalize" the unitarizable $\overline{P}$-modules into P-modules. There is a well known theory of globalizations which we review here following [Vog08]. For technical reasons, we will actually need to use so called *formal globalization*[3] which is not strictly speaking globalization as it is not representation of G. It is, however, representation of P.

Fix a maximal compact subgroup K_0 of a real Lie group G_0. Suppose we have a linear action of K_0 on a complex vector space M. A vector $m \in M$ is called K_0-*finite* if the span of the K_0-orbit of m is finite-dimensional and if the action of K_0 on this subspace is continuous. The linear action of K_0 on M is called K_0-*finite* when every vector is K_0-finite. By definition, *Harish-Chandra module* for G_0 is a finite length $\mathfrak{g}-$module equipped with a compatible, K_0-finite, linear action. One knows that an irreducible K_0-module has finite multiplicity in a Harish-Chandra module. For our purposes, it will also be useful to refer to a category of *good* K_0-modules. A *good* K_0-module is a locally finite module such that each irreducible K_0-module has finite multiplicity therein.

A representation of G_0 on a complete locally convex topological vector space V is called *admissible* if V has finite length (with respect to closed invariant subspaces) and if each irreducible K_0-module has finite multiplicity in V. When V is admissible, then each K_0-finite vector in V is differentiable and the subspace of K_0-finite vectors is a Harish-Chandra module. The representation is called *smooth* if every vector in V is differentiable. In this case, V is a $\mathfrak{g}-$module. For

[3] Denoted by $\mathbb{V}^{-K}$ in [Vog08].

example, it is known that an admissible representation in a Banach space is smooth if and only if the representation is finite-dimensional.

Given a Harish-Chandra module M, *a globalization* M_{glob} *of* M is an admissible representation of G_0 whose underlying $(\mathfrak{g}, K_0)-$module of K_0-finite vectors is isomorphic to M. By now, four canonical globalizations of Harish-Chandra modules are known to exist. These are: the smooth globalization of Casselman and Wallach, its dual (called: the distribution globalization), Schmid's minimal globalization and its dual (the maximal globalization). All four globalizations are smooth and functorial.

The *minimal globalization* M_{min} of a Harish-Chandra module M is uniquely characterized by the property that any $(\mathfrak{g}, K_0)-$equivariant linear map of M onto the K_0-finite vectors of an admissible representation V lifts to a unique, continuous G_0-equivariant linear map of M_{min} into V. In particular, M_{min} embeds G_0-equivariantly and continuously into any globalization of M. The construction of the minimal globalization shows that it's realized on a *DNF space*. This means that its continuous dual, in the strong topology, is a nuclear Frechét space. One knows that M_{min} consists of analytic vectors and that it surjects onto the analytic vectors in any Banach space globalization. Like each of the canonical globalizations, the minimal globalization is functorially exact. In particular, a closed G_0-invariant subspace of a minimal globalization is the minimal globalization of its underlying Harish-Chandra module and a continuous G_0-equivariant linear map between minimal globalizations has closed range.

To characterize the maximal globalization, we introduce the K_0-finite dual on the category of Harish-Chandra modules. In particular, let M be a Harish-Chandra module. Then the algebraic dual M^* of M is a $\mathfrak{g}-$module and a K_0-module, but in general not K_0-finite. We define $M^\vee$, *the* K_0-*finite (or Harish-Chandra) dual to* M, to be the subspace of K_0-finite vectors in M^*. Thus $M^\vee$ is a Harish-Chandra module. In fact, the functor $M \mapsto M^\vee$ is exact on the category of good K_0-modules. We also have the formula

$$(M^\vee)^\vee \cong M.$$

The maximal globalization M_{max} of M can be defined by the equation

$$M_{\mathrm{max}} = ((M^\vee)_{\mathrm{min}})'$$

where the last prime denotes the continuous dual equipped with the strong topology. In particular, M_{max} is a globalization of M. Observe that the maximal globalization is an exact functor, since all functors used in the definition are exact. Because of the minimal property of M_{min}, it follows that any globalization of M embeds continuously and equivariantly into M_{max}. Note that the continuous dual of a maximal globalization is the minimal globalization of the dual Harish-Chandra module.

Writing the elements of a Harish-Chandra module as almost everywhere zero sequences we can actually identify these globalizations as vector spaces of sequences with infinitely many nonzero elements which moreover satisfy certain growth conditions. Namely, fixing an K-invariant scalar product on each K-representation, the minimal globalization consists of those sequences whose K-norms decrease exponentially fast and the maximal globalization consists of sequences whose K-norms are less than exponentially increasing.

From now on consider the Hermitian symmetric situation, where $\mathfrak{k}$ is actually the Levi subalgebra of a parabolic subalgebra with abelian radical. Consider the simple quotient $L(\lambda)$ of the parabolic Verma module $M(\lambda)$ induced from the opposite parabolic algebra $\overline{\mathfrak{p}}$. It is not a P-representation but we would like induce from it a P-representation in order to be able to use it in geometrical applications. Let $L(\lambda) = \bigoplus_{\mu\in\widehat{K_{\mathbb{C}}}} L_\mu$ be the decomposition of $L(\lambda)$ into $K_{\mathbb{C}}$-types. Each L_μ is contained in some $S^k(\mathfrak{p}_+, F(\lambda))$ modulo the maximal submodule of $M(\lambda)$ and the algebra $\mathfrak{p}_+$ acts as multiplication by a variable, while $\mathfrak{p}_-$ acts basically as differentiation. To formalize this write $L(\lambda) = \bigoplus_{\mu\in\widehat{K_{\mathbb{C}}},k\in\mathbb{N}} L_{\mu,k}$ where $L_{\mu,k}$ is the $K_{\mathbb{C}}$-type contained in $S^k(\mathfrak{p}_+, F(\lambda))$ and note that $\mathfrak{p}_+(L_{\mu,k}) \subset L_{\mu,k+1}$.

The *formal globalization* or rather *formal completion* of $L(\lambda)$ is defined as $\overline{L(\lambda)} = \prod_{\mu\in\widehat{K_{\mathbb{C}}}} L_\mu$ (product of topological vector spaces). Since each $K_{\mathbb{C}}$-type is finite-dimensional and each $S^k(\mathfrak{p}_+, F(\lambda))$ contains only finitely many irreducible $K_{\mathbb{C}}$-representations, we can write it as

$$\overline{L(\lambda)} = \prod_{k\in\mathbb{N}} L_k,$$

where $L_k = \bigoplus_{\mu\in\widehat{K_c}} L_{\mu,k}$ is a finite sum. The action of $\mathfrak{p}_+$ works as a right shift: $\mathfrak{p}_+(L_k) \subset L_{k+1}$. The $(\mathfrak{g}, K)$-module $L(\lambda)$ can be realized as a space of polynomials with values in a finite dimensional K-representation F_λ and it's formal completion can be thought of as a space of formal power series with values in F_λ.

Now it is easy to see that the formal globalization is representation of P, since the action of $\mathfrak{p}_+$ on $\overline{L(\lambda)}$ integrates without any problems. Let X be an arbitrary element of $\mathfrak{p}_+$. The exponential $u := \exp(tX)v$ is defined for $v \in \overline{L(\lambda)}$ as usual by $\sum_{i=0}^\infty \frac{t^i}{i!} X^i v$. This is well defined for v if and only if each component u_μ is well defined. If u_μ is contained in some $S^k(\mathfrak{p}_+, F(\lambda))$, then it is a sum of at most k elements of lower or equal geometric weight.

Proposition 1.4.1. *Suppose that the Lie algebra homology and cohomology differentials are disjoint for* $\mathbb{V}$*, i.e.* $\ker\delta\cap\operatorname{im}\partial = 0$ *and* $\ker\partial\cap\operatorname{im}\delta = 0$*. The differentials are disjoint for the formal completion of* $\mathbb{V}$ *as well. Moreover, if the homology (or equivalently cohomology) of* $\mathbb{V}$ *is finite-dimensional, then it is equal to the cohomology of the formal completion.*

Proof. Let $\mathbb{W}$ be the direct sum of chain spaces of δ. Elements of the formal completion (of an infinite-dimensional module) $\mathbb{W}$ can be written as infinite tuples of elements of $\mathbb{W}$

$$u \in \overline{\mathbb{W}} \leftrightarrow u = (u_i)_{i\in I},$$

where the index set I runs over all eigenvalues of the action of E on $\mathbb{W}$.

For a contradiction, suppose that there is nonzero u in $\ker\delta\cap\operatorname{im}\partial$. Without a loss of generality, we can consider u to be an element of the k-th chain space. Since u is nonzero, there must exists i such that $u_i \neq 0$. Both differentials are natural, which means that they commute with restriction from $\overline{\mathbb{W}}$ to $\mathbb{W}$. They are moreover $K_{\mathbb{C}}$-invariant, which in particular means that they commute with the action of the grading element, which means that they act component-wise on $(u_i)_{i\in I}$. It follows that if we take $\widetilde{u} = (\widetilde{u_j}) \in \mathbb{W} \subseteq \overline{\mathbb{W}}$ to be zero for $j \neq i$ and $\widetilde{u_i} = u_i$, then we have $\delta\,\widetilde{u} = 0$. Since $u \in \operatorname{im}\partial$, there must exists a $v \in \overline{\mathbb{W}}$ such that $\partial\, v = u$ and we can define in the same way as before $\widetilde{v} \in \mathbb{W}$ such that

$\partial \tilde{v} = \tilde{u}$. Thus we see that there is a non-zero element in $\ker \delta \cap \operatorname{im} \partial \cap \Lambda^k \otimes \mathbb{W}$. This is a contradiction with the assumption that δ and ∂ are disjoint for $\mathbb{V}$.

Basically, the same argument proves that the cohomology must stay the same if it is finite-dimensional. From the Hodge theory (which is implied by the disjointness) we know that any element of the cohomology group can be uniquely represented by an element in $\ker(\delta + \partial)$. Let $D = \delta + \partial$ and suppose for a contradiction that there is a nonzero element (u_i) in $\ker D \setminus \ker D_{|\mathbb{W}}$, in the k-th chain space of $\mathbb{V}$. Pick i_0 such that $u_{i_0} \neq 0$ and i_0 is bigger than any eigenvalue of E on $\ker D_{|\mathbb{W}}$.[4] Since D is again $K_{\mathbb{C}}$-invariant we proceed as before and construct $\tilde{u} \in \mathbb{W} \cap \ker D$. The choice of i_0 now leads to the desired contradiction.

□

[4]This is well defined because of the finite-dimensionality of $\ker D_{|\mathbb{W}}$.

2. Hermitian symmetric spaces

2.1 Real Lie algebras

The *complexification* of a real reductive Lie algebra $\mathfrak{g}_0$ is defined as $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. With Lie bracket defined naturally by complex linear extension as follows

$$[A \otimes z, B \otimes w] = [A, B] \otimes zw.$$

As a real vector space we have isomorphism $\mathfrak{g} \simeq \mathfrak{g}_0 \oplus \imath\mathfrak{g}_0$ defined by $X \otimes (a + \imath b) \mapsto aX \oplus \imath bX$. Writing elements $Z, Z' \in \mathfrak{g}$ as $Z = X + \imath Y$ $Z' = X' + \imath Y'$ we get

$$[Z, Z'] = [X, X'] - [Y, Y'] + \imath \left([X, Y'] + [X', Y] \right).$$

A *real form* of a complex Lie algebra $\mathfrak{g}$ is a real Lie algebra $\mathfrak{g}_0$ such that $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$. A complex Lie algebra usually has many nonisomorphic real forms. For a complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ we will denote by σ the conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$.

Example 2.1.1. *The complexification of the classical Lie algebra of trace-free real matrices $\mathfrak{sl}(n, \mathbb{R})$ is naturally isomorphic to the Lie algebra of trace-free complex matrices $\mathfrak{sl}(n, \mathbb{C})$. Another real form of $\mathfrak{sl}(n, \mathbb{C})$ is for example the algebra of trace-free skew Hermitian matrices $\mathfrak{su}(n)$.*

An *involution* of a Lie algebra $\mathfrak{g}$ is Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{g}$ which squares to identity. An involution θ of a real semisimple Lie algebra $\mathfrak{g}_0$ is called *Cartan involution* if the symmetric bilinear form $B_\theta(X, Y) = -B(X, \theta Y)$ is positive definite. We will denote the complex linear extension of θ to the complexification of $\mathfrak{g}_0$ by the same symbol and we will still call it the Cartan involution. Any real semisimple Lie algebra has a Cartan involution and it is unique up to inner automorphism.

Take a matrix Lie algebra, i.e. a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$, that is closed under transposition and define $\theta X = -X^\dagger$ where $X^\dagger = \overline{X}^t$. Then

$$\theta[X, Y] = -[X, Y]^\dagger = -[Y^\dagger, X^\dagger] = [-X^\dagger, -Y^\dagger] = [\theta X, \theta Y]$$

shows that θ is involution. Using the fact that B is invariant with respect to automorphisms we get that

$$\begin{aligned} B_\theta(X, Y) &= -B(X, \theta Y) = -B(\theta X, \theta^2 Y) \\ &= -B(\theta X, Y) = B(Y, -\theta X) = B_\theta(Y, X), \end{aligned}$$

which demonstrates that B_θ is symmetric. To show that it is also positive definite, we first need to show that for a scalar product $\langle X, Y \rangle = \Re \operatorname{Tr}(XY^\dagger)$ we have that the adjoint of $\operatorname{ad} X$ is $\operatorname{ad} X^\dagger$.

$$\begin{aligned} \langle [X, Y], Z \rangle &= \Re \operatorname{Tr}(XYZ^\dagger - YXZ^\dagger) = \\ &= \Re \operatorname{Tr}(YZ^\dagger X - YXZ^\dagger) = \\ &= \Re \operatorname{Tr}(Y(X^\dagger Z - ZX^\dagger)^\dagger) = \langle Y, [X^\dagger, Z] \rangle. \end{aligned}$$

Now we can show that B_θ is in fact positive definite as follows

$$\begin{aligned} B_\theta(X,X) &= -B(X,\theta X) = -\operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} \theta X) \\ &= \operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad}(X^\dagger)) = \operatorname{Tr}(\operatorname{ad} X \circ (\operatorname{ad} X)^*) \geq 0. \end{aligned}$$

In fact, any real semisimple Lie algebra $\mathfrak{g}_0$ is isomorphic to a Lie algebra of real matrices that is closed under transpose and the isomorphism can be chosen in such a way that the Cartan involution of $\mathfrak{g}_0$ is carried to negative transpose (Proposition VI.6.28 of [Kna96]).

A Cartan involution θ of $\mathfrak{g}_0$ yields an eigenspace decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of $\mathfrak{g}_0$ into $+1$ and -1 eigenspaces of θ. Since θ is a Lie algebra homomorphism, it follows that

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0, \quad [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, \quad [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0. \tag{2.1}$$

From these relations it is easy to derive that $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are orthogonal with respect to B_θ and B. If X is in $\mathfrak{k}_0$ and Y is in $\mathfrak{p}_0$, then $\operatorname{ad} X \circ \operatorname{ad} Y$ sends $\mathfrak{k}_0$ to $\mathfrak{p}_0$ and $\mathfrak{p}_0$ to $\mathfrak{k}_0$; i.e. as a matrix in block form subordinated to the decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ it has the form $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$. Thus it has trace 0 and $B(X,Y) = 0$ and since $\theta Y = -Y$ also $B_\theta(X,Y) = 0$. Because B_θ is positive definite, the eigenspaces have the property that

$$B \text{ is } \begin{cases} \text{negative definite on } \mathfrak{k}_0 \\ \text{positive definite on } \mathfrak{p}_0. \end{cases} \tag{2.2}$$

A decomposition of $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ that satisfies (2.1) and (2.2) is called *Cartan decomposition* of $\mathfrak{g}_0$. Conversely any Cartan decomposition defines a Cartan involution by

$$\theta = \begin{cases} +\operatorname{Id} \text{ on } \mathfrak{k}_0 \\ -\operatorname{Id} \text{ on } \mathfrak{p}_0 \end{cases}$$

and we see that Cartan involutions are in bijective correspondence with Cartan decompositions.

Example 2.1.2. *Cartan involution on $\mathfrak{sl}(n,\mathbb{R})$ is given by negative transpose, i.e. $\theta X = -X^t$. The subalgebra $\mathfrak{k}_0$ is then $\mathfrak{so}(n,\mathbb{R})$ and the ideal $\mathfrak{p}_0$ is the space of symmetric matrices.*

For $\mathfrak{su}(n)$ is a Cartan involution given by a negative conjugate transpose $\theta X = -\overline{X}^t$. The subalgebra $\mathfrak{k}_0$ is the whole $\mathfrak{su}(n)$ and $\mathfrak{p}_0$ is of course empty.

Cartan involutions and decompositions can be even pushed to the Lie group level.

Theorem 2.1.3 (Theorem VI.6.31 of [Kna96])**.** *Let G be a semisimple Lie group, let θ be a Cartan involution of its Lie algebra $\mathfrak{g}_0$, let $\mathfrak{g}_0 = liek_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition, and let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}_0$. Then*

1. *there exists a Lie group automorphism Θ of G with differential θ, and $\Theta^2 = \operatorname{Id}$*

2. *the subgroup of G fixed by Θ is K*

3. *the mapping $K \times \mathfrak{p}_0 \to G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism onto*

4. *K is closed and contains the center Z of G*

5. *K is compact if and only if Z is finite in which case it is a maximal compact subgroup of G.*

This theorem justifies the following definition. We will call a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ *compact* if $\mathfrak{g}_0 = \mathfrak{k}_0$, i.e. if the Killing form B is negative definite.

Take a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and consider $\mathfrak{g}_0$ as a subset of its complexification $\mathfrak{g}$. Inspecting the signature of B_θ we easily see that $\mathfrak{k}_0 \oplus \imath\mathfrak{p}_0$ is a compact form, say $\mathfrak{u}_0$, of $\mathfrak{g}$. Denote by σ the conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$ and let τ denote the conjugation with respect to $\mathfrak{u}_0$. Both σ and τ are either Id or $-\operatorname{Id}$ on $\mathfrak{k}_0, \imath\mathfrak{k}_0, \mathfrak{p}_0, \imath\mathfrak{p}_0$. This immediately implies that the two involutions commute $\sigma \circ \tau = \tau \circ \sigma$. In particular $\tau(\mathfrak{g}_0) \subseteq \mathfrak{g}_0$ and the restriction of τ to $\mathfrak{g}_0$ is the Cartan involution θ corresponding to the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$.

Conversely, given a compact form $\mathfrak{u}_0$ of a complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ such that the corresponding conjugations τ and σ commute, we get a Cartan involution for $\mathfrak{g}_0$ by restriction of the involution τ that corresponds to the compact form $\mathfrak{u}_0$. Indeed, since $\mathfrak{g}_0 \subseteq \mathfrak{g} = \mathfrak{u}_0 \oplus \imath\mathfrak{u}_0$, we get for any $X + \imath Y \in \mathfrak{g}_0$

$$B_\theta(X + \imath Y, X + \imath Y) = -B(X + \imath Y, X - \imath Y) = -B(X, X) - B(Y, Y),$$

which is positive definite since X, Y are elements from the compact Lie algebra $\mathfrak{u}_0$.

A semisimple Lie algebra has up to inner isomorphism only one compact real form and it can be even shown that the compact form can be chosen in such a way that the corresponding conjugation τ commutes with any a priori chosen involution σ of $\mathfrak{g}$. This is the core of the proof of existence of Cartan involution for arbitrary real semisimple Lie algebra $\mathfrak{g}_0$.

To summarize, if $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$, then $\mathfrak{k}_0 \oplus \imath\mathfrak{p}_0$ is a compact real form of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$. Conversely, if $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are the $+1$ and -1 eigenspaces of an involution σ of $\mathfrak{g}$ then σ is Cartan involution if and only if the real form $\mathfrak{k}_0 \oplus \imath\mathfrak{p}_0$ is compact.

In the following we will use this simple lemma.

Lemma 2.1.4. *If $\mathfrak{g}_0$ is a real semisimple Lie algebra and θ is a Cartan involution, then the adjoint operator to $\operatorname{ad} X$ with respect to B_θ is $-\operatorname{ad} \theta X$, i.e.*

$$(\operatorname{ad} X)^* = -\operatorname{ad} \theta X, \quad \forall X \in \mathfrak{g}_0$$

Proof. We have for all $Y, Z \in \mathfrak{g}$

$$\begin{aligned} B_\theta((\operatorname{ad} X)Y, Z) &= -B([X, Y], \theta Z) = B(Y, [X, \theta Z]) \\ &= B(Y, [\theta^2 X, \theta Z]) = B(Y, \theta[\theta X, Z]) \\ &= -B_\theta(Y, [\theta X, Z]) = B_\theta(Y, (-\operatorname{ad} \theta X)Z). \end{aligned}$$

□

In addition to compact form, there is always another real form (also unique up to conjugation) of a complex semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ and consider its complexification $\mathfrak{h}$ which is (by definition) a Cartan subalgebra of $\mathfrak{g}$. We call a real Lie algebra $\mathfrak{g}_0$ a *split form* of $\mathfrak{g}$ if the restrictions of elements of $\mathfrak{h}^*$ to $\mathfrak{h}_0$ are real valued.

To construct a split real form we can just take a suitable basis of $\mathfrak{g}$ and its real span. In detail, take a complex Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$ and define $\mathfrak{h}_0$ as the subset of $\mathfrak{h}$ on which all the roots take only real values. There is always a choice (see Theorem VI.6.6 of [Kna96]) of root vectors $X_\alpha \in \mathfrak{g}_\alpha$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$ and

$$\beta \neq -\alpha \,\&\, \alpha + \beta \notin \Phi \Longrightarrow [X_\alpha, X_\beta] = 0$$
$$\alpha + \beta \in \Phi \Longrightarrow [X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta},$$

where $N_{\alpha,\beta} \in \mathbb{R}$ and $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. Now the split form of $\mathfrak{g}$ is obtained as

$$\mathfrak{g}_{\text{split}} = \mathfrak{h}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathbb{R}X_\alpha$$

and a compact form of $\mathfrak{g}$ can be then given as

$$\mathfrak{k}_0 = \imath\mathfrak{h}_0 \oplus \bigoplus_{\alpha\in\Phi^+} \mathbb{R}(X_\alpha - X_{-\alpha}) \oplus \imath\mathbb{R}(X_\alpha + X_{-\alpha}).$$

Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and let θ be its Cartan involution. A Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is called *θ-stable* if $\theta(\mathfrak{h}_0) = \mathfrak{h}_0$. In such a case we have $\mathfrak{h}_0 = (\mathfrak{h}_0 \cap \mathfrak{k}_0) \oplus (\mathfrak{h}_0 \cap \mathfrak{p}_0$, where $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the Cartan decomposition. We call the dimension of $\mathfrak{t} = \mathfrak{h}_0 \cap \mathfrak{k}_0$ the *compact dimension* of a θ-stable Cartan subalgebra $\mathfrak{h}_0$ and similarly the dimension of $\mathfrak{a} = \mathfrak{h}_0 \cap \mathfrak{p}_0$ is called the *noncompact dimension* of $\mathfrak{h}_0$. A θ-stable Cartan subalgebra $\mathfrak{h}_0 \leq \mathfrak{g}_0$ is called *maximally compact* or *maximally noncompact* if and only if its compact (respectively noncompact) dimension is maximal possible.

In contrast to the complex case, Cartan subalgebras of real Lie algebras are not unique up to conjugation. Up to conjugation by inner automorphism, there is a finite number of Cartan subalgebras and any Cartan subalgebra is conjugated via an inner automorphism to a θ-stable Cartan subalgebra. Moreover there is up to conjugation by an element of K only one maximally compact (or maximally noncompact) θ-stable Cartan subalgebra of $\mathfrak{g}_0$.

Let $\mathfrak{g}$ be a complexification of $\mathfrak{g}_0$ and let $\mathfrak{h}$ be a complexification of a θ-stable maximally noncompact Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$. Let σ be a complex conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$. For $\alpha \in \Phi$ we define $\sigma^*\alpha$ by $(\sigma^*\alpha)(H) = \overline{\alpha(\sigma H)}$ for all $H \in \mathfrak{h}$. The mapping σ^* is an involutive automorphism of the root system $\Phi(\mathfrak{g}, \mathfrak{h})$. A root $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ is called *compact root* if $\sigma^*\alpha = -\alpha$. The set of compact roots is denoted by Φ_c.

Proposition 2.1.5 (Proposition 2.3.8 of [ČS09]). *The set of compact roots is given by $\Phi_c = \{\alpha \in \Phi \,|\, \alpha$ restricted to $\mathfrak{a}$ is $0\}$. It is an abstract root system on the Euclidean subspace of $\imath\mathfrak{t} \oplus \mathfrak{a}$ spanned by its elements. For $\alpha \in \Phi_c$, the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{k} \leq \mathfrak{g}$.*

Choose a set of positive roots Φ^+ such that $\sigma^*\alpha \in \Phi^+$ for all $\alpha \in \Phi^+ \setminus \Phi_c$. One way to obtain such a system of positive roots is to choose ordering of Φ by choosing a basis $\{H_1, \dots, H_p\}$ of $\mathfrak{a}$ and extending it to a basis $\{H_1, \dots, H_r\}$ of $\imath\mathfrak{t} \oplus \mathfrak{a}$. By definition $\alpha \in \Phi^+$ if and only if there is an index j such that $\alpha(H_j) > 0$ and $\alpha(H_i) = 0$ for all $i < j$. By definition $\alpha(H_i) \neq 0$ for some $i \leq p$ for $\alpha \in \Phi^+ \setminus \Phi_c$ and since all $\alpha(H_j)$ are real, $\sigma H_i = H_i$ for $i \leq p$ and $\sigma(H_i) = -H_i$ for $i > p$.

Let Δ be the set of simple roots of Φ^+ and put $\Delta_c = \Delta \cap \Phi_c$. Then Δ_c are a system of simple roots for Φ_c and we order our system of simple roots Δ in such a way that elements of Δ_c come last. The following lemma is due to Ichiro Satake.

Lemma 2.1.6. *1. The element $\sigma^*\alpha - \alpha$ is not a root for any $\alpha \in \Phi$.*

2. For $\alpha \in \Delta \setminus \Delta_c$, there is a unique element $\alpha' \in \Delta \setminus \Delta_c$ such that $\sigma^*\alpha - \alpha'$ is a linear combination of compact roots.

The *Satake diagram* of the real Lie algebra $\mathfrak{g}_0$ is defined as follows. In the Dynkin diagram associated to the simple system Δ, represent compact roots by a black dot and roots in $\Delta \setminus \Delta_c$ by a white dot. Moreover, for any $\alpha \in \Delta \setminus \Delta_c$ such that $\sigma^*\alpha \neq \alpha$, connect α by an arrow to the unique simple root $\alpha' \in \Delta \setminus \Delta_c$ such that $\sigma^*\alpha - \alpha'$ is a linear combination of compact roots.

Definition 2.1.7 (page 274, [KV95])**.** *Let $\mathfrak{q}$ be a parabolic subalgebra of the complexification of a real semisimple Lie algebra $\mathfrak{g}_0$ and let θ be the corresponding Cartan involution. We will call $\mathfrak{q}$ to be θ-stable if $\theta\mathfrak{q} = \mathfrak{q}$. It follows then that also the opposite parabolic subalgebra $\mathfrak{q}^-$ is θ-stable and that the Levi part is θ-stable $\theta\mathfrak{l} = \mathfrak{l}$.*

For a real Lie algebra $\mathfrak{g}_0$ we define $\mathfrak{q}_0$ to be a parabolic subalgebra if and only if the complexification $\mathfrak{q}$ is a parabolic subalgebra of the complexification $\mathfrak{g}$. Similarly to the complex case, there is an equivalence between parabolic subalgebra $\mathfrak{g}$ and gradings on $\mathfrak{g}$ also in the real category. The standard parabolic subalgebras are given by crossed roots in Satake diagrams, where we are allowed to cross only white roots (i.e. those that are not in Φ_c.

Let's look at some specific examples of real Lie algebras and groups.

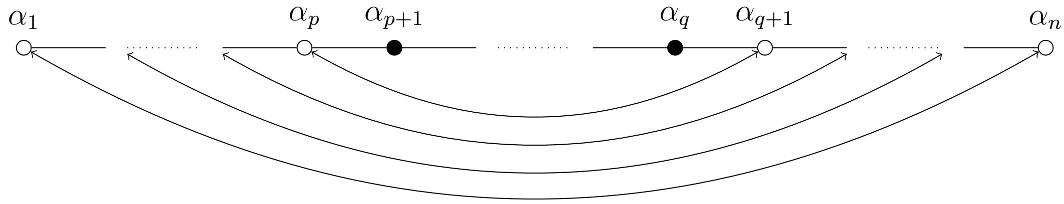
$$I_{m,n} = \begin{pmatrix} \mathrm{Id}_m & 0 \\ 0 & -\mathrm{Id}_n \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$$

$$X^\dagger = \overline{X}^t$$

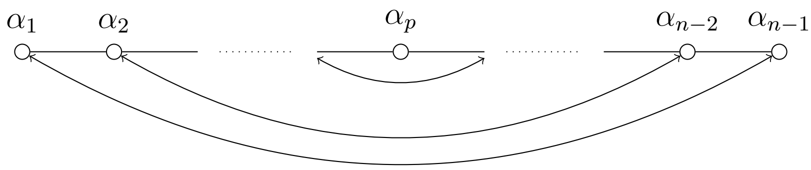
$$\begin{aligned}
\mathfrak{sl}(n,\mathbb{C}) &= \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid \operatorname{Tr} X = 0\} && \text{for } n \geq 1 \\
\mathfrak{so}(m,n) &= \{X \in \mathfrak{gl}(m+n,\mathbb{R}) \mid X^\dagger I_{m,n} + I_{m,n}X = 0\} && \text{for } m+n \geq 3 \\
\mathfrak{su}(m,n) &= \{X \in \mathfrak{sl}(m+n,\mathbb{C}) \mid X^\dagger I_{m,n} + I_{m,n}X = 0\} && \text{for } m+n \geq 2 \\
\mathfrak{so}^\dagger(2n) &= \{X \in \mathfrak{su}(n,n) \mid X^t I_{n,n} J + I_{n,n} J X = 0\} && \text{for } n \geq 2 \\
\mathfrak{sp}(n,\mathbb{R}) &= \{X \in \mathfrak{gl}(2n,\mathbb{R}) \mid X^t J + JX = 0\} && \text{for } n \geq 1
\end{aligned}$$

$$\begin{aligned}
\mathrm{SU}(p,q) &= \{g \in \mathrm{GL}(p+q,\mathbb{C} \mid g I_{p,q} g^\dagger = I_{p,q}\} \\
\mathrm{Sp}(n,\mathbb{C}) &= \{g \in \mathrm{GL}(2n,\mathbb{C} \mid g^t J g = J\} \\
\mathrm{Sp}(n,\mathbb{R}) &= \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{SU}(n,n) \\
\mathrm{O}(2n,\mathbb{C}) &= \{g \in \mathrm{GL}(n,\mathbb{C} \mid g^t J I_{n,n} g = J I_{n,n}\} \\
\mathrm{SO}^*(2n,\mathbb{C}) &= \mathrm{O}(2n,\mathbb{C}) \cap \mathrm{SU}(n,n)
\end{aligned}$$

The Satake diagram of $\mathfrak{su}(p,q)$ for $p+q=n+1$, $1 \le p \le \frac{n-1}{2}$.

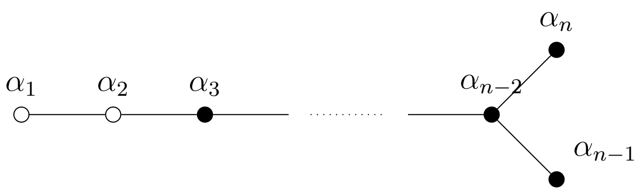


The Satake diagram of $\mathfrak{su}(p,p)$ for $n=2p+1$, $p \le 2$.



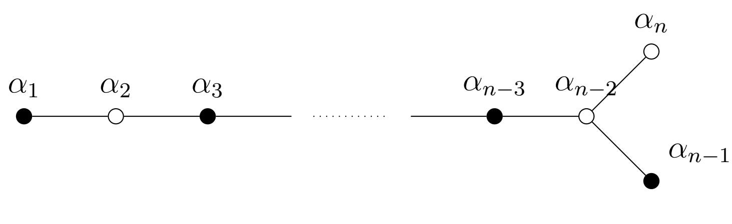
The Satake diagram for $\mathfrak{so}(2,2n-1)$.

The Satake diagram for $\mathfrak{so}(2,2n-2)$.

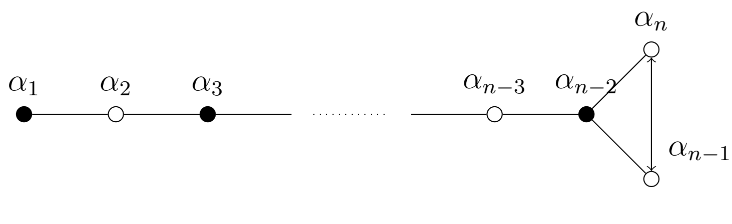


The Satake diagram for $\mathfrak{sp}(n,\mathbb{R})$.

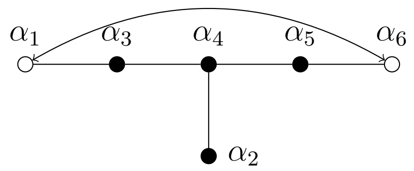
The Satake diagram for $\mathfrak{so}^*(2n)$ for even n.



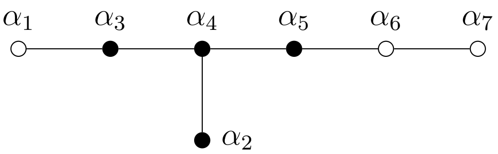
The Satake diagram for $\mathfrak{so}^*(2n)$ for odd n.



The Satake diagram for $\mathfrak{e}_6^{-14}$ - EIII.



The Satake diagram for $\mathfrak{e}_7^{-25}$ - EVII.



While Satake diagrams show best the structure of a real Lie algebra they classify, there is an alternative approach which starts with a maximally compact θ-stable Cartan subalgebra. We won't go into details and refer reader to [Kna96]. In the following we will actually use the Vogan diagrams to denote the Hermitian symmetric pairs. The reason is that there is always only one white node corresponding to the unique simple noncompact root and it is the same simple root that defines the complex parabolic subalgebra of the Hermitian pair. I.e. by turning all white nodes in these Vogan diagrams into crossed nodes one gets the usual notation for the parabolic pair.

2.2 Classical Hermitian symmetric spaces

Let G be simply connected, connected simple Lie group, let Z be its center and let K be a closed maximal subgroup of G such that K/Z is compact. A unitary representation $(\rho, \mathbb{V})$ of G such that the underlying $(\mathfrak{g}, K)$-module is an irreducible highest weight module is called unitary highest weight module. From the universal property of (generalized) Verma modules it follows that any unitarizable highest weight module is the unique irreducible quotient of a (generalized) Verma module. It is a result of Harish-Chandra that nontrivial unitarizable highest weight modules occur only when G/K is a noncompact Hermitian symmetric space. The table 2.1 of all such Hermitian pairs (G, K) is given below.

Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the corresponding Lie algebras of G and K and let $\mathfrak{g}$ and $\mathfrak{k}$ denote their complexifications. By $G_{\mathbb{C}}$ and K_C we denote the complexifications of G and K. The Cartan decomposition gives us $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{q}_0$ and upon complexification we get a splitting of $\mathfrak{q} = \mathfrak{p}_- \oplus \mathfrak{p}_+$. There is a choice of a Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{h} \leq \mathfrak{k}$. With respect to this Cartan subalgebra we have a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ with $\mathfrak{h} \leq \mathfrak{k}$ and $\mathfrak{p}_- \leq \mathfrak{n}^-$, $\mathfrak{p}_+ \leq \mathfrak{n}$.

Both algebras $\mathfrak{p} := \mathfrak{k} \oplus \mathfrak{p}_+$ and $\overline{\mathfrak{p}} := \mathfrak{k} \oplus \mathfrak{p}_-$ are parabolic subalgebras of $\mathfrak{g}$. Moreover their nilradicals $\mathfrak{p}_-$ and $\mathfrak{p}_+$ are not only nilpotent but even abelian. By P and $\overline{P}$ we denote the corresponding parabolic subgroups of $G_{\mathbb{C}}$. The homogeneous space $G_{\mathbb{C}}/P$ is diffeomorphic to the compact Hermitian symmetric space and $\mathfrak{p}_-$ is naturally mapped via exponential map and projection to an open and dense subset of this compact manifold. The so called Harish-Chandra embedding gives us a realization of the noncompact dual G/K as an orbit in this embedded $\mathfrak{p}_-$. This realizes G/K as a bounded Hermitian symmetric domain in $\mathfrak{p}_-$ and it manifests the Hermitian structure on G/K.

$G_{\mathbb{C}}$	G	K
$SL(p+q,\mathbb{C})$	$SU(p,q)$	$S(U(p)\times U(q))$
$SO(p+2,\mathbb{C})$	$SO(2,p)$	$S(O(p)\times O(2))$
$SO(2n,\mathbb{C})$	$SO^*(2n)$	$U(n)$
$Sp(2n,\mathbb{C})$	$Sp(n,\mathbb{R})$	$U(n)$
$E_6^{\mathbb{C}}$	E_6^{-14}	$\mathrm{Spin}(10)\times SO(2)$
$E_7^{\mathbb{C}}$	E_7^{-25}	$E_6\times SO(2)$

Table 2.1: Hermitian symmetric pairs

Let Φ be the set of roots of $(\mathfrak{g},\mathfrak{h})$ and let Φ_c denote the set of roots of $(\mathfrak{k},\mathfrak{h})$. We call elements of Φ_c the compact roots and the remaining roots in $\Phi_n=\Phi\setminus\Phi_c$ are called noncompact. We define the positive roots Φ^+ in such a way that elements of $\Phi_n^+=\Phi^+\cap\Phi_n$ span $\mathfrak{p}_-$. We denote the positive compact roots by $\Phi_c^+=\Phi_c\cap\Phi^+$. By ω_i we denote the i-th fundamental weight in the standard ordering.

Let Δ denote the set of simple roots of Φ. From now on we will denote the set of compact roots by Δ_c. The set $\Delta\setminus\Delta_c$ contains a single element, the unique *non-compact simple root*, which we will denote by γ_1. We define a totally ordered sequence $\xi_1,\xi_2,\dots\xi_t$ of *strongly orthogonal* non-compact positive roots inductively by γ_i being the unique minimal element in $\{\alpha\in\Phi_n^+\,|\,(\alpha,\gamma_j)=0, 1\le j<i\}$. We define

$$\mu_j:=\sum_{i=j}^{t}\xi_i$$

and note that they are all dominant weights. This result as well as the following one is due to Schmid, Kostant and Hua. It plays an important role in the proof of classification of unitarizable highest weight modules [EJ90] which are the topic of chapter 3.

Theorem 2.2.1. *Let I denote the set of integral multiindices $\underline{i}=(i_1,\dots,l_t)$ with $i_1\ge i_2\ge\dots\ge i_t\ge 0$. Let $F_{\underline{i}}$ denote the irreducible finite-dimensional $\mathfrak{k}$-module with highest weight $\sum_{j=1}^t i_j\xi_i$. Then $S(\mathfrak{p}_+)$ has a multiplicity free decomposition*

$$S(\mathfrak{p}_+)\simeq\bigoplus_{\underline{i}\in I}F_{\underline{i}}. \tag{2.3}$$

The table 2.2 gives sets of strongly orthogonal roots for Hermitian symmetric pair.

$\mathfrak{g}_0$	β	t	ξ_{t-i}	μ_{t-i}
$\mathfrak{su}(p,q)$	α_p	p	$\epsilon_i-\epsilon_{t-i+1}$	$\omega_i+\omega_{t-i+1}$
$\mathfrak{so}(2,2n-1)$	α_1	2	$\epsilon_1+\epsilon_2,\epsilon_1-\epsilon_2$	$\omega_2, 2\omega_1$
$\mathfrak{sp}(n,\mathbb{R})$	α_n	n	$2\epsilon_i$	$2\omega_1$
$\mathfrak{so}(2,2n-2)$	α_1	2	$\epsilon_1+\epsilon_2,\epsilon_1-\epsilon_2$	$\omega_2, 2\omega_1$

Table 2.2: Strongly orthogonal roots

Moreover, up to a scalar multiple, there exists for each j a unique non-zero $v_j \in S(\mathfrak{p}_+)^{\mathfrak{n}}$ of weight μ_j. Furthemore $S(\mathfrak{p}_+)^{\mathfrak{n}}$ is just the polynomial algebra on the $v_j, j = 1, \ldots, t$. We will use this in the section 2.4.1 and we refer to [GW09] for details.

2.3 Octonionic planes

The classical Hermitian symmetric spaces (i.e. those corresponding to the classical Lie groups) are treated thoroughly in [FK94]. The exceptional cases were descibed in [Dru78; Dru81]. The smaller of the two exceptional cases can be seen as a complexification of the octonionic projective plane [LM01; LM03; AB03]. It's hyperplane section was described in [PTF11]. The real octonionic projective plane is a symmetric Riemannian space $\mathrm{F}_4/\mathrm{Spin}(9)$ with exceptional holonomy Spin(9). Elementary description of this space as well as it's other noncompact (pseudo)Riemannian cousins was given in [HSV09]. It required a lot of case by case calculations and classification of Osserman manifolds to identify the manifolds with the appropriate homogeneous spaces. In this section we show how to obtain the Riemannian metric and curvature in a uniform and elemental way.

2.3.1 Octonions and the exceptional Jordan algebra

Let $\mathbb{O}$ denote the normed algebra of octonions over a field $\Bbbk$ of characteristic 0 (we will consider only $\Bbbk = \mathbb{R}$ and $\Bbbk = \mathbb{C}$) and let $N \colon \mathbb{O} \to \mathbb{R}$ denote the corresponding norm that satisfies $N(xy) = N(x)N(y)$. Let $(\,|\,)$ denote the scalar product of octonions defined by polarization $(x\,|\,y) = \frac{1}{2}(N(x+y) - N(x) - N(y))$. Sometimes it is defined without the factor of 1/2, because then some formulas are simpler and one can also work over a field of characteristic 2. The conjugation is defined by $\overline{x} = 2\,(x\,|\,1)\,1 - x$, where 1 is the unit of the octonionic algebra $\mathbb{O}$. We define the real part of x as $\Re(x) = \frac{x+\overline{x}}{2}$ and the imaginary part as $\Im(x) = \frac{x-\overline{x}}{2}$.

One can construct $\mathbb{O}$ e.g. by the Cayley-Dickson process. Basic relations concerning the scalar product are

$$\begin{aligned} (x\,|\,y) &= \frac{x\overline{y} + y\overline{x}}{2} \\ (x\,|\,y) &= (\overline{x}\,|\,\overline{y})\,. \end{aligned} \tag{2.4}$$

Another useful identities one gets via polarizations (see [SV00, p. 5])

$$(x_1 y\,|\,x_2 y) = (x_1\,|\,x_2)\,N(y), \quad (x y_1\,|\,x y_2) = N(x)\,(y_1\,|\,y_2) \tag{2.5}$$

$$(x_1 y_1\,|\,x_2 y_2) + (x_1 y_2\,|\,x_2 y_1) = 2\,(x_1\,|\,x_2)\,(y_1\,|\,y_2)\,. \tag{2.6}$$

Combining the second equation of (2.4) with (2.6) we get

$$2\,(a\,|\,c)\,(b\,|\,d) = \left(a\overline{b}\,|\,c\overline{d}\right) + \left(a\overline{d}\,|\,c\overline{b}\right). \tag{2.7}$$

Finally, we will need

$$(ab\,|\,c) = (b\,|\,\overline{a}c)\,, \quad (a\,|\,bc) = (a\overline{c}\,|\,b)\,. \tag{2.8}$$

The octonionic multiplication can be "decomposed" using scalar product and cross product similarly as in the case of quaternions. Namely, we have $(x \,|\, y) = \Re(x\overline{y})$ and we define the *cross product* as

$$x \times y = \Im(x\overline{y}) = \frac{1}{2}(x\overline{y} - y\overline{x}).$$

It is not really the seven-dimensional cross product, but its restriction to the space of imaginary octonions is. A bit more obscure is the *triple cross product*

$$u \times v \times w = \frac{1}{2}\Big(u(\overline{v}w) - w(\overline{v}u)\Big)$$

that appears in the theory of calibrations (and in fact defines what is called the Cayley calibration), see [Har90] for details. Finally, the associator is defined as

$$\{x, y, z\} = (xy)z - x(yz).$$

The associator is completely antisymmetric. This property is actually equivalent to the alternativity of the octonionic algebra which in turn implies by the Artin theorem that any subalgebra generated by two elements is associative. It is easy to see that if any entry is a multiple of unit in $\mathbb{O}$, then the associator is zero.

The projective octonionic plane $\mathbb{O}P^2$ can be defined via the exceptional formally real Jordan algebra $J_3(\mathbb{O}) = \mathrm{Herm}(3, \mathbb{O})$. The points of this projective geometry are then idempotents of trace one. The automorphism group of this Jordan algebra is the compact real group F_4 whose action preserves the trace and determinant of these octonionic matrices. One can define F_4-invariant positive definite scalar product on $J_G(\mathbb{O})$ as

$$\langle A \,|\, B\rangle = \mathrm{Tr}(A \circ B),$$

where $A \circ B$ is the Jordan product of A and B defined by

$$A \circ B = \frac{1}{2}\left(AB + BA\right).$$

We can generalize this as follows. Take a matrix $G = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix}$ such that $G^2 = \mathrm{Id}$ with all γ_i are from the ground field $\Bbbk$. Then define G-Hermitian matrices as those satisfying $GA = A^\dagger G$ and on the space of those matrices we still have the same Jordan algebra structure given by the anticommutator. All G-symmetric matrices have the following form

$$\begin{pmatrix} \gamma_1 r_1 & \gamma_2\overline{x_1} & \gamma_3\overline{x_2} \\ \gamma_1 x_1 & \gamma_2 r_2 & \gamma_3\overline{x_3} \\ \gamma_1 x_2 & \gamma_2 x_3 & \gamma_3 r_3 \end{pmatrix}. \tag{2.9}$$

The scalar product on such matrices is defined by $\langle A \,|\, B\rangle = \mathrm{Tr}\,(A \circ B)$ and for a general G-symmetric matrix this gives us the quadratic form

$$\langle A \,|\, A\rangle = 2\left(\gamma_1\gamma_2 N(x_1) + \gamma_1\gamma_3 N(x_2) + \gamma_2\gamma_3 N(x_3)\right) + \sum_{i=1}^{3} r_i^2$$

and by polarization this yields

$$\langle A \mid B \rangle = 2\left(\gamma_1\gamma_2 \left(x_1 \mid y_1\right) + \gamma_1\gamma_3 \left(x_2 \mid y_2\right) + \gamma_2\gamma_3 \left(x_3 \mid y_3\right)\right) + \sum_{i=1}^{3} r_i s_i. \tag{2.10}$$

From classification of Jordan algebras (see [SV00] for details) we know that up to isomorphism there is only one exceptional Jordan algebra over the complex numbers given by $J_3(\mathbb{O}_{\mathbb{C}})$ which corresponds to $G = \mathrm{Id}$. Over the real numbers there are actually three isomorphism classes represented by $J_3(\mathbb{O})$, $J_{1,2}(\mathbb{O})$, corresponding to $G = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$ and the Jordan algebra over the split octonions $J_3(\mathbb{O}')$. The automorphism group of an exceptional Jordan algebra is a group of type F_4. In the complex case we get of course the complex Lie group F_4, the automorphism group of $J_G(\mathbb{O})$ is the compact Lie group of type F_4, the automorphism group of $J_3(\mathbb{O}')$ is the split real Lie group of type F_4 and finally the $J_{1,2}(\mathbb{O})$ has the only remaining real Lie group F_4^{-20} as its automorphism group.

We also define scalar product on $\mathbb{O}^3$ by

$$(x \mid y) = \frac{1}{2}(x^\dagger G y + y^\dagger G x) = \sum_{i=1}^{3} \gamma_i \left(x_i \mid y_i\right).$$

Now we define the affine patches / coordinate charts.

$$\mathbb{O}P^2: \quad U_i = \left\{(x_1, x_2, x_3)^T \in \mathbb{O}^3 \,\middle|\, x_i = 1\right\} \tag{2.11}$$

$$\mathbb{O}P^{1,1}: \quad \begin{aligned} U_1 &= \left\{(1, x_2, x_3)^T \in \mathbb{O}^3 \,\middle|\, 1 + N(x_2) - N(x_3) > 0\right\} \\ U_2 &= \left\{(x_1, 1, x_3)^T \in \mathbb{O}^3 \,\middle|\, N(x_1) + 1 - N(x_3) > 0\right\} \end{aligned} \tag{2.12}$$

$$\mathbb{O}H^2: \quad U_3 = \left\{(x_1, x_2, 1)^T \in \mathbb{O}^3 \,\middle|\, N(x_1) + N(x_2) - 1 < 0\right\} \tag{2.13}$$

$$\mathbb{O}P_s^2: \quad U_i = \left\{(x_1, x_2, x_3)^T \in \mathbb{O}_s^3 \,\middle|\, x_i = 1 \,\&\, \textstyle\sum_i N(x_i) > 0\right\} \tag{2.14}$$

We would like to identify points corresponding to the same lines from different patches. Over an associative algebra it is quite easy, because we can just use the equivalence relation $(x_1, x_2, x_3)^T \sim (\lambda x_1, \lambda x_2, \lambda x_3)^T$ for any nonzero λ from the algebra. This relation is however not transitive when we are working over the octonions. Nevertheless it is transitive on our affine coordinate patches as was shown in [HSV09]. There is only one complication, in the case of the split octonions we actually demand λ to have positive norm $N(\lambda) > 0$.

With these preliminaries behind us we can finally define the $\mathbb{R}$ (or $\mathbb{C}$) linear mapping $\varphi : \mathbb{O}^3 \to J_G(\mathbb{O})$ by

$$\varphi(a) = \frac{a(Ga)^\dagger}{a^\dagger G a}.$$

The idea for such a mapping is borrowed from [All97; Asl91]. Its matrix form looks like this

$$\varphi(a) = \frac{1}{\sum_i \gamma_i N(a_i)} \begin{pmatrix} \gamma_1 N(a_1) & \gamma_2 a_1 \overline{a_2} & \gamma_3 a_1 \overline{a_3} \\ \gamma_1 a_2 \overline{a_1} & \gamma_2 N(a_2) & \gamma_3 a_2 \overline{a_3} \\ \gamma_1 a_3 \overline{a_1} & \gamma_2 a_3 \overline{a_2} & \gamma_3 N(a_3) \end{pmatrix} \tag{2.15}$$

Four octonionic planes were defined in the article [HSV09] by giving maps and transition functions. The domains used for maps are actually just the analogs of classical affine coordinates of the projective or hyperbolic plane. We show that these octonionic planes can be actually defined as the space of idempotent matrices of trace one in some exceptional Jordan algebra $J_G(\mathbb{O})$ which makes the F_4-symmetry quite manifest. Now we define our affine coordinate patches which are completely analogous to the classical picture from $\mathbb{R}^{1,2}$. The only novelty is the split projective plane $\mathbb{O}P^2_s$.

Lemma 2.3.1. *The equations for a matrix of the form* (2.9) *to be in the octonionic plane are*

$$\gamma_1 r_1 + \gamma_2 r_2 + \gamma_3 r_3 = 1, \tag{2.16}$$

$$\begin{aligned} r_1 &= \gamma_1 r_1^2 + \gamma_2 N(x_1) + \gamma_3 N(x_2) && (2.17)\\ r_2 &= \gamma_2 r_2^2 + \gamma_1 N(x_1) + \gamma_3 N(x_3) && (2.18)\\ r_3 &= \gamma_3 r_3^2 + \gamma_1 N(x_2) + \gamma_2 N(x_3) && (2.19)\end{aligned}$$

$$\begin{aligned} r_3 x_1 &= \overline{x_3} x_2 && (2.20)\\ r_2 x_2 &= x_3 x_1 && (2.21)\\ r_1 x_3 &= x_2 \overline{x_1}. && (2.22)\end{aligned}$$

Proof. Straightforward, we have just used the equation $\operatorname{Tr} A = 1$ to rewrite the off-diagonal terms of $A^2 = A$:

$$\begin{aligned}\gamma_1 x_1 &= (\gamma_1 r_1 + \gamma_2 r_2)\gamma_1 x_1 + \gamma_1\gamma_3 \overline{x_3} x_2\\ \gamma_1 x_2 &= (\gamma_1 r_1 + \gamma_3 r_3)\gamma_1 x_2 + \gamma_1\gamma_2 x_3 x_1\\ \gamma_2 x_3 &= (\gamma_2 r_2 + \gamma_3 r_3)\gamma_2 x_3 + \gamma_1\gamma_2 x_2 \overline{x_1}.\end{aligned}$$

□

Lemma 2.3.2. *The map φ restricted to the affine coordinate patches has a well-defined smooth inverse on an octonionic plane, i.e. for any $A \in J_G(\mathbb{O})$ satisfying $A^2 = A$ and $\operatorname{Tr} A = 1$ there exists $a \in \mathbb{O}^3$ whose one coordinate is 1 and such that $\varphi(a) = A$.*

Proof. The equation (2.16) implies that at least one r_i is nonzero. Without loss of generality, we will treat the case $r_1 \neq 0$ and the coordinate patch with first coordinate being 1 as the other cases follow by permuting indices.

Comparing (2.9) with (2.15) and imposing $a_1 = 1$ we see that we must have $r_1 = {}^1/_{(a\,|\,a)}$ and $\gamma_1 x_1 = \frac{1}{(a\,|\,a)}\gamma_1 a_2 \overline{a_1} = r_1\gamma_1 a_2$ which leads us to defining $a_2 = {}^{x_1}/_{r_1}$. Similarly, we obtain $a_3 = {}^{x_2}/_{r_1}$. Now we need to check whether these choices for a_2, a_3 satisfy all the remaining equations of (2.9) = (2.15).

First of all $(a\,|\,a) = \gamma_1 + \gamma_2\frac{N(x_1)}{r_1^2} + \gamma_3\frac{N(x_2)}{r_1^2} = \frac{\gamma_1 r_1^2 + \gamma_2 N(x_1) + \gamma_3 N(x_2)}{r_1^2}$. By the equation (2.17) this is $1/r_1$ as it should be. The remaining antidiagonal term of (2.9) = (2.15) is $x_3 = {}^{a_3\overline{a_2}}/_{(a\,|\,a)} = {}^{x_2\overline{x_1}}/_{r_1}$ which is equivalent to the equation (2.22).

The diagonal terms pose a slightly bigger challenge as they are equivalent to $r_1 r_2 = N(x_1)$ and $r_1 r_3 = N(x_2)$. For a nonzero x_1, x_2 these equations can be derived from (2.20) and (2.21).[1] Let us show the case $x_2 = 0$. The equation

[1] In the associative case, these are two of the equations for $\varphi(a)$ to be of rank 1 and this actually remains true even in the non-associative case.

(2.22) yields $x_3 = 0$ as we suppose $r_1 \neq 0$ from the beginning. The equation (2.20) implies either $r_3 = 0$ or $x_1 = 0$. The latter case leading to A being a zero everywhere except at the upper left position and $a = (1,0,0)^t$. In the former case we see that the equations (2.17), (2.18) turn into

$$\gamma_1 r_1 = r_1^2 + \gamma_1\gamma_2 N(x_1), \qquad \gamma_2 r_2 = r_2^2 + \gamma_1\gamma_2 N(x_1)$$

and their sum yields after a rearrangement

$$N(x_1) = \frac{1 - r_1^2 - r_2^2}{2\gamma_1\gamma_2}.$$

If we take the square of the condition $1 = \operatorname{Tr} A = \gamma_1 r_1 + \gamma_2 r_2$, we obtain $r_1^2 + 2\gamma_1\gamma_2 r_1 r_2 + r_2^2 = 1$ and subsequently

$$r_1 r_2 = \frac{1 - r_1^2 - r_2^2}{2\gamma_1\gamma_2}.$$

□

Lemma 2.3.3. *The directional derivative of φ is given by*

$$\partial_u \varphi(x) = \frac{\psi(x,u) - 2\,(x \,|\, u)\,\varphi(x)}{(x \,|\, x)},$$

where $\psi(x,u) = u(Gx)^\dagger + x(Gu)^\dagger$ or in matrix form

$$\psi(x,u) = \begin{pmatrix} 2\gamma_1\,(x_1 \,|\, u_1) & \gamma_2(x_1\overline{u_2} + u_1\overline{x_2}) & \gamma_3(x_1\overline{u_3} + u_1\overline{x_3}) \\ \gamma_1(x_2\overline{u_1} + u_2\overline{x_1}) & 2\gamma_2\,(x_2 \,|\, u_2) & \gamma_3(x_2\overline{u_3} + u_2\overline{x_3}) \\ \gamma_1(x_3\overline{u_1} + u_3\overline{x_1}) & \gamma_2(x_3\overline{u_2} + u_3\overline{x_2}) & 2\gamma_3\,(x_3 \,|\, u_3) \end{pmatrix}.$$

Proof. The directional derivative $\partial_u\varphi(x) = \lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}t}\varphi(x+tu)$ and using

$$\partial_u \left(\frac{1}{x^\dagger G x}\right) = -2\frac{(x \,|\, u)}{(x \,|\, x)^2}$$

we obtain the result after a short calculation. □

Now it is straightforward to calculate the pullback $\langle \partial_u\varphi(x) \,|\, \partial_v\varphi(x)\rangle$ of the Jordan algebra scalar product and calculate the curvature using Gauss equation. The main point here is that the scalar products on the Jordan algebras are invariant with respect to their automorphism groups and so one obtains Riemannian metrics on these coordinate patches which are invariant with respect to these groups. What remains is to prove that these patches actually cover whole orbit in each case and determine the automorphism groups. But these matters are well known [SV00; SY03].

2.4 Invariant differential operators

There is a well known correspondence between homomorphisms of parabolic Verma modules and invariant differential operators acting on sections of associated bundles over G/P [ČSS01]. For $\mathfrak{g}$-dominant integral weights there is actually a whole complex of invariant differential operators called Bernstein-Gelfand-Gelfand resolution [BGG75]. These operators can be actually defined on any parabolic geometry modeled on (G, P) [ČSS01; CD01]. Using the results of [HPR06] it can be shown that construction of [CD01] works also for bundles associated to formal completions of unitarizable highest weight modules [Tuč12]. For regular weights all Kostant modules admit a BGG resolution over G/P. See [EHP14] and references therein.

Connection between unitarizable highest weight modules and invariant differential operators is one of the main topcis of [DES91]. To explain it here, we have to change notation a little bit and let α denote multiindices of natural numbers.

Let $P(V, W)$ denote the space of polynomials between two complex vector spaces V and W which are endowed with a Hermitian inner product. For $p \in P(V, W)$ define $p(\partial) : P(V, W) \to P(V, \mathbb{C})$ by equality $p(\partial)e^{(s|t)_V} = p(\overline{s})e^{(s|t)_V}$. In coordinates we get that for $p(t) = \sum_\alpha a_\alpha t^\alpha$ the resulting linear differential operator has the expression $p(\partial) = \sum_\alpha \overline{a_\alpha} t^\alpha$. The *Fischer inner product* on $P(V, W)$ is defined by $\langle p, q\rangle := (q(\partial)p)(0)$. Explicitly in coordinates we have for $p(t) = \sum_\alpha a_\alpha t^\alpha$ and $q(t) = \sum_\alpha b_\alpha t^\alpha$ that $\langle p, q\rangle = \sum_\alpha \alpha!(a_\alpha, b_\alpha)_W$.

Lemma 2.4.1. *Let $p, q \in P(V, W)$ and let $f \in P(V, \mathbb{C})$. Then*

$$\langle p, fq\rangle = \langle f(\partial)p, q\rangle = \langle q(\partial)p, f\rangle.$$

Proof. All three expression are equal to $((fq)(\partial)p)(0)$. □

As was argued in [DES91] this inner product can be used to define a nondegenerate pairing between $M(\lambda)$ and its conjugate dual $M(\lambda)^*$. In this duality one gets that the maximal submodule $J(\lambda)$ is orthogonal to $L(\lambda)$.

Theorem 2.4.2 (Theorem 2.9 of [DES91])**.** *Let $m_1, \dots, m_k$ be any set of generators for the maximal submodule $J(\lambda)$ of the Verma module $M(\lambda)$ which we view as an $S(\mathfrak{p}_+)$-module. Then the simple submodule $L(\lambda)$ of the conjugate dual $M(\lambda)^*$ is the kernel of the constant coefficient operators $m_i(\partial)$, $1 \leq i \leq t$.*

Proof. A polynomial f is in L if and only if $\langle f, pm_i\rangle = 0$ for all $p \in P(\mathfrak{p}_-, \mathbb{C}) = S(\mathfrak{p}_+)$, $1 \leq i \leq t$. According to the previous lemma this is equivalent to $p(\partial)m_i(\partial)f)(0)$ for all relevant p and i. From the polynomiality of f follows that $m_i(\partial)f = 0$ for all $1 \leq i \leq t$. □

Of course, the simple submodule $L(\lambda)$ is isomorphic to the simple quotient of $M(\lambda)$ by $J(\lambda)$. Reader interested in explicit form of the generators m_i can consult chapter 10 of [DES91]. The relationship between the Fischer inner product and the Shapovalov form was calculated by [Wac99].

2.4.1 Explicit singular vectors for $\mathfrak{su}(m, n)$

Since Verma modules are in particular highest weight modules, any homomorphism between them is determined by the image of the generating vector. This image is called singular vector and it is, up to duality, precisely the invariant differential operator we seek. In this section we apply the recently developed F-method [Kob14; Kob+15] to find (and classify) all singular vectors in Verma modules for $\mathfrak{su}(m, n)$ which are induced from scalar representation.

Let us consider the connected complex simple Lie group $G = \mathrm{SL}(n+m, \mathbb{C})$, $n, m \in \mathbb{N}$, and its Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+m, \mathbb{C})$ given by

$$\mathfrak{sl}(n+m, \mathbb{C}) = \{X \in M_{n+m,n+m}(\mathbb{C});\ \mathrm{Tr}\, X = 0\}. \tag{2.23}$$

A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is given by diagonal matrices

$$\mathfrak{h} = \{\mathrm{diag}(a_1, a_2, \dots, a_{n+m});\ a_1, a_2, \dots, a_{n+m} \in \mathbb{C},\ \textstyle\sum_{i=1}^{n+m} a_i = 0\}. \tag{2.24}$$

For $i = 1, 2, \dots, n+m$ we define $\varepsilon_i \in \mathfrak{h}^*$ by $\varepsilon_i(\mathrm{diag}(a_1, a_2, \dots, a_{n+m})) = a_i$. Then the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is $\Delta = \{\varepsilon_i - \varepsilon_j;\ 1 \leq i \neq j \leq n+m\}$. A positive root system in Δ is $\Delta^+ = \{\varepsilon_i - \varepsilon_j;\ 1 \leq i < j \leq n+m\}$ with the set of simple roots $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_{n+m-1}\}$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, 2, \dots, n+m-1$. Finally, the fundamental weights are $\omega_i = \sum_{j=1}^{i} \varepsilon_j$ for $i = 1, 2, \dots, n+m-1$.

The subset $\Sigma = \{\alpha_1, \dots, \alpha_{n-1}, \alpha_{n+1}, \dots, \alpha_{n+m-1}\}$ of Π generates the root subsystem Δ_Σ in $\mathfrak{h}^*$. We associate to Σ the standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ by $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$. By subscript μ we denote the $\mathfrak{l}$-isotypical component of a module. The Σ-height $\mathrm{ht}_\Sigma(\alpha)$ of $\alpha \in \Delta$ is

$$\mathrm{ht}_\Sigma\Big(\textstyle\sum_{i=1}^{n+m-1} a_i \alpha_i\Big) = a_n, \tag{2.25}$$

so $\mathfrak{g}$ is a $|1|$-graded Lie algebra with respect to the grading given by $\mathfrak{g}_i = \bigoplus_{\alpha \in \Delta,\, \mathrm{ht}_\Sigma(\alpha) = i} \mathfrak{g}_\alpha$ for $0 \neq i \in \mathbb{Z}$, and $\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta,\, \mathrm{ht}_\Sigma(\alpha) = 0} \mathfrak{g}_\alpha$. Moreover, we have $\mathfrak{l} = \mathfrak{g}_0 \simeq \mathbb{C} \oplus \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(m, \mathbb{C})$, $\mathfrak{u} = \mathfrak{g}_1 \simeq \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$, and $\bar{\mathfrak{u}} = \mathfrak{g}_{-1} \simeq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$.

The basis $\{f_{ij};\ i = 1, 2, \dots, m,\ j = 1, 2, \dots, n\}$ of the root spaces in the opposite nilradical $\bar{\mathfrak{u}}$ is given by

$$f_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix}, \tag{2.26}$$

where E_{ij} is the $(m \times n)$-matrix having 1 at the intersection of the i-th row and j-th column and 0 elsewhere. Analogously, the basis $\{e_{ij};\ i = 1, 2, \dots, n,\ j = 1, 2, \dots, m\}$ of the root spaces in the nilradical $\mathfrak{u}$ is given by

$$e_{ij} = \begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix}, \tag{2.27}$$

where E_{ij} is the $(n \times m)$-matrix having 1 at the intersection of the i-th row and j-th column and 0 elsewhere. The Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ is the linear span of

$$h_c = \begin{pmatrix} \frac{1}{n} I_n & 0 \\ 0 & \frac{-1}{m} I_m \end{pmatrix}, \qquad h_{A,B} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{2.28}$$

where $A \in M_{n,n}(\mathbb{C})$, $B \in M_{m,m}(\mathbb{C})$ satisfy $\mathrm{Tr}\, A = 0$ and $\mathrm{Tr}\, B = 0$. Moreover, the element h_c is a basis of the center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$.

Finally, the parabolic subgroup P of G with the Lie algebra $\mathfrak{p}$ is defined by $P = N_G(\mathfrak{p})$, where the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is given by

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix};\ A \in M_{n,n}(\mathbb{C}),\ B \in M_{m,m}(\mathbb{C}),\ C \in M_{n,m}(\mathbb{C}),\ \mathrm{Tr}\, A + \mathrm{Tr}\, B = 0 \right\}. \tag{2.29}$$

Any character $\lambda \in \mathrm{Hom}_P(\mathfrak{p}, \mathbb{C})$ is given by

$$\lambda = \alpha \widetilde{\omega}_n \tag{2.30}$$

for some $\alpha \in \mathbb{C}$, where $\widetilde{\omega}_n \in \mathrm{Hom}_P(\mathfrak{p}, \mathbb{C})$ is equal to $\omega_n \in \mathfrak{h}^*$ regarded as trivially extended to $\mathfrak{p} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta_\Sigma} \mathfrak{g}_\alpha) \oplus \mathfrak{u}$. The character $\rho \in \mathrm{Hom}_P(\mathfrak{p}, \mathbb{C})$ is defined as

$$\rho = \tfrac{n+m}{2}\, \widetilde{\omega}_n. \tag{2.31}$$

For $\alpha \in \mathbb{C}$ we denote the character $\alpha\widetilde{\omega}_n$ by $\lambda_\alpha \in \mathrm{Hom}_P(\mathfrak{p}, \mathbb{C})$.

Let us introduce the notation

$$E^x = \sum_{k=1}^m \sum_{\ell=1}^n x_{k\ell} \partial_{x_{k\ell}}, \qquad E^y = \sum_{k=1}^m \sum_{\ell=1}^n y_{k\ell} \partial_{y_{k\ell}} \tag{2.32}$$

for the Euler homogeneity operators and

$$R_A^x = \sum_{i,j=1}^n \sum_{k=1}^m a_{ij} x_{ki} \partial_{x_{kj}}, \qquad L_B^x = -\sum_{i,j=1}^m \sum_{k=1}^n b_{ij} x_{jk} \partial_{x_{ik}}, \tag{2.33}$$

$$R_A^y = -\sum_{i,j=1}^n \sum_{k=1}^m a_{ij} y_{kj} \partial_{y_{ki}}, \qquad L_B^y = \sum_{i,j=1}^m \sum_{k=1}^n b_{ij} y_{ik} \partial_{y_{jk}} \tag{2.34}$$

for $A = (a_{ij}) \in M_{n,n}(\mathbb{C})$, $B = (b_{ij}) \in M_{m,m}(\mathbb{C})$. We set $L_{ij}^y = L_{E_{ij}}^y$ for $1 \leq i, j \leq m$ and $R_{ij}^y = R_{E_{ij}}^y$ for $1 \leq i, j \leq n$.

Theorem 2.4.3. *Let $\lambda \in \mathrm{Hom}_P(\mathfrak{p}, \mathbb{C})$ and let $(\sigma, \mathbb{V})$ be a $\mathfrak{p}$-module. Let $\mathcal{A}_{\overline{\mathfrak{u}}}$ be the Weyl algebra of the vector space $\overline{\mathfrak{u}}$ generated by $\{x_\alpha, \partial_{x_\alpha};\, \alpha \in \Delta(\mathfrak{u})\}$ and $\mathcal{A}_{\overline{\mathfrak{u}^*}}$ be the Weyl algebra of the vector space $\overline{\mathfrak{u}^*}$ generated by by $\{y_\alpha, \partial_{y_\alpha};\, \alpha \in \Delta(\mathfrak{u})\}$. Then the embedding of $\mathfrak{g}$ into $\mathcal{A}_{\overline{\mathfrak{u}}} \otimes_{\mathbb{C}} \mathrm{End}\, \mathbb{V}_{\lambda+\rho}$ and into it's Fourier dual $\mathcal{A}_{\overline{\mathfrak{u}}^*} \otimes_{\mathbb{C}} \mathrm{End}\, \mathbb{V}_{\lambda-\rho}$ is given by*

1)

$$\begin{aligned} \pi_{\sigma_\lambda}(f_{ij}) &= -\partial_{x_{ij}}, \\ \hat{\pi}_{\sigma_\lambda}(f_{ij}) &= -y_{ij} \end{aligned} \tag{2.35}$$

for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$;

2)

$$\begin{aligned} \pi_{\sigma_\lambda}(h_c) &= \left(\frac{1}{n} + \frac{1}{m}\right) E^x + \left(\lambda(h_c) + \frac{n+m}{2}\right) + \sigma(h_c), \\ \pi_{\sigma_\lambda}(h_{A,B}) &= R_A^x + L_B^x + \sigma(h_{A,B}), \\ \hat{\pi}_{\sigma_\lambda}(h_c) &= -\left(\frac{1}{n} + \frac{1}{m}\right) E^y + \left(\lambda(h_c) - \frac{n+m}{2}\right) + \sigma(h_c), \\ \hat{\pi}_{\sigma_\lambda}(h_{A,B}) &= R_A^y + L_B^y + \sigma(h_{A,B}) \end{aligned} \tag{2.36}$$

for all $A \in M_{n,n}(\mathbb{C})$, $B \in M_{m,m}(\mathbb{C})$ satisfying $\mathrm{Tr}\, A = 0$ and $\mathrm{Tr}\, B = 0$;

3)

$$\begin{aligned}\pi_{\sigma_\lambda}(e_{ij}) &= \sum_{k=1}^m \sum_{\ell=1}^n x_{ki}x_{j\ell}\partial_{x_{k\ell}} + \bigg(\lambda(h_c) + \frac{n+m}{2}\bigg)x_{ji} + \sigma(e_{ij}) + x_{ji}\sigma(h_c) \\ &\quad + \sum_{\ell=1}^n x_{j\ell}\sigma(h_{E_{i\ell}-\frac{1}{n}I_n\delta_{i\ell},0}) - \sum_{k=1}^m x_{ki}\sigma(h_{0,E_{kj}-\frac{1}{m}I_m\delta_{kj}}), \\ \hat{\pi}_{\sigma_\lambda}(e_{ij}) &= \sum_{k=1}^m \sum_{\ell=1}^n y_{k\ell}\partial_{y_{ki}}\partial_{y_{j\ell}} - \bigg(\lambda(h_c) - \frac{n+m}{2}\bigg)\partial_{y_{ji}} + \sigma(e_{ij}) - \partial_{y_{ji}}\sigma(h_c) \\ &\quad - \sum_{\ell=1}^n \partial_{y_{j\ell}}\sigma(h_{E_{i\ell}-\frac{1}{n}I_n\delta_{i\ell},0}) + \sum_{k=1}^m \partial_{y_{ki}}\sigma(h_{0,E_{kj}-\frac{1}{m}I_m\delta_{kj}})\end{aligned} \tag{2.37}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Proof. It follows immediately by straightforward caculations from [KS17]. □

Let $\alpha \in \mathbb{C}$ and let $(\sigma, \mathbb{C})$ be the trivial $\mathfrak{p}$-module. Then we have an isomorphism

$$\tau \circ \Phi_{\lambda_\alpha} \colon M(\mathbb{C}_{\lambda_\alpha-\rho}) \xrightarrow{\sim} \mathbb{C}[\bar{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_\alpha-\rho} \tag{2.38}$$

of $\mathfrak{g}$-modules, where the action of $\mathfrak{g}$ on $\mathbb{C}[\bar{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_\alpha-\rho}$ is given by Theorem 2.4.3.

If we introduce an $\mathfrak{l}$-module

$$\mathrm{Sol} = \{f \in \mathbb{C}[\bar{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{V}_{\lambda-\rho};\ \hat{\pi}_{\sigma_\lambda}(a)f = 0 \text{ for all } a \in \mathfrak{u}\}, \tag{2.39}$$

then by (2.38) we obtain an isomorphism

$$\tau \circ \Phi_\lambda \colon M(\mathbb{V}_{\lambda-\rho})^{\mathfrak{u}} \xrightarrow{\sim} \mathrm{Sol} \tag{2.40}$$

of $\mathfrak{l}$-modules where $\mathcal{F}$ is the algebraic Fourier duality mapping uniquely determined by

$$\mathcal{F}(x_\alpha) = -\partial_{y_\alpha}, \qquad \mathcal{F}(\partial_{x_\alpha}) = y_\alpha.$$

Proposition 2.4.4. *Let $\alpha \in \mathbb{C}$ and let $r = \min\{n, m\}$. Then we have*

$$\mathbb{C}[\bar{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_\alpha-\rho} \simeq \bigoplus_{\mu \in \Lambda_\alpha^+(\mathfrak{p})} \mathbb{F}_\mu \tag{2.41}$$

as $\mathfrak{l}$-modules, where $\mathbb{F}_\mu$ is the simple $\mathfrak{l}$-module with highest weight $\mu \in \mathfrak{h}^*$ and

$$\Lambda_\alpha^+(\mathfrak{p}) = \Bigg\{\sum_{i=1}^r a_i\omega_{n-i} + \sum_{i=1}^r a_i\omega_{n+i} + \bigg(\alpha - \frac{n+m}{2} - 2\sum_{i=1}^r a_i\bigg)\omega_n;\ a_i \in \mathbb{N}_0\Bigg\}. \tag{2.42}$$

Moreover, the corresponding highest weight vector v_μ of $\mathbb{F}_\mu$ in $\mathbb{C}[\bar{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_\alpha-\rho}$ with

$$\mu = \sum_{i=1}^r a_i\omega_{n-i} + \sum_{i=1}^r a_i\omega_{n+i} + \bigg(\alpha - \frac{n+m}{2} - 2\sum_{i=1}^r a_i\bigg)\omega_n \tag{2.43}$$

is $v_\mu = q_1^{a_1}q_2^{a_2}\ldots q_r^{a_r}$, where

$$q_k = \det\begin{pmatrix} y_{1,n-k+1} & y_{1,n-k+2} & \cdots & y_{1,n} \\ y_{2,n-k+1} & y_{2,n-k+2} & \cdots & y_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{k,n-k+1} & y_{k,n-k+2} & \cdots & y_{k,n}\end{pmatrix} \tag{2.44}$$

for $k = 1, 2, \ldots, r$.

Proof. First of all, we need to verify that the polynomials q_k are the highest weight vectors with highest weights $\omega_{n-k}+\omega_{n+k}+\big(\alpha-\frac{n+m}{2}-2\big)\omega_n$ for $k=1,2,\dots,r$. Using (2.34) it easily follows that $L^y_{ij}q_k=0$ for $j=k+1,k+2,\dots,m$ and $L^y_{ij}q_k$ is the polynomial which arises from q_k replacing the j-th row $(y_{j,n-k+1},y_{j,n-k+2},\dots,y_{j,n})$ with the i-th row $(y_{i,n-k+1},y_{i,n-k+2},\dots,y_{i,n})$ in the corresponding matrix (2.44) for $j=1,2,\dots,k$. Analogously, we have that $-R^y_{ij}q_k=0$ for $i=1,2,\dots,n-k$ and $-R^y_{ij}q_k$ is the polynomial which arises from q_k replacing the i-th column $(y_{1,i},y_{2,i},\dots,y_{k,i})^{\mathrm{T}}$ with the j-th column $(y_{1,j},y_{2,j},\dots,y_{k,j})^{\mathrm{T}}$ in the corresponding matrix (2.44) for $i=n-k+1,n-k+2,\dots,n$. Therefore, we obtain $R^y_{ij}q_k=0$ for $1\leq i<j\leq n$ and $L^y_{ij}q_k=0$ for $1\leq i<j\leq m$, which gives us that q_k is annihilated by $\mathfrak{l}\cap\mathfrak{n}$. Furthermore, since we have

$$\begin{aligned}\hat{\pi}_{\lambda_\alpha}(h_{A,B})q_k &= \sum_{i=1}^n a_iR^y_{ii}q_k+\sum_{j=1}^m b_jL^y_{jj}q_k=-\sum_{i=n-k+1}^n a_iq_k+\sum_{j=1}^k b_jq_k\\ &=\Bigg(\sum_{i=1}^{n-k}a_i+\sum_{j=1}^k b_j\Bigg)q_k\\ &=\omega_{n-k}(h_{A,B})q_k+\omega_{n+k}(h_{A,B})q_k+\bigg(\alpha-\frac{n+m}{2}-2\bigg)\omega_n(h_{A,B})q_k\end{aligned}$$

for all $A=\operatorname{diag}(a_1,a_2,\dots,a_n)$, $B=\operatorname{diag}(b_1,b_2,\dots,b_m)$ satisfying $\operatorname{Tr}A=0$, $\operatorname{Tr}B=0$, and

$$\begin{aligned}\hat{\pi}_{\lambda_\alpha}(h_c)q_k &= -\bigg(\frac{1}{n}+\frac{1}{m}\bigg)E^yq_k+\bigg(\alpha-\frac{n+m}{2}\bigg)q_k\\ &=-k\bigg(\frac{1}{n}+\frac{1}{m}\bigg)q_k+\bigg(\alpha-\frac{n+m}{2}\bigg)q_k\\ &=\omega_{n-k}(h_c)q_k+\omega_{n+k}(h_c)q_k+\bigg(\alpha-\frac{n+m}{2}-2\bigg)\omega_n(h_c)q_k,\end{aligned}$$

we obtain that q_k is a weight vector with respect to $\mathfrak{l}\cap\mathfrak{h}$. Therefore, the polynomial q_k is the highest weight vector with highest weight $\omega_{n-k}+\omega_{n+k}+\big(\alpha-\frac{n+m}{2}-2\big)\omega_n$.

As an immediate consequence of the previous result, we get that the polynomial $q_1^{a_1}q_2^{a_2}\dots q_r^{a_r}$ with $a_1,a_2,\dots,a_r\in\mathbb{N}_0$ is the highest weight vector with highest weight

$$\mu=\sum_{k=1}^r a_k\omega_{n-k}+\sum_{k=1}^r a_k\omega_{n+k}+\Bigg(\alpha-\frac{n+m}{2}-2\sum_{k=1}^n a_k\Bigg)\omega_n$$

and generates a finite-dimensional simple $\mathfrak{l}$-submodule of $\mathbb{C}[\bar{\mathfrak{u}}^*]\otimes_{\mathbb{C}}\mathbb{C}_{\lambda_\alpha-\rho}$ isomorphic to $\mathbb{F}_\mu$, since $\mathbb{C}[\bar{\mathfrak{u}}^*]\otimes_{\mathbb{C}}\mathbb{C}_{\lambda_\alpha-\rho}$ is a semisimple $\mathfrak{l}$-module decomposing into finite-dimensional simple $\mathfrak{l}$-submodules. This gives us an injective homomorphism

$$\bigoplus_{\mu\in\Lambda^+_\alpha(\mathfrak{p})}\mathbb{F}_\mu\to\mathbb{C}[\bar{\mathfrak{u}}^*]\otimes_{\mathbb{C}}\mathbb{C}_{\lambda_\alpha-\rho}$$

of $\mathfrak{l}$-modules, which is an isomorphism as follows from [GW09]. □

Lemma 2.4.5. *Let $\alpha \in \mathbb{C}$ and $\mu \in \Lambda^+_\alpha(\mathfrak{p})$. Then we have*

$$\mathrm{Sol}_\mu \simeq \begin{cases} \mathbb{F}_\mu & \text{for } \alpha \in \mathbb{C},\ \mu = \left(\alpha - \frac{n+m}{2}\right)\omega_n, \\ \mathbb{F}_\mu & \text{for } \alpha = a - k + \frac{n+m}{2},\ \mu = a\omega_{n-k} + a\omega_{n+k} - (a+k)\omega_n, \\ & a \in \mathbb{N},\ k \in \{1,2,\dots,r\} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Using Proposition 2.4.4 we know that $\mathbb{C}[\bar{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_\alpha - \rho}$ is a multiplicity free $\mathfrak{l}$-module. Moreover, since we have $\mathrm{Sol}_\mu \subset (\mathbb{C}[\bar{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_\alpha - \rho})_\mu \simeq \mathbb{F}_\mu$ for $\mu \in \Lambda^+_\alpha(\mathfrak{p})$, we obtain that Sol_μ is either the zero $\mathfrak{l}$-module or isomorphic to the $\mathfrak{l}$-module $\mathbb{F}_\mu$. Therefore, it is enough to verify, whether the highest weight vector v_μ of $(\mathbb{C}[\bar{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_\alpha - \rho})_\mu$ is contained in Sol or not. Let $\mu \in \Lambda^+_\alpha(\mathfrak{p})$ and let us assume that the highest weight vector

$$v_\mu = q_1^{a_1} q_2^{a_2} \dots q_r^{a_r}$$

of $(\mathbb{C}[\bar{\mathfrak{u}}^*] \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_\alpha - \rho})_\mu$ with $a_1, a_2, \dots a_r \in \mathbb{N}_0$ belongs to Sol. For greater clarity, we denote by q^a the polynomial $q_1^{a_1} q_2^{a_2} \dots q_r^{a_r}$ for $a = (a_1, a_2, \dots, a_r) \in \mathbb{N}_0^r$ and by $1_k \in \mathbb{N}_0^r$ the r-tuple having 1 at the k-th position and 0 otherwise. Furthermore, we have

$$\begin{aligned} \hat{\pi}_{\lambda_\alpha}(e_{ij}) &= \sum_{k=1}^m \sum_{\ell=1}^n y_{k\ell} \partial_{y_{ki}} \partial_{y_{j\ell}} - \left(\alpha - \frac{n+m}{2}\right) \partial_{y_{ji}} \\ &= \sum_{k=1}^m \partial_{y_{ki}} L^y_{kj} - \left(\alpha - \frac{n-m}{2}\right) \partial_{y_{ji}} \end{aligned}$$

for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

If $v_\mu = 1$, then $\hat{\pi}_{\lambda_\alpha}(e_{ij}) v_\mu = 0$ for all $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ and $\alpha \in \mathbb{C}$. Therefore, we have $\mathrm{Sol}_\mu \simeq \mathbb{F}_\mu$ for $\mu = \left(\alpha - \frac{n+m}{2}\right)\omega_n$ and $\alpha \in \mathbb{C}$.

Now, if $v_\mu \neq 1$, then there exists an index $s \in \{1, 2, \dots, r\}$ satisfying $a_s \neq 0$ and $a_\ell = 0$ for $\ell = s+1, s+2, \dots, r$. Since $\partial_{y_{k,n-s+1}} q_\ell = 0$ and $L^y_{ks} q_\ell = 0$ for $\ell = 1, 2, \dots, s-1$ and $k = 1, 2, \dots, m$, we may write

$$\begin{aligned} \hat{\pi}_{\lambda_\alpha}(e_{n-s+1,s}) v_\mu &= \sum_{k=1}^m \partial_{y_{k,n-s+1}} L^y_{ks} q^a - \left(\alpha - \tfrac{n-m}{2}\right) \partial_{y_{s,n-s+1}} q^a \\ &= \sum_{k=1}^m a_s \partial_{y_{k,n-s+1}} q^{a-1_s} L^y_{ks} q_s - \left(\alpha - \tfrac{n-m}{2}\right) \partial_{y_{s,n-s+1}} q^a \\ &= \sum_{k=s}^m a_s \partial_{y_{k,n-s+1}} q^{a-1_s} L^y_{ks} q_s - \left(\alpha - \tfrac{n-m}{2}\right) \partial_{y_{s,n-s+1}} q^a, \end{aligned}$$

where we used that $L^y_{ks} q_s = 0$ for $k < s$ in the last equality. Further, using $\partial_{y_{k,n-s+1}} q_\ell = 0$ for $k > \ell$, we obtain

$$\begin{aligned} \hat{\pi}_{\lambda_\alpha}(e_{n-s+1,s}) v_\mu &= a_s \partial_{y_{s,n-s+1}} q^{a-1_s} L^y_{ss} q_s + \sum_{k=s+1}^m a_s q^{a-1_s} \partial_{y_{k,n-s+1}} L^y_{ks} q_s \\ &\quad - \left(\alpha - \tfrac{n-m}{2}\right) \partial_{y_{s,n-s+1}} q^a \\ &= a_s \partial_{y_{s,n-s+1}} q^a + (m-s) a_s q^{a_s - 1_s} \partial_{y_{s,n-s+1}} q_s - \left(\alpha - \tfrac{n-m}{2}\right) \partial_{y_{s,n-s+1}} q^a \\ &= \left(a_s - s - \alpha + \tfrac{n+m}{2}\right) \partial_{y_{s,n-s+1}} q^a, \end{aligned}$$

where we used $L^y_{ss}q_s = q_s$ and $\partial_{y_{k,n-s+1}} L^y_{ks} q_s = \partial_{y_{s,n-s+1}} q_s$ for $k > s$. As $\partial_{y_{s,n-s+1}} q^a \neq 0$ and $\hat{\pi}_{\lambda_\alpha}(e_{n-s+1,s})v_\mu = 0$, we get

$$\alpha = a_s - s + \frac{n+m}{2}. \tag{2.45}$$

Our next step is to show that $a_\ell = 0$ for $\ell = 1, 2, \ldots, s-1$. Let us assume that there exists an index $t \in \{1, 2, \ldots, s-1\}$ satisfying $a_t \neq 0$ and $a_\ell = 0$ for $\ell = t+1, t+2, \ldots, s-1$. Analogously as in the previous part, we may write

$$\begin{aligned}
\hat{\pi}_{\lambda_\alpha}(e_{n-t+1,t})v_\mu &= \sum_{k=1}^{m} \partial_{y_{k,n-t+1}} L^y_{kt} q^a - \Big(\alpha - \tfrac{n-m}{2}\Big) \partial_{y_{t,n-t+1}} q^a \\
&= \sum_{k=1}^{m} \Big(a_s \partial_{y_{k,n-t+1}} q^{a-1_s} L^y_{kt} q_s + a_t \partial_{y_{k,n-t+1}} q^{a-1_t} L^y_{kt} q_t \Big) \\
&\quad - \Big(\alpha - \tfrac{n-m}{2}\Big) \partial_{y_{t,n-t+1}} q^a \\
&= \sum_{k=s+1}^{m} a_s \partial_{y_{k,n-t+1}} q^{a-1_s} L^y_{kt} q_s + \sum_{k=t+1}^{m} a_t \partial_{y_{k,n-t+1}} q^{a-1_t} L^y_{kt} q_t \\
&\quad + a_s \partial_{y_{t,n-t+1}} q^{a-1_s} L^y_{tt} q_s + a_t \partial_{y_{t,n-t+1}} q^{a-1_t} L^y_{tt} q_t \\
&\quad - \Big(\alpha - \tfrac{n-m}{2}\Big) \partial_{y_{t,n-t+1}} q^a \\
&= \sum_{k=s+1}^{m} a_s q^{a-1_s} \partial_{y_{k,n-t+1}} L^y_{kt} q_s + \sum_{k=t+1}^{m} a_t q^{a-1_t} \partial_{y_{k,n-t+1}} L^y_{kt} q_t \\
&\quad + \sum_{k=t+1}^{s} a_t (\partial_{y_{k,n-t+1}} q^{a-1_t}) L^y_{kt} q_t + \Big(a_s + a_t - \alpha + \tfrac{n-m}{2}\Big) \partial_{y_{t,n-t+1}} q^a.
\end{aligned}$$

Using $\partial_{y_{k,n-t+1}} L^y_{kt} q_s = \partial_{y_{t,n-t+1}} q_s$ for $k > s$ and $\partial_{y_{k,n-t+1}} L^y_{kt} q_t = \partial_{y_{t,n-t+1}} q_t$ for $k > t$, we get

$$\begin{aligned}
\hat{\pi}_{\lambda_\alpha}(e_{n-t+1,t})v_\mu &= (m-s) a_s q^{a-1_s} \partial_{y_{t,n-t+1}} q_s + (m-t) a_t q^{a-1_t} \partial_{y_{t,n-t+1}} q_t \\
&\quad + \sum_{k=t+1}^{s} a_s a_t q^{a-1_s-1_t} (\partial_{y_{k,n-t+1}} q_s) L^y_{kt} q_t \\
&\quad + \Big(a_s + a_t - \alpha + \tfrac{n-m}{2}\Big) \partial_{y_{t,n-t+1}} q^a \\
&= a_s a_t q^{a-1_s} \partial_{y_{t,n-t+1}} q_s + a_t (a_t + s - t) q^{a-1_t} \partial_{y_{t,n-t+1}} q_t \\
&\quad + \sum_{k=t+1}^{s} a_s a_t q^{a-1_s-1_t} (\partial_{y_{k,n-t+1}} q_s) L^y_{kt} q_t,
\end{aligned}$$

where we used (2.45) in the last equality. Since $\hat{\pi}_{\lambda_\alpha}(e_{n-t+1,t})v_\mu = 0$, we obtain the equation

$$q^{a-1_s-1_t} \Bigg(a_t a_s q_t \partial_{y_{t,n-t+1}} q_s + a_t (a_t + s - t) q_s \partial_{y_{t,n-t+1}} q_t + \sum_{k=t+1}^{s} a_s a_t (\partial_{y_{k,n-t+1}} q_s) L^y_{kt} q_t \Bigg) = 0,$$

which gives us

$$a_t a_s q_t \partial_{y_{t,n-t+1}} q_s + a_t (a_t + s - t) q_s \partial_{y_{t,n-t+1}} q_t + \sum_{k=t+1}^{s} a_s a_t (\partial_{y_{k,n-t+1}} q_s) L^y_{kt} q_t = 0.$$

Applying the operator $\partial_{y_{t,n-t+1}}$ to the previous equation, we get

$$a_t(a_s + a_t + s - t)(\partial_{y_{t,n-t+1}} q_s)(\partial_{y_{t,n-t+1}} q_t) = 0,$$

because $\partial_{y_{t,n-t+1}}(\partial_{y_{k,n-t+1}} q_s) L^y_{kt} q_t = 0$ for $k = t+1, t+2, \dots, s$. Moreover, we have $\partial_{y_{t,n-t+1}} q_s \neq 0$, $\partial_{y_{t,n-t+1}} q_t \neq 0$ and $a_s + a_t + s - t > 0$, hence we obtain $a_t = 0$, which is a contradiction with the choice of the index t. Therefore, we have $v_\mu = q_s^{a_s}$ for $\mu = a_s\omega_{n-s} + a_s\omega_{n+s} - (a_s + s)\omega_n$ and $\alpha = a_s - s + \frac{n+m}{2}$. It remains to show that $\hat{\pi}_{\lambda_\alpha}(e_{ij})v_\mu = 0$ for all $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ and $\alpha = a_s - s + \frac{n+m}{2}$. We may write

$$\begin{aligned}
\hat{\pi}_{\lambda_\alpha}(e_{ij})v_\mu &= \sum_{k=1}^{m} \partial_{y_{ki}} L^y_{kj} q_s^{a_s} - (a_s - s + m)\partial_{y_{ji}} q_s^{a_s} \\
&= \sum_{k=1}^{m} a_s \partial_{y_{ki}} q_s^{a_s - 1} L^y_{kj} q_s - (a_s - s + m)\partial_{y_{ji}} q_s^{a_s} \\
&= a_s \partial_{y_{ji}} q_s^{a_s-1} L^y_{jj} q_s + \sum_{k=s+1}^{m} a_s \partial_{y_{ki}} q_s^{a_s-1} L^y_{kj} q_s - (a_s - s + m)\partial_{y_{ji}} q_s^{a_s} \\
&= a_s \partial_{y_{ji}} q_s^{a_s} + \sum_{k=s+1}^{m} a_s q_s^{a_s-1} \partial_{y_{ki}} L^y_{kj} q_s - (a_s - s + m)\partial_{y_{ji}} q_s^{a_s} \\
&= \sum_{k=s+1}^{m} a_s q_s^{a_s-1} \partial_{y_{ji}} q_s - (m - s)\partial_{y_{ji}} q_s^{a_s} \\
&= (m - s)\partial_{y_{ji}} q_s^{a_s} - (m - s)\partial_{y_{ji}} q_s^{a_s} = 0,
\end{aligned}$$

where we used $L^y_{kj} q_s = 0$ if $j > s$ or if $k \leq s$ and $k \neq j$ in the third equality and $\partial_{y_{ki}} L^y_{kj} q_s = \partial_{y_{ji}} q_s$ for $k > s$ in the fifth equality. Hence, we have $\mathrm{Sol}_\mu \simeq \mathbb{F}_\mu$ for $\mu = a\omega_{n-k} + a\omega_{n+k} - (a + k)\omega_n$ and $\alpha = a - k + \frac{n+m}{2}$. □

Theorem 2.4.6. *We have*

$$\tau \circ \Phi_{\lambda_\alpha + \rho} \colon M(\mathbb{C}_{\lambda_\alpha})^{\mathfrak{u}} \xrightarrow{\sim} \begin{cases} \mathbb{F}_{\mu_{\alpha,r}} & \text{if } \alpha + r \notin \mathbb{N}, \\ \mathbb{F}_{\mu_{\alpha,r}} \oplus \bigoplus\limits_{k=0}^{\min\{r-1,\alpha+r-1\}} \mathbb{F}_{\mu_{\alpha,k}} & \text{if } \alpha + r \in \mathbb{N}, \end{cases}$$

where $\mu_{\alpha,k} = (\alpha + r - k)\omega_{n-r+k} + (\alpha + r - k)\omega_{n+r-k} - (\alpha + 2r - 2k)\omega_n$ *for* $k = 0, 1, \dots, r$.

Proof. The decomposition of the space of singular vectors $M(\mathbb{C}_{\lambda_\alpha})^{\mathfrak{u}}$ is a straightforward consequence of Lemma 2.4.5. □

If $\alpha + r \in \mathbb{N}$, then Theorem 2.4.6 ensures the existence of a non-trivial homomorphism

$$\varphi \colon M(\mathbb{F}_{\mu_{\alpha,k}}) \to M(\mathbb{C}_{\lambda_\alpha}) \tag{2.46}$$

of generalized Verma modules for $k = 0, 1, \dots, \min\{r - 1, \alpha + r - 1\}$, where the simple $\mathfrak{l}$-module $\mathbb{F}_{\mu_{\alpha,k}}$ is extended to a $\mathfrak{p}$-module by $\mathfrak{u}$ acting trivially, uniquely determined by a homomorphism

$$\varphi_0 \colon \mathbb{F}_{\mu_{\alpha,k}} \to M(\mathbb{C}_{\lambda_\alpha}) \tag{2.47}$$

of $\mathfrak{p}$-modules, which is defined by

$$\varphi_0(v_{\mu_{\alpha,k}}) = q_{r-k}^{\alpha+r-k}(f_{ij})v_\alpha, \tag{2.48}$$

where $v_\alpha \in M(\mathbb{C}_{\lambda_\alpha})$ is the highest weight vector of $M(\mathbb{C}_{\lambda_\alpha})$ and $v_{\mu_{\alpha,k}} \in \mathbb{F}_{\mu_{\alpha,k}}$ is the highest weight vector of $\mathbb{F}_{\mu_{\alpha,k}}$. Finally, let us remark that the same approach works also in the $\mathfrak{sp}$ and $\mathfrak{so}^*$ cases [KT17].

3. Unitarizable highest weight modules

One of the main open problem of representation theory is classification of unitary modules of Lie groups. Apart from the finite-dimensional representations (which are unitary for compact Lie groups) one of the classes of modules where a complete classification was obtained is the class of unitarizable modules of highest weight. It is precisely the intersection of BGG category $\mathcal{O}$ and Harish-Chandra $(\mathfrak{g}, K)$-modules. It was proved already by Harish-Chandra that these modules can occur only in the Hermitian symmetric case, see [Har55; Har56a] and [Har56b] for details.

A sufficient and necessary condition for unitarizability of a highest weight module appears in [GZ81]. Independently, these modules were classified by Jakobsen in [Jak81; Jak83] with later development in [Jak96]. Another, yet independent, approach was given by [EHW83] with later simplification in [Jos92]. In [Ada87] it was shown that all unitarizable highest weight modules can be obtained via the derived functor construction from either a one-dimensional or a unipotent representation. This parametrization given here fits in nicely with the Langlands classification and the coadjoint orbit picture.

In this chapter we follow mainly the article [EHW83] and reorganization of the unitarizable weight into integral cones that was presented in [DES91].

3.1 Classification

The algebra $\mathfrak{k}$ has a one-dimensional center which is complementary to the span of Φ_c. We pick a generator ζ of the center by requiring $\frac{2\langle\zeta,\beta\rangle}{\langle\beta,\beta\rangle} = 1$, where β is the unique maximal noncompact root[1] of Φ^+. Now any line $\lambda + z\zeta$ for $z \in \mathbb{C}$ can be uniquely written as $\lambda_0 + z\zeta$ with

$$\langle \lambda_0 + \rho, \beta \rangle = 0, \quad \rho = \sum_{\alpha \in \Phi^+} \alpha.$$

If $L(\lambda)$ is an irreducible unitarizable module for $\lambda = \lambda_0 + z\zeta$, then z must be real and the $K_{\mathbb{C}}$-finiteness implies that λ must be Φ_c^+-dominant integral. Write the generalized Verma module $M(\lambda)$ as $S(\mathfrak{p}_+) \otimes F(\lambda_0) \otimes \mathbb{C}_z\zeta$. If we fix a basis of $\mathfrak{g}$ and $F(\lambda_0)$, we may view the modules $M(\lambda)$ as defined on the same vector space $S(\mathfrak{p}_+) \otimes F(\lambda_0)$ where the action of $\mathfrak{g}$ is given by polynomial expressions in z. Likewise, we may view the Shapovalov form on $M(\lambda)$ as a bilinear form on $S(\mathfrak{p}_+) \otimes F(\lambda_0)$ with values in complex polynomials $\mathbb{C}[z]$.

A $\mathfrak{g}$-module M is called unitarizable if there exists (necessarily unique up to a multiple) positive definite contravariant form on M. The following theorem explains the structure of the set of $z \in \mathbb{R}$ such that the module $M(\lambda)$ is unitarizable.

[1] Equivalently, β is the highest weight of $\mathfrak{k}$-representation $\mathfrak{p}_+$.

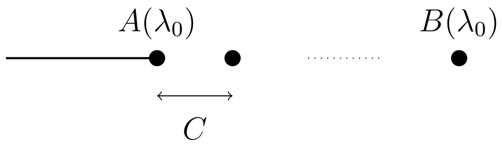


Figure 3.1: Structure of unitarizable weights

Theorem 3.1.1 (Theorem 2.4 of [EHW83]). *The set of real numbers z with $L(\lambda_0 + z\zeta)$ a unitarizable $\mathfrak{g}$-module is given by the set*

$$\{z \in \mathbb{R} \,|\, z \leq A(\lambda_0)\} \cup \{z = A(\lambda_0) + kC(\lambda_0) \,|\, z \leq B(\lambda_0) \,\&\, k \in \mathbb{N}_0\},$$

where $A(\lambda_0)$, $B(\lambda_0)$ and $C(\lambda_0)$ are real numbers expressible in terms of certain root systems $Q(\lambda_0)$ and $R(\lambda_0)$ associated to λ_0. Moreover $C(\lambda_0)$ is independent of λ_0 and depends only on the type of $\mathfrak{g}_0$. The values of C are listed in Table 3.1 and the structure of the set of unitarizable highest weights is depicted in Figure 3.1.

The discrete series representation corresponds to $z < 0$ and limit of discrete series representation corresponds to $z = 0$. For $z < A(\lambda_0)$ we have $L(\lambda) = M(\lambda)$ (i.e. the generalized Verma modules are irreducible) and for all $z \geq A(\lambda_0)$ such that $L(\lambda)$ are unitarizable the generalized Verma modules are reducible.

$\mathfrak{g}_0$	$\mathrm{SU}(p,q)$	$\mathrm{Sp}(n,\mathbb{R})$	$\mathrm{SO}^*(2n)$	$\mathrm{SO}(2,2n-2)$	$\mathrm{SO}(2,2n-1)$	E_6	E_7
C	1	$\frac{1}{2}$	2	$n-2$	$n-\frac{3}{2}$	3	4

Table 3.1: Distance between points of reducibility

Let $\Phi_c(\lambda_0) := \{\alpha \in \Phi_c | \langle \alpha, \lambda_0 \rangle = 0\}$ and recall the definition of β the maximal non-compact root. Take the root subsystem of Φ generated by $\pm\beta$ and $\Phi_c(\lambda_0)$ and decompose it into a disjoint union of simple root systems. Let $Q(\lambda_0)$ be the root system in this union which contains β.

If Φ has two root lengths and if there are short compact roots α not orthogonal to $Q(\lambda_0)$ with $\frac{2\langle \lambda_0, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 1$, then let Ψ be the root system generated by $\pm\beta, \Phi_c(\lambda_0)$ and all such α. Let $R(\lambda_0)$ be the simple component of Ψ which contains β. If Φ has only one root length or if no such α exists, then put $R(\lambda_0) = Q(\lambda_0)$.

Since these root systems are subsystems of Φ and since each has compact and noncompact roots, each is a root system of a Hermitian symmetric pair.

There is a convenient way to construct $Q(\lambda_0)$ by means of Dynkin diagrams. Draw a Dynkin diagram of $\mathfrak{g}$ and delete the unique node corresponding to simple noncompact root. Now adjoin to the resulting diagram $-\beta$ by the usual rules as when constructing extended Dynkin diagrams. The maximal connected subdiagram containing $-\beta$ such that its every compact simple root is orthogonal to λ_0 is the Dynkin diagram of $Q(\lambda_0)$. We illustrate on the case of $\mathfrak{su}(p,q)$. Here the noncompact root β is α_p and we get the following extended Dynkin diagram.

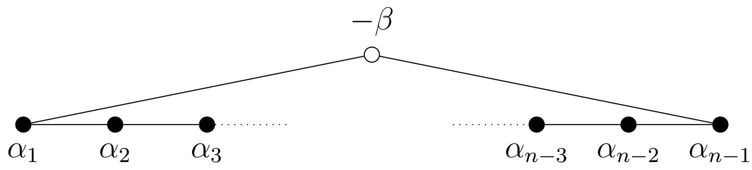


Figure 3.2: The Dynkin diagram of $Q(\lambda_0)$

If we write down λ_0 with respect to the basis of fundamental weights $\lambda_0 = \sum_i a_i\omega_i$ we see that $Q(\lambda_0)$ is a root system of $\mathfrak{su}(p', q')$ where p' and q' are maximal such that the coefficients $a_1, a_2, \ldots, a_{p'}$ and $a_{n-q'+1}, a_{n-q'+2}, \ldots, a_n$ are nonzero.

The root system $R(\lambda_0)$ is different from $Q(\lambda_0)$ only in two cases. The first one is $\mathfrak{g}_0 = \mathfrak{sp}(n, \mathbb{R})$ where $Q(\lambda_0) = \mathfrak{sp}(n', \mathbb{R})$ and $R(\lambda_0) = \mathfrak{sp}(n'', \mathbb{R})$ with $n' < n'' \leq n$. The second one is $\mathfrak{g}_0 = \mathfrak{so}(2, 2n-1)$ where $Q(\lambda_0) = \mathfrak{su}(1, n-1)$ and $R(\lambda_0) = \mathfrak{so}(2, 2n-1)$ with $\lambda_0 = (\lambda_1, \frac{1}{2}, \ldots, \frac{1}{2})$ in the ϵ-basis.

The following two theorems finish the general classification of unitarizable highest weight modules.

Theorem 3.1.2 (Theorem 2.8 of [EHW83])**.** *Let* $\Phi^+_{c,1} := \Phi^+_c \cap Q(\lambda_0)$ *and let* $\Phi^+_{c,2} := \Phi^+_c \cap R(\lambda_0)$*. Denote by* $\rho_{c,i}$ *half of the sum of roots in* $\Phi^+_{c,i}$*.*

If $\mathfrak{g}_0 = \mathfrak{so}(2, 2n-1)$ *and* $Q(\lambda_0) \neq R(\lambda_0)$ *then*

$$B(\lambda_0) = 1 + \frac{\langle \rho_{c,2}, \beta \rangle}{\langle \beta, \beta \rangle}.$$

In all other cases

$$B(\lambda_0) = 1 + \frac{\langle \rho_{c,1} + \rho_{c,2}, \beta \rangle}{\langle \beta, \beta \rangle}.$$

Theorem 3.1.3 (Theorem 2.10 of [EHW83])**.** *The first reduction point* $A(\lambda_0)$ *is given by*[2]

$$A(\lambda_0) = B(\lambda_0) - (\text{split rank of } Q(\lambda_0) - 1)C.$$

Now let's see what we can tell about the maximal module $J(\lambda)$ of the generalized Verma module $M(\lambda)$.

Theorem 3.1.4 (Theorem 3.1 of [DES91])**.** *Suppose* $L(\lambda) = M(\lambda)/J(\lambda)$ *is unitarizable and* $J(\lambda) \neq 0$*. Then*

1. $H^1(\mathfrak{p}_-, L(\lambda))$ *is an irreducible* $\mathfrak{k}$*-module*
2. $J(\lambda)$ *is generated over* $S(\mathfrak{p}_+)$ *by an irreducible* $\mathfrak{k}$*-submodule* $J(\lambda)^0$ *isomorphic to* $H^1(\mathfrak{p}_-, L(\lambda))$*.*

In particular, the generator of $J(\lambda)$ must be contained in some component $S^k(\mathfrak{p}_+) \otimes F(\lambda)$. This k is called *level of reduction* of $L(\lambda)$ and is denoted by $l(\lambda)$.

Definition 3.1.5. *The set of reduction points* Λ_r *is the union of all reduction points. Explicitly*

$$\Lambda_r := \{\lambda = z\zeta + \lambda_0 \in \mathfrak{h}^* | z = A(\lambda_0) + kC, k \in \mathbb{N}_0, z \leq B(\lambda_0)\}.$$

[2]Or in other words, the number of reduction points equals the split rank of $Q(\lambda_0)$.

For $\lambda \in \Lambda_r$ let $a(\lambda) := (Q(\lambda_0), R(\lambda_0), l(\lambda))$ and let $\mathcal{A}$ denote the set of all such triples as λ ranges over Λ_r. For $a \in \mathcal{A}$, let Λ_a denote the set of all $\lambda \in \Lambda_r$ with $a(\lambda) = a$.

Now we can look more closely at the structure of the set of reduction points.

Corollary 3.1.6 (of 3.1.3)**.** *Let $a = (Q, R, l) \in \mathcal{A}$ and let $\lambda \in \Lambda_a$. If we write $\lambda = z\zeta + \lambda_0$, then $z = B(\lambda) - (l-1)C$ and on the other hand for $\lambda = (B - nC)\zeta + \lambda_0 \in \Lambda_r$ we have $l(\lambda) = n + 1$.*

Let $\mathfrak{h}^*_{\mathbb{R}}$ denote the real span of the roots. A *cone* with vertex zero (in $\mathfrak{h}^*_{\mathbb{R}}$) is the intersection of a (nonempty) collection of closed half spaces. Each cone C is thus determined by a finite set $\{h_i \in \mathfrak{h} | i = 1, \ldots, k\}$ with $C = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} | \lambda(h_i) \geq 0, i = 1, \ldots k\}$. An *integral cone* will be the intersection of a cone with the set of all $\mathfrak{k}$-integral points of $\mathfrak{h}^*$. For an integral element $\nu \in \mathfrak{h}^*$, a translated cone with vertex ν is a set of the form $\nu + C$ with C some integral cone.

Definition 3.1.7. *For $a = (Q, R, l) \in \mathcal{A}$, let C_a be the integral cone of $\mathfrak{k}$-dominant integral elements in $\mathfrak{h}^*_{\mathbb{R}}$ which are orthogonal to elements in R.*

For concrete calculations the following lemma can be useful.

Lemma 3.1.8 (Section 4.3 of [EH04a])**.** *The cone $C_{Q,R,l}$ consists of positive integral multiples of weights of the form $\omega_i - (\omega_i, \beta^\vee)\zeta$, where ω_i is a fundamental weight corresponding to simple root α_i that does not belong to R.*

Proof. Any $\mathfrak{k}$-dominant integral weight can be written in the form $\mu = \sum_i a_i\omega_i + b\zeta$, where a_i are non-negative integers. Such a weight is perpendicular to R if and only if it is perpendicular to all compact simple roots contained in R and to the noncompact root $-\beta$, i.e. to the simple roots of the root system R. The crucial observation here is that for each Hermitian symmetric space the weight ζ is in fact the fundamental weight corresponding to the simple noncompact root. Hence we have

$$0 = \left(\sum a_i\omega_i + b\zeta, \alpha_i^\vee\right) = a_i$$

for all compact simple roots of R and

$$0 = \left(\sum a_i\omega_i + b\zeta, \beta^\vee\right)$$

for the noncompact root. Recalling the definition of ζ and solving for b we see that the cone consists of vectors of the form $\sum a_i(\omega_i - (\omega_i, \alpha_i^\vee)\zeta)$ where the sum is only indices whose simple roots are not in R. □

Proposition 3.1.9 (Proposition 6.6 of [DES91])**.** *The set of reduction points Λ_r is the disjoint union of the sets $\Lambda_a, a \in \mathcal{A}$.*

Each set Λ_a is a translated integral cone with vertex $\lambda_a + C_a$. We list the vertices λ_a in sections 3.2.1 through 3.2.6.

Proof. The first statement is trivial and the second one follows by case by case computations. □

The next proposition gives an alternative way to compute the highest weight of the maximal submodule.

Proposition 3.1.10 (Proposition 6.8 of [DES91]). *Suppose $\lambda \in \Lambda_a$ with $a = (Q, R, l)$. Let u and v denote respectively the unique elements of maximal length in the Weyl groups for the positive root systems $Q \cap \Phi_c^+$ and $R \cap \Phi_c^+$ and let μ denote the highest weight of the maximal submodule $J(\lambda)$. Then*

$$\mu = \lambda + \frac{1}{2}(u\mu_l + v\mu_l)$$

in all cases except when $\mathfrak{g}_0 = \mathfrak{so}(2, 2n-1)$ and $Q \neq R$. In this case

$$\mu = \lambda + \frac{1}{2}(\mu_l + v\mu_l).$$

Moreover, in all cases $F(\mu)$ occurs with multiplicity one in $M(\lambda + \rho)$ and in all cases $F(\mu)$ is a PRV component of a tensor product in $S(\mathfrak{p}_+) \otimes F(\lambda)$.

The next theorem deals with effect of a sort of 'translation functor' on unitarizable highest weight modules.

Theorem 3.1.11 (Factorization theorem 6.15 of [DES91]). *Fix $a \in \mathcal{A}$ and let $\lambda = \lambda_a + \lambda' \in \Lambda_a$. Let $J(\lambda)^0$ and $J(\lambda_a)^0$ be the $\mathfrak{k}$-modules that generate $J(\lambda)$ and $J(\lambda_a)$. Extend the $\mathfrak{k}$-equivariant projection*

$$P : F(\lambda_a) \otimes F(\lambda') \to F(\lambda)$$

to a mapping $\widetilde{P} : M(\lambda_a) \otimes F(\lambda') \to M(\lambda)$ by

$$\widetilde{P}(F \otimes v)(T) := P(F(T) \otimes v),$$

where we have used that $M(\lambda_a) = S(\mathfrak{p}_+) \otimes F(\lambda_a)$.

Let μ and μ_a denote the highest weights of $J(\lambda)^0$ and $J(\lambda_a)^0$. Then

1. *$\mu = \mu_a + \lambda'$*
2. *$P(J(\lambda_a)^0 \otimes F(\lambda')) = J(\lambda)^0$ and*
3. *$P(J(\lambda_a) \otimes F(\lambda')) = J(\lambda)$.*

3.2 Nilpotent cohomology of unitarizable highest weight modules

The convention employed in this section is that we omit the terms whose indices are outside natural boundaries.

Definition 3.2.1. *Let Ψ_λ be the set if roots in Φ which are orthogonal to $\lambda + \rho$ and let $\Psi_\lambda^+ = \Psi_\lambda \cap \Phi^+$. Denote by $\Phi_{n,\lambda}^+$ the roots which satisfy the following conditions*

1. *$\alpha \in \Phi_n^+$ and $(\lambda + \rho, \alpha^\vee)$ is a positive integer;*
2. *α is orthogonal to Ψ_λ;*
3. *α is short if there exist a long root in Ψ_λ.*

Let W_λ be the subgroup of W which is generated by reflections s_α for $\alpha \in \Phi^+_{n,\lambda}$.

Let Φ_λ be the subset of Φ of elements β with $s_\beta \in W_\lambda$ and let $\Phi_{\lambda,c} = \Phi_c \cap \Phi_\lambda$, $\Phi^+_{\lambda,c} = \Phi_{\lambda,c} \cap \Phi^+$.

Finally, define $W^{c,i}_\lambda = \{w \in W_\lambda \,|\, w\rho$ is $\Phi^+_{\lambda,c}$-dominant and $l_\lambda(w) = i\}$.

Theorem 3.2.2 (3.7 [DES91]). *Let L be unitarizable with highest weight λ. Then for $i \in \mathbb{N}$ we have*

$$H^i(\mathfrak{p}_+, L) \simeq \bigoplus_{w \in W^{c,i}_\lambda} F(\overline{w(\lambda+\rho)} - \rho) \tag{3.1}$$

where $\overline{\lambda}$ is the unique Φ^+_c-dominant element in the W_c orbit of λ.

It turns out that Φ_λ is actually a root system of a simple Lie algebra and moreover it's intersection with noncompact roots is nonempty and gives a decomposition of Φ_λ into compact and noncompact roots. In other words, we obtain a smaller Hermitian symmetric pair defined by λ.

Definition 3.2.3. *The Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$ attached to $(\Phi_{\lambda,c}, \Phi_\lambda \cap \Phi_n)$ is called* reduced Hermitian symmetric pair *associated to λ.*

A priori, it is not clear that W_λ is a Weyl group. The following theorem deals with this issue.

Theorem 3.2.4 (Theorem 3.3 of [Dye90], [Deo89]). *Let (W, R) be a Weyl group generated by a set of simple reflections R and let $T = \bigcup_{w \in W} wRw^{-1}$ be the set of all reflections. If G is any subgroup of a Weyl group W that is generated by reflections, then it is a Coxeter group. Let $N(w) = \{t \in T \,|\, l(tw) < l(w)\}$ where l denotes the length function of (W, R). The set $\{t \in T \,|\, N(t) \cap G = \{t\}\}$ is a set of Coxeter generators for G.*

We can use this theorem to find the simple roots of the reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$. The proof of the formula (3.1) is based on the fact that for unitarizable highest weights the Enright Shelton equivalences 1.2.3 translate the problem to calculation of nilpotent Lie algebra cohomology of the reduced Hermitian pair with values in a finite dimensional representation. Hence one can just calculate the BGG diagram of minimal representatives of the pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$ using the classical algorithm and then apply the embedding of $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$ into $(\mathfrak{g}, \mathfrak{k})$.

Remark 3.2.5. *The $\mathfrak{k}$-weight of the first cohomology is given by $\lambda_0 - \rho$ where λ_0 is the unique Φ^+_c dominant element in the W_c orbit of $s_{\gamma_0}\lambda$ for the unique noncompact simple root $\gamma_0 \in \Phi^+_\lambda$.*

Lemma 3.2.6. *Let λ be a highest weight of a unitarizable highest weight module. If a positive root is orthogonal to $\lambda + \rho$ then it must be noncompact.*

$$\alpha \in \Phi^+ : \alpha \perp \lambda + \rho \Longrightarrow \alpha \in \Phi^+_n$$

Proof. Every positive roots can be written as a positive linear combination of simple roots, i.e. $\alpha = \sum_i c_i\alpha_i$ where $c_i \geq 0$. The fundamental weights form a basis of $\mathfrak{h}^*$ and thus $\lambda = \sum_i k_i \omega_i$. Now we just use the defining property of fundamental weights $\frac{2(\alpha_i,\omega_j)}{(\alpha_i,\alpha_i)} = \delta_{ij}$ to compute

$$\begin{aligned}(\alpha, \lambda+\rho) &= \sum_{i,j} (c_i k_j (\alpha_i, \omega_j) + c_i(\alpha_i, \omega_j)) \\ &= \sum_{i,j} \left(c_i k_j \frac{(\alpha_i,\alpha_i)}{2} \delta_{ij} + c_i \frac{(\alpha_i,\alpha_i)}{2} \delta_{ij} \right) \\ &= \sum_i c_i \frac{(\alpha_i,\alpha_i)}{2} (k_i+1).\end{aligned}$$

If λ is a highest weight of a unitarizable module, then all but one of the coefficients k_i are non-negative and the only possibly negative coefficient corresponds to the fundamental weight dual to the coroot of the unique noncompact simple root - let's denote it's index by i_0. If the scalar product $(\alpha, \lambda+\rho)$ is zero, then c_{i_0} must be nonzero – all the remaining terms in the sum are non-negative. But $c_{i_0} \neq 0$ is equivalent to α being noncompact. $\square$

Example 3.2.7. *Let us take $\mathfrak{g} = \mathfrak{so}(2, 2n-2)$ with $\lambda = (2-n)\omega_1$. Then in the epsilon basis we have $\lambda + \rho = (1, n-2, \ldots, 1, 0)$ and $\Psi_\lambda^+ = \{\epsilon_1 - \epsilon_{n-1}\}$. The only noncompact root that is orthogonal to $\epsilon_1 - \epsilon_{n-1}$ and whose scalar product with $\lambda+\rho$ is positive integral is $\alpha = \epsilon_1 + \epsilon_{n-1}$. Thus we get $\Phi_{n,\lambda}^+ = \epsilon_1 + \epsilon_{n-1} = \Phi_\lambda$. It follows that*

$$\begin{aligned}H^0(\mathfrak{p}_-, L((2-n)\omega_1)) &= F((2-n)\omega_1) \\ H^1(\mathfrak{p}_-, L((2-n)\omega_1)) &= F(-n\omega_1) \\ H^i(\mathfrak{p}_-, L((2-n)\omega_1)) &= 0 \textit{ for } i \geq 2.\end{aligned}$$

Similarly there is only one root generating W_λ for $\mathfrak{g} = \mathfrak{so}(2, 2n-1)$ and $\lambda = (\frac{3}{2} - n)\omega_1$ and we get that in that case

$$\begin{aligned}H^0(\mathfrak{p}_-, L((\frac{3}{2}-n)\omega_1)) &= F((\frac{3}{2}-n)\omega_1) \\ H^1(\mathfrak{p}_-, L((\frac{3}{2}-n)\omega_1)) &= F((-\frac{1}{2}-n)\omega_1) \\ H^i(\mathfrak{p}_-, L((\frac{3}{2}-n)\omega_1)) &= 0 \textit{ for } i \geq 2.\end{aligned}$$

Moreover, by inspecting the tables 3.3 and 3.4, we see that in both of these cases the cone C_a is empty.

Remark 3.2.8. *The formula* (3.1) *is actually stated a little bit differently in [Enr88]. Namely, the finite dimensional modules appearing in the cohomology are $F(\overline{w} \cdot lambda)$ where $\overline{w}$ is the minimal length representant of w. The same formula appears in a recent article [EHP14]. However, this formula is wrong as the following example shows.*

Consider $\mathfrak{su}(1,2)$ and weight $\lambda = -(a+2)\omega_1 + (a+1)\omega_2$. The reduced Hermitian pair is of type A_1 and it's given by $\{\epsilon_1 - \epsilon_3\}$. The associated reflection is $s_{\epsilon_1-\epsilon_3} = s_1 s_2 s_1$ and its minimal coset representative is $s_1 s_2$. It's affine action on λ gives $-2\omega_1 - (a+2)\omega_2$ which is not Φ_c^+-dominant. On the other hand the (normal) action of $s_1 s_2 s_1$ on $\lambda+\rho$ gives $-(a+2)\omega_1 + (a+1)\omega_2$ which is Φ_c^+-dominant and hence the first cohomology is $F(-(a+3)\omega_1 + a\omega_2)$.

Cohomologies of all unitarizable modules for the two expceptional types are explicitely computed in [EH04b]. The papers [EW04], [EH04a] treat also certain weights for the classical types $\mathfrak{su}(p,q)$, $\mathfrak{sp}(n,\mathbb{R})$ and $\mathfrak{so}^*(2n)$. The orthogonal cases $\mathfrak{so}(2,2n-2)$, $\mathfrak{so}(2,2n-1)$ are rather easy and are calculated completely in sections 3.2.4 and 3.2.5. For the other classical types we calculate some of the data that go into the formula (3.1) and calculate the cohomologies completely in small ranks in appendix A. We borrow idea from [EH04b] and calculate possible Ψ_λ^+ by labeling the poset of noncompact roots by scalar products of the associated coroots with $\lambda+\rho+\mu$ for $\mu \in C$. We show only nodes / roots where the scalar product is not always negative.

In general the combinatorics behind the calculations is rather involved. In most cases the reduced Hermitian pair doesn't stay the same on the cone which is due to the fact that these cones can intersect facets in nontrivial ways.

Example 3.2.9. *To illustrate the situation, here is an example for* $\mathrm{SO}^*(16)$ *and cone of unitarizable weights* $(a_5+1)\,\omega_5 + a_6\omega_6 + a_7\omega_7 - (2\,a_5 + 2\,a_6 + a_7 + 8)\,\omega_8$.

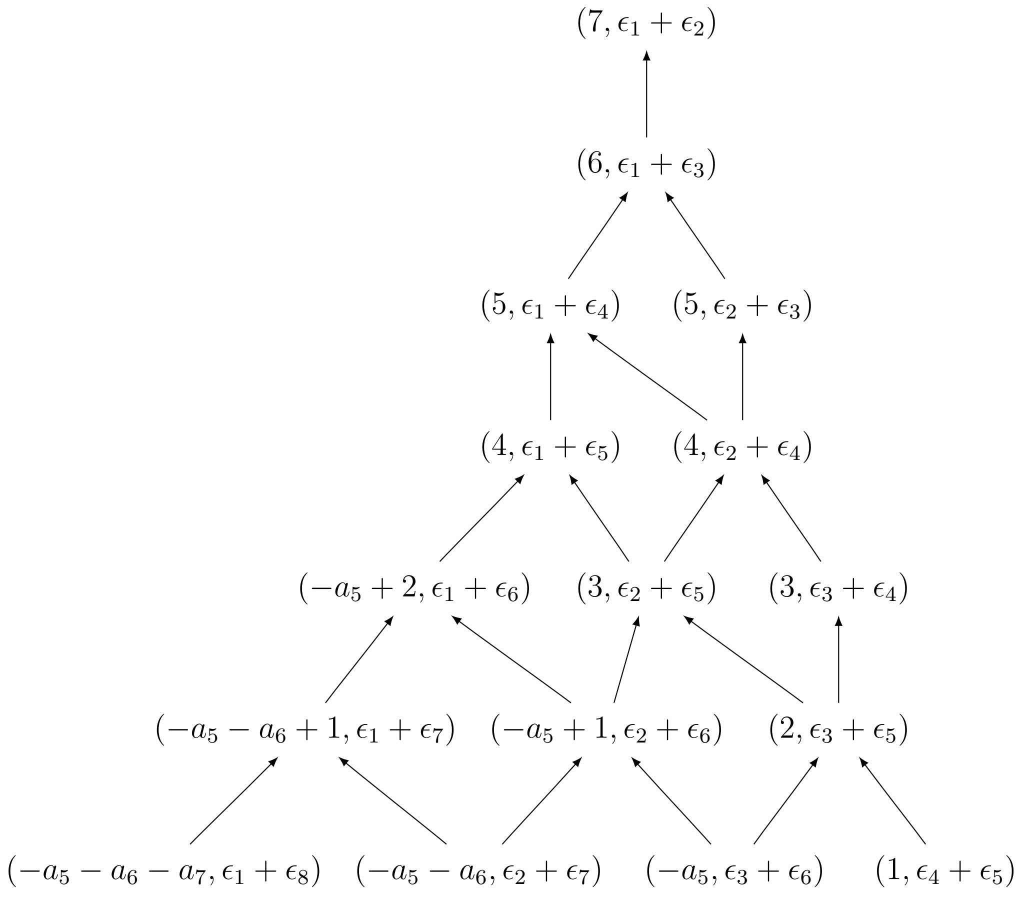


Figure 3.3: Nonnegative scalar products with noncompact roots

We can see that the set of singular weights Ψ_λ^+ *is generically empty, but for* $a_5 = a_6 = a_7$ *it contains the roots* $\epsilon_1 - \epsilon_8$ *and* $\epsilon_2 + \epsilon_7$ *and for small values of* a_5 *even more. It is however clear that the cone can be written as a union of smaller cones such that* Ψ_λ^+ *remains the same on each of them.*

3.2.1 $\mathrm{SU}(p,q), p+q=n \sim A_{n-1}, n \geq 2$

Root system data

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad \omega_i = \epsilon_1 + \cdots + \epsilon_i$$

$$\begin{aligned}
\Phi &= \{\epsilon_i - \epsilon_j | i \neq j, i, j = 1 \ldots n\} \\
\Phi_c^+ &= \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq p \text{ or } p+1 \leq i < j \leq n\} \\
\Phi_n^+ &= \{\epsilon_i - \epsilon_j | 1 \leq i \leq p, p+1 \leq j \leq n\}
\end{aligned}$$

$$\beta = \epsilon_1 - \epsilon_n, \quad 2\rho = (n-1, n-3, \ldots -n+3, -n+1), \quad \zeta = (\underbrace{\frac{q}{n}, \ldots, \frac{q}{n}}_{p \text{ times}}, \underbrace{\frac{-p}{n}, \ldots, \frac{-p}{n}}_{q \text{ times}})$$

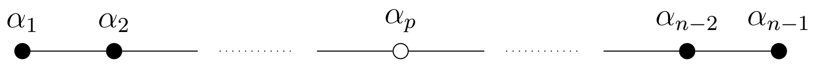


Figure 3.4: Marked Dynkin diagram of $\mathfrak{su}(p,q)$ for $p+q=n$

The reduction points of unitarizable highest weight modules are the following integral translated cones $\lambda_a + C_a$: Let $a = (Q, R, l)$, $Q = R = \mathrm{SU}(p', q')$, $1 \leq p' \leq p$, $1 \leq q' \leq q$. Then

$$C_a = \{a_{p'}\omega_{p'} + \cdots + a_p\omega_p + \cdots + a_{n-q'}\omega_{n-q'} \,|\, a_p = -a_{p'} - \cdots - a_{p-1} - a_{p+1} - \cdots - a_{n-q'}\}.$$

$$\begin{gathered}
\lambda_a = \omega_{p'} + \omega_{n-q'} - (n+l+1-p'-q')\omega_p \\
\mu_a = \omega_{p'-l} + \omega_{n-q'+l} - (n+l+1-p'-q')\omega_p \\
1 \leq p' \leq p, \quad 1 \leq q' \leq q, \quad 1 \leq l \leq \min(p', q') \\
Q(\lambda_a) = R(\lambda_a) = \mathrm{SU}(p', q')
\end{gathered}$$

Nilpotent cohomology in detail

Now we compute the cohomology for $\lambda = \omega_{p'} + \omega_{n-q'} - (n+l+1-p'-q')\omega_p$. We have for $k = 2 - (n+l+1-p'-q') = 1 + p' + q' - n - l$ that

$$(\epsilon_i, \lambda) = \begin{cases} k, & 1 \leq i \leq p' \\ k-1, & p' < i \leq p \\ 1, & p < i \leq n-q' \\ 0, & n-q' < i \leq n. \end{cases}$$

The positive roots of $\mathfrak{su}(p,q)$ are $\epsilon_i - \epsilon_j$, $i < j$ and we have

$$(\epsilon_i - \epsilon_j, \rho) = \frac{n+1-2i}{2} - \frac{n+1-2j}{2} = j - i > 0.$$

Now, in order to determine the set Ψ_λ^+, we compute all possible values of $(\epsilon_i - \epsilon_j, \lambda + \rho)$ for $i < j$

	$1 \le j \le p'$	$p' < j \le p$	$p < j \le n-q'$	$n-q' < j \le n$
$1 \le i \le p'$	$j-i$	$1+j-i$	$k-1+j-i$	$k+j-i$
$p' < i \le p$		$j-i$	$k-2+j-i$	$k-1+j-i$
$p < i \le n-q'$			$j-i$	$1+j-i$
$n-q' < i \le n$				$j-i$

We see that only terms containing k can be zero, because $j-i > 0$ and $2-n \le k \le 0$. Thus the only singular roots are the noncompact ones. Substituting for k and solving for j we get

	$p < j \le n-q'$	$n-q' < j \le n$
$1 \le i \le p'$	$j = i+m$	$j = i+m-1$
$p' < i \le p$	$j = i+m+1$	$j = i+m$

where

$$m = 1-k = n+l-p'-q'.$$

The case $j = i+m+1$ doesn't lead to any solution, since $j-i-m-1 = j-i-n-l+p'+q'-1$ is strictly negative for $p' < i \le p < j \le n-q'$. The case $j = i+m$ for $i \le p'$ leads to

$$\begin{gathered}\max\{1, q'-q+1+p'-l\} \le i \le p'-l \\ \max\{1+n+l-p'-q', p+1\} \le j \le n-q',\end{gathered}$$

which results in an empty set of singular roots if and only if $l = p'$. The case of $j = i+m$ for $i > p'$ yields

$$\begin{gathered}p'+1 \le i \le \min\{p, p'+q'-l\} \\ n-q'+l+1 \le j \le \min\{n+l-p'-q'+p, n\},\end{gathered}$$

which gives an empty set of singular roots if and only if $l = q'$ or $p = p'$. And finally the case $j = i+m-1$ gives

$$\begin{gathered}p'+2-l \le i \le p' \\ n-q'+1 \le j \le n-q'-1+l\end{gathered}$$

which doesn't contribute to singular roots if and only if $l = 1$.

Two roots $\epsilon_i - \epsilon_j$, $\epsilon_a - \epsilon_b$ are orthogonal if and only if $\{i,j\} \cap \{a,b\} = \emptyset$. Thus the set of positive noncompact roots orthogonal to Ψ_λ^+ is

$$\begin{aligned}\Big\{\epsilon_i - \epsilon_j \,|\, i \in \{1,\dots,q'-q+p'-l\} \cup \{p'-l+1\} \cup \{p'+q'-l+1,\dots,p\} \\ j \in \{p+1,\dots,m\} \cup \{n+l-q'\} \cup \{m+p+1,\dots,n\}\Big\}, \quad (3.2)\end{aligned}$$

where a set $\{a,\dots,b\}$ is considered empty if $a > b$.

Positive noncompact roots $\alpha = \epsilon_i - \epsilon_j$ satisfying $(\lambda+\rho,\alpha) \in \mathbb{Z}^+$ are given by constraints

$$\begin{aligned}1 \le i \le p' &\Longrightarrow \max\{i+m, p\} < j \le n-q' \vee \max\{i+m, n-q'+1\} \le j \le n \\ p' < i \le p &\Longrightarrow \max\{p, i+m+1\} < j \le n-q' \vee \max\{n-q', i+m\} < j \le n.\end{aligned} \quad (3.3)$$

If a positive noncompact root $\epsilon_i - \epsilon_j$ with $i > p'$ is orthogonal to all singular roots and it's scalar product with $\lambda + \rho$ is positive, then necessarily

$$i \in \{p' + q' - l + 1, \dots, p\} \text{ and } i + m < j \leq n.$$

But for $i = p' + q' - l + 1$ we get that $i + m = n + 1$ and hence we see that there are no such roots. Thus, we have to look only at the noncompact roots satisfying the first constraint of (3.3).

If a noncompact root $\epsilon_i - \epsilon_j$ with $i \leq p'$ is orthogonal to the singular roots Ψ_λ^+, then either $i = p' - l + 1$ or $i \leq q' - q + p' - l$. If $i = p' - l + 1$, then we get $i + m = n - q' + 1$ which means that we have to look only at

$$\max\{i + m, n - q' + 1\} = n - q' + 1 \leq j \leq n.$$

Taking into account (3.2) we see that the roots with indices given by

$$i = p' - l + 1, \quad j \in \{n - q' + l\} \cup \{m + p + 1, \dots, n\}$$

are included in $\Phi^+_{n,\lambda}$. For $i \leq q' - q + p' - l$ we have $i + m \leq p$ and the constraints of (3.3) reduce to

$$p < j \leq n$$

and thus the remaining roots of $\Phi^+_{n,\lambda}$ have indices given by

$$i \in \{1, \dots, q' - q + p' - l\}, \quad j \in \{p + 1, \dots, m\} \cup \{n + l - q'\} \cup \{m + p + 1, \dots, n\}.$$

Because all roots have the same length $\sqrt{2}$ we finally arrive at $\Phi^+_{n,\lambda}$ being equal to the set of roots $\epsilon_i - \epsilon_j$ with indices given by the following constraints

$$\begin{aligned} i = p' - l + 1 \quad & \& \quad j \in \{n - q' + l\} \cup \{m + p + 1, \dots, n\} \\ i \in \{1, \dots, q' - q + p' - l\} \quad & \& \\ & j \in \{p + 1, \dots, m\} \cup \{n + l - q'\} \cup \{m + p + 1, \dots, n\}. \end{aligned} \tag{3.4}$$

The Weyl group of $\mathfrak{sl}(n)$ is generated by root reflections and is isomorphic to the symmetric group S_n on the set $\{1, \dots, n\}$. Indeed, the reflection $s_{\epsilon_i - \epsilon_j}$ acts as a transposition of the ith and jth coordinate in ϵ-basis. Since there are overlaps in the ranges for j for various i, it follows that the group W_λ is isomorphic to the permutation subgroup of S_n which permutes only a subset M_λ of $\{1, \dots, n\}$ and leaves all other elements fixed. This subset is

$$M_\lambda = \{1, \dots, q' - q + p' - l, p' - l + 1, p + 1, \dots, m, n - q' + l, m + p + 1, \dots, n\}$$

in the case of $q' - q + p' - l > 0$ and

$$M_\lambda = \{p' - l + 1, n - q' + l, m + p + 1, \dots, n\}$$

if $q' - q + p' - l \leq 0$. We remind, that a range $a, \dots, b$ is considered empty if $a > b$. The root subsystem Φ_λ is then given by

$$\Phi_\lambda = \{\epsilon_i - \epsilon_j \mid i \neq j, i, j \in M_\lambda\}.$$

The pair $(\Phi_\lambda, \Phi_{\lambda,c})$ is a pair of root systems such that the corresponding Lie algebras $(\mathfrak{g}_\lambda, \mathfrak{g}_{\lambda,c})$ form a Hermitian symmetric pair of type $\mathfrak{su}(a, b)$ for some

a, b. The formula of the theorem 3.2.2 is basically coming (via Enright Shelton equivalences) from the classical Kostant formula for Lie algebra cohomology of the nilradical coming from $\mathfrak{g}_\lambda$. Hence the cohomology groups can be computed in a classical way just by restriction to the indices contained in M_λ.

Let us look at some specific cases. For $p = 1$ the only possible value for l is 1 and $1 \leq q' \leq n-1$. The weights are $\lambda = (q'-n)\omega_1 + \omega_{n-q'}$ and the corresponding set $M_\lambda = \{1, n-q'+1, \ldots, n\}$.

Now let's look what happens when $p = 2$. There are three possibilities

1. $p' = 1, l = 1$
   The weight is $\lambda = \omega_1 + \omega_{n-q'} - (n+1-q')\omega_2$ and the resulting set of admissible indices $M_\lambda = \{1, n-q'+1, \ldots, n\}$.

2. $p' = 2, l = 1$
   The weight is $\lambda = \omega_2 + \omega_{n-q'} - (n-q')\omega_2 = \omega_{n-q'} + (q'-n+1)\omega_2$ and the resulting set of admissible indices $M_\lambda = \{1, \ldots, n\}$ if $q' = q$ (in which case $\lambda = 0$) and $M_\lambda = \{2, n-q'+1, \ldots, n\}$ for $q' < q$.

3. $p' = 2, l = 2$
   The weight is $\lambda = \omega_2 + \omega_{n-q'} - (n+1-q')\omega_2 = \omega_{n-q'} + (q'-n)\omega_2$ and the resulting set of admissible indices $M_\lambda = \{1, n-q'+2, \ldots, n\}$.

The most singular case occurs for $\mathfrak{su}(k,k)$ and $p' = q' = l = k$. Then the set $M_\lambda =$ is $\{1, 2k\}$. For $p' = q' = k$ and $l < k$ we get $M_\lambda = \{1, \ldots, k-l+1, k+l, \ldots, 2k\}$.

Examples of posets of minimal length representatives

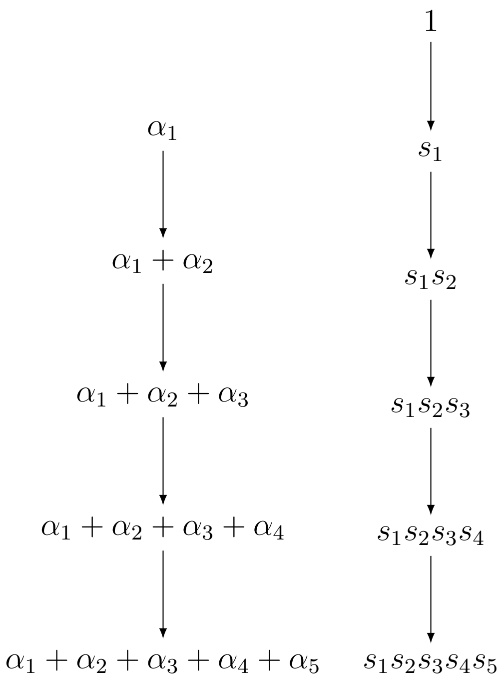


Figure 3.5: Poset of noncompact roots and the Bruhat graph for $\mathrm{SU}(1,5)$

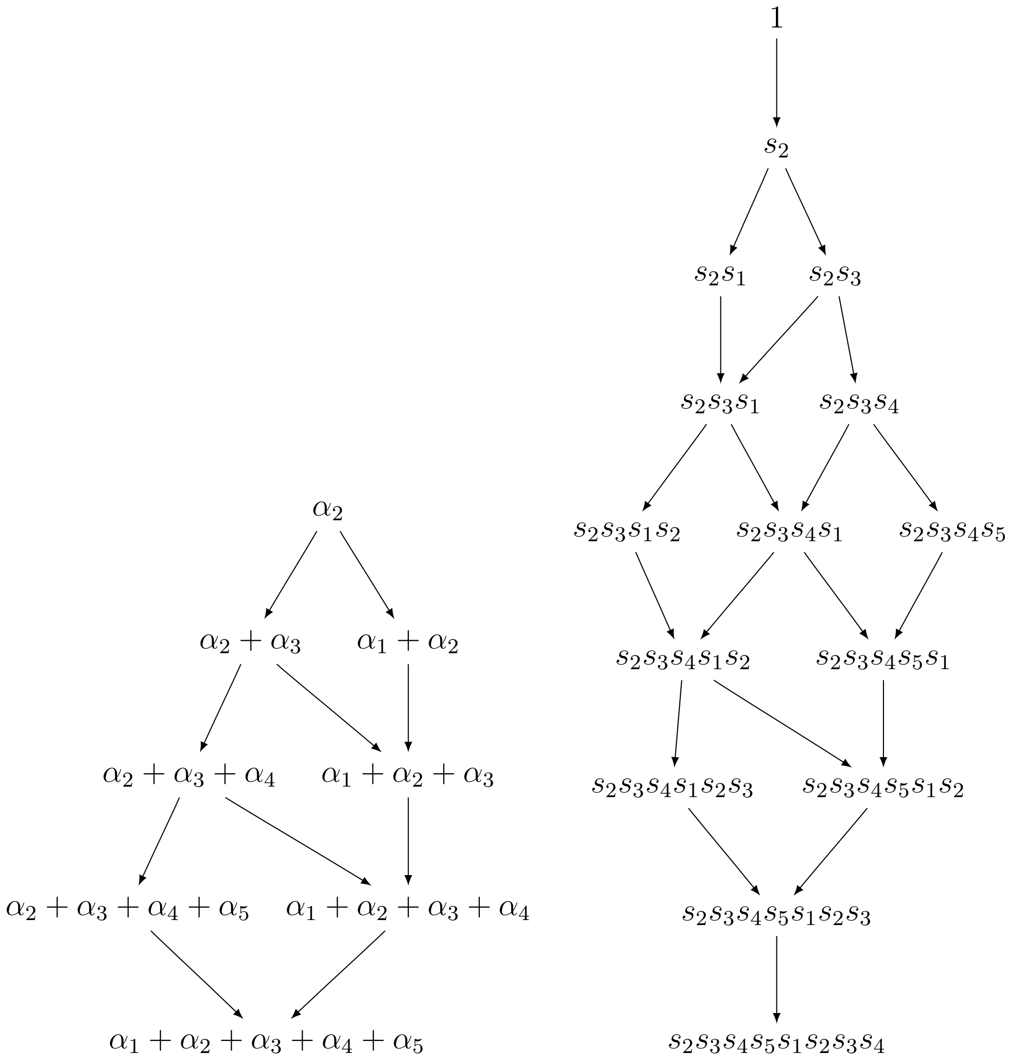


Figure 3.6: Poset of noncompact roots and the Bruhat graph for $\mathrm{SU}(2,4)$

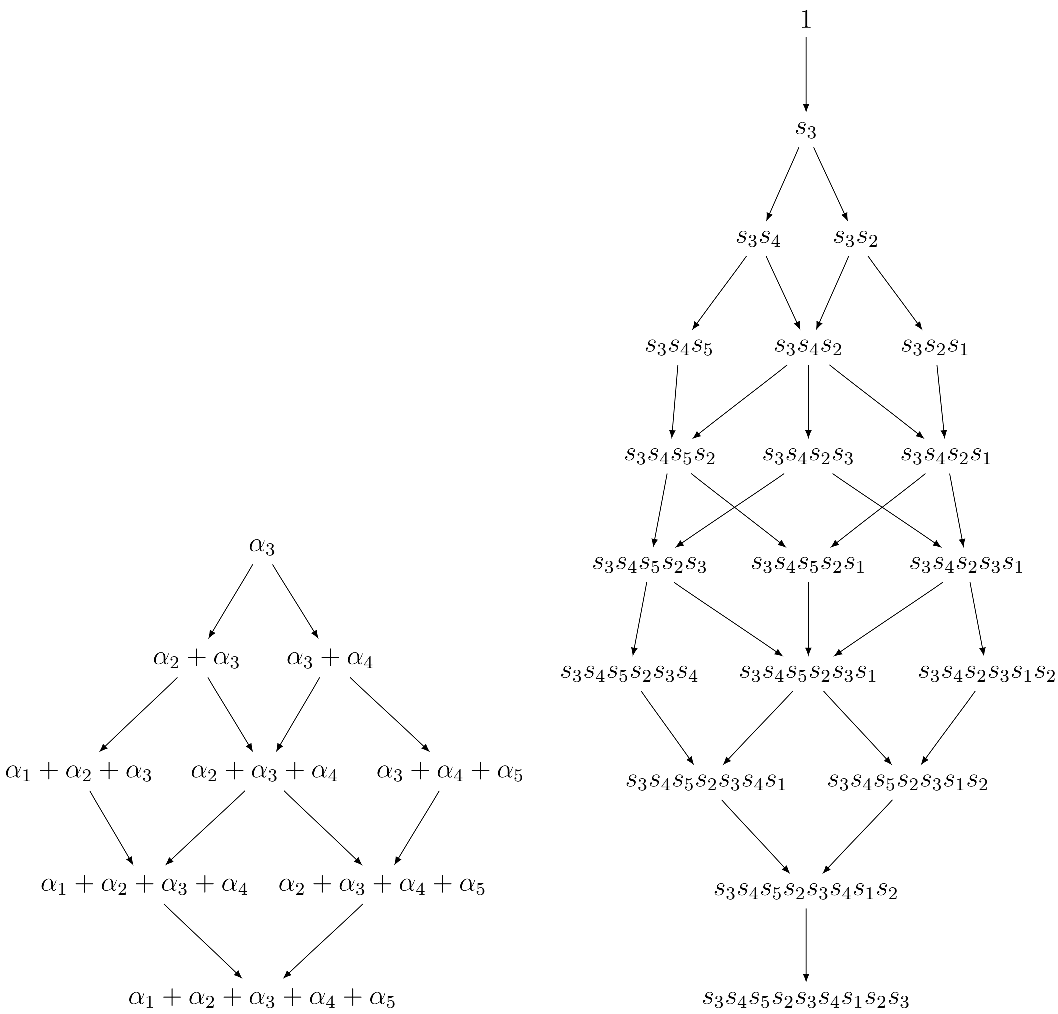


Figure 3.7: Poset of noncompact roots and the Bruhat graph for $\mathrm{SU}(3,3)$

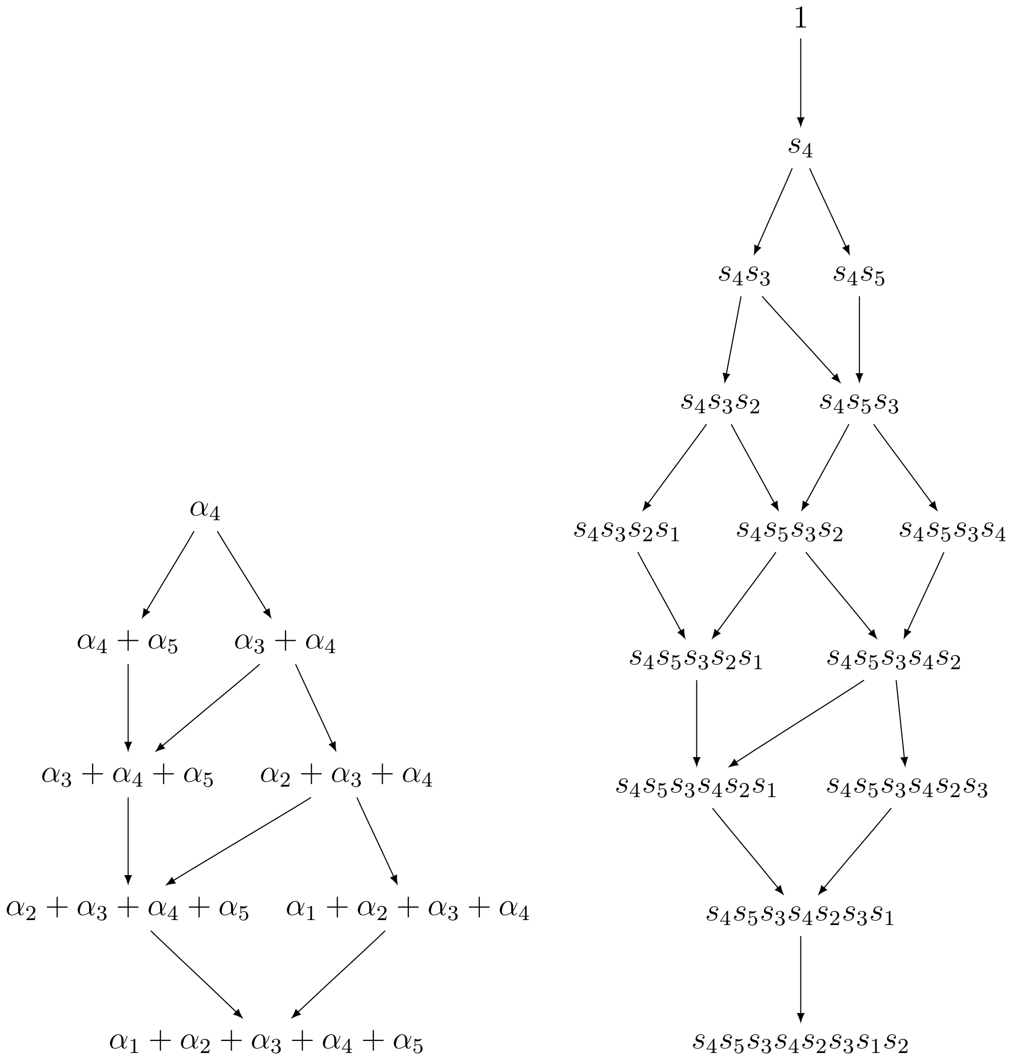


Figure 3.8: Poset of noncompact roots and the Bruhat graph for $\mathrm{SU}(4,2)$

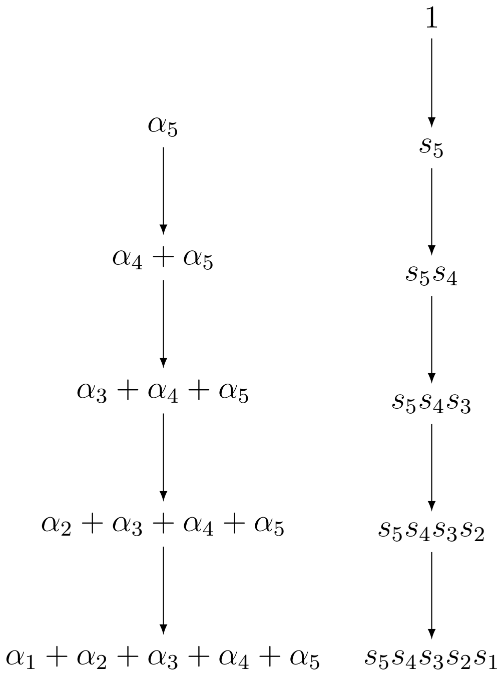


Figure 3.9: Poset of noncompact roots and the Bruhat graph for $\mathrm{SU}(5,1)$

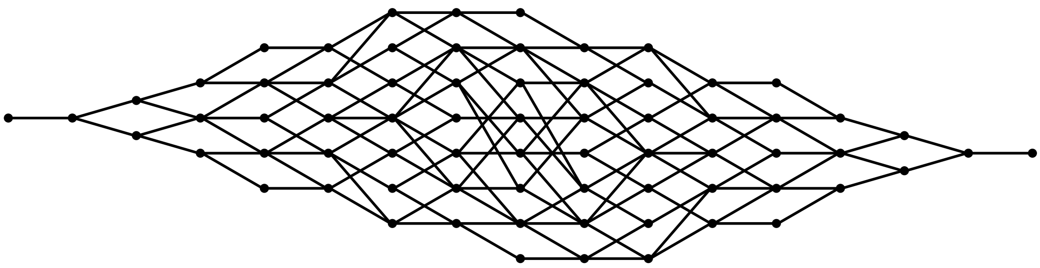


Figure 3.10: The BGG graph of type $(A_7, A_3 \times A_3)$

Low rank examples

3.2.2 $\mathrm{Sp}(n, \mathbb{R}) \sim C_n, n \geq 2$

Root system data

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, i < n, a_n = 2\epsilon_n, \quad \omega_i = \epsilon_1 + \cdots + \epsilon_i$$

$$\begin{aligned} \Phi &= \{\pm\epsilon_i \pm \epsilon_j | i, j = 1, \dots, n\} \setminus \{0\} \\ \Phi_c^+ &= \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq n\} \\ \Phi_n^+ &= \{\epsilon_i + \epsilon_j | 1 \leq i \leq j \leq n\} \end{aligned}$$

$$\beta = 2\epsilon_1, \quad \rho = (n, \dots, 1), \quad \zeta = (1, 1, \dots, 1)$$

Figure 3.11: Marked Dynkin diagram of $\mathrm{Sp}(n, \mathbb{R})$

The reduction points of unitarizable highest weight modules are the following integral translated cones $\lambda_a + C_a$: Let $a = (Q, R, l)$, $R = \mathrm{Sp}(r, \mathbb{R})$ and $1 \leq l \leq r \leq n$. Then

$$C_a = \{a_r\omega_r + \cdots + a_n\omega_n \,|\, a_n = -(a_r + \cdots + a_{n-1})\}$$

and

$$\begin{aligned} \lambda_a &= \omega_q + \omega_r - (2 + n - \frac{1}{2}(r + q - l + 1))\omega_n \\ \mu_a &= \omega_{q-l} + \omega_{r-l} - (2 + n - \frac{1}{2}(r + q - l + 1))\omega_n \\ & 1 \leq q \leq r \leq n, \quad 1 \leq l \leq q \\ & Q(\lambda_a) = \mathrm{Sp}(q, \mathbb{R}), \quad R(\lambda_a) = \mathrm{Sp}(r, \mathbb{R}) \end{aligned}$$

Nilpotent cohomology in detail

Scalar products of ρ with noncompact roots

$$\begin{aligned} (\epsilon_i - \epsilon_j, \rho) &= n - i + 1 - (n - j + 1) = j - i \\ (\epsilon_i + \epsilon_j, \rho) &= n - i + 1 + (n - j + 1) = 2n + 2 - i - j. \end{aligned}$$

The ith coordinate of λ with respect to the ϵ-basis is

$$(\epsilon_i, \lambda) = \begin{cases} 2 + \frac{1}{2}(r + q - l + 1) - n - 2, & 1 \leq i \leq q \\ 1 + \frac{1}{2}(r + q - l + 1) - n - 2, & q < i \leq r \\ \frac{1}{2}(r + q - l + 1) - n - 2, & r < i. \end{cases}$$

Computation of $(\epsilon_i - \epsilon_j, \lambda + \rho)$ for $j > i$ leads to the following table

	$1 < j \leq q$	$q < j \leq r$	$r < j \leq n$
$1 \leq i \leq q$	$j - i$	$1 + j - i$	$2 + j - i$
$q < i \leq r$		$j - i$	$1 + j - i$
$r < i \leq n$			$j - i$

and scalar products of the remaining positive roots $\epsilon_i + \epsilon_j$, $j \geq i$ are

	$1 < j \leq q$	$q < j \leq r$	$r < j \leq n$
$1 \leq i \leq q$	$3+m-i-j$	$2+m-i-j$	$1+m-i-j$
$q < i \leq r$		$1+m-i-j$	$m-i-j$
$r < i \leq n$			$m-1-i-j$

where

$$m = r + q - l.$$

We see again that only noncompact roots can be orthogonal to $\lambda+\rho$ in accordance with the lemma 3.2.6. If a positive noncompact root $\epsilon_i + \epsilon_j$ belongs to Ψ^+_λ, then

1. $1 \leq i \leq q$
   (a) $1 \leq j \leq q$:
   $$3 + r - l \leq i \leq \min\left\{q, \frac{3+m}{2}\right\}$$
   (b) $q < j \leq r$:
   $$2 + q - l \leq i \leq \min\{1 + r - l, q\}$$
   (c) $r < j \leq n$:
   $$\max\{1, 1 + m - n\} \leq i \leq q - l$$
2. $q < i \leq r$
   (a) $q < j \leq r$:
   $$1 + q \leq i \leq \frac{1+m}{2}$$
   (b) $r < j \leq n$:
   $$m - n \leq i < q - l$$
3. $r < i \leq n, \quad r < j \leq n$:
   $$\max\{r+1, m-n-1\} \leq i \leq \frac{m-1}{2}.$$

The set of indices 2a is empty, because $q - l < q$; similarly the third set is empty since $r + 1 > \frac{m-1}{2}$.

A singular long root exists if and only if

$$m \text{ is odd and } (3 + r \leq q + l \text{ or } q + l < 1 + r)$$

or alternatively a singular root doesn't exist if and only if

$$m \text{ is even or } m \text{ is odd and either } q + l = 2 + r \text{ or } q + l = 1 + r.$$

Two positive noncompact roots are orthogonal if and only if the intersection of their indices is empty, i.e.

$$(\epsilon_i + \epsilon_j, \epsilon_k + \epsilon_l) = 0 \Longleftrightarrow \{i, j\} \cap \{k, l\} = \emptyset.$$

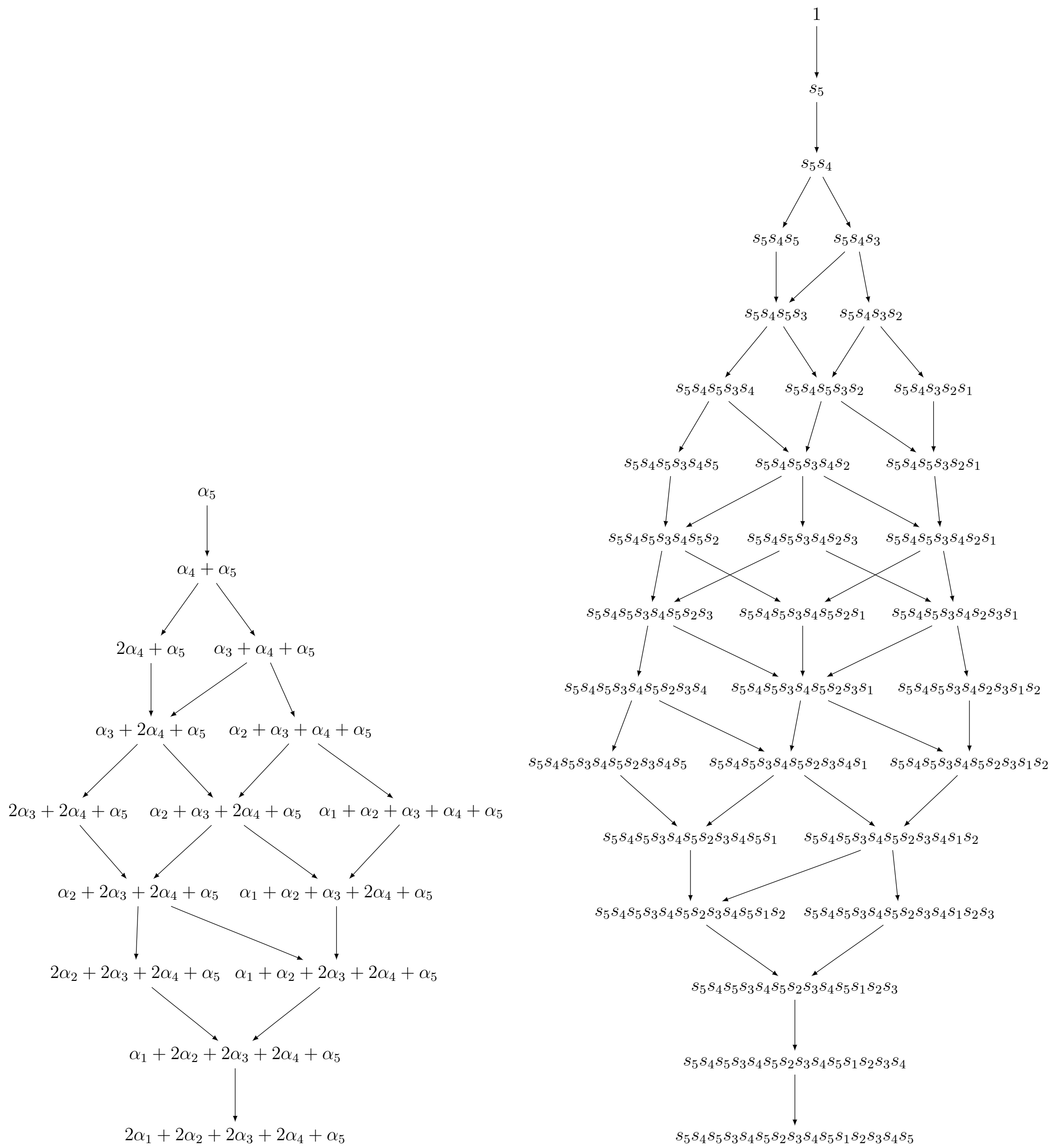


Figure 3.12: Poset of noncompact roots and the BGG graph for $\mathrm{Sp}(5,\mathbb{R})$

3.2.3 $\mathrm{SO}^*(2n) \sim D_n, n \geq 4$

Root system data

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, i < n, \alpha_n = \epsilon_{n-1} + \epsilon_n$$

$$\omega_i = \epsilon_1 + \cdots \epsilon_i, i < n-1, \quad \omega_{n-1} = \frac{1}{2}(\epsilon_1 + \cdots \epsilon_{n-1} - \epsilon_n), \quad \omega_n = \frac{1}{2}(\epsilon_1 + \cdots \epsilon_{n-1} + \epsilon_n)$$

$$\begin{aligned} \Phi &= \{\pm\epsilon_i \pm \epsilon_j | i \neq j, i, j = 1 \ldots n\} \\ \Phi_c^+ &= \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq n\} \\ \Phi_n^+ &= \{\epsilon_i + \epsilon_j | 1 \leq i < j \leq n\} \end{aligned}$$

$$\beta = \epsilon_1 + \epsilon_2, \quad \rho = (n-1, \ldots, 1, 0), \quad \zeta = \frac{1}{2}(1, 1, \ldots, 1)$$

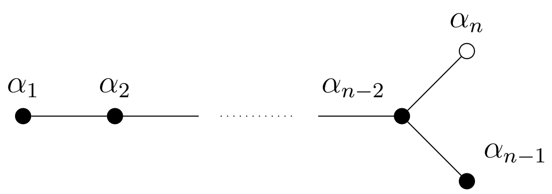


Figure 3.13: Marked Dynkin diagram for $\mathrm{SO}^*(2n)$

Vertex λ_a	Weight μ_a	$Q(\lambda_a) = R(\lambda_a)$	$l(\lambda_a)$
$\omega_2 - (2n-2)\omega_n$	$-(2n-2)\omega_n$	$\mathrm{SU}(1,1)$	1
$\omega_p - 2(n-p+l)\omega_n$	$\omega_{p-2l} - 2(n-p+l)\omega_n$	$\mathrm{SO}^*(2p)$[1]	$1 \leq l \leq \left[\frac{p}{2}\right]$
$\omega_{n-1} - (1+2l)\omega_n$	$\omega_{n-1-2l} - 2(1+l)\omega_n$	$\mathrm{SO}^*(2n-2)$	$1 \leq l \leq \left[\frac{n-1}{2}\right]$
$-(2l-2)\omega_n$	$\omega_{n-2l} - 2l\omega_n$	$\mathrm{SO}^*(2n)$	$1 \leq l \leq \left[\frac{n}{2}\right]$
$\omega_1 + \omega_{q+1} - (2n-q)\omega_n$	$\omega_q - (2n-q)\omega_n$	$\mathrm{SU}(1,q)$[2]	1
$\omega_1 + \omega_{n-1} - (n+1)\omega_n$	$\omega_{n-2} - (n+2)\omega_n$	$\mathrm{SU}(1,n-2)$	1
$\omega_1 - (n-1)\omega_n$	$\omega_{n-1} - n\omega_n$	$\mathrm{SU}(1,n-1)$	1

[1] $3 \leq p \leq n-2$
[2] $2 \leq q \leq n-3$

Table 3.2: Vertices and root systems for $\mathrm{SO}^*(2n)$, $n \geq 4$

Let $a = (Q, R, l)$, $Q = R$. Then for $R = \mathrm{SO}^*(2p)$, $3 \leq p \leq n$

$$C_a = \{a_p\omega_p + \cdots + a_n\omega_n \,|\, a_n = -2a_p - \cdots - 2a_{n-2} - a_{n-1}\}$$

and for $R = \mathrm{SU}(1,q)$, $1 \leq q \leq n-1$

$$C_a = \{a_1\omega_1 + a_{q+1}\omega_{q+1} + \cdots + a_n\omega_n \,|\, a_n = -(a_1 + 2a_{q+1} + \cdots + 2a_{n-2} + a_{n-1})\}.$$

Nilpotent cohomology in detail

Scalar products of positive noncompact roots with ρ are

$$(\epsilon_i+\epsilon_j,\rho)=2n-i-j.$$

1. $\lambda=\omega_2-(2n-2)\omega_n$

$$(\epsilon_i+\epsilon_j,\lambda+\rho)=\begin{cases}1, & i=1,j=2\\ 2-j, & i=1,2<j\le n\\ 1-j, & i=2,2<j\le n\\ 2-i-j, & 2<i<j\end{cases}$$

$$\Psi^+_\lambda=\emptyset,\quad \Phi^+_{n,\lambda}=\{\epsilon_1+\epsilon_2\}$$

Since the reduced pair is of rank 1, the whole cohomology is given in the table 3.3.

2. $\lambda=\omega_p-2(n-p+l)\omega_n$, $3\le p\le n-2$, $1\le l\le\left[\frac{p}{2}\right]$

$$(\epsilon_i+\epsilon_j,\lambda+\rho)=\begin{cases}2(p-l+1)-i-j, & 1\le i<j\le p\\ 2(p-l)+1-i-j, & 1\le i\le p<j\le n\\ 2(p-l)-i-j, & p<i<j\le n\end{cases}$$

$$\begin{aligned}\Psi^+_\lambda=\{\epsilon_i+\epsilon_{2(p-l)+1-i}\,|\,&\max\{1,1+2(p-l)-n\}\le i<p-2l+1\}\cup\\ &\{\epsilon_i+\epsilon_{2(p-l)+2-i}\,|\,p-2l+2\le i<p-l+1\}\end{aligned}$$

$$\begin{aligned}M_\lambda&=\{1,\dots,\max\{0,2(p-l)-n\},p-2l+1,p-l+1\}\\ \Phi^+_{n,\lambda}&=\{\epsilon_i+\epsilon_j\,|\,i,j\in M_\lambda\,\&\,i<j\}\end{aligned}$$

3. $\lambda=\omega_{n-1}-(1+2l)\omega_n$, $1\le l\le\left[\frac{n-1}{2}\right]$

$$(\epsilon_i+\epsilon_j,\lambda+\rho)=\begin{cases}2(n-l)-i-j, & 1\le i<j<n\\ n-2l-1-i, & 1\le i<n=j\end{cases}$$

   (a) n is odd and $l=\frac{n-1}{2}$

$$\Psi^+_\lambda=\left\{\epsilon_i+\epsilon_{n+1-i}\,\middle|\,1<i<\frac{n+1}{2}\right\}$$

$$\Phi^+_{n,\lambda}=\left\{\epsilon_1+\epsilon_{\frac{n+1}{2}}\right\}$$

   (b) n even or n odd and $l<\frac{n-1}{2}$

$$\Psi^+_\lambda=\{\epsilon_i+\epsilon_{2(n-l)-i}\,|\,n-2l<i<n-l\}\cup\{\epsilon_{n-2l-1}+\epsilon_n\}$$

$$\Phi^+_{n,\lambda}=\{\epsilon_i+\epsilon_j\,|\,i<j\,\&\,i,j\in\{1,\dots,n-2l-2,n-2l,n-l\}\}$$

4. $\lambda = -(2l-2)\omega_n$, $1 \le l \le \left[\frac{n}{2}\right]$

$$(\epsilon_i + \epsilon_j, \lambda + \rho) = 2(n-l+1) - i - j$$

$$\Psi_\lambda^+ = \{\epsilon_i + \epsilon_{2(n-l+1)-i} \,|\, n + 2(1-l) \le i \le n - l\}$$

$$\Phi_{n,\lambda}^+ = \{\epsilon_i + \epsilon_j \,|\, i < j \,\&\, i, j \in \{1, \ldots, n-2l+1, n-l+1\}\}$$

5. $\lambda = \omega_1 + \omega_{q+1} - (2n-q)\omega_n$, $2 \le q \le n-3$

$$(\epsilon_i + \epsilon_j, \lambda + \rho) = \begin{cases} 2 + q - j, & i = 1, 2 \le j \le q+1 \\ 1 + q - j, & i = 1, q+1 < j \le n \\ 2 + q - i - j, & 2 \le i < j \le q+1 \\ 1 + q - i - j, & 2 \le i \le q+1 < j \le n \\ q - i - j, & q+1 < i < j \le n \end{cases}$$

$$\Psi_\lambda^+ = \left\{\epsilon_i + \epsilon_{2+q-i} \,\middle|\, 1 < i < \frac{q}{2} + 1\right\}$$

$$\Phi_{n,\lambda}^+ = \begin{cases} \{\epsilon_1 + \epsilon_{q+1}\}, & q \text{ is odd} \\ \left\{\epsilon_1 + \epsilon_{q+1}, \epsilon_1 + \epsilon_{\frac{q}{2}+1}\right\}, & q \text{ is even} \end{cases}$$

6. $\lambda = \omega_1 + \omega_{n-1} - (n+1)\omega_n$

$$(\epsilon_i + \epsilon_j, \lambda + \rho) = \begin{cases} n - j, & i = 1 < j < n \\ -1, & i = 1, j = n \\ n - i - j, & 1 < i < j < n \\ -1 - i, & 1 < i < n = j \end{cases}$$

$$\Psi_\lambda^+ = \left\{\epsilon_i + \epsilon_{n-i} \,\middle|\, 1 < i < \frac{n}{2}\right\}, \quad \Phi_{n,\lambda}^+ = \begin{cases} \{\epsilon_1 + \epsilon_{n-1}\}, & n \text{ is odd} \\ \{\epsilon_1 + \epsilon_{n-1}, \epsilon_1 + \epsilon_{\frac{n}{2}}\}, & n \text{ is even} \end{cases}$$

7. $\lambda = \omega_1 - (n-1)\omega_n$

$$(\epsilon_i + \epsilon_j, \lambda + \rho) = \begin{cases} n + 1 - j, & i = 1 < j \le n \\ n + 1 - i - j, & 1 < i < j \le n \end{cases}$$

$$\Psi_\lambda^+ = \left\{\epsilon_i + \epsilon_{n+1-i} \,\middle|\, 1 < i < \frac{n+1}{2}\right\}$$

$$\Phi_{n,\lambda}^+ = \begin{cases} \{\epsilon_1 + \epsilon_n\}, & n \text{ is even} \\ \{\epsilon_1 + \epsilon_n, \epsilon_1 + \epsilon_{\frac{n+1}{2}}\}, & n \text{ is odd} \end{cases}$$

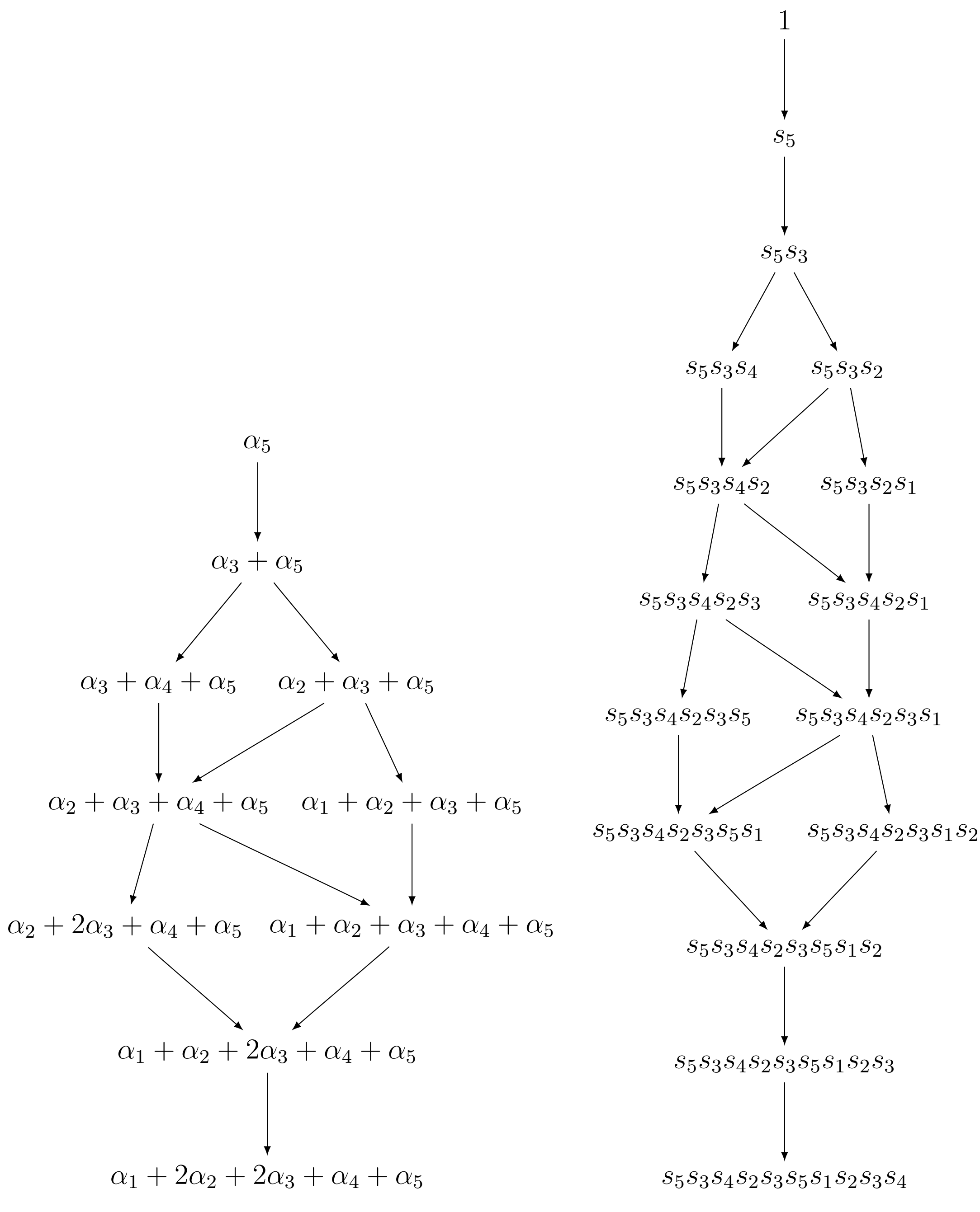


Figure 3.14: Poset of noncompact roots and the BGG graph for $SO^*(10)$

3.2.4 $\mathrm{SO}(2,2n-2) \sim D_n, n \geq 3$

Root system data

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, i < n, \alpha_n = \epsilon_{n-1} + \epsilon_n$$

$$\omega_i = \epsilon_1 + \cdots \epsilon_i, i < n-1, \quad \omega_{n-1} = \frac{1}{2}(\epsilon_1 + \cdots \epsilon_{n-1} - \epsilon_n), \quad \omega_n = \frac{1}{2}(\epsilon_1 + \cdots \epsilon_{n-1} + \epsilon_n)$$

$$\begin{aligned} \Phi &= \{\pm\epsilon_i \pm \epsilon_j | i \neq j, i, j = 1 \dots n\} \\ \Phi_c^+ &= \{\epsilon_i \pm \epsilon_j | 2 \leq i < j \leq n\} \\ \Phi_n^+ &= \{\epsilon_1 \pm \epsilon_j | 2 \leq j \leq n\} \end{aligned}$$

$$\beta = \epsilon_1 + \epsilon_2, \quad \rho = (n-1, \dots, 1, 0), \quad \zeta = (1, 0, \dots, 0)$$

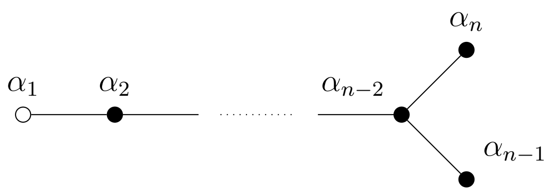
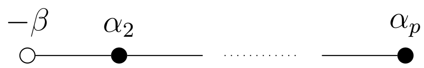


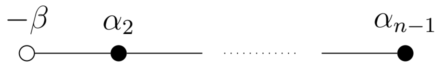
Figure 3.15: Marked Dynkin diagram for $\mathrm{SO}(2,2n-2)$

Vertex λ_a	Weight μ_a	$Q(\lambda_a) = R(\lambda_a)$	$l(\lambda_a)$
$-(2n-p-1)\omega_1 + \omega_{p+1}$	$-(2n-p)\omega_1 + \omega_p$	$\mathrm{SU}(1,p)$[1]	1
$-(n+1)\omega_1 + \omega_{n-1} + \omega_n$	$-(n+2)\omega_1 + \omega_{n-2}$	$\mathrm{SU}(1,n-2)$	1
$-(n-1)\omega_1 + \omega_n$[2]	$-n\omega_1 + \omega_{n-1}$	$\mathrm{SU}(1,n-1)$	1
$-(n-2)\omega_1$	$-n\omega_1$	$\mathrm{SO}(2,2n-2)$	2
0	$-2\omega_1 + \omega_2$	$\mathrm{SO}(2,2n-2)$	1
$-(n-1)\omega_1 + \omega_{n-1}$[3]	$-n\omega_1 + \omega_n$	$\mathrm{SU}(1,n-1)$	1

[1] $1 \leq p \leq n-3$ with Dynkin diagram of $R(\lambda_a)$:



[2] Dynkin diagram of $R(\lambda_a)$:



[3] Dynkin diagram of $R(\lambda_a)$:

Table 3.3: Vertices and root systems for $\mathrm{SO}(2,2n-2)$, $n \geq 3$

Nilpotent cohomology in detail

Scalar products of ρ with positive noncompact roots

$$(\epsilon_1 + \epsilon_j, \rho) = 2n - 1 - j, \quad (\epsilon_1 - \epsilon_j, \rho) = j - 1. \tag{3.5}$$

1. $\lambda = -(2n-p-1)\omega_1 + \omega_{p+1}$
   Scalar products of positive noncompact roots with $\lambda + \rho$ are

$$(\epsilon_1 + \epsilon_j, \lambda + \rho) = \begin{cases} p+2-j, & 1 < j \le p+1 \\ p+1-j, & p+1 < j \le n \end{cases}$$

$$(\epsilon_1 - \epsilon_j, \lambda + \rho) = \begin{cases} p-2n+j, & 1 < j \le p+1 \\ p-2n+1+j, & p+1 < j \le n. \end{cases}$$

   This gives an empty set of singular roots $\Psi_\lambda^+ = \emptyset$ and the set of generating roots is $\Phi_{n,\lambda}^+ = \{\epsilon_1 + \epsilon_j \,|\, 1 < j \le p+1\}$. The generated root subsystem is

$$\Phi_\lambda = \{\pm(\epsilon_1 + \epsilon_j \,|\, 1 < j \le p+1\} \cup \{\epsilon_i - \epsilon_j \,|\, 1 < i, j \le p+1 \,\&\, i \ne j\}$$

   and is of type A_p.

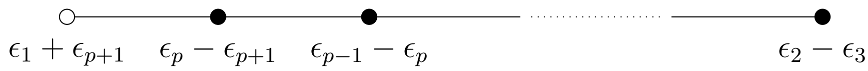


Figure 3.16: The reduced Hermitian symmetric pair for $\lambda = -(2n-p-1)\omega_1 + \omega_{p+1}$

   The integral cone is in this case

$$C = \{a_1\omega_1 + a_{p+1}\omega_{p+1} + \cdots + a_n\omega_n \,|\, a_1 + 2(a_{p+1} + \cdots + a_{n-2}) + a_{n-1} + a_n = 0\}$$

   and one can easily check that $\Psi_\lambda^+ = \Psi_{\lambda+\mu}^+$ for all $\mu \in C$ and thus the translation principle from the section 1.2.1 applies.

$$(A - a_{n-1} - a_n - 2n + p + 1,\ 0,\ 0, \ldots,\ 0,\ 0,\ a_{p+1} + 1,\ a_{p+2}, \ldots,\ a_n)$$

$$\downarrow$$

$$(A - a_{n-1} - a_n - 2n + p,\ 0,\ 0, \ldots,\ 0,\ 1,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$

$$\downarrow$$

$$(A - a_{n-1} - a_n - 2n + p - 1,\ 0,\ 0, \ldots,\ 1,\ 0,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$

$$\vdots$$

$$(A - a_{n-1} - a_n - 2n + 1,\ 0,\ 1, \ldots,\ 0,\ 0,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$

$$\downarrow$$

$$(A - a_{n-1} - a_n - 2n + 2,\ 1,\ 0, \ldots,\ 0,\ 0,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$

$$\downarrow$$

$$(A - a_{n-1} - a_n - 2n + 2,\ 0,\ 0, \ldots,\ 0,\ 0,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$

Figure 3.17: Nilpotent cohomology / BGG resolution, $A = -2(a_{p+1} + \cdots + a_{n-2})$

2. $\lambda = -(n+1)\omega_1 + \omega_{n-1} + \omega_n$
   Scalar products of positive noncompact roots with $\lambda + \rho$

$$(\epsilon_1 + \epsilon_j, \lambda + \rho) = \begin{cases} n - j, & 1 < j < n \\ -1, & j = n \end{cases}$$

$$(\epsilon_1 - \epsilon_j, \lambda + \rho) = \begin{cases} -n - 2 + j, & 1 < j < n \\ -1, & j = n \end{cases}$$

   show that there are no singular roots $\Psi_\lambda^+ = \emptyset$ and that the set of generating roots is $\Phi_{n,\lambda}^+ = \{\epsilon_1 + \epsilon_j \,|\, 1 < j < n\}$. The generated root subsystem of Φ is

$$\Phi_\lambda = \{\pm(\epsilon_1 + \epsilon_j \,|\, 1 < j < n\} \cup \{\epsilon_i - \epsilon_j \,|\, 1 < i, j \leq n \,\&\, i \neq j\}$$

   and is of type A_{n-2}. The integral cone is

$$C = \{-(a+b)\omega_1 + a\omega_{n-1} + b\omega_n) \,|\, a, b \in \mathbb{N}_0\}$$

   and an easy calculation gives $\Psi_\lambda^+ = \Psi_{\lambda+\mu}^+$ for all $\mu \in C$. Thus we can use the translation principle from the section 1.2.1 and get the same shape of cohomology on the whole cone.

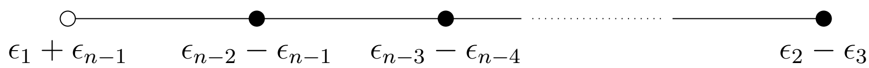


Figure 3.18: The reduced Hermitian symmetric pair for $\lambda = -(n+1)\omega_1 + \omega_{n-1} + \omega_n$

$$(-(a+b+n+1), 0, 0, \ldots, 0, 0, a+1, b+1)$$
$$\downarrow$$
$$(-(a+b+n+2), 0, 0, \ldots, 0, 1, a, b)$$
$$\downarrow$$
$$(-(a+b+n+3), 0, 0, \ldots, 1, 0, a, b)$$
$$\vdots$$
$$(-(a+b+2n-3), 0, 1, \ldots, 0, 0, a, b)$$
$$\downarrow$$
$$(-(a+b+2n-2), 1, 0, \ldots, 0, 0, a, b)$$
$$\downarrow$$
$$(-(a+b+2n-2), 0, 0, \ldots, 0, 0, a, b)$$

Figure 3.19: Nilpotent cohomology / BGG resolution for $\lambda = -(a+b+n+1)\omega_1 + (a+1)\omega_{n-1} + (b+1)\omega_n$

3. $\lambda = -(n-1)\omega_1 + \omega_n$
   Scalar products of positive noncompact roots with $\lambda + \rho$ are

$$(\epsilon_1 + \epsilon_j, \lambda + \rho) = n + 1 - j$$
$$(\epsilon_1 - \epsilon_j, \lambda + \rho) = j - n.$$

   The integral cone is in this case

$$C = \{t(-\omega_1 + \omega_n) \,|\, t \in \mathbb{N}_0\}$$

   and there is a scalar product that depends on the value of t, namely the scalar product with the root $\epsilon_1 - \epsilon_n$.

   For $t = 0$ we obtain set of singular roots $\Psi^+_\lambda = \{\epsilon_1 - \epsilon_n\}$ and set generating roots $\Phi^+_{n,\lambda} = \{\epsilon_1 + \epsilon_n\}$. This gives the subsystem of type A_1

$$\Phi_\lambda = \{\epsilon_1 + \epsilon_n, -\epsilon_1 - \epsilon_n\}$$

   and the resulting weights for nontrivial cohomology groups are all in the table 3.3.

   For $t \geq 1$ we get no singular roots $\Psi^+_\lambda = \emptyset$ and the generated subsystem is of type A_{n-1}.

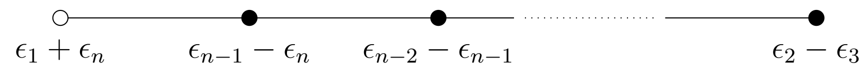


Figure 3.20: The reduced Hermitian symmetric pair for $\lambda = -(n-1)\omega_1 + \omega_n$

$$(-t-n+1, 0, 0, \ldots, 0, 0, t+1)$$
$$\downarrow$$
$$(-t-n, 0, 0, \ldots, 0, 1, t)$$
$$\downarrow$$
$$(-t-n-1, 0, 0, \ldots, 1, 0, t)$$
$$\vdots$$
$$(-t-2n+4, 0, 1, \ldots, 0, 0, t)$$
$$\downarrow$$
$$(-t-2n+3, 1, 0, \ldots, 0, 0, t)$$
$$\downarrow$$
$$(-t-2n+3, 0, 0, \ldots, 0, 0, t)$$

Figure 3.21: Nilpotent cohomology / BGG resolution for
$\lambda = -(t+n-1)\omega_1 + (t+1)\omega_n$

4. $\lambda = -(n-2)\omega_1$
   Scalar products of positive noncompact roots with $\lambda + \rho$ are

$$(\epsilon_1 + \epsilon_j, \lambda + \rho) = 1 + n - j$$
$$(\epsilon_1 - \epsilon_j, \lambda + \rho) = 1 - n + j.$$

   The set of singular positive roots is $\Psi^+_\lambda = \{\epsilon_1 - \epsilon_{n-1}\}$ and the set of generating roots $\Phi^+_{n,\lambda} = \{\epsilon_1 + \epsilon_{n-1}\}$. It follows that

$$\Phi_\lambda = \{\epsilon_1 + \epsilon_{n-1}, -\epsilon_1 - \epsilon_{n-1}\},$$

   which is of type A_1 and all the information on cohomology is already contained in the table 3.3.

5. $\lambda = 0$
   Scalar products of positive noncompact roots with $\lambda + \rho$ are given by (3.5) and we see that there are no singular roots $\Psi_\lambda = \emptyset$ since all scalar products are positive integers. It follows that $\Phi^+_{n,\lambda} = \Phi^+_n$ and after a moment of thought we realize that

$$\Phi_\lambda = \Phi.$$

   We conclude that the cohomology is in this case computed by the orginial Kostant's formula.

6. $\lambda = -(n-1)\omega_1 + \omega_{n-1}$
   Scalar products of positive noncompact roots with $\lambda + \rho$ are

$$(\epsilon_1 + \epsilon_j, \lambda + \rho) = \begin{cases} n + 1 - j, & 1 < j < n \\ 0, & j = n \end{cases}$$
$$(\epsilon_1 - \epsilon_j, \lambda + \rho) = \begin{cases} j - n, & 1 < j < n \\ 1, & j = n \end{cases}$$

   which gives $\Psi^+_\lambda = \{\epsilon_1 + \epsilon_n\}$ and $\Phi^+_{n,\lambda} = \{\epsilon_1 - \epsilon_n\}$. This generates the roots subsystem of type A_1

$$\Phi_\lambda = \{\epsilon_1 - \epsilon_n, \epsilon_n - \epsilon_1\}$$

   and all the cohomology information is contained in table 3.3. The integral cone is in this case

$$C = \{x(-\omega_1 + \omega_{n-1}) \,|\, x \in \mathbb{N}_0\}$$

   and one for $\mu \in C$, $\mu = -x\omega_1 + x\omega_{n-1}$ one has

$$(\epsilon_1 + \epsilon_j, \mu) = \begin{cases} 0, & 1 < j < n \\ -x, & j = n \end{cases}$$
$$(\epsilon_1 - \epsilon_j, \mu) = \begin{cases} -x, & 1 < j < n \\ 0, & j = n. \end{cases}$$

Since x is a positive integer, it follows that

$$\Psi^+_{\lambda+\mu} = \emptyset, \quad \Phi^+_{n,\lambda+\mu} = \{\epsilon_1 - \epsilon_n\} \cup \{\epsilon_1 + \epsilon_j \,|\, 1 < j < n\}.$$

The root subsystem generated by reflections is

$$\Phi_{\lambda+\mu} = \{\pm(\epsilon_1 - \epsilon_n)\} \cup \{\pm(\epsilon_1 + \epsilon_j \,|\, 1 < j < n\} \cup \{\epsilon_i - \epsilon_j \,|\, 1 < i, j < n \,\&\, i \neq j\}$$

and is of type A_{n-1}.

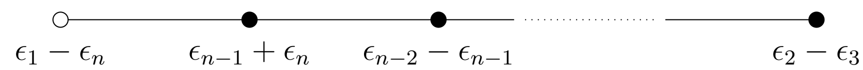


Figure 3.22: The reduced Hermitian symmetric pair for $\lambda = -(n-1)\omega_1 + \omega_{n-1}$

$$(-t-n+1, 0, 0, \ldots, 0, t+1, 0)$$
$$\downarrow$$
$$(-t-n, 0, 0, \ldots, 0, t, 1)$$
$$\downarrow$$
$$(-t-n-1, 0, 0, \ldots, 1, t, 0)$$
$$\vdots$$
$$(-t-2n+4, 0, 1, \ldots, 0, t, 0)$$
$$\downarrow$$
$$(-t-2n+3, 1, 0, \ldots, 0, t, 0)$$
$$\downarrow$$
$$(-t-2n+3, 0, 0, \ldots, 0, t, 0)$$

Figure 3.23: Nilpotent cohomology / BGG resolution for
$\lambda = -(t+n-1)\omega_1 + (t+1)\omega_{n-1}$

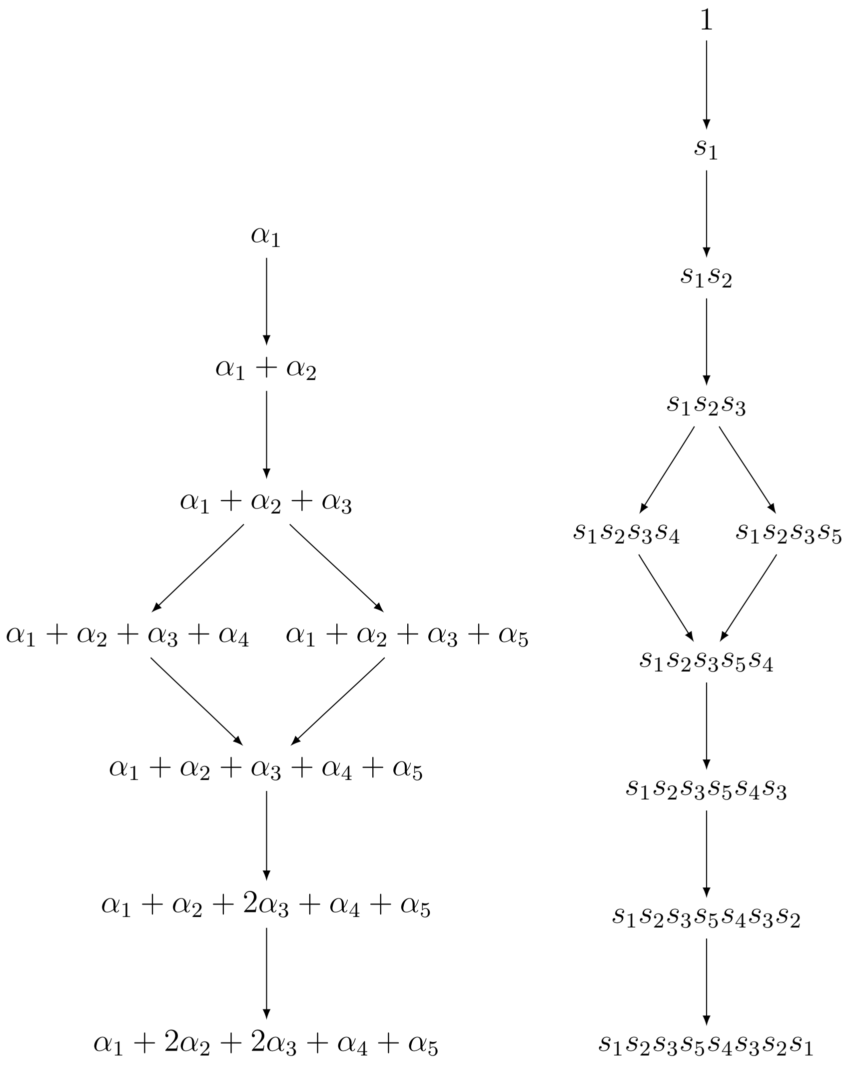


Figure 3.24: Poset of noncompact roots and the BGG graph for $\mathrm{SO}(2,8)$

3.2.5 $\mathrm{SO}(2, 2n-1) \sim B_n, n \geq 2$

Root system data

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, i < n, \quad \alpha_n = \epsilon_n$$
$$\omega_i = \epsilon_1 + \cdots + \epsilon_i, i < n, \quad \omega_n = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)$$

$$\Phi = \{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j | i \neq j, i, j = 1 \ldots n\}$$
$$\Phi_c^+ = \{\epsilon_i \pm \epsilon_j | 2 \leq i < j \leq n\} \cup \{\epsilon_j | 2 \leq j \leq n\}$$
$$\Phi_n^+ = \{\epsilon_1 \pm \epsilon_j | 2 \leq j \leq n\} \cup \{\epsilon_1\}$$

$$\beta = \epsilon_1 + \epsilon_2, \quad \rho = (n - \frac{1}{2}, \ldots, \frac{1}{2}), \quad \zeta = (1, 0, \ldots, 0)$$

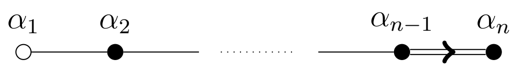


Figure 3.25: Marked Dynkin diagram for $\mathrm{SO}(2, 2n-1)$

Vertex λ_a	Weight μ_a	$Q(\lambda_a) = R(\lambda_a)$[1]	$l(\lambda_a)$
$-(2n-p)\omega_1 + \omega_{p+1}$	$-(2n-p+1)\omega_1 + \omega_p$	$\mathrm{SU}(1,p)$[2]	1
$-(n+1)\omega_1 + 2\omega_n$	$-(n+2)\omega_1 + \omega_{n-1}$	$\mathrm{SU}(1, n-1)$	1
0	$-2\omega_1 + \omega_2$	$\mathrm{SO}(2, 2n-1)$	1
$-(n - \frac{3}{2})\omega_1$	$-(n + \frac{1}{2})\omega_1$	$\mathrm{SO}(2, 2n-1)$	2
$-(n - \frac{1}{2})\omega_1 + \omega_n$	$-(n + \frac{1}{2})\omega_1 + \omega_n$	$\mathrm{SU}(1, n-1)$	1

[1] Except in the last row, where $R(\lambda_a) = \mathrm{SO}(2, 2n-1)$.
[2] $1 \leq p \leq n-2$

Table 3.4: Vertices and root systems for $\mathrm{SO}(2, 2n-1)$, $n \geq 2$

Nilpotent cohomology in detail

Scalar products of of ρ with positive noncompact roots

$$(\epsilon_1, \rho) = n - \frac{1}{2}, \quad (\epsilon_1 + \epsilon_j, \rho) = 2n - j, \quad (\epsilon_1 - \epsilon_j, \rho) = j - 1. \tag{3.6}$$

1. $\lambda = (p - 2n)\omega_1 + \omega_{p+1}$
   The scalar products of positive noncompact roots with $\lambda + \rho$

$$(\epsilon_1, \lambda + \rho) = p - n + \frac{1}{2}$$
$$(\epsilon_1 + \epsilon_j, \lambda + \rho) = \begin{cases} p + 2 - j, & 1 < j \leq p + 1 \\ p + 1 - j, & p + 1 < j \leq n \end{cases}$$
$$(\epsilon_1 - \epsilon_j, \lambda + \rho) = \begin{cases} p - 2n + j - 1, & 1 < j \leq p + 1 \\ p - 2n + j, & p + 1 < j \leq n \end{cases}$$

reveal that the set of singular roots is empty $\Psi_\lambda^+ = \emptyset$ and that the set of generating roots is $\Phi_{n,\lambda}^+ = \{\epsilon_1 + \epsilon_j \,|\, 1 < j \leq p+1\}$. The generated root subsystem of type A_p is

$$\Phi_\lambda = \{\pm(\epsilon_1 + \epsilon_j \,|\, 1 < j \leq p+1\} \cup \{\epsilon_i - \epsilon_j \,|\, 1 < i, j \leq p+1 \,\&\, i \neq j\}.$$

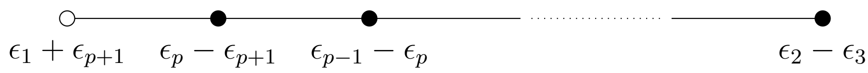


Figure 3.26: The reduced Hermitian symmetric pair for $\lambda = (p-2n)\omega_1 + \omega_{p+1}$

The integral cone is in this case

$$C = \{a_1\omega_1 + a_{p+1}\omega_{p+1} + \cdots + a_n\omega_n \,|\, a_1 + 2(a_{p+1} + \cdots + a_{n-1}) + a_n = 0\}$$

and one can easily check that $\Psi_\lambda^+ = \Psi_{\lambda+\mu}^+$ for all $\mu \in C$ and thus the translation principle from the section 1.2.1 applies.

$$(A - a_n - 2n + p,\ 0,\ 0, \ldots,\ 0,\ 0,\ a_{p+1} + 1,\ a_{p+2}, \ldots,\ a_n)$$
$$\downarrow$$
$$(A - a_n - 2n + p - 1,\ 0,\ 0, \ldots,\ 0,\ 1,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$
$$\downarrow$$
$$(A - a_n - 2n + p - 2,\ 0,\ 0, \ldots,\ 1,\ 0,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$
$$\vdots$$
$$(A - a_n - 2n + 2,\ 0,\ 1, \ldots,\ 0,\ 0,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$
$$\downarrow$$
$$(A - a_n - 2n + 1,\ 1,\ 0, \ldots,\ 0,\ 0,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$
$$\downarrow$$
$$(A - a_n - 2n + 1,\ 0,\ 0, \ldots,\ 0,\ 0,\ a_{p+1},\ a_{p+2}, \ldots,\ a_n)$$

Figure 3.27: Nilpotent cohomology / BGG resolution, $A = -2(a_{p+1} + \cdots + a_{n-1})$

2. $\lambda = -(n+1)\omega_1 + 2\omega_n$
   Scalar products of positive noncompact roots with $\lambda + \rho$

$$(\epsilon_1, \lambda + \rho) = -\frac{1}{2}$$
$$(\epsilon_1 + \epsilon_j, \lambda + \rho) = n + 1 - j$$
$$(\epsilon_1 - \epsilon_j, \lambda + \rho) = -n - 2 + j$$

show that the set of singular roots is again empty $\Psi_\lambda^+ = \emptyset$ and the set of generating roots is $\Phi_{n,\lambda}^+ = \{\epsilon_1 + \epsilon_j \,|\, 1 < j \leq n\}$. The generated root subsystem of type A_{n-1} is

$$\Phi_\lambda = \{\pm(\epsilon_1 + \epsilon_j \,|\, 1 < j \leq p+1\} \cup \{\epsilon_i - \epsilon_j \,|\, 1 < i, j \leq p+1 \,\&\, i \neq j\}.$$

The integral cone is in this case

$$C = \{t(-\omega_1 + \omega_n) \,|\, t \in \mathbb{N}_0\}$$

and $\Psi_\lambda^+ = \Psi_{\lambda+\mu}^+$ for all $\mu \in C$.

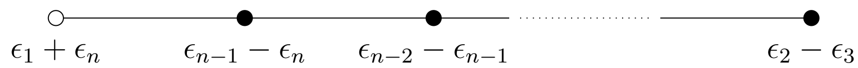


Figure 3.28: The reduced Hermitian symmetric pair for $\lambda = -(n+1)\omega_1 + 2\omega_n$

$$(-t-n-1, 0, 0, \ldots, 0, 0, t+2)$$
$$\downarrow$$
$$(-t-n-2, 0, 0, \ldots, 0, 1, t)$$
$$\downarrow$$
$$(-t-n-3, 0, 0, \ldots, 1, 0, t)$$
$$\vdots$$
$$(-t-2n+2, 0, 1, \ldots, 0, 0, t)$$
$$\downarrow$$
$$(-t-2n+1, 1, 0, \ldots, 0, 0, t)$$
$$\downarrow$$
$$(-t-2n+1, 0, 0, \ldots, 0, 0, t)$$

Figure 3.29: Nilpotent cohomology / BGG resolution for
$\lambda = -(t+n+1)\omega_1 + (t+2)\omega_n$

3. $\lambda = 0$
   In this case the scalar products of positive roots with $\lambda + \rho$ are of course given by (3.6) and there are no singular roots $\Psi_\lambda^+ = \emptyset$. We have $\Phi_{n,\lambda}^+ = \Phi_n^+$, the root subsystem is

$$\Phi_\lambda = \Phi$$

   and the Kostant's formula applies.

4. $\lambda = (\frac{3}{2} - n)\omega_1$
   The scalar products of positive noncompact roots with $\lambda + \rho$ are

$$(\epsilon_1, \lambda + \rho) = 1$$
$$(\epsilon_1 + \epsilon_j, \lambda + \rho) = n + \frac{3}{2} - j$$
$$(\epsilon_1 - \epsilon_j, \lambda + \rho) = -n + \frac{1}{2} + j.$$

   The set of singular roots is empty $\Psi_\lambda^+ = \emptyset$ and the integrality conditions of the definition 3.2.1 imply that $\Phi_{n,\lambda}^+ = \{\epsilon_1\}$. It follows that

$$\Phi_\lambda = \{\pm\epsilon_1\}$$

   and that all the nontrivial cohomologies are contained in the table 3.4.

5. $\lambda = -(n - \frac{1}{2})\omega_1 + \omega_n$
   Sscalar products of positive noncompact roots with $\lambda + \rho$

$$(\epsilon_1, \lambda + \rho) = \frac{1}{2}$$
$$(\epsilon_1 + \epsilon_j, \lambda + \rho) = n + \frac{3}{2} - j$$
$$(\epsilon_1 - \epsilon_j, \lambda + \rho) = -n - \frac{1}{2} + j$$

   again show that there are no singular roots $\Psi_\lambda^+ = \emptyset$ and since $\epsilon_1^\vee = 2\epsilon_1$ we have $\Phi_{n,\lambda}^+ = \{\epsilon_1\}$. This yields the subsystem

$$\Phi_\lambda = \{\pm\epsilon_1\}$$

   which is of type A_1 and the nontrivial cohomologies are given by table 3.4.

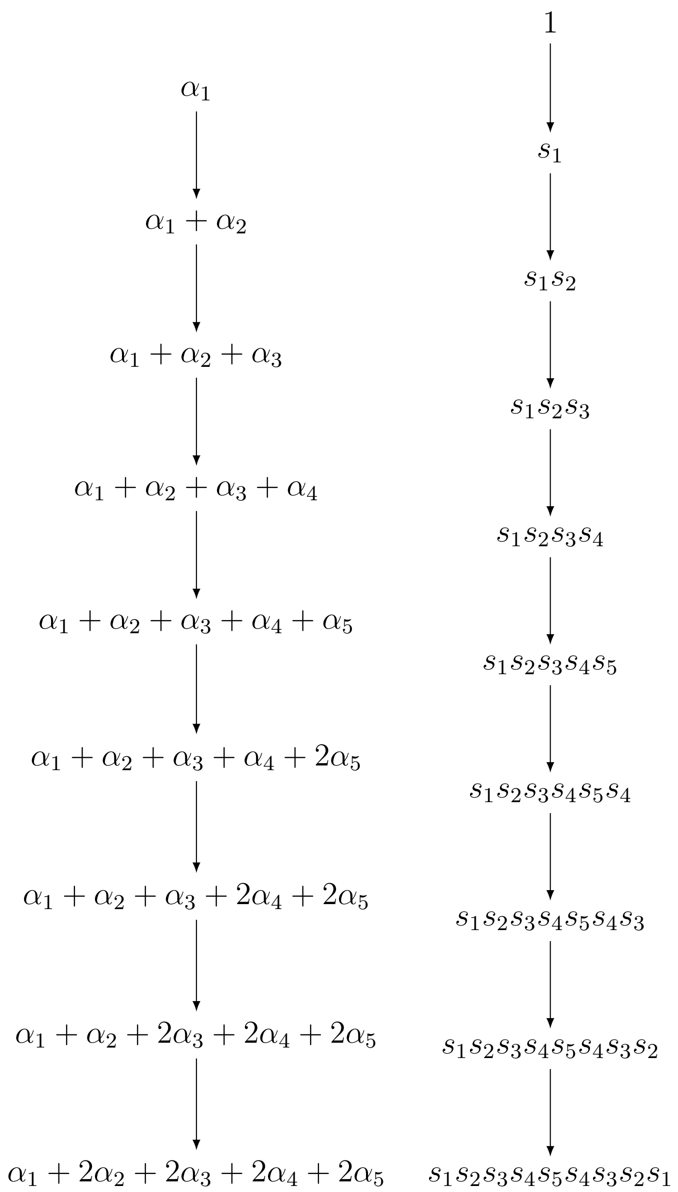
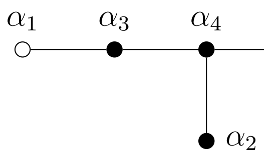


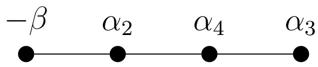
Figure 3.30: Poset of noncompact roots for $\mathrm{SO}(2,9)$

3.2.6 Exceptional cases

E_6



Vertex λ_a	Weight μ_a	$Q(\lambda_a) = R(\lambda_a)$	$l(\lambda_a)$
$-12\omega_1 + \omega_2$	$-12\omega_1$	SU(1, 1)	1
$-12\omega_1 + \omega_4$	$-12\omega_1 + \omega_2$	SU(1, 2)	1
$-12\omega_1 + \omega_3 + \omega_5$	$-12\omega_1 + \omega_4$	SU(1, 3)	1
$-9\omega_1 + \omega_5$ [1]	$-10\omega_1 + \omega_3$	SU(1, 4)	1
$-10\omega_1 + \omega_3 + \omega_6$ [2]	$-10\omega_1 + \omega_5$	SU(1, 4)	1
$-8\omega_1 + \omega_3$	$-8\omega_1 + \omega_6$	SU(1, 5)	1
$-5\omega_1 + \omega_6$	$-6\omega_1 + \omega_2$	SO(2, 8)	1
$-8\omega_1 + \omega_6$	$-9\omega_1$	SO(2, 8)	2
0	$-2\omega_1 + \omega_3$	$EIII$	1
$-3\omega_1$	$-5\omega_1 + \omega_6$	$EIII$	2

[1] Dynkin diagram 

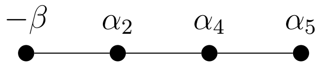
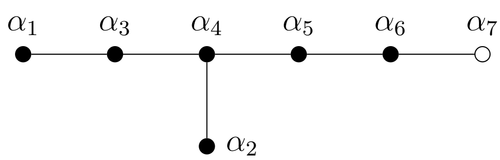
[2] Dynkin diagram 

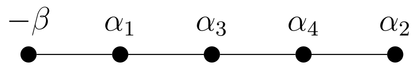
Table 3.5: Vertices and root systems for E_6

For full cohomologies of unitarizable modules see [EH04b].

E_7



Vertex λ_a	Weight μ_a	$Q(\lambda_a) = R(\lambda_a)$	$l(\lambda_a)$
$\omega_1 - 18\omega_7$	$-18\omega_7$	$\mathrm{SU}(1,1)$	1
$\omega_3 - 18\omega_7$	$\omega_1 - 18\omega_7$	$\mathrm{SU}(1,2)$	1
$\omega_4 - 18\omega_7$	$\omega_3 - 18\omega_7$	$\mathrm{SU}(1,3)$	1
$\omega_2 + \omega_5 - 18\omega_7$	$\omega_4 - 18\omega_7$	$\mathrm{SU}(1,4)$	1
$\omega_5 - 15\omega_7$[1]	$\omega_2 - 15\omega_7$	$\mathrm{SU}(1,5)$	1
$\omega_2 + \omega_6 - 16\omega_7$[2]	$\omega_5 - 16\omega_7$	$\mathrm{SU}(1,5)$	1
$\omega_2 - 13\omega_7$	$\omega_6 - 14\omega_7$	$\mathrm{SU}(1,6)$	1
$\omega_6 - 10\omega_7$	$\omega_1 - 10\omega_7$	$\mathrm{SO}(2,10)$	1
$\omega_6 - 14\omega_7$	$-14\omega_7$	$\mathrm{SO}(2,10)$	2
0	$\omega_6 - 2\omega_7$	$EVII$	1
$-4\omega_7$	$\omega_1 - 6\omega_7$	$EVII$	2
$-8\omega_7$	$-10\omega_7$	$EVII$	3

[1] Dynkin diagram 

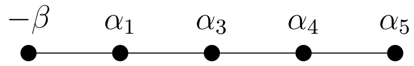
[2] Dynkin diagram 

Table 3.6: Vertices and root systems for E_7

For full cohomologies of unitarizable modules see [EH04b].

Conclusion

We have calculated cohomology of all unitarziable weights in the conformal Hermitian symmetric case and obtained partial results in the remaining ones. We found explicit singular vectors in some cases and we have shown in general that these modules give rise to sequences of differential operators similarly to finite-dimensional $\mathfrak{g}$ representations. The formula for cohomology of unitarizable weights is based on certain equivalence of categories that transports the results from the finite-dimensional representations to the unitarizable ones. What does one obtain by transporting unitarizable modules from the smaller rank?

The class of modules for which there exists the BGG resolution (in the flat case) is strictly bigger than the class of unitarizable highest weight modules. It is not clear whether there are curved analogues for these nonunitarizable Kostant modules. Another natural question is whether our modification of the Calderbank–Diemer construction works also for some other globalization apart from the formal one. And of course, having explicit expression for these operators is also highly desirable.

We have made some progress towards elementary description of the smaller of the two exceptional Hermitian symmetric cases. Description of this space using octonions in a similar vein to [PTF11] based on [BH10] is current work in progress.

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List of Figures

List of Tables

A. Cohomology of unitarizable modules for low ranks

A.1 Cohomology for A_n, $2 \leq n \leq 5$

A.1.1 $\mathfrak{su}(1,1)$: $p' = 1$, $q' = 1$, $l = 1$

Cone of unitarizable weights: 0

$$(1, \epsilon_1 - \epsilon_2)$$

Figure A.1: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

$$\underset{\epsilon_1 - \epsilon_2}{\circ}$$

Figure A.2: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0) \longrightarrow (-2)$$

Figure A.3: Nilpotent cohomology / BGG resolution

A.1.2 $\mathfrak{su}(1,2)$: $p' = 1$, $q' = 1$, $l = 1$

Cone of unitarizable weights: $-(a_2 + 2)\,\omega_1 + (a_2 + 1)\,\omega_2$

$$(1, \epsilon_1 - \epsilon_3)$$

Figure A.4: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

$$\underset{\epsilon_1 - \epsilon_3}{\circ}$$

Figure A.5: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(-a_2 - 2,\ a_2 + 1) \longrightarrow (-a_2 - 3,\ a_2)$$

Figure A.6: Nilpotent cohomology / BGG resolution

A.1.3 $\mathfrak{su}(1,2)$: $p' = 1$, $q' = 2$, $l = 1$

Cone of unitarizable weights: 0

$$(1, \epsilon_1 - \epsilon_2) \longrightarrow (2, \epsilon_1 - \epsilon_3)$$

Figure A.7: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

Figure A.8: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ 0) \longrightarrow (-2,\ 1) \longrightarrow (-3,\ 0)$$

Figure A.9: Nilpotent cohomology / BGG resolution

A.1.4 $\mathfrak{su}(1,3)$: $p' = 1$, $q' = 1$, $l = 1$

Cone of unitarizable weights: $-(a_2 + a_3 + 3)\,\omega_1 + a_2\omega_2 + (a_3 + 1)\,\omega_3$

$$(1, \epsilon_1 - \epsilon_4)$$

Figure A.10: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

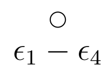

Figure A.11: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(-a_2 - a_3 - 3,\ a_2,\ a_3 + 1) \longrightarrow (-a_2 - a_3 - 4,\ a_2,\ a_3)$$

Figure A.12: Nilpotent cohomology / BGG resolution

A.1.5 $\mathfrak{su}(1,3)$: $p' = 1$, $q' = 2$, $l = 1$

Cone of unitarizable weights: $-(a_2 + 2)\,\omega_1 + (a_2 + 1)\,\omega_2$

$$(1, \epsilon_1 - \epsilon_3) \longrightarrow (2, \epsilon_1 - \epsilon_4)$$

Figure A.13: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

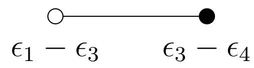

Figure A.14: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(-a_2 - 2,\ a_2 + 1,\ 0) \longrightarrow (-a_2 - 3,\ a_2,\ 1) \longrightarrow (-a_2 - 4,\ a_2,\ 0)$$

Figure A.15: Nilpotent cohomology / BGG resolution

A.1.6 $\mathfrak{su}(1,3)$: $p' = 1$, $q' = 3$, $l = 1$

Cone of unitarizable weights: 0

$$(1, \epsilon_1 - \epsilon_2) \longrightarrow (2, \epsilon_1 - \epsilon_3) \longrightarrow (3, \epsilon_1 - \epsilon_4)$$

Figure A.16: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

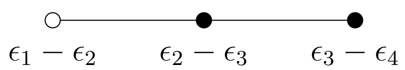

Figure A.17: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ 0,\ 0) \longrightarrow (-2,\ 1,\ 0) \longrightarrow (-3,\ 0,\ 1) \longrightarrow (-4,\ 0,\ 0)$$

Figure A.18: Nilpotent cohomology / BGG resolution

A.1.7 $\mathfrak{su}(2,2)$: $p' = 1$, $q' = 1$, $l = 1$

Cone of unitarizable weights: $(a_1 + 1)\,\omega_1 - (a_1 + a_3 + 4)\,\omega_2 + (a_3 + 1)\,\omega_3$

$$(1, \epsilon_1 - \epsilon_4)$$

Figure A.19: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

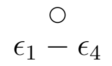

Figure A.20: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(a_1+1,\ -a_1-a_3-4,\ a_3+1) \longrightarrow (a_1,\ -a_1-a_3-4,\ a_3)$$

Figure A.21: Nilpotent cohomology / BGG resolution

A.1.8 $\mathfrak{su}(2,2)$: $p' = 1$, $q' = 2$, $l = 1$

Cone of unitarizable weights: $(a_1+1)\,\omega_1 - (a_1+2)\,\omega_2$

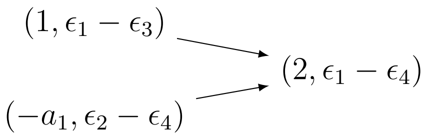

Figure A.22: Non-negative scalar products with noncompact roots

$\lambda = \omega_1 - 2\omega_2$
Set of singular roots: $\{\epsilon_2 - \epsilon_4\}$

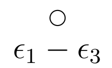

Figure A.23: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(1,\ -2,\ 0) \longrightarrow (0,\ -3,\ 1)$$

Figure A.24: Nilpotent cohomology / BGG resolution

$\lambda = (a_1+1)\,\omega_1 - (a_1+2)\,\omega_2,\ a_1 \geq 1$
Set of singular roots: $\emptyset$

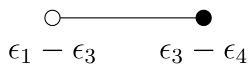

Figure A.25: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(a_1+1,\ -a_1-2,\ 0) \longrightarrow (a_1,\ -a_1-3,\ 1) \longrightarrow (a_1-1,\ -a_1-3,\ 0)$$

Figure A.26: Nilpotent cohomology / BGG resolution

A.1.9 $\mathfrak{su}(2,2)$: $p' = 2$, $q' = 1$, $l = 1$

Cone of unitarizable weights: $-(a_3+2)\,\omega_2 + (a_3+1)\,\omega_3$

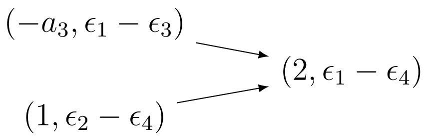

Figure A.27: Non-negative scalar products with noncompact roots

$\lambda = -2\omega_2 + \omega_3$
Set of singular roots: $\{\epsilon_1 - \epsilon_3\}$

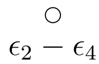

Figure A.28: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ -2,\ 1) \longrightarrow (1,\ -3,\ 0)$$

Figure A.29: Nilpotent cohomology / BGG resolution

$\lambda = -(a_3+2)\,\omega_2 + (a_3+1)\,\omega_3,\ a_3 \geq 1$
Set of singular roots: $\emptyset$

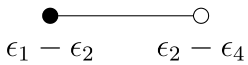

Figure A.30: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ -a_3-2,\ a_3+1) \longrightarrow (1,\ -a_3-3,\ a_3) \longrightarrow (0,\ -a_3-3,\ a_3-1)$$

Figure A.31: Nilpotent cohomology / BGG resolution

A.1.10 $\mathfrak{su}(2,2)$: $p' = 2$, $q' = 2$, $l = 1$

Cone of unitarizable weights: 0

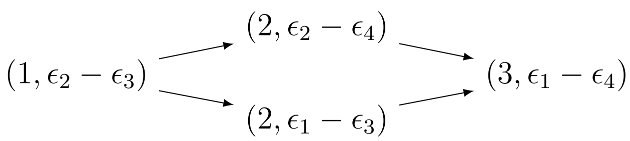

Figure A.32: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

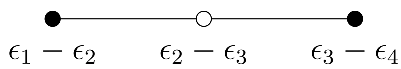

Figure A.33: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

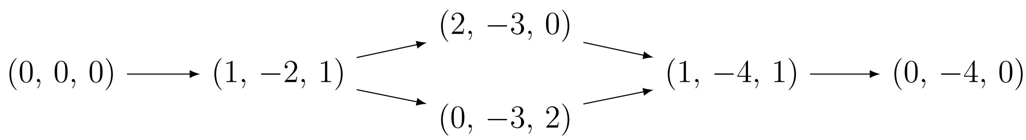

Figure A.34: Nilpotent cohomology / BGG resolution

A.1.11 $\mathfrak{su}(2,2)$: $p' = 2$, $q' = 2$, $l = 2$

Cone of unitarizable weights: $-\omega_2$

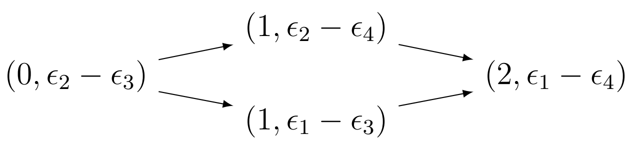

Figure A.35: Non-negative scalar products with noncompact roots

Set of singular roots: $\{\epsilon_2 - \epsilon_3\}$

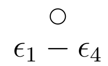

Figure A.36: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ -1,\ 0) \longrightarrow (2,\ -5,\ 2)$$

Figure A.37: Nilpotent cohomology / BGG resolution

A.1.12 $\mathfrak{su}(1,4)$: $p' = 1$, $q' = 1$, $l = 1$

Cone of unitarizable weights: $-(a_2 + a_3 + a_4 + 4)\,\omega_1 + a_2\omega_2 + a_3\omega_3 + (a_4 + 1)\,\omega_4$

$$(1, \epsilon_1 - \epsilon_5)$$

Figure A.38: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

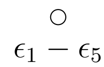

Figure A.39: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(-a_2 - a_3 - a_4 - 4,\ a_2,\ a_3,\ a_4 + 1) \to (-a_2 - a_3 - a_4 - 5,\ a_2,\ a_3,\ a_4)$$

Figure A.40: Nilpotent cohomology / BGG resolution

A.1.13 $\mathfrak{su}(1,4)$: $p' = 1$, $q' = 2$, $l = 1$

Cone of unitarizable weights: $-(a_2 + a_3 + 3)\,\omega_1 + a_2\omega_2 + (a_3 + 1)\,\omega_3$

$$(1, \epsilon_1 - \epsilon_4) \longrightarrow (2, \epsilon_1 - \epsilon_5)$$

Figure A.41: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

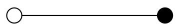

Figure A.42: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(-a_2 - a_3 - 3,\ a_2,\ a_3 + 1,\ 0) \to (-a_2 - a_3 - 4,\ a_2,\ a_3,\ 1) \to (-a_2 - a_3 - 5,\ a_2,\ a_3,\ 0)$$

Figure A.43: Nilpotent cohomology / BGG resolution

A.1.14 $\mathfrak{su}(1,4)$: $p' = 1$, $q' = 3$, $l = 1$

Cone of unitarizable weights: $-(a_2 + 2)\,\omega_1 + (a_2 + 1)\,\omega_2$

$$(1, \epsilon_1 - \epsilon_3) \longrightarrow (2, \epsilon_1 - \epsilon_4) \longrightarrow (3, \epsilon_1 - \epsilon_5)$$

Figure A.44: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

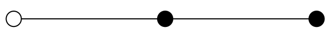

Figure A.45: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(-a_2-2,\ a_2+1,\ 0,\ 0) \to (-a_2-3,\ a_2,\ 1,\ 0) \to (-a_2-4,\ a_2,\ 0,\ 1) \to (-a_2-5,\ a_2,\ 0,\ 0)$$

Figure A.46: Nilpotent cohomology / BGG resolution

A.1.15 $\mathfrak{su}(1,4)$: $p' = 1$, $q' = 4$, $l = 1$

Cone of unitarizable weights: 0

$$(1, \epsilon_1 - \epsilon_2) \longrightarrow (2, \epsilon_1 - \epsilon_3) \longrightarrow (3, \epsilon_1 - \epsilon_4) \longrightarrow (4, \epsilon_1 - \epsilon_5)$$

Figure A.47: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

Figure A.48: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ 0,\ 0,\ 0) \to (-2,\ 1,\ 0,\ 0) \to (-3,\ 0,\ 1,\ 0) \to (-4,\ 0,\ 0,\ 1) \to (-5,\ 0,\ 0,\ 0)$$

Figure A.49: Nilpotent cohomology / BGG resolution

A.1.16 $\mathfrak{su}(2,3)$: $p' = 1$, $q' = 1$, $l = 1$

Cone of unitarizable weights: $(a_1+1)\,\omega_1 - (a_1+a_3+a_4+5)\,\omega_2 + a_3\omega_3 + (a_4+1)\,\omega_4$

$$(1, \epsilon_1 - \epsilon_5)$$

Figure A.50: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

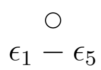

Figure A.51: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(a_1+1,\, -a_1-a_3-a_4-5,\, a_3,\, a_4+1) \rightarrow (a_1,\, -a_1-a_3-a_4-5,\, a_3,\, a_4)$$

Figure A.52: Nilpotent cohomology / BGG resolution

A.1.17 $\mathfrak{su}(2,3)$: $p'=1$, $q'=2$, $l=1$

Cone of unitarizable weights: $(a_1+1)\,\omega_1-(a_1+a_3+4)\,\omega_2+(a_3+1)\,\omega_3$

$$(-a_1, \epsilon_2-\epsilon_5) \rightarrow (2, \epsilon_1-\epsilon_5), \quad (1, \epsilon_1-\epsilon_4) \rightarrow (2, \epsilon_1-\epsilon_5)$$

Figure A.53: Non-negative scalar products with noncompact roots

$\lambda = \omega_1-(a_3+4)\,\omega_2+(a_3+1)\,\omega_3$
Set of singular roots: $\{\epsilon_2-\epsilon_5\}$

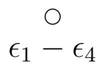

Figure A.54: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(1,\, -a_3-4,\, a_3+1,\, 0) \longrightarrow (0,\, -a_3-4,\, a_3,\, 1)$$

Figure A.55: Nilpotent cohomology / BGG resolution

$\lambda = (a_1+1)\,\omega_1-(a_1+a_3+4)\,\omega_2+(a_3+1)\,\omega_3,\ a_1 \geq 1$
Set of singular roots: $\emptyset$

Figure A.56: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(a_1+1,\, -a_1-a_3-4,\, a_3+1,\, 0) \rightarrow (a_1,\, -a_1-a_3-4,\, a_3,\, 1) \rightarrow (a_1-1,\, -a_1-a_3-4,\, a_3,\, 0)$$

Figure A.57: Nilpotent cohomology / BGG resolution

A.1.18 $\mathfrak{su}(2,3)$: $p' = 1$, $q' = 3$, $l = 1$

Cone of unitarizable weights: $(a_1 + 1)\,\omega_1 - (a_1 + 2)\,\omega_2$

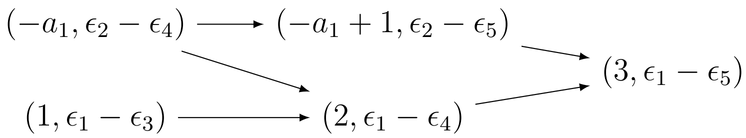

Figure A.58: Non-negative scalar products with noncompact roots

$\lambda = \omega_1 - 2\omega_2$
Set of singular roots: $\{\epsilon_2 - \epsilon_4\}$

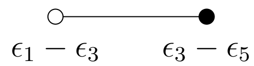

Figure A.59: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(1,\ -2,\ 0,\ 0) \longrightarrow (0,\ -3,\ 1,\ 0) \longrightarrow (-2,\ -4,\ 1,\ 1)$$

Figure A.60: Nilpotent cohomology / BGG resolution

$\lambda = 2\omega_1 - 3\omega_2$
Set of singular roots: $\{\epsilon_2 - \epsilon_5\}$

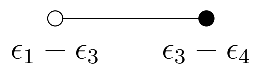

Figure A.61: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(2,\ -3,\ 0,\ 0) \longrightarrow (1,\ -4,\ 1,\ 0) \longrightarrow (0,\ -4,\ 0,\ 1)$$

Figure A.62: Nilpotent cohomology / BGG resolution

$\lambda = (a_1 + 1)\,\omega_1 - (a_1 + 2)\,\omega_2$, $a_1 \geq 2$
Set of singular roots: $\emptyset$

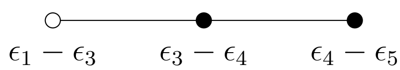

Figure A.63: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(a_1+1,\ -a_1-2,\ 0,\ 0) \to (a_1,\ -a_1-3,\ 1,\ 0) \to (a_1-1,\ -a_1-3,\ 0,\ 1) \to (a_1-2,\ -a_1-3,\ 0,\ 0)$$

Figure A.64: Nilpotent cohomology / BGG resolution

A.1.19 $\mathfrak{su}(2,3)$: $p'=2$, $q'=1$, $l=1$

- Cone of unitarizable weights: $-\left(a_3+a_4+3\right)\omega_2+a_3\omega_3+\left(a_4+1\right)\omega_4$

$$(-a_4, \epsilon_1-\epsilon_4) \to (2, \epsilon_1-\epsilon_5) \leftarrow (1, \epsilon_2-\epsilon_5)$$

Figure A.65: Non-negative scalar products with noncompact roots

$\lambda=-\left(a_3+3\right)\omega_2+a_3\omega_3+\omega_4$
Set of singular roots: $\{\epsilon_1-\epsilon_4\}$

Figure A.66: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ -a_3-3,\ a_3,\ 1) \longrightarrow (1,\ -a_3-4,\ a_3,\ 0)$$

Figure A.67: Nilpotent cohomology / BGG resolution

$\lambda=-\left(a_3+a_4+3\right)\omega_2+a_3\omega_3+\left(a_4+1\right)\omega_4,\ a_4\geq 1$
Set of singular roots: $\emptyset$

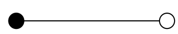

Figure A.68: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ -a_3-a_4-3,\ a_3,\ a_4+1) \to (1,\ -a_3-a_4-4,\ a_3,\ a_4) \to (0,\ -a_3-a_4-4,\ a_3,\ a_4-1)$$

Figure A.69: Nilpotent cohomology / BGG resolution

A.1.20 $\mathfrak{su}(2,3)$: $p' = 2$, $q' = 2$, $l = 1$

Cone of unitarizable weights: $-(a_3+2)\,\omega_2 + (a_3+1)\,\omega_3$

$$\begin{array}{ccccc} (-a_3, \epsilon_1-\epsilon_3) & \longrightarrow & (2, \epsilon_1-\epsilon_4) & \searrow & \\ & \nearrow & & & (3, \epsilon_1-\epsilon_5) \\ (1, \epsilon_2-\epsilon_4) & \longrightarrow & (2, \epsilon_2-\epsilon_5) & \nearrow & \end{array}$$

Figure A.70: Non-negative scalar products with noncompact roots

$\lambda = -2\omega_2 + \omega_3$
Set of singular roots: $\{\epsilon_1 - \epsilon_3\}$

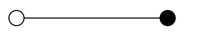

Figure A.71: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ -2,\ 1,\ 0) \longrightarrow (1,\ -3,\ 0,\ 1) \longrightarrow (2,\ -4,\ 0,\ 0)$$

Figure A.72: Nilpotent cohomology / BGG resolution

$\lambda = -(a_3+2)\,\omega_2 + (a_3+1)\,\omega_3,\ a_3 \geq 1$
Set of singular roots: $\emptyset$

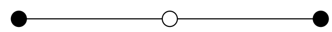

Figure A.73: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

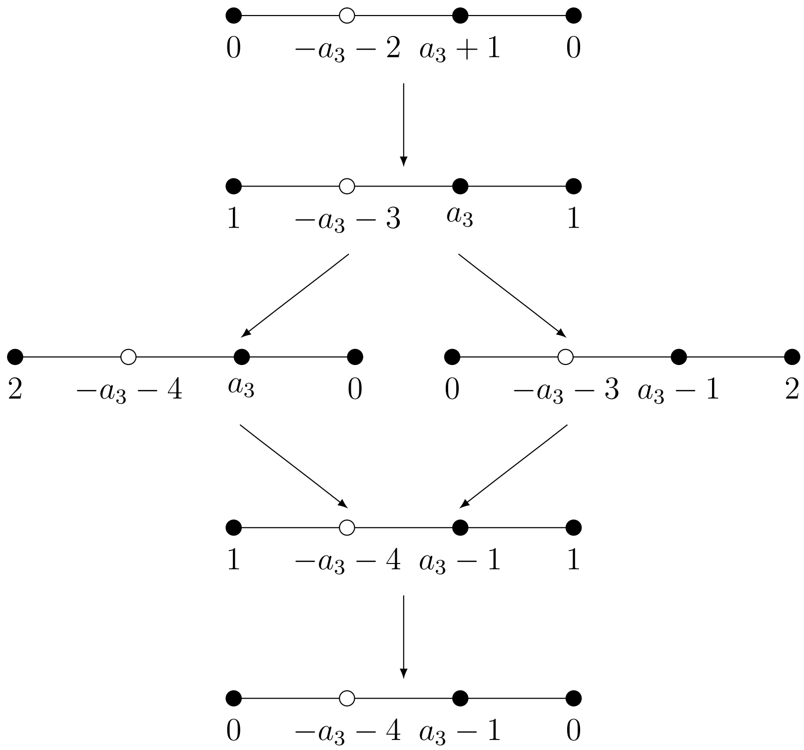

Figure A.74: Nilpotent cohomology / BGG resolution

A.1.21 $\mathfrak{su}(2,3)$: $p'=2$, $q'=2$, $l=2$

Cone of unitarizable weights: $-\left(a_3+3\right)\omega_2+\left(a_3+1\right)\omega_3$

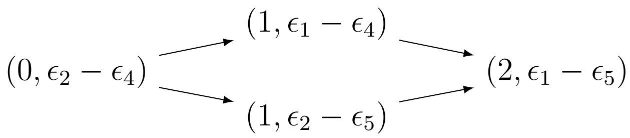

Figure A.75: Non-negative scalar products with noncompact roots

Set of singular roots: $\{\epsilon_2-\epsilon_4\}$

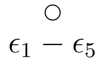

Figure A.76: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda,\mathfrak{k}_\lambda)$

$$(0,\,-a_3-3,\,a_3+1,\,0)\longrightarrow(2,\,-a_3-5,\,a_3-1,\,2)$$

Figure A.77: Nilpotent cohomology / BGG resolution

A.1.22 $\mathfrak{su}(2,3)$: $p' = 2$, $q' = 3$, $l = 1$

Cone of unitarizable weights: 0

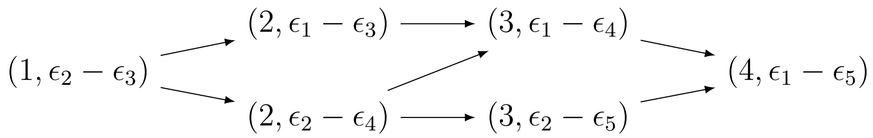

Figure A.78: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

Figure A.79: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

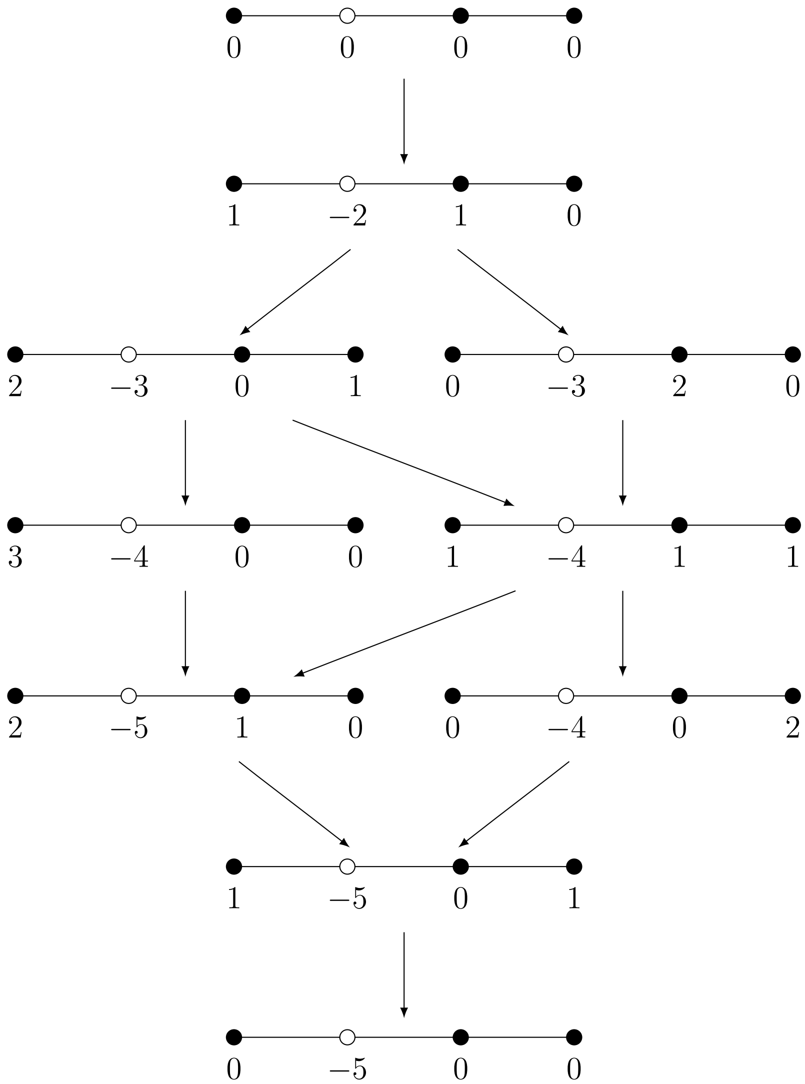

Figure A.80: Nilpotent cohomology / BGG resolution

A.1.23 $\mathfrak{su}(2,3)$: $p' = 2$, $q' = 3$, $l = 2$

Cone of unitarizable weights: $-\omega_2$

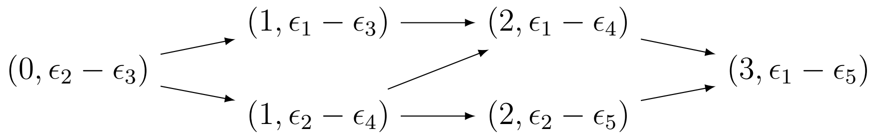

Figure A.81: Non-negative scalar products with noncompact roots

Set of singular roots: $\{\epsilon_2 - \epsilon_3\}$

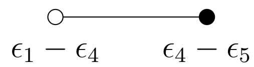

Figure A.82: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ -1,\ 0,\ 0) \longrightarrow (2,\ -5,\ 2,\ 0) \longrightarrow (3,\ -6,\ 1,\ 1)$$

Figure A.83: Nilpotent cohomology / BGG resolution

A.2 Cohomology for C_n, $2 \leq n \leq 3$

A.2.1 $\mathfrak{sp}(2)$: $r = 1$, $q = 1$, $l = 1$

Cone of unitarizable weights: $(a_1 + 2)\,\omega_1 - (a_1 + 3)\,\omega_2$

$$(1, 2\epsilon_1)$$

Figure A.84: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

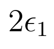

Figure A.85: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(a_1 + 2,\ -a_1 - 3) \longrightarrow (a_1,\ -a_1 - 3)$$

Figure A.86: Nilpotent cohomology / BGG resolution

A.2.2 $\mathfrak{sp}(2)$: $r = 2$, $q = 1$, $l = 1$

Cone of unitarizable weights: $\omega_1 - \frac{3}{2}\omega_2$

$$(1, \epsilon_1 + \epsilon_2) \longrightarrow (\tfrac{3}{2}, 2\epsilon_1)$$

Figure A.87: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

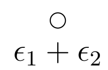

Figure A.88: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$\left(1,\ -\tfrac{3}{2}\right) \longrightarrow \left(1,\ -\tfrac{5}{2}\right)$$

Figure A.89: Nilpotent cohomology / BGG resolution

A.2.3 $\mathfrak{sp}(2)$: $r = 2$, $q = 2$, $l = 1$

Cone of unitarizable weights: 0

$$(1, 2\epsilon_2) \longrightarrow (3, \epsilon_1 + \epsilon_2) \longrightarrow (2, 2\epsilon_1)$$

Figure A.90: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

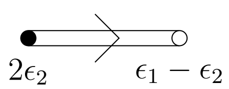

Figure A.91: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ 0) \longrightarrow (2,\ -1) \longrightarrow (4,\ -3) \longrightarrow (4,\ -4)$$

Figure A.92: Nilpotent cohomology / BGG resolution

A.2.4 $\mathfrak{sp}(2)$: $r = 2$, $q = 2$, $l = 2$

Cone of unitarizable weights: $-\frac{1}{2}\omega_2$

$$\left(\tfrac{1}{2}, 2\epsilon_2\right) \longrightarrow \left(2, \epsilon_1 + \epsilon_2\right) \longrightarrow \left(\tfrac{3}{2}, 2\epsilon_1\right)$$

Figure A.93: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

$$\underset{\epsilon_1 + \epsilon_2}{\bigcirc}$$

Figure A.94: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$\left(0,\ -\tfrac{1}{2}\right) \longrightarrow \left(0,\ -\tfrac{5}{2}\right)$$

Figure A.95: Nilpotent cohomology / BGG resolution

A.2.5 $\mathfrak{sp}(3)$: $r = 1$, $q = 1$, $l = 1$

Cone of unitarizable weights: $(a_1 + 2)\,\omega_1 + a_2\omega_2 - (a_1 + a_2 + 4)\,\omega_3$

$$(1, 2\epsilon_1)$$

Figure A.96: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

$$\underset{2\epsilon_1}{\bigcirc}$$

Figure A.97: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(a_1 + 2,\ a_2,\ -a_1 - a_2 - 4) \longrightarrow (a_1,\ a_2,\ -a_1 - a_2 - 4)$$

Figure A.98: Nilpotent cohomology / BGG resolution

A.2.6 $\mathfrak{sp}(3)$: $r = 2$, $q = 1$, $l = 1$

Cone of unitarizable weights: $\omega_1 + (a_2 + 1)\,\omega_2 - \left(a_2 + \frac{7}{2}\right)\omega_3$

$$\left(1, \epsilon_1 + \epsilon_2\right) \longrightarrow \left(\tfrac{3}{2}, 2\epsilon_1\right)$$

Figure A.99: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

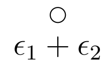

Figure A.100: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$\left(1,\ a_2+1,\ -a_2-\tfrac{7}{2}\right) \longrightarrow \left(1,\ a_2,\ -a_2-\tfrac{7}{2}\right)$$

Figure A.101: Nilpotent cohomology / BGG resolution

A.2.7 $\mathfrak{sp}(3)$: $r = 2$, $q = 2$, $l = 1$

Cone of unitarizable weights: $(a_2+2)\,\omega_2 - (a_2+3)\,\omega_3$

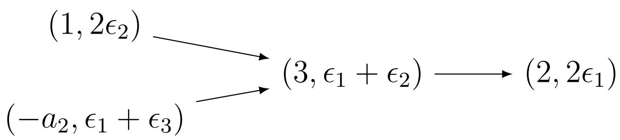

Figure A.102: Non-negative scalar products with noncompact roots

$\lambda = 2\omega_2 - 3\omega_3$
Set of singular roots: $\{\epsilon_1 + \epsilon_3\}$

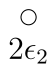

Figure A.103: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ 2,\ -3) \longrightarrow (2,\ 0,\ -3)$$

Figure A.104: Nilpotent cohomology / BGG resolution

$\lambda = (a_2+2)\,\omega_2 - (a_2+3)\,\omega_3,\ a_2 \geq 1$
Set of singular roots: $\emptyset$

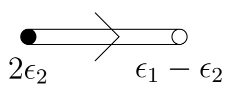

Figure A.105: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\, a_2+2,\, -a_2-3) \searrow (2,\, a_2+1,\, -a_2-3) \searrow (4,\, a_2-1,\, -a_2-3) \searrow (4,\, a_2-2,\, -a_2-3)$$

Figure A.106: Nilpotent cohomology / BGG resolution

A.2.8 $\mathfrak{sp}(3)$: $r=2$, $q=2$, $l=2$

Cone of unitarizable weights: $(a_2+2)\,\omega_2 - \left(a_2+\frac{7}{2}\right)\omega_3$

$$\left(\tfrac{1}{2}, 2\epsilon_2\right) \longrightarrow (2, \epsilon_1+\epsilon_2) \longrightarrow \left(\tfrac{3}{2}, 2\epsilon_1\right)$$

Figure A.107: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

$$\underset{\epsilon_1+\epsilon_2}{\circ}$$

Figure A.108: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$\left(0,\, a_2+2,\, -a_2-\tfrac{7}{2}\right) \longrightarrow \left(0,\, a_2,\, -a_2-\tfrac{7}{2}\right)$$

Figure A.109: Nilpotent cohomology / BGG resolution

A.2.9 $\mathfrak{sp}(3)$: $r=3$, $q=1$, $l=1$

Cone of unitarizable weights: $\omega_1 - 2\omega_3$

$$\begin{matrix} (1, \epsilon_1+\epsilon_3) \searrow & & \\ & (2, \epsilon_1+\epsilon_2) \longrightarrow (2, 2\epsilon_1) \\ (0, 2\epsilon_2) \nearrow & & \end{matrix}$$

Figure A.110: Non-negative scalar products with noncompact roots

Set of singular roots: $\{2\epsilon_2\}$

$$\underset{\epsilon_1+\epsilon_3}{\circ}$$

Figure A.111: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(1,\ 0,\ -2) \longrightarrow (0,\ 1,\ -3)$$

Figure A.112: Nilpotent cohomology / BGG resolution

A.2.10 $\mathfrak{sp}(3)$: $r = 3$, $q = 2$, $l = 1$

Cone of unitarizable weights: $\omega_2 - \frac{3}{2}\omega_3$

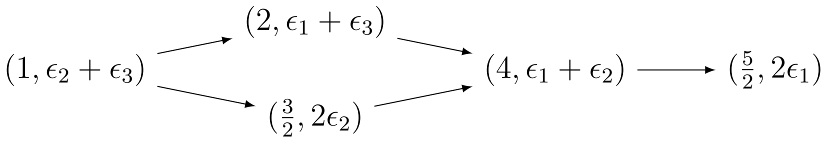

Figure A.113: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

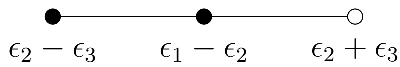

Figure A.114: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$\left(0,\ 1,\ -\tfrac{3}{2}\right) \longrightarrow \left(1,\ 1,\ -\tfrac{5}{2}\right) \longrightarrow \left(0,\ 2,\ -\tfrac{7}{2}\right) \longrightarrow \left(0,\ 0,\ -\tfrac{7}{2}\right)$$

Figure A.115: Nilpotent cohomology / BGG resolution

A.2.11 $\mathfrak{sp}(3)$: $r = 3$, $q = 2$, $l = 2$

Cone of unitarizable weights: $\omega_2 - 2\omega_3$

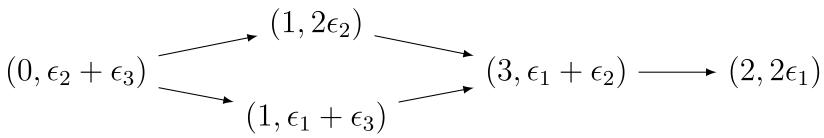

Figure A.116: Non-negative scalar products with noncompact roots

Set of singular roots: $\{\epsilon_2 + \epsilon_3\}$

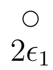

Figure A.117: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ 1,\ -2) \longrightarrow (1,\ 3,\ -5)$$

Figure A.118: Nilpotent cohomology / BGG resolution

A.2.12 $\mathfrak{sp}(3)$: $r = 3$, $q = 3$, $l = 1$

Cone of unitarizable weights: 0

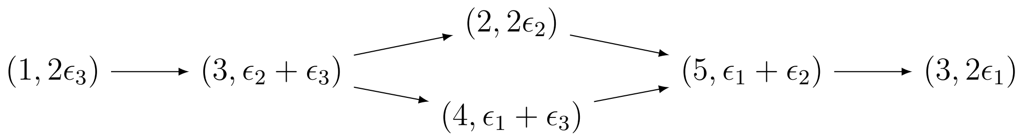

Figure A.119: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

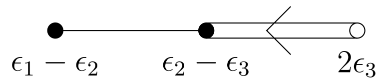

Figure A.120: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

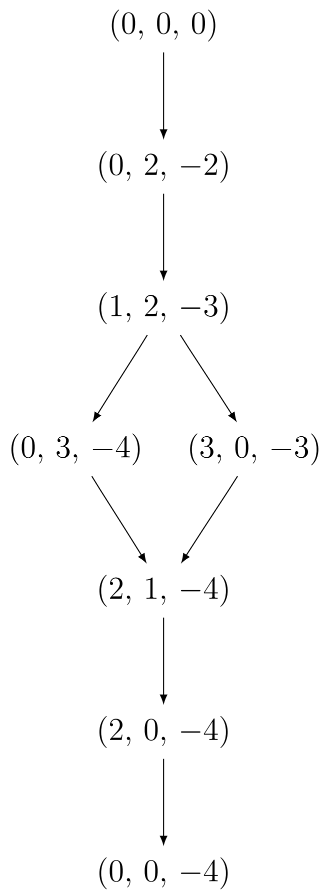

Figure A.121: Nilpotent cohomology / BGG resolution

A.2.13 $\mathfrak{sp}(3)$: $r = 3$, $q = 3$, $l = 2$

Cone of unitarizable weights: $-\frac{1}{2}\omega_3$

$$(\tfrac{1}{2}, 2\epsilon_3) \longrightarrow (2, \epsilon_2 + \epsilon_3) \longrightarrow (3, \epsilon_1 + \epsilon_3) \longrightarrow (4, \epsilon_1 + \epsilon_2) \longrightarrow (\tfrac{5}{2}, 2\epsilon_1)$$
$$(2, \epsilon_2 + \epsilon_3) \longrightarrow (\tfrac{3}{2}, 2\epsilon_2) \longrightarrow (4, \epsilon_1 + \epsilon_2)$$

Figure A.122: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

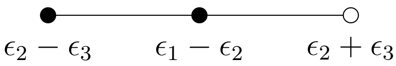

Figure A.123: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$\left(0,\ 0,\ -\tfrac{1}{2}\right) \longrightarrow \left(2,\ 0,\ -\tfrac{5}{2}\right) \longrightarrow \left(1,\ 1,\ -\tfrac{7}{2}\right) \longrightarrow \left(1,\ 0,\ -\tfrac{7}{2}\right)$$

Figure A.124: Nilpotent cohomology / BGG resolution

A.2.14 $\mathfrak{sp}(3)$: $r = 3$, $q = 3$, $l = 3$

Cone of unitarizable weights: $-\omega_3$

$$(0, 2\epsilon_3) \longrightarrow (1, \epsilon_2 + \epsilon_3) \longrightarrow (1, 2\epsilon_2) \longrightarrow (3, \epsilon_1 + \epsilon_2) \longrightarrow (2, 2\epsilon_1)$$
$$(1, \epsilon_2 + \epsilon_3) \longrightarrow (2, \epsilon_1 + \epsilon_3) \longrightarrow (3, \epsilon_1 + \epsilon_2)$$

Figure A.125: Non-negative scalar products with noncompact roots

Set of singular roots: $\{2\epsilon_3\}$

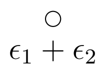

Figure A.126: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ 0,\ -1) \longrightarrow (3,\ 0,\ -4)$$

Figure A.127: Nilpotent cohomology / BGG resolution

A.3 Cohomology for $\mathrm{SO}^*(8)$

A.3.1 $\mathfrak{so}^*(8)$: $\omega_2 - 6\omega_4$

Cone of unitarizable weights: $a_1\omega_1 + (a_2 + 1)\,\omega_2 + a_3\omega_3 - (a_1 + 2\,a_2 + a_3 + 6)\,\omega_4$

$$(1, \epsilon_1 + \epsilon_2)$$

Figure A.128: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

$$\circ$$
$$\epsilon_1 + \epsilon_2$$

Figure A.129: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(a_1,\ a_2 + 1,\ a_3,\ -a_1 - 2\,a_2 - a_3 - 6)$$
$$\downarrow$$
$$(a_1,\ a_2,\ a_3,\ -a_1 - 2\,a_2 - a_3 - 6)$$

Figure A.130: Nilpotent cohomology / BGG resolution

A.3.2 $\mathfrak{so}^*(8)$: $\omega_3 - 3\omega_4$

Cone of unitarizable weights: $(a_3 + 1)\,\omega_3 - (a_3 + 3)\,\omega_4$

$$(1, \epsilon_2 + \epsilon_3) \longrightarrow (2, \epsilon_1 + \epsilon_3)$$
$$(-a_3, \epsilon_1 + \epsilon_4) \longrightarrow (2, \epsilon_1 + \epsilon_3)$$
$$(2, \epsilon_1 + \epsilon_3) \longrightarrow (3, \epsilon_1 + \epsilon_2)$$

Figure A.131: Non-negative scalar products with noncompact roots

$\lambda = \omega_3 - 3\omega_4$
Set of singular roots: $\{\epsilon_1 + \epsilon_4\}$

$$\circ$$
$$\epsilon_2 + \epsilon_3$$

Figure A.132: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ 0,\ 1,\ -3) \longrightarrow (1,\ 0,\ 0,\ -4)$$

Figure A.133: Nilpotent cohomology / BGG resolution

$\lambda = (a_3 + 1)\,\omega_3 - (a_3 + 3)\,\omega_4,\ a_3 \geq 1$
Set of singular roots: $\emptyset$

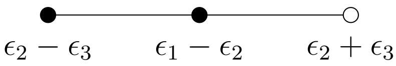

Figure A.134: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ 0,\ a_3+1,\ -a_3-3) \to (1,\ 0,\ a_3,\ -a_3-4) \to (0,\ 1,\ a_3-1,\ -a_3-5) \to (0,\ 0,\ a_3-1,\ -a_3-5)$$

Figure A.135: Nilpotent cohomology / BGG resolution

A.3.3 $\mathfrak{so}^*(8)$: 0

Cone of unitarizable weights: 0

$$(1, \epsilon_3+\epsilon_4) \longrightarrow (2, \epsilon_2+\epsilon_4) \rightrightarrows \begin{matrix} (3, \epsilon_1+\epsilon_4) \\ (3, \epsilon_2+\epsilon_3) \end{matrix} \rightrightarrows (4, \epsilon_1+\epsilon_3) \longrightarrow (5, \epsilon_1+\epsilon_2)$$

Figure A.136: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

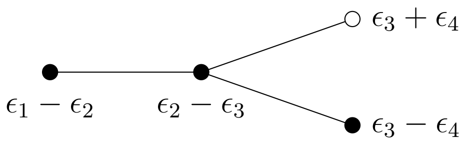

Figure A.137: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

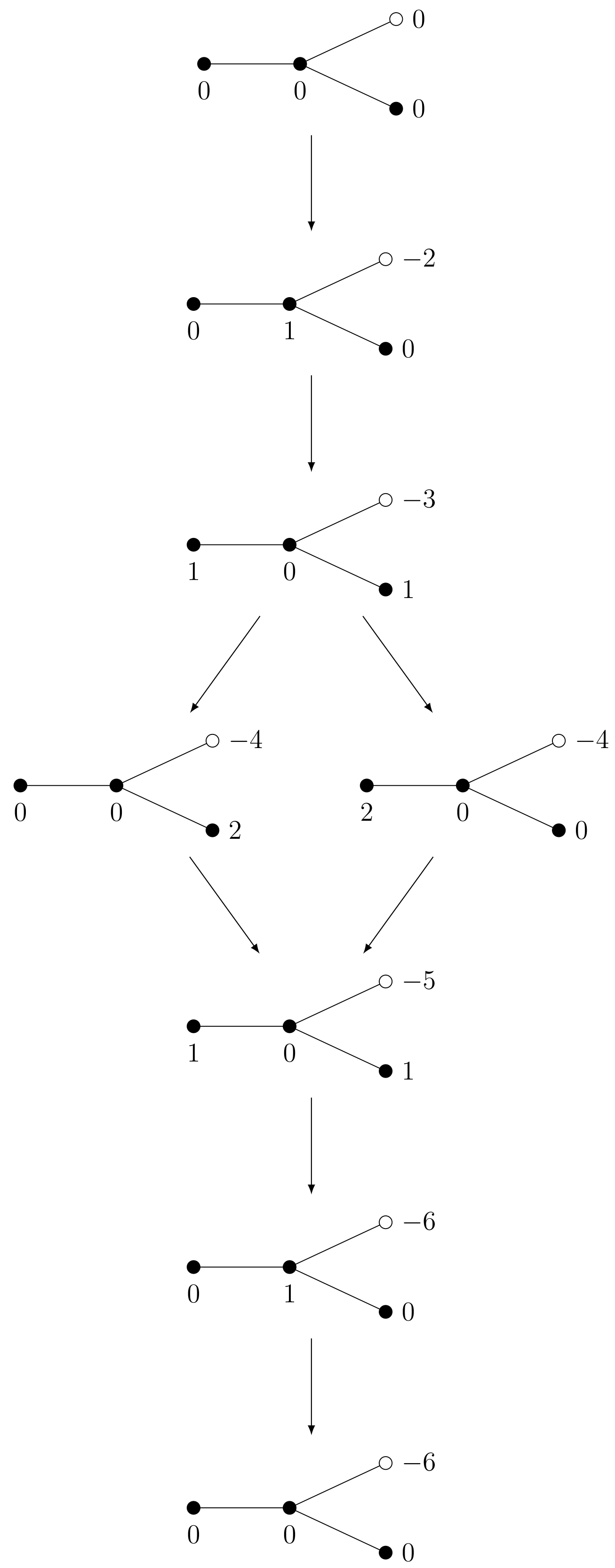

Figure A.138: Nilpotent cohomology / BGG resolution

A.3.4 $\mathfrak{so}^*(8)$: $-2\omega_4$

Cone of unitarizable weights: $-2\omega_4$

$$
\begin{array}{ccccccc}
 & & (1, \epsilon_1 + \epsilon_4) & & & & \\
 & \nearrow & & \searrow & & & \\
(0, \epsilon_2 + \epsilon_4) & & & & (2, \epsilon_1 + \epsilon_3) & \longrightarrow & (3, \epsilon_1 + \epsilon_2) \\
 & \searrow & & \nearrow & & & \\
 & & (1, \epsilon_2 + \epsilon_3) & & & &
\end{array}
$$

Figure A.139: Non-negative scalar products with noncompact roots

Set of singular roots: $\{\epsilon_2 + \epsilon_4\}$

$$
\begin{array}{c} \circ \\ \epsilon_1 + \epsilon_3 \end{array}
$$

Figure A.140: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(0,\ 0,\ 0,\ -2) \longrightarrow (0,\ 2,\ 0,\ -6)$$

Figure A.141: Nilpotent cohomology / BGG resolution

A.3.5 $\mathfrak{so}^*(8)$: $\omega_1 + \omega_2 - 7\omega_4$

Cone of unitarizable weights: $(a_1 + 1)\,\omega_1 + (a_2 + 1)\,\omega_2 + a_3\omega_3 - (a_1 + 2\,a_2 + a_3 + 7)\,\omega_4$

$$(1, \epsilon_1 + \epsilon_2)$$

Figure A.142: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

$$
\begin{array}{c} \circ \\ \epsilon_1 + \epsilon_2 \end{array}
$$

Figure A.143: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(a_1 + 1,\ a_2 + 1,\ a_3,\ -a_1 - 2\,a_2 - a_3 - 7) \rightarrow (a_1 + 1,\ a_2,\ a_3,\ -a_1 - 2\,a_2 - a_3 - 7)$$

Figure A.144: Nilpotent cohomology / BGG resolution

A.3.6 $\mathfrak{so}^*(8)$: $\omega_1 + \omega_3 - 5\omega_4$

Cone of unitarizable weights: $(a_1+1)\,\omega_1 + (a_3+1)\,\omega_3 - (a_1+a_3+5)\,\omega_4$

$$(1, \epsilon_1+\epsilon_3) \longrightarrow (2, \epsilon_1+\epsilon_2)$$

Figure A.145: Non-negative scalar products with noncompact roots

Set of singular roots: $\emptyset$

Figure A.146: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(a_1+1,\ 0,\ a_3+1,\ -a_1-a_3-5) \to (a_1,\ 1,\ a_3,\ -a_1-a_3-6) \to (a_1,\ 0,\ a_3,\ -a_1-a_3-6)$$

Figure A.147: Nilpotent cohomology / BGG resolution

A.3.7 $\mathfrak{so}^*(8)$: $\omega_1 - 3\omega_4$

Cone of unitarizable weights: $(a_1+1)\,\omega_1 - (a_1+3)\,\omega_4$

$$(-a_1, \epsilon_2+\epsilon_3) \to (2, \epsilon_1+\epsilon_3), \quad (1, \epsilon_1+\epsilon_4) \to (2, \epsilon_1+\epsilon_3), \quad (2, \epsilon_1+\epsilon_3) \longrightarrow (3, \epsilon_1+\epsilon_2)$$

Figure A.148: Non-negative scalar products with noncompact roots

$\lambda = \omega_1 - 3\omega_4$
Set of singular roots: $\{\epsilon_2+\epsilon_3\}$

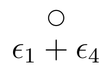

Figure A.149: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

$$(a_1+1,\ 0,\ 0,\ -a_1-3) \longrightarrow (a_1,\ 0,\ 1,\ -a_1-4)$$

Figure A.150: Nilpotent cohomology / BGG resolution

$\lambda = (a_1+1)\,\omega_1 - (a_1+3)\,\omega_4,\ a_1 \geq 1$
Set of singular roots: $\emptyset$

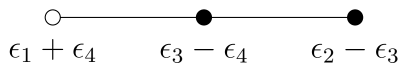

Figure A.151: The reduced Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$

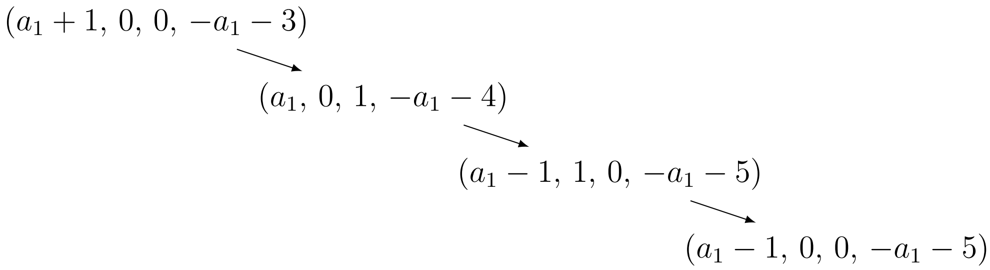

Figure A.152: Nilpotent cohomology / BGG resolution

B. Source code

The Bruhat graphs and calculations in low rank in this work were produced with the help of the following code written for the mathematical software called Sage (`www.sagemath.org`). Most of the code will be submitted for inclusion in the official distribution.

```
#######################################################################
##############               HELPER FUNCTIONS             ##############
#######################################################################

from sage.graphs.graph_latex import setup_latex_preamble
setup_latex_preamble()

def one_graded_from_dynkin(D, crossed_node):
    ct = D.cartan_type() # D might be relabeled, we need to get rid of that
    ct = CartanType([ct.type(), ct.rank()])
    index_set = list(ct.index_set())
    index_set.remove(crossed_node)
    return ct, index_set

def setup(CT, index_set):
    W = WeylGroup(CT, prefix="s")
    AS = W.domain()
    FW = AS.fundamental_weights()
    vFW = FW.map(lambda v: v.to_vector())
    RP = AS.root_poset(facade=True)
    nonparabolic_roots = [x.to_ambient() for x in
    ↪   AS.root_system.root_lattice().positive_roots_nonparabolic(index_set=index_set)]
    nRP = RP.subposet(nonparabolic_roots)
    rho = AS.rho()
    return W, AS, FW, vFW, rho, nRP

def inject_positive_integer_variables(names, p=0, q=0):
    """
    If names is a string (e.g. "A") this function injects into the workspace n
↪   variables named Ap till Aq inclusive.
    Otherwise it is assumed that names is a list of variables that will be
↪   injected.
    They are assumed to take only nonnegative integral values.
    """
    def inject_name(name, i=None):
        if i is not None:
            name = name + str(i)
        var(name)
        eval("assume({s} >= 0, ({s}, 'integer'))".format(s=name))

    if isinstance(names, basestring):
        for i in range(p, q+1):
            inject_name(names, i)
    else:
        for name in names:
            inject_name(name)

def setup_cone(variable_list, cone_str):
    """
```

```
    Deletes all assumptions, defines variables and declares them to be
↪   integral and nonnegative.

    variable_list is either a list of strings or 3-tuple containing name and
↪   range for the variables (see inject_positive_integer_variables)
    """
    forget()
    global vFW
    print("\n Initial assumptions:")
    print(assumptions())
    if isinstance(variable_list, list):
        inject_positive_integer_variables(variable_list)
    else:
        inject_positive_integer_variables(*variable_list)
    print("Cone: %s with assumptions:" % cone_str)
    print(assumptions())
    print("\n")
    exec(cone_str, globals())
    #exec(cone_str, globals(), locals())
    #exec(cone_str, locals(), globals())

class RootWithScalarProduct:
    """
    Use this to relabel graphs of positive roots with scalar product with
↪   given weight v.
    """
    def __init__(self, r, v):
        self.root = r
        self.scalarproduct = v.dot_product(r.associated_coroot().to_vector())

    def _latex_(self):
        return "(%s, %s)" % (latex(self.scalarproduct), latex(self.root))

    def __str__(self):
        return str(self.scalarproduct)

    def __repr__(self):
        return repr(self.scalarproduct)

def poset_scalar_product(poset, v, only_nonnegative=True):
    """
    Returns LaTeX code of poset of roots whose nodes were labeled by inner
↪   product of those roots with give weight v.
    """
    p = poset.relabel(lambda r: RootWithScalarProduct(r, v))
    if only_nonnegative:
        p = p.subposet([x for x in p if not(x.scalarproduct < 0)])
    if p.is_empty():
        print("Poset of scalar products is empty.")
    return p

def _fix_basis_latex(string):
    """
    Basis of Ambient spaces are indexed by e_0, ..., e_{n-1} instead of
↪   conventional \epsilon_1, ..., \epsilon_n.
    This functions returns string with fixed LaTeX source. You can render its
↪   output in notebook by calling latex.eval(...)
    """
```

```
    import re
    def shift_number(matchobj):
        return "e_{%d}" % (int(matchobj.group(1)) + 1)

    index_re = re.compile("e_{(\d+)}")
    return index_re.sub(shift_number, string).replace("e_{", "\epsilon_{")

def fix_basis_latex(obj):
    return _fix_basis_latex(str(latex(obj))).replace("0000000000000", "")

def get_poset_latex(poset, orientation="up"):
    hd = poset.hasse_diagram()
    if orientation != "up":
        hd.set_latex_options(rankdir=orientation)

    return fix_basis_latex(latex(hd))

def save_diagram(name, poset, orientation="up"):
    import os
    with open(os.path.join("diagrams", name + ".tikz"), 'w') as f:
        f.write(get_poset_latex(poset, orientation=orientation))

#######################################################################
##############  Parabolic enhancements for Weyl groups   ##############
#######################################################################

import sage.combinat.root_system.weyl_group as wg

def parabolic_bruhat_graph(self, index_set = None, side="right"):
    """
    Returns the Hasse graph of the poset ``self.bruhat_poset(index_set,side)``
↪   with edges labeled by the cover relation
    """
    elements = self.minimal_representatives(index_set, side)
    covers =[(x,y)  for y in elements for x in y.bruhat_lower_covers() if x in
    ↪   elements]
    res = DiGraph()
    for u,v in covers:
        res.add_edge(u,v,v.inverse()*u)
    return res

def parabolic_weight_graph(self, weight, index_set=None,side="right"):
    elements = self.minimal_representatives(index_set,side)
    wl0 = self.long_element(index_set)
    #covers =[(wl0*x,wl0*y)  for y in elements for x in
    ↪   y.bruhat_lower_covers() if x in elements] # funguje jen pro "right"
    covers =[(x,y)  for y in elements for x in y.bruhat_lower_covers() if x in
    ↪   elements]
    res = DiGraph()
    rho = weight.parent().rho()
    v = weight + rho
    def act_on_weight(v,x):
        return str((v.weyl_action(x) -
        ↪   rho).to_dominant_chamber(index_set).to_vector())
    for x,y in covers:
        #a = v.weyl_action(x) - rho
        #b = v.weyl_action(y) - rho
        #res.add_edge(str(a.to_vector()),str(b.to_vector()))
```

```
        a = act_on_weight(v,x)
        b = act_on_weight(v,y)
        res.add_edge(a,b)
    return res

def parabolic_weight_graph_enum(self, weight, index_set=None, side="right"):
    elements = [x for x in
    ↪   enumerate(self.minimal_representatives(index_set,side))]
    covers =[(x,y)  for y in elements for x in elements if x[1] in
    ↪   y[1].bruhat_lower_covers()]
    res = DiGraph()
    rho = weight.parent().rho()
    v = weight + rho
    def act_on_weight(v,x):
        return str((v.weyl_action(x) -
        ↪   rho).to_dominant_chamber(index_set).to_vector())
    for x,y in covers:
        a = str(x[0]) + ":" + act_on_weight(v,x[1])
        b = str(y[0]) + ":" + act_on_weight(v,y[1])
        res.add_edge(a,b)
    return res

def parabolic_poset(self, Levi_indices, side="right"):
    # returns a poset of minimal representatives of W_S \ W
    # self is a finite-dimensional Weyl group
    # first we compute orbit of the characteristic vector of our parabolic
    ↪   subalgebra
    # this is for representatives of left cosets; to obtain representatives
    ↪   for right cosets just take the inverse
    elements = self.minimal_representatives(Levi_indices, side)
    #since our Weyl elements should be already reduced (?), we could optimize
    ↪   this step by constructing the cover relations directly thus reducing
    ↪   quadratic complexity to linear
    covers = tuple([x,y]  for y in elements for x in y.bruhat_lower_covers()
    ↪   if x in elements)
    return Poset( (elements, covers), cover_relations = True)

def parabolic_weight_poset(self, weight, Levi_indices, side="right",
↪   relative_index_set=None):
    rho = weight.parent().rho()
    v = weight + rho
    elements = self.minimal_representatives(Levi_indices, side,
    ↪   relative_index_set=relative_index_set)
    covers = tuple([x,y]  for y in elements for x in y.bruhat_lower_covers()
    ↪   if x in elements)
    labels = {}
    for x in elements:
        labels[x] = str((v.weyl_action(x) -
        ↪   rho).to_dominant_chamber(Levi_indices).to_vector())
    return Poset( (elements, covers), cover_relations = True,
    ↪   element_labels=labels)

def minimal_representatives(self, index_set=None, side="right",
↪   relative_index_set=None):
    """
    Returns the set of minimal coset representatives of ``self`` by a
↪   parabolic subgroup.
```

```
    INPUT:

    - ``index_set`` - a subset (or iterable) of the nodes of the Dynkin
↪   diagram, empty by default, denotes the generators of the Levi part
    - ``side`` - 'left' or 'right' (default)
    - ``relative_index_set`` - superset of index_set for the relative Case,
↪   again determines the Levi part

    See documentation of ``self.bruhat_poset`` for more details.

    The output is equivalent to
↪   ``set(w.coset_representative(index_set,side))``

    but this routine is much faster. For explanation of the algorithm see e.g.
↪   Cap, Slovak:
    Parabolic geometries, p. 332

    EXAMPLES::

        sage: G = WeylGroup(CartanType("A4"),prefix="s")
        sage: index_set = [1,3,4]
        sage: side = "left"
        sage: a = set(w for w in G.minimal_representatives(index_set,side))
        sage: b = set(w.coset_representative(index_set,side) for w in G)
        sage: print a.difference(b)
        set([])
    """
    from sage.combinat.root_system.root_system import RootSystem
    from copy import copy

    if side != 'right' and side != 'left':
        raise ValueError, "%s is neither 'right' nor 'left'"%(side)

   #TODO check for relative_index_set being a superset of index_set

    weight_space = RootSystem(self.cartan_type()).weight_space()
    if index_set == None:
        crossed_nodes = set(self.index_set())
        relative_crossed_nodes = set()
    else:
        crossed_nodes = set(self.index_set()).difference(index_set)
        if not relative_index_set:
            relative_index_set = self.index_set()
        relative_crossed_nodes =
        ↪   set(self.index_set()).difference(relative_index_set)
    # the characteristic vector
    rhop = sum([weight_space.fundamental_weight(i) for i in crossed_nodes if
    ↪   not(i in relative_crossed_nodes)])
    '''

    The variable "todo" serves for traversing the orbit of rhop, while the
↪   directory "known" serves
elements in the orbit of rhop while known[vec] are paths of simple reflections
↪   from rhop to vec.

    '''
    todo = [rhop]
    known = dict()
```

```
    known[rhop] = []
    if rhop == 0:
        return set( [self.one()] )
    else:
        while len(todo) > 0:
            vec = todo.pop()
            nonzero_coeffs = [i for i in self.index_set() if
            ↪  (vec.coefficient(i) > 0) and (i in relative_index_set)]
            for i in nonzero_coeffs:
                new_vec = vec.simple_reflection(i)
                new_reflections = copy(known[vec])
                new_reflections.append(i)
                todo.append(new_vec)
                known[new_vec] = new_reflections
        if side =='left':
            return set(self.from_reduced_word(w) for w in known.values())
        else:
            #here we could just take the inverses of w but reversing the list
            ↪  of simple reflections
            return set(self.from_reduced_word(w[::-1]) for w in
            ↪  known.values())

def bruhat_poset(self, index_set = None, side="right", facade = False):
    from sage.combinat.posets.posets import Poset
    elements = self.minimal_representatives(index_set, side)
    # Since our Weyl elements should be already reduced (?), we could
    # optimize this step by constructing the cover relations directly (see Cap,
    ↪  Slovak: Parabolic
    # thus reducing quadratic complexity of the next step to linear. On the
    ↪  other hand, we would
    covers = tuple([x,y] for y in elements for x in  y.bruhat_lower_covers()
                   if x in elements)
    return Poset((self, covers), cover_relations = True, facade=facade)


wg.WeylGroup_gens.minimal_representatives = minimal_representatives
#wg.WeylGroup_gens.bruhat_poset = bruhat_poset
wg.WeylGroup_gens.parabolic_poset = parabolic_poset
wg.WeylGroup_gens.parabolic_bruhat_graph = parabolic_bruhat_graph
wg.WeylGroup_gens.parabolic_weight_graph = parabolic_weight_graph
wg.WeylGroup_gens.parabolic_weight_graph_enum = parabolic_weight_graph_enum
wg.WeylGroup_gens.parabolic_weight_poset = parabolic_weight_poset

######################################################################
##############   Cohomology given by root embeddings    ##############
######################################################################

def get_P_lambda(CT, index_set, side):
    """
    Returns parabolic poset for CT with Levi part given by index_set that
↪   consists of minimal representative of `side' cosets.
    """
    W = WeylGroup(CT, prefix="s")
    return W.parabolic_poset(index_set, side)

def W_lambda_weight_poset(P_lambda, embedding):
    """
    Returns poset P_lambda embedded to a bigger Weyl group through embedding
```

```
    :embedding:  tuple (simple_roots, W) where simple_roots are embeddings of
↪   simple_roots into reflections in W.
    """
    simple_roots, W = embedding
    AS = W.domain()
    for i in simple_roots.keys():
        simple_roots[i] = AS.from_vector(vector(simple_roots[i]))

#       reflections = w.parent().reflections()
#       embedding = {}
#       for i  in simple_roots:
#           embedding[i] = reflections()[simple_roots[i]]
    def embedd(w):
        reflections_word = map(lambda i: simple_roots[i], w.reduced_word())
        return W.from_reduced_word(reflections_word, word_type="all")
    print "Embedding: ", simple_roots
    return P_lambda.relabel(embedd)

def to_fundamental_weights(AS, u):
    """
    u is a dense vector over symbolic ring epsilon basis and this will return
↪   its coordinates wrt basis of fundamental weights
    should work better than matrix M as some Cartan Types (i.e. E_6) are
↪   implemented in Ambient space of higher dimension than the rank
    """
    return vector([u.dot_product(AS.simple_coroot(i).to_vector()) for i in
    ↪   AS.index_set()])


def get_dominant(AS, u, index_set):
    """
    Returns as provably dominant element in the orbit of u as possible. The
↪   orbit is taken with respect to Weyl subgroup generated by indices from
↪   index_set. If all coefficients of u are numbers it will return the
↪   dominant element.

    If there is no dominant element ends up in infinite loop! TODO BUG?
    """
    negative = True
    while negative:
        for i in index_set:
            c = u.dot_product(AS.simple_coroot(i).to_vector())
            if c < 0:
                break
        else: # we haven't found provably negative coefficient wrt fundamental
        ↪   weights
            negative = False
        if negative:
            u = AS.weyl_group().simple_reflection(i)*u
    return u

def get_W_action(AS, index_set, side, v, node_dist=1.5, with_dynkins=False):
    """
    index_set is index_set determines the Levi part of the space we are
↪   embedding into
    """
    class W_action:
        ct = AS.cartan_type()
```

```
        #M = Matrix([AS.fundamental_weight(i).to_vector() for i in
        ↪  AS.index_set()]).transpose().inverse() # change of basis from
        ↪  "epsilons" to fundamental weights

        def to_fundamental_weights(self, u):
            """
            u is a dense vector over symbolic ring epsilon basis and this will
↪   return its coordinates wrt basis of fundamental weights
            should work better than matrix M as some Cartan Types (i.e. E_6)
↪   are implemented in Ambient space of higher dimension than the rank
            """
            return vector([u.dot_product(AS.simple_coroot(i).to_vector()) for
            ↪  i in AS.index_set()])

        def __init__(self, w):
            #AS = w.domain()
            #nw = w.coset_representative(index_set, side) # Enright is wrong!
            nw = w

            self.result = self.to_fundamental_weights(get_dominant(AS,
            ↪  nw.matrix()*v - AS.rho().to_vector(), index_set))

        def __repr__(self):
            return repr(w)

        def _latex_(self):
            if with_dynkins:
                global parabolic_index_set
                parabolic_index_set = map(lambda i: latex(self.result[i-1]),
                ↪  index_set) # TODO can give wrong node fill in case there
                ↪  are repeating labels
                def labeling(i):
                    return latex(self.result[i-1])
                dynkin_latex = "\n\n \\begin{tikzpicture}\n" +
                ↪  _fix_basis_latex(self.ct._latex_dynkin_diagram(label=labeling,
                ↪  node_dist=node_dist)) + "\\end{tikzpicture}\n\n "
                return dynkin_latex
            else:
                return latex(self.result)

    return W_action

def to_fundamental_weights(AS, u):
            """
            u is a dense vector over symbolic ring epsilon basis and this will
↪   return its coordinates wrt basis of fundamental weights
            should work better than matrix M as some Cartan Types (i.e. E_6)
↪   are implemented in Ambient space of higher dimension than the rank
            """
            return vector([u.dot_product(AS.simple_coroot(i).to_vector()) for
            ↪  i in AS.index_set()])

def cohomology_poset(small_CT, small_index_set, simple_roots_embedding, W,
↪   big_index_set, v, with_dynkins=False):
    P = get_P_lambda(small_CT, small_index_set, "left")
    eP = W_lambda_weight_poset(P, (simple_roots_embedding, W))
    AS = W.domain()
```

```
    action = get_W_action(AS, big_index_set, "left", v,
    ↪  with_dynkins=with_dynkins)
    return eP.relabel(action)


#######################################################################
##############    Cohomology of unitarizable modules    ##############
#######################################################################

def WG_action(w, v):
    """
    Action of weyl group element w on vector v.
    Workaround for subgroups not containing elements of the supergroup.
    """
    AS = v.parent()
    return AS.from_vector(w.matrix()*v.to_vector())

def get_length_function(positive_roots):
    positive_roots = set(positive_roots)
    @cached_function
    def l(w):
        return len([a for a in positive_roots if WG_action(w.inverse(), -a) in
        ↪  positive_roots])
    return l

def get_generating_roots(weight, index_set):
    """
    Returns a list of roots that generate the reflection subgroup which
↪   governs cohomology of unitarizable hihgest weight modules.
    First part of Enright's formula from his paper on u-cohomology.
    The convention is that Verma modules are induced from lambda (i.e. no
↪   rho-shift)
    """
    AS = weight.parent() # the ambient space
    rho = AS.rho()
    Psi = [r for r in AS.positive_roots() if r.scalar(rho + weight) == 0]

    if AS.cartan_type()[0] in "BCG":
        print [x.is_short_root() for x in Psi]
        is_there_long_root = any(not(x.is_short_root()) for x in Psi)
    else:
        is_there_long_root = False
    print "Is there long root:", is_there_long_root
    def test_root(r):
        n = r.associated_coroot().scalar(weight + rho) # TODO check coroot
        ↪  calculations
        if is_there_long_root:
            short = r.is_short_root()
        else:
            short = True
        #print r, long_root, short
        return n.is_integer() and n > 0 and short

    nonparabolic_roots = [x.to_ambient() for x in
    ↪  AS.root_system.root_lattice().positive_roots_nonparabolic(index_set=index_set)]
    parabolic_roots = [x.to_ambient() for x in
    ↪  AS.root_system.root_lattice().positive_roots_parabolic(index_set=index_set)]
    #print("Nonparabolic roots: %s" % sorted(nonparabolic_roots))
```

```
    Phi = [r for r in nonparabolic_roots if test_root(r) and all(r.scalar(s)
    ↪   == 0 for s in Psi)]
    return Phi, Psi, parabolic_roots, nonparabolic_roots

def generate_subgroup(generators):
    """
    Keep multiplying and taking inverses as long as new elements are
↪   constructed.
    Unfortunately, this routine takes too much time in practice.
    """
    new = set(a*b for (a,b) in cartesian_product([generators,
    ↪   generators])).union(set(g.inverse() for g in generators))
    if new == generators:
        return new
    else:
        return generate_subgroup(new)

def DyerN(w):
    W = w.parent()
    return [t for t in W.reflections() if (t*w).length() < w.length()]

def DyerCoxeterGenerators(H):
    return [w for w in H if set(DyerN(w)) == set([w])]

def get_subsystem_data(weight, index_set):
    AS = weight.parent()
    W = AS.weyl_group()
    generating_roots, Psi, parabolic_roots, nonparabolic_roots =
    ↪   get_generating_roots(weight, index_set)
    reflections = W.reflections()
    generators = set(reflections[r] for r in generating_roots)

    #W_lambda = list(generate_subgroup(generators)) # subgroup generates H as
    ↪   a matrix group and we lose all the WeylGroupElement methods # too slow
    #W_lambda = [W.element_class(W, h) for h in W.subgroup(generators)] # too
    ↪   slow
    W_lambda = W.subgroup(generators)
    W_lambda_reflections = []
    for x in W_lambda:
        g = W.element_class(W, x)
        if g in reflections:
            W_lambda_reflections.append(g)

    # calculate Coxeter generators of the reflection subgroup
    # see [Deodhar] or [Dyer] for proof
    def DyerCoxeterGenerators(H_reflections):
        # optimized version
        #H_reflections = [W.element_class(W, x) for x in reflections if x in
        ↪   H] # WARNING switching H and reflections leads to empty set!
        # H_reflections = [W.element_class(W, x) for x in H if
        ↪   W.element_class(W, x) in reflections] # the previous stopped
        ↪   working in Sage 7.6 # refactored shortly thereafter to assume that
        ↪   we have only reflections at input
        W_length = get_length_function(AS.positive_roots())
        def DyerN(w):
            w = W.element_class(W, w)
            return set(t for t in H_reflections if  W_length(t*w) <
            ↪   W_length(w))
```

```
        return [w for w in H_reflections if DyerN(w) == set([w])]

    coxeter_generators = DyerCoxeterGenerators(W_lambda_reflections)

    lambda_positive_roots = [r for r in reflections.keys() if reflections[r]
    ↪  in W_lambda_reflections]
    lambda_simple_roots = [r for r in reflections.keys() if reflections[r] in
    ↪  coxeter_generators]
    lambda_parabolic_roots = [r for r in lambda_positive_roots if r in
    ↪  parabolic_roots]
    lambda_nonparabolic_roots = [r for r in lambda_positive_roots if r in
    ↪  nonparabolic_roots]

    # decompose coset representative according to their length
    from collections import defaultdict
    def is_dominant(v, positive_roots):
        return all(v.scalar(r) > 0 for r in positive_roots)
    lambda_W_c = defaultdict(list)
    rho = AS.rho()
    lambda_length = get_length_function(lambda_positive_roots)
    for w in W_lambda:
        if is_dominant(WG_action(w, rho), lambda_parabolic_roots):
            lambda_W_c[lambda_length(w)].append(w)

    return Psi, generating_roots, lambda_simple_roots, lambda_positive_roots,
    ↪  lambda_parabolic_roots, lambda_nonparabolic_roots, W_lambda,
    ↪  lambda_W_c

# small hack for LaTeXing DynkinDiagrams of generalized flag manifolds
from sage.combinat.root_system.cartan_type import CartanType_abstract as cta
def _my_latex_draw_node(self, x, y, label, position="below=4pt",
↪  fill='white'):
        r"""
        Draw (possibly marked [crossed out]) circular node ``i`` at the
        position ``(x,y)`` with node label ``label`` .
        - ``position`` -- position of the label relative to the node
        - ``anchor`` -- (optional) the anchor point for the label
        EXAMPLES::
            sage: CartanType(['A',3])._latex_draw_node(0, 0, 1)
            '\\draw[fill=white] (0 cm, 0 cm) circle (.25cm)
↪  node[below=4pt]{$1$};\n'
        """
        global parabolic_index_set
        #print parabolic_index_set, label
        fill = "black" if label in parabolic_index_set else "white"
        return "\\draw[fill={}] ({} cm, {} cm) circle (.1cm)
        ↪  node[{}]{{${}$}};\n".format(fill, x, y, position, label)
cta._latex_draw_node =  _my_latex_draw_node

def examine(weight, index_set, cone=None, only_nonnegative=True,
↪  show_diagrams=False, show_latex=False, orientation="up", cartan_type="",
↪  with_dynkins=False, **kwargs):
    Psi, generating_roots, lambda_simple_roots, lambda_positive_roots,
    ↪  lambda_parabolic_roots, lambda_nonparabolic_roots, H, lambda_W_c =
    ↪  get_subsystem_data(weight, index_set)
    print("Weight: %s" % weight)
    print("Singular roots: %s" % sorted(Psi))
```

```
    print("Set of generating roots: %s" % sorted(generating_roots))
    print("Set of generated roots: %s" % sorted(lambda_positive_roots))
    #show("Set of generated roots:", lambda_positive_roots)
    print("Simple roots: %s" % sorted(lambda_simple_roots))
    #print("Scalar products of pairs of distinct simple roots: %s" %
    ↪   set(u.scalar(v) for (u,v) in cartesian_product([lambda_simple_roots,
    ↪   lambda_simple_roots]) if u != v))

    print("Noncompact lambda-roots: %s" % lambda_nonparabolic_roots)

    AS = weight.parent()
    weight = weight
    if cone is not None:
        print("Translated cone: %s" % cone)
        v = weight.to_vector() + cone + AS.rho().to_vector() # we induce Verma
        ↪   modules from lambda and hence we need to test scalar products with
        ↪   weight shifted by rho
    else:
        v = weight.to_vector() + AS.rho().to_vector() # we induce Verma
        ↪   modules from lambda and hence we need to test scalar products with
        ↪   weight shifted by rho

    nonparabolic_roots = [x.to_ambient() for x in
    ↪   AS.root_system.root_lattice().positive_roots_nonparabolic(index_set=index_set)]
    nRP = AS.root_poset(facade=True).subposet(nonparabolic_roots)
    sP = poset_scalar_product(nRP, v, only_nonnegative=only_nonnegative)
    if sP.is_empty():
        print "Poset of nonnegative scalar products is empty."
        sP_latex = ""
    else:
        sP_latex = get_poset_latex(sP, orientation)
    if  sP_latex != "":
        if show_diagrams:
            print("Scalar products (possibly only the nonnegative ones) of the
            ↪   weight (in the cone) with noncompact roots:")
            _ = latex.eval(sP_latex)
        if show_latex:
            print sP_latex

### BUG
# DynkinDiagram calls CartanMatrix with all its arguments (*args) see line 181
↪   in dynkin_diagram.py
    # this causes error with relabeling for A1, i.e.
    ↪   DynkinDiagram(CartanMatrix([[2]]), ["ahoj"])
    # workaround here is to use Matrix instead of CartanMatrix, but note that
    ↪   the label in Dynkin diagram is wrong!
    # also, DynkinDiagram(CM, index_set=...) doesn't work as intended and
    ↪   produces labeling range(order(CM))
    # print("Cartan matrix:\n%s" % CM)
    # if len(lambda_simple_roots) > 0:
#         D = DynkinDiagram(CM, lambda_simple_roots)
#     else:
#         D = DynkinDiagram(CM, index_set=lambda_simple_roots)

# in the end, we relabel explicitely in case we have rank 1
    CM = Matrix([[rj.scalar(ri.associated_coroot()) for rj in
    ↪   lambda_simple_roots] for ri in lambda_simple_roots])
    D = DynkinDiagram(CM, lambda_simple_roots)
```

```
if len(lambda_simple_roots) == 1:
    D = D.relabel({1: lambda_simple_roots[0]})


# let's calculate the cohomology of the "shape" given by vertex of the
↪   cone
e = set(D.index_set()).intersection(lambda_nonparabolic_roots).pop() # we
↪   know that there is only one simple noncompact root in the
↪   lambda-subsystem
print("Noncompact root in the lambda-subsystem: %s" % e)
crossed_node = D.index_set().index(e) + 1
# simple_roots_embedding = dict(enumerate(D.index_set(), 1)) #BUG returns
↪   sorted index_set
simple_roots_embedding = D.cartan_type()._relabelling

ct, small_index_set = one_graded_from_dynkin(D, crossed_node)
# print ct, small_index_set, simple_roots_embedding,
↪   AS.weyl_group(),index_set, v

# There is a BUG in DynkinDiagram._latex_ which causes not very nice
↪   labeling of nodes in the Dynkin diagram
#latex.eval(fix_basis_latex(latex(D)))
#print fix_basis_latex("\n" + latex(D) + "\n")
# This is a workaround
labeling = lambda i: latex(i)

global parabolic_index_set
parabolic_index_set = [latex(simple_roots_embedding[s]) for s in
↪   small_index_set]

dynkin_latex = "\\begin{tikzpicture}\n" +
↪   _fix_basis_latex(D.cartan_type()._latex_dynkin_diagram(labeling)) +
↪   "\\end{tikzpicture}"
#dynkin_latex = dynkin_latex.replace(".25cm",
↪   ".15cm").replace("fill=white", "fill=black") # make nodes smaller and
↪   black TODO automatically make noncompact root white

if show_diagrams:
    _ =latex.eval(dynkin_latex)
if show_latex:
    print dynkin_latex

cP = cohomology_poset(
        ct, # to get rid of relabeling, we want this to be labeled by
        ↪   integers
        small_index_set,
        simple_roots_embedding,
        AS.weyl_group(prefix="s"),
        index_set, # not used right now, uniform "projection" on minimal
        ↪   representatives doesn't seem to work (i.e. Enright has a
        ↪   mistake in his paper, we take k-dominant weight in the orbit)
        v,
        with_dynkins=with_dynkins
        )
bgg_poset = get_poset_latex(cP, orientation)
if show_diagrams:
    _ = latex.eval(bgg_poset)
if show_latex:
```

```
        print bgg_poset
        #print(latex(cP))

    from string import Template

    scalar_poset_template = Template(r"""
\begin{figure}[H]
  \centering
      $scalar_poset
  \caption{Nonnegative scalar products with noncompact roots}
\end{figure}
    """)
    if sP_latex == "":
        scalar_poset = ""
    else:
        scalar_poset = scalar_poset_template.substitute(scalar_poset=sP_latex)

    template = Template(r"""

\subsubsection{$cartan_type}

Cone of unitarizable weights: $weight \\

$scalar_poset

%\noindent $$\lambda = $$ $weight \\
\noindent Set of singular roots: $singular_roots \\

\begin{figure}[H]
  \centering
  $reduced_dynkin
  \caption{The reduced hermitian symmetric pair $$(\mathfrak{g}_\lambda,
↪   \mathfrak{k}_\lambda)$$}
\end{figure}

\begin{figure}[H]
  \centering
      $bgg_poset
  \caption{Nilpotent cohomology / BGG resolution}
\end{figure}

        """)

    if Psi:
        singular_roots_latex = "$\{" + ", ".join(_fix_basis_latex(latex(r))
        ↪   for r in Psi) + "$\}"
    else:
        singular_roots_latex = "$\emptyset$"
    coefficients = [c for c in to_fundamental_weights(AS, v -
    ↪   AS.rho().to_vector())]
    F = CombinatorialFreeModule(SR, ["omega_{%d}" % i for i in range(1,
    ↪   len(coefficients)+1)], prefix="omega", latex_prefix="\\omega")
    terms = latex(sum(c*F.monomial(i) for (i,c) in enumerate(coefficients,
    ↪   1)))
    #terms = " + ".join("{c}\\omega_{i}".format(c=c, i=i) for (i, c) in
    ↪   enumerate())
    return  template.substitute(weight="$" + terms.strip() + "$",
                 singular_roots=singular_roots_latex,
```

```
 reduced_dynkin=dynkin_latex,
 scalar_poset=scalar_poset,
 bgg_poset=bgg_poset,
 cartan_type=cartan_type
)
```

C. Published articles

Construction of conformally invariant differential operators

Vít Tuček

Abstract. We present a method of computation of the explicit form of conformally invariant differential operators on $\mathbb{R}^n$ defined using the ambient metric construction. The action of the conformal group on the conformal compactification of $\mathbb{R}^n$ is realised as the action of $\mathrm{SO}(n+1,1)$ on the projectivisation of the null cone in the ambient space $\mathbb{R}^{n+1,1}$. We first review a class of differential operators on the ambient space, which give rise to the conformally invariant differential operators on $\mathbb{R}^n$, and then we show a method how to write down the explicit coefficients of the induced operator by means of a suitable adapted frame on the ambient space. The procedure gives an alternative and direct method how to compute the so called higher symmetry operators for the Laplace equation introduced by M. Eastwood.

Mathematics Subject Classification (2000). Primary 53A30; Secondary 58J70.

Keywords. symmetry operators, Laplace, conformally invariant operators.

1. Introduction

It is easily checked that the Laplace operator is invariant with respect to the group of Euclidean transformations (rotations & translations) in the sense that

$$\Delta(f \circ A) = (\Delta f) \circ A, \ \ A \in \mathrm{Iso}(\mathbb{R}^n)$$

holds for any smooth function f. Another classical result is, that one can use the spherical inversion to 'translate' solutions of the Dirichlet problem from the inside of the unit ball to the outside and vice versa. This is based on the fact that for $x' = x/\|x\|^2$ we have

$$\Delta_{(x)} u(x) = \|x\|^{2-n} \Delta_{(x')} v(x')$$

where $u(x) = \|x\|^{2-n} v(x'(x))$ and $\Delta_{(x)}$ denotes the Laplacian in coordinates x. The spherical inversion of course doesn't preserve lengths; however it preserves angles

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which makes it a conformal transformation. The problem is that it is not defined on the whole $\mathbb{R}^n$. To remedy this situation one can either work with pseudogroups or take a suitable conformal compactification. The latter approach, which we will follow in this article, goes in a similar spirit as adding the point at infinity to the complex plane to get the Riemann sphere.

Another example of an operator with similar nice properties with respect to the group of conformal transformations is the Dirac operator. Thus one is led naturally to the study of the conformally invariant (also called covariant) differential operators. There is a complete classification of conformally invariant operators on a sphere given by [7] and [1]. The articles [2] and [13] contain an extensive summary of the results. Generalisation to geometrical structures other than conformal is possible and desirable. For example the operators arising in a resolution of the Dirac operator in k-variables are invariant with respect to the group $\mathrm{Spin}(n+k,k)$ (see [9, 10]). Another example is the symplectic analogue of the Dirac operator, which belongs to a class of operators treated in [11] using the representation theoretical results of [12]. The classification of invariant operators usually boils down to decompositions of various tensor products of representations into irreducibles under the appropriate structure group. The construction of majority of these operators has been carried out also in the curved setting ([4, 3, 8]), but their coefficients are difficult to determine.

In the paper [6], M. Eastwood studied higher symmetries of the conformally invariant Laplace operator on the sphere and he constructed them using the ambient construction. In the paper, he also showed that they have curved analogues and computed their explicit form using various additional tools. The main aim of the article is to develop methods how to compute the family of these conformally invariant operators directly from their definition on the ambient space.

We use the so called abstract index notation introduced by Penrose, which is extremely convenient for performing coordinate-free computations with tensors. In this notation the indices represent the kind of object they're attached to, rather than the coordinates with respect to some basis. The upper indices denote vectors (or vector fields) while lower ones represent one-forms. Repetition of indices denotes contraction and thus one writes the natural pairing between vectors and forms as $x^a y_a$. The round and square brackets around indices stand for the symmetrisation and antisymmetrization respectively. A hat over an index or a symbol implies it's omission in an expression. In a presence of a metric tensor g_{ab} we lower and raise the indices as usual $x^a = x_b g^{ab}$. For vectors and forms on $\mathbb{R}^n$ we use lower case indices while for the ambient space $\mathbb{R}^{n+2}$ we use the upper case.

The symbol ∂_a denotes the operator which to any smooth function f assigns the one-form $\partial_a f$ for which $x^a \partial_a f$ is the derivation of f in the direction x^a. The Leibniz rule applies and consequently we have $x^a \partial_a (y^b \partial_b f) = x^a y^b \partial_a \partial_b f + x^a (\partial_a y^b) \partial_b f$. In long expressions we use $\partial_{a_1 \cdots a_s}$ as a shorthand for $\partial_{a_1} \cdots \partial_{a_s}$.

In the next section we review the classical description of the group of conformal transformations on $\mathbb{R}^{p,q}$ and we will use this description in the third section to provide a method for an explicit construction of the conformally invariant differential operators. In the last section we compute as an example the coefficients of the so called higher symmetry operators of the Laplace operator.

2. Conformal geometry and the ambient construction

There exists two equivalent approaches to conformal geometry in the setting of Riemannian manifolds. One of them uses the rather advanced notion of Cartan geometry modelled on a parabolic pair of Lie groups $(SO(n+1,1),P)$, while the other approach defines the conformal transformation as those diffeomorphisms φ of a Riemannian manifold (M,g) which preserve the metric up to a scalar multiple – i.e. $\varphi^* g = \Omega^2 g$ for some $\Omega \in \mathcal{C}^\infty(M)$ such that $\forall m \in M : \Omega(m) \neq 0$. The conformal class $[g]$ determined by a metric g is then the equivalence class of the relation $(g \simeq \tilde{g} \leftrightarrow \exists \Omega : \tilde{g} = \Omega^2 g)$. The ambient model provides a nice way to easily identify these approaches in the case of the Euclidean space.

In what follows we will work with a pseudoeuclidean space $\mathbb{R}^{p,q}$ equipped with a symmetric nondegenerate bilinear form g_{ab} of signature (p,q), $p+q=n$. The local conformal transformations are those diffeomorphisms of open sets of $\mathbb{R}^{p,q}$ which preserve angles of curves. The Liouville theorem states (see e.g. [13]) that, in the case of $n \geq 3$, every local conformal transformation on $\mathbb{R}^n$ is a composition of translations, rotations, dilatations or special conformal transformations[1] As a consequence, the conformal group is generated by these four kinds of mappings. The situation for $n=2$ is quite different since one has uncountably many local conformal transformations – the group generated by these four mappings is then sometimes called the Möbius group.

The *ambient space* of $\mathbb{R}^{p,q}$ is the direct sum $\mathbb{R}^{p+1,q+1} = \mathbb{R} \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}$ with non-degenerate symmetric bilinear form g_{AB} defined by

$$g_{AB} x^A y^B = x^0 y^\infty + x^\infty y^0 + g_{ab} x^a y^b$$

for $x^A = \left(x^0, x^a, x^\infty\right)^{\mathrm{T}}$ and $y^A = \left(y^0, y^a, y^\infty\right)^{\mathrm{T}}$.

The term ambient will be used when referring to the objects defined on some open subset of $\mathbb{R}^{n+2}$ and ambient objects will be distinguished by a tilde. For example the ambient Laplace operator is $\widetilde{\Delta} = g^{AB}\partial_A \partial_B$.

If we want to lower the index of x^A, we have

$$x_A = g_{AB} x^B = \left(x_\infty, x_a, x_0\right)$$

and as a block matrix the ambient metric takes the form $g_{AB} = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & g_{ab} & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$. It is a matter of an elementary calculation to show that the signature of the ambient metric is indeed $(p+1,q+1)$.

---

[1]Special conformal transformation is a generalisation of the circle inversion. It is given by a map $x \mapsto (x-x_0)/\|x-x_0\|^2$.

Let $r = g_{AB}x^A x^B$ be the quadratic form associated to the ambient metric g_{AB}. The *null cone* $\mathcal{N} = \{x \in \mathbb{R}^{p+1,q+1} \,|\, r(x) = 0\}$ is the zero set of r. Consider the mapping $\phi : \mathbb{R}^{p,q} \to \mathcal{N} \subset \mathbb{R}^{p+1,q+1}$ given by

$$x^a \mapsto \begin{pmatrix} 1 \\ x^a \\ -x^a x_a/2 \end{pmatrix} =: \phi^A.$$

The rays of the null cone intersect the embedded $\mathbb{R}^n$ at exactly one point and we have the following dense subset of $\mathcal{N}$

$$\mathcal{N}_0 = \left\{ \begin{pmatrix} t \\ tx^a \\ -t\frac{x^a x_a}{2} \end{pmatrix} : t \in \mathbb{R} \setminus \{0\} \right\} \cong \mathbb{R} \setminus \{0\} \times \phi(\mathbb{R}^n)$$

with an associated projection map $\pi : \mathcal{N}_0 \to \mathbb{R}^{p,q}$ defined as $\pi : \begin{pmatrix} t \\ tx^a \\ -t\frac{x^a x_a}{2} \end{pmatrix} \mapsto x^a$.

Lemma 2.1. *The triple $(\phi, \mathcal{N}, g_{AB})$ determines the conformal class of $\mathbb{R}^{p,q}$.*

Proof. For any smooth nowhere zero function Ω on $\mathbb{R}^n$ consider the subset of $\mathcal{N}$ given by $\Omega(x^a)\phi(x^a)$. This can be viewed as another embedding of $\mathbb{R}^n$ into the ambient space. The claim is that the ambient metric induces the metric $\Omega^2 g_{ab}$ on this embedded $\mathbb{R}^n$.

The tangent map of this embedding is

$$\partial_a \phi^B = \begin{pmatrix} \partial_a \Omega(x^b) \\ (\partial_a \Omega(x^b))x^b + \Omega(x^b)\delta_a{}^b \\ -(\partial_a \Omega(x^b))\frac{x^b x_b}{2} - \Omega(x^b)x_a \end{pmatrix}$$

and, denoting $\Omega_a = \partial_a \Omega(x^b)$, the pullback of the ambient metric at a point $x \in \mathbb{R}^n$ is computed as follows

$$\begin{aligned} g_{AB}\partial_c\phi^A\partial_d\phi^B &= \partial_c\phi^{(0}\partial_d\phi^{\infty)} + g_{ab}\partial_c\phi^a\partial_d\phi^b \\ &= -\Omega_c(\Omega_d\tfrac{x^b x_b}{2} + \Omega x_d) - \Omega_d(\Omega_c\tfrac{x^b x_b}{2} + \Omega x_c) + \\ &\quad + g_{ab}(\Omega_c x^a + \Omega\delta_c{}^a)(\Omega_d x^b + \Omega\delta_d{}^b) \\ &= -\Omega_c\Omega_d x^b x_b - 2\Omega\Omega_{(c}x_{d)} + \Omega^2 g_{cd} + \Omega_c\Omega_d g_{ab}x^a x^b + \\ &\quad + g_{ab}(\Omega_c x^a\Omega\delta_d{}^b + \Omega\delta_c{}^a\Omega_d x^b) \\ &= \Omega^2 g_{cd} - 2\Omega\Omega_{(c}x_{d)} + \Omega_c\Omega g_{ad}x^a + \Omega\Omega_d g_{cb}x^b \\ &= \Omega^2 g_{cd}. \end{aligned}$$

We can conclude that $\Omega\phi$ is isometrical embedding of $(\mathbb{R}^n, \Omega^2 g_{ab})$ into the ambient space and that the ambient metric induces g_{ab} in the case of $\Omega = 1$. □

It is obvious that $\mathcal{N}$ is preserved by the defining action of $\mathrm{SO}(p+1, q+1)$. The fact that is of the most importance to us is that every local conformal transformation of $\mathbb{R}^{p,q}$ is given by this action – one just multiplies the vector $\phi(x^a)$ by a $\mathrm{SO}(p+1,q+1)$ matrix and takes the projection π of the result. The projectivisation of $\mathcal{N}$ is the conformal compactification of $\mathbb{R}^{p,q}$ and the stabiliser of a line in $\mathcal{N}$ is a parabolic subgroup P of $\mathrm{SO}(p+1,q+1)$. This identifies the conformal compactification as a flat Cartan geometry of type $(\mathrm{SO}(p+1,q+1),P)$. For details see [13].

Following the idea from [5] we introduce *the adapted frame* on $\mathbb{R}^{p+1,q+1}$ in order to be able to perform efficient calculations.

Definition 2.2. For $t,\rho\in\mathbb{R}$ and $x^a\in\mathbb{R}^{p,q}$ define three vectors

$$X^A=\begin{pmatrix} t\\ tx^a\\ t(\rho-\frac{x^ax_a}{2})\end{pmatrix}\quad Y^A_b=\partial_bX^A=\begin{pmatrix}0_b\\ t\delta^a_b\\ -tx_b\end{pmatrix}\quad Z^A=-\frac{1}{n}\partial^bY^A_b=\begin{pmatrix}0\\ 0^a\\ t\end{pmatrix}.$$

If we lower the indices via the ambient metric we have

$$X_A=(t(\rho-\frac{x^ax_a}{2}),tx_a,1)\quad Y_{bA}=(-tx_b,tg_{ab},0_b)\quad Z_A=(t,0_a,0).$$

One immediately sees that X^A is the embedding $\phi(x^a)$ when $t=1$ and $\rho=0$.

Lemma 2.3. *For X^A, Y^A_b and Z^A as above we have*

$$t^2(\delta_A{}^B+2\rho Z_AZ^B)=X_AZ^B+Z_AX^B+Y^c_AY^B_c. \tag{2.1}$$

Proof. The equation (2.1) is an analogue of the standard decomposition of the identity mapping on $\mathbb{R}^n$ into the projectors to some orthonormal basis. As such it follows from the following straightforward computations in coordinates.

$$\begin{aligned}
g^{AB}X_AX_B&=2\cdot t^2(\rho-x^ax_a/2)+t^2g^{ab}x_ax_b=-t^2r+t^2r+2t^2\rho=2t^2\rho\\
g^{AB}X_AZ_B&=t\cdot t+0\cdot(-tx^ax_a/2)+tg^{ab}x_a0_b=t^2\\
g^{AB}Y^c_AY^d_B&=0^d\otimes(-tx^c)+0^c\otimes(-tx^d)+g^{ab}t\delta^c_at\delta^d_b=t^2g^{cd}\\
g^{AB}X_AY^c_B&=(-tx^ax_a/2)\cdot 0^c+(-tx^c)\cdot t+g^{ab}tx_at\delta^c_b=-t^2x^c+t^2x^c=0^c\\
g^{AB}Z_AY^c_B&=t\cdot 0^c+0\cdot(-tx^c)+g^{ab}t0_at\delta^c_b=0^c\\
g^{AB}Z_AZ_B&=2\cdot 0+g^{ab}0_a0_b=0,
\end{aligned} \tag{2.2}$$

□

If we differentiate vector field X^A with respect to the real parameter t we get $\partial_tX^A=\phi(x^a)$. Because $Y^B_cY^d_B=t^2\delta^d_c$, we only need to compute $\partial_\rho X^A=Z^A$ to see that the formula for X^A defines in fact a change of coordinates on the open half-space $\{t>0\}$ of $\mathbb{R}^{p+1,q+1}$. Consequently, the symbols X^A and x^A represent the same object – the identity vector field on $\mathbb{R}^{p+1,q+1}$. Let's explicitly define the mapping of the coordinate change:

$$\Phi(t,x^a,\rho)=\begin{pmatrix} t\\ tx^a\\ t(\rho-\frac{x^ax_a}{2})\end{pmatrix}=\begin{pmatrix}y^0\\ y^a\\ y^\infty\end{pmatrix}.$$

We see that $\phi(x^a) = \Phi(1, x^a, 0)$ and the identity (2.1) simplifies on the image of ϕ to

$$\delta_A{}^B = X_A Z^B + Z_A X^B + Y_A^c Y_c^B.$$

This identity will be of a great use later on.

Lemma 2.4. *The Euler operator in the new coordinates is equal to $\mathbb{E} = t\partial_t$.*

Proof. For $f(y^A) \in \mathcal{C}^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned} t\partial_t f(y^A(t, x^a, \rho)) &= t(\frac{\partial f}{\partial y^0} + x^a \frac{\partial f}{\partial y^a} + (\rho - \frac{x^b x_b}{2})\frac{\partial f}{\partial y^\infty}) \\ &= y^0(\frac{\partial f}{\partial y^0} + \frac{y^a}{y^0}\frac{\partial f}{\partial y^a} + \frac{y^\infty}{y^0}\frac{\partial f}{\partial y^\infty}) \\ &= (y^A \partial_A f) \circ \Phi(t, x^a, \rho) \end{aligned}$$

□

Since we want to deal with differential operators we need to incorporate smooth functions on $\mathbb{R}^{p,q}$ into the picture. Suppose that f is a smooth function defined on the neighbourhood of origin in $\mathbb{R}^{p,q}$. Then for any $w \in \mathbb{C}$

$$\tilde{f}(\Phi(t, x^a, 0)) = t^w f(x^a) \tag{2.3}$$

defines a smooth function on a 'conical neighbourhood' of $(1, 0, 0)$ inside the null cone $\mathcal{N}$. Moreover it is a homogeneous function of degree w because $\tilde{f}(\lambda y^A) = \lambda^w \tilde{f}(y^A)$ for $\lambda > 0$. Conversely f may be recovered from $\tilde{f}$ by setting $t = 1$. In order to be able to apply ambient differential operators to $\tilde{f}$ we need to extend it from the null cone to the whole space or at least to some open (in $\mathbb{R}^{p+1,q+1}$) neighbourhood of $(1, 0, 0)$. We will call any such extension *ambient extension.* There are infinitely many choices for such an extension even if we stick to the homogeneous ones. Nevertheless, any two such extensions will differ by a very convenient factor.

Lemma 2.5. *Let $\tilde{f}$ and $\hat{f}$ be two smooth w-homogeneous extensions of f on some open neighbourhood of $(1, 0, 0)$ not containing zero. Then there exist a smooth $(w-2)$-homogeneous function h such that $(\tilde{f} - \hat{f})(y^A) = r(y^A)h(y^A)$ where r is the defining quadric of the null cone.*

Proof. For any smooth function k on $\mathbb{R}^{p+1,q+1}$ holds

$$k(t, x^a, \rho) = k(t, x^a, 0) + \int_0^1 \frac{d}{ds} k(t, x^a, s\rho)\mathrm{d}s = k(t, x^a, 0) + \rho \int_0^1 \frac{\partial k}{\partial \rho}(t, x^a, s\rho)\mathrm{d}s.$$

According to the first equation of (2.2) we have $\rho = r/2ti^2$. For $k = (\tilde{f} - \hat{f}) \circ \Phi$ we have $k(t, x^a, 0) = 0$ and the result follows by substitution with Φ^{-1}. □

Remark 2.6. The classical chain rule formula, with regard to (2.3), gives

$$\partial_a f = \partial_a(\tilde{f} \circ \phi) = (\partial_a \phi^B)(\partial_B \tilde{f}) \circ \phi = (Y_a^B \partial_B \tilde{f}) \circ \phi \tag{2.4}$$

for any ambient extension $\tilde{f}$ of f because $\partial_a \phi^B$ equals Y_a^B on the image of ϕ. Using this expression for $f = Y_b^A \circ \phi$ we get

$$(-t g_{cb} Z^A) \circ \phi = \partial_c (Y_b^A \circ \phi) = (Y_c^D \partial_D Y_b^A) \circ \phi. \tag{2.5}$$

3. Construction of conformally invariant differential operators

Let $(\mathbb{V}_1, \varrho_1), (\mathbb{V}_2, \varrho_2)$ be two representations of a Lie group G and let G have a smooth action on a manifold M. One defines the induced action of G on smooth functions $\mathcal{C}^\infty(M, \mathbb{V}_i)$ by $(g \cdot f)(x) = \varrho_i(g)(f(g^{-1})x)$. The G-invariant differential operators are defined as those differential operators which are equivariant with respect to the induced action of G.

In our case, the Lie group in question is $\mathrm{SO}(p+1, q+1)$ and the underlying manifold is $\mathcal{N}$. As a consequence, we may find the conformally invariant operators among the orthogonally invariant ones. An ambient differential operator $\tilde{D}$ induces an operator on $\mathbb{R}^n$ if and only if the value of $(\tilde{D}\tilde{f})\circ\phi$ doesn't depend on the ambient extension $\tilde{f}$ of f (i.e. $\tilde{D}\tilde{f} = \tilde{D}\hat{f}$). For a linear operator, this condition is equivalent to $\tilde{D}(\tilde{f} - \hat{f}) = \tilde{D}(rh) = 0$. Of course such a condition can hold only for some weights w of the extension and not for the other weights. The conclusion is that a conformally invariant linear differential operator is induced by a $\mathrm{SO}(p+1, q+1)$-invariant operator $\tilde{D}$ for which there exists a weight $w \in \mathbb{C}$ such that

$$[\tilde{D}, r]\tilde{f} = \tilde{D} r \tilde{f} - r \tilde{D} \tilde{f} = 0.$$

Suppose we are given such an operator $\tilde{D}$ with leading term $S^{A_1 \cdots A_i}(x^A)\partial_{A_1 \cdots A_i}$. In order to find the coefficients of the induced operator we use the adapted frame (2.1) and write $\tilde{D} = S^{A_1 \cdots A_i}(x^A) \delta_{A_1}{}^{B_1} \cdots \delta_{A_i}{}^{B_i} \partial_{B_1} \cdots \partial_{B_i} + \mathrm{LOT}$ which on the image of ϕ simplifies to

$$S^{A_1 \cdots A_i}(x^a)(X_{A_1} Z^{B_1} + Z_{A_1} X^{B_1} + Y^c_{A_1} Y_c^{B_1}) \cdots (X_{A_i} Z^{B_i} + Z_{A_i} X^{B_i} + Y^c_{A_i} Y_c^{B_i}) \partial_{B_1 \cdots B_i},$$

since we already assume that the result doesn't depend on the ambient extension and hence we can drop all the terms containing ϱ. Also the terms containing Z^{B_i} can be omitted as well, because they represent derivatives in the direction transversal to the null cone and the result doesn't depend on the ambient extension.

Let us compute the expression for the operator induced by the ambient Laplace operator as an illustration of this method,

$$\begin{aligned}
\left(g^{AB}\partial_A \partial_B \tilde{f}\right) \circ \phi &= \left(g^{AB} \delta_A{}^C \delta_B{}^D \partial_C \partial_D \tilde{f}\right) \circ \phi \\
&= \Big[g^{AB} \frac{1}{t^2} (X_A Z^C + Z_A X^C + Y^q_A Y^C_q - \rho U_A{}^C) \cdot \\
&\quad \cdot \frac{1}{t^2} (X_B Z^D + Z_B X^D + Y^r_B Y^D_r - \rho U_B{}^D) \partial_C \partial_D \tilde{f} \Big] \circ \phi \\
&= [(Z^C X^D + X^C Z^D) \partial_C \partial_D f + g^{qr} Y^C_q Y^D_r \partial_C \partial_D \tilde{f}] \circ \phi. \qquad \text{by (2.2)}
\end{aligned}$$

Because we have $[\partial_A, \partial_B] = 0$, the first term in the last expression equals

$$\begin{aligned} 2Z^C X^D \partial_C \partial_D &= 2\left(Z^C \partial_C X^D \partial_D - Z^C (\partial_C X^D)\partial_D\right) \\ &= 2\left(Z^C \partial_C X^D \partial_D - Z^C ({\delta_C}^D)\partial_D\right) \\ &= 2Z^C \partial_C (\mathbb{E} - 1) \end{aligned}$$

applied to $\tilde{f}$ and evaluated on the image of ϕ. The second term is

$$\begin{aligned} \left(g^{qr} Y_q^C Y_r^D \partial_C \partial_D \tilde{f}\right) \circ \phi &= \left(g^{qr} [Y_q^C \partial_C Y_r^D \partial_D - Y_q^C (\partial_C Y_r^D)\partial_D] \tilde{f}\right) \circ \phi \\ &= \left(g^{qr} Y_q^C \partial_C (Y_r^D \partial_D \tilde{f})\right) \circ \phi + \left(g^{qr} g_{qr} Z^D \partial_D \tilde{f}\right) \circ \phi \quad \text{by (2.5)} \\ &= \Delta f + n(Z^D \partial_D \tilde{f}) \circ \phi. \end{aligned}$$

Hence

$$(\widetilde{\Delta} \tilde{f}) \circ \phi = \Delta f + \left(Z^D \partial_D (n + 2\mathbb{E} - 2)\tilde{f}\right) \circ \phi$$

and we see that, for $w = 1 - n/2$, the ambient Laplace operator on $\mathbb{R}^{p+1,q+1}$ induces the Laplace operator on $\mathbb{R}^{p,q}$.

We see that the computation of the coefficients boils down to two parts – calculation of the contractions with the symbol of the operator and calculation of the contractions with the differentials. For the latter part we can record here the following lemma.

Lemma 3.1. *Let $D(k, s)$ be an operator defined as*

$$D(k, s) = X^{D_1} \cdots X^{D_k} Y_{c_{k+1}}^{D_{k+1}} \cdots Y_{c_s}^{D_s} \partial_{D_1} \cdots \partial_{D_s}.$$

Then modulo the terms depending on the ambient extension we have for w-homogeneous functions

$$(D(k, s)\tilde{f}) \circ \phi = \left[\prod_{i=1}^{k} (w - s + i)\right] \partial_{c_{k+1}} \cdots \partial_{c_s} f.$$

Proof. Let $T(k)$ denote the operator $X^{D_1} \cdots X^{D_k} \partial_{D_1} \cdots \partial_{D_k}$. We have

$$\begin{aligned} T(k) &= X^{D_1} \cdots X^{D_{k-2}} X^{D_{k-1}} X^{D_k} \partial_{D_1} \cdots \partial_{D_k} \\ &= X^{D_1} \cdots X^{D_{k-2}} X^{D_{k-1}} (\partial_{D_1} X^{D_k} - {\delta_{D_1}}^{D_k}) \partial_{D_2} \cdots \partial_{D_k} \\ &= X^{D_1} \cdots X^{D_{k-2}} X^{D_{k-1}} \partial_{D_1} X^{D_k} \partial_{D_2} \cdots \partial_{D_k} f - T(k-1) \\ &= X^{D_1} \cdots X^{D_{k-2}} (\partial_{D_1} X^{D_{k-1}} - {\delta_{D_1}}^{D_{k-1}}) X^{D_k} \partial_{D_2} \cdots \partial_{D_k} - T(k-1) \\ &= X^{D_1} \cdots X^{D_{k-2}} \partial_{D_1} X^{D_{k-1}} X^{D_k} \partial_{D_2} \cdots \partial_{D_k} - 2T(k-1) \\ &\;\;\vdots \\ &= X^{D_1} \partial_{D_1} X^{D_2} \cdots X^{D_k} \partial_{D_2} \cdots \partial_{D_k} - (k-1)T(k-1) \\ &= \mathbb{E}\, T(k-1) - (k-1)T(k-1) = (\mathbb{E} - k + 1)T(k-1) \end{aligned}$$

Since $T(1) = X^{D_1}\partial_{D_1} = \mathbb{E}$ we see that

$$T(k) = X^{D_1}\cdots X^{D_k}\partial_{D_1}\cdots\partial_{D_k} = (\mathbb{E}-k+1)(\mathbb{E}-k+2)\cdots(\mathbb{E}-1)\,\mathbb{E}\,.$$

We can view Y_b^A as a 1-homogeneous function on $\mathbb{R}^n$ with values in $\mathbb{R}^{n+2}$ because the homogeneity in the standard coordinates translates to homogeneity in t by the lemma (2.4). Therefore the Euler operator acts as the identity on Y_c^D which implies $[\mathbb{E}, Y_{c_i}^{D_i}] = 0$. Using this fact we can write $D(k,s) = T(k)Y_{c_{k+1}}^{D_{k+1}}\cdots Y_{c_s}^{D_s}\partial_{D_{k+1}\cdots D_s}$.

Iterating the formula (2.4) we get

$$\partial_{c_1}\cdots\partial_{c_k} f = (Y_{c_1}^{D_1}\partial_{D_1}(Y_{c_2}^{D_2}\partial_{D_2}(\cdots\partial_{D_{k-1}}(Y_{c_k}^{D_k}\partial_{D_k}\tilde{f}))\cdots)\circ\phi.$$

Since $Y_c^D\partial_D Y_b^A = -tg_{cb}Z^A$ by (2.5), we see that the difference between the above expression and the formula $Y_{c_1}^{D_1}\cdots Y_{c_k}^{D_k}\partial_{c_1}\cdots\partial_{c_k}$ yields a differentiation of f in the direction transversal to the embedding ϕ that clearly depends on the choice of an ambient extension.

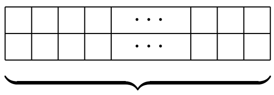
Since each differentiation lowers the homogeneity by one, $T(k)$ acts in the expression for $(D(k,s)\tilde{f})\circ\phi$ on $w-(s-k)$-homogeneous function and the result follows. □

4. Symmetry operators of the Laplace equation

As an application of the just presented method we compute the so called higher symmetry operators for the Laplace equation. We say that a linear differential operator D is symmetry operator of the Laplace equation if there exists another linear differential operator δ such that $\Delta D = \delta\Delta$. It is easy to see that these operators preserve the space of harmonic functions. It was shown in [6] that, modulo the trivial symmetry operators of the form $D\Delta$, all the symmetry operators are induced from the ambient operators of the form $\mathcal{D}_V := V^{A_1B_1\cdots A_sB_s}X_{A_1}\cdots X_{A_s}\partial_{B_1}\cdots\partial_{B_s}$ where

$$V^{A_1B_1\cdots A_sB_s} \in \bigotimes^{2s}\mathbb{R}^{n+2}$$

is a tensor that is skew in each pair of indices A_iB_i, is totally trace-free, and such that skewing over any three indices gives zero. It follows that $V^{A_1B_1\cdots A_sB_s}$ is symmetric with respect to a change of the form $A_iB_i \leftrightarrow A_jB_j$ and that symmetrising over any $s+1$ indices gives zero. These symmetries can be expressed by a Young tableau



trace-free part.

s boxes in each row

From these symmetry properties it is easy to prove that $\mathcal{D}_V$ commutes with r and with $\widetilde{\Delta}$. Since it also preserves homogeneity it follows that the operator induced on $(1-n)/2$-homogeneous functions is a symmetry operator with δ being the operator induced on $(-3-n)/2$-homogeneous functions.

Theorem 4.1. *Let $V^{A_1B_1\cdots A_sB_s}$ be a tensor with the aforementioned symmetries and let*

$$V^{c_1\cdots c_s} = \left(V^{A_1B_1\cdots A_sB_s} X_{A_1}\cdots X_{A_s} Y^{c_1}_{B_1}\cdots Y^{c_s}_{B_s}\right)\circ\phi.$$

Let f be a smooth function on $\mathbb{R}^n$ and let $\tilde{f}$ be its w-homogeneous extension on some open neighbourhood of $\phi(\mathbb{R}^n)$. The operator on $\mathbb{R}^n$ defined by $\mathrm{D}^w_V f = (\mathcal{D}_V\tilde{f})\circ\phi$ equals to

$$\mathrm{D}^w_V f = \sum_{k=0}^{s}(-1)^k\tbinom{s}{k}\left(\prod_{i=1}^{k}\tfrac{w-s+i}{n+2s-1-i}\right)(\partial_{c_1}\cdots\partial_{c_k}V^{c_1\cdots c_s})\partial_{c_{k+1}}\cdots\partial_{c_s}f. \tag{4.1}$$

Proof. We apply the method described in the previous section and discard the terms containing ϱ and Z^{D_i}. We arrive at the following equality on the image of ϕ

$$\mathcal{D}_V\tilde{f} = V^{A_1B_1\cdots A_sB_s}X_{A_1}\cdots X_{A_s}\sum_{\substack{I,J\subseteq\{1,\dots s\}\\ I\cap J=\emptyset}}\left(\prod_{i\in I}Z_{B_i}X^{D_i}\prod_{j\in J}Y^{c_j}_{B_j}Y^{D_j}_{c_j}\right)\partial_{D_1\cdots D_s}\tilde{f}.$$

Because the tensor $X_{A_1}\cdots X_{A_s}$ is symmetric and $V^{A_1B_1\cdots A_sB_s}$ is symmetric in pairs A_iB_i, we can write

$$\begin{aligned}V^{A_1B_1\cdots A_sB_s}X_{A_1}\cdots X_{A_s}&\left(\prod_{i\in I}Z_{B_i}X^{D_i}\prod_{j\in J}Y^{c_j}_{B_j}Y^{D_j}_{c_j}\right) = \\ &= V^{A_1B_1\cdots A_sB_s}X_{A_1}\cdots X_{A_s}\left(\prod_{i=1}^{k}Z_{B_i}X^{D_i}\prod_{j=k+1}^{s}Y^{c_j}_{B_j}Y^{D_j}_{c_j}\right)\end{aligned}$$

for any two disjoint subsets I, J of $\{1,\dots,s\}$ whose union is the whole set and where I has cardinality k. For brevity we introduce the symbols $X_{A_1\cdots A_s}$, $Y^{c_1\cdots c_s}_{B_1\cdots B_s}$, $Z_{B_1\cdots B_s}$ as shorthands for $X_{A_1}\cdots X_{A_s}$ etc.

So far, we've got the following expression for $\mathcal{D}_V$ on the image of ϕ

$$\sum_{k=0}^{s}\binom{s}{k}V^{A_1B_1\cdots A_sB_s}X_{A_1\cdots A_s}Z_{B_1\cdots B_k}Y^{c_{k+1}\cdots c_s}_{B_{k+1}\cdots B_s}X^{D_1\cdots D_k}Y^{D_{k+1}\cdots D_s}_{c_{k+1}\cdots c_s}\partial_{D_1}\cdots\partial_{D_s}\tilde{f},$$

where it is understood that for $k=0$ the term under the sum equals to

$$V^{A_1B_1\cdots A_sB_s}X_{A_1\cdots A_s}Y^{c_1}_{B_1}\cdots Y^{c_s}_{B_s}X^{D_1}\cdots X^{D_k}Y^{D_{k+1}}_{c_{k+1}}\cdots Y^{D_s}_{c_s}\partial_{D_1}\cdots\partial_{D_s}f.$$

and analogously there are only 'Z terms' for $k=s$.

Let $S(k) = V^{A_1B_1\cdots A_sB_s}X_{A_1\cdots A_s}Z_{B_1\cdots B_k}Y^{c_{k+1}\cdots c_s}_{B_{k+1}\cdots B_s}$ be the symbol part of the operator. Using the chain rule and the Leibniz rule we get

$$\begin{aligned}\partial_{c_{k+1}}(S(k)\circ\phi) &= (Y^D_{c_{k+1}}\partial_D S(k))\circ\phi \\ &= \Big[V^{A_1B_1\cdots A_sB_s}Y^D_{c_{k+1}}\big(\partial_D(X_{A_1\cdots A_s})Z_{B_1\cdots B_k}Y^{c_{k+1}\cdots c_s}_{B_{k+1}\cdots B_s} + \\ &\quad + X_{A_1\cdots A_s}Z_{B_1\cdots B_k}\partial_D(Y^{c_{k+1}\cdots c_s}_{B_{k+1}\cdots B_s})\big)\Big]\circ\phi,\end{aligned}$$

because $Y_a^D \partial_D Z_B$ equals zero by a straightforward computation.

Using the identity $Y_{c_1}^D \partial_D X_{A_i} = Y_{c_1 A_i}$ and the Leibniz rule again we write the first summand as

$$\sum_{i=1}^{s} (V^{A_1 B_1 \cdots A_s B_s} X_{A_1 \cdots \widehat{A_i} \cdots A_s} (Y_{c_1}^D \partial_D X_{A_i}) Z_{B_1 \cdots B_k} Y_{B_{k+1} \cdots B_s}^{c_{k+1} \cdots c_s}) \circ \phi.$$

Plugging in the identity $Y_{c_1}^D \partial_D X_{A_i} = Y_{c_1 A_i}$ we can simplify it to

$$\sum_{i=1}^{s} \Big(V^{A_1 B_1 \cdots A_s B_s} X_{A_1 \cdots \widehat{A_i} \cdots A_s} (g_{A_i B_1} - X_{A_i} Z_{B_1} - Z_{A_i} X_{B_1}) Z_{B_1 \cdots B_k} Y_{B_{k+1} \cdots B_s}^{c_{k+1} \cdots c_s} \Big) \circ \phi,$$

because $Y_{c_1 A_i} Y_{B_1}^{c_1}$ equals to $g_{A_i B_1} - X_{A_i} Z_{B_1} - Z_{A_i} X_{B_1}$ on the image of ϕ due to the (2.1). Using trace-freeness of $V^{A_1 B_1 \cdots A_s B_s}$ and its antisymmetry in $A_1 B_1$, we see that for $i \neq 1$ the only nontrivial contraction is with the term $X_{A_i} Z_{B_1}$; for $i = 1$ we contract V with the tensor field $Z_{A_1} X_{B_1} + X_{A_1} Z_{B_1}$ which is symmetric in its indices. Therefore the first summand equals to $-(s-1)S(k+1)$.

With the help of the identity (2.5), the second summand yields

$$\sum_{i=k+1}^{s} \Big(V^{A_1 B_1 \cdots A_s B_s} X_{A_1 \cdots A_s} Z_{B_1 \cdots B_k} Y_{B_{k+1}}^{c_{k+1}} \cdots \widehat{Y_{B_i}^{c_i}} \cdots Y_{B_s}^{c_s} (-\delta_{c_1}{}^{c_i} Z_{B_i}) \Big) \circ \phi.$$

For $i = 1$ the contraction results in $-nS(k+1)$ whereas for $i \neq 1$ we get $-S(k+1)$ because of the symmetry in pairs $A_1 B_1 \leftrightarrow A_i B_i$. Thus the second summand equals to $-(n+s-k-1)S(k+1)$. Putting it all together we arrive at

$$\partial_{c_{k+1}} (S(k) \circ \phi) = -(n + 2s - 2 - k) S(k+1) \circ \phi.$$

Since $S(0) \circ \phi = V^{c_1 \cdots c_s}$, we see that

$$S(k+1) = \frac{(-1)^{k+1}}{(n+2s-2) \cdots (n+2s-2-k)} \partial_{c_{k+1}} \cdots \partial_{c_1} V^{c_1 \cdots c_s}$$

and using the lemma 3.1 we finally conclude

$$\begin{aligned}
\mathrm{D}_V^w f &= (\mathcal{D}_V \tilde{f}) \circ \phi \\
&= \left(\sum_{k=0}^{s} \binom{s}{k} S(k) D(k,s) \tilde{f} \right) \circ \phi \\
&= \sum_{k=0}^{s} \binom{s}{k} S(k) \circ \phi \cdot (D(k,s) \tilde{f}) \circ \phi \\
&= \sum_{k=0}^{s} \binom{s}{k} A(k,s,w) (\partial_{c_k} \cdots \partial_{c_1} V^{c_1 \cdots c_k}) \partial_{c_{k+1}} \cdots \partial_{c_s} f
\end{aligned}$$

where

$$A(k,s,w) = (-1)^k \frac{(w-s+1) \cdots (w-s+k)}{(n+2s-2) \cdots (n+2s-1-k)}.$$

□

Remark 4.2. The formula (4.1) agrees with the formula on the page 1659 of [6], where the author uses a rather sophisticated notion of naturality in order to obtain the result.

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Vít Tuček
MÚUK
Sokolovská 49/83
Praha
186 00
e-mail: `vit.tucek@gmail.com`

Hyperplane section $\mathbb{OP}_0^2$ of the complex Cayley plane as the homogeneous space $\mathrm{F}_4/\mathrm{P}_4$

Karel Pazourek, Vít Tuček, Peter Franek

Abstract. We prove that the exceptional complex Lie group F_4 has a transitive action on the hyperplane section of the complex Cayley plane $\mathbb{OP}^2$. Although the result itself is not new, our proof is elementary and constructive. We use an explicit realization of the vector and spin actions of $\mathrm{Spin}(9,\mathbb{C}) \leq \mathrm{F}_4$. Moreover, we identify the stabilizer of the F_4-action as a parabolic subgroup P_4 (with Levi factor $\mathrm{B}_3\mathrm{T}_1$) of the complex Lie group F_4. In the real case we obtain an analogous realization of $\mathrm{F}_4{}^{(-20)}/\mathrm{P}_4$.

Keywords: Cayley plane, octonionic contact structure, twistor fibration, parabolic geometry, Severi varieties, hyperplane section, exceptional geometry

Classification: Primary 32M12; Secondary 14M17

1. Introduction

The real octonionic projective plane $\mathbb{OP}^2_{\mathbb{R}}$, also called Cayley plane or octave plane, has been thoroughly treated in the literature. It appears in numerous contexts. It is a projective plane where the Desargues axiom does not hold. It was firstly considered by Ruth Moufang [21], who found a relation of the so called little Desargues axiom and the alternativity of the coordinate ring. It is well known that $\mathbb{OP}^2_{\mathbb{R}}$ is a Riemanian symmetric manifold $\mathrm{F}_4/\mathrm{Spin}(9)$. Due to its relation to the exceptional Jordan algebra $\mathcal{J}_3(\mathbb{O})$, there is also a connection of this plane to a model of quantum mechanics considered by Neumann, Jordan and Wigner [14]. More recently, the authors of [7] show that the Cayley plane consists of normalized solutions of a Dirac equation. For more details and connections with physics we refer to the article by Baez [3].

It is possible to mimic the construction of classical projective plane $\mathbb{RP}^2$ via equivalence classes of triples (see [11]) also in the case of $\mathbb{OP}^2_{\mathbb{R}}$, but usually Freudenthal's approach via the exceptional Jordan algebra $\mathcal{J}_3(\mathbb{O})$ is used. The idea is that lines in space correspond to projectors with one-dimensional image. Hence the Cayley plane can be defined as elements of (real) projectivization of $\mathcal{J}_3(\mathbb{O})$ of rank

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one. Now the rank for octonionic matrices is a bit tricky due to the nonassociativity and requires the definition of Jordan cross product of these matrices. For details we refer to Jacobson's monograph [13]. There one can also find a classification of orbits of the automorphism group of $\mathcal{J}_3(\mathbb{O})$ (which is F_4) from which it follows that $\mathbb{OP}^2_{\mathbb{R}}$ is a homogeneous space. (The isotropy subgroup is determined for example in [10], [22].)

In fact, Jacobson's book [13] treats octonionic algebras over general field and hence we get the definition of the complex Cayley plane $\mathbb{OP}^2$ as well. This space is also of geometric interest, as it is an exceptional member of the Severi varieties — the unique extremal varieties for secant defects. For details see [18], [19].

Now, let us consider the intersection of the complex Cayley plane $\mathbb{OP}^2$ with the hyperplane given by traceless matrices $\mathcal{J}_0 := \{A \in \mathcal{J}_3(\mathbb{O}_{\mathbb{C}}) \mid \operatorname{Tr} A = 0\}$. The resulting space is studied in [18], [19], where the authors call it the generic hyperplane section and denote it by $\mathbb{OP}^2_0$. It is a total space of a certain twistor fibration over the real Cayley plane (see [2], [8]). Because $\mathbb{OP}^2_0$ is a complex projective variety, the stabilizer is a parabolic subgroup of F_4. The authors of [18] state that the isomorphism $\mathbb{OP}^2_0 = \mathrm{F}_4/\mathrm{P}_4$ is suggested by 'geometric folding'. A rigorous proof of this isomorphism can be gleaned from [13]. This proof however requires a lot of the theory of nonassociative algebras, most notably the Jordan coordinization theorem. Quite a short proof can be given using the Borel fixed point theorem. In a hope to make $\mathbb{OP}^2_0$ more accessible to geometrically inclined audience, we present a *constructive* proof of the transitivity of the action of F_4 on $\mathbb{OP}^2_0$ based on the representation theory of complex spin groups. From the theory of nonassociative algebras only Artin's theorem is needed. Following the approach of [10], we explicitly realize the spin groups $\mathrm{Spin}(9,\mathbb{C})$ and $\mathrm{Spin}(8,\mathbb{C})$ as subgroups of F_4 and we use the description of their actions to find the reduction of an arbitrary element to a previously chosen one.

It is well known that the Cartan geometry modeled on the pair $(\mathrm{F}_4, \mathrm{P}_4)$ is rigid, i.e. any regular normal Cartan geometry of this type is locally isomorphic to the homogeneous model. The real version of this pair corresponding to the group $\mathrm{F}_4^{(-20)}$ appears as a conformal infinity of the Einstein space $\mathbb{O}\mathrm{H}^2$ [4]. The geometry obtained is called 'octonionic-contact', because there is a naturally defined eight-dimensional maximally nonintegrable subbundle of the tangent bundle. The contact geometry in the classical sense (studied for example in [15], [16]) is also present among the homogeneous spaces of the group F_4 — namely the one whose isotropy group is the parabolic subgroup corresponding to the other 'outer' simple root of the Lie algebra of $\mathfrak{f}_4$.

After some necessary definitions in Section 2, we describe explicitly the presentations of $\mathrm{Spin}(9,\mathbb{C})$ and $\mathrm{Spin}(8,\mathbb{C})$ inside of $\mathrm{End}(\mathbb{O}^2) \otimes_{\mathbb{R}} \mathbb{C}$ in Section 3. We also explicitly describe vector and spinor representations of $\mathrm{Spin}(9,\mathbb{C})$ in such a way that their image is inside F_4. Section 3 continues with the proof of the transitivity of the action of F_4 on $\mathbb{OP}^2_0$. We conclude by dealing with the real case. In the last section we compute the stabilizer of a point.

2. Notations and definitions

2.1 Complexified octonions and the hyperplane section. For a comprehensive reference on octonionic algebras over any field we refer to [22]. We denote by $\mathbb{O}$ the octonionic algebra over the field of complex numbers. The complex-valued 'norm' on $\mathbb{O}$ is denoted by N. The algebra $\mathbb{O}$ is normed ($N(ab) = N(a)N(b)$) but it fails to be a division ring, since N is isotropic. This algebra is not associative. Nevertheless, it is alternative, which means that the trilinear form (called the associator) $[u, v, w] \mapsto (uv)w - u(vw)$ is completely skew-symmetric. Later on we will use the so called *Artin's theorem* which states that any subalgebra of an alternative algebra generated by two elements is associative. It follows that products involving only two elements can be written without parenthesis unambiguously.

The symbol L_u denotes the operator of left multiplication by u, i.e. $L_u(v) := uv$ for any $v \in \mathbb{O}$. Note that $L_u L_v \neq L_{uv}$ in general due to the nonassociativity of octonionic algebras.

Since there is up to isomorphism only one octonionic algebra over $\mathbb{C}$ we can think of $\mathbb{O}$ in the following way: $\mathbb{O} = \mathbb{O}_{\mathbb{R}} \otimes \mathbb{C} = \mathbb{O}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathbb{O}_{\mathbb{R}}$ is the classical real algebra of octonions ([3]). The multiplication on this tensor product is canonically defined by

$$(o_1 \otimes z_1)(o_2 \otimes z_2) := o_1 o_2 \otimes z_1 z_2 \text{ for } o_1,\, o_2 \in \mathbb{O}, z_1, z_2 \in \mathbb{C}$$

and conjugation is given by $\overline{o \otimes z} := \bar{o} \otimes z$.

The multiplication of an arbitrary element $o \otimes z \in \mathbb{O}$ by a complex number w is understood in the sense of multiplication by element $1 \otimes w$, i.e. $w(o \otimes z) := o \otimes (wz)$. We identify the elements of $\mathbb{R} \otimes \mathbb{C}$ with complex numbers under the canonical isomorphism $r \otimes w \mapsto rw$, for $r \in \mathbb{R}$, $w \in \mathbb{C}$. The real and imaginary parts of $o \otimes z$ are defined to be $(\Re\, o) \otimes z$ and $(\Im\, o) \otimes z$, where $\Re\, o$ and $\Im\, o$ are the real and purely imaginary part of o respectively.

The mentioned complex valued quadratic form N is given by

$$N(o \otimes z) := o\bar{o}zz, \qquad o \in \mathbb{O}, z \in \mathbb{C}.$$

Following Springer [22], we denote by $\langle \cdot, \cdot \rangle$ the double of the bilinear form associated to N, $\langle x, y \rangle = N(x+y) - N(x) - N(y)$. An octonion $u \in \mathbb{O}$ is pure imaginary if and only if $\langle u, 1 \rangle = 0$.

For later use, we will record here several useful identities which hold in any octonionic algebra and whose proof can also be found in [22]

$$\langle xy, z \rangle = \langle y, \bar{x}z \rangle$$

$$x(\bar{x}y) = N(x)y \tag{1a}$$

$$u(\bar{x}y) + x(\bar{u}y) = \langle u, x \rangle y \tag{1b}$$

$$u(\bar{x}(uy)) = ((u\bar{x})u)y. \tag{1c}$$

Due to the nonassociativity of the algebras involved we need to make clear distinction between associative algebras of $\mathbb{C}$-linear endomorphisms, which we denote by End, and the possibly nonassociative algebras of $n \times n$ matrices with entries in some algebra $\mathbb{F}$ which are denoted by $M(n, \mathbb{F})$.

The conjugation on $\mathbb{O}$ naturally defines the conjugation on $M(n, \mathbb{O})$. The conjugate of an element $A \in M(n, \mathbb{O})$ is denoted by $\bar{A}$. The symbol $\mathrm{Herm}(n, \mathbb{O})$ stands for the set of $n \times n$ hermitian matrices over $\mathbb{O}$, i.e.

$$\mathrm{Herm}(n, \mathbb{O}) = \{A \in M(n, \mathbb{O}) | \bar{A}^T = A\}.$$

We denote the subspace of trace-free matrices by lower index $\mathrm{Herm}_0(n, \mathbb{O})$. All tensor products in this article are taken over the real numbers.

The complex exceptional Jordan algebra $\mathcal{J}_3(\mathbb{O})$ is the vector space $\mathrm{Herm}(3, \mathbb{O})$ endowed with the symmetric product $\circ : \mathrm{Herm}(3, \mathbb{O}) \times \mathrm{Herm}(3, \mathbb{O}) \to \mathrm{Herm}(3, \mathbb{O})$ defined by $A \circ B := \frac{1}{2}(AB + BA)$.

Now we define the basic object of our interest.

Definition 2.1.1. The hyperplane section of the complex Cayley plane $\mathbb{OP}_0^2$ is the projectivization over $\mathbb{C}$ of the following subset of $\mathcal{J}_3(\mathbb{O})$

$$\widehat{\mathbb{OP}}_0^2 := \left\{ A \in \mathrm{Herm}(3,\ \mathbb{O}) \middle|\ A^2 = 0,\ \mathrm{tr} A = 0,\ A \neq 0 \right\}.$$

2.2 The spin groups. For an n-dimensional complex vector space $\mathbb{V}$ and a nondegenerate quadratic form N on $\mathbb{V}$, we denote the corresponding Clifford algebra by $C\ell(\mathbb{V}, N)$ (our convention is $vv = -N(v)$). The spin group of $C\ell(\mathbb{V}, N)$ is denoted by $\mathrm{Spin}(\mathbb{V}, N)$. It is generated inside $C\ell(\mathbb{V}, N)$ by products uv, $u, v \in \mathbb{V}$ where $N(u) = N(v) = 1$. By $\mathrm{Spin}(n, \mathbb{C})$ we denote the spin group associated to the standard quadratic form $\sum_{i=1}^n z_i^2$ on $\mathbb{C}^n$.

For $w \in \mathbb{C}$ we define *the generalized complex sphere*

$$S^{n-1}(w) = \{0 \neq z \in \mathbb{V} \,|\, N(z) = w^2\}.$$

As a consequence of Witt's theorem we have

Lemma 2.2.1. *The group* $\mathrm{Spin}(n, \mathbb{C})$ *acts transitively via the vector representation on the generalized complex spheres.*

2.3 Complex Lie algebra $\mathfrak{f}_4$. The complex exceptional Lie group F_4 can be defined as the automorphism group of the complex exceptional Jordan algebra $(\mathcal{J}_3(\mathbb{O}), \circ)$ (see [22]). In other words F_4 is the subgroup of $\mathrm{GL}(27, \mathbb{C})$ such that $g \in \mathrm{F}_4$ if and only if $g(A \circ B) = gA \circ gB$ for every $A, B \in \mathrm{Herm}(3, \mathbb{O})$.

The action of F_4 preserves the trace on $\mathrm{Herm}(3, \mathbb{O})$. This can be easily seen from the equality

$$\mathrm{Tr}\, A = \frac{1}{3} \mathrm{Tr}\,(B \mapsto A \circ B).$$

It is easy to verify that the action of $\mathrm{O}(3,\mathbb{C})$ on $\mathrm{Herm}(3,\mathbb{O})$ given by

$$\mathrm{O}(3,\,\mathbb{C}) \ni g \longmapsto (A \mapsto gAg^T), \quad A \in \mathrm{Herm}(3,\mathbb{O})$$

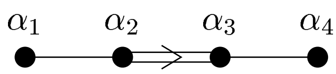
defines an injective group homomorphism $\mathrm{O}(3,\mathbb{C}) \hookrightarrow \mathrm{F}_4$.

Now we present basic facts about the complex simple Lie algebra $\mathfrak{f}_4$ of the group F_4. We shall use these facts as well as the properties of the root system of the Lie algebra $\mathfrak{f}_4$ in the last section of this text. Details can be found in [5].

There exist a choice of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{f}_4$, an orthonormal (with respect to the Killing form of $\mathfrak{f}_4$) basis $\{\epsilon_i\}_{i=1}^4$ of $\mathfrak{h}^*$ and a choice of simple roots

$$\Delta = \left\{\alpha_1 = \epsilon_2 - \epsilon_3,\ \alpha_2 = \epsilon_3 - \epsilon_4,\ \alpha_3 = \epsilon_4,\ \alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\right\}.$$

In this convention the Dynkin diagram is



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The set Δ determines the set of positive roots Φ^+. For any root α, we define the coroot $H_\alpha \in \mathfrak{h}$ by $\lambda(H_\alpha) = 2\langle\lambda,\alpha\rangle/2\langle\alpha,\alpha\rangle$, where $\langle\ ,\ \rangle$ is the Killing form.

The fundamental weights $\{\varpi_i\}_{i=1}^4$ are defined as the dual basis to the simple coroots. We denote the irreducible representation of $\mathfrak{f}_4$ with the highest weight λ by ϱ_λ.

3. Action of F_4 on $\widehat{\mathbb{OP}_0^2}$

In this section we explicitly describe the group $\mathrm{Spin}(9,\mathbb{C})$ as a multiplicative subgroup of $\mathrm{End}(\mathbb{O}^2)\otimes\mathbb{C}$ and construct its representation on $\mathrm{Herm}(3,\mathbb{O})$. Using this representation, we prove that F_4 acts transitively on the hyperplane section $\widehat{\mathbb{OP}_0^2}$. The scalar multiplication on the algebra $\mathrm{End}(\mathbb{O}^2)\otimes\mathbb{C}$ acts only on the first part of the tensor product, i.e. $w\cdot(A\otimes z) = (wA)\otimes z$ for $w, z \in \mathbb{C}$, $A \in \mathrm{End}(\mathbb{O}^2)$.

3.1 Realisation of $\mathrm{Spin}(9,\mathbb{C})$**.** First we need an auxiliary result concerning the Clifford algebra $C\ell(\mathbb{O},N)$.

Lemma 3.1.1. *The map* $\mu : \mathbb{O} \to \mathrm{End}(\mathbb{O}^2)$ *given by*

$$u \longmapsto \begin{pmatrix} 0 & L_u \\ -L_{\bar{u}} & 0 \end{pmatrix}$$

can be uniquely extended to the isomorphism of complex associative algebras $C\ell(\mathbb{O},N) \simeq \mathrm{End}(\mathbb{O}^2)$.

Proof: Easy calculation and (1a) shows that $\mu(u)\mu(u) = -N(u)\,\mathrm{Id}$. Using the universal property of Clifford algebras and the fact that the algebra $C\ell(8,\mathbb{C})$ is simple (see [9]), we immediately get the result. □

Let $\mathbb{V}_9$ be the complex vector space $\mathbb{C} \oplus \mathbb{O}$. We define the quadratic form N' by $(r, u) \mapsto r^2 + N(u)$. Let $\kappa : \mathbb{V}_9 \to \operatorname{End}(\mathbb{O}^2) \otimes \mathbb{C}$ be the homomorphism of vector spaces given by

$$\kappa : (r, u) \longmapsto \begin{pmatrix} r & L_u \\ L_{\bar{u}} & -r \end{pmatrix} \otimes \imath,$$

where $\imath$ denotes the imaginary unit in $\mathbb{C}$.

Proposition 3.1.2. *The Clifford algebra* $C\ell(\mathbb{V}_9, N')$ *is isomorphic* (*as an associative algebra*) *to* $\operatorname{End}(\mathbb{O}^2) \otimes \mathbb{C}$.

Proof: It is known (see e.g. [9]) that $C\ell(\mathbb{V}_9, N') \simeq M(16, \mathbb{C}) \oplus M(16, \mathbb{C})$. Calculation and (1a) shows that $\kappa(r, u)\kappa(r, u) = -N'(r, u) \operatorname{Id}$. The universal mapping property of Clifford algebras gives us the following commutative diagram

$$\begin{array}{ccc} \mathbb{V}_9 & \xrightarrow{\ i\ } & M(16, \mathbb{C}) \oplus M(16, \mathbb{C}) \\ & \searrow^{\kappa} & \Big\downarrow f \\ & & \operatorname{End}(\mathbb{O}^2) \otimes \mathbb{C}\ . \end{array}$$

Because $\kappa(-1, 0)\kappa(0, u) = \mu(u) \otimes 1$, we see that the image of f generates the subalgebra $\operatorname{End}(\mathbb{O}^2) \otimes 1$. The equality

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \imath = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \imath \cdot \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \otimes 1$$

implies that the image of f generates the whole algebra $\operatorname{End}(\mathbb{O}^2) \otimes \mathbb{C}$. Since the dimensions of the considered algebras are the same, it follows that f is an isomorphism. □

Lemma 3.1.3. *The spin group* $\operatorname{Spin}(\mathbb{V}_9, N')$ *is generated* (*inside* $\operatorname{End}(\mathbb{O}^2) \otimes \mathbb{C}$) *by elements of the form*

$$g_{r,u} := \begin{pmatrix} r & -L_u \\ L_{\bar{u}} & r \end{pmatrix} \otimes 1, \quad r \in \mathbb{C},\ u \in \mathbb{O}, \quad r^2 + u\bar{u} = 1.$$

Proof: The spin group is by definition generated by products of the form $\kappa(r, u)\kappa(s, v)$, where $N'(r, u) = N'(s, v) = 1$. Since $g_{r,u} = \kappa(r, u)\kappa(-1, 0)$ and $\kappa(r, u)\kappa(s, v) = g_{r,u}g_{-s,v}$, the lemma follows. □

For brevity we will identify $A \otimes 1 \in \operatorname{End}(\mathbb{O}^2) \otimes \mathbb{C}$ with $A \in \operatorname{End}(\mathbb{O}^2)$ from now on; i.e.

$$g_{r,u} = \begin{pmatrix} r & -L_u \\ L_{\bar{u}} & r \end{pmatrix} .$$

3.2 Representations of $\mathrm{Spin}(\mathbb{V}_9, N')$**.** We will use the following decomposition of $\mathrm{Herm}(3, \mathbb{O})$

$$\begin{pmatrix} r_1 & \bar{x}_1 & \bar{x}_2 \\ x_1 & r_2 & x_3 \\ x_2 & \bar{x}_3 & r_3 \end{pmatrix} = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \bar{x}_1 & \bar{x}_2 \\ x_1 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & s & x_3 \\ 0 & \bar{x}_3 & -s \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}$$

in order to define the action of $\mathrm{Spin}(\mathbb{V}_9, N')$ on it. In other words — we take the $\mathbb{C}$-linear isomorphism $\mathrm{Herm}(3, \mathbb{O}) \to \mathbb{C} \oplus \mathbb{O}^2 \oplus \mathrm{Herm}_0(2, \mathbb{O}) \oplus \mathbb{C}$ and we endow each of the spaces in the decomposition with an action of $\mathrm{Spin}(\mathbb{V}_9, N')$. The $\mathbb{O}^2$ summand will be *the spinor part* and we will call the $\mathrm{Herm}(2, \mathbb{O})_0$ summand *the vector part*.

Lemma 3.2.1. *Let* Φ *be the linear isomorphism between the space of trace-free hermitian matrices* $\mathrm{Herm}_0(2, \mathbb{O})$ *and* $\kappa(\mathbb{V}_9)$ *defined by*

$$\Phi : \begin{pmatrix} s & x \\ \bar{x} & -s \end{pmatrix} \mapsto \begin{pmatrix} s & L_x \\ L_{\bar{x}} & -s \end{pmatrix} \otimes \imath$$

and let ϱ_V *be the vector representation of* $\mathrm{Spin}(\mathbb{V}_9, N')$*.*

If we define the representation of $\mathrm{Spin}(\mathbb{V}_9, N')$ *on* $\mathrm{Herm}_0(2, \mathbb{O})$ *by* $\xi_V(g)a := \Phi^{-1}(\varrho_V(g)\Phi(a))$*, the following formula holds for the generators* $g_{r,u}$ *of* $\mathrm{Spin}(\mathbb{V}_9, N')$

$$\xi_V(g_{r,u}) \begin{pmatrix} s & x \\ \bar{x} & -s \end{pmatrix} = \left[\begin{pmatrix} r & -u \\ \bar{u} & r \end{pmatrix} \begin{pmatrix} s & x \\ \bar{x} & -s \end{pmatrix} \right] \begin{pmatrix} r & u \\ -\bar{u} & r \end{pmatrix}$$

$$= \begin{pmatrix} s\big(r^2 - N(u)\big) - r\langle x, u\rangle & 2rsu + r^2 x - u\bar{x}u \\ 2rs\bar{u} + r^2\bar{x} - \bar{u}x\bar{u} & -s\big(r^2 - N(u)\big) + r\langle \bar{x}, \bar{u}\rangle \end{pmatrix}. \tag{2}$$

Proof: The vector representation of $\mathrm{Spin}(\mathbb{V}_9, N')$ is given by $v \mapsto gvg^{-1}$ where v is an element of $\kappa(\mathbb{V}_9)$ and $g \in \mathrm{Spin}(\mathbb{V}_9, N')$. For $g_{r,u} = \kappa(r, u)\kappa(-1, 0)$ we get $g_{r,u}^{-1} = g_{r,-u}$.

Thus we have the following formula for $\rho_V(g_{r,u})$ evaluated on $v = \left(\begin{smallmatrix} s & L_x \\ L_{\bar{x}} & -s \end{smallmatrix}\right) \otimes \imath$

$$\begin{pmatrix} s\big(r^2 - N(u)\big) - r(L_u L_{\bar{x}} + L_x L_{\bar{u}}) & 2rsL_u + r^2 L_x - L_u L_{\bar{x}} L_u \\ 2rsL_{\bar{u}} + r^2 L_{\bar{x}} - L_{\bar{u}} L_x L_{\bar{u}} & -s\big(r^2 - N(u)\big) + r(L_{\bar{u}} L_x + L_{\bar{x}} L_u) \end{pmatrix} \otimes \imath.$$

From (1b) we have $L_u L_{\bar{x}} + L_x L_{\bar{u}} = L_{\langle x,u\rangle}$. With the help of the first Moufang identity (1c) we may substitute $L_u L_{\bar{x}} L_u = L_{(u\bar{x})u}$. Applying the isomorphism Φ to the result gives the expression for $\xi_V(g_{r,u})\Phi^{-1}(v)$ which agrees with (2). □

The spinor representation of $\mathrm{Spin}(\mathbb{V}_9, N')$ acts on $\mathbb{O}^2$ by (see Chapter 6 of [9] for details)

$$\xi_S(g_{r,u})(x_1, x_2) = \begin{pmatrix} r & -L_u \\ L_{\bar{u}} & r \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} rx_1 - ux_2 \\ \bar{u}x_1 + rx_2 \end{pmatrix}.$$

We let the $\mathrm{Spin}(\mathbb{V}_9, N')$ act on the rest of the summands of $\mathrm{Herm}(3, \mathbb{O})$ trivially and denote the resulting action by ξ.

Proposition 3.2.2. *The representation ξ is faithful and preserves the Jordan product. In other words* $\mathrm{Spin}(\mathbb{V}_9, N') \simeq \mathrm{Im}(\xi)$ *is a subgroup of* F_4.

Proof: Since the spinor representation ξ_S is faithful, the representation ξ is faithful as well. In order to prove that this action preserves the Jordan product we introduce the following three by three hermitian matrix

$$G_{r,u} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & -u \\ 0 & \bar{u} & r \end{pmatrix} \in \mathrm{Herm}(3, \mathbb{O}),$$

where $(r, u) \in \mathbb{V}_9$ is of unit norm. Straightforward calculations reveal that $G_{r,u}^{-1} = G_{r,-u}$ and that $G_{r,u} A G_{r,u}^{-1}$ gives the expression for the action of $\xi(g_{r,u})$ on A. Moreover the expression $G_{r,u} A G_{r,u}^{-1}$ is unambiguous for any $A \in \mathrm{Herm}(3, \mathbb{O})$.

Put $g = g_{r,u}$, $G = G_{r,u}$ for simplicity. For each $A \in \mathrm{Herm}(3, \mathbb{O})$ we have

$$(\xi(g)A)(\xi(g)A) = (GAG^{-1})(GAG^{-1}).$$

Let us suppose for a moment that $(GAG^{-1})(GAG^{-1}) = G(A(G^{-1}G)A)G^{-1}$. Then we would have

$$(\xi(g)A)(\xi(g)A) = \xi(g)(A^2) \tag{3}$$

for any $A \in \mathrm{Herm}(3, \mathbb{O})$. Using this equality for $A + B$ instead of A we would get on the left hand side

$$\begin{aligned} (\xi(g)(A+B))\,(\xi(g)(A+B)) &= \big(\xi(g)A + \xi(g)B\big)\big(\xi(g)A + \xi(g)B\big) \\ &= (\xi(g)A)^2 + (\xi(g)A)(\xi(g)B) \\ &\quad + (\xi(g)B)(\xi(g)A) + (\xi(g)B)^2, \end{aligned}$$

while the right hand side would equal

$$\xi(g)\left((A+B)^2\right) = \xi(g)(A^2) + \xi(g)(AB) + \xi(g)(BA) + \xi(g)(B^2).$$

Using (3) for $\xi(g)(A^2)$ and $\xi(g)(B^2)$ we would get that

$$(\xi(g)A)(\xi(g)B) + (\xi(g)B)(\xi(g)A) = \xi(g)(AB + BA).$$

So we only need to prove that we can rearrange the brackets in the expression $(GAG^{-1})(GAG^{-1})$. From the Artin's theorem it follows that

$$(u_1 a u_2)(u_3 a u_4) = u_1(a(u_2 u_3)a)u_4,$$

where u_i are elements of the linear span of $\{r, u, \bar{u}\}$ and $a \in \mathbb{O}$ is arbitrary. Using the same trick as above and writing this equality for $a + b$ instead of a we get

$$(u_1 a u_2)(u_3 b u_4) + (u_1 b u_2)(u_3 a u_4) = u_1(a(u_2 u_3)b)u_4 + u_1(b(u_2 u_3)a)u_4.$$

The equation

$$((GAG^{-1})(GAG^{-1}))_{a,b} = \frac{1}{2}\sum_{i,j,\dots,m} (G_{a,i}A_{i,j}G^{-1}_{j,k})(G_{k,l}A_{l,m}G^{-1}_{m,b}) + (G_{a,l}A_{l,m}G^{-1}_{m,k})(G_{k,i}A_{i,j}G^{-1}_{j,b})$$

and the fact that $G_{i,j}$ are from the linear span of $\{r, u, \bar{u}\}$ imply

$$(GAG^{-1})(GAG^{-1}) = G(A(G^{-1}G)A)G^{-1} = GA^2G^{-1}.$$

□

Remark 3.2.3. One could define the representation ξ directly using the matrix $G_{r,u}$. It is however not clear that the expression $G_{r,u}AG^{-1}_{r,u}$ defines a representation due to the nonassociativity of the product of $\mathrm{Herm}(3,\mathbb{O})$.

3.3 The subgroup $\mathrm{Spin}(8,\mathbb{C})$**.** The usual presentation of spin groups gives (see Lemma 3.1.1) the following set of generators of $\mathrm{Spin}(\mathbb{O}, N)$

$$\left\{ \begin{pmatrix} -L_uL_{\bar{v}} & 0 \\ 0 & -L_{\bar{u}}L_v \end{pmatrix} \middle| \; u, v \in \mathbb{O},\ N(u) = N(v) = 1 \right\}.$$

One can obtain matrices of this form as products $g_{0,u}g_{0,v}$ which means that these generators are in fact elements of $\mathrm{Spin}(\mathbb{V}_9, N')$. The formula for the restriction of ξ_V to the subgroup $\mathrm{Spin}(\mathbb{O}, N)$

$$\xi_V\left(\begin{pmatrix} L_uL_{\bar{v}} & 0 \\ 0 & L_{\bar{u}}L_v \end{pmatrix}\right)\begin{pmatrix} s & x_3 \\ \bar{x}_3 & -s \end{pmatrix} = \begin{pmatrix} s & u(\bar{v}x_3\bar{v})u \\ \bar{u}(v\bar{x}_3v)\bar{u} & -s \end{pmatrix} \tag{4}$$

is easily proved using (2).

Analogously, the action of $\mathrm{Spin}(\mathbb{O}, N)$ on $\mathbb{O}^2$ is given by

$$\xi_S\left(\begin{pmatrix} L_uL_{\bar{v}} & 0 \\ 0 & L_{\bar{u}}L_v \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u(\bar{v}x_1) \\ \bar{u}(vx_2) \end{pmatrix}, \tag{5}$$

which is the direct sum of two inequivalent spinor representations of $\mathrm{Spin}(\mathbb{O}, N)$. Please note that the quadratic form N is invariant with respect to all the three inequivalent actions of $\mathrm{Spin}(\mathbb{O}, N)$ on the vector space $\mathbb{O}$.

3.4 Transitivity of the F_4 action on $\widehat{\mathbb{OP}}{}_0^2$.

Lemma 3.4.1. *Let*

$$A = \begin{pmatrix} -2t & \bar{x}_1 & \bar{x}_2 \\ x_1 & t+s & \bar{x}_3 \\ x_2 & x_3 & t-s \end{pmatrix}$$

be an element of $\widehat{\mathbb{OP}}{}_0^2$*. Then the vector part of* A *is isotropic* (*i.e.* $s^2 + N(x_3) = 0$) *if and only if* $N(x_1) = N(x_2) = 0$ *and if and only if* $t = 0$*.*

Proof: The statement is a straightforward consequence of the fact that diagonal elements of A^2 must equal zero. □

Theorem 3.4.2. *The group* F_4 *acts transitively on* $\widehat{\mathbb{O}\mathbb{P}}^2_0$. *For every* $A \in \widehat{\mathbb{O}\mathbb{P}}^2_0$ *there exists* $g \in \mathrm{F}_4$ *such that*

$$g \cdot A = \begin{pmatrix} \imath & 1 & 0 \\ 1 & -\imath & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

Proof: First we suppose that $A \in \widehat{\mathbb{O}\mathbb{P}}^2_0$ has nonisotropic vector part. In such case we can use Lemma 2.2.1 to prove that there exists an element $h_1 \in \mathrm{Spin}(\mathbb{V}_9, N')$ such that

$$\xi(h_1)A = \begin{pmatrix} r_1 & \bar{x}_1 & \bar{x}_2 \\ x_1 & r_2 & 0 \\ x_2 & 0 & r_3 \end{pmatrix}, \text{ with } r_1, r_2, r_3 \in \mathbb{C},\ x_1, x_2 \in \mathbb{O}.$$

Let us denote $\xi(h_1) =: g_1 \in \mathrm{F}_4$. The matrix $(g_1 \cdot A)^2$ has the form

$$\begin{pmatrix} r_1^2 + N(x_1) + N(x_2) & \bar{x}_1(r_1 + r_2) & \bar{x}_2(r_1 + r_3) \\ x_1(r_1 + r_2) & r_2^2 + N(x_2) & x_1\bar{x}_2 \\ x_2(r_1 + r_3) & x_2\bar{x}_1 & r_3^2 + N(x_2) \end{pmatrix}. \tag{7}$$

This is a zero matrix, in particular $N(x_1)N(x_2) = N(x_1\bar{x}_2) = 0$, so x_1 and x_2 cannot be both non-isotropic. On the other hand, they cannot be both isotropic because of Lemma 3.4.1.

Assume first that $N(x_1) \neq 0$ and $N(x_2) = 0$. The action of $\mathrm{Spin}(\mathbb{O}, N)$ preserves the vector part $\left(\begin{smallmatrix} r_2 & 0 \\ 0 & r_3 \end{smallmatrix}\right)$ of $g_1 \cdot A$ because of (4). Let

$$h_2 := \kappa(0, -1)\kappa(0, \frac{x_1}{\sqrt{N(x_1)}}) \in \mathrm{Spin}(\mathbb{O}, N)$$

and $\xi(h_2) =: g_2 \in \mathrm{F}_4$. By (5), g_2 sends the spinor part $x_1 \oplus x_2$ of $g_1 \cdot A$ to $x_1' \oplus x_2'$ where $x_1' = \sqrt{N(x_1)} \in \mathbb{C}$ and $x_2' = \frac{1}{\sqrt{N(x_1)}} x_1 x_2$. The matrix $(g_2 g_1 \cdot A)^2$ has the same form as (7) with x_1 and x_2 substituted by x_1' and x_2'. It is still a zero matrix and its $(2,3)$-position $0 = x_1'\bar{x}_2'$ implies $x_2' = 0$ (x_1' is a nonzero complex number). The other positions of this matrix imply $0 = r_3^2 + N(x_2')$, so $r_3 = 0$, and $r_1^2 + N(x_1') = r_1^2 + (x_1')^2 = 0$, so

$$g_2 g_1 \cdot A = \begin{pmatrix} \pm\imath w & w & 0 \\ w & \mp\imath w & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $0 \neq w \in \mathbb{C}$.

The case $N(x_1) = 0$, $N(x_2) \neq 0$ leads in a similar way to a matrix of the form $\begin{pmatrix} \pm\imath w & 0 & w \\ 0 & 0 & 0 \\ w & 0 & \mp\imath w \end{pmatrix}$, $0 \neq w \in \mathbb{C}$, which can be transformed by the orthogonal matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ to the previous one. One can get rid of the sign ambiguity with $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and the matrix $\begin{pmatrix} \imath w & w & 0 \\ w & -\imath w & 0 \\ 0 & 0 & 0 \end{pmatrix}$ can be transformed to the canonical form (6) by conjugating by the orthogonal matrix

$$\begin{pmatrix} \frac{1}{\sqrt{w}} & 0 & \frac{-\imath\sqrt{1-w}}{\sqrt{w}} \\ \frac{-\imath(1-w)}{\sqrt{w}} & \sqrt{w} & -\frac{\sqrt{1-w}}{\sqrt{w}} \\ \imath\sqrt{1-w} & \sqrt{1-w} & 1 \end{pmatrix}.$$

So, $g_3g_2g_1 \cdot A$ has the canonical form (6), where g_3 is some element in the image of the embedding $O(3,\mathbb{C}) \hookrightarrow \mathrm{F}_4$ defined in Section 2.3.

If A has isotropic but nonzero vector part, then the preceding lemma implies that the topleft element of A is 0. Using Lemma 2.2.1 we can find an element $g' \in \xi(\mathrm{Spin}(\mathbb{V}_9, N')) \leq \mathrm{F}_4$ such that $g' \cdot A = \begin{pmatrix} 0 & \bar{x}_1 & \bar{x}_2 \\ x_1 & \imath w & w \\ x_2 & w & -\imath w \end{pmatrix}$ where $w \neq 0$. Conjugation by $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ leads to a matrix whose top left element is $\imath w \neq 0$. By the previous lemma, such a matrix has nonisotropic vector part and we have reduced this case to the already solved one.

Finally, suppose that A has zero vector part, $A = \begin{pmatrix} 0 & \bar{x}_1 & \bar{x}_2 \\ x_1 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix}$. This matrix is nonzero by definition. If $x_2 \neq 0$, then the action of $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ transforms it to a matrix with nonzero vector part. The case $x_1 \neq 0$ is treated similarly. $\square$

Remark 3.4.3. We see from the proof that in order to prove transitivity of F_4 on $\mathbb{OP}_0^2$, it is sufficient to consider only discrete subgroup of $\mathrm{O}(3,\mathbb{C})$ isomorphic to S_3 — a permutation group on three letters. This is a manifestation of the triality principle.

Now we prove that the cone $\widehat{\mathbb{OP}_0^2}$ over $\mathbb{OP}_0^2$ is a smooth manifold.

Proposition 3.4.4. *The space* $\widehat{\mathbb{OP}_0^2}$ *is a smooth manifold of dimension* 32.

Proof: Let as define the smooth map $f : \mathrm{Herm}(3,\mathbb{O})_0 \to \mathrm{Herm}(3,\mathbb{O})_0$ by $f(A) := A^2$. We use the implicit function theorem to show that $\widehat{\mathbb{OP}_0^2} = f^{-1}(0)\setminus\{0\}$ is a smooth manifold. The differential of f at A is easily proved to be $B \mapsto 2A \circ B$. We already know that F_4 acts transitively on $f^{-1}(0) \setminus \{0\} = \widehat{\mathbb{OP}_0^2}$ and so we have

$$\dim\ker(B \mapsto A \circ B) = \dim\ker\big(B \mapsto g \cdot (A \circ (g^{-1} \cdot B))\big) = \dim\ker\big(B \mapsto (g \cdot A) \circ B\big)$$

for any $g \in \mathrm{F}_4$. So, the differential df of f has constant rank on the set $f^{-1}(0)\setminus\{0\}$ and $\widehat{\mathbb{OP}_0^2}$ is a smooth manifold.

The kernel of the differential of f at the canonical point (6) equals

$$\left\{ \begin{pmatrix} \imath\Re(x_1) & x_1 & x_2 \\ \bar{x}_1 & -\imath\Re(x_1) & -\imath x_2 \\ \bar{x}_2 & -\imath\bar{x}_2 & 2\Re(x_1) \end{pmatrix} \middle| \, x_1, x_2 \in \mathbb{O} \right\}$$

and is isomorphic to the tangent space of $\widehat{\mathbb{OP}}^2_0$ at that point. □

3.5 The real case. By choosing an appropriate involution on $\mathcal{J}_3(\mathbb{O}_\mathbb{C})$ we get a model for $\mathrm{F}_4^{(-20)}/\mathrm{P}_4$ — i.e. the conformal infinity of the Einstein space $\mathbb{O}H^2$. According to Yokota [23] the following real subalgebra of $\mathcal{J}_3(\mathbb{O}_\mathbb{C})$

$$\left\{ A \in \mathcal{J}_3(\mathbb{O}_\mathbb{C}) : I_1 \overline{A}^T I_1 = A, I_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$
$$= \left\{ \begin{pmatrix} r_1 & x_1 & x_2 \\ -\overline{x}_1 & r_2 & x_3 \\ -\overline{x}_2 & \overline{x}_3 & r_3 \end{pmatrix} : x_i \in \mathbb{O}_\mathbb{R}, r_i \in \mathbb{R} \right\}$$

has $\mathrm{F}_4^{(-20)}$ as its automorphism group. By restricting the map κ to $\mathbb{R} \oplus \mathbb{O}_\mathbb{R}$ we get presentation of $\mathrm{Spin}(9,\mathbb{R})$ and the restriction of our representation ξ maps $\mathrm{Spin}(9,\mathbb{R})$ into $\mathrm{F}_4^{(-20)}$. Instead of $\mathrm{O}(3,\mathbb{C})$ we have the compact orthogonal group $\mathrm{O}(3,\mathbb{R})$.

The model of $\mathrm{F}_4^{(-20)}/\mathrm{P}_4$ is given by the same equations as in the complex case. Since there are no isotropic elements in the vector part, the proof of transitivity is now much simpler. By transitivity of $\mathrm{SO}(9,\mathbb{R})$ on spheres we can map any element of our model to a matrix of the form $\left(\begin{smallmatrix} -2t & x_1 & x_2 \\ -\overline{x}_1 & t+s & 0 \\ -\overline{x}_2 & 0 & t-s \end{smallmatrix}\right)$. The square of this matrix has to be zero by definition which for diagonal elements gives three equations that yield easily $t^2 - s^2 = 0$. The case $t = -s$ leads to $x_1 = 0$ and can be reduced to the case of $t = s$ by conjugation with $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$.

The case $t = s$ gives $x_2 = 0$ and we can easily find an action of $\mathrm{Spin}(8,\mathbb{R})$ that maps x_1 to a positive real number which gives us a matrix in the form $\left(\begin{smallmatrix} -r & x & 0 \\ -x & r & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, where all the entries are real and $r^2 = x^2$. We can reduce the case $r = -x$ to the case $r = x$ by conjugation with $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Thus we can map an arbitrary element A from our real Jordan algebra, such that $\mathrm{Tr}\, A = 0$ and $A^2 = 0$, to a matrix of the form $\left(\begin{smallmatrix} -x & x & 0 \\ -x & x & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ where x is a positive real number. This shows that $\mathrm{F}_4^{(-20)}$ has transitive action on the real projectivization of the appropriate set.

4. Description of the stabilizer of the F_4 action

In this section we will identify the stabilizer of $\mathbb{OP}_0^2$ as a concrete parabolic subgroup of F_4.

Lemma 4.0.1. *There exists up to isomorphism only one irreducible representation ϱ of the group F_4 such that*

$$1 < \dim_{\mathbb{C}} \varrho \leq 26.$$

The highest weight of this representation is $\varpi_4 = \epsilon_1$.

PROOF: Let $\lambda, \mu \in \mathfrak{h}^*$ be two integral dominant weights, $\mu \neq 0$. By a direct application of the Weyl dimensional formula (see Goodman, Wallach [9]), we obtain that $\dim \varrho_{\lambda+\mu} > \dim \varrho_\lambda$. Using the program LiE [20], we get $\dim \rho_{\varpi_1} = 52$, $\dim \varrho_{\varpi_2} = 1274$, $\dim \varrho_{\varpi_3} = 273$ and $\dim \varrho_{\varpi_4} = 26$. By the previous inequality, we see that there is only one irreducible 26-dimensional representations of the Lie algebra $\mathfrak{f}_4$. □

Since $\dim \mathcal{J}_0 = 26$ and all finite dimensional representation of the simple Lie group F_4 are completely reducible, we obtain immediately the following.

Proposition 4.0.2. *The restriction to the defining representation of F_4 on $\mathcal{J}_0 = \mathrm{Herm}(3, \mathbb{O})_0$ is isomorphic to the 26-dimensional irreducible representation ϱ_{ϵ_1}.*

It is clear from definition that $\mathbb{OP}_0^2$ is a projective variety. According to Humphreys [12] this implies that the stabilizer group of any point is a parabolic subgroup of F_4. Since any parabolic subgroup contains Borel subgroup, it follows that the points of the variety are lines spanned by highest weight vectors.

For a fixed choice of the Cartan subalgebra $\mathfrak{h}$ and simple roots Δ there is a $1-1$ correspondence between isomorphism classes of parabolic subalgebras $\mathfrak{p} \subseteq \mathfrak{g}$ and subsets $\Sigma \subseteq \Delta$ of the set Δ of simple roots described e.g. in [6, Chapter 3]. We will denote the parabolic subalgebra corresponding to $\Sigma = \{\alpha_i\}$ by $\mathfrak{p}_i$.

Because the highest weight of $\mathcal{J}_0$ is ϵ_1, the following theorem follows directly from [6, Theorem 3.2.5]. Its proof is not difficult — it is based on the fact that for each $X \in \mathfrak{g}_\alpha$ one can find $Y \in \mathfrak{g}_{-\alpha}$ such that $[Y, X] = H_\alpha$, where $H_\alpha(\lambda) = \langle \lambda, \alpha \rangle$ and the fact that the set of weights is invariant under the action of Weyl group.

Theorem 4.0.3. *Let P be the stabilizer of a point $p \in \mathbb{OP}_0^2$ with respect to the action of the group F_4. Then the Lie algebra $\mathfrak{p}$ of the group P is isomorphic to $\mathfrak{p}_4$.*

Remark 4.0.4. We see that $\widehat{\mathbb{OP}_0^2}$ is the F_4-orbit of the highest weight vector in $\mathcal{J}_0$. Points in $\widehat{\mathbb{OP}_0^2}$ are exactly all possible highest weight vectors for this representation, corresponding to different choices of $\mathfrak{h}$ and Φ^+. The real case can be treated in similar manner with analogous results. See [6] for details.

Remark 4.0.5. From the computation of the harmonic curvature (as done for example in [17], also see [6]) one can prove that the homogeneous space does not admit curved deformations in the sense of regular normal Cartan geometries.

However, if one relaxes the regularity condition there are some deformations of this structure [1].

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Charles University, Department of Mathematics Education, Sokolovská 83,
186 75 Prague 8, Czech Republic

Charles University, Mathematical Institute, Sokolovská 83,
186 75 Prague 8, Czech Republic

Institute of Computer Science, Academy of Sciences of the Czech Republic,
Pod Vodárenskou věží 271/2, 182 07 Prague 8, Czech Republic

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YAMABE OPERATOR VIA BGG SEQUENCES

Vít Tuček

Abstract. We show that the conformally invariant Yamabe operator on a complex conformal manifold can be constructed as a first BGG operator by inducing from certain infinite-dimensional representation.

1. Introduction

The general problem is to find all natural differential operators between sections of natural vector bundles on some geometric category (e.g. on conformal or projective manifolds). The naturality implies that on an open subset of a homogeneous space (which is the canonical model for the 'geometry' under consideration) the natural vector bundles are the homogeneous vector bundles and natural differential operators are invariant differential operators. However not all invariant differential operators are natural. The counterexample is a power of the Laplace operator (see [9] and [8]).

It was proven in [4] that there is a large class of invariant operators which are natural. Among these so-called BGG operators are various interesting 'geometric' operators whose kernels give e.g. conformal Killing tensor fields. Consequently, the BGG operators were much studied – see e.g. [2] for recent applications. The construction of these operators was much simplified by Calderbank and Diemer in [1].

The construction of BGG operators starts with a general parabolic geometry $(\mathcal{G}, \omega)$ over a manifold M modeled on a parabolic pair (G, P) and a finite-dimensional $(\mathfrak{g}, P)$-representation $\mathbb{V}$ and its output is a sequence of differential operators between associated bundles associated to representations whose weight is given by the Kostant formula [11]. In the flat case this actually yields a complex, which computes the sheaf cohomology of the sheaf of constant sections of the bundle $\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$. The Cartan connection of the geometry induces an affine connection on $\mathcal{V}$ and one of the crucial features of BGG operators is that the kernel of the first operator in the sequence contains the so called normal solutions, which arise in an explicit way from parallel sections of this affine connection. In particular, in the flat case all solutions are normal and hence we can realize all solutions of first BGG

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operator as parallel sections of $\mathcal{V}$ with respect to the affine connection induced by ω. In this article we show that the BGG construction by Calderbank and Diemmer basically works for a certain infinite-dimensional representation in which case it yields the conformally invariant Yamabe operator. The Yamabe operator is just the Laplace – Beltrami operator with a scalar curvature term

$$Y = \Delta_{LP} - \frac{n-2}{4(n-1)} R\,.$$

Let us denote the Levi part of P by L and the Lie algebra of the nilradical by $\mathfrak{p}_+$. The original construction of Calderbank and Diemer shows actually much more than the existence of BGG operators. In fact, they modify Hodge theory for a finite-dimensional representation $\mathbb{V}$ developed in [11] in order to get a homotopy transfer between twisted deRham sequence on $\mathcal{V}$ and bundles induced from Lie algebra homology of $\mathfrak{p}_+$ with values in $\mathbb{V}$. This homotopy transfer is basically P-equivariant modification of the homotopy transfer data coming from the L-equivariant Hodge decomposition of $\mathbb{V}$. This modification is not possible without introducing differential terms to the algebraic Hodge Laplacian of [11] and hence the resulting BGG operators can have order higher than one. To be a little bit more concrete, one has from [11] that for a certain algebraic operator $\Box$ there is a Hodge decomposition $\mathbb{V} = \ker\Box \oplus \operatorname{im}\Box$ which is L-equivariant and one would like to extend this to a Hodge decomposition of the sections of $\mathcal{V}$. To this end, one introduces in a straightforward way a differential operator $\Box_{\mathfrak{g}}$ and tries to show that $\Gamma(M, \mathcal{V}) = \operatorname{im}\Box_{\mathfrak{g}} \oplus \ker\Box_{\mathfrak{g}}$. In particular, the operator $\Box_{\mathfrak{g}}$ must be invertible on its own image.

In order to get the Yamabe operator as a resulting BGG: operator, it is necessary to consider an infinite-dimensional representation, since it is known that the kernel of the Yamabe operator is inifite-dimensional on $\mathbb{R}^{p,q}$ and the kernel of the first BGG operator has, in the flat case, the same dimension as $\mathbb{V}$. The representation we will consider is a formal globalization of a unitarizable highest weight module. Unitarizable highest weight module is a module which is both a $(\mathfrak{g}, K)$-module and a highest weight module for $\mathfrak{g}$. The unitarizability of the module ensures that the Hodge decomposition of $\mathbb{V}$ is still valid, while the formal globalization enables us to proceed with the proof of invertibility of $\Box_{\mathfrak{g}}$. The original unitarizable highest weight representation is basically a certain subspace of polynomials, whereas its formal globalization is a subspace of formal power series.

2. Modules, homology & formal globalization

The unitarizable highest weight modules occur only for noncompact Hermitian symmetric pairs and hence we have to restrict the signature. Throughout this article we will use symbol G to denote the group $SO_0(2,p)$. The maximal compact subgroup of this group is $SO(2) \times SO(p)$ and we will denote it by K. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the corresponding Lie algebras and let $\mathfrak{g}$ and $\mathfrak{k}$ denote their complexifications. The homogeneous manifold G/K is a noncompact Hermitian space. The Cartan decomposition gives us $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{r}$ and upon complexification we get a splitting of $\mathfrak{r}_{\mathbb{C}} = \mathfrak{p}_- \oplus \mathfrak{p}_+$ into eigenspaces of the complex structure that is defined on the tangent space $T_{eK}G/K \simeq \mathfrak{r}$. Both algebras $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{p}_+$ and $\overline{\mathfrak{p}} = \mathfrak{k} \oplus \mathfrak{p}_-$ are parabolic subalgebras of $\mathfrak{g}$. Moreover their nilradicals $\mathfrak{p}_-$ and $\mathfrak{p}_+$ are not only nilpotent but even abelian. By P and $\overline{P}$ we denote the corresponding parabolic subgroups of $G_{\mathbb{C}} = SO(p+2,\mathbb{C})$. The homogoneous space $G_{\mathbb{C}}/P$ is diffeomorphic to compact Hermitian symmetric space and $\mathfrak{p}_-$ is naturally mapped via composition of exponential map and projection to an open and dense subset of this compact manifold. The so called Harish-Chandra embedding gives us a realization of the noncompact dual G/K as an orbit in this embedded $\mathfrak{p}_-$. This realizes G/K as a bounded Hermitian symmetric domain in $\mathfrak{p}_-$ and shows where does the Hermitian structure on G/K comes from. We will denote by $K_{\mathbb{C}}$ the complexification of K.

We will use notation ϵ^i, $i = 1,\dots,2p$ for the elements of the basis of the nilradical $\mathfrak{p}_+$ and by e_i we will denote the dual basis defined by the Killing form. The elements e_i then span the nilradical of the opposite parabolic subalgebra.

There is a choice of a Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{h} \leq \mathfrak{k}$. Let Δ be the set of roots of $(\mathfrak{g},\mathfrak{h})$ and let Δ_c denote the set of roots of $(\mathfrak{k},\mathfrak{h})$. We call elements of Δ_c the compact roots and the remaining roots in $\Delta_n = \Delta \setminus \Delta_c$ are called noncompact. We define the positive roots Δ^+ in such a way that elements of $\Delta^+_n = \Delta^+ \cap \Delta_n$ span $\mathfrak{p}_-$[1]. We denote the positive compact roots by $\Delta^+_c = \Delta_c \cap \Delta^+$. By ω_i we denote the i-th fundamental weight in the standard ordering.

Let λ be a Δ^+_c dominant and integral weight and denote by $F(\lambda)$ the finite dimensional irreducible $\mathfrak{k}$-module. We extend any irreducible representation of $K_{\mathbb{C}}$ to P and to $\overline{P}$ by letting $\mathfrak{p}_+$ and $\mathfrak{p}_-$ act trivially. The generalized Verma module $M(\lambda)$ is defined as $M(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\overline{\mathfrak{p}})} F(\lambda)$. It is well known and easy to prove that $M(\lambda)$ contains a maximal nontrivial submodule and we denote by $L(\lambda)$ the corresponding irreducible quotient of $M(\lambda)$. Since the nilradical $\mathfrak{p}_-$ of $\overline{\mathfrak{p}}$ is abelian, we have that $M(\lambda) \simeq S(\mathfrak{p}_+) \otimes F(\lambda)$ as $K_{\mathbb{C}}$ representations, where $S(\mathfrak{p}_+)$ is the symmetric algebra over the Lie algebra $\mathfrak{p}_+$.

There is a distinguish element in the center of $\mathfrak{k}$ called *grading element* which acts by zero on $\mathfrak{k}$, by 1 on $\mathfrak{p}_+$ and by -1 on $\mathfrak{p}_-$. This elements acts by a scalar on any irreducible representation of K. We call this scalar the *geometric weight*. In the case of $M(\lambda)$, the geometric weight corresponds to the polynomial degree shifted by the weight of $F(\lambda)$.

The chain space of Lie algebra homology $C_k(\mathfrak{p}_+,\mathbb{V})$ of the algebra $\mathfrak{p}_+$ with values in $\mathbb{V}$ is defined as $\Lambda^k\mathfrak{p}_+ \otimes \mathbb{V}$. The Lie algebra homology differential $\partial^* : C_{k+1}(\mathfrak{p}_+,\mathbb{V}) \to C_k(\mathfrak{p}_+,\mathbb{V})$ is defined for a general nilpotent subalgebra $\mathfrak{p}_+$ by

$$\begin{aligned}\partial^*(Z_0 \wedge \cdots Z_k \otimes v) = &\sum_{i=0}^{k} (-1)^{i+1} Z_0 \wedge \cdots \wedge \widehat{Z_i} \wedge \cdots \wedge Z_k \otimes Z_i \cdot v + \\ &+ \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_0 \wedge \cdots \wedge \widehat{Z_i} \wedge \cdots \wedge \widehat{Z_j} \wedge \cdots \wedge Z_k \otimes v\,,\end{aligned}$$

[1]One would expect that the positive roots would span $\mathfrak{p}_+$. However, we choose this condition in order to be consistent with cohomological formulas in [7]. Alternatively, one could work with lowest weight modules instead.

where $Z_i \in \mathfrak{p}_+$ for $i = 0, \dots, k$. Since our algebra $\mathfrak{p}_+$ is abelian, the second term in the sum is zero. The Lie algebra homology differential is P-equivariant.

The k^{th} chain space $C^k(\mathfrak{p}_-, \mathbb{V})$ of the Lie algebra cohomology $H^\bullet(\mathfrak{p}_-, \mathbb{V})$ of $\mathfrak{p}_-$ with coefficients in $\mathbb{V}$ is the space of antisymmetric linear mappings from the k^{th} tensor power of $\mathfrak{p}_-$ to $\mathbb{V}$. The Lie algebra cohomology differential is defined by the following general formula

$$(\partial\, \psi)(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i X_i \cdot \psi(X_0, \dots, \widehat{X_i}, \dots, X_l) + \\ + \sum_{i<j} (-1)^{i+j} \psi([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k)\,,$$

where $\psi \in \operatorname{Hom}(\Lambda^k \mathfrak{p}_-, \mathbb{V})$ and $X_i \in \mathfrak{p}_-$ for $i = 0, \dots, k$. Again, we can forget the second term in our case. We can identify $C^k(\mathfrak{p}_-, \mathbb{V}) = \operatorname{Hom}(\Lambda^k \mathfrak{p}_-, \mathbb{V})$ with $\Lambda^k {\mathfrak{p}_-}^* \otimes \mathbb{V}$. Since the Killing form induces an isomorphism ${\mathfrak{p}_-}^* \simeq \mathfrak{p}_+$, we can consider the Lie algebra cohomology differential ∂ as an operator on the chain spaces of Lie algebra homology $\partial : \Lambda^k {\mathfrak{p}_-}^* \otimes V \to \Lambda^{k+1} {\mathfrak{p}_-}^* \otimes V$. After these identifications we get the formula

$$\partial(Z_1 \wedge \dots \wedge Z_k \otimes v) = \sum_{i=1}^{2p} \epsilon^i \wedge Z_1 \wedge \dots \wedge Z_k \otimes e_i \cdot v\,.$$

The Lie algebra cohomology differential is $\overline{P}$-equivariant. In particular, both ∂ and ∂^* are $K_{\mathbb{C}}$-equivariant and consequently they preserve the geometric weight.

The Kostant Laplacian is defined on each $C_k(\mathfrak{p}_+, \mathbb{V})$ as $\Box = \partial^*\, \partial + \partial\, \partial^*$. It was proven in [11] that for a finite-dimensional $\mathfrak{g}$-representation $\mathbb{V}$ there is a positive definite scalar product on $C_\bullet(\mathfrak{p}_+, \mathbb{V})$ with respect to which are ∂ and ∂^* adjoint. It follows that there is a direct sum *Hodge decomposition* of $K_{\mathbb{C}}$-modules $C_\bullet(\mathfrak{p}_+, \mathbb{V}) = \operatorname{im} \partial \oplus \ker \Box \oplus \operatorname{im} \partial^*$ and moreover $\ker \partial^* = \ker \Box \oplus \operatorname{im} \partial^*$ and $\ker \partial = \ker \Box \oplus \operatorname{im} \partial$. It follows that $H_\bullet(\mathfrak{p}_+, \mathbb{V}) \simeq \ker \Box \simeq H^\bullet(\mathfrak{p}_-, \mathbb{V})$.

The authors of [10] proved that if the representation $\mathbb{V}$ is a unitarizable $(\mathfrak{g}, K)$-module, then the Hodge decomposition of $C_\bullet(\mathfrak{p}_+, \mathbb{V})$ is still valid.

Inspecting the classification of unitarizable highest weight modules in [6], we see that we can take for $\mathbb{V}$ the module $L(\lambda)$ with $\lambda = (2 - n)\omega_1$ in the case $p = 2n - 2$ and $\lambda = (3/2 - n)\omega_1$ in the case $p = 2n - 1$. From now on, when we write $L(\lambda)$ we will mean either $L((2 - n)\omega_1)$ or $L((\frac{3}{2} - n)\omega_1)$ depending on the value of $p = \dim G - 2$.

The cohomology groups $H^i(\mathfrak{p}_-, L(\lambda))$ can be computed from the formula given in [7]. We get in the even case

$$H^0(\mathfrak{p}_-, L((2 - n)\omega_1)) = F((2 - n)\omega_1)$$
$$H^1(\mathfrak{p}_-, L((2 - n)\omega_1)) = F(-n\omega_1)$$
$$H^i(\mathfrak{p}_-, L((2 - n)\omega_1)) = 0 \text{ for } i \geq 2$$

and in the odd case

$$\begin{aligned}
H^0(\mathfrak{p}_-, L((\tfrac{3}{2}-n)\omega_1)) &= F((\tfrac{3}{2}-n)\omega_1)\\
H^1(\mathfrak{p}_-, L((\tfrac{3}{2}-n)\omega_1)) &= F((-\tfrac{1}{2}-n)\omega_1)\\
H^i(\mathfrak{p}_-, L((\tfrac{3}{2}-n)\omega_1)) &= 0 \quad \text{for } i \geq 2.
\end{aligned}$$

The problem is that $L(\lambda)$ is not a P-representation. This is not surprising because the generalized Verma module $M(\lambda)$ was induced from a $\overline{P}$-representation. Let $L(\lambda) = \bigoplus_{\mu \in \widehat{K_{\mathbb{C}}}} L_\mu$ be the decomposition of $L(\lambda)$ into $K_{\mathbb{C}}$-types. Each L_μ is contained in some $S^k(\mathfrak{p}_+, F(\lambda))$ modulo the maximal submodule of $M(\lambda)$ and the algebra $\mathfrak{p}_+$ acts as a multiplication by a variable, while $\mathfrak{p}_-$ acts basically as a differentiation. To formalize this write $L(\lambda) = \bigoplus_{\mu \in \widehat{K_{\mathbb{C}}}, k \in \mathbb{N}} L_{\mu,k}$ where $L_{\mu,k}$ is the $K_{\mathbb{C}}$-type contained in $S^k(\mathfrak{p}_+, F(\lambda))$ and note that $\mathfrak{p}_+(L_{\mu,k}) \subset L_{\mu,k+1}$.

The *formal globalization* ([13]) of $L(\lambda)$ is defined as $\overline{L(\lambda)} = \prod_{\mu \in \widehat{K_{\mathbb{C}}}} L_\mu$ (product of topological vector spaces). Since each $K_{\mathbb{C}}$-type is finite-dimensional and each $S^k(\mathfrak{p}_+, F(\lambda))$ contains only finitely many irreducible $K_{\mathbb{C}}$-representations, we can write it as

$$\overline{L(\lambda)} = \prod_{k \in \mathbb{N}} L_k \,,$$

where $L_k = \bigoplus_{\mu \in \widehat{K_c}} L_{\mu,k}$ is a finite sum. The action of $\mathfrak{p}_+$ works as a right shift: $\mathfrak{p}_+(L_k) \subset L_{k+1}$.

Now it is easy to see that the formal globalization is a representation of P, because the action of $\mathfrak{p}_+$ on $\overline{L(\lambda)}$ integrates without any problems. The component of degree k of $\exp(X)v$, $X \in \mathfrak{p}_+$ is given by a sum of $k+1$ elements involving components of v of degree $\leq k$. The P-invariant filtration is given by

$$\overline{L(\lambda)}^{\,l} = \prod_{k=l}^{\infty} L_k \,.$$

Theorem 2.1. *There is a Hodge decomposition for* $C^\bullet(\mathfrak{p}, \overline{L(\lambda)})$ *and*

$$H^\bullet(\mathfrak{p}_-, L(\lambda)) = H^\bullet(\mathfrak{p}_-, \overline{L(\lambda)}) \,.$$

Proof. The article [11] proves that for a Hodge decomposition to exist it is sufficient to have ∂^* and ∂ disjoint, meaning that $\ker \partial^* \cap \operatorname{im} \partial = 0$ and $\ker \partial \cap \operatorname{im} \partial^* = 0$. We claim that the operators ∂^* and ∂ are disjoint even on the level of formal globalization and thus the Hodge decomposition is still valid. The proof is simple — since both ∂^* and ∂ are $K_{\mathbb{C}}$-equivariant, they act element-wise on $\prod_{\mu \in K_C} L_\mu$.

Explicitly, suppose for contradiction that there exists $u \in \ker \partial^* \cap \operatorname{im} \partial$ such that the μ-component u_μ of u is not zero. Then since ∂^* and ∂ act component-wise, we get that $\partial^* u_\mu = 0$ and there must be $v_\mu \in L_\mu$ such that $\partial v_\mu = u_\mu$. But that means that the nonzero element $u_\mu \in L(\lambda)$ is contained in $\ker \partial^* \cap \operatorname{im} \partial = 0$.

Similarly, since $H^\bullet(\mathfrak{p}_-, L(\lambda)) = \ker \partial^* \cap \ker \partial$ is *finite-dimensional* graded vector space, we get that the cohomology remains the same. ◻

3. Construction

First we need to recall some fundamental notions of parabolic geometries. The canonical reference is the book by Čap and Slovák [3]. For reader's convenience we repeat the Calderbank – Diemer construction for $\mathbb{V}$ finite-dimensional, however in order to get the Yamabe operator we treat also the case when $\mathbb{V} = L(\lambda)$ and the parabolic pair (G, P) is the pair of complex conformal geometry — i.e. $G = SO(p+2, \mathbb{C})$ and P parabolic subgroup with Levi part $K_{\mathbb{C}}$.

Let G be a Lie group and let P be its parabolic subgroup. A *Cartan geometry* modeled on the pair (G, P) is a P-principal bundle $\pi : \mathcal{G} \to M$ with a P-equivariant one-form $\omega : T\mathcal{G} \to \mathfrak{g}$ such that for each $u \in \mathcal{G}$, $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is an isomorphism restricting to the canonical isomorphism between $T_u(\mathcal{G}_{\pi(u)})$ and $\mathfrak{p}$ (the so called *Cartan connection*). The *curvature function* $\kappa : \mathcal{G} \to \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ of a Cartan geometry is defined by

$$\kappa(u)(\xi, \chi) = [\xi, \chi] - \omega([\omega^{-1}(\xi), \omega^{-1}(\chi)])(u)\,,$$

where the first bracket is the bracket of $\mathfrak{g}$ while the second bracket is just the bracket of vector fields on $\mathcal{G}$.

For any (possibly infinite dimensional) continuous representation $\mathbb{V}$ of P we can form an associated topological vector bundle $\mathcal{V} := \mathcal{G} \times_P \mathbb{V} \to M$. The bundle associated to $\mathfrak{p}_+$ is the cotangent bundle T^*M and the bundle associated to $\mathfrak{g}/\mathfrak{p}$ is the tangent bundle TM. It what follows we identify the P-representation $\mathfrak{g}/\mathfrak{p}$ with $\mathfrak{p}_-$ via the Killing form of $\mathfrak{g}$. The bundle associated to the adjoint representation on $\mathfrak{g}$ is called *adjoint tractor bundle* and is denoted by $\mathcal{A}M$.

It can be checked that κ is in fact horizontal and P-equivariant and hence it induces a section of $\Omega^2 M \otimes \mathcal{A}M$ which we will denote by the same symbol.

Sections of associated bundles are in bijective correspondence with P-equivariant functions on the total space $\mathcal{G}$. For an infinite-dimensional representation $\mathbb{V}$ we define smooth sections as smooth P-equivariant functions on the total space with values in $\mathbb{V}$. It directly follows that a smooth section can have values only in the subspace of smooth vectors of $\mathbb{V}$.

The *invariant derivative* on V is defined by

$$\begin{gathered}\nabla^\omega : \mathcal{C}^\infty(\mathcal{G}, \mathbb{V}) \to \mathcal{C}^\infty(\mathcal{G}, \mathfrak{g}^* \otimes \mathbb{V}) \\ \nabla^\omega_\xi f = \mathrm{d}f(\omega^{-1}(\xi))\end{gathered}$$

for all $\xi \in \mathfrak{g}$. It is P-equivariant and so maps $\mathcal{C}^\infty(\mathcal{G}, \mathbb{V})^P$ into $\mathcal{C}^\infty(\mathcal{G}, \mathfrak{g}^* \otimes \mathbb{V})^P$ and thus we get a linear map $\nabla^\omega : \Gamma(M, \mathcal{V}) \to \Gamma(M, \mathcal{A}M \otimes \mathcal{V})$. Note that from the definition of the fundamental derivative it follows that $\nabla^\omega_{e_i} s$ has the same geometric weight as s.

We define the *tractor connection* by

$$\begin{gathered}\nabla^{\mathfrak{g}} : \mathcal{C}^\infty(\mathcal{G}, \mathbb{V}) \to \mathcal{C}^\infty(\mathcal{G}, \mathfrak{g}^* \otimes \mathbb{V}) \\ \nabla^{\mathfrak{g}}_\xi f = \nabla^\omega_\xi f + \xi \cdot f\,.\end{gathered}$$

It is easily checked that for P-equivariant f and for any $\xi \in \mathfrak{p}$ we get $\nabla^{\mathfrak{g}}_\xi f = 0$ and hence $\nabla^{\mathfrak{g}}$ induces a covariant derivative on $\mathcal{V}$.

We define the associated twisted deRham differential

$$\mathrm{d}^{\mathfrak{g}} : \Gamma(M, \Omega^k \mathcal{V}) \to \Gamma(M, \Omega^{k+1}\mathcal{V})$$

by the usual formula. We will need the expression for $\mathrm{d}^{\mathfrak{g}}$ in local coordinates. Let ϵ^i be elements of the basis of $\mathfrak{p}_+$, let e_i be the elements of the dual basis and denote the corresponding sections on M by the same symbols. Then

$$(\mathrm{d}^{\mathfrak{g}}\, s)(u) = \sum_i \epsilon^i \wedge (\nabla^{\omega}_{e_i} s)(u) + \partial\, s(u) - \sum_{i<j} \epsilon^i \wedge \epsilon^j \wedge \kappa(e_i, e_j) \lrcorner\, s(u)\,,$$

where only the first term depends on the one-jet of $s \in \Gamma(M, \mathcal{V})$ and the remaining two terms act algebraically on the values of s. Note that only the $\mathfrak{p}_-$-component of $\kappa(e_i, e_j)$ (the *torsion component*) contributes to the contraction.

The Lie algebra homology differential $\partial^* : \Lambda^i \mathfrak{p}_+ \otimes \mathbb{V} \to \Lambda^{i-1}\mathfrak{p}_+ \otimes \mathbb{V}$ is P-equivariant and hence it induces operator (denoted by the same symbol) on the bundles associated to the chain spaces. These bundles are of course exterior forms with values in $\mathcal{V}$ and we will denote them by $\Omega^i\mathcal{V}$. Let $B_i\mathcal{V}$ denote the image of ∂^* on i-forms with values in $\mathcal{V}$, let $Z_i\mathcal{V}$ denote its kernel and let $H_i\mathcal{V}$ denote the corresponding quotients $Z_i\mathcal{V}/B_i\mathcal{V}$. Again, from P-equivariance it follows that there are natural identifications

$$Z_\bullet \mathcal{V} = \mathcal{G} \times_P \ker \partial^* \qquad B_\bullet \mathcal{V} = \mathcal{G} \times_P \operatorname{im} \partial^* \qquad H_i\mathcal{V} = \mathcal{G} \times_P H_i(\mathfrak{p}_+, \mathbb{V})\,.$$

The BGG operators were constructed in [1] by using a family of differential operators $\Pi^{\mathfrak{g}}_k : \Gamma(M, \Omega^k\mathcal{V}) \to \Gamma(M, \Omega^{k+1}\mathcal{V})$ which vanish on im ∂^* and map into $\ker \partial^*$. The *BGG operator* $D_k : \Gamma(\mathcal{H}_k(\mathfrak{p}_+, \mathbb{V})) \to \Gamma(\mathcal{H}_{i+1}(\mathfrak{p}_+, \mathbb{V}))$ is then defined as

$$D_k s := \operatorname{proj} \circ \Pi^{\mathfrak{g}}_{k+1} \circ \mathrm{d}^{\mathfrak{g}} \circ \Pi^{\mathfrak{g}}_k \circ \operatorname{rep},$$

where proj is the algebraic projection on homology and rep is a choice of representative in the homology class.

The idea for constructing $\Pi^{\mathfrak{g}}_k$ comes from the expression for the algebraic projection onto $\ker \Box$ which is given by $\mathrm{Id} - \Box^{-1}\, \Box$. Because ∂^* commutes with $\Box^{-1}$ this equals to $\mathrm{Id} - \Box^{-1}\, \partial^*\, d - d\, \Box^{-1}\, \partial^*$. We need a P-equivariant operator and since the Lie algebra cohomology differential is the only thing that is not P-equivariant in this formula, we can try to restore the P-equivariance by adding a differential term. This reasoning leads to the following definitions

$$\Box_{\mathfrak{g}} = \partial^*\, \mathrm{d}^{\mathfrak{g}} + \mathrm{d}^{\mathfrak{g}}\, \partial^*, \quad Q = {\Box_{\mathfrak{g}}}^{-1}\, \partial^*$$
$$\Pi^{\mathfrak{g}}_k = \mathrm{Id} - Q\, \mathrm{d}^{\mathfrak{g}} - \mathrm{d}^{\mathfrak{g}}\, Q.$$

Now the problem arises how to compute the inverse of $\Box_{\mathfrak{g}}$ at least on the image of ∂^*. Once this inverse is provided, the desired properties of $\Pi^{\mathfrak{g}}_k$ follow immediately as algebraic consequences.

There always exists a reduction of our P-bundles to its Levi subgroup L, which means that we can construct the sought inverse by using L-equivariant operators. Since the inverse must be unique it follows that it doesn't depend on the choice of reduction from P to L and hence it is P-equivariant.

Lemma 3.1. *Let $\mathbb{V}$ be either a finite-dimensional $(\mathfrak{g}, P)$-module or the representation $\overline{L(\lambda)}$. Then the operator $\square_{\mathfrak{g}}$ is invertible on $B_i\mathcal{V}$ and the inverse is given by*

$$\square_{\mathfrak{g}}{}^{-1} = \Big(\sum_{k=0}^{\infty} N^k\Big)\,\square^{-1}\,,$$

where $N = -\,\square^{-1}(\square_{\mathfrak{g}} - \square)$.

Proof. We need to prove that the infinite sum makes sense for any section $s \in \Gamma(M, B_i\mathcal{V})$. Let us compute the local expression for $N(s)$, where we consider s to have values in some irreducible L-type.

$$\begin{aligned} -\,\square\, N(s)(u) &= (\square_{\mathfrak{g}} - \square)s(u) \\ &= \partial^*(\mathrm{d}^{\mathfrak{g}} - d)s(u) \quad \text{ because } \; s \in \Gamma(M, B_i\mathcal{V}) \\ &= \partial^*\Big(\sum_i \epsilon^i \wedge \nabla^{\omega}_{e_i} s - \sum_{i<j} \epsilon^i \wedge \epsilon^j \wedge \kappa(e_i, e_j) \lrcorner\, s\Big)(u)\,. \end{aligned}$$

The first term increases the geometric weight, because fundamental derivative doesn't change w, wedging with an element from $\mathfrak{p}_+$ increases it by one and ∂^* preserves the geometric weight.

The second term also increases the weight, because the contraction with $\kappa(e_i, e_j)$ lowers it by one and wedging with two elements from $\mathfrak{p}_+$ increases it by two.

For a finite-dimensional representation $\mathbb{V}$ it follows that the operator N is nilpotent and in the infinite sum there is only finitely many terms nonzero. Thus the operator $\Pi^{\mathfrak{g}}_k$ is a differential operator of finite order.

If we start with $\mathbb{V} = \overline{L(\lambda)}$, then it is sufficient to consider only the case $i = 0$, because all higher homologies are zero. For a section with values in the representation $\overline{L(\lambda)} = \prod_{k=0}^{\infty} L_k$ we get that N works as a component-wise derivation composed with right shift. It follows that the sum is well defined and the components of $\Pi^{\mathfrak{g}}_k$ are differential operators of increasing order — the component corresponding to L_k of the sum $\sum_{k=0}^{\infty} N(s)(u)$ has at most $k+1$ nonzero terms. $\square$

The following proposition is one of the main results of [1]. Since the (easy) proof was left to the reader there and since the original statement contained some irrelevant sign errors, we write down all the details here.

Proposition 3.2 ([1], Proposition 5.5)**.** *The operator $\Pi^{\mathfrak{g}}_k \colon \Gamma(M, \Omega^k\mathcal{V}) \to \Gamma(M, \Omega^k\mathcal{V})$ has the following properties.*

(1) *The operator $\Pi^{\mathfrak{g}}_k$ vanishes on $\operatorname{im}\partial^*$ and maps into $\ker\partial^*$:*

$$\Pi^{\mathfrak{g}}_k \circ \partial^* = 0 \quad \& \quad \partial^* \circ \Pi^{\mathfrak{g}}_k = 0\,.$$

(2) *The operator $\Pi^{\mathfrak{g}}_k$ induces identity on the homology $\mathcal{H}_k(\mathfrak{p}_+, \mathbb{V})$:*

$$\Pi^{\mathfrak{g}}_k = \mathrm{Id} \mod \operatorname{im}\partial^*\,.$$

(3) *The commutator of $\mathrm{d}^{\mathfrak{g}}$ and $\Pi^{\mathfrak{g}}$ equals to the commutator of Q and R*

$$\mathrm{d}^{\mathfrak{g}} \circ \Pi^{\mathfrak{g}}_k - \Pi^{\mathfrak{g}}_{k+1} \circ \mathrm{d}^{\mathfrak{g}} = Q \circ R - R \circ Q\,,$$

where R is the curvature operator defined by $R(s) = (\mathrm{d}^{\mathfrak{g}} \circ \mathrm{d}^{\mathfrak{g}})(s)$.

(4) *For $k = 0$ and in the flat case, the operator is actually a projection:*

$$\left(\Pi_k^{\mathfrak{g}}\right)^2 = \Pi_k^{\mathfrak{g}} + Q \circ R \circ Q\,.$$

(5)

$$\Pi_k^{\mathfrak{g}} \circ \Box_{\mathfrak{g}} = -Q \circ R \circ \partial^* \quad \& \quad \Box_{\mathfrak{g}} \circ \Pi_k^{\mathfrak{g}} = -\,\partial^* \circ R \circ Q\,.$$

Thus in the flat case we have a projection $\Pi^{\mathfrak{g}}$ onto a subspace of $\ker \partial^*$ *complementary to* $\operatorname{im} \partial^*$ *and moreover, this projection is actually a chain map between twisted deRham complexes $\mathrm{d}^{\mathfrak{g}} \colon \Omega^\bullet \mathcal{V} \to \Omega^{\bullet+1}\mathcal{V}$ which is homotopic to the identity, the chain-homotopy being the operator $Q \colon \Omega^\bullet \mathcal{V} \to \Omega^{\bullet-1}\mathcal{V}$.*

Proof. We will prove these point one by one by easy algebraic manipulations.

The first point is proven by the following two calculations:

$$\begin{aligned}
\Pi^{\mathfrak{g}}\, \partial^* &= \left(\mathrm{Id} - {\Box_{\mathfrak{g}}}^{-1}\, \partial^*\, \mathrm{d}^{\mathfrak{g}} - \mathrm{d}^{\mathfrak{g}}\, {\Box_{\mathfrak{g}}}^{-1}\, \partial^*\right) \partial^* \\
&= \partial^* - {\Box_{\mathfrak{g}}}^{-1}\, \partial^*\, \mathrm{d}^{\mathfrak{g}}\, \partial^* \\
&= \partial^* - {\Box_{\mathfrak{g}}}^{-1}\, \Box_{\mathfrak{g}}\, \partial^* && \text{because } \partial^*\, \mathrm{d}^{\mathfrak{g}}\, \partial^* = \Box_{\mathfrak{g}}\, \partial^*
\end{aligned}$$

proves the first half and

$$\begin{aligned}
\partial^*\, \Pi^{\mathfrak{g}} &= \partial^* - \partial^*\, {\Box_{\mathfrak{g}}}^{-1}\, \partial^*\, \mathrm{d}^{\mathfrak{g}} - \partial^*\, \mathrm{d}^{\mathfrak{g}}\, {\Box_{\mathfrak{g}}}^{-1}\, \partial^* \\
&= \partial^* - \partial^*\, \mathrm{d}^{\mathfrak{g}}\, {\Box_{\mathfrak{g}}}^{-1}\, \partial^* && \text{because } [\partial^*, {\Box_{\mathfrak{g}}}^{-1}] = 0 \text{ on } \operatorname{im} \partial^* \\
&= \partial^* - \Box_{\mathfrak{g}}\, {\Box_{\mathfrak{g}}}^{-1}\, \partial^* && \text{since } \Box_{\mathfrak{g}} = \partial^*\, \mathrm{d}^{\mathfrak{g}} \text{ on } \operatorname{im} \partial^*
\end{aligned}$$

proves the second half of the first point.

The second point is a direct consequence of definitions, because for a section s with values in $Z_k\mathcal{V}$ we get $\Pi_k^{\mathfrak{g}}(s) = s - {\Box_{\mathfrak{g}}}^{-1}\, \partial^*\, \mathrm{d}^{\mathfrak{g}}\, s$ and ${\Box_{\mathfrak{g}}}^{-1}$ maps $B_k\mathcal{V}$ to $B_k\mathcal{V}$.

Proof of the next point of the proposition is also just unwinding the definitions and trivial algebra:

$$[\mathrm{d}^{\mathfrak{g}}, \Pi^{\mathfrak{g}}] = [\mathrm{d}^{\mathfrak{g}}, -Q\, \mathrm{d}^{\mathfrak{g}} - \mathrm{d}^{\mathfrak{g}}\, Q] = -\,\mathrm{d}^{\mathfrak{g}}\, Q\, \mathrm{d}^{\mathfrak{g}} - \mathrm{d}^{\mathfrak{g}}\, \mathrm{d}^{\mathfrak{g}}\, Q + Q\, \mathrm{d}^{\mathfrak{g}}\, \mathrm{d}^{\mathfrak{g}} + \mathrm{d}^{\mathfrak{g}}\, Q\, \mathrm{d}^{\mathfrak{g}}\ .$$

To prove the fourth point, it is good to note first that from the already proven fact $\Pi^{\mathfrak{g}}\, \partial^* = 0$ it follows that also $\Pi^{\mathfrak{g}} Q = 0$. Moreover even Q^2 equals zero. Now we have

$$\begin{aligned}
(\Pi^{\mathfrak{g}})^2 &= \Pi^{\mathfrak{g}}(\mathrm{Id} - Q\, \mathrm{d}^{\mathfrak{g}} - \mathrm{d}^{\mathfrak{g}}\, Q) \\
&= \Pi^{\mathfrak{g}} - \Pi^{\mathfrak{g}} Q\, \mathrm{d}^{\mathfrak{g}} - \Pi^{\mathfrak{g}}\, \mathrm{d}^{\mathfrak{g}}\, Q \\
&= \Pi^{\mathfrak{g}} - \mathrm{d}^{\mathfrak{g}}\, \Pi^{\mathfrak{g}} Q + [\mathrm{d}^{\mathfrak{g}}, \Pi^{\mathfrak{g}}] Q \\
&= \Pi^{\mathfrak{g}} + [Q, R] Q && \text{by the third point of the proposition} \\
&= \Pi^{\mathfrak{g}} + QRQ + RQQ.
\end{aligned}$$

The last point requires two calculations:

$$\begin{aligned}
\Box_{\mathfrak{g}} \Pi^{\mathfrak{g}} &= \partial^* \mathrm{d}^{\mathfrak{g}} \Pi^{\mathfrak{g}} + \mathrm{d}^{\mathfrak{g}} \partial^* \Pi^{\mathfrak{g}} && \text{the second term here is zero by the first point} \\
&= \partial^* \Pi^{\mathfrak{g}} \mathrm{d}^{\mathfrak{g}} + \partial^* [\mathrm{d}^{\mathfrak{g}}, \Pi^{\mathfrak{g}}] && \text{here the first term is zero by the first point} \\
&= \partial^* [Q, R] && \text{by point three} \\
&= \partial^* QR - \partial^* RQ \\
&= -\partial^* RQ && \text{because } \partial^* Q = \partial^* \Box_{\mathfrak{g}}{}^{-1} \partial^* = 0
\end{aligned}$$

and a similar one

$$\begin{aligned}
\Pi^{\mathfrak{g}} \Box_{\mathfrak{g}} &= \Pi^{\mathfrak{g}} (\partial^* \mathrm{d}^{\mathfrak{g}} + \mathrm{d}^{\mathfrak{g}} \partial^*) = \Pi^{\mathfrak{g}} \partial^* \mathrm{d}^{\mathfrak{g}} + \Pi^{\mathfrak{g}} \mathrm{d}^{\mathfrak{g}} \partial^* \\
&= \mathrm{d}^{\mathfrak{g}} \Pi^{\mathfrak{g}} \partial^* + [\Pi^{\mathfrak{g}}, \mathrm{d}^{\mathfrak{g}}] \partial^* = -[Q, R] \partial^* \\
&= -QR \partial^* + RQ \partial^* = -QR \partial^* \ .
\end{aligned}$$

Finally, we deal with $\operatorname{im} \partial^* \cap \operatorname{im} \Pi^{\mathfrak{g}}$. By the fourth point of the proposition we get

$$\partial^* u = \Pi^{\mathfrak{g}} v = \Pi^{\mathfrak{g}} \Pi^{\mathfrak{g}} v - QRQv = \Pi^{\mathfrak{g}} \partial^* u - QRQv$$

which by the first point of the proposition implies that $\operatorname{im} \partial^* \cap \operatorname{im} \Pi^{\mathfrak{g}} = \operatorname{im} QRQ$. Thus in the flat case we see that $\operatorname{im} \partial^*$ and $\operatorname{im} \Pi^{\mathfrak{g}}$ are complementary. Finally, in the flat case, the statement that the projection operator $\Pi^{\mathfrak{g}}$ is chain-homotopic to the identity via Q is a direct consequence of definitions. $\square$

Because $\Pi^{\mathfrak{g}}$ maps into $\ker \partial^*$ we have that $\Box_{\mathfrak{g}} \Pi^{\mathfrak{g}} = \partial^* \mathrm{d}^{\mathfrak{g}} \Pi^{\mathfrak{g}}$. Combining this equality with the fifth point of the previous proposition gives us $\partial^* \mathrm{d}^{\mathfrak{g}} \Pi^{\mathfrak{g}} = \Box_{\mathfrak{g}} \Pi^{\mathfrak{g}} = -\partial^* RQ$. Since $Q = 0$ on $\ker \partial^*$, we see see that $\mathrm{d}^{\mathfrak{g}} \Pi^{\mathfrak{g}}$ maps $\ker \partial^*$ to $\ker \partial^*$. This allows us to write the BGG operator as $D_k = \operatorname{proj} \circ \mathrm{d}^{\mathfrak{g}} \circ \Pi^{\mathfrak{g}}_k \circ \operatorname{rep}$.

The operator $\Pi^{\mathfrak{g}} \circ \operatorname{rep} \colon H_\bullet \mathcal{V} \to \Omega^\bullet \mathcal{V}$ gives us the unique representatives of the homology classes in $\ker \Box_{\mathfrak{g}}$. Indeed $\ker \Box_{\mathfrak{g}} \cap \operatorname{im} \partial^* = 0$, because for $u = \partial^* v \in \ker \Box_{\mathfrak{g}}$ we get $0 = \Box_{\mathfrak{g}} u = \Box_{\mathfrak{g}} \partial^* v$ and we know that $\Box_{\mathfrak{g}}$ is invertible on $\operatorname{im} \partial^*$.

Proposition 3.3. *The operator* $\operatorname{proj} \circ \Pi^{\mathfrak{g}}_0$ *maps injectively parallel sections of* $\mathcal{V}$ *into the kernel of* D_0. *In the flat case, the operator is even surjective with inverse being* $\Pi^{\mathfrak{g}}_0 \circ \operatorname{rep}$.

Proof. The restriction of $\Pi^{\mathfrak{g}}_0$ to $\ker \mathrm{d}^{\mathfrak{g}}$ equals identity and it is easily computed that the operator $\operatorname{proj} \circ \Pi^{\mathfrak{g}}_0$ maps parallel sections into the solution space of D_0. For a parallel section $s \in \ker \mathrm{d}^{\mathfrak{g}} \cap \operatorname{im} \partial^*$ we have $s = \Pi^{\mathfrak{g}}_0(s)$ and thus $s \in \operatorname{im} \partial^* \cap \operatorname{im} \Pi^{\mathfrak{g}}_0$. Since this space equals to the image of QRQ, which is a zero mapping on 0-forms, we see that the map $\operatorname{proj} \circ \Pi^{\mathfrak{g}}_0$ is indeed injective on parallel section of $\mathcal{V}$.

On the other hand, for $u \in \ker D_0$ we get $\big(\mathrm{d}^{\mathfrak{g}} \Pi^{\mathfrak{g}}_0 \circ \operatorname{rep}\big)(u) \in \operatorname{im} \partial^*$ and in the flat case we can commute $\mathrm{d}^{\mathfrak{g}}$ and $\Pi^{\mathfrak{g}}$ according to the third point of the proposition. It follows that $\big(\mathrm{d}^{\mathfrak{g}} \Pi^{\mathfrak{g}}_0 \circ \operatorname{rep}\big)(u) = \big(\Pi^{\mathfrak{g}}_1 \mathrm{d}^{\mathfrak{g}} \circ \operatorname{rep}\big)(u)$ and because $\operatorname{im} \partial^* \cap \operatorname{im} \Pi^{\mathfrak{g}}_1 = 0$ we get that in the flat case the operator $\Pi^{\mathfrak{g}}_0 \circ \operatorname{rep}$ maps $\ker D_0$ to parallel sections of $\mathrm{d}^{\mathfrak{g}}$. $\square$

4. Conclusion

We have constructed a nontrivial differential operator of finite order over a complex parabolic geometry of conformal type. This operator acts between sections of bundles associated to one-dimensional representations, whose geometric weights are precisely those for the conformal Yamabe operator. It is well known that there is essentially unique differential operator acting between such bundles, i.e. the Yamabe operator.

Of course, it would be desirable to investigate this construction in real cases, i.e. for classical (pseudo)conformal structures. However, as the example in [12] shows, one must be extremely careful to draw any conclusions. The article [10] shows that the Hodge decomposition is valid for all unitarizable highest weight representations, which were classified e.g. in [6], and the same trick with formal globalization goes through. Moreover, the results cover not only the cohomology of $\mathfrak{p}_+$ but also of its appropriate subalgebras. The resulting operators are to be identified not as easily as the Yamabe operator, however there is a close connection of unitarizable highest weight modules and $\mathfrak{k}$-invariant differential operators presented in [5]. Consulting the weights of appropriate zeroth and first homology modules, there doesn't appear to be a case which would yield higher GJMS operators as first BGG operators.

All of these matters are currently investigated by the author of this note and are to be part of his dissertation thesis.

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Mathematical Institute, Charles University,
Sokolovská 83, 186 75 Prague,
Czech Republic
E-mail: vit.tucek@gmail.com