FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

DOCTORAL THESIS

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Tilting theory of commutative rings

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Study programme: Mathematics
Study branch: Algebra, number theory, and mathematical logic

Prague 2017

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Title: Tilting theory of commutative rings

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Abstract: The thesis compiles my contributions to the tilting theory, mainly in the setting of a module category over a commutative ring. We give a classification of tilting classes over an arbitrary commutative ring in terms of data of geometrical flavor - certain filtrations of the Zariski spectrum. This extends and connects the results known previously for the noetherian case, and for Prüfer domains. Also, we show how the classes can be expressed using the local and Čech homology theory. For 1-tilting classes, we explicitly construct the associated tilting modules, generalizing constructions of Fuchs and Salce. Furthermore, over any commutative ring we classify the silting classes and modules. Amongst other results, we exhibit new examples of cotilting classes, which are not dual to any tilting classes - a phenomenon specific to non-noetherian rings.

Keywords: commutative ring, tilting module, silting module, representation theory, local cohomology

Název: Vychylující teorie nad komutativními okruhy

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Abstrakt: Práce shrnuje mé příspěvky k vychylující teorii, především pro kategorii modulů nad komutativním okruhem. V práci klasifikujeme vychylující třídy nad libovolným komutativním okruhem pomocí údajů s geometrickou příchutí - jisté filtrace Zariskiho spektra. Tento výsledek zobecňuje a dává jednotný rámec výsledkům do té doby známým v noetherovském případě a pro Prüferovské obory. Dále ukážeme, jak lze tyto třídy vyjádřit pomocí lokální či Čechovy homologické teorie. Pro 1-vychylující třídy zkonstruujeme explicitně příslušné vychylující moduly, čímž zobecníme konstrukci Fuchse a Salceho. Navíc, nad libovolným komutativním okruhem popíšeme silting třídy i moduly. Mezi dalšími výsledky zmiňme nové příklady kovychylujících tříd, které nejsou duální žádné vychylující třídě - fenomén specifický pro nenoetherovské okruhy.

Klíčová slova: komutativní okruh, vychylující modul, silting modul, teorie reprezentací, lokální homologie

I would like to thank my advisor Jan Trlifaj for his patient guidance, unfaltering support, and continuous encouragement during my studies, as well as for all the time spent on my manuscripts. My thanks belong to both co-authors of the papers included in this thesis - to Jan Šťovíček for all the discussions and his endless patience whenever I lacked to comprehend, and to Lidia Angeleri Hügel for the warm hospitality and all the time she reserved for our work.

My gratitude is definitely due to my family, for always supporting me, to my friends, for keeping me sane, and to Radka, for surviving my studies.

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Chapter 1

Introduction: Tilting theory of commutative rings

The core of this thesis consists of the following three papers, two of them published:

[i] Michal Hrbek, *One-tilting classes and modules over commutative rings*, Journal of Algebra, 462:1-22, 2016.

DOI: 10.1016/j.jalgebra.2016.05.014

[ii] Lidia Angeleri Hügel and Michal Hrbek, *Silting modules over commutative rings*, International Mathematics Research Notices, 2016.

DOI: 10.1093/imrn/rnw147

[iii] Michal Hrbek and Jan Šťovíček, *Tilting classes over commutative rings*, preprint, arXiv:1701.05534, 2017.

Tilting theory originated in the work of Brenner-Butler and Happel-Ringel in the early 1980's. Since then, it crawled out of its original habitat of the category of finite-dimensional representations of finite-dimensional algebras, and was generalized in various directions. In this thesis we are concerned with the setting of modules over an arbitrary commutative ring. Paper [i] extends the classification of 1-tilting classes, at the time known for commutative noetherian ring, and for Prüfer and almost perfect domains, to an arbitrary commutative ring. Also, the associated tilting modules are constructed explicitly. In [ii], we extend these results further to the rather new setting of silting modules, introduced by Angeleri-Marks-Vitória in 2014. Finally, in the preprint [iii], we classify all n-tilting classes over a commutative ring. The general theme is that the classes, be it n-tilting or silting, are parametrized by data of geometrical flavor - finite filtrations of subsets of the Zariski spectrum of the ring (or rather, its "Hochster" dual). Also, we show how the n-tilting classes can expressed alternatively as classes of modules vanishing in certain degrees of local, or Čech homology theory.

Before presenting these three papers, the rest of this chapter will gather relevant aspects of tilting theory and commutative algebra, as well as attempt to put our results into a wider context. Also, in the last section, we prove some new results on flat cover closure of tilting classes.

A short disclaimer concerning the structure of the thesis:

- Each of the four chapters, be it this introductory one, or the three chapters containing the papers [i]-[iii], has its own bibliography at its end. Therefore, any citation in square brackets is to be located at the reference list at the end of the chapter the citation is situated in. The only exceptions are the references [i]-[iii].
- Although the thesis is a compilation of standalone papers, it has a continuous numbering of chapters, sections, and theorems.

1.1 Cotorsion pairs and approximations

We start by recalling the fundaments of the theory of cotorsion pairs and approximations, and on the way, we introduce the notation used in this chapter. Let R be an associative and unital ring. By Mod-R we denote the category of right R-modules. Whenever we talk about a module without specifying the hand and/or the ring, we always mean right R-modules. Sometimes, the duality will also take us to the category of left R-modules, which we denote by R-Mod. Whenever F is an additive functor (possibly contravariant), we denote by $\operatorname{Ker} F$ the class of all objects X from the domain category such that $F(X) = 0$. Given a class $\mathcal{C}$ of right R-modules, we define the following subclasses of Mod-R.

$$\mathcal{C}^{\perp_0} = \operatorname{Ker}\operatorname{Hom}_R(\mathcal{C}, -) := \bigcap_{C \in \mathcal{C}} \operatorname{Ker}\operatorname{Hom}_R(C, -),$$

$$\mathcal{C}^{\perp_1} = \operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{C}, -),$$
$$\mathcal{C}^{\perp_\infty} = \operatorname{Ker}\operatorname{Ext}^{>0}_R(\mathcal{C}, -) := \bigcap_{i>0} \operatorname{Ker}\operatorname{Ext}^i_R(\mathcal{C}, -).$$

Similarly, we define the following subclasses of R-Mod:

$$\mathcal{C}^{\intercal_1} = \operatorname{Ker}\operatorname{Tor}^R_1(\mathcal{C}, -),$$

$$\mathcal{C}^{\intercal_\infty} = \operatorname{Ker}\operatorname{Tor}^R_{>0}(\mathcal{C}, -).$$

The version for $\mathcal{C}$ being a class of left modules will be used too and is defined analogously. If $\mathcal{C} = \{M\}$ is a singleton, we write just $M^{\perp_1}$, and analogically for the other symbols.

Remark 1.1.1. In papers [i], [ii], and [iii] we use the indexless symbol $^\perp$. We warn the reader that in papers [i] and [ii], we use the convention $^\perp := {}^{\perp_1}$, but in the paper [iii] we use $^\perp := {}^{\perp_\infty}$.

Definition 1.1.2. A *cotorsion pair* in Mod-R is a pair $(\mathcal{A}, \mathcal{B})$ of classes of R-modules, such that $\mathcal{B} = \mathcal{A}^{\perp_1}$ and $\mathcal{A} = {}^{\perp_1}\mathcal{B}$. If $\mathcal{S}$ is a subclass of Mod-R, we define the cotorsion pair $(\mathcal{A}, \mathcal{B})$ *generated* by $\mathcal{S}$ by setting $\mathcal{B} = \mathcal{S}^{\perp_1}$ and $\mathcal{A} = {}^{\perp_1}\mathcal{B}$.

Next we recall the right and left module approximations - the preenvelopes and precovers, and their minimal versions - envelopes and covers.

Definition 1.1.3. Let $\mathcal{C}$ be a class of modules. A map $f : M \to C$ in Mod-R is a $\mathcal{C}$*-preenvelope* of M if $C \in \mathcal{C}$, and for any map $f' : M \to C'$ with $C' \in \mathcal{C}$ there is a (not necessarily unique) map $g : C \to C'$ such that $f' = gf$. A $\mathcal{C}$-preenvelope is a $\mathcal{C}$*-envelope*, if, moreover, any endomorphism h of C such that $f = hf$ is necessarily an automorphism.

A monic map $f : M \to C$ is a special $\mathcal{C}$-preenvelope, if $\mathrm{Coker}(f) \in {}^{\perp_1}\mathcal{C}$ (and this is easily seen to be a $\mathcal{C}$-preenvelope).

A class of modules $\mathcal{C}$ is said to be *preenveloping (enveloping, special preenveloping)* if any module M has a $\mathcal{C}$-preenvelope (envelope, special preenvelope).

The notions of $\mathcal{C}$-precover, $\mathcal{C}$-cover, and special $\mathcal{C}$-precover are defined dually.

The following lemma due to Wakamatsu shows that the minimal approximations are usually special.

Lemma 1.1.4. [23, Lemma 5.13] *Let $\mathcal{C}$ be a class of modules closed under extensions. Then:*

(i) a monic $\mathcal{C}$-envelope is a special $\mathcal{C}$-preenvelope, and dually

(ii) a surjective $\mathcal{C}$-cover is a special $\mathcal{C}$-precover.

We will mostly consider approximations by left or right classes of a cotorsion pair. The following result was proven by Salce.

Lemma 1.1.5. [23, Lemma 5.20] *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in* Mod-R. *Then the following are equivalent:*

(i) $\mathcal{A}$ is a special precovering class,

(ii) $\mathcal{B}$ is a special preenveloping class.

Cotorsion pairs satisfying the equivalent conditions of the Salce Lemma are called *complete*. The following very important result shows that any cotorsion pair generated by a set is complete. Also, the left class of such cotorsion pairs can be described in terms of the generating set. Recall that given a class $\mathcal{S}$ of modules and a module M, we say that M is $\mathcal{S}$*-filtered* provided that there is a continuous and increasing chain of submodules $(M_\alpha, \alpha < \lambda)$ such that $M_0 = 0$, $\bigcup_{\alpha<\lambda} M_\alpha = M$ and $M_{\alpha+1}/M_\alpha \simeq S$ for some $S \in \mathcal{S}$ for any $\alpha < \lambda$.

Theorem 1.1.6. [23, Theorem 6.11 and Corollary 6.14] *Let $\mathcal{S}$ be a set of modules. Then the cotorsion pair $(\mathcal{A}, \mathcal{B})$ generated by $\mathcal{S}$ is complete. Moreover, the class $\mathcal{A}$ coincides with the class of all direct summands of all $\mathcal{S} \cup \{R\}$-filtered modules.*

1.2 Tilting and cotilting theory essentials

1.2.1 Tilting modules and finite type

If M is a module, denote by $\mathrm{Add}(M)$ the class of all direct summands of arbitrary direct sums of copies of M. The following definition of a tilting module is due to Colpi-Trlifaj [20] in case of pd ≤ 1 and to Angeleri-Coelho [10] in general:

Definition 1.2.1. Let R be a ring and $n \geq 0$. A right R-module T is $(n-)$*tilting* provided that the following three conditions hold:

(T1) $\mathrm{pd}\, T \leq n$,

(T2) $\mathrm{Ext}^i_R(T, T^{(\varkappa)}) = 0$ for any $i > 0$ and any cardinal $\varkappa$,

(T3) there is an exact sequence

$$0 \to R \to T_0 \to T_1 \to \cdots \to T_n \to 0,$$

where $T_i \in \mathrm{Add}(T)$ for each $i = 0, 1, \ldots, n$.

Definition 1.2.2. Given a tilting module T, the induced cotorsion pair $(\mathcal{A}, \mathcal{T})$, with

$$\mathcal{T} = T^{\perp_\infty}, \mathcal{A} = {}^{\perp_1}\mathcal{T}$$

is called a *tilting cotorsion pair*. The class $\mathcal{A}$ is the *left tilting class* and $\mathcal{T}$ the *right tilting class* (or just the *tilting class*) associated to T.

Two tilting modules T and T' are said to be *equivalent* if they induce the same tilting class.

Tilting modules and their equivalences can be characterized nicely:

Lemma 1.2.3. [23, Lemma 13.16], [13, Theorem 3.11]

- *Two tilting modules T and T' are equivalent if and only if* $\mathrm{Add}(T) = \mathrm{Add}(T')$ *if and only if* $\mathrm{Add}(T) \subseteq \mathrm{Add}(T')$.
- *A module T is n-tilting if and only if*

$$T^{\perp_\infty} = \mathrm{Gen}_n(T),$$

where $\mathrm{Gen}_n(T)$ *is the class of all modules M possessing an exact sequence of form*

$$T^{(\varkappa_n)} \to T^{(\varkappa_{n-1})} \to \cdots \to T^{(\varkappa_1)} \to M \to 0$$

for some cardinals $\varkappa_1, \ldots, \varkappa_n$.

Even in the case of large tilting modules, any tilting class is of finite type - that is, it is an Ext-orthogonal class to a set of modules, which are finite in the strong sense, as defined below. This was proved first by Bazzoni-Herbera for the case of 1-tilting, and then by Bazzoni-Šťovíček in general. Before stating the Finite type theorem for tilting classes, we need some terminology first.

Definition 1.2.4. A module M is said to be *strongly finitely presented* if it admits a projective resolution consisting of finitely generated projectives. We fix the following notation:

- By mod-R we denote the full subcategory of Mod-R consisting of strongly finitely presented modules.
- For any $n \geq 0$, we denote by $\mathrm{mod_n}$-R the full subcategory of mod-R consisting of modules of projective dimension at most n.

Definition 1.2.5. Let $M \in$ Mod-R and let

$$\cdots \to P_n \xrightarrow{f_n} P_{n+1} \xrightarrow{f_n} \cdots \xrightarrow{f_2} P_0 \xrightarrow{f_1} M \to 0$$

be a projective resolution of M. An *n-th syzygy* of M, denoted $\Omega^n M$, is defined as the kernel of the map f_n for any $n > 0$. Instead of $\Omega^1 M$ we often write just ΩM, and we use the convention $\Omega^0 M = M$. We immediately remark that the syzygies of M are not well-defined as R-modules, indeed, they depend on the choice of the projective resolution of M. However, $\Omega^n M$ is well-defined up to *projective equivalence*, that is, up to adding or removing a projective direct summand. Usually, the arguments will not depend on the choice of representative of the projective equivalence class, and in such cases, if not specified otherwise, we assume that $\Omega^n M$ is some fixed choice of the representative module.

Let now

$$0 \to M \xrightarrow{g_1} E_0 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} E_n \xrightarrow{g_n} E_{n+1} \to \cdots$$

be a minimal injective coresolution of M. For any $n > 0$, we let $\Omega^{-n} = \mathrm{Coker}(g_n)$ denote the *n-th minimal cosyzygy* of M. This time, $\Omega^{-n} M$ is uniquely determined up to isomorphism.

The following definition goes back to Auslander-Bridger ([9]).

Definition 1.2.6. Let $\mathcal{A}$ be an abelian category with enough projectives. We say that a full subcategory $\mathcal{S}$ of mod-R is a *resolving subcategory* if the following conditions hold:

(i) $\mathcal{S}$ contains all projectives,

(ii) $\mathcal{S}$ is closed under extensions and direct summands,

(iii) $\mathcal{S}$ is closed under taking syzygies, i.e. $A \in \mathcal{S}$ whenever there is an exact sequence $0 \to A \to P \to C \to 0$ in $\mathcal{A}$ with $C \in \mathcal{S}$ and P projective.

We will mostly be interested in resolving subcategories of $\mathrm{mod_n}$-R for some $n \geq 0$. In this case, the condition (i) in the above definition translates as: $\mathcal{S}$ contains all finitely generated projective R-modules. Also, note that condition (iii) can be equivalently replaced by the following condition: if $0 \to A \to B \to C \to 0$ is an exact sequence such that $B, C \in \mathcal{S}$, then $A \in \mathcal{S}$.

Theorem 1.2.7. ([15], [17]) *Let R be a ring. There is a 1-1 correspondence*

$$\{\textit{resolving subcategories } \mathcal{S} \textit{ of } \mathrm{mod}_n\text{-R}\} \leftrightarrow \{\textit{n-tilting classes } \mathcal{T} \textit{ in } \text{Mod-R}\}$$

given by $\mathcal{S} \mapsto \mathcal{S}^{\perp_1} = \mathcal{S}^{\perp_\infty}$ and $\mathcal{T} \mapsto ({}^{\perp_1}\mathcal{T}) \cap \text{mod-R}$.

As a consequence, all tilting classes are definable in a sense we explain now. A formula in the first-order language of right R-modules is called a *pp-formula* if it is of form

$$\exists \bar{x} \in R^n : \bar{y}A = \bar{x}B$$

for some $\bar{y} \in R^m$ and some R-matrices A and B of appropriate size. For example, one can express divisibility of a module by a finitely generated ideal $\langle r_1, \dots, r_n \rangle$ by setting $\bar{y} = y \in R$, $\bar{x} \in R^n$, $A = 1$, and

$$B = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}.$$

Theorem 1.2.8. [27, §2.3], [28, Theorem 3.4.7] *Let $\mathcal{C}$ be a class of right R-modules, with R being an arbitrary ring. Then the following are equivalent:*

1. *$\mathcal{C}$ is a class of all modules satisfying some prescribed set of pp-formulas, or implications between pp-formulas,*
2. *$\mathcal{C}$ is a class of all modules M such that $\operatorname{Hom}_R(f, M)$ is surjective for any f from a prescribed set of homomorphisms between finitely generated projective R-modules,*
3. *$\mathcal{C}$ is a class closed under products, direct limits, and pure submodules.*

Definition 1.2.9. A subclass $\mathcal{C}$ of Mod-R is said to be *definable* if it satisfies the equivalent conditions of Theorem 1.2.8.

For convenience, let us state explicitly how the definability of tilting classes follows from them being of finite type. Let $\mathcal{T}$ be a tilting class and $\mathcal{S}$ a resolving subcategory of mod_n-R for some $n \geq 0$ such that $\mathcal{T} = \mathcal{S}^{\perp_1}$. Fixing a projective presentation $P_1^S \xrightarrow{f_S} P_0^S \to S \to 0$ with P_0^S, P_1^S finitely generated projectives for every $S \in \mathcal{S}$, we infer that $\mathcal{T}$ consists precisely of those modules M, for which $\operatorname{Hom}_R(f_S, M)$ is surjective for any $S \in \mathcal{S}$.

Theorem 1.2.10. *[17] Any tilting class is definable.*

1.2.2 Cotilting modules and duality

The cotilting classes and modules are defined completely dually to their tilting counterparts. If M is a module, we let $\operatorname{Prod}(M)$ denote the class of all direct summands of arbitrary direct products of copies of M.

Definition 1.2.11. A left R-module C is *(n-)cotilting* if the following three conditions are satisfied:

(C1) $\operatorname{id} C \leq n$,

(C2) $\operatorname{Ext}_R^i(C^\varkappa, C) = 0$ for each $i > 0$ and any cardinal $\varkappa$,

(C3) there is an exact sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to W \to 0,$$

where W is an injective cogenerator of R-Mod, and $C_i \in \operatorname{Prod}(C)$ for each $i \leq n$.

Definition 1.2.12. Given a cotilting module C in R-Mod, the (co)induced cotorsion pair $(\mathcal{C}, \mathcal{W})$, where

$$\mathcal{C} = {}^{\perp_\infty}C, \mathcal{W} = \mathcal{C}^{\perp_1},$$

is called a *cotilting cotorsion pair*. The class $\mathcal{C}$ is the *(left) cotilting class* and $\mathcal{W}$ the *right cotilting class* associated to C.

Two cotilting modules C, C' are said to be *equivalent* if they induce the same cotilting classes. This happen precisely when $\operatorname{Prod}(C) = \operatorname{Prod}(C')$ ([23, Remark 15.6]).

As their tilting cousins, the cotilting classes turn out to be definable. Alas, in general this does not follow from the finite type of tilting classes directly by a duality argument, but as a consequence of the fact that any cotilting module is pure-injective. Recall that a module U is *pure-injective*, if $\mathrm{Hom}_R(f, M)$ is surjective for any pure embedding f. The pure-injectivity of cotilting modules was proved first for the case of injective dimension 1 by Bazzoni [18], and the general case is due to Šťovíček.

Theorem 1.2.13. *[37] Any cotilting module is pure-injective. In particular, any cotilting class is definable.*

Here we remark the approximation properties of tilting and cotilting classes.

Lemma 1.2.14. *Any tilting class is special preenveloping. Any cotilting class is covering.*

Proof. The first statement is a direct consequence of Theorem 1.1.6. The second statement needs a different deduction, we refer the reader to [23, Theorem 15.9]. □

Cotilting classes can be characterized by their closure properties among the definable classes in the following, useful way.

Theorem 1.2.15. [3, Proposition 3.14] *Let $n \geq 0$ and let $\mathcal{C}$ be a class of left modules over any ring R. Then $\mathcal{C}$ is n-cotilting if and only if the following conditions hold:*

1. *$\mathcal{C}$ is definable,*
2. *$\mathcal{C}$ is a resolving subcategory of* R-Mod*,*
3. *any n-th syzygy of any R-module belongs to $\mathcal{C}$.*

In particular, 1*-cotilting classes are precisely the torsion-free classes closed under direct limits and containing R.*

Proof. For convenience, we prove how the final claim follows from [3, Proposition 3.14]. Let $\mathcal{C}$ be a 1-cotilting class, and we need to show that $\mathcal{C}$ is closed under submodules. Let $M \in \mathcal{C}$ and choose a submodule A of M, inducing an exact sequence

$$0 \to A \to M \xrightarrow{\pi} C \to 0.$$

Consider the pullback X of π with some surjective $P \to C$ map with P projective.

This induces the standard pullback diagram:

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & A \arrow[r] & M \arrow[r, "\pi"] \arrow[u] & C \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, equal] & X \arrow[r] \arrow[u] & P \arrow[r] \arrow[u] & 0 \\
 & & \Omega C \arrow[r, equal] \arrow[u] & \Omega C \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since $\mathcal{C}$ contains all (first) syzygies, we infer that $\Omega C \in \mathcal{C}$. As $\mathcal{C}$ is extension-closed, also $X \in \mathcal{C}$. Since P is projective, the middle row short exact sequence splits, and therefore A is a direct summand of X. Since $\mathcal{C}$ is definable, and thus closed under direct summands, we infer that $A \in \mathcal{C}$, as desired. □

Cotilting modules not are not only defined dually to tilting modules, there is also an explicit duality in the game. Suppose that R is an S-algebra, where S is a commutative ring. This is not a special setting in any sense - we can choose $S = \mathbb{Z}$ for any ring R, but it allows for a more comfortable choices in special cases (if R is commutative, we often choose $R = S$, while for an k-algebra over a field k, we are likely to put $S = k$). Choosing an injective cogenerator W of Mod-S, we define the *elementary duality functor* $(-)^+ = \mathrm{Hom}_S(-, W)$. This duality will take any tilting module to a cotilting module, and makes the associated tilting and cotilting class *dual definable* (in the sense of [28, §3.4.2]):

Proposition 1.2.16. ([23, Theorem 15.18], Lemma 4.4) *Let T be an n-tilting right R-module. Then its dual T^+ is an n-cotilting left R-module.*

Furthermore, let us denote the associated tilting and cotilting class by $\mathcal{T}$ and $\mathcal{C}$, respectively. Then for any right R-module M and any left R-module N we have:

- *$M \in \mathcal{T}$ if and only if $M^+ \in \mathcal{C}$,*
- *$N \in \mathcal{C}$ if and only if $N^+ \in \mathcal{T}$.*

In the situation of Proposition 1.2.16, let us say that the cotilting class $\mathcal{C}$ is *dual* to the tilting class $\mathcal{T}$. However, not every cotilting class fits in this picture - there are cotilting classes not dual to any tilting class (see §4.9). The finite type of tilting classes gives a nice characterization of those cotilting classes which come from the dual side:

Proposition 1.2.17. *There is a 1-1 correspondence*

$$\{\textit{resolving subcategories } \mathcal{S} \textit{ of } \mathrm{mod}_n\text{-R}\} \leftrightarrow$$

$$\{n\textit{-cotilting classes } \mathcal{C} \textit{ in } \text{R-Mod} \textit{ dual to some tilting class in } \text{Mod-R}\}$$

given by $\mathcal{S} \mapsto \mathcal{S}^{\intercal_1}$ and $\mathcal{C} \mapsto ({}^{\intercal_1}\mathcal{C}) \cap \text{mod-R}$.

We say that a class $\mathcal{C}$ of left R-modules is of *cofinite type* if there is a resolving subcategory $\mathcal{S}$ of $\mathrm{mod_n\text{-}R}$ for some $n \geq 0$ such that $\mathcal{C} = \mathcal{S}^{\intercal_1}$. By Proposition 1.2.17, any class of cofinite type is cotilting, and cotilting classes of cofinite type are precisely those cotilting classes, which are dual to some tilting class. A cotilting module C is said to be of *cofinite type* if the induced cotilting class is of cofinite type.

Theorem 1.2.18. *The assignment $T \mapsto T^+$ induces a bijection between the equivalence classes of right n-tilting and left n-cotilting R-modules of cofinite type. Hence, it also induces a bijection between n-tilting classes in* Mod-R *and n-cotilting classes of cofinite type in* R-Mod*.*

1.2.3 Cofinite type in the commutative case

In our approach, the first step towards the classification results over a commutative ring was noticing that the cotilting classes of cofinite type are closed under injective envelopes, which is the point where hereditary torsion pairs enter the game. At this point, some remarks are due. First, such a claim is specific to the commutative setting (see Example 1.4.9). Secondly, this seems to be the reason, why working in the dual setting of cotilting classes proved to be a good strategy - there is no apparent dual closure property for the tilting classes. The most obvious analog - the statement “tilting classes over commutative rings are closed under flat covers” - is seldom true, as we discuss in §1.6.

The closure under injective envelopes can even be used to characterize the cofinite type classes amongst the cotilting classes.

Definition 1.2.19. Let $\mathcal{C}$ be a cotilting class induced by a cotilting module C, i.e. $\mathcal{C} = {}^{\perp_\infty}C$. For any $i \geq 0$, we define the class

$$\mathcal{C}_{(i)} = {}^{\perp_\infty}(\Omega^{-i}C).$$

Note that $\mathcal{C}_{(i)} = \{M \in \text{R-Mod} \mid \Omega^i M \in \mathcal{C}\}$, and that by [3, Lemma 3.5], $\mathcal{C}_{(i)}$ is an $(n-i)$-cotilting class for any $i = 0, 1, \dots, n$.

Theorem 1.2.20. (Proposition 4.5.5, Theorem 4.9.1) *Let R be a commutative ring and $\mathcal{C}$ a cotilting class in* Mod-R*. Then the following are equivalent:*

1. *$\mathcal{C}$ is of cofinite type,*
2. *$\mathcal{C}_{(i)}$ is closed under taking injective envelopes for all $i = 0, 1, \dots, n-1$.*

In particular, a $1-$cotilting class is of cofinite type if and only if it is closed under injective envelopes.

In [3], the authors proved that over a noetherian commutative ring, all cotilting classes are actually of cofinite type. The first example of a cotilting class **not** of cofinite type was showed by Bazzoni in [11]. We refer the reader to section §4.9 for a substantial generalization of Bazzoni’s construction. In particular, we construct for any $n > 1$ a cotilting class $\mathcal{C}$ over a certain commutative ring such that $\mathcal{C}_{(i)}$ is closed under injective envelopes for all $i = 0, 1, 2, \dots, n-2$, but so that $\mathcal{C}$ is **not** of cofinite type, showing that condition (ii) of Theorem 1.2.20 cannot be weakened.

1.3 Silting and cosilting modules

Silting modules were introduced by Angeleri, Marks, and Vitória in [2] as a module-theoretic counterpart of the 2-term silting complexes. The definition generalizes 1-tilting modules (the common framework encompassing both silting and n-tilting modules would be the n-term silting complexes, which we do not discuss in this thesis). In this section, we quickly gather the relevant facts about silting and cosilting theory, which are analogous to the (1-)tilting setting (but the proofs are necessarily not).

Definition 1.3.1. [2] A right R-module T is *silting* (with respect to σ), provided that there is a projective presentation

$$P_{-1} \xrightarrow{\sigma} P_0 \to T \to 0,$$

such that $\mathcal{D}_\sigma = \mathrm{Gen}(T)$, where

$$\mathcal{D}_\sigma = \{M \in \text{Mod-R} \mid \mathrm{Hom}_R(\sigma, M) \text{ is surjective}\}.$$

In this case we call $\mathcal{D}_\sigma$ a *silting* class. Two silting modules T, T' are *equivalent* if they induce the same silting class in this way.

Remark 1.3.2. Any $1-$tilting module is a silting module with respect to any of its monic projective presentations. Indeed, if σ is such presentation, $\mathcal{D}_\sigma = T^{\perp_1}$, which is equal to $\mathrm{Gen}(T)$ by Lemma 1.2.3.

Example 1.3.3. The choice of the projective presentation σ matters. As a simple example, consider a ring with nontrivial decomposition $R = R_1 \times R_2$. Then R_1 is a silting module with respect to the projective presentation

$$R_2 \xrightarrow{\sigma} R_1 \xrightarrow{\cong} R_1 \to 0,$$

where $\sigma = 0$, but not with respect to projective presentation

$$0 \xrightarrow{\sigma'} R_1 \xrightarrow{\cong} R_1 \to 0.$$

Indeed, $\mathcal{D}_\sigma = \text{Mod-R}_1 = \mathrm{Gen}(R_1)$, but $\mathcal{D}_{\sigma'} = \text{Mod-R}$.

Let $\sigma : P_{-1} \to P_0$ be a homomorphism of two projective modules. We say that σ is of *finite type* provided that there is a set I and homomorphisms σ_i between finitely generated projective modules for each $i \in I$, such that $\mathcal{D}_\sigma = \bigcap_{i\in I} \mathcal{D}_{\sigma_i}$. The following finite type theorem for silting modules is analogous to the tilting case:

Theorem 1.3.4. ([4, Theorem 6.3], Theorem 3.2.3) *Let σ be a map between projective modules. Then the following are equivalent:*

1. *σ is of finite type,*
2. *$\mathcal{D}_\sigma$ is definable,*
3. *$\mathcal{D}_\sigma$ is a silting class.*[1]

[1] This does not in general imply that $\mathrm{Coker}\,\sigma$ is a silting module!

The dual version of silting modules was introduced in [19].

Definition 1.3.5. A left R-module C is *cosilting* if it admits an injective copresentation

$$0 \to C \to E_0 \xrightarrow{\lambda} E_1$$

such that $\mathcal{C}_\lambda = \mathrm{Cogen}(C)$, where

$$\mathcal{C}_\lambda = \{M \in \text{R-Mod} \mid \mathrm{Hom}_R(M, \lambda) = 0\}.$$

Definition 1.3.6. Let λ be a map between left injective R-modules. We say that λ (or $\mathcal{C}_\lambda$) is of *cofinite type*, if there is a set $\{\sigma_i, i \in I\}$ of maps between finitely generating projectives such that $\mathcal{C}_\lambda = \bigcap_{i\in I} \mathcal{T}_{\sigma_i}$, where

$$\mathcal{T}_\sigma = \{M \in \text{R-Mod} \mid \sigma_i \otimes_R M \text{ is injective}\},$$

for any map σ of right R-modules.

Theorem 1.3.7. (Lemma 3.3.3, Proposition 3.3.4)

1. *If T is a silting right R-module with respect to a projective presentation σ, then T^+ is cosilting left R-module with respect to the injective copresentation σ^+. In this situation, σ^+ is of cofinite type, and $\mathcal{C}_{\sigma^+} = \mathcal{T}_\sigma$.*

2. *In the same situation as in (1), the classes $\mathcal{D}_\sigma$ and $\mathcal{C}_{\sigma^+}$ are dual definable. That is, both those classes are definable, and given $M \in$ Mod-R and $N \in$ R-Mod, we have:*
$$M \in \mathcal{D}_\sigma \Leftrightarrow M^+ \in \mathcal{C}_{\sigma^+},$$
$$N \in \mathcal{C}_{\sigma^+} \Leftrightarrow N^+ \in \mathcal{D}_\sigma.$$

3. *A map λ of left injective R-modules is of cofinite type if and only if there is a map σ of right projective R-modules of finite type such that $\mathcal{C}_\lambda = \mathcal{C}_{\sigma^+}$. In particular, in this situation $\mathcal{C}_\lambda$ is a cosilting class.*

Generalizing the 1-tilting case from [23, Theorem 15.31], over a left noetherian ring, this duality is "onto".

Theorem 1.3.8. (Theorem 3.3.7) *Let R be a left noetherian ring. Then every cosilting class in* R-Mod *is of cofinite type. In this case, silting classes in* Mod-R *are precisely the definable torsion classes.*

The cosilting classes on the other hand are precisely the definable torsion-free classes over any ring. This was proved independently by Wei-Zhang ([34]) and Breaz-Žemlička ([30]). As in the case of cotilting classes, we can now characterize the cofinite type classes amongst the cosilting classes as those closed under injective envelopes.

Theorem 1.3.9. *Let R be a commutative ring and $\mathcal{C}$ a cosilting class in* Mod-R. *Then $\mathcal{C}$ is of cofinite type if and only if it is closed under injective envelopes (i.e., it is a torsion-free class of a hereditary torsion pair, see §1.4).*

Proof. If $\mathcal{C}$ is of cofinite type, then it is closed under injective envelopes by Lemma 3.4.2. To prove the converse, let $\mathcal{C}$ be a cosilting class closed under injective envelopes. By the preceding paragraph, $\mathcal{C}$ is a definable torsion-free class, and thus $\mathcal{C}$ is a torsion-free class in a hereditary torsion pair $(\mathcal{T}, \mathcal{C})$ of finite type. With respect to Theorem 1.4.6, there is a set $\mathcal{I}$ of finitely generated ideals such that $\mathcal{C} = \bigcap_{I \in \mathcal{I}} (R/I)^{\perp_0}$. For each $I \in \mathcal{I}$, there is $n_I \geq 0$ and a projective presentation

$$R^{n_I} \xrightarrow{\sigma_I} R \to R/I \to 0.$$

Then $\mathrm{Hom}_R(R/I, M) = 0$ if and only if $\mathrm{Hom}_R(\sigma_I, M)$ is injective. By [1, Proposition 20.10], we have a natural isomorphism

$$\mathrm{Hom}_R(\sigma_I, M) \simeq \mathrm{Hom}_R(\sigma_I, R) \otimes_R M.$$

Denoting $\sigma_I^* = \mathrm{Hom}_R(\sigma_I, R)$, we see that $\mathcal{C} = \bigcap_{I \in \mathcal{I}} \mathcal{T}_{\sigma_I^*}$, where $\sigma_I^* : R \to R^n$ is a map between finitely generated projectives. Hence, $\mathcal{C}$ is of cofinite type. □

Corollary 1.3.10. *Over a commutative ring, cosilting classes of cofinite type are precisely the torsion-free classes of hereditary torsion pairs of finite type.*

Proof. By [30, Theorem 3.5], cosilting classes over an arbitrary ring are precisely the torsion-free classes closed under direct limits. Theorem 1.3.9 shows that the cofinite-type classes amongst these are precisely the ones closed under injective envelopes. This concludes the proof. □

1.4 Hereditary torsion pairs, Gabriel topologies, Thomason sets

The aim of this section is to briefly recall the theory of hereditary torsion pairs in the category of modules over a commutative ring. Large portion of the results is known, but be the results may be too fragmented in the literature to be easily referenced.

Definition 1.4.1. Let R be a ring. By a *torsion pair* we mean a pair $(\mathcal{T}, \mathcal{F})$ of subclasses of Mod-R such that $\mathrm{Hom}_R(T, F) = 0$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$, and such that for any right R-module there is a short exact sequence

$$0 \to T(M) \to M \to F(M) \to 0$$

with $T(M) \in \mathcal{T}$ and $F(M) \in \mathcal{F}$. Such an exact sequence is uniquely determined (up to a unique isomorphism of exact sequences), and the assignment $M \mapsto T(M)$ is a subfunctor called the *torsion subfunctor* with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$. The class $\mathcal{T}$ is called the *torsion class* of this torsion pair, and $\mathcal{F}$ the torsion-free class of this torsion pair.

A class $\mathcal{C}$ of modules is called simply a *torsion class* (*torsion-free class*) if it fits as a torsion (torsion-free) class of some torsion pair.

These classes are well-known to be characterized by the following closure properties:

- Class $\mathcal{T}$ is torsion if and only if it is closed under extensions, direct sums, and epimorphic images.

- Class $\mathcal{F}$ is torsion-free if and only if it is closed under extensions, direct products, and submodules.

1.4.1 Hereditary torsion pairs and Gabriel topologies

Definition 1.4.2. A torsion pair $(\mathcal{T}, \mathcal{F})$ is *hereditary* if $\mathcal{T}$ is closed under submodules (equivalently, if $\mathcal{F}$ is closed under injective envelopes).

A theorem due to Gabriel shows that hereditary torsion pairs are determined precisely by those cyclic modules belonging to the torsion class. In order to make this precise, we need to introduce the notion of Gabriel topology of ideals. We will use the following notation: If I is a right ideal of a ring R, and $t \in R$ an element, we set $(I : t) = \{r \in R \mid tr \in I\}$. It is easily seen that $(I : t)$ is always a right ideal.

Definition 1.4.3. Let R be a ring. A (right) *Gabriel topology (or a Gabriel filter*[2]*)* of R is a filter $\mathcal{G}$ on the set of all right ideals of R satisfying the following conditions:

1. if $I \in \mathcal{G}$ and $t \in R$, then $(I : t) \in \mathcal{G}$, and
2. if J is a right ideal of R, and $I \in \mathcal{G}$ such that $(J : t) \in \mathcal{G}$ for any $t \in \mathcal{I}$, then necessarily $J \in \mathcal{G}$.

Theorem 1.4.4. [31, §VI Proposition 4.2 and Theorem 5.1] *There is a 1-1 correspondence*

$$\{\textit{hereditary torsion pairs } (\mathcal{T}, \mathcal{F}) \textit{ in } \text{Mod-R}\} \leftrightarrow \{\textit{right Gabriel topologies } \mathcal{G} \textit{ of } R\}$$

given by the assignments

$$\mathcal{G} \mapsto (\{M \in \text{Mod-R} \mid \text{Ann}(m) \in \mathcal{G}\ \forall m \in M\}, (\bigoplus_{I \in \mathcal{G}} R/I)^{\perp_0})$$

$$(\mathcal{T}, \mathcal{F}) \mapsto \{I \textit{ right ideal of } R \mid R/I \in \mathcal{T}\}.$$

In this correspondence, the hereditary torsion pairs with torsion-free classes definable are easily detected.

Definition 1.4.5. A hereditary torsion pair $(\mathcal{T}, \mathcal{F})$ is said to be *of finite type* if $\mathcal{F}$ is closed under taking direct limits.[3]

A right Gabriel topology is called *finitely generated*, if it has a filter basis consisting of finitely generating right ideals.

Theorem 1.4.6. *The correspondence from Theorem 1.4.4 restricts to bijective correspondence*

$$\{\textit{hereditary torsion pairs } (\mathcal{T}, \mathcal{F}) \textit{ of finite type in } \text{Mod-R}\} \leftrightarrow$$

$$\{\textit{right finitely generated Gabriel topologies } \mathcal{G} \textit{ of } R\}.$$

[2]We preferred this alternative terminology in paper [ii].

[3]In paper [i] we imposed this definition also on possibly non-hereditary torsion pairs. Although this is not unseen in literature, it is not standard. To avoid confusion, in this chapter we speak of finite type only in the hereditary case.

Proof. This can be deduced from [31, §XIII Proposition 1.2], where it is proved that $\mathcal{G}$ is finitely generated if and only if the torsion functor $T(-)$ associated to the hereditary torsion pair $(\mathcal{T}, \mathcal{F})$ corresponding to $\mathcal{G}$ commutes with direct limits. If $T(-)$ commutes with direct limits, then $\mathcal{F}$ is clearly closed under direct limits. To prove the converse, let $M = \varinjlim_{i \in I} M_i$ be a direct limits of some direct system. For each $i \in I$ let

$$0 \to T(M_i) \to M_i \to F(M_i) \to 0$$

be the exact sequence induced by the torsion pair $(\mathcal{T}, \mathcal{F})$. Consider the direct limit map $\psi : \varinjlim_{i \in I} T(M_i) \to M$. Since the direct limit functor is exact, we have that ψ is monic. Also, as torsion classes are always closed under direct limits, ψ factors through the inclusion $T(M) \subseteq M$, and therefore ψ corestricts to a map $\varphi : \varinjlim_{i \in I} T(M_i) \to T(M)$. Then we have $\mathrm{Coker}(\psi) \simeq \varinjlim_{i \in I} F(M_i)$. By the hypothesis, the latter module belongs to $\mathcal{F}$. Therefore, the map φ has to be onto $T(M)$, proving finally that φ is an isomorphism.

Alternatively, this follows from the (proof of) Lemma 2.2.4. $\square$

In the commutative case, the definition of a finitely generated Gabriel topology simplifies considerably.

Lemma 1.4.7. (Lemma 2.2.3) *Let R be a commutative ring. Suppose that $\mathcal{G}$ is a filter of ideals of R with a filter basis consisting of finitely generated ideals. Then $\mathcal{G}$ is a (finitely generated) Gabriel topology if and only if it is closed under ideal product.*

Theorem 1.4.6 says in particular that torsion-free classes of hereditary torsion pairs of finite type are right Hom-orthogonals to sets of finitely presented modules. The following results shows that, over a commutative ring, the converse is also true. That is, any class given as a right Hom-orthogonal to a set of finitely presented modules fits as a torsion-free class into some hereditary torsion pair. This is the principle behind the close connection of cofinite-type cotilting classes with hereditary torsion pairs (cf. Corollary 2.3.6).

Lemma 1.4.8. (Proposition 4.2.6) *Let R be a commutative ring, and $\mathcal{F}$ a class such that $\mathcal{F} = \mathcal{S}^{\perp_0}$ for a set $\mathcal{S}$ of finitely presented modules. Then $\mathcal{F}$ is a torsion-free class of some hereditary torsion pair of finite type.*

Proof. The proof of this is completely dual to that of Lemma 3.4.2. For convenience however, we write the proof here in detail. Clearly, $\mathcal{F}$ is a torsion-free class closed under direct limits. It remains to prove that $\mathcal{F}$ is closed under taking injective envelopes. Fix $S \in \mathcal{S}$ and a presentation

$$P_{-1} \xrightarrow{\sigma} P_0 \to S \to 0$$

with P_{-1}, P_0 being finitely generated projectives. Let $M \in \mathcal{F}$ and consider its injective envelope

$$0 \to M \xrightarrow{\iota} E(M).$$

Applying $\operatorname{Hom}_R(\sigma,-)$ onto ι, we obtain the following diagram:

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & \\
\operatorname{Hom}_R(P_{-1},M) \arrow[d] & & \operatorname{Hom}_R(P_0,M) \arrow[ll, "\operatorname{Hom}_R(\sigma,M)"'] \arrow[d, "\operatorname{Hom}_R(P_0,\iota)"'] & 0 \arrow[l] \\
\operatorname{Hom}_R(P_{-1},E(M)) & & \operatorname{Hom}_R(P_0,E(M)) \arrow[ll, "\operatorname{Hom}_R(\sigma,E(M))"'] &
\end{tikzcd}$$

The diagram has the following properties:

- It is a commutative diagram of R-modules, not just abelian groups. This point is subtle, but it is precisely where we use (and need) the commutativity of R!
- All drawn columns and rows are exact. This follows from P_{-1}, P_0 being projective, and from the assumption of $\operatorname{Hom}_R(S,M)=0$.
- $\operatorname{Ker}\operatorname{Hom}_R(\sigma,E(M)) = \operatorname{Hom}_R(S,E(M))$.

It is enough to show that the map $\operatorname{Hom}_R(\sigma,E(M))$ is injective. Indeed, then $\operatorname{Hom}_R(S,E(M))=0$, which is the goal. Without loss of generality, we can assume both σ and M non-zero. First, note that $\operatorname{Hom}_R(P_0,\iota)$ is an injective envelope of the R-module $\operatorname{Hom}_R(P_0,M)$. This can be observed as follows: If $P_0 \simeq R^n$ for some $n>0$, then $\operatorname{Hom}_R(R^n,\iota)$ is isomorphic to the direct sum map $\iota^n : M^n \to E(M)^n$, which is clearly an injective envelope of M^n, as $E(M)^n$ is injective and ι^n is essential (see [1, Proposition 6.17]). If P_0 is not free, then the claim follows from the decomposition $P_0 \oplus P' \simeq R^n$ for some finitely generated projective module P'.

Towards a contradiction, suppose now that $\operatorname{Ker}\operatorname{Hom}_R(\sigma,E(M))$ is non-zero. By the established essentiality of the map $\operatorname{Hom}_R(P_0,\iota)$, there is a non-zero element $x \in \operatorname{Hom}_R(P_0,M)$ such that $\operatorname{Hom}_R(\sigma,E(M)) \circ \operatorname{Hom}_R(P_0,\iota)(x) = 0$. By commutativity of the square, and the monicity of the left vertical map of the square, this yields that $\operatorname{Hom}_R(\sigma,M)(x)=0$. But $\operatorname{Hom}_R(\sigma,M)$ is monic, a contradiction. $\square$

Example 1.4.9. Lemma 1.4.7 is in general not true if R is not commutative. For example, consider any right semihereditary and left perfect ring R, that is not left self-injective. A class of such rings is easy to obtain - if $R=KQ$ is a quiver algebra of an acyclic quiver Q with at least one arrow, then R is semihereditary and artinian from both sides, and it is not self-injective on either side. Let $\mathcal{C}$ be the class of all projective left R-modules. Then $\mathcal{C}$ is a 1-cotilting class of cofinite type. Indeed, since R is left perfect, $\mathcal{C}$ coincides with the class of all flat left R-modules. Therefore, $\mathcal{C} = \mathcal{S}^{\intercal_1}$, where $\mathcal{S} = \{R/I \mid I \text{ a finitely generated right ideal of } R\}$. Since R is right hereditary, $\mathcal{S} \subseteq \text{mod}_1\text{-R}$, and thus $\mathcal{C}$ is a 1-cotilting class of cofinite type by Proposition 1.2.17. Also, by Corollary 2.3.6, we can express this class as $\mathcal{C} = \{S^\dagger \mid S \in \mathcal{S}\}^{\perp_0}$, where $S^\dagger = \operatorname{Ext}^1_R(S,R)$, a finitely presented left R-module.

We claim that $\mathcal{C}$ is not closed under injective envelopes. Indeed, if that was the case, then $E(R) \in \mathcal{C}$ would be a left projective R-module. By injectivity, we can extend the identity on R to a split epimorphism $E(R) \to R$. But then R is a left injective R-module, a contradiction with R not being left self-injective.

1.4.2 Hochster duality and Thomason sets

Both of our main classification results, that is for silting and n-tilting classes over a commutative ring, parametrize the classes by certain subsets of the Zariski spectrum of the ring. These subsets are in general neither closed or open in the Zariski topology; they are unions of certain Zariski-closed subsets. However, it turns out that they are precisely the open sets of the so-called Hochster dual of the Zariski spectrum. As is customary in algebraic geometry, we use the term *quasi-compact* instead of compact to emphasize that the spaces we deal with are usually not Hausdorff.

Definition 1.4.10. A quasi-compact topological space X is *spectral* if the following two conditions hold:

- the set of quasi-compact open subsets of X is closed under finite intersections and forms an open basis for the topology of X, and
- X is *sober*, that is, every irreducible closed set is the closer of a unique point of X.

A continuous map between two spectral spaces is called *spectral* if the full inverse image of any quasi-compact open set is quasi-compact.

The spectrum of any commutative ring with the Zariski topology is spectral. On the other hand, Hochster laboriously constructed to any spectral space X a commutative ring R such that X is homeomorphic to $\mathrm{Spec}(R)$. In other words:

Theorem 1.4.11. ([35]) *Up to homeomorphism, spectral spaces are precisely the Zariski spectra of commutative rings.*

Now we discuss the Hochster duality. One short survey on this topic (of course, from the point of view of our application) is contained in paper [***iii***] - Section §4.2.1. Here we sum up the approach following the recent paper [5] by Kock and Pitsch - in the language of frames from the area of pointless topology. Apart from closely following [5], our basic reference for the theory of frames (and their dual counterparts locales) is [7, II].

Definition 1.4.12. A *frame* is a joint-complete lattice in which finite meets distribute over arbitrary joins. A lattice homomorphism between frames is a *frame map* if it preserves arbitrary joints. This defines the category of frames.

In this paragraph, we follow [7, II.1]. Given any topological space X, the lattice $\Omega(X)$ of all open sets of X forms a frame. The assignment $X \mapsto \Omega(X)$ induces a contravariant functor from the category of topological spaces to the category of frames. This functor admits a right adjoint functor which assigns to a frame F the topological space (called the *point space*)

$$\mathrm{Pt}(F) = \{\varphi : F \to \{0,1\} \mid \varphi \text{ is a frame map}\},$$

with topology having as open sets precisely the sets of form

$$\{\varphi : F \to \{0,1\} \mid \varphi(a) = 1\}$$

for some element a of the frame F. It turns out that the topological spaces homeomorphic to $\mathrm{Pt}(F)$ are precisely the sober spaces. A frame F is called *spatial* if for any elements $a, b \in F$ such that $a \not\leq b$ there is a frame map $\varphi : F \to \{0,1\}$ (a "point") such that $\varphi(a) = 1$ and $\varphi(b) = 0$ (intuitively, frame F is spatial if "it has enough points").

Theorem 1.4.13. [7, II 1.7] *There is a contravariant equivalence between the category of sober topological spaces and the category of spatial frames.*

An element c in a frame F is *finite* if whenever $c \leq \bigvee_{a \in A} a$ for some subset $A \subseteq F$, then $c \leq \bigvee_{a \in B} a$ for some finite subset B of A. Frame F is called *coherent* if the finite elements form a sublattice, and every element is expressible as a joint of finite elements (in particular, by definition this means that the maximal element of F is finite). By [7, II 3.4], any coherent frame is spatial.

A frame map between coherent frames is a *coherent* frame map, if it sends finite elements to finite elements. By restricting the functor $X \mapsto \Omega(X)$ to spectral spaces and spectral maps, we obtain:

Theorem 1.4.14. [7, II 3.4] *The category of spectral spaces and spectral maps is contravariantly equivalent to the category of coherent frames and coherent frame maps.*

The final ingredient is the relation of distributive lattices and coherent frames. Given a coherent frame F, the sublattice K of all finite elements of F is a distributive lattice. On the other hand, if K is a distributive lattice, then the lattice $\mathrm{Idl}(K)$ of all ideals of F is a frame ([7, II 3.3]). It is not hard to see that any coherent frame is uniquely determined by the lattice of its finite elements, and that these two assignments are mutually inverse equivalences of the corresponding categories.

Theorem 1.4.15. [7, II 3.3] *The assignment $K \mapsto Idl(K)$ induces an equivalence from the category of distributive lattices and lattice maps to the category of coherent frames and coherent frame maps. The inverse equivalence assigns to a coherent frame the sublattice of all finite elements.*

Now we are ready to describe the Hochster duality. We start with a spectral space X, and its frame of open sets $\Omega(X)$. Denote by $K(X)$ the distributive sublattice of $\Omega(X)$ consisting of all finite elements, that is, of quasi-compact open sets of X. By the previous discussion, the frame $\Omega(X)$ is precisely the complete lattice of all ideals of $K(X)$; in symbols, $\Omega(X) = \mathrm{Idl}(K(X))$. Now we can form the dual lattice $K(X)^{op}$ of $K(X)$ - this is again a distributive lattice. Then the lattice $F^* := \mathrm{Idl}(K(X)^{op})$ is a coherent frame. The spectral space $X^* = \mathrm{Pt}(F^*)$ associated to this frame is called the *Hochster dual* of X.

Since F^* is a coherent frame, the Hochster dual X^* of a spectral space X is again a spectral space. Moreover, the underlying set of X^* is naturally identified with the original space X (but the topology is different!). Indeed, the points of X^* correspond to prime ideals of $K(X)^{op}$ (see [7, II 3.4 Paragraph and Lemma]).

These are precisely the prime filters in $K(X)$, which correspond by taking complements to prime ideals in $K(X)$ - the points of $\mathrm{Pt}(\Omega(X)) \simeq X$. One of the benefits of this frame-theoretic approach is that now just from the simple fact that $(D^{op})^{op} = D$ for any distributive lattice D, we can infer that Hochster duality is an honest duality: $(X^*)^*$ is homeomorphic to X for any spectral space X.

Starting with $X = \mathrm{Spec}(R)$ for some commutative ring R, the Hochster dual $\mathrm{Spec}(R)^*$ of the spectrum is (homeomorphic to) a topological space with underlying set $\mathrm{Spec}(R)$, where the topology is given by defining a basis of closed sets consisting of precisely those sets, which are open and quasi-compact in $\mathrm{Spec}(R)$. Therefore, the open sets of $\mathrm{Spec}(R)^*$ are precisely the arbitrary unions $\bigcup_{i \in I} V_i$ of sets V_i, such that the complement of V_i is open and quasi-compact in $\mathrm{Spec}(R)$. These sets have an easy characterization. As usual, given an ideal I of a commutative ring R, we let $V(I) = \{p \in \mathrm{Spec}(R) \mid I \subseteq p\}$ be the Zariski closed set defined by ideal I.

Lemma 1.4.16. [6, 00F6 Lemma 10.28.1] *A subset V of $\mathrm{Spec}(R)$ has a quasi-compact open complement if and only if $V = V(I)$ for some finitely generated ideal I.*

In 1997 work [38], Thomason showed that the open sets of the Hochster dual $\mathrm{Spec}(R)^*$ correspond bijectively to the thick subcategories of the derived category of perfect complexes over an arbitrary commutative ring. This explains the following name for the open sets of the space $\mathrm{Spec}(R)^*$ often found in literature, even though Hochster's works predates that of Thomason by almost three decades.

Definition 1.4.17. Let R be a commutative ring. A subset X of $\mathrm{Spec}(R)$ is called *Thomason*, if it satisfies any of the three following equivalent conditions:

(i) X is an open set in $\mathrm{Spec}(R)^*$,

(ii) X is a union of Zariski closed sets with quasi-compact complements,

(iii) there is a set $\mathcal{I}$ of finitely generated ideals of R such that $X = \bigcup_{I \in \mathcal{I}} V(I)$.

1.4.3 The three points of viewing Thomason sets and the vaguely associated primes

We finish this section by writing out in detail the tight connection between the three kinds of objects over a commutative ring R - hereditary torsion pairs of finite type, finitely generated Gabriel topologies, and Thomason sets. Apart from the vaguely associated prime business, all the parts are either salvageable in literature, or folklore to experts. However, since it proved highly useful to switch between these three viewpoints in the body of the thesis (mainly in papers [**i**] and [**iii**]), we feel it is worthy to make the correspondence clear. The following generalization of the notion of associated primes over a noetherian ring was introduced in paper [**i**]:

Definition 1.4.18. Let R be a commutative ring and M an R-module. We say that a prime $p \in \mathrm{Spec}(R)$ is *vaguely associated* to M if R/p belongs to the smallest full subcategory of Mod-R containing M and closed under direct limits and submodules. By $\mathrm{VAss}(M)$, we denote the set of all primes which are vaguely associated to M.

Let us gather the basic properties of this notion, proved in [**i**]:

Lemma 1.4.19. *Let R be a commutative ring.*

1. *If M is a non-zero R-module, then $\mathrm{VAss}(M)$ is non-empty.*
2. *If R is noetherian, then $\mathrm{VAss}(M)$ coincides with the set of standard associated primes $\mathrm{Ass}(M)$.*
3. *For any R-module M, $\mathrm{VAss}(M) \subseteq \mathrm{Supp}(M) = \{p \in \mathrm{Spec}(R) \mid M \otimes_R R_p \neq 0\}$.*
4. *If M is a finitely generated R-module, then $p \in \mathrm{VAss}(M)$ implies that $\mathrm{Hom}_R(M, R/p) \neq 0$.*

Proof. Follows directly from Lemmas 2.3.8 - 2.3.10. □

Theorem 1.4.20. *Let R be a commutative ring. Then there are bijective correspondences between the following collections:*

(i) Thomason subsets X of $\mathrm{Spec}(R)$,

(ii) finitely generated Gabriel topologies $\mathcal{G}$ of R, and

(iii) hereditary torsion pairs $(\mathcal{T}, \mathcal{F})$ of finite type.

The mutually inverse assignments are given as follows:

$$
\begin{aligned}
&(i) \to (ii) : X \mapsto \{I \text{ ideal of } R \mid V(I) \subseteq X\},\\
&(i) \to (iii) : X \mapsto (\{M \mid \mathrm{Supp}(M) \subseteq X\}, \{M \mid \mathrm{VAss}(M) \cap X = \emptyset\}),\\
&(ii) \to (i) : \mathcal{G} \mapsto \bigcup_{I \in \mathcal{G}} V(I) = \mathcal{G} \cap \mathrm{Spec}(R),\\
&(iii) \to (i) : (\mathcal{T}, \mathcal{F}) \mapsto \bigcup_{M \in \mathcal{T}} \mathrm{Supp}(M) = \mathrm{Spec}(R) \setminus \mathrm{Ass}(\mathcal{F}).
\end{aligned}
$$

For correspondences between (ii) and (iii) we refer to Theorem 1.4.6.

Proof. $(i) \leftrightarrow (ii)$: This is proved in Lemma 4.2.10.

$(i) \to (iii)$: Combine Propositions 4.2.11 and 4.2.13.

$(iii) \to (i)$: The assignment of the torsion class $\mathcal{T}$ is handled in Proposition 4.2.11. Then $\mathcal{F} = \mathcal{T}^{\perp_0}$. Since $\mathcal{T}$ is closed under submodules and direct limits, we have

$$X := \bigcup_{M \in \mathcal{T}} \mathrm{Supp}(M) = \bigcup_{F \in \mathcal{T},\ \text{F finitely generated}} \mathrm{Supp}(F).$$

From Lemma 2.3.10, we infer equivalences for a prime p:

$$p \in X \iff \exists M \in \mathcal{T} : \mathrm{Hom}_R(M, R/p) \neq 0 \iff R/p \notin \mathcal{F} \iff p \notin \mathrm{Ass}(\mathcal{F}).$$

That concludes the proof. □

We conclude this section by a homological characterization of vaguely associated primes. The condition (i) of the following Proposition would in the setting of a noetherian ring R and a finitely generated module M read "the grade of M with respect to I is at least n". The connection with the grade theory is made more apparent in Sections §4.3.3 and §4.3.4, where Koszul cohomology characterization is appended.

Proposition 1.4.21. (Proposition 4.3.13) *Let R be a commutative ring and I a finitely generated ideal. Given a module M and $n > 0$, the following are equivalent:*

(i) $\operatorname{Ext}^i_R(R/I, M) = 0$ *for all* $i = 0, 1, \dots, n-1$,

(ii) $\operatorname{VAss}(M) \cap V(I) = \emptyset$ *for all* $i = 0, 1, \dots, n-1$.

In particular, the hereditary torsion-free pair of finite type with the torsion-free class $(R/I)^{\perp_0}$ (cf. Lemma 1.4.8) corresponds to the Thomason set $V(I)$ in the sense of Theorem 1.4.20.

1.5 Classification results

Now we are ready to sum our main results - the classifications theorem for tilting and silting classes over an arbitrary commutative ring. In this section, let R be always a commutative ring.

1.5.1 Classes

The following notion will give us the right parametrizing set for the n-tilting classes. It is a direct generalization of the notion of the same name used in [3].

Definition 1.5.1. Let $n \geq 0$. We say that a sequence $\bar{\mathfrak{t}} = (\mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_{n-1})$ of hereditary torsion pairs $\mathfrak{t}_i = (\mathcal{T}(\mathfrak{t}_i), \mathcal{F}(\mathfrak{t}_i))$ of finite type is *characteristic (of length n)* if the following two conditions are satisfied:

- The torsion-free classes form a (not necessarily strictly) increasing sequence: $\mathcal{F}(\mathfrak{t}_0) \subseteq \mathcal{F}(\mathfrak{t}_1) \subseteq \cdots \subseteq \mathcal{F}(\mathfrak{t}_{n-1})$, and
- $\Omega^{-i} R \in \mathcal{F}(\mathfrak{t}_i)$ for each $i = 0, 1, \dots, n-1$.

Via Theorem 1.4.20, to each hereditary torsion pair $\mathfrak{t} = (\mathcal{T}(\mathfrak{t}), \mathcal{F}(\mathfrak{t}))$ corresponds a finitely generated Gabriel topology $\mathcal{G}(\mathfrak{t})$, and a Thomason set $X(\mathfrak{t})$. Then we have:

Lemma 1.5.2. *A sequence $(\mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_{n-1})$ of hereditary torsion pairs of finite type is characteristic if and only if*

- $\mathcal{G}(\mathfrak{t}_0) \supseteq \mathcal{G}(\mathfrak{t}_1) \supseteq \cdots \supseteq \mathcal{G}(\mathfrak{t}_{n-1})$, *and*
- $\operatorname{Ext}^i_R(R/I, R) = 0$ *for each* $I \in \mathcal{G}(\mathfrak{t}_i)$ *and each* $i = 0, 1, \dots, n-1$

if and only if

- $X(\mathfrak{t}_0) \supseteq X(\mathfrak{t}_1) \supseteq \cdots \supseteq X(\mathfrak{t}_{n-1})$, *and*
- $\mathrm{VAss}(\Omega^{-i}R) \cap X(\mathfrak{t}_i) = \emptyset$ *for each* $i = 0, 1, \ldots, n-1$.

Proof. A direct consequence of Theorem 1.4.20 and Lemma 1.4.21. □

Hence, we can view a characteristic sequence equivalently as a sequence of hereditary torsion pairs of finite type, finitely generated Gabriel topologies, or Thomason sets satisfying the conditions above. Indeed, from now on we will freely switch betweens those three versions of characteristic sequences in order to make the formulations of the classification theorems easier. Using the closure under injective envelopes, the classification on the cotilting side has the following nice form, for which we do not have an analogous version on the tilting side.

Theorem 1.5.3. (Theorem 4.5.3. and Theorem 4.6.1) *There is a 1-1 correspondence*

$$\{\textit{characteristic sequences } \bar{\mathfrak{t}} \textit{ of length } n\} \leftrightarrow \{n\textit{-cotilting classes } \mathcal{C} \textit{ of cofinite type}\}$$

given by the mutually inverse assignments

$$\bar{\mathfrak{t}} \mapsto \{M \in \text{Mod-R} \mid \Omega^{-i}M \in \mathcal{F}(\mathfrak{t}_i)\ \forall i = 0, 1, \ldots, n-1\}$$

$$\mathcal{C} \mapsto (\mathrm{Ass}(\mathcal{C})^c, \mathrm{Ass}(\mathcal{C}_{(1)})^c, \ldots, \mathrm{Ass}(\mathcal{C}_{(n-1)})^c),$$

where ^{c} *stands for the complement of a set in* $\mathrm{Spec}(R)$.

By the character duality, we then automatically have a bijection between characteristic sequences and tilting classes, which can be stated explicitly in the following way. Given a finitely generated Gabriel topology $\mathcal{G}$, let $\mathcal{G}^f$ denote the collection of all finitely generated ideals from $\mathcal{G}$ (and indeed, the following theorem would work if we replaced $\mathcal{G}^f$ by any subset $\mathcal{G}'$ of $\mathcal{G}$ consisting of finitely generated ideals such that the closure of $\mathcal{G}'$ under ideal product is a filter basis of $\mathcal{G}$).

Theorem 1.5.4. (Theorem 4.6.2) *Given* $n \geq 0$, *the following three families are in bijective correspondences:*

(i) *characteristic sequences* $\bar{\mathfrak{t}}$ *of length* n,

(ii) n*-tilting classes* $\mathcal{T}$ *in* Mod-R, *and*

(iii) n*-cotilting classes* $\mathcal{C}$ *of cofinite type in* Mod-R

given by assignments

$$(i) \to (ii) : \bar{\mathfrak{t}} \mapsto \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}(\mathfrak{t}_i)^f} \mathrm{Ker}\,\mathrm{Tor}_i^R(R/I, -),$$

$$(i) \to (iii) : \bar{\mathfrak{t}} \mapsto \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}(\mathfrak{t}_i)^f} \mathrm{Ker}\,\mathrm{Ext}_R^i(R/I, -).$$

In paper [*iii*], we prove versions of Theorem 1.5.4 replacing the Tor and Ext by Koszul, local, and Čech homology and cohomology, and we also provide a description of the corresponding resolving subcategories of $\mathrm{mod_n}$-R. For this, see Sections §4.6 and §4.7.

Finally, we extract the main result from paper [*ii*] on silting classes in the following way.

Theorem 1.5.5. *The following collections are in bijective correspondences*

(i) Thomason subsets X of $\mathrm{Spec}(R)$*,*

(ii) silting classes $\mathcal{D}$ in Mod-R*,*

(iii) cosilting classes $\mathcal{C}$ of cofinite type in Mod-R*.*

These are given by the following mutually inverse bijections

$$(i) \to (ii) : X \mapsto \{M \in \text{Mod-R} \mid M = IM \textit{ for any ideal } I \textit{ such that } V(I) \subseteq X\},$$

$$(ii) \to (i) : \mathcal{D} \mapsto \{p \in \mathrm{Spec}(R) \mid M = pM \textit{ for any } M \in \mathcal{D}\},$$

and

$$(i) \to (iii) : X \mapsto \{M \in \text{Mod-R} \mid \mathrm{VAss}(M) \cap X = \emptyset\},$$

$$(iii) \to (i) : \mathcal{C} \mapsto (\mathrm{Spec}(R) \setminus \mathrm{Ass}(\mathcal{C})).$$

Proof. $(i) \leftrightarrow (ii)$: This follows by straightforward combination of Theorem 3.4.7 and Theorem 1.4.20.

$(i) \leftrightarrow (iii)$: By Corollary 1.3.10, the cosilting classes of cofinite type in Mod-R are precisely the hereditary torsion-free classes of finite type. The correspondence is then established by Theorem 1.4.20. □

Note that in the light of Theorem 1.5.4, the 1-tilting class amongst the silting classes in Theorem 1.5.5 are precisely the ones which correspond to a Thomason set X such that $\mathrm{VAss}(R) \cap X = \emptyset$.

1.5.2 Modules

Generally speaking, given an n-tilting class $\mathcal{T}$ we can always construct an n-tilting module T such that $T^{\perp_\infty} = \mathcal{T}$ by iterating special $\mathcal{T}$-preenvelopes, starting with R:

$$0 \to R \to T_0 \to T_1 \to \cdots \to T_{n-1} \to T_n \to 0,$$

where T_n is the cokernel of the n-iteration of the special $\mathcal{T}$-preenvelope. Then $T = T_0 \oplus T_1 \oplus \cdots \oplus T_n$ is the desired n-tilting module (see [23, Chapter 13]). However, the special $\mathcal{T}$-preenvelopes, although the small object argument ensures their existence, are in general hard to construct explicitly. In Section §2.4, we provide for any 1-tilting class over a commutative ring a rather explicitly constructed tilting module δ generating it. We call δ a *Fuchs-Salce tilting module*, as its construction generalizes that of the Fuchs and Salce modules introduced by Facchini [24] for divisible modules over domains, by Fuchs-Salce [25] for S-divisible module for any multiplicative set S of a domain, and by Salce [26] for localizing sets of a Prüfer domain. Also, this construction is generalized to the silting case in Construction 3.4.5.

One application of this construction is an alternative elementary proof of the commutative version of the so-called Saorín's problem, solved originally by Bazzoni ([14]) - see Theorem 2.4.6. Here, we remark another consequence of this construction. In Section §2.5, we discuss the cases when the 1-tilting class $\mathcal{T}$ over a commutative ring R admits a tilting module T induced by a flat ring epimorphisms - meaning that $T = S \oplus S/R$, where $R \to S$ is a flat epimorphism of rings. Such situation is characterized in terms of the associated Gabriel topology in Theorem 2.5.4, and it is a rather special situation. For example if $R = k[x, y]$ is a ring of polynomials in variables x, y over a field k, I is the ideal generated by $\{x, y\}$, then the class $\mathcal{T} = \{M \in \text{Mod-R} \mid M = IM\}$ of I-divisible modules is 1-tilting. But, there is no flat ring epimorphism $R \to S$ such that $\mathrm{Gen}(S) = \mathcal{T}$ (see [31, §IX Exercise 3 and Theorem 2.5.4]). However, if carried out carefully, the Fuchs-Salce construction yields a tilting module for this class of form $M \oplus M/R$, where M is a flat module with R as a submodule.

Proposition 1.5.6. *Let $\mathcal{I}$ be a countable set of finitely generated ideals of R. Then there is an inclusion $R \overset{\iota}{\hookrightarrow} M$ of R into a flat R-module M such that $M \oplus M/\operatorname{Im}(\iota)$ is a 1-tilting module generating the tilting class $\{M \in \text{Mod-R} \mid M = IM \ \forall I \in \mathcal{I}\}$.*

Proof. Since $\mathcal{I}$ is countable, there is an ω-sequence $(I_n \mid 0 < n < \omega)$ of ideals of $\mathcal{I}$ such that every ideal of $\mathcal{I}$ occurs in this sequence infinitely many times. For each $n > 0$ we fix a finite generating set $x_n^1, x_n^2, \dots, x_n^{k_n}$. Then the construction of the Fuchs-Salce module in Definition 2.4.1 can be adjusted as follows (and in the case of $\mathcal{I}$ being a singleton, the construction will be exactly the same).

For each $n > 0$, let $A_n = \begin{pmatrix} x_n^1 \\ x_n^2 \\ \vdots \\ x_n^{k_n} \end{pmatrix}$ be the matrix inducing a map $R \overset{A_n}{\hookrightarrow} R^{k_n}$.

Note that since $\mathrm{Hom}_R(R/I_n, R) = 0$, this map is a monomorphism. The module $M = M_{\mathcal{I}}$ is obtained as a direct limit of the following well-ordered ω-sequence of monomorphisms

$$R \overset{A_1}{\hookrightarrow} R^{k_1} \xhookrightarrow{A_2^{\oplus k_2}} R^{k_1 \cdot k_2} \xhookrightarrow{A_3^{\oplus k_3}} R^{k_1 \cdot k_2 \cdot k_3} \xhookrightarrow{A_4^{\oplus k_4}} \cdots$$

where $A^{\oplus k}$ denotes the diagonal-block matrix of k copies of A. As a direct limit of free modules, M is flat. Since the sequence I_n contains each ideal of $\mathcal{I}$ infinitely many times, we infer that $M = IM$ for any $I \in \mathcal{I}$. Also, because $\mathrm{Coker}(A_n)$ is an Auslander-Bridger transpose of R/I_n for each $n > 0$, we conclude that the inclusion $A_1 : R \hookrightarrow M$ is a special $\mathcal{I}$-divisible preenvelope of R (see Proposition 2.4.3 for details). By the same argument as in Proposition 2.4.3, we have that $M \oplus M/\operatorname{Im} A_1$ is a 1-tilting module generating $\{M \in \text{Mod-R} \mid M = IM \ \forall I \in \mathcal{I}\}$. □

1.5.3 Example - von Neumann regular rings

To further illustrate our results, we apply them to a particular class of commutative rings. Recall that a ring R is *von Neumann regular*, if any one of the following equivalent conditions is satisfied by R (see [32, Theorem 1.1, Corollary 1.13]):

(i) for any $a \in R$ there is $x \in R$ such that $a = axa$,

(ii) any finitely generated ideal right (or equivalently, left) ideal of R is generated by an idempotent,

(iii) all right R-modules (or equivalently, or left R-modules) are flat.

Proposition 1.5.7. *Let R be a commutative von Neumann regular ring. Then there are bijections between the following collections:*

(i) ideals I of R,

(ii) silting classes $\mathcal{D}$ in Mod-R*,*

(iii) cosilting classes $\mathcal{C}$ in Mod-R*,*

(iv) epiclasses of flat ring epimorphisms $R \to S$.

The bijections from (i) to $(ii), (iii)$, and (iv) are given as follows:

$$(i) \to (ii) : I \mapsto \mathcal{D} = \mathrm{Gen}(R/I),$$

$$(i) \to (iii) : I \mapsto \mathcal{C} = \mathrm{Gen}(R/I),$$

$$(i) \to (iv) : I \mapsto (R \to S) = (R \twoheadrightarrow R/I).$$

In particular, any definable torsion class in Mod-R *is silting.*

Furthermore, a silting module generating the silting class $\mathrm{Gen}(R/I)$ *for some ideal I can be chosen as R/I.*

Proof. $(i) \to (ii)$: Let $\mathcal{D}$ be a silting class in Mod-R. By Theorem 1.5.5, there is a finitely generated Gabriel topology $\mathcal{G}$ such that $\mathcal{D} = \{M \in \text{Mod-R} \mid M = JM \ \forall J \in \mathcal{G}\}$. Since R is von Neumann regular, any finitely generated ideal J is generated by some idempotent. Therefore, there is a set $\mathcal{E}$ of idempotents of R such that $\mathcal{D} = \{M \in \text{Mod-R} \mid M = eM \ \forall e \in \mathcal{E}\}$. Define I to be the ideal generated by the set $\{1 - e \mid e \in \mathcal{E}\}$ of all the complements of elements from $\mathcal{E}$. Then it is easily seen that $IM = 0$ for any $M \in \mathcal{D}$, establishing $\mathcal{D} \subseteq \mathrm{Gen}(R/I) = \text{Mod-}R/I$. On the other hand, R/I is clearly divisible by e for any $e \in \mathcal{E}$, showing that $R/I \in \mathcal{D}$. As $\mathcal{D}$ is closed under direct sums and epimorphic images, we conclude that $\mathcal{D} = \mathrm{Gen}(R/I)$.

Therefore, the assignment $I \mapsto \mathrm{Gen}(R/I)$ induces a surjection from the set of ideals to silting classes. As R is commutative, different choice of ideals yield different classes, and therefore this assignment is a bijection, as desired.

$(i) \to (iii)$: First, we show that any cosilting class over R is of cofinite type. Since R is von Neumann regular, the notions of pure-injective envelope and injective envelope coincide. If $\mathcal{C}$ is a cosilting class, it is definable by [19, Corollary 4.8], and thus $\mathcal{C}$ is closed under pure-injective envelopes by [23, Lemma 6.9]. Together, any cosilting class in Mod-R is closed under injective envelopes, and thus is of cofinite type by Theorem 1.3.9.

Therefore, by Theorem 1.3.7, the assignment

$$\mathcal{D} \mapsto \mathcal{C} = \{M \in \text{Mod-R} \mid M^+ \in \mathcal{D}\}$$

is a bijection between silting classes $\mathcal{D}$ and cosilting classes $\mathcal{C}$ in Mod-R. But it is easy to check that the R-module elementary duality functor $(-)^+$ restricted to Mod-R/I is naturally isomorphic to an elementary duality functor in the module category Mod-R/I. Since $\mathcal{D} = \text{Mod-}R/I$ for some ideal I, we infer that $\mathcal{C} = \text{Mod-}R/I$. Indeed, if $M \in \text{Mod-R}$ is such that $IM^+ = 0$, then $IM^{++} = 0$, and therefore by the evaluation monomorphism $M \hookrightarrow M^{++}$ also $IM = 0$.

Furthermore, since cosilting classes coincide with definable torsion-free classes by [30], any definable torsion class is dual definable to a cosilting class in the sense of Theorem 1.3.7, and thus is silting.

$(i) \to (iv)$: Since any R-module is flat, the surjective ring morphism $R \to R/I$ is a flat ring epimorphism. On the other hand, any (epiclass of) a flat ring epimorphism $R \to S$ is uniquely determined by a (finitely generated[4]) Gabriel topology $\mathcal{G}$ by the assignment $(R \to S) \mapsto \mathcal{G} = \{J \subseteq R \mid S = JS\}$ (see [31, §XI, Proposition 3.4]). If $\mathcal{G}$ is a finitely generated Gabriel topology, let I be the ideal generated by complements of idempotents from $\mathcal{G}$ as in the proof $(i) \to (ii)$ above. It follows that $\mathcal{G} = \{J \subseteq R \mid J(R/I) = R/I\}$, establishing the correspondence.

Finally, we prove that R/I is a silting module in Mod-R for any ideal I. To that end, we need to construct a map σ between projective modules such that $\text{Coker}(\sigma) \simeq R/I$, and such that $\mathcal{D}_\sigma = \text{Gen}(R/I)$. Let $\mathcal{E}$ be a set of idempotents of R such that $\text{Gen}(R/I) = \{M \in \text{Mod-R} \mid M = eM \ \forall e \in \mathcal{E}\}$, see the proof of $(i) \to (ii)$. For each $e \in \mathcal{E}$, let $\pi_e : R \twoheadrightarrow R/(1-e)R$ be the canonical split projection, and let $\pi : R^{(\mathcal{E})} \twoheadrightarrow \bigoplus_{e\in\mathcal{E}} R/(1-e)R$ be the (block) direct sum of maps π_e. Then $\mathcal{D}_\pi$ consists precisely of those modules annihilated by $(1-e)$ for each $e \in \mathcal{E}$, and thus $\mathcal{D}_\pi = \text{Gen}(R/I)$.

Next, consider the inclusion map $\tau_e : (1-e)R \hookrightarrow R$, and let $\tau : \bigoplus_{e\in\mathcal{E}}(1-e)R \to R$ be the summing map induced by τ_e's. Since I is generated by elements $(1-e), e \in \mathcal{E}$, we have $\text{Coker}(\tau) = R/I$. Now because $\text{Hom}_R((1-e)R, M) = 0$ for any $e \in \mathcal{E}$ and $M \in \text{Gen}(R/I)$, we see that $\text{Gen}(R/I) \subseteq \mathcal{D}_\tau$.

Finally, let

$$\sigma = \pi \oplus \tau : R^{(\mathcal{E})} \oplus \bigoplus_{e\in\mathcal{E}}(1-e)R \xrightarrow{\begin{pmatrix}\pi & 0\\ 0 & \tau\end{pmatrix}} (\bigoplus_{e\in\mathcal{E}} R/(1-e)R) \oplus R$$

be the (block) direct sum of maps π and τ. Then $\text{Coker}(\sigma) = \text{Coker}(\tau) = R/I$ and $\mathcal{D}_\sigma = \mathcal{D}_\pi \cap \mathcal{D}_\tau = \text{Gen}(R/I)$, as desired. □

We add that the only tilting class over a (possibly non-commutative) von Neumann regular ring R is the whole Mod-R. Indeed, a tilting class contains all injectives, and thus all pure-injectives. Since tilting classes are definable, such a class has to contain all R-modules (see [23, Lemma 6.9]).

Also, some behaviour of silting modules not found amongst tilting modules can be demonstrated using Proposition 1.5.7. First, whenever I is not finitely

[4]Indeed, epiclasses of flat ring epimorphisms are in 1-1 correspondence with certain finitely generated Gabriel topologies called *perfect*, and over semihereditary rings (such as the von Neumann regular rings), any finitely generated Gabriel topology is perfect. See [31, §XI].

generated, we exhibited an example of a finitely generated silting module which is not projective, generalizing Example 3.4.9 (cf. [23, Lemma 13.2]). Also, if I is projective, but not finitely generated[5], then R/I is a silting module of projective dimension 1 which is not tilting. Therefore, it admits a monomorphic projective presentation, but non of these witnesses R/I being silting.

1.6 Integral domains and closure under flat covers

The goal of this section could be viewed as an explanation for why the process of classification of the tilting classes, be it the noetherian case in [3] or in our approach in [i] or [iii], was easier handled by working in the dual setting of cotilting classes first, and only then transferring the results back by the elementary duality. Namely, the crucial initial observation that cotilting (or cosilting) classes over commutative rings are closed under taking injective envelopes does not have an obvious analog on the tilting side. With respect to Theorem 1.5.4, the natural question to consider is:

Question 1.6.1. Are tilting classes over commutative rings closed under flat covers?

If this was true, we could for example easily infer a dual version of Theorem 1.5.3, describing n-tilting classes homologically by ideal divisibility of their "yokes" (that is, kernels of maps in their minimal flat resolution). Alas, as we show in this section, the answer to Question 1.6.1 is a rather resolute **NO**.

By a flat cover, we mean a $\mathcal{FLAT}$-cover, where $\mathcal{FLAT}$ denotes the class of all flat R-modules. Say that a class of modules $\mathcal{C}$ is *closed under flat covers* if for any flat cover $F \to M$, $M \in \mathcal{C}$ implies $F \in \mathcal{C}$. The following lemma allows for a useful characterization of tilting classes closed under flat covers.

Lemma 1.6.2. *Let $\mathcal{C}$ be a preenveloping class closed under direct summands. Then $\mathcal{C}$ is closed under flat covers if and only if any flat module admits a $\mathcal{C}$-preenvelope, which is also flat.*

Proof. ($\Rightarrow$) Let F be a flat module and $f : F \to C$ a $\mathcal{C}$-preenvelope. By the assumption, the domain of the flat cover $h : F' \to C$ of C is in $\mathcal{C}$. Then there is a map $g : F \to F'$ such that $f = hg$. It is easy to see that g is a $\mathcal{C}$-preenvelope of F, and it is flat.

($\Leftarrow$) Let $C \in \mathcal{C}$ and consider its flat cover $h : F \to C$. Using the assumption, there is a flat $\mathcal{C}$-preenvelope of F, say $f : F \to L$. Since f is a $\mathcal{C}$-preenvelope, there is a factorization $g : L \to C$ with $h = gf$. Because h is a flat precover, there is a factorization $l : L \to F$ such that $g = hl$. The situation is encaptured in the following diagram:

[5] Such an ideal is available whenever R is not noetherian, that is, not semisimple. Indeed, by [29, p. 45], any countably generated ideal is projective.

$$\begin{tikzcd} F \arrow[r, "h"] \arrow[d, shift left, "f"] & C \\ L \arrow[u, dashed, shift left, "l"] \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Hence, $hlf = gf = h$, and thus lf is an automorphism of F. Therefore, f is a split monomorphism, and therefore $F \in \mathcal{C}$, since $\mathcal{C}$ is closed under direct summands. $\square$

Corollary 1.6.3. *A tilting class $\mathcal{T}$ is closed under flat covers if and only if any flat R-module has a $\mathcal{T}$-preenvelope which is flat.*[6]

From this point we confine ourselves to the case of R being an integral domain, and consider the class $\mathcal{D} = \{M \in \text{Mod-R} \mid rM = M \ \forall 0 \neq r \in R\}$ of all classical divisible R-modules. Then $\mathcal{D} = (\bigoplus_{0 \neq r \in R} R/rR)^{\perp_\infty}$ is a 1-tilting class in Mod-R. We prove that $\mathcal{D}$ is closed under flat covers if and only if R is an almost perfect domain, and in this case, all tilting classes are closed under flat covers. We start by recalling several useful kinds of integral domains.

During the rest of the section, let Q always denote the field of quotients of integral domain R.

Lemma 1.6.4. *Suppose that F is a flat R-module such that F admits a flat $\mathcal{D}$-preenvelope. Than the canonical map $F \to F \otimes_R Q$ is a $\mathcal{D}$-envelope of F.*

Proof. Let $f : F \to L$ be a $\mathcal{D}$-preenvelope of F such that L is flat, and let $\iota : F \to F \otimes_R Q$ denote the canonical map. Since L is flat (and thus torsion-free), and divisible, we have a natural isomorphism $L \otimes_R Q \simeq L$. Then tensoring f by Q yields that f factors through ι. This already shows that ι is a $\mathcal{D}$-preenvelope of F. Because ι is an essential monomorphism, we easily deduce that it has to be a $\mathcal{D}$-envelope. $\square$

Recall that an integral domain R is called *Matlis*, if $\operatorname{pd} Q \leq 1$.

Lemma 1.6.5. *Let R be a Matlis domain. If there is a flat module F such that* $\operatorname{pd} F > 1$*, then the class $\mathcal{D}$ of all divisible R-modules is not closed under flat covers.*

Proof. Using Lemma 1.6.2, it is enough to show that F has no flat $\mathcal{D}$-preenvelope. Towards a contradiction, suppose otherwise. Therefore, by Lemma 1.6.4, the canonical map $\iota : F \to F \otimes_R Q$ is a $\mathcal{D}$-envelope. By the Wakamatsu Lemma ([23, Lemma 5.13]), ι is a special $\mathcal{D}$-preenvelope, and thus $\operatorname{pd} \operatorname{Coker}(\iota) \leq 1$. But since $F \otimes_R Q$ is isomorphic to a direct sum of copies of Q, we conclude from R being a Matlis domain that $\operatorname{pd}(F \otimes_R Q) \leq 1$, and thus $\operatorname{pd} F \leq 1$, a contradiction. $\square$

Lemma 1.6.6. *Let R be an integral domain. If $\mathcal{D}$ is closed under flat covers, then R is Matlis.*

[6] Note that this is not the same thing as a $(\mathcal{T} \cap \mathcal{FLAT})$-preenvelope.

Proof. If $\mathcal{D}$ is closed under flat covers, then again by Lemma 1.6.2 and Lemma 1.6.4, the canonical map $\iota : R^{(\varkappa)} \to Q^{(\varkappa)}$ is a $\mathcal{D}$-preenvelope for all cardinals $\varkappa$. But then Q generates $\mathcal{D}$, which implies $\operatorname{pd} Q \leq 1$ by [12, Theorem 1.1]. □

An integral domain is said to be *almost perfect* if any proper factor of R is a perfect ring (i.e., a ring such that any flat module is projective). These domains were introduced by Bazzoni and Salce and have many equivalent characterizations (see e.g. [22]). We list just a few of them, which will be useful in what follows. Recall that integral domain R is *h-local* if the following two conditions hold:

1. R has finite character, that is, each non-zero element belongs to only finitely many maximal ideals of R, and
2. every non-zero prime ideal of R is contained in precisely one maximal ideal of R.

Theorem 1.6.7. [22, Main Theorem] *For an integral domain R, the following conditions are equivalent:*

1. *R is almost perfect,*
2. *R is h-local, and every localization of R is almost perfect,*
3. *every R-module of weak dimension ≤ 1 has projective dimension ≤ 1.*

We also note some properties of almost perfect domains:

Lemma 1.6.8. *Let R be an almost perfect domain. Then R is Matlis, and has Krull dimension at most 1. A noetherian domain of Krull dimension at most 1 is almost perfect.*

Proof. The first two properties are proved in [22, Proposition 3.5]. The last claim is [22, Proposition 5.1]. □

Before proceeding with studying flat covers, we prove some properties of h-local domains of Krull dimension 1 concerning the Hochster topology of the spectrum and also the tilting modules. By $\operatorname{mSpec}(R)$ we denote the subset of $\operatorname{Spec}(R)$ consisting of all maximal ideals. Reader interesting only in flat cover closure only needs to be concerned with Proposition 1.6.9, and then skip to Lemma 1.6.13. Also, given a maximal ideal $\mathfrak{m}$ we denote by $R_\mathfrak{m}$ the localization of R in $\mathfrak{m}$, and for any subset X of $\operatorname{mSpec}(R)$ let $R_X = \bigcap_{\mathfrak{m} \in X} R_\mathfrak{m}$. If $\mathfrak{m} \in \operatorname{mSpec}(R)$, we denote for convenience by $[\mathfrak{m}] = \operatorname{mSpec}(R) \setminus \{\mathfrak{m}\}$ the complement of $\mathfrak{m}$ in $\operatorname{mSpec}(R)$. Given a module M, we adopt the shorthand notation $M_\mathfrak{m} = M \otimes_R R_\mathfrak{m}$.

Proposition 1.6.9. *Let R be an integral domain of Krull dimension 1. Then any subset of* $\operatorname{mSpec}(R)$ *is Thomason if and only if R is h-local.*

Proof. Let R be of Krull dimension 1. First, let R be h-local and let us show that any set of maximal ideals is Thomason. It is clearly enough to show that any singleton set $\{\mathfrak{m}\} \subseteq \operatorname{mSpec}(R)$ is Thomason, that is, find a finitely generated ideal I such that $V(I) = \{\mathfrak{m}\}$. Pick any non-zero element $x_0 \in \mathfrak{m}$. Since R is h-local, $V(x_0)$ is a finite set containing $\mathfrak{m}$. Now for any $\mathfrak{n} \in V(x_0)$ not equal to

$\mathfrak{m}$, there is an element $x_\mathfrak{n} \in \mathfrak{m}$ such that $x_\mathfrak{n} \notin \mathfrak{n}$. Hence, the ideal I generated by x_0 and all the $x_\mathfrak{n}$'s is a finitely generated ideal with $V(I) = \{\mathfrak{m}\}$.

Conversely, suppose that for any maximal ideal $\mathfrak{m}$ there is an ideal I generated by $x_1, \dots, x_n$ such that $V(I) = \{\mathfrak{m}\}$, and let us show that R is h-local. By [21, IV.3. Theorem 3.7], it is enough to show that for each $\mathfrak{m} \in \mathrm{mSpec}(R)$, the module $A = R_\mathfrak{m} \otimes_R R_{[\mathfrak{m}]}$ is divisible (and thus, isomorphic to Q). There is a disjoint partition of $[\mathfrak{m}] = X_1 \cup X_2 \ \cup \ldots \cup X_n$ such that for each $i = 1, \dots, n$, x_i is not contained in any maximal ideal from the set X_i. It follows that R_{X_i} is divisible by x_i; indeed, $\frac{1}{x_i} \in R_\mathfrak{n}$ for each $\mathfrak{n} \in X_i$. As localizations commute with finite intersections, we have that $A = \bigcap_{1 \leq i \leq n} A_i$, where $A_i = R_\mathfrak{m} \otimes_R R_{X_i}$. Let $S_i = \{x_i^n \mid n \geq 0\}$ be the multiplicative set generated by x_i. Then $A_i S_i^{-1} = (R_\mathfrak{m} S_i^{-1}) \otimes_R R_{X_i}$ is a divisible R-module, because $x_i \in \mathfrak{m}$, and $R_\mathfrak{m}$ is a 1-dimensional local domain. Because $x_i \notin X_i$, the module A_i is divisible by S_i, that is $A_i = x_i A_i$. On the other hand, we showed that the localization $A_i S_i^{-1}$ is divisible (by any non-zero element of R). It follows that A_i is divisible for each i. Therefore, $A = \bigcap_{1 \leq i \leq n} A_i$ is an intersection of torsion-free divisible modules, which is always divisible, as desired. □

Lemma 1.6.10. *Let R be an integral domain of Krull dimension at most 1. Then any tilting class is 1-tilting.*

Proof. With respect to Theorem 1.5.4, it is enough to show that $\mathrm{Ext}^1_R(R/I, R) \neq 0$ for any non-zero finitely generated ideal I (in the noetherian setting, this follows directly from the classical grade theory). Applying $\mathrm{Hom}_R(R/I, -)$ to the exact sequence $0 \to R \to Q \to Q/R \to 0$ yields $\mathrm{Ext}^1_R(R/I, R) \simeq \mathrm{Hom}_R(R/I, Q/R)$. Since R/I is finitely presented, the vanishing of the latter abelian group is equivalent to $\mathrm{Hom}_{R_\mathfrak{m}}(R_\mathfrak{m}/I_\mathfrak{m}, Q/R_\mathfrak{m}) = 0$ for all $\mathfrak{m} \in \mathrm{mSpec}(R)$. But whenever $\mathfrak{m} \in V(I)$, this would by 1-dimensionality and locality[7] imply that $Q/R_\mathfrak{m}$ is a (classical) torsion-free $R_\mathfrak{m}$-module, and therefore $Q/R_\mathfrak{m} = 0$. This can happen only in the situation $Q = R$, where the whole claim is trivially true. □

Definition 1.6.11. Let R be a Matlis domain with quotient field Q. Denote by $\pi : Q \to Q/R$ the canonical projection, and let A be any direct summand of Q/R. Then the module $A \oplus \pi^{-1}[A]$ is the *Bass tilting module* associated to A.

Since R is Matlis, that is $\mathrm{pd}\, Q \leq 1$, and using the fact that, denoting by $\mathcal{P}_1$ the class of all R-modules of projective dimension ≤ 1, $(\mathcal{P}_1, \mathcal{D})$ is a cotorsion pair (see [16, Proposition 6.3]), it is easy to check properties (T1)-(T3) in order to infer that any Bass tilting module is a 1-tilting R-module.

Proposition 1.6.12. *Let R be a Matlis domain. Then any tilting module is equivalent to a Bass tilting module if and only if R is h-local and of Krull dimension at most 1.*

Proof. Let R be a 1-dimensional h-local domain. By the previous lemma, any tilting class is 1-tilting. Let $\mathcal{T}$ be a 1-tilting class, and let $X \subseteq \mathrm{mSpec}(R)$ be the Thomason set corresponding to $\mathcal{T}$ in the sense of Theorem 1.5.4 and Theorem 1.4.20. Explicitly, with respect to Proposition 1.6.9, there is a finitely

[7] Indeed, over a 1-dimensional local integral domain there are only three hereditary torsion pairs of finite type - $(0, \text{Mod-R})$, $(\text{Mod-R}, 0)$, and the classical one, cf. Theorem 1.4.20.

generated ideal $I_\mathfrak{m}$ such that $V(I_\mathfrak{m}) = \{\mathfrak{m}\}$, and $\mathcal{T} = \{M \in \text{Mod-R} \mid M = I_\mathfrak{m} M \ \forall \mathfrak{m} \in X\}$. Since R is h-local, there is by [21, Theorem 3.7] a natural isomorphism $Q/R \simeq \bigoplus_{\mathfrak{m}\in \text{mSpec}(R)} Q/R_\mathfrak{m} \simeq \bigoplus_{\mathfrak{m}\in \text{mSpec}(R)} R_{[\mathfrak{m}]}/R$. We let $A_X = \bigoplus_{\mathfrak{m}\in X} Q/R_\mathfrak{m}$ and let $T_X = A_X \oplus \pi^{-1}[A_X]$ be the Bass tilting module corresponding to A_X. It is easy to see that T_X is divisible by a finitely generated ideal I if and only if $V(I) \subseteq X$ (as $\pi^{-1}[A_X] = R_{\text{mSpec}(R)\setminus X}$). Since T_X is a 1-tilting module, it has to generate the tilting class $\mathcal{T}$ (see Theorem 2.3.16).

Suppose, on the other hand, that any tilting class in Mod-R is generated by a Bass tilting module. First, let us show that the Krull dimension of R is at most 1. Indeed, if $\mathfrak{p}$ is a non-zero and non-maximal prime ideal, consider the tilting class $\mathcal{T}_S$ of all S-divisible modules, where $S = R \setminus \mathfrak{p}$ (any multiplicative set is a filter basis of a finitely generated Gabriel topology). Suppose $\mathcal{T}_S$ is generated by a Bass tilting module T. Localizing at any maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$, we can assume without loss of generality that R is local - clearly a localization of a Bass tilting module at $\mathfrak{m}$ is a Bass tilting module over $R_\mathfrak{m}$, and by [23, Proposition 13.50], the tilting class generated by $T_\mathfrak{m}$ consists of all $S_\mathfrak{m} = (R_\mathfrak{m} \setminus \mathfrak{p}_\mathfrak{m})$-divisible modules. Since R is now a assumed to be local, Q/R is indecomposable, and thus there are only two Bass tilting modules over - the trivial one generating Mod-R, and then $Q \oplus Q/R$ generating $\mathcal{D}$. As none of these generates $\mathcal{T}_S$, we established a contradiction, and thus R is indeed 1-dimensional.

Finally we prove that R is h-local. Given any maximal ideal $\mathfrak{m}$, the cofinite subset $[\mathfrak{m}]$ of mSpec(R) is easily seen to be Thomason. Therefore, there is a tilting class $\mathcal{T}$ corresponding to $[\mathfrak{m}]$ as in the first part of this proof. By the hypothesis, $\mathcal{T}$ is generated by a Bass tilting module $T_A = A \oplus \pi^{-1}[A]$ for some direct summand A of Q/R. By [21, §IV, Lemma 4.2b], necessarily $A \simeq R_\mathfrak{m}/R$ and Q/R is naturally isomorphic to $R_\mathfrak{m}/R \oplus R_{[\mathfrak{m}]}/R$. Then the complement $B = R_{[\mathfrak{m}]}/R$ is non-zero, and if $T_B = B \oplus \pi^{-1}[B]$ is the Bass tilting module corresponding to B, the tilting class is not equal to Mod-R. Furthermore, any ideal I such that $T_B = IT_B$ is not contained in any maximal ideal other than $\mathfrak{m}$. Then the Thomason set corresponding to T_B needs to be the singleton $\{\mathfrak{m}\}$. We proved that any subset of mSpec(R) is Thomason, and therefore R is h-local by Proposition 1.6.9. $\square$

Lemma 1.6.13. *Let R be an integral domain. If $\mathcal{D}$ is closed under flat covers, then R is almost perfect.*

Proof. By combining Lemmas 1.6.5 and 1.6.6, we know that the hypothesis implies that all flat R-modules have projective dimension at most 1. Let M be of flat dimension 1, that is, we have an exact sequence

$$0 \to F_1 \to F_0 \to M \to 0$$

with F_0, F_1 flat. Let Z denote the cokernel of the obvious map $F_1 \to F_0 \otimes_R Q$. Consider the exact sequence

$$0 \to (F_1 \otimes_R Q)/F_1 \to Z \to M \otimes_R Q \to 0.$$

The leftmost term has projective dimension ≤ 1, because it is a cokernel of a special $\mathcal{D}$-preenvelope (Lemma 1.6.4), while the rightmost term has projective dimension ≤ 1 because R is Matlis. Hence, $\operatorname{pd} Z \leq 1$. Now, let us focus our attention on the exact sequence

$$0 \to M \to Z \to (F_0 \otimes_R Q)/F_0 \to 0.$$

We already know that $\operatorname{pd} Z \leq 1$. The rightmost term of the sequence is again a cokernel of a special $\mathcal{D}$-preenvelope, whence it is of projective dimension at most 1 too. Therefore, $\operatorname{pd} M \leq 1$ as desired. We proved that R-modules of weak dimension ≤ 1 coincide with the R-modules of projective dimension ≤ 1, proving that R is almost perfect. $\square$

In the final step, we prove that all tilting classes over an almost perfect domain are closed under flat covers. Before that, we recall some required basics of the fractional ideal theory. Given an ideal I, let $I^{-1} = \{r \in Q \mid rI \subseteq R\}$. Ideal I is *invertible* if $II^{-1} = R$. For any $n > 0$, we set $I^{-n} = (I^n)^{-1}$ and $I^0 = R$.

Lemma 1.6.14. *Let R be an integral domain and I a non-zero ideal. Then:*

(i) I is projective if and only if it is invertible, and in this case, I is finitely generated.

(ii) If I is invertible, the quotient of fractional ideals $I^{-(n+1)}/I^{-n}$ is a projective generator in Mod-R/I for any $n \geq 0$.

(iii) If I is invertible, then $\operatorname{Ext}^1_R(P, D) = 0$ for any projective R/I-module P and any D such that $D = ID$.

Proof. (i) See [33, Proposition 7.2 and Lemma 7.1].

(ii) Consider the tensor product $I^{-(n+1)} \otimes_R R/I$. By right exactness of tensoring, we have natural isomorphisms

$$I^{-(n+1)} \otimes_R R/I \simeq (I^{-(n+1)} \otimes_R R)/(I^{-(n+1)} \otimes_R I) \simeq I^{-(n+1)}R/I^{-(n+1)}I.$$

If I is invertible, then I^n is invertible too, and $I^{-(n+1)} = (I^{-1})^{n+1}$. This shows that $I^{-(n+1)} \otimes_R R/I \simeq I^{-(n+1)}/I^{-n}$. Therefore, $I^{-(n+1)}/I^n$ is an R/I-module, and since $I^{-(n+1)}$ is a projective R-module, $I^{-(n+1)}/I^{-n}$ is a projective R/I-module. Since R is a domain, $I^{-(n+1)}$ is a projective generator (as the trace is a pure ideal of R), and thus $I^{-(n+1)} \otimes_R R/I$ is also a projective generator in Mod-R/I.

(iii) Since I is projective, we have that $I \to R \to R/I \to 0$ is a projective presentation of R/I. Therefore, I^{-1}/R is an Auslander-Bridger transpose of R/I, and thus $(I^{-1}/R)^{\perp_1} = \{D \in \text{Mod-R} \mid D = ID\}$ (for details, see §2.3.1 and Theorem 2.3.16). By (ii), I^{-1}/R is a projective generator of Mod-R/I, and thus $\operatorname{Add}(R/I) = \operatorname{Add}(I^{-1}/R)$, and therefore $\operatorname{Ext}^1_R(P, D) = 0$ for any $P \in \operatorname{Add}(R/I)$ and any $D \in \text{Mod-R}$ such that $D = ID$.

$\square$

Lemma 1.6.15. *Let R be an almost perfect domain. Then:*

(i) every finitely generated Gabriel topology has a filter basis of projective ideals,

(ii) every 1-tilting class is closed under flat covers.

Proof. (i) First, by Lemma 1.6.9, there is for each $\mathfrak{m} \in \mathrm{mSpec}(R)$ a finitely generated ideal $I^{\mathfrak{m}}$, such that $V(I^{\mathfrak{m}}) = \{\mathfrak{m}\}$. It is then enough to settle (i) for the Gabriel topology generated by $I^{\mathfrak{m}}$, that is, for the case $\mathcal{G} = \{J \mid \exists n > 0 : (I^{\mathfrak{m}})^n \subseteq J\}$). Indeed, a filter basis of finitely generated Gabriel topology corresponding to some general Thomason set $X \subseteq \mathrm{mSpec}(R)$ via Theorem 1.4.20 can be obtained as the ideal-product closure of the set $\{I^{\mathfrak{m}} \mid \mathfrak{m} \in X\}$, and a product of projective (=invertible) ideals is clearly projective.

An easy observation shows that by localizing at $\mathfrak{m}$, the obtained set $\mathcal{G}_{\mathfrak{m}} = \{J_{\mathfrak{m}} \mid J \in \mathcal{G}\}$ is a finitely generated Gabriel topology over $R_{\mathfrak{m}}$. Since $I^{\mathfrak{m}} \in \mathcal{G}$, $\mathcal{G}_{\mathfrak{m}}$ contains a non-trivial ideal. As $R_{\mathfrak{m}}$ is local and 1-dimensional, $\mathcal{G}_{\mathfrak{m}}$ has to consist of all non-zero ideals of $R_{\mathfrak{m}}$. As $\mathcal{G}_{\mathfrak{n}} = \{R_{\mathfrak{n}}\}$ for all $\mathfrak{n} \in (\mathrm{mSpec}(R) \setminus \{\mathfrak{m}\})$, it follows that there is a filter basis of $\mathcal{G}$ consisting of locally projective ideals. In particular, there is $J \in \mathcal{G}$ with $J \neq R$ and J flat. As R is almost perfect, this implies $\mathrm{pd}\, J \leq 1$.

Finally, by [36, Proposition 3.2], any R-module of finite projective dimension has projective dimension at most 1. Therefore, $\mathrm{pd}\, R/J = 1$, and thus J is actually projective, and hence also finitely generated. The finitely generated Gabriel topology generated by J needs to coincide with $\mathcal{G}$, and thus $\{J^n \mid n \in \omega\}$ is a filter basis of $\mathcal{G}$.

(ii) Let $\mathcal{T}$ be a 1-tilting class in Mod-R. By Theorem 2.3.16, there is a finitely generated Gabriel topology $\mathcal{G}$ such that $\mathcal{T} = \mathcal{G}$-Div. Let $\mathcal{I}$ be a filter basis of $\mathcal{G}$ consisting of (finitely generated) projective ideals. Using (i) and the h-locality of R again, it is enough to show that $\mathcal{T}$ is closed under flat covers if $\mathcal{G}$ is generated by a projective ideal I such that $V(I) = \{\mathfrak{m}\}$ for some $\mathfrak{m} \in \mathrm{mSpec}(R)$ (every non-trivial 1-tilting class is an intersection of those).

Let $Q_I = \bigcup_{n\in\mathbb{N}} I^{-n}$ be the union of powers of the fractional ideal I^{-1}. Then the inclusion $\varphi : R \to Q_I$ is a flat ring epimorphism, and $Q_I \in \mathcal{T}$. Let $Y = \mathrm{Coker}(\varphi)$. Then Y is filtered by modules $I^{-(n+1)}/I^{-n}, n \geq 0$. By Lemma 1.6.14 and by Eklof Lemma ([23, Lemma 6.2]), we conclude that φ is a flat special $\mathcal{T}$-preenvelope of R. Let now F be a flat R-module, and consider the exact sequence:

$$0 \to F \to F \otimes_R Q_I \to X \to 0.$$

Any presentation of F as a direct limit of free modules yields that X is a direct limit of copies of Y, so there is a pure exact sequence:

$$\epsilon : 0 \to K \xrightarrow{*} Y^{(\varkappa)} \to X \to 0.$$

Denote by $Y_n = \mathrm{Hom}_R(R/I^n, Y)$ the I^n-socle of Y. Applying $\mathrm{Hom}_R(R/I^n, -)$ yields an exact sequence

$$\epsilon' : 0 \to K_n \to (Y_n)^{(\varkappa)} \to X_n \to 0,$$

where K_n and X_n are the I^n-socles of K and X, accordingly. This exact sequence is pure in R/I^n-Mod. Indeed, if G is a finitely presented R/I^n-module, the sequence $\mathrm{Hom}_{R/I^n}(G, \epsilon')$ is naturally isomorphic to $\mathrm{Hom}_{R/I^n}(G, \mathrm{Hom}_R(R/I^n, \epsilon))$. We have a natural isomorphism (of complexes)

$$\mathrm{Hom}_{R/I^n}(G, \mathrm{Hom}_R(R/I^n, \epsilon)) \simeq \mathrm{Hom}_R(G, \epsilon).$$

Since I^n is finitely generated, G is a finitely presented R-module, and because ϵ is an exact sequence in Mod-R, the resulting sequence is exact.

Now apply the tensor functor $R/I \otimes_{R_{I^n}} -$ onto the pure exact sequence ϵ', which yields a pure exact sequence

$$0 \to K_n/IK_n \xrightarrow{*} (Y_n/IY_n)^{(\varkappa)} \to X_n/IX_n \to 0.$$

Since $IY_n = Y_{n-1}$, it follows that $X_n/IX_n = X_n/X_{n-1}$. As this sequence is pure, and Y_n/Y_{n-1} is a projective R/I-module by Lemma 1.6.14, it follows that X_n/X_{n-1} is a flat R/I-module. But R/I is a perfect ring, and thus X_n/X_{n-1} is actually a projective R/I-module, and therefore belongs to ${}^{\perp_1}\mathcal{T}$ by Lemma 1.6.14. As X is filtered by the set $\{X_n/X_{n-1} \mid n \in \mathbb{N}\}$, where $X_0 = 0$, we apply again the Eklof Lemma in order to infer that $X \in {}^{\perp_1}\mathcal{T}$. Therefore, the monomorphism $F \to F \otimes_R Q_I$ is a special $\mathcal{T}$-preenvelope of F, which is flat. Using Lemma 1.6.2, we infer that $\mathcal{T}$ is closed under flat covers. □

Together, this yields the following characterization:

Theorem 1.6.16. *Let R be an integral domain. Then the following conditions are equivalent:*

(i) R is an almost perfect domain,

(ii) the class $\mathcal{D}$ of all divisible modules is closed under flat covers,

(iii) all 1-tilting classes in Mod-R *are closed under flat covers, and*

(iv) all tilting classes in Mod-R *are closed under flat covers.*

Proof. $(i) \to (iii)$: Lemma 1.6.15.

$(iii) \to (iv)$: The hypothesis implies in particular that the class $\mathcal{D}$ is closed under flat covers, which by Lemma 1.6.13 implies that R is almost perfect, and thus 1-dimensional. Therefore, any tilting class is 1-tilting by Lemma 1.6.10.

$(iv) \to (ii)$: Trivial.

$(ii) \to (i)$: Lemma 1.6.13. □

We remark that Bazzoni proved in [8] the following related characterization: An integral domain R is almost perfect if and only if the class $\mathcal{D}$ is enveloping.

Bibliography for Chapter 1

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One-tilting classes and modules over commutative rings

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Published: Journal of Algebra, 462:1-22, 2016

Abstract. We classify 1-tilting classes over an arbitrary commutative ring. As a consequence, we classify all resolving subcategories of finitely presented modules of projective dimension at most 1. Both these collections are in 1-1 correspondence with faithful Gabriel topologies of finite type, or equivalently, with Thomason subsets of the spectrum avoiding a set of primes associated in a specific way to the ring. We also provide a generalization of the classical Fuchs and Salce tilting modules, and classify the equivalence classes of all 1-tilting modules. Finally we characterize the cases when tilting modules arise from perfect localizations.

Chapter 2

One-tilting classes and modules over commutative rings

2.1 Introduction

The classification of tilting classes and modules was done gradually, starting with abelian groups ([GT00]), then small Dedekind domains, first assuming V=L ([TW02],[TW03]), and then in ZFC ([BET05]), for Prüfer domains ([Baz07]), and almost perfect domains ([AJ11]). Recently, in [AHPŠT14] the authors classified tilting classes of a commutative noetherian ring in terms of finite sequences of subsets of the Zariski spectrum of R. In particular, they proved that 1-tilting classes correspond bijectively to specialization closed subsets of $\mathrm{Spec}(R)$ that do not contain associated primes of R. We generalize this result to arbitrary commutative rings by showing that there is a one-to-one correspondence between 1-tilting classes and Thomason subsets of $\mathrm{Spec}(R)$ that avoid primes "associated" to R in certain sense. Thomason subsets of the spectrum coincide with specialization closed subsets in the noetherian case, and seem to be the correct generalization in various classification theorems. The prime example of this phenomenon is the classification of compactly generated localizing subcategories of the unbounded derived category of R done first by Neeman for noetherian rings and then in general by Thomason ([Tho97]).

As in the noetherian case in ([AHPŠT14]), we start working in the dual setting of cotilting classes. Even though there is an explicit duality between tilting modules and cotilting modules of cofinite type, the one way nature of the duality makes the tilting side harder to approach. For example, cotilting modules over commutative noetherian case are described in [ŠTH14], but tilting modules were described only for special classes of noetherian rings. The crucial step in our approach is to show that a 1-cotilting class is of cofinite type if and only if it is closed under injective envelopes (Corollary 2.3.13).

The author is partially supported by the Grant Agency of the Czech Republic under the grant no. 14-15479S and by the project SVV-2015-260227 of the Charles University in Prague.

Alternatively, 1-tilting classes over a commutative ring R correspond bijectively to faithful finitely generated Gabriel topologies over R. From this point of view, our classification generalizes directly results for Prüfer domains from [Baz07]. If R is not semihereditary, one has to replace the cyclic generators of the hereditary torsion class by their Auslander-Bridger transposes in order to describe the resolving subcategories of finitely presented modules of projective dimension at most 1. In the second part of the paper, we use this idea and construct an associated tilting module for each 1-tilting class over a commutative ring. This construction generalizes the Fuchs and Salce tilting modules introduced by Facchini, Fuchs-Salce, and Salce ([Fac88], [FS92], [Sal05]) from multiplicative sets over a domain and finitely generated Gabriel topology over a Prüfer domain to general faithful finitely generated Gabriel topology over a commutative ring.

In the rest of the second section we use the "minimality" of the constructed 1-tilting modules and provide an elementary proof of the commutative version of the recently solved Saorín's problem ([BHP+15]). Finally, in the last section we show that a 1-tilting module arises from a perfect localization if and only if the associated Gabriel topology is perfect and the induced perfect localization has projective dimension 1.

2.2 Preliminaries

2.2.1 Basic notation and cotorsion pairs

Given an (associative, unital) ring R, we denote by Mod-R the category of all right R-modules and by mod-R the full subcategory of Mod-R consisting of all finitely presented right R-modules.

For a class of right R-modules $\mathcal{S}$, we will use the following notation:

$$\mathcal{S}^{\perp} = \{M \in \text{Mod-R} \mid \operatorname{Ext}^1_R(S, M) = 0 \text{ for all } S \in \mathcal{S}\},$$

$$^{\perp}\mathcal{S} = \{M \in \text{Mod-R} \mid \operatorname{Ext}^1_R(M, S) = 0 \text{ for all } S \in \mathcal{S}\}.$$

Similarly, if $\mathcal{S}$ is a class of left R-modules we let:

$$\mathcal{S}^{\intercal} = \{M \in \text{Mod-R} \mid \operatorname{Tor}^1_R(M, S) = 0 \text{ for all } S \in \mathcal{S}\}.$$

Given a class $\mathcal{S}$, a chain of submodules of an R-module M

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq M_{\alpha+1} \subseteq \cdots \subseteq M_\lambda = M$$

indexed by ordinal $\lambda + 1$, with the property that $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ for each limit ordinal $\beta \leq \lambda$ and $M_{\alpha+1}/M_\alpha$ is isomorphic to some module from $\mathcal{S}$ for each $\alpha < \lambda$, is called an *$\mathcal{S}$-filtration* of M. We say that M is *$\mathcal{S}$-filtered* if it possesses an $\mathcal{S}$-filtration.

A couple of full subcategories $(\mathcal{A}, \mathcal{B})$ of Mod-R is called a *cotorsion pair* provided that $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$. Given a class $\mathcal{S}$ of modules, the cotorsion pair $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ is *generated* by $\mathcal{S}$. The following important result about cotorsion pairs generated by sets of modules will be used freely throughout the paper.

Lemma 2.2.1. *([GT12, Corollary 6.14]) Let $\mathcal{S}$ be a set of modules and $(\mathcal{A}, \mathcal{B})$ the cotorsion pair generated by $\mathcal{S}$. Then $\mathcal{A}$ consists precisely of all direct summands of all $\mathcal{S}$-filtered modules.*

2.2.2 Gabriel topologies, torsion pairs, and divisibility

Given a right ideal I and an element t of a ring R, we denote $(I : t) = \{r \in R \mid tr \in I\}$.

Definition 2.2.2. A filter $\mathcal{G}$ of right ideals of R is called a *Gabriel topology* provided that:

- if $I \in \mathcal{G}$ and $t \in R$, then $(I : t) \in \mathcal{G}$,
- if J is a right ideal and $I \in \mathcal{G}$ is such that $(J : t) \in \mathcal{G}$ for any $t \in I$, then $J \in \mathcal{G}$.

A Gabriel topology is *finitely generated* if it has a basis of finitely generated right ideals. A right ideal I of R is *faithful* if $\operatorname{Ann} I = 0$ (if R is commutative, this is equivalent to $\operatorname{Hom}_R(R/I, R) = 0$). We say that a Gabriel topology is *faithful* if it has a basis consisting of faithful ideals (and thus all ideals in $\mathcal{G}$ are faithful).

There is an easier description of finitely generated Gabriel topologies over commutative rings.

Lemma 2.2.3. *Suppose R is commutative. A filter $\mathcal{G}$ of ideals of R with a basis of finitely generated ideals is a (finitely generated) Gabriel topology iff it is closed under ideal products.*

Proof. If $\mathcal{G}$ is a Gabriel topology, then it is closed under products, since for any $i \in I$ we have $(IJ : i) \supseteq J$, and thus $IJ \in \mathcal{G}$, provided that $I, J \in \mathcal{G}$. Suppose that $\mathcal{G}$ is closed under products. Let $I \in \mathcal{G}$ and $t \in R$. Since R is commutative, $I \subseteq (I : t)$ and thus the latter ideal is in $\mathcal{G}$. Let now J be any ideal and $I \in \mathcal{G}$ such that $(J : t) \in \mathcal{G}$ for each $t \in I$. We want to show that $J \in \mathcal{G}$. By the hypothesis, we can assume that I is finitely generated, say with a generating set $\{i_1, i_2, \ldots, i_n\}$. We have $I_k = (J : i_k) \in \mathcal{G}$ for all $k = 1, 2, \ldots, n$. It follows that $II_1I_2 \cdots I_k \subseteq J$, and thus $J \in \mathcal{G}$, as claimed. $\square$

We say that a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of Mod-R is a *torsion pair* provided that $\mathcal{T} = \{M \in \text{Mod-R} \mid \operatorname{Hom}_R(M, F) = 0 \text{ for all } F \in \mathcal{F}\}$ and $\mathcal{F} = \{M \in \text{Mod-R} \mid \operatorname{Hom}_R(T, M) = 0 \text{ for all } T \in \mathcal{T}\}$. The class $\mathcal{T}$ (resp. $\mathcal{F}$) is called a *torsion* (resp. *torsion-free*) class. A class of modules fits into a torsion pair as a torsion (resp. torsion-free) class iff it is closed under extensions, direct sums, and homomorphic images (resp. under extensions, direct products, and submodules). Such torsion pair is said to be:

- *hereditary* provided that $\mathcal{T}$ is closed under submodules (or, equivalently, $\mathcal{F}$ is closed under injective envelopes),
- *faithful* provided that $R \in \mathcal{F}$,
- *of finite type* provided that $\mathcal{F}$ is closed under direct limits.

With any torsion pair $(\mathcal{T}, \mathcal{F})$ in Mod-R there is an associated idempotent subfunctor t on Mod-R called the *torsion radical*, defined by the property that for any module M, we have $t(M) \in \mathcal{T}$ and $M/t(M) \in \mathcal{F}$. It is easy to see that $(\mathcal{T}, \mathcal{F})$ is of finite type iff t commutes with direct limits. The following observation will be useful in characterizing cotilting torsion-free classes of cofinite type.

Lemma 2.2.4. *A hereditary torsion pair $(\mathcal{T}, \mathcal{F})$ is of finite type iff there is a set $\mathcal{S}$ of finitely presented modules such that $\mathcal{F} = \{M \in \text{Mod-R} \mid \text{Hom}_R(S, M) = 0$ for all $S \in \mathcal{S}\}$.*

Proof. The if-part follows from the fact that $\text{Hom}_R(S, -)$ commutes with direct limit for any finitely presented module S.

Let us prove the other implication. Since the pair is hereditary, there is a set $\mathcal{E}$ of finitely generated modules such that $\mathcal{F} = \text{Ker}\,\text{Hom}_R(\mathcal{E}, -)$ (e.g. the set of all cyclic modules from $\mathcal{T}$). We are left to show that we can find such set consisting of finitely presented modules. Fix $M \in \mathcal{E}$. Let $0 \to K \to R^n \to M \to 0$ be a free presentation of M. We can write K as a directed union $K = \bigcup_{i \in I} K_i$ of its finitely generated submodules. Then M is a direct limit of finitely presented modules $R^n/K_i, i \in I$ in a way that all the maps of this direct system are projections. Since the torsion radical t of the torsion pair $(\mathcal{T}, \mathcal{F})$ commutes with direct limits, we have that $M = t(M) = t(\varinjlim_I R^n/K_i) = \varinjlim_I t(R^n/K_i)$. Let $J_i, i \in I$ be submodules of R^n containing K_i such that the torsion-free part of R^n/K_i is isomorphic to R^n/J_i for each $i \in I$. Since $\varinjlim_I R^n/J_i$ is isomorphic to the torsion-free part of M, it is zero, and thus $\varinjlim_I J_i = \bigcup_I J_i = R^n$. As R^n is finitely generated, there is $k \in I$ with $J_k = R^n$. It follows that R/K_k is in $\mathcal{T}$, and thus M is a direct limit of finitely presented modules $R^n/K_i, i \geq k$, which all belong to $\mathcal{T}$, because the directed system consisted of projections. Put $\mathcal{S}_M = \{R^n/K_i, i \geq k\}$. Because $\mathcal{S}_M \subseteq \mathcal{T}$ generates M, we infer that $\mathcal{T} = \text{Ker}\,\text{Hom}_R(\mathcal{E} \setminus \{M\} \cup \mathcal{S}_M, -)$.

Constructing the set of finitely presented modules $\mathcal{S}_M$ for each $M \in \mathcal{E}$ and putting $\mathcal{S} = \bigcup_{M \in \mathcal{E}} \mathcal{S}_M$, we infer that $\mathcal{T} = \text{Ker}\,\text{Hom}_R(\mathcal{S}, -)$ as desired. $\square$

Given a Gabriel topology $\mathcal{G}$, there is a hereditary torsion pair induced by $\mathcal{G}$ with the torsion class $\{M \mid \text{Ann}(m) \in \mathcal{G}$ for all $m \in M\}$. Also, there is another torsion pair (usually not hereditary) with the torsion class $\{M \in \text{Mod-R} \mid M = MI$ for all $I \in \mathcal{G}\}$.

Theorem 2.2.5. *([Ste75, §VI.Theorem 5.1]) Let R be a ring R. There is a 1-1 correspondence between hereditary torsion pairs $(\mathcal{T}, \mathcal{F})$ in* Mod-R *and Gabriel topologies $\mathcal{G}$ given by*

$$\mathcal{T} \mapsto \{I \textit{ right ideal} \mid R/I \in \mathcal{T}\},$$

$$\mathcal{G} \mapsto \{M \in \text{Mod-R} \mid \text{Ann}(m) \in \mathcal{G} \textit{ for all } m \in M\}.$$

Notation 2.2.6. Given a set of (right) ideals $\mathcal{I}$, we denote by $\mathcal{I}$-Div the class of all $\mathcal{I}$-divisible right modules, that is, the class

$$\mathcal{D}_{\mathcal{I}} = \{M \in \text{Mod-R} \mid M = MI \text{ for all } I \in \mathcal{I}\}.$$

2.2.3 Prime spectrum

Given a commutative ring R, we denote by $\text{Spec}\,R$ the prime spectrum of R. Set $\text{Spec}\,R$ is endowed with the Zariski topology, i.e. the topology with closed sets being the sets of form

$$V(I) = \{\mathfrak{p} \in \text{Spec}\,R \mid I \subseteq \mathfrak{p}\},$$

for some ideal I of R. Following the work of Thomason ([Tho97]), we say that a subset of $\operatorname{Spec} R$ is *Thomason* if it is a union of sets $V(I)$ with I being finitely generated (equivalently, if it is a union of Zariski closed sets with quasi-compact complements). It is well-known that Thomason subsets of $\operatorname{Spec} R$ correspond bijectively to finitely generated Gabriel topologies (by assigning to a finitely generated Gabriel topology the set of all primes contained in it). We will prove a "faithful" version of this fact in Theorem 2.3.16 for convenience.

For any $M \in$ Mod-R, symbol $\operatorname{Ass} M$ stands for the set of all associated primes of M, that is, all primes $\mathfrak{p} \in \operatorname{Spec} R$ such that $R/\mathfrak{p}$ embeds into M. Similarly, for a subclass $\mathcal{C}$ of Mod-R we fix a notation $\operatorname{Ass} \mathcal{C} = \bigcup_{M \in \mathcal{C}} \operatorname{Ass} M$.

We denote the localization of R at prime $\mathfrak{p}$ by $R_\mathfrak{p}$. For any $M \in$ Mod-R we put $M_\mathfrak{p} = M \otimes_R R_\mathfrak{p}$. The set of all primes $\mathfrak{p}$ such that $M_\mathfrak{p}$ is non-zero is called the *support* of M and denoted by $\operatorname{Supp} M$. It is well-known that $\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{Ann} M \subseteq \mathfrak{p}\}$ provided that M is finitely generated.

2.2.4 Tilting and cotilting

We use the following definition of an (infinitely generated) right 1-tilting module over an arbitrary ring R ([CT95],[HC01]).

Definition 2.2.7. An R-module T is said to be *1-tilting* if

- $\operatorname{pd} T \leq 1$,
- $\operatorname{Ext}^1_R(T, T^{(X)}) = 0$ for any set X,
- there is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$, where T_i is a direct summand of a direct sum of copies of T for each $i = 0, 1$.

The class $\mathcal{T} = T^\perp$ is called a *1-tilting class*, and the induced cotorsion pair $(\mathcal{A}, \mathcal{T})$ a *1-tilting cotorsion pair*. Two 1-tilting modules T and T' are said to be equivalent if $T^\perp = T'^\perp$.

Remark 2.2.8. Given a module M, denote by $\operatorname{Gen}(M)$ the class of all homomorphic images of direct sums of copies of M. We remark that module T is 1-tilting if and only if $\operatorname{Gen}(T) = T^\perp$ (see [GT12, Lemma 14.2]), providing an easier alternative definition. In particular, note that a 1-tilting class is a torsion class. Also, a class $\mathcal{T}$ is 1-tilting if and only if it is a special preenveloping torsion class ([GT12, Theorem 14.4]).

Classical tilting theory of artin algebras focuses on finitely presented tilting modules, which is in stark contrast with the commutative setting, where only infinitely generated ones are interesting:

Lemma 2.2.9. *([PT11, Lemma 1.2]) Let R be a commutative ring. Then any 1-tilting module equivalent to a finitely generated one is projective.*

On the other hand, infinitely generated tilting modules share a lot of properties of their classical finitely presented counterparts. In particular, they still serve as a generalization of progenerators from the classical Morita equivalence, as they induce equivalences of subcategories of module categories, or derived equivalences between triangulated subcategories of derived module categories ([Baz10], [BMT11]).

Even though tilting modules over commutative rings are almost always infinitely generated, the tilting classes can be fully described in terms of the small module category mod-R. Using the Small Object Argument, one can show that given a set $\mathcal{S}$ of finitely presented modules of projective dimension at most 1, the class $\mathcal{S}^\perp$ is 1-tilting. Crucial results by Bazzoni-Herbera (and Bazzoni-Šťovíček for general n-tilting classes) show that the converse is also true. We recall that a subcategory $\mathcal{S}$ of mod-R is *resolving*, if all finitely generated projectives are contained in $\mathcal{S}$, and $\mathcal{S}$ is closed under extensions, direct summands, and syzygies.

Theorem 2.2.10. *([BH08], [BŠ07]) Let R be a ring. There is a 1-1 correspondence between 1-tilting classes $\mathcal{T}$ in* Mod-R*, and resolving subcategories $\mathcal{S}$ of* mod-R *contained in $\{M \in \text{mod-R} \mid \operatorname{pd} M \leq 1\}$. The corespondence is given by the assignments $\mathcal{S} \mapsto \mathcal{S}^\perp$ and $\mathcal{T} \mapsto ({}^\perp\mathcal{T}) \cap \text{mod-R}$.*

Definition 2.2.11. An R-module C is said to be *1-cotilting* if

- $\operatorname{id} C \leq 1$,
- $\operatorname{Ext}^1_R(C^X, C) = 0$ for any set X,
- there is an exact sequence $0 \to C_1 \to C_0 \to W \to 0$, where W is an injective cogenerator in Mod-R, and C_i is a direct summand of a direct product of copies of C for each $i = 0, 1$.

The class $\mathcal{C} = {}^\perp C$ is called a *1-cotilting class induced by C* and 1-cotilting modules C, C' are said to be *equivalent* if their induced cotilting classes coincide.

Unlike tilting classes, 1-cotilting classes do not in general come from a set of finitely presented modules unless the ring is noetherian (see a counter-example due to Bazzoni in [Baz07, Proposition 4.5]).

Definition 2.2.12. A 1-cotilting class $\mathcal{C}$ is of *cofinite type* provided there is a set of finitely presented modules $\mathcal{S}$ of projective dimension at most 1 such that $\mathcal{C} = \mathcal{S}^\intercal$.

Given a 1-tilting right module T, its character module $T^+ = \operatorname{Hom}_\mathbb{Z}(T, \mathbb{Q}/\mathbb{Z})$ is a 1-cotilting left R-module. Furthermore, if $\mathcal{S}$ is a subset of mod-R such that $T^\perp = \mathcal{S}^\perp$, then the cotilting class ${}^\perp(T^+)$ equals $\mathcal{S}^\intercal$, and thus is of cofinite type. In fact, the assigment $T \mapsto T^+$ induces a 1-1 correspondence between equivalence classes of 1-tilting right R-modules and equivalence classes of 1-cotilting left modules of cofinite type (meaning that the induced 1-cotilting class is of cofinite type). For details, see [GT12, §15].

2.3 Tilting and cofinite-type cotilting classes

2.3.1 General formulas

We start with recalling the notion of transpose from [AB69]. Although this idea was originally used mostly in the artin algebra setting, it has proven useful in classifying tilting classes over commutative noetherian rings in [AHPŠT14]. In fact, it will serve the same purpose over a general commutative ring.

Definition 2.3.1. Let R be a ring and M a finitely presented left R-module. Let $P_1 \xrightarrow{f} P_0 \to M \to 0$ be a presentation of M with both P_0 and P_1 finitely generated projectives. We use the notation $(-)^*$ for the regular module duality functor $\mathrm{Hom}_R(-, R)$. The *(Auslander-Bridger) transpose* of M is obtained as the cokernel of the map of right R-modules $f^* : P_0^* \to P_1^*$. We denote the transpose by $\mathrm{Tr}\, M$.

Remark 2.3.2. It is important to note that the right R-module $\mathrm{Tr}\, M$ is uniquely determined only up to *stable equivalence*, that is, up to splitting off or adding a projective direct summand ([AB69, §2.1]). We will use the notation $M \overset{\mathrm{st}}{\simeq} N$ for M being stably equivalent to N.

However, there is a nice choice of a concrete representative module for $\mathrm{Tr}\, M$ if $\mathrm{pd}_R\, M \leq 1$. Indeed, then $\mathrm{Ext}^1_R(M, R) \overset{\mathrm{st}}{\simeq} \mathrm{Tr}\, M$ (see Lemma 2.3.4 below).

We gather several well-known homological formulae for the transpose we will need later on, and reprove them in our setting for convenience.

Lemma 2.3.3. ([AHPŠT14, Lemma 2.9]) *Let R be a ring, M a non-zero left finitely presented R-module, such that* $\mathrm{Hom}_R(M, R) = 0$. *Then:*

(i) $\mathrm{pd}_R\, \mathrm{Tr}\, M = 1$ *and* $\mathrm{Tr}\, M$ *is finitely presented,*

(ii) $\mathrm{Hom}_R(M, -)$ *and* $\mathrm{Tor}^R_1(\mathrm{Tr}\, M, -)$ *are isomorphic functors,*

(iii) $\mathrm{Ext}^1_R(\mathrm{Tr}\, M, -)$ *and* $(- \otimes_R M)$ *are isomorphic functors.*

Proof. (i) Since M is finitely presented, there is a part of a projective resolution of M

$$P_1 \to P_0 \to M \to 0, \tag{2.1}$$

consisting of finitely generated projectives. Applying $(-)^*$ we get a complex

$$0 \leftarrow \mathrm{Tr}\, M \leftarrow P_1^* \leftarrow P_0^* \leftarrow 0,$$

which is exact by our hypothesis on M, showing that $\mathrm{pd}\, \mathrm{Tr}\, M \leq 1$, and that $\mathrm{Tr}\, M$ is finitely presented. If $\mathrm{Tr}\, M$ was projective, then M is projective, which together with $M^* = 0$ implies that $M = 0$, a contradiction. Hence, $\mathrm{pd}\, \mathrm{Tr}\, M = 1$.

(ii) Let N be a left R-module. By definition, $\mathrm{Ext}^1_R(M, N)$ is the first homology of the complex obtained by applying $\mathrm{Hom}_R(-, N)$ on (2.1). We now use the natural isomorphism $\mathrm{Hom}_R(P, N) \simeq P^* \otimes_R N$ where P is a finitely generated projective (see [AF92, Proposition 20.6]) to infer the desired isomorphism.

(iii) Analogous.

□

A sort of converse for Lemma 2.3.3 also holds, if we choose a representative of the transpose well enough. Unlike Lemma 2.3.3, this result does not generalize to higher projective dimension in a straightforward way.

Lemma 2.3.4. *Let $S \in$* mod-R *be such that* $\mathrm{pd}_R\, S \leq 1$. *Put* $S^\dagger = \mathrm{Ext}^1_R(S, R)$. *Then $S^\dagger$ is a finitely presented left R-module with $(S^\dagger)^* = 0$, and* $S^\dagger \overset{\mathrm{st}}{\simeq} \mathrm{Tr}\, S$.

Proof. Let
$$0 \to P_1 \to P_0 \to S \to 0, \tag{2.2}$$
be a projective resolution of S consisting of finitely generated projectives. Applying $(-)^*$ we get an exact sequence
$$0 \leftarrow S^\dagger \leftarrow P_1^* \leftarrow P_0^* \leftarrow S^* \leftarrow 0,$$
showing that $S^\dagger$ is finitely presented, and by the definition $S^\dagger \overset{\text{st}}{\simeq} \operatorname{Tr} S$. Applying $(-)^*$ again we get back to the exact sequence (2.2), proving that $(S^\dagger)^* = 0$. ☐

Notation 2.3.5. We fix the notation $S^\dagger = \operatorname{Ext}_R^1(S, R)$ for any $S \in$ Mod-R.

Combining Lemma 2.3.3 and Lemma 2.3.4 we obtain:

Corollary 2.3.6. *Let $\mathcal{S}$ be a set of finitely presented right R-modules of projective dimension 1. Let $\mathcal{T} = \mathcal{S}^\perp$ and $\mathcal{C} = \mathcal{S}^\intercal$ be the induced 1-tilting and 1-cotilting class of cofinite type. Then:*

- $\mathcal{T} = \bigcap_{S \in \mathcal{S}} \operatorname{Ker}(- \otimes_R S^\dagger)$,
- $\mathcal{C} = \bigcap_{S \in \mathcal{S}} \operatorname{Ker} \operatorname{Hom}_R(S^\dagger, -)$.

Proof. By Lemma 2.3.4, the module $S^\dagger$ satisfies $S^* = 0$ and $\operatorname{pd} S \leq 1$ for each $S \in \mathcal{S}$. Therefore, we can use Lemma 2.3.3(3) to infer that $\bigcap_{S \in \mathcal{S}} \operatorname{Ker} \operatorname{Ext}_R^1(\operatorname{Tr} S^\dagger, -) = \bigcap_{S \in \mathcal{S}} \operatorname{Ker}(- \otimes_R S^\dagger)$. As $\operatorname{Tr} S^\dagger \overset{\text{st}}{\simeq} S$ by Lemma 2.3.4, this class is equal to $\mathcal{T}$ as desired. The formula for the cotilting class is derived analogously, using Lemma 2.3.3(2). ☐

2.3.2 Commutative rings

From now on, let R be a **commutative ring**. We begin by proving that any cofinite type 1-cotilting torsion pair is hereditary. Note that this in general fails for non-commutative rings[1]. If R is not noetherian, the classical theory of associated primes does not function well. Indeed, there can be non-zero modules with no associated primes[2]. The following notion will prove useful in the cotilting setting.

Definition 2.3.7. Given a class $\mathcal{C}$ of modules, let $\operatorname{SubLim}(\mathcal{C})$ denote the smallest (isomorphism-closed) subclass of Mod-R containing $\mathcal{C}$ closed under direct limits and submodules.

We say that a prime $\mathfrak{p}$ is *vaguely associated* to a module M if $R/\mathfrak{p}$ is contained in $\operatorname{SubLim}(\{M\})$. Denote the set of all vaguely associated primes of M by $\operatorname{VAss} M$.

[1] Counter-example (communicated to the author by Jan Šťovíček) can be obtained as follows. Let A be a left hereditary right artinian ring such that the only projective injective right module is zero (e.g. the Kronecker algebra over a field). Then the class of all projective (equally, flat) right A-modules is equal to $(\text{R-mod})^\intercal$, and thus is a 1-cotilting class of cofinite type not closed under injective envelopes.

[2] Easy example can be obtained as follows. Let R be a valuation domain of Krull dimension 1 with idempotent radical (e.g. the ring of all Puiseux series over a field). Then it is an easy exercise to show that any cyclic module of form R/rR for $r \in R$ non-zero has zero socle, and thus it has no associated primes.

First, we note that this is indeed a generalization of the concept of associated primes over noetherian rings.

Lemma 2.3.8. *Let R be noetherian, then* $\operatorname{VAss} M = \operatorname{Ass} M$ *for any R-module M.*

Proof. Let $\mathfrak{p} \in \operatorname{VAss} M$ and let us show that $\mathfrak{p} \in \operatorname{Ass} M$. By definition, we have that $R/\mathfrak{p} \in \operatorname{SubLim}(\{M\})$. Since taking submodules does not introduce any new associated primes, it is enough to show that whenever L is a direct limit of a direct system $L_i, i \in I$ such that $\mathfrak{p} \notin \operatorname{Ass} L_i$ for each $i \in I$, then $\mathfrak{p} \notin \operatorname{Ass} L$. Using [GT12, Corollary 2.9], there is a pure epimorphism $\pi : \bigoplus_{i\in I} L_i \to L$. Since R is noetherian, the module $R/\mathfrak{p}$ is finitely presented, and thus we can factorize the inclusion $R/\mathfrak{p} \hookrightarrow L$ through π. Therefore, $R/\mathfrak{p} \in \operatorname{Ass} \bigoplus_{i\in I} L_i$. This already shows that there is $i \in I$, such that $\mathfrak{p} \in L_i$, a contradiction. We showed that $\operatorname{VAss} M \subseteq \operatorname{Ass} M$; the inverse inclusion is trivially true. □

Lemma 2.3.9. *If M is non-zero, then* $\operatorname{VAss} M$ *is non-empty. Also,* $\operatorname{VAss} M \subseteq \operatorname{Supp} M$*.*

Proof. Define

$$X = \{I \text{ ideal} \mid R \neq I \text{ and } R/I \in \operatorname{SubLim}(\{M\})\}.$$

Since M is non-zero, X is non-empty. We claim that X is inductive (with respect to inclusion). Indeed, let $\mathfrak{c}$ be an increasing chain of ideals from X and put $I = \bigcup \mathfrak{c}$. As $R \notin \mathfrak{c}$, also $I \neq R$. The cyclic module R/I can be obtained as a direct limit of the modules $R/J, J \in \mathfrak{c}$, from $\operatorname{SubLim}(\{M\})$. Then R/I is an element of $\operatorname{SubLim}(\{M\})$, and thus $I \in X$, proving that X is inductive.

We can thus use Zorn's Lemma to find a maximal element $\mathfrak{p}$ of X, which is easily seen to be prime, hence $\mathfrak{p} \in \operatorname{VAss} M$.

Finally, any $\mathfrak{p} \in \operatorname{VAss}(M)$ has to be in the support of M, because $R/\mathfrak{p} \in \operatorname{SubLim}(\{M\})$, and the localization functor $- \otimes_R R_\mathfrak{p}$ commutes with submodules and direct limits. □

Lemma 2.3.10. *For a finitely generated module M and prime $\mathfrak{p}$,* $\operatorname{Hom}_R(M, R/\mathfrak{p}) \neq 0$ *iff* $\mathfrak{p} \in \operatorname{Supp} M$*.*

Proof. Suppose first that there is a non-zero map $f : M \to R/\mathfrak{p}$. If $\mathfrak{p} \notin \operatorname{Supp} M$, then $R/\mathfrak{p}$ contains a non-zero $R/\mathfrak{p}$-torsion submodule, a contradiction.

Let $\mathfrak{p} \in \operatorname{Supp} M$. Consider the quotient $M/\mathfrak{p}M$. Localizing at $\mathfrak{p}$ we obtain $M_\mathfrak{p}/(\mathfrak{p}_\mathfrak{p} M_\mathfrak{p})$. As $M_\mathfrak{p}$ is a non-zero finitely generated $R_\mathfrak{p}$-module, the latter quotient is non-zero by Nakayama. It follows that the torsion-free quotient of $M/\mathfrak{p}M$ (considered now as a module over the integral domain $R/\mathfrak{p}$) is non-zero. This module is well-known to embed into a finite product of $R/\mathfrak{p}$ (see [GT12, Lemma 16.1]). Hence, $\operatorname{Hom}_R(M, R/\mathfrak{p})$ is non-zero as claimed. □

Proposition 2.3.11. *Any 1-cotilting class of cofinite type is closed under taking injective envelopes. That is, any 1-cotilting torsion pair is hereditary.*

Proof. Let $\mathcal{C}$ be a 1-cotilting class and let C be a 1-cotilting module cogenerating $\mathcal{C}$. Using [Man01, Lemma 1.3], it is enough to show that $E(C) \in \mathcal{C}$. Since $\mathcal{C}$ is of cofinite type, there is a set $\mathcal{S}$ of finitely presented modules of projective dimension

1 such that $\mathcal{C} = \mathcal{S}^{\intercal}$. By Corollary 2.3.6, putting $\mathcal{E} = \{S^{\dagger} \mid S \in \mathcal{S}\}$ we get $\mathcal{C} = \bigcap_{M \in \mathcal{E}} \operatorname{Ker} \operatorname{Hom}_R(M, -)$. Suppose that there is a non-zero map $M \to E(C)$ for some $M \in \mathcal{E}$. Its image has to intersect C non-trivially. It follows that there is an ideal $J \neq R$ containing $\operatorname{Ann} M$ such that $R/J \in \mathcal{C}$. By Lemma 2.3.9, there is a prime $\mathfrak{p} \in \operatorname{VAss}(R/J)$. Since $\mathcal{C}$ is closed under submodules and direct limits, we have that $R/\mathfrak{p} \in \mathcal{C}$. On the other hand, since M is finitely generated and $J \subseteq \mathfrak{p}$, we have that $\mathfrak{p} \in \operatorname{Supp} M$, and thus $\operatorname{Hom}_R(M, R/\mathfrak{p}) \neq 0$ by Lemma 2.3.3. This is a contradiction. □

Corollary 2.3.12. *Let R be a commutative ring. Then 1-cotilting classes of cofinite type in* Mod-R *coincide with torsion-free classes of faithful hereditary torsion pairs of finite type.*

Proof. Any 1-cotilting class of cofinite type is a torsion-free class of a faithful hereditary torsion-pair of finite type by Corollary 2.3.6 and Proposition 2.3.11.

A torsion-free class of a faithful hereditary torsion pair of finite type is of form $\operatorname{Ker} \operatorname{Hom}_R(\mathcal{S}, -)$ for a set of finitely presented modules $\mathcal{S}$ by Lemma 2.2.4. Since the pair is faithful, we have $S^* = 0$ for each $S \in \mathcal{S}$, and thus the torsion-free class equals $\{\operatorname{Tr} S \mid S \in \mathcal{S}\}^{\intercal}$ by Lemma 2.3.3, proving that it is a 1-cotilting class of cofinite type. □

Given a module M, we let $\operatorname{Prod}(M)$ denote the class of all modules isomorphic to a direct summand of product of copies of M.

Corollary 2.3.13. *Let R be a commutative ring, and $\mathcal{C}$ a 1-cotilting class. Then the following conditions are equivalent:*

1. *$\mathcal{C}$ is of cofinite type,*
2. *$\mathcal{C}$ is closed under injective envelopes,*
3. *for any 1-cotilting module C with $\mathcal{C} = {}^{\perp}C$, we have $E(C) \in \operatorname{Prod}(C)$.*

Proof. (1) $\Rightarrow$ (2): Proposition 2.3.11.

(2) $\Rightarrow$ (3): Let $\mathcal{W} = \mathcal{C}^{\perp}$. Since $E(C)$ is injective, we have $E(C) \in \mathcal{C} \cap \mathcal{W}$, and $\mathcal{C} \cap \mathcal{W} = \operatorname{Prod}(C)$ by [GT12, Lemma 15.4].

(3) $\Rightarrow$ (1): As $E(C) \in \mathcal{C}$, the class $\mathcal{C}$ is closed under injective envelopes by [Man01, Lemma 1.3]. Then the induced torsion pair $(\mathcal{E}, \mathcal{C})$ is faithful, hereditary, and of finite type, and thus $\mathcal{C}$ is of cofinite type by Corollary 2.3.12. □

Before classifying all 1-tilting classes, we distinguish the following two steps.

Lemma 2.3.14. *Let $\mathcal{T}$ be a 1-tilting class and J an ideal such that $M = JM$ for each $M \in \mathcal{T}$. Then there is a finitely generated ideal $I \subseteq J$ such that $M = IM$ for each $M \in \mathcal{T}$.*

Proof. Let T be a 1-tilting module such that $\mathcal{T} = T^{\perp} = \operatorname{Gen}(T)$. Since $\mathcal{T}$ is closed under direct products, we have that $T^T = JT^T$. Let $\mathbf{t} = (t)_{t \in T} \in T^T$ be the sequence of all elements of T. Since $\mathbf{t} \in T^T = JT^T$, there is a finitely generated ideal $I \subseteq J$ such that $\mathbf{t} \in IT^T$. Looking at the canonical projections, we infer that $t \in IT$ for each $t \in T$, showing that $T = IT$. But since T generates $\mathcal{T}$, this means that $\mathcal{T} \subseteq \{M \in \text{Mod-R} \mid M = IM\}$ as claimed. □

Lemma 2.3.15. *Let S be a finitely presented module of projective dimension 1. Then there is a finitely generated ideal I such that $S^\perp = \{I\}$-Div $= \{M \in$ Mod-R $\mid M = IM\}$.*

Proof. We first show that there is an ideal J such that $S^\perp = \{J\}$-Div. By Corollary 2.3.6, $\mathcal{T} = \operatorname{Ker} - \otimes_R S^\dagger$. Also let $\mathcal{C} = \operatorname{Ker}\operatorname{Hom}_R(S^\dagger, -)$ be the induced 1-cotilting class and let $C = T^+$ be the 1-cotilting module dual to T. Let us fix a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = S^\dagger$ of $S^\dagger$ by cyclic modules, that is, such that there is an ideal J_i with $M_{i+1}/M_i \simeq R/J_i$ for each $i = 0, \dots, n-1$. We have shown in Proposition 2.3.11 that $E(C) \in \mathcal{C}$, and thus $\operatorname{Hom}_R(S^\dagger, E(C)) = 0$. Since the functor $\operatorname{Hom}_R(-, E(C))$ is exact, it follows that $\operatorname{Hom}_R(R/J_i, E(C)) = 0$ for each $i = 1, \dots, n$, and thus also $\operatorname{Hom}_R(R/J_i, C) = 0$ for each $i = 1, \dots, n$. Using the standard isomorphism $(R/J_i \otimes_R T)^+ \simeq \operatorname{Hom}_R(R/J_i, T^+)$, we get that $(R/J_i \otimes_R T)^+ = 0$, and thus $R/J_i \otimes_R T = 0$. In other words, $T = J_iT$ for each $i = 1, \dots, n$, and thus $T = JT$, where we put $J = J_1J_2\cdots J_n$. We have proved that $\mathcal{T} \subseteq \{J\}$-Div. The other inclusion follows easily, as $\mathcal{S}^\dagger$ is filtered by $\{R/J_i \mid i = 1, \dots, n\}$.

Since $S^\perp$ is a 1-tilting class, by Lemma 2.3.14 there is a finitely generated ideal $I \subseteq J$ such that $S^\perp \subseteq \{I\}$-Div. The latter inclusion must be an equality, because $\{I\}$-Div $\subseteq \{J\}$-Div $= S^\perp$. $\square$

2.3.3 Main theorem

Theorem 2.3.16. *Let R be a commutative ring. There are bijections between the following collections:*

1. *1-tilting classes $\mathcal{T}$,*
2. *1-cotilting classes of cofinite type $\mathcal{C}$,*
3. *faithful finitely generated Gabriel topologies $\mathcal{G}$,*
4. *Thomason subsets X of* $\operatorname{Spec}(R) \setminus \operatorname{VAss}(R)$,
5. *faithful hereditary torsion pairs $(\mathcal{E}, \mathcal{F})$ of finite type in* Mod-R,
6. *resolving subcategories of* mod-R *consisting of modules of projective dimension at most 1.*

The bijections are given as follows:

Bijection	*Formula*
(1) → (2)	$\mathcal{T} \mapsto ({}^\perp\mathcal{T} \cap \text{mod-R})^\intercal$
(1) → (3)	$\Psi : \mathcal{T} \mapsto \{I \textit{ ideal} \mid M = IM \textit{ for all } M \in \mathcal{T}\}$
(3) → (1)	$\Phi : \mathcal{G} \mapsto \mathcal{G}\text{-Div} = (\bigoplus_{I\in\mathcal{G},\ I\ f.g.} \operatorname{Tr}(R/I))^\perp$
(3) → (4)	$\Xi : \mathcal{G} \mapsto \mathcal{G} \cap \operatorname{Spec}(R)$
(4) → (3)	$\Theta : X \mapsto \{J \textit{ ideal} \mid \exists I \subseteq J \textit{ finitely generated such that } V(I) \subseteq X\}$
(5) → (2)	$(\mathcal{E}, \mathcal{F}) \mapsto \mathcal{F}$
(2) → (4)	$\mathcal{C} \mapsto (\operatorname{Spec}(R) \setminus \operatorname{Ass}\mathcal{C})$
(3) → (2)	$\mathcal{G} \mapsto \{M \in \text{Mod-R} \mid \operatorname{Ann}(m) \notin \mathcal{G} \textit{ for all non-zero } m \in M\}$
(3) → (6)	$\mathcal{G} \mapsto \mathcal{S} = \{M \in \text{mod-R} \mid M \textit{ is isomorphic to a direct summand of a finitely } \{R\} \cup \{\operatorname{Tr}(R/I) \mid I \in \mathcal{G} \textit{ f.g.}\}\textit{-filtered module}\}$

Proof. (1) $\leftrightarrow$ (2): Follows by Theorem 2.2.10 and using the character duality (see the last paragraph of Section 2).

(1) $\leftrightarrow$ (3): First let us prove that the prescribed maps $\Psi : \mathcal{T} \mapsto \mathcal{G}$ and $\Phi : \mathcal{G} \mapsto \mathcal{T}$ are well-defined. Let $\mathcal{T}$ be a 1-tilting class. By Lemma 2.3.14, whenever $J \in \Psi(\mathcal{T})$, there is a finitely generated ideal $I \subseteq J$ with $I \in \Psi(\mathcal{T})$. Also, any ideal in $\Psi(\mathcal{T})$ is faithful. Indeed, otherwise the special $\mathcal{T}$-preenvelope of R would have a non-zero annihilator, which is not the case. As $\Psi(\mathcal{T})$ is evidently a filter closed under products, we infer from Lemma 2.2.3 that it is a faithful finitely generated Gabriel topology. On the other hand, if $\mathcal{G}$ is a faithful finitely generated Gabriel topology with basis of finitely generated ideals $\mathcal{I}$, then $\Phi(\mathcal{G}) = \{M \in \text{Mod-R} \mid M \otimes_R R/I = 0 \text{ for each } I \in \mathcal{I}\}$, and thus $\Phi(\mathcal{G}) = \mathcal{E}^{\perp}$, where $\mathcal{E} = \{\operatorname{Tr} R/I \mid I \in \mathcal{I}\}$ by Lemma 2.3.3 (explicitly, we use the isomorphism of functors $- \otimes_R R/I \simeq \operatorname{Ext}^1_R(\operatorname{Tr}(R/I), -)$). This is a 1-tilting class by the same lemma.

We need to prove that Ψ and Φ are mutually inverse. Let $\mathcal{T}$ be a 1-tilting class. It is easy to see that $\mathcal{T} \subseteq \Phi(\Psi(\mathcal{T}))$. Let $\mathcal{S}$ be a set of finitely presented modules of projective dimension 1 such that $\mathcal{T} = \mathcal{S}^{\perp}$. By Lemma 2.3.15, there is for each $S \in \mathcal{S}$ a (again, necessarily faithful) finitely generated ideal I_S such that $S^{\perp} = \{M \in \text{Mod-R} \mid M = I_S M\}$. Put $\mathcal{I} = \{I_S \mid S \in \mathcal{S}\}$. It follows that $\mathcal{T} = \bigcap_{S \in \mathcal{S}} S^{\perp} = \mathcal{I}\text{-Div}$. Then $\mathcal{I} \subseteq \Psi(\mathcal{T})$, and therefore $\Phi(\Psi(\mathcal{T})) \subseteq \mathcal{T}$, proving that $\Phi(\Psi(\mathcal{T})) = \mathcal{T}$.

Finally, let $\mathcal{G}$ be a faithful finitely generated Gabriel topology and let $\mathcal{I}$ be some basis of $\mathcal{G}$ of finitely generated ideals. Let $J \in \Phi(\Psi(\mathcal{G}))$ be a finitely generated ideal. Denote by $(\mathcal{A}, \mathcal{T})$ the tilting cotorsion pair $({}^{\perp}\Psi(\mathcal{G}), \Psi(\mathcal{G}))$. Note that since $\mathcal{T} = \mathcal{I}\text{-Div}$, this cotorsion pair is generated (in the sense of [GT12, Definition 5.15]) by the set $\mathcal{S} = \{\operatorname{Tr}(R/I) \mid I \in \mathcal{I}\}$. Since $\mathcal{T} \subseteq \{M \in \text{Mod-R} \mid M = JM\}$, we have that $\operatorname{Tr}(R/J)$[3] $\in \mathcal{A}$. By [GT12, Corollary 6.14] and the Hill Lemma ([GT12, Theorem 7.10]), we infer that $\operatorname{Tr}(R/J)$ is a direct summand of a module N possessing a finite $\mathcal{S} \cup \{R\}$-filtration.

Therefore, there is a filtration $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_l = N$ with $N_{i+1}/N_i \in \mathcal{S} \cup \{R\}$ for each $i = 0, 1, \dots, l-1$. Let us apply the functor $\operatorname{Ext}^1_R(-, R) = (-)^{\dagger}$ to this filtration. Since all modules in $\mathcal{S} \cup \{R\}$ have projective dimension at most 1, this functor will act as a right exact functor on this filtration. As the filtration of N was finite, we obtain a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_l = N^{\dagger}$ such that M_{i+1}/M_i is isomorphic to a homomorphic image of $X^{\dagger}$ for some $X \in \mathcal{S} \cup \{R\}$ for each $i = 0, 1, \dots, l-1$.

Since $\operatorname{pd}(\operatorname{Tr}(R/I)) = 1$ for any $I \in \mathcal{I}$, we can apply Lemma 2.3.4 in order to see that $\operatorname{Tr}(R/I)^{\dagger} \overset{\text{st}}{\simeq} \operatorname{Tr}\operatorname{Tr}(R/I)$, and that $(\operatorname{Tr}(R/I)^{\dagger})^* = 0$. The only possibility is that $\operatorname{Tr}(R/I)^{\dagger} \simeq R/I$. As $R^{\dagger} = 0$, we conclude that $N^{\dagger}$ admits a filtration $0 = M'_0 \subseteq M'_1 \subseteq \cdots \subseteq M'_k = N^{\dagger}$ such that $M'_{i+1}/M'_i \simeq R/L_i$ for an ideal L_i containing some ideal $I_i \in \mathcal{I}$ for each $i = 0, 1, \dots, k-1$. Put $L = L_0 L_1 \cdots L_{k-1}$. Then $L \subseteq \operatorname{Ann} N^{\dagger}$, and as $I_0 I_1 \cdots I_{k-1} \subseteq L$, we have that $L \in \mathcal{G}$. But $R/J \simeq \operatorname{Tr}(R/J)^{\dagger}$ is a direct summand of $N^{\dagger}$, whence $L \subseteq \operatorname{Ann} N^{\dagger} \subseteq \operatorname{Ann}(R/J) = J$. Therefore, $J \in \mathcal{G}$, proving that $\Phi(\Psi(\mathcal{G})) = \mathcal{G}$.

(3) $\leftrightarrow$ (4): Let us again first prove that prescribed maps $\Xi : \mathcal{G} \mapsto X$ and $\Theta : X \mapsto \mathcal{G}$ are well-defined. Let $\mathcal{G}$ be a faithful finitely generated Gabriel topology with basis $\mathcal{I}$ of finitely generated ideals. Then $\Xi(\mathcal{G})$ is equal to $\bigcup_{I \in \mathcal{I}} V(I)$, and

[3] The stable equivalence representative choices do not matter in this argument.

therefore is a Thomason set. Suppose that there is $\mathfrak{p} \in \Xi(\mathcal{G}) \cap \mathrm{VAss}(R)$. Let $\mathcal{C}$ be the 1-cotilting class of cofinite type associated to $\Phi(\mathcal{G})$. Since $\mathfrak{p} \in \mathrm{VAss}(R)$, and $R \in \mathcal{C}$, we have that $R/\mathfrak{p} \in \mathcal{C}$. But this is a contradiction, because $\mathcal{C} = \bigcap_{J \in \mathcal{G}} \mathrm{Ker}\,\mathrm{Hom}_R(R/J, -)$ by the previous bijection and Lemma 2.3.4 and Corollary 2.3.6.

Let X be a Thomason subset of $\mathrm{Spec}(R) \setminus \mathrm{VAss}(R)$. It is easy to see that $\Theta(X)$ is a finitely generated Gabriel topology. Suppose that there is an ideal $I \in \Theta(X)$ and a non-zero map $R/I \to R$. Then there is $\mathfrak{p} \in \mathrm{VAss}(R)$ such that $I \subseteq \mathfrak{p}$, and therefore $\mathfrak{p} \in \Theta(X)$. But then $\mathfrak{p} \in X$, a contradiction.

Now we prove that Ξ and Θ are mutually inverse. That $\Xi(\Theta(X)) = X$ is easy to see. Let us show that $\Theta(\Xi(\mathcal{G})) = \mathcal{G}$. Clearly $\mathcal{G} \subseteq \Theta(\Xi(\mathcal{G}))$. Suppose that there is $I \in \Theta(\Xi(\mathcal{G})) \setminus \mathcal{G}$. Since $\mathcal{G}$ has a basis of finitely generated ideals, by Zorn's Lemma there is a maximal ideal with this property, let I' be maximal such. Then I' is necessarily prime. Since $\Theta(\Xi(\mathcal{G})) \cap \mathrm{Spec}(R) = \mathcal{G} \cap \mathrm{Spec}(R)$, we arrived at a contradiction.

(3) $\to$ (2): Correctness of this bijection follows from Lemma 2.3.4 and Corollary 2.3.6. Indeed, the cotilting class dual to the tilting class $\Phi(\mathcal{G})$ is equal to $\bigcap_{I \in \mathcal{G}} \mathrm{Ker}\,\mathrm{Hom}_R(R/I, -)$.

(2) $\to$ (4): Using the already established bijections, and that $\mathcal{C}$ is closed under submodules and direct limits, it is enough to show that $\mathcal{C} = \{M \in \text{Mod-R} \mid \mathrm{VAss}(M) \cap \mathcal{G} = \emptyset\}$, where $\mathcal{G}$ is the finitely generated Gabriel topology such that $\mathcal{C} = \bigcap_{I \in \mathcal{G}} \mathrm{Ker}\,\mathrm{Hom}_R(R/I, -)$. It is easily seen that $\mathfrak{p} \notin \mathrm{VAss}(M)$ for any prime $\mathfrak{p} \in \mathcal{G}$ and $M \in \mathcal{C}$. To prove the converse, suppose that $\mathrm{VAss}(M) \cap \mathcal{G} = \emptyset$. If there was a non-zero map in $\mathrm{Hom}_R(R/I, M)$ with $I \in \mathcal{G}$, there would exist by Lemma 2.3.9 a prime ideal $\mathfrak{p} \in \mathrm{VAss}(M)$ such that $I \subseteq \mathfrak{p}$ (see the proof of Proposition 2.3.11), a contradiction. Therefore, we can conclude that $M \in \mathcal{C}$.

(5) $\to$ (2): Direct consequence of Corollary 2.3.12.

(3) $\to$ (6): By Theorem 2.2.10, there is a 1-1 correspondence between 1-tilting classes and resolving subcategories of projective dimension at most 1 given by $\mathcal{T} \mapsto \mathcal{S} = ({}^{\perp}\mathcal{T}) \cap \text{mod-R}$. If $\mathcal{G}$ is a Gabriel topology with $\mathcal{G} = \Psi(\mathcal{T})$, we know from above that the cotorsion pair $({}^{\perp}\mathcal{T}, \mathcal{T})$ is generated by the set $\{R\} \cup \{\mathrm{Tr}(R/I) \mid I \in \mathcal{G} \text{ f.g.}\}$. Then $\mathcal{S} = ({}^{\perp}\mathcal{T}) \cap \text{mod-R}$ has the desired form, and we established the correspondence. $\square$

2.4 Tilting modules

2.4.1 Fuchs-Salce tilting modules

In the previous part we have proved that 1-tilting classes coincide with the classes of all modules divisible by all ideals of a faithful finitely generated Gabriel topology. The purpose of this section is to construct 1-tilting modules generating those classes, and hence classify all 1-tilting modules over commutative rings up to equivalence. Of course we can always construct such modules using the Small Object Argument (see [ET01] or [GT12, Theorem 6.11, Remark 13.47]). However, the following construction is "minimal" in the sense that the resulting module has a filtration of length only ω by direct sums of finitely presented modules. Also, the explicit contruction allows for direct computations, as we will demonstrate in the next second subsection.

The following construction generalizes the tilting modules generating the class of all divisible modules over a domain due to Fachini ([Fac88]), of all S-divisible modules for a multiplicative set S due to Fuchs-Salce ([FS92]), and of all $\mathcal{F}$-divisible modules for a finitely generated Gabriel topology $\mathcal{F}$ over a Prüfer domain due to Salce ([Sal05]).

Definition 2.4.1. Let R be a commutative ring, and let $\mathcal{I}$ be a set of faithful finitely generated ideals of R. For each $I \in \mathcal{I}$ fix a finite set of generators $\{x_1^I, x_2^I, \dots, x_{n_I}^I\}$ of I. Let Λ denote the set consisting of all finite sequences of pairs of the form (I, k), where $I \in \mathcal{I}$, and $k < n_I$ (including the empty sequence denoted by w). Let F be a free R-module with the basis Λ. Given two sequences $\lambda, \lambda' \in \Lambda$, we denote their concatenation by $\lambda \sqcup \lambda'$. In particular, symbol $\lambda \sqcup (I, k)$ will stand for appending pair (I, k) to sequence $\lambda \in \Lambda$.

Define a submodule G of F as the span of all elements of the form

$$\lambda - \sum_{k \in n_I} x_k^I (\lambda \sqcup (I, k)),$$

for each $\lambda \in \Lambda$ and $I \in \mathcal{I}$. Put $M_{\mathcal{I}} = F/G$. Let us call the module $\delta_{\mathcal{I}} = M_{\mathcal{I}} \oplus M_{\mathcal{I}}/\operatorname{Span}(w)$ the *Fuchs-Salce tilting module.*

If $\mathcal{G}$ is a faithful finitely generated Gabriel topology, we will abuse the notation by writing $\delta_{\mathcal{G}}$ instead of $\delta_{\mathcal{I}}$, where $\mathcal{I}$ is the set of all finitely generated ideals from $\mathcal{G}$.

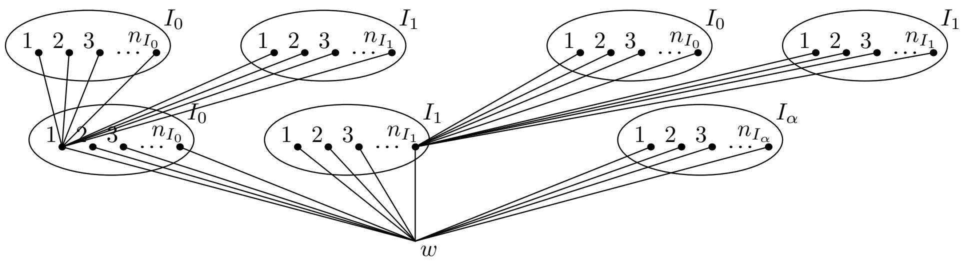

Figure 2.1: Construction of the Fuchs-Salce tilting module

Picture illustrates the first three levels of the homogeneous tree Λ from Definition 2.4.1. The basis of the module $M_{\mathcal{I}}$ consists of all vertices of this tree. For each “bubble” we add one relation identifying the parent vertex with the linear combination of the vertices in the bubble with scalar coefficients being the chosen generators $x_1^I, x_2^I, \dots, x_{n_I}^I$ of the ideal I.

We fix a concrete representative in the stable equivalence class $\operatorname{Tr}(R/I)$:

Notation 2.4.2. In the setting as in Definition 2.4.1, we define for each $I \in \mathcal{I}$ a module $\operatorname{tr}(R/I) = R^{n_I}/(x_1^I, x_2^I, \dots, x_{n_I}^I)R$. Note that $\operatorname{tr}(R/I) \overset{\text{st}}{\simeq} \operatorname{Tr}(R/I)$.

Proposition 2.4.3. *The module $\delta_{\mathcal{I}}$ defined above is a 1-tilting module generating the 1-tilting class $\mathcal{I}$-*Div*.*

Proof. Put $\mathcal{T} = \mathcal{I}$-Div. First note that by the hypothesis that we have imposed on the ideals in $\mathcal{I}$, we see that $\operatorname{tr}(R/I)$ is a finitely presented module of projective dimension 1 for each $I \in \mathcal{I}$, and whence $(\bigoplus_{I \in \mathcal{I}} \operatorname{tr}(R/I))^\perp = \mathcal{T}$ is a 1-tilting class. Put $\mathcal{A} = {}^\perp\mathcal{T}$.

Let $M_{\mathcal{I}}$ be the module from Definition 2.4.1. For each $n \in \omega$ let M_n be the submodule of $M_{\mathcal{I}}$ generated by (the images of) all sequences in Λ of length smaller then or equal to n. In particular, $M_0 = \mathrm{Span}(w)$ is isomorphic to R. The quotient M_{n+1}/M_n is generated by the cosets of all sequences in Λ of length $n+1$. We claim that M_{n+1}/M_n is isomorphic to a direct sum of a suitable number of copies of the modules $\mathrm{tr}(R/I)$ with $I \in \mathcal{I}$.

In order to prove this claim, let us fix more notation: For each $n \in \omega$ denote by Λ_n (resp. $\Lambda_{<n}$) a subset of Λ consisting of all sequences of length n (resp. smaller than n). Put $F_n = \mathrm{Span}(\Lambda_{<n+1}) \subseteq F$ and $G_n = F_n \cap G$, so that $M_n \simeq F_n/G_n$. Let $X = \{\lambda - \sum_{k \in n_I} x_k^I(\lambda \sqcup (I,k)) \mid \lambda \in \Lambda, I \in \mathcal{I}\}$ be the prescribed set of generators of G. Let $X_n = X \cap F_n$ for each $n \in \omega$. Observe that X is actually a free basis of G, and furthermore, that $X \setminus X_n$ is linearly independent in F modulo F_n. Indeed, our hypothesis of $\mathrm{Hom}_R(R/I, R) = 0$ assures that I has no non-trivial annihilator in R for each $I \in \mathcal{I}$, and thus the elements of the form $\sum_{k \in n_I} x_k^I(\lambda \sqcup (I,k))$ are torsion-free, and hence they are linearly independent in F/F_n. It follows that $G_n = \mathrm{Span}(X_n)$, that is, G_n is generated by elements

$$\lambda - \sum_{k \in n_I} x_k^I(\lambda \sqcup (I,k)),$$

where $\lambda \in \Lambda_{<n}, I \in \mathcal{I}$. From this it is easily seen that M_{n+1}/M_n can be viewed as a module with generators Λ_{n+1} and relations of the form

$$\sum_{k \in n_I} x_k^I(\lambda \sqcup (I,k)) = 0,$$

where $\lambda \in \Lambda_n$ and $I \in \mathcal{I}$. It follows that $M_{n+1}/M_n \simeq \bigoplus_{\lambda \in \Lambda_n} \bigoplus_{I \in \mathcal{I}} \mathrm{tr}(R/I)$, and the claim is proved.

As $M_0 = wR \simeq R$, we have that $M_{\mathcal{I}}$ is filtered by the set $\{R\} \cup \{\mathrm{tr}(R/I) \mid I \in \mathcal{I}\} \subseteq \mathcal{A}$, and so $M_{\mathcal{I}} \in \mathcal{A}$, and also $M_{\mathcal{I}}/wR \in \mathcal{A}$. On the other hand, $M_{\mathcal{I}}$ is generated by Λ, and from the construction we have that for each $\lambda \in \Lambda$ and each $I \in \mathcal{I}$, $\lambda \in IM_{\mathcal{I}}$. It follows that $M_{\mathcal{I}} \in \mathcal{T}$.

Altogether we have that the inclusion $R \simeq M_0 \to M_{\mathcal{I}}$ is a special $\mathcal{T}$-preenvelope of R. An argument [GT12, Remark 13.47] then shows that $\delta_{\mathcal{I}} = M_{\mathcal{I}} \oplus M_{\mathcal{I}}/wR$ is a 1-tilting module in $\mathcal{A} \cap \mathcal{T}$, and thus generating the class $\mathcal{T}$. □

Combining Theorem 2.3.16 and Proposition 2.4.3 we obtain the following.

Theorem 2.4.4. *Let R be a commutative ring. Then*

$$\{\delta_{\mathcal{G}} \mid \mathcal{G} \textit{ a faithful finitely generated Gabriel topology}\}$$

is the set of representatives of equivalence classes of all 1-tilting modules over R.

2.4.2 An application

As an application, we present an alternative proof of the positive solution of the so-called Saorín's problem for commutative rings. The Saorín's problem is the following statement.

Problem 2.4.5. *([PS13])* Let R be a ring and T a 1-tilting module such that the induced torsion-free class $\mathcal{F} = \mathrm{Ker}\,\mathrm{Hom}_R(T, -)$ is closed under direct limits. Is then T equivalent to a finitely generated 1-tilting module?

The motivation of the problem is the recent result of Parra and Saorín [PS13, Theorem 4.9], which states that heart of the t-structure associated to a tilting torsion pair $(\mathcal{T}, \mathcal{F})$ is a Grothendieck category if and only if $\mathcal{F}$ is closed under direct limits.

If R is commutative, then any finitely generated tilting module is projective, so a positive answer implies that $\mathcal{F} = \{0\}$. Problem 2.4.5 has a negative answer in general, a very involved counter-example was found by Herzog, and further counter-examples that are non-commutative, but two-sided noetherian were constructed by Příhoda ([BHP+15]). On the other hand, Problem 2.4.5 has a positive answer whenever R is commutative, as proved by Bazzoni in [BHP+15]. We can now reprove the latter result in an elementary way using our classification of 1-tilting classes and 1-tilting modules.

Theorem 2.4.6. ([BHP+15]) *Let R be a commutative ring and T a 1-tilting module such that $\mathcal{F} = \operatorname{Ker}\operatorname{Hom}_R(T, -)$ is closed under direct limits. Then T is projective.*

Proof. By Theorem 2.3.16 and Theorem 2.4.4, there is a faithful finitely generated Gabriel topology $\mathcal{G}$ such that $\mathcal{T} = T^{\perp} = \mathcal{G}$-Div and we can without loss of generality assume that $T = \delta_{\mathcal{G}}$. Suppose that T is not projective. Then necessarily $\mathcal{G}$ contains a non-trivial ideal. Let $\mathcal{I}$ be a basis of finitely generated ideals of $\mathcal{G}$. Let $M_{\mathcal{I}}$ be the module from the construction of $\delta_{\mathcal{G}}$, that is, $\delta_{\mathcal{G}} = M_{\mathcal{I}} \oplus M_{\mathcal{I}}/\operatorname{Span}(w)$, and $M_{\mathcal{I}} = \bigcup_{n\in\omega} M_n$ with $M_0 = \operatorname{Span}(w) \simeq R$ and M_{n+1}/M_n isomorphic to a direct sum of copies of $\operatorname{tr}(R/I), I \in \mathcal{I}$ for each $n \in \omega$.

Denote by $t(-)$ the torsion radical of the torsion pair $(\mathcal{T}, \mathcal{F})$. Since $\mathcal{F}$ is closed under direct limits, the direct limit functor is exact, and $M \in \mathcal{T}$, we have $M_{\mathcal{I}} = \bigcup_{n\in\omega} t(M_n)$. It follows that there is $n \in \omega$ such that $w \in t(M_n)$. In other words, there is a submodule X of M_n containing w such that $X = IX$ for each $I \in \mathcal{I}$. It is clear that $n \neq 0$, since then it would follow that $R = t(R)$, which cannot be the case since $\mathcal{G}$ contains non-trivial ideals.

Suppose that $n > 0$. From now on we adopt the notation of Definition 2.4.1 for the generators of M_n. For each n-tuple of ideals $\bar{I} = (I_1, I_2, \dots, I_n) \in \mathcal{I}^n$, we define a finite subset $Y_{\bar{I}}$ of M_n as follows: Let $Y_{\bar{I},1} = \{(w, (I_1, k)) \mid k \in n_{I_1}\}$. For $1 < j \leq n$, we put $Y_{\bar{I},j} = \{\lambda \sqcup (I_j, k) \mid \lambda \in Y_{\bar{I},j-1}, k \in n_{I_j}\}$. Finally, we set $Y_{\bar{I}} = \bigcup_{1\leq j\leq n} Y_{\bar{I},j}$. Note that $\operatorname{Span}(Y_{\bar{I}})$ is a free R-module with basis consisting of sequences from $Y_{\bar{I}}$ of maximal length (that is, of length n). We can index this basis as follows: Denote by $\kappa_{\bar{I}}$ the set of all sequences $\bar{k} = (k_1, k_2, \dots, k_n) \in \omega^n$ such that $k_j \in n_{I_k}$ for all $1 \leq j \leq n$. For each $\bar{k} \in \kappa_{\bar{I}}$ let $\lambda^{\bar{I}}_{\bar{k}}$ denote the element $w \sqcup (I_1, k_1) \sqcup (I_2, k_2) \sqcup \dots \sqcup (I_n, k_n)$. Then $\operatorname{Span}(Y_{\bar{I}})$ is a free module with basis $\{\lambda^{\bar{I}}_{\bar{k}} \mid \bar{k} \in \kappa_{\bar{I}}\}$. Also, note that $w = \sum_{\bar{k}\in\kappa_{\bar{I}}} x^{I_1}_{k_1} x^{I_2}_{k_2} \cdots x^{I_n}_{k_n} \lambda^{\bar{I}}_{\bar{k}}$. Denote $x^{\bar{I}}_{\bar{k}} = x^{I_1}_{k_1} x^{I_2}_{k_2} \cdots x^{I_n}_{k_n}$ for each $\bar{I} \in \mathcal{I}^n$ and $\bar{k} \in \kappa_{\bar{I}}$. Finally, it is easy to see that $M_n = \operatorname{Span}(Y_{\bar{I}} \mid \bar{I} \in \mathcal{I}^n)$.

As X is the direct limit of all its finitely generated submodules containing w, we again use the hypothesis of $\mathcal{F}$ being closed under direct limits in order to find a finitely generated submodule N of X such that $w \in t(N)$. As N is finitely generated, there are $\bar{I}^1, \bar{I}^2, \dots, \bar{I}^m \in \mathcal{I}^m$ such that $t(N) \subseteq \operatorname{Span}(Y_{\bar{I}^1}, Y_{\bar{I}^2}, \dots, Y_{\bar{I}^m})$. Denote this span by S. Then S is a module with generators

$$\{\lambda^{\bar{I}^j}_{\bar{k}} \mid 1 \leq j \leq m, \bar{k} \in \kappa_{\bar{I}^j}\}$$

subject to the following relations:

$$w = \sum_{\bar{k}\in\kappa_{\bar{I}^1}} x_{\bar{k}}^{\bar{I}^1}\lambda_{\bar{k}}^{\bar{I}^1} = \sum_{\bar{k}\in\kappa_{\bar{I}^2}} x_{\bar{k}}^{\bar{I}^2}\lambda_{\bar{k}}^{\bar{I}^2} = \cdots = \sum_{\bar{k}\in\kappa_{\bar{I}^m}} x_{\bar{k}}^{\bar{I}^m}\lambda_{\bar{k}}^{\bar{I}^m}. \tag{2.3}$$

This leads to a contradiction. Indeed, since $t(N)$ is divisible by each ideal in $\mathcal{I}$, and $\mathcal{I}$ consists of finitely generated faithful (and therefore not idempotent) ideals, we infer that there is an ideal $J \in \mathcal{I}$ such that $J \subsetneq \prod_{1\le i\le m, 1\le j\le n} I_j^i$. Hence, there are elements $s_{\bar{k}}^i \in J$ for each $1 \le i \le m$ and $\bar{k} \in \kappa_{\bar{I}^i}$, such that $w = \sum_{i=1}^m \sum_{k\in\kappa_{\bar{I}^i}} s_{\bar{k}}^i \lambda_{\bar{k}}^{\bar{I}^i}$. Note that $S/\operatorname{Span}(w)$ decomposes as follows:

$$S/\operatorname{Span}(w) \simeq \bigoplus_{1\le i\le m} R^{(\kappa_{\bar{I}^i})}/\operatorname{Span}(\sum_{\bar{k}\in\kappa_{\bar{I}^i}} x_{\bar{k}}^{\bar{I}^i}\bar{k}).$$

Projecting S onto the i-th summand in this decomposition yields that there is $t_i \in R$ such that $s_{\bar{k}}^i = t_i x_{\bar{k}}^{\bar{I}^i}$ for all $1 \le i \le m$ and $\bar{k} \in \kappa_{\bar{I}^i}$. Since $\operatorname{Span}(x_{\bar{k}}^{\bar{I}^i} \mid \bar{k} \in \kappa_{\bar{I}^i}) = I_1^i I_2^i \cdots I_n^i$, we infer that $t_i \in (I_1^i I_2^i \cdots I_n^i : J)$. Using the relations (2.3) several times, we get that $w = (t_1 + t_2 + \cdots + t_m)\sum_{\bar{k}\in\kappa_{\bar{I}^1}} x_{\bar{k}}^{\bar{I}^1}\lambda_{\bar{k}}^{\bar{I}^1}$. Since $\operatorname{Ann}(w) = 0$, this implies that $t_1 + t_2 + \cdots + t_m = 1$. But $t_i \in (I_1^i I_2^i \cdots I_n^i : J) \subseteq (\prod_{1\le i'\le m, 1\le j\le n} I_j^{i'} : J) \neq R$ for each $i = 1, \ldots, m$ by the assumption on J, making the assertion $t_1 + t_2 + \cdots + t_j = 1$ a contradiction. □

2.5 Perfect localizations

As hereditary torsion classes coincide with localizing subcategories of Mod-R, each hereditary torsion class $\mathcal{E}$ in Mod-R gives rise to a (Serre) localization Mod-R $\to$ Mod-R $/\mathcal{E}$. Therefore, each 1-tilting class over a commutative ring corresponds naturally to some localization functor. The localized category is not in general a module category, and so it is not induced by a ring homomorphism. In this section, we focus on the case when this localization is induced by a flat ring epimorphism. In particular, we describe when this so-called *perfect localization* allows to replace the Fuchs-Salce module by a much nicer tilting module, arisen from a ring of quotients.

Given a Gabriel topology $\mathcal{G}$, recall that a module M is *$\mathcal{G}$-closed* if the inclusion $I \subseteq R$ induces an isomorphism $\operatorname{Hom}_R(R, M) \to \operatorname{Hom}_R(I, M)$ for any ideal $I \in \mathcal{G}$. Denote the full subcategory of all $\mathcal{G}$-closed modules by $\mathcal{X}(\mathcal{G})$. This subcategory is *Giraud*, that is, a full subcategory of Mod-R such that its inclusion into Mod-R has a left adjoint which is exact (and, in fact, all Giraud subcategories of Mod-R are of form $\mathcal{X}(\mathcal{G})$ for some Gabriel topology $\mathcal{G}$). The composition of this left adjoint and the original inclusion yields a *localization functor* L : Mod-R $\to$ Mod-R. The unit of the adjunction $\eta_R : R \to L(R)$ then induces a ring structure on $Q_{\mathcal{G}} = L(R)$, with the unit η_R being a ring homomorphism. For all details we refer to Chapters VII.-XI. in [Ste75], as for the proof of the following:

Theorem 2.5.1. *([Ste75, XI, Proposition 3.4]) Let R be a commutative ring, and $\mathcal{G}$ a Gabriel topology. Then the following are equivalent:*

1. *$\eta_R : R \to Q_{\mathcal{G}}$ is a flat ring epimorphism, and $\{I \subseteq R \mid Q_{\mathcal{G}} = IQ_{\mathcal{G}}\} = \mathcal{G}$,*

2. *$\mathcal{X}(\mathcal{G})$ is naturally equivalent to* Mod-Q$_{\mathcal{G}}$,

3. *the R-module $Q_{\mathcal{G}}$ is $\mathcal{G}$-divisible.*

If $\mathcal{G}$ satisfies conditions of this theorem, we call it *perfect*. Say that a perfect localization $\lambda : R \to S$ is *faithful*, if the map λ is injective. Say that two ring epimorphisms $\lambda : R \to S, \lambda' : R \to S'$ are *equivalent* if there is a ring isomorphism $\varphi : S \to S'$, such that $\lambda' = \varphi\lambda$. The equivalence classes of ring epimorphism under this equivalence are called *epiclasses* of R. By [Ste75, XI, Theorem 2.1], the ring maps $R \to Q_{\mathcal{G}}$, with $\mathcal{G}$ running through perfect Gabriel topologies, parametrize all epiclasses of flat ring epimorphisms, which justifies the terminology *perfect localization* instead of flat ring epimorphism.

We recall that a ring is right *semihereditary*, if any finitely generated right ideal is projective.

Theorem 2.5.2. (cf. [BŠ14, Proposition 7.4]) *Let R be a commutative semihereditary ring, $\mathcal{T}$ a 1-tilting class, and $\mathcal{G}$ a Gabriel topology associated to this class (via Theorem 2.3.16). Then $\mathcal{G}$ is perfect, and the perfect localization $\eta : R \to Q_{\mathcal{G}}$ is faithful. Furthermore, there is a 1-1 correspondence between* 1*-tilting classes $\mathcal{T}$ and epiclasses of faithful perfect localizations $R \hookrightarrow S$; the correspondence given by*

$$\Gamma : \mathcal{T} = \mathcal{G}\text{-Div} \mapsto (R \hookrightarrow Q_{\mathcal{G}}),$$

$$\Delta : (R \hookrightarrow S) \mapsto \{I \subseteq R \mid S = IS\}\text{-Div}.$$

Proof. By Theorem 2.3.16, the Gabriel topology $\mathcal{G}$ is necessarily finitely generated and faithful, and hence perfect by [Ste75, XI, Corollary 3.5] and [Ste75, IX, Proposition 5.2], and any perfect Gabriel topology inducing a faithful perfect localization arises in this way. The map $\eta_R : R \to Q_{\mathcal{G}}$ is injective again by faithfulness of $\mathcal{G}$ and [Ste75, IX, Lemma 1.2]. Together with Theorem 2.5.1, this shows that Γ is well-defined. By Theorem 2.5.1 and [Ste75, XI, Theorem 2.1], $\Delta(R \hookrightarrow S)$ is equal to some perfect Gabriel topology $\mathcal{G}$, which is finitely generated by [Ste75, XI, Proposition 3.4], and faithful by [Ste75, IX, Lemma 1.2], and thus Δ is well-defined by Theorem 2.3.16. Finally, Γ and Δ are mutually inverse, for checking which it is now enough to use the fact that epiclasses of faithful perfect localizations are parametrized by the set $\{R \to Q_{\mathcal{G}} \mid \mathcal{G}$ a perfect faithful Gabriel topology$\}$. □

Definition 2.5.3. We say that a tilting module T *arises from a perfect localization*, if there is an faithful perfect localization $R \hookrightarrow S$ such that $S \oplus S/R$ is a 1-tilting module equivalent to T.

We can now prove the following generalization of (part of) [AHA12, Theorem 4.10] and [HHT05, Theorem 1.1].

Theorem 2.5.4. *Let R be a commutative ring, T a 1-tilting module, and $\mathcal{G}$ a Gabriel topology associated to $\mathcal{T} = T^{\perp}$ in the sense of Theorem 2.3.16. Then the following are equivalent:*

1. *$\mathcal{G}$ is perfect, and* $\operatorname{pd}_R Q_{\mathcal{G}} \leq 1$,
2. *T arises from a perfect localization,*
3. $\operatorname{Gen}(Q_{\mathcal{G}}) = \mathcal{G}\text{-Div}$.

Proof. (1) ⇒ (2): By [AHA12, Lemma 1.10], the module $T' = Q_\mathcal{G} \oplus Q_\mathcal{G}/R$ is a 1-tilting module. Therefore, there is by Theorem 2.3.16 a faithful finitely generated Gabriel topology $\mathcal{G}'$ such that $T'^{\perp} = \mathcal{G}'$-Div. Using Theorem 2.5.1, we conclude that $\mathcal{G} = \mathcal{G}'$, proving that $\mathcal{T} = T^{\perp}$, and therefore T is equivalent to T'.

(2) ⇒ (3): Follows quickly from $\mathrm{Gen}(Q_\mathcal{G}) = \mathrm{Gen}(Q_\mathcal{G} \oplus Q_\mathcal{G}/R) = \mathcal{T} = \mathcal{G}$-Div.

(3) ⇒ (1): That $\mathcal{G}$ is perfect follows directly from Theorem 2.5.1. By (3), there is an epimorphism $Q_\mathcal{G}^{(X)} \to \delta_\mathcal{G}$ for some set X. Since $Q_\mathcal{G} \in \mathcal{T}$, there is also an epimorphism $\delta_\mathcal{G}^{(Y)} \to Q_\mathcal{G}$ for some set Y. Together we get an epimorphism $Q_\mathcal{G}^{(X\times Y)} \to Q_\mathcal{G}$ in Mod-R. As $R \to Q_\mathcal{G}$ is a ring epimorphism of R, Mod-$Q_\mathcal{G}$ is a full subcategory of Mod-R, and thus the epimorphism from last sentence is actually a map of $Q_\mathcal{G}$-modules, and hence it splits. But then also the epimorphism $Q_\mathcal{G}^{(X\times Y)} \to \delta_\mathcal{G}^{(Y)}$ splits, and thus $Q_\mathcal{G}$ has projective dimension at most 1 over R. □

Acknowledgement

I am grateful to Jan Trlifaj for many fruitful discussions, as well as for careful reading of my manuscripts. I would like to thank Jan Šťovíček for our discussions, and for introducing me to the work of Thomason.

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[ii]

Silting modules over commutative rings

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Published: International Mathematics Research Notices, DOI: 10.1093/imrn/rnw147, 2016

Abstract. Tilting modules over commutative rings were recently classified in [i]: they correspond bijectively to faithful Gabriel topologies of finite type. In this note we extend this classification by dropping faithfulness. The counterpart of an arbitrary Gabriel topology of finite type is obtained by replacing tilting with the more general notion of a silting module.

Chapter 3

Silting modules over commutative rings

3.1 Introduction

Silting modules were introduced in [3] as a common generalisation of tilting modules and of the support τ-tilting modules from [1]. They are in bijection with 2-term silting complexes and with certain t-structures and co-t-structures in the derived module category. For certain rings, they are also known to parametrize universal localisations and wide subcategories of finitely presented modules [14, Theorem 4.5],[4, Corollary 5.15].

In this note, we give a classification of silting modules over commutative rings, establishing a bijective correspondence with Gabriel filters of finite type. This extends the results in [13] from the tilting to the silting case, and it is a further piece of evidence for the close relationship between silting modules and localisation theory.

Our result is achieved by investigating the dual notion of a cosilting module recently introduced in [9] as a generalisation of cotilting modules. Indeed, the dual of a silting module T is a cosilting module T^+, and there is a duality between the modules in the silting class $\operatorname{Gen} T$ and the cosilting class $\operatorname{Cogen} T^+$. When R is commutative, $\operatorname{Cogen} T^+$ turns out to be the torsionfree class of a hereditary torsion pair of finite type. We can thus interpret the modules in $\operatorname{Cogen} T^+$ as the $\mathcal{G}$-torsionfree modules with respect to a Gabriel filter of finite type $\mathcal{G}$. The silting class $\operatorname{Gen} T$ is then the class of $\mathcal{G}$-divisible modules. This defines a map assigning a Gabriel filter $\mathcal{G}$ to every silting class $\operatorname{Gen} T$. We show that this assignment is a bijection by constructing explicitly, for any $\mathcal{G}$, a silting module T which generates

The first named author is partially supported by Fondazione Cariparo, Progetto di Eccellenza ASATA. The second named author is partially supported by the Grant Agency of the Czech Republic under the grant no. 14-15479S.

the $\mathcal{G}$-divisible modules (Construction 3.4.5). We also provide a construction for a cosilting module cogenerating the $\mathcal{G}$-torsionfree modules (Construction 3.5.2).

In general, not all cosilting modules arise as duals of silting modules. This is a phenomenon that already occurs for cotilting modules [6], see Example 3.5.1 for a cosilting example. If R is a commutative noetherian ring, however, our classification yields bijections between silting classes, cosilting classes, Gabriel filters, and subsets of $\operatorname{Spec}(R)$ closed under specialisation (Theorem 3.5.1). This generalises the classification of tilting and cotilting modules in [5, Theorem 2.11].

In fact, silting and cosilting classes are in bijection also over non-commutative noetherian rings. As a consequence, every definable torsion class of right modules over a left noetherian ring is generated by a silting module (Corollary 3.3.7). Finally, extending a result from [8], we show that the only silting torsion pair of finite type over a commutative ring is the trivial one (Proposition 3.4.8).

The note is organized as follows. In Section 2 we investigate a finiteness condition which is shown to hold for silting classes, recovering a recent result from [15]. Section 3 is devoted to the duality between silting and cosilting classes. In Sections 4 and 5 we turn to commutative rings and prove our classification results. In 3.5.1 we further exhibit an example showing that the inclusion of silting modules in the class of finendo quasitilting modules proved in [3, Proposition 3.10] is proper.

3.2 Definability and finite type

Let R be a ring, and let Mod-R (respectively, R-Mod) denote the category of all right (respectively, left) R-modules. Denote by Proj-R and proj-R the full subcategory of Mod-R consisting of all projective and all finitely generated projective modules, respectively. Given a subcategory $\mathcal{C}$ of Mod-R, write $\operatorname{Mor}(\mathcal{C})$ for the class of all morphisms in Mod-R between objects in $\mathcal{C}$, and denote

$$\mathcal{C}^{\perp} = \{M \in \text{Mod-}R \mid \operatorname{Ext}^1_R(C, M) = 0\}.$$

Given a map σ in Mor(Proj-R), we are going to investigate the class

$$\mathcal{D}_\sigma := \{X \in \text{Mod–}R \mid \operatorname{Hom}_R(\sigma, X) \text{ is surjective}\}.$$

We say that $\mathcal{D}_\sigma$ is of *finite type* if it is determined by a set of morphisms between finitely generated projective modules, i. e. there are $\sigma_i \in \text{Mor(proj-R)}, i \in I$, such that $\mathcal{D}_\sigma = \bigcap_{i\in I} \mathcal{D}_{\sigma_i}$. As a shorthand, we say that $\sigma \in$ Mor(Proj-R) is of *finite type* if the class $\mathcal{D}_\sigma$ is of finite type.

Recall that a class is said to be *definable* if it is closed under direct limits, direct products and pure submodules. We are going to see that $\mathcal{D}_\sigma$ is definable if and only if it is of finite type.

Lemma 3.2.1. *Let $\sigma \in \operatorname{Mor}(\textit{Proj-R})$. Then*

$$\mathcal{D}_\sigma = (\operatorname{Coker}\sigma)^{\perp} \cap \mathcal{D}_{\sigma'},$$

where $\sigma' : P_{-1} \to \operatorname{Im}\sigma$ is given by restricting the codomain of σ to its image.

Proof. It is clear that $\mathcal{D}_\sigma \subseteq \mathcal{D}_{\sigma'}$. Then for any $M \in \mathcal{D}_{\sigma'}$, a standard long exact sequence argument shows that $M \in \mathcal{D}_\sigma$ if and only if $\mathrm{Ext}^1_R(\mathrm{Coker}\sigma, M) = 0$, finishing the proof. $\square$

Lemma 3.2.2. *Let $\sigma \in \mathrm{Mor}(\textit{Proj-R})$ be a map between projective modules. Then $\mathcal{D}_\sigma = \mathcal{D}_\varphi$, where φ is a map between free modules.*

Proof. Suppose that $\sigma : P_{-1} \to P_0$. Let P' be a projective module such that $P_{-1} \oplus P$ is free, and let P'' be a projective module such that $P_0 \oplus P' \oplus P''$ is free. We then let φ be the direct sum of the maps $\sigma : P_{-1} \to P_0, P' \xrightarrow{=} P'$, and $0 \xrightarrow{0} P''$. It is a routine check that $\mathcal{D}_\sigma = \mathcal{D}_\varphi$. $\square$

Theorem 3.2.3. *Let $\sigma \in \mathrm{Mor}(\textit{Proj-R})$. Then the following are equivalent:*

(i) $\mathcal{D}_\sigma$ is of finite type,

(ii) $\mathcal{D}_\sigma$ is definable.

Proof. In the whole proof, let $\sigma : P_{-1} \to P_0$, and $C = \mathrm{Coker}\sigma$.

(i) $\to$ (ii): As an intersection of definable classes is a definable class, it is enough to show that $\mathcal{D}_\sigma$ is definable if $\sigma \in \mathrm{Mor}$(proj-R). By [3, Lemma 3.9], $\mathcal{D}_\sigma$ is closed under direct products and epimorphic images, it is thus enough to show that it is closed under direct sums and pure submodules. By Lemma 3.2.1, we have that $\mathcal{D}_\sigma = \mathcal{D}_{\sigma'} \cap C^\perp$, where $\sigma' : P_{-1} \to \mathrm{Im}\,\sigma$ is σ with codomain restricted to its image. As C is finitely presented, the class $C^\perp$ is definable by [11, Theorem 13.26]. We finish the proof by showing that $\mathcal{D}_{\sigma'}$ is closed under direct sums and submodules.

Let $(M_i \mid i \in I)$ be a family of modules from $\mathcal{D}_{\sigma'}$, and $f : P_{-1} \to \bigoplus_{i\in I} M_i$ a map. As P_{-1} is finitely generated, there is a finite subset $J \subseteq I$ such that f factors through the direct summand $\bigoplus_{i\in J} M_i \simeq \prod_{i\in J} M_i$. Since $\mathcal{D}_{\sigma'}$ is clearly closed under products, f factorizes through σ'.

Let $M \in \mathcal{D}_{\sigma'}$ and $\iota : N \subseteq M$ be an inclusion. Applying $\mathrm{Hom}_R(-, \iota)$ on the exact sequence $0 \to K \to P_{-1} \xrightarrow{\sigma'} \mathrm{Im}\,\sigma \to 0$ yields

$$\begin{CD}
0 @>>> \mathrm{Hom}_R(\mathrm{Im}\,\sigma, M) @>{\mathrm{Hom}_R(\sigma', M)}>> \mathrm{Hom}_R(P_{-1}, M) @>{\varphi}>> \mathrm{Hom}_R(K, M)\\
@. @AAA @AAA @A{\theta}AA\\
0 @>>> \mathrm{Hom}_R(\mathrm{Im}\,\sigma, N) @>{\mathrm{Hom}_R(\sigma', N)}>> \mathrm{Hom}_R(P_{-1}, N) @>{\psi}>> \mathrm{Hom}_R(K, N).
\end{CD}$$

By the assumption, the map $\varphi = 0$, and thus $\theta\psi = 0$. By left-exactness, all the vertical maps are injective, and therefore $\psi = 0$, showing that $\mathrm{Hom}_R(\sigma', N)$ is surjective. Therefore, $N \in \mathcal{D}_{\sigma'}$.

(ii) $\to$ (i): Using Lemma 3.2.2, we can without loss of generality assume that P_{-1} and P_0 are free modules. Fix a free basis X of P_{-1}, and write the set X as a direct union $X = \bigcup_{i\in I} X_i$ of its finite subsets, inducing a presentation of P_{-1} as a direct union of direct summands $F_i = R^{(X_i)}$. Denote $G_i = \sigma(F_i)$. Fix a free basis Y of P_0. As G_i is finitely generated for each $i \in I$, there is a finite subset Y_i' of Y spanning G_i. Moreover, there are finite subsets Y_i of Y such that $Y_i' \subseteq Y_i$ for each $i \in I$, and $(Y_i \mid i \in I)$ forms a directed system. Indeed, we can find such sets by setting $Y_i = Y_i' \cup \bigcup_{j\leq i} Y_j'$. As this is clearly a finite union of finite sets, we have that $(Y_i \mid i \in I)$ is a directed system of finite subsets of Y. We let $F_i' = R^{(Y_i)}$, a free direct summand of P_0 for each $i \in I$.

The directed union $\bigcup_{i \in I} F_i' = R^{(\bigcup_{i \in I} Y_i)}$ is a direct summand of P_0, and the projection of the image of σ onto the complement $R^{(Y \setminus \bigcup_{i \in I} Y_i)}$ is necessarily zero. Therefore, we can without loss of generality assume that $P_0 = \bigcup_{i \in I} F_i'$. Let $\sigma_i : F_i \to F_i'$ be the restriction of σ onto F_i, with codomain restricted to F_i'. We claim that $\mathcal{D}_\sigma \subseteq \mathcal{D}_{\sigma_i}$ for each $i \in I$. To prove this, let $M \in \mathcal{D}_\sigma$ and fix a map $f_i : F_i \to M$. As F_i is a direct summand of P_{-1}, we can extend f_i to a map $f : P_{-1} \to M$. As $M \in \mathcal{D}_\sigma$, there is a map $g : P_0 \to M$ such that $f = g\sigma$. Let g_i be the restriction of g to F_i'. Then $f_i = g_i \sigma_i$, proving that $M \in \mathcal{D}_{\sigma_i}$. Denoting $\mathcal{D} = \bigcap_{i \in I} \mathcal{D}_{\sigma_i}$, we have $\mathcal{D}_\sigma \subseteq \mathcal{D}$.

Finally, we show that $\mathcal{D}_\sigma$ is of finite type by proving $\mathcal{D} \subseteq \mathcal{D}_\sigma$. The class $\mathcal{D}_\sigma$ is definable by the assumption, and the definability of the class $\mathcal{D}$ is proved by implication $(i) \to (ii)$ of this Theorem. By [11, Lemma 6.9], it is enough to show that $M \in \mathcal{D}$ implies $M \in \mathcal{D}_\sigma$ for M pure-injective.

Let $M \in \mathcal{D}$ be pure-injective. Denote $C_i = \mathrm{Coker}\sigma_i$ for all $i \in I$. By Lemma 3.2.1, we have that $\mathrm{Ext}^1_R(C_i, M) = 0$ and $M \in \mathcal{D}_{\sigma_i'}$, where σ_i' is given by restricting the codomain of σ_i to G_i. Since M is pure-injective, we have by [11, Lemma 6.28] the following isomorphism:

$$\mathrm{Ext}^1_R(C, M) \simeq \mathrm{Ext}^1_R(\varinjlim_{i \in I} C_i, M) \simeq \varprojlim_{i \in I} \mathrm{Ext}^1_R(C_i, M).$$

This shows that $M \in C^\perp$. Applying $\mathrm{Hom}_R(-, M)$ to the exact sequence $F_i \xrightarrow{\sigma_i'} G_i \to 0$ we obtain that $\mathrm{Hom}_R(\sigma_i', M)$ is an isomorphism for all $i \in I$. As inverse limit of a directed system of isomorphisms is an isomorphism, we obtain that

$$\varprojlim_{i \in I} \mathrm{Hom}_R(\sigma_i', M) \simeq \mathrm{Hom}_R(\varinjlim_{i \in I} \sigma_i', M) \simeq \mathrm{Hom}_R(\sigma', M)$$

is an isomorphism, where again $\sigma' : P_{-1} \to G$ is given by restricting the codomain of σ to its image $G = \bigcup_{i \in I} G_i$. In other words, $M \in \mathcal{D}_{\sigma'}$. As $M \in C^\perp \cap \mathcal{D}_{\sigma'}$, Lemma 3.2.1 yields $M \in \mathcal{D}_\sigma$ as desired. $\square$

3.3 Silting and cosilting modules

According to [3], an R-module T is said to be *silting* if it admits a projective presentation $P_{-1} \xrightarrow{\sigma} P_0 \longrightarrow T \to 0$ such that the class $\mathrm{Gen}\, T$ of T-generated modules coincides with the class $\mathcal{D}_\sigma$. The class $\mathrm{Gen}\, T$ is then called a *silting class.*

It is shown in [3, 3.5 and 3.10] that silting classes are definable torsion classes. From Theorem 3.2.3 we obtain that every silting class is of finite type. This reproves the if-part in the following recent result due to Marks and Šťovíček.

Theorem 3.3.1. *[15, Theorem 6.3] A map σ in* $\mathrm{Mor}(\textit{Proj-}R)$ *is of finite type if and only if the class $\mathcal{D}_\sigma$ is a silting class.*

Remark 3.3.2. The only-if-part of Theorem 3.3.1 could be employed in the proof of our main Theorem 3.4.7, but we will show that each Gabriel filter of finite type gives rise to a silting class directly, by constructing the silting modules explicitly in Construction 3.4.5 and Proposition 3.4.6.

Let us now turn to the dual notion. Following [9], an R-module C is said to be *cosilting* if it admits an injective copresentation $0 \to C \longrightarrow E_0 \xrightarrow{\sigma} E_1$ such that the class $\operatorname{Cogen} C$ of C-cogenerated modules coincides with the class

$$\mathcal{C}_\sigma := \{X \in \text{Mod–}R \mid \operatorname{Hom}_R(X, \sigma) \text{ is surjective}\}.$$

The class $\operatorname{Cogen} C$ is then called a *cosilting class*.

It is shown in [9] that every cosilting module is pure-injective and that cosilting classes are definable torsionfree classes. In fact, there is a duality between the silting classes in Mod-R and certain cosilting classes in R-Mod (see also [9, 3.7 and 3.9]). These cosilting classes will be characterized by the property below.

Definition 3.3.1. For any map $\sigma \in \text{Mor(Proj-R)}$, denote

$$\mathcal{T}_\sigma = \{X \in R\text{-Mod} \mid \sigma \otimes_R X \text{ is injective}\}.$$

Given a map λ between injective left R-modules, we say that the class $\mathcal{C}_\lambda$ (or, the map λ) is of *cofinite type*, if there is a set of maps $\sigma_i \in \text{Mor(proj-R)}, i \in I$, such that $\mathcal{C}_\lambda = \bigcap_{i\in I} \mathcal{T}_{\sigma_i}$.

Let us investigate the duality. Assume that R is a k-algebra over some commutative ring k. Given an R-module M, we denote by M^+ its dual with respect to an injective cogenerator of Mod–k, for example we can take $k = \mathbb{Z}$ and M^+ the character dual of M. To every definable category $\mathcal{C}$ of right (left) R-modules we can now associate a *dual definable category* of left (right) R-modules $\mathcal{C}^\vee$ which is determined by the property that a module M belongs to $\mathcal{C}$ if and only if its dual module $M^+ \in \mathcal{C}^\vee$. This assignment defines a bijection between definable subcategories of Mod-R and R-Mod, which restricts to a bijection between definable torsion classes and definable torsionfree classes and maps tilting classes to cotilting classes of cofinite type, see [7, Propositions 5.4 and 5.7 and Theorem 7.1]. We are now going to prove the analogous result for silting and cosilting classes.

Lemma 3.3.3.

1. *Let $\sigma \in \text{Mor}(\textit{Proj-R})$. Then $\mathcal{T}_\sigma = \mathcal{C}_{\sigma^+}$, and a left R-module X belongs to $\mathcal{C}_{\sigma^+}$ if and only if $X^+ \in \mathcal{D}_\sigma$.*

2. *If $\sigma \in \text{Mor}(\textit{Proj-R})$ has finite type, then $\mathcal{D}_\sigma$ and $\mathcal{C}_{\sigma^+}$ are dual definable categories, and a right R-module Y belongs to $\mathcal{D}_\sigma$ if and only if $Y^+ \in \mathcal{C}_{\sigma^+}$.*

3. *A map λ between injective left R-modules has cofinite type if and only if there is a map $\sigma \in \text{Mor}(\textit{Proj-R})$ of finite type such that $\mathcal{C}_\lambda = \mathcal{C}_{\sigma^+}$.*

Proof. (1),(2) By Hom-$\otimes$-adjunction, for any left R-module X there is a commutative diagram linking the maps $\operatorname{Hom}_R(X, \sigma^+)$, $(\sigma \otimes_R X)^+$ and $\operatorname{Hom}_R(\sigma, X^+)$. This shows that $X \in \mathcal{C}_{\sigma^+}$ if and only if $X^+ \in \mathcal{D}_\sigma$, which in turn means that $(\sigma \otimes_R X)^+$ is surjective, or equivalently, $\sigma \otimes_R X$ is injective.

Furthermore, if σ is of finite type, the definable class $\mathcal{D}_\sigma$ contains a right R-module Y if and only if it contains its double dual Y^{++}, see e.g. [16, 3.4.21]. This implies that $Y \in \mathcal{D}_\sigma$ if and only if $Y^+ \in \mathcal{C}_{\sigma^+}$.

(3) Let $\sigma_i \in \text{Mor(proj-R)}, i \in I$, be a set of maps such that $\mathcal{C}_\lambda = \bigcap_{i\in I} \mathcal{T}_{\sigma_i}$, and let $\sigma = \bigoplus_{i\in I} \sigma_i$. Then $\mathcal{C}_\lambda = \bigcap_{i\in I} \mathcal{T}_{\sigma_i} = \mathcal{T}_\sigma = \mathcal{C}_{\sigma^+}$ by (1), and $\mathcal{D}_\sigma = \bigcap_{i\in I} \mathcal{D}_{\sigma_i}$, so the map σ is of finite type. Conversely, if $\mathcal{C}_\lambda = \mathcal{C}_{\sigma^+}$ for a map σ of finite type, there are maps $\sigma_i \in \text{Mor(proj-R)}, i \in I$, such that $\mathcal{D}_\sigma = \bigcap_{i\in I} \mathcal{D}_{\sigma_i}$, and $\mathcal{C}_\lambda = \mathcal{C}_{\sigma^+} = \bigcap_{i\in I} \mathcal{C}_{\sigma_i^+} = \bigcap_{i\in I} \mathcal{T}_{\sigma_i}$. □

Proposition 3.3.4. *Let $\sigma \in \mathrm{Mor}(\textit{Proj-R})$, and let $T = \mathrm{Coker}\sigma$ be a silting module with respect to σ. Then T^+ is a cosilting left R-module with respect to the injective copresentation σ^+. Moreover,* $\mathrm{Gen}\, T$ *and* $\mathrm{Cogen}\, T^+$ *are dual definable classes, and* $\mathrm{Cogen}\, T^+$ *is a cosilting class of cofinite type.*

Proof. We have to verify $\mathrm{Cogen}\, T^+ = \mathcal{C}_{\sigma^+}$. The class $\mathcal{C}_{\sigma^+}$ is closed under submodules by [9, 3.5], so for the inclusion $\subset$ it is enough to show that $\mathcal{C}_{\sigma^+}$ contains the direct product $(T^+)^\alpha$ for any cardinal α. Notice that the definable class $\mathcal{D}_\sigma$ contains $T^{(\alpha)}$. The claim then follows from Lemma 3.3.3 as $T^{(\alpha)\,+} \cong (T^+)^\alpha$. For the inclusion $\supset$, take $X \in \mathcal{C}_{\sigma^+}$. Then $X^+ \in \mathcal{D}_\sigma = \mathrm{Gen}\, T$, so there is an epimorphism $T^{(\alpha)} \to X^+$ for some cardinal α. This yields a monomorphism $X \hookrightarrow X^{++} \to (T^+)^\alpha$, showing that $X \in \mathrm{Cogen}\, T^+$. □

From Theorem 3.3.1 and Lemma 3.3.3 we obtain

Corollary 3.3.5. *The assignment* $\mathrm{Gen}\, T \mapsto \mathrm{Cogen}\, T^+$ *is a 1-1-correspondence between silting classes in Mod-R and cosilting classes of cofinite type in R-Mod.*

We now give a criterion for a torsionfree definable class to be of cofinite type.

Lemma 3.3.6. *Let $\mathcal{U}$ be a set of finitely presented left R-modules, and let $(\mathcal{T}, \mathcal{F})$ be the torsion pair in R-Mod generated by $\mathcal{U}$, that is, $\mathcal{F} = \{M \in R\text{-Mod} \mid \mathrm{Hom}_R(U, M) = 0$ for all $U \in \mathcal{U}\}$. Then $\mathcal{F}$ is a cosilting class of cofinite type.*

Proof. For every $U \in \mathcal{U}$ we choose a projective presentation $\alpha_U \in \mathrm{Mor}(\text{R-proj})$, and we denote $\sigma_U = {\alpha_U}^*$ and $\sigma = \bigoplus_{U \in \mathcal{U}} \sigma_U$. Then, using that for any $P \in$ R-proj and any $X \in R$-Mod there is a natural isomorphism $P^* \otimes_R X \cong \mathrm{Hom}_R(P, X)$, we see that $\mathcal{F} = \bigcap_{U \in \mathcal{U}} \mathcal{T}_{\sigma_U} = \mathcal{C}_{\sigma^+}$ is a cosilting class of cofinite type. □

Corollary 3.3.7. *If R is a left noetherian ring, the definable torsionfree classes in R-Mod coincide with the cosilting classes of cofinite type, and the assignment* $\mathrm{Gen}\, T \mapsto \mathrm{Cogen}\, T^+$ *defines a 1-1-correspondence between silting classes in Mod-R and cosilting classes in R-Mod. Moreover, the definable torsion classes in Mod-R coincide with the silting classes.*

Proof. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in R-Mod with $\mathcal{F}$ being definable. By [11, Lemma 4.5.2], there is a torsion pair $(\mathcal{U}, \mathcal{V})$ in R-mod such that $\mathcal{T}$ and $\mathcal{F}$ consist of the direct limits of modules in $\mathcal{U}$ and $\mathcal{V}$, respectively, and $\mathcal{F} = \{M \in R\text{-Mod} \mid \mathrm{Hom}_R(U, M) = 0$ for all $U \in \mathcal{U}\}$. Then $\mathcal{F}$ is a cosilting class of cofinite type by Lemma 3.3.6. In particular, every cosilting class is of cofinite type, and Corollary 3.3.5 yields the second statement.

For the last statement, recall from [7, Proposition 5.7] that the bijection in Corollary 3.3.5 extends to a bijection between definable torsion classes and definable torsionfree classes. By the discussion above, if $\mathcal{T}$ is a definable torsion class, its dual definable class $\mathcal{T}^+$ coincides with the dual definable class of a silting class. Now use that the assignment is injective. □

In general, a definable torsion class need not be silting, cf. Example 3.5.1. As for the dual result, it was recently shown in [19] that the definable torsionfree classes over an arbitrary ring are precisely the cosilting classes. But in general these classes are not of cofinite type, see again Example 3.5.1.

3.4 Silting classes over commutative rings

In this section, we classify silting classes over commutative rings, proving that they coincide precisely with the classes of divisibility by sets of finitely generated ideals.

The key to our classification are the following results relating cosilting modules of cofinite type with hereditary torsion pairs. Recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ is *hereditary* if the torsion class $\mathcal{T}$ is closed under submodules, or equivalently, the torsionfree class $\mathcal{F}$ is closed under injective envelopes. Moreover, $(\mathcal{T}, \mathcal{F})$ has *finite type* if $\mathcal{F}$ is closed under direct limits.

First of all, combining Lemma 3.3.6 with [13, Lemma 2.4], we obtain

Corollary 3.4.1. *Let R be a ring. If $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion pair of finite type in R-Mod, then $\mathcal{F}$ is a cosilting class of cofinite type.*

For a commutative ring, also the converse holds true.

Lemma 3.4.2. *Let R be a commutative ring. Let λ be a map between injective R-modules. If $\mathcal{C}_\lambda$ is of cofinite type, then it is a torsionfree class in a hereditary torsion pair of finite type.*

In particular, if R is a commutative noetherian ring, a torsion pair has finite type if and only if it is hereditary.

Proof. By assumption $\mathcal{C}_\lambda = \bigcap_{i \in I} \mathcal{T}_{\sigma_i}$ for a set of maps $\sigma_i \in \mathrm{Mor}(\text{proj-R}), i \in I$. It is then enough to prove the claim for each $\mathcal{T}_{\sigma_i}$, or in other words, we can assume w.l.o.g. that $\mathcal{C}_\lambda = \mathcal{T}_\sigma$ for some $\sigma \in \mathrm{Mor}(\text{proj-R})$. By Lemma 3.3.3, $\mathcal{T}_\sigma = \mathcal{C}_{\sigma^+}$ is a definable category, which is closed under submodules and extensions by [9, Lemma 2.3], so it is a torsion-free class closed under direct limits. It remains to show that is is also closed under injective envelopes.

Let $M \in \mathcal{T}_\sigma$, and consider the exact sequence induced by an injective envelope $0 \to M \xrightarrow{\iota} E(M) \to C \to 0$. Tensoring this sequence with σ yields a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & P_{-1} \otimes_R M \arrow[r, "\sigma \otimes_R M"] \arrow[d, "P_{-1} \otimes_R \iota"] & P_0 \otimes_R M \arrow[d] \\
 & P_{-1} \otimes_R E(M) \arrow[r, "\sigma \otimes_R E(M)"] & P_0 \otimes_R E(M).
\end{tikzcd}$$

The exactness of the columns follows from the projectivity of P_{-1}, P_0, while the exactness of the first row follows by definition of $\mathcal{T}_\sigma$. Since R is commutative, this is a commutative diagram in Mod-R (this is where we need the commutativity of R). First, we claim that the inclusion $P_{-1} \otimes_R \iota$ is an injective envelope of $P_{-1} \otimes_R M$. Indeed, let P be a finitely generated projective such that $P_{-1} \oplus P \simeq R^n$ for some n. Then $(P_{-1} \oplus P) \otimes_R \iota = R^n \otimes_R \iota$ is essential by [2, Proposition 6.17(2)], and since $E(M)^n \simeq R^n \otimes_R E(M)$ is injective, it is an injective envelope of $M^n = R^n \otimes_R M$. As $R^n \otimes_R \iota = (P_{-1} \otimes_R \iota) \oplus (P \otimes_R \iota)$, we conclude that $P_{-1} \otimes_R \iota$ is an injective envelope of $P_{-1} \otimes_R M$.

If $P_{-1} \otimes_R M$ is zero, then its injective envelope $P_{-1} \otimes_R E(M)$ is also zero, and thus $\sigma \otimes_R E(M)$ is injective. Towards a contradiction, suppose that $P_{-1} \otimes_R M$ is non-zero, and the kernel of $\sigma \otimes_R E(M)$ is non-zero. By the essentiality of $P_{-1} \otimes_R \iota$, there is a non-zero element $x \in P_{-1} \otimes_R M$ such that $(\sigma \otimes_R E(M))(P_{-1} \otimes_R \iota)(x) = 0$, which by a simple diagram chasing yields $(\sigma \otimes_R M)(x) = 0$, a contradiction to $\sigma \otimes_R M$ being injective. Therefore, the kernel of $\sigma \otimes_R E(M)$ is zero. In both cases, we showed that $E(M) \in \mathcal{T}_\sigma$.

The last statement follows from Corollary 3.3.7. $\square$

It is well known that hereditary torsion pairs correspond bijectively to Gabriel filters. This will allow to establish a correspondence between silting classes and Gabriel filters. We first review the relevant notions.

Reminder 3.4.3. A filter $\mathcal{G}$ of right ideals of R is a *(right) Gabriel filter*, if the following conditions hold true:

(i) if $I \in \mathcal{G}$, then for any $x \in R$ the ideal $(I : x) = \{r \in R \mid xr \in I\}$ belongs to $\mathcal{G}$,

(ii) if J is a right ideal such that there is $I \in \mathcal{G}$ with $(J : x) \in \mathcal{G}$ for all $x \in I$, then $J \in \mathcal{G}$.

Further, $\mathcal{G}$ is of *finite type* if it has a filter basis consisting of finitely generated ideals. We remark that a filter of ideals of a commutative ring with a filter basis of finitely generated ideals is a Gabriel filter (of finite type) if and only if it is closed under ideal multiplication, see e.g. [13, Lemma 2.3].

Every Gabriel filter $\mathcal{G}$ induces a hereditary torsion pair $(\mathcal{T}_\mathcal{G}, \mathcal{F}_\mathcal{G})$ where

$$\mathcal{F}_\mathcal{G} = \bigcap_{I \in \mathcal{G}} \operatorname{Ker} \operatorname{Hom}_R(R/I, -)$$

is the class of *$\mathcal{G}$-torsionfree modules*. The assignment $\mathcal{G} \mapsto (\mathcal{T}_\mathcal{G}, \mathcal{F}_\mathcal{G})$ defines a bijection between Gabriel filters (of finite type) and hereditary torsion pairs (of finite type), see [18, Chapter VI, Theorem 5.1, and Chapter XIII, Proposition 1.2].

Given a Gabriel filter $\mathcal{G}$, we say that a module $M \in$ Mod-R is *$\mathcal{G}$-divisible* if $MI = M$ for all $I \in \mathcal{G}$. If Div-$\mathcal{G}$ denotes the class of $\mathcal{G}$-divisible modules, then

$$\text{Div-}\mathcal{G} = \bigcap_{I \in \mathcal{G}} \operatorname{Ker}(- \otimes_R R/I).$$

By Hom-$\otimes$ adjunction, a module M is $\mathcal{G}$-divisible if and only if its dual M^+ is $\mathcal{G}$-torsionfree (cf. [18, Chapter VI, Proposition 9.2]). So, if the Gabriel filter $\mathcal{G}$ is of finite type, Div-$\mathcal{G}$ and $\mathcal{F}_\mathcal{G}$ are dual definable classes, and it follows from Corollary 3.4.1 that Div-$\mathcal{G}$ is a silting class.

Again, in the commutative case, we also have the converse.

Proposition 3.4.4. *Let R be a commutative ring, and let $\sigma \in \operatorname{Mor}($Proj-$R)$ be of finite type. Then there is a Gabriel filter of finite type $\mathcal{G}$ such that $\mathcal{D}_\sigma =$ Div-$\mathcal{G}$.*

Proof. By assumption $\mathcal{D}_\sigma = \bigcap_{i \in I} \mathcal{D}_{\sigma_i}$ for a set of maps $\sigma_i \in \operatorname{Mor}(\text{proj-R}), i \in I$. If each $\mathcal{D}_{\sigma_i} =$ Div-$\mathcal{G}_i$ for some Gabriel filter of finite type $\mathcal{G}_i$, then $\mathcal{D}_\sigma = \bigcap_{i \in I}$ Div-$\mathcal{G}_i =$ Div-$\mathcal{G}$, where $\mathcal{G} = \{J \subseteq R \mid I_1 I_2 \cdots I_n \subseteq J$ whenever $I_1, I_2, \ldots, I_n \in \bigcup_{i \in I} \mathcal{G}_i\}$ is the smallest Gabriel filter of finite type containing $\mathcal{G}_i$ for all $i \in I$. So we can again assume w.l.o.g. that $\sigma \in \operatorname{Mor}(\text{proj-R})$.

In Lemma 3.4.2 we showed that $\mathcal{C}_{\sigma^+}$ is a hereditary torsionfree class of finite type. So there is a Gabriel filter $\mathcal{G}$ of finite type such that $\mathcal{C}_{\sigma^+} = \mathcal{F}_{\mathcal{G}}$, which amounts to $\mathcal{D}_\sigma = \text{Div-}\mathcal{G}$. This completes the proof. $\square$

Combining the results above, we obtain the desired classification of silting classes over commutative rings. Here we give a direct proof by providing an explicit construction for a silting module corresponding to a Gabriel filter of finite type. It generalises the construction of a Fuchs-Salce tilting module in [13].

Construction 3.4.5. Let R be a commutative ring and $\mathcal{G}$ a Gabriel filter of finite type. Let $\mathcal{I}$ be the collection of all finitely generated ideals from $\mathcal{G}$. For each $I \in \mathcal{G}$, we fix a finite set of generators $\{x_0^I, x_1^I, \ldots, x_{n_I-1}^I\}$. The projective presentation

$$R^{n_I} \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

induces a projective presentation

$$R \xrightarrow{\sigma_I} R^{n_I} \longrightarrow \mathrm{Tr}(R/I) \longrightarrow 0,$$

where $\sigma_I : R \to R^{n_I}$ is given by $\sigma_I(1) = (x_0^I, x_1^I, \ldots, x_{n_I-1}^I)$ and Tr denotes the Auslander-Bridger transpose of R/I (which is uniquely determined only up to stable equivalence). It is easy to check that $\mathcal{D}_{\sigma_I} = \{M \in \text{Mod-}R \mid M = IM\}$, and thus $\text{Div-}\mathcal{G} = \bigcap_{I \in \mathcal{I}} \mathcal{D}_{\sigma_I}$.

Let now Λ be the set of all finite sequences of pairs (I, k), with $I \in \mathcal{I}$ and $0 \leq k < n_I$. The set includes the empty sequence denoted by w, and it is equipped with the operation of concatenation of sequences, for which we use the symbol $\sqcup$. Let F be the free module on basis Λ, F' the free module on basis $\Lambda \setminus \{w\}$, and K the free module on basis $\Lambda \times \mathcal{I}$.

We define a map $\varphi_{\mathcal{G}} : K \to F$ by its values on the designated basis elements: for any $(\lambda, I) \in \Lambda \times \mathcal{I}$ we set

$$\varphi_{\mathcal{G}}((\lambda, I)) = \lambda - \sum_{k<n_I} x_k^I(\lambda \sqcup (I, k)).$$

We also define a map $\varphi'_{\mathcal{G}} : K \to F'$ by the commutative diagram

$$\begin{tikzcd} K \arrow[r, "\varphi_{\mathcal{G}}"] \arrow[d, equal] & F \arrow[r] \arrow[d, "p"'] & C_{\mathcal{G}} \arrow[r] \arrow[d] & 0 \\ K \arrow[r, "\varphi'_{\mathcal{G}}"] & F' \arrow[r] & C'_{\mathcal{G}} \arrow[r] & 0, \end{tikzcd}$$

where p denotes the canonical projection $F \to F'$ killing the coordinate w.

Let now $P_{-1} = K \oplus K$ and $P_0 = F \oplus F'$ and consider

$$P_{-1} \xrightarrow{\sigma_{\mathcal{G}}} P_0 \longrightarrow T_{\mathcal{G}} \longrightarrow 0,$$

where $\sigma_{\mathcal{G}}$ is the direct sum of the maps $\varphi_{\mathcal{G}}$ and $\varphi'_{\mathcal{G}}$, and $T_{\mathcal{G}} = C_{\mathcal{G}} \oplus C'_{\mathcal{G}}$.

Proposition 3.4.6. *The module $T_{\mathcal{G}}$ is a silting module with respect to the map $\sigma_{\mathcal{G}}$, and* $\mathrm{Gen}\,(T_{\mathcal{G}}) = \text{Div-}\mathcal{G}$.

Proof. We divide the proof into several steps. Let us first fix some notation. Let $A = \sum_{I\in\mathcal{I}} \mathrm{Ann}(I)$. Further, for every ideal $I \in \mathcal{I}$, we define $S_I = \mathrm{Tr}(R/I)\otimes_R R/A$, and we set $\mathcal{S} = \{S_I \mid I \in \mathcal{I}\}$.

Step I: Every $I \in \mathcal{I}$ gives rise to a faithful ideal $(I + A)/A$ in the ring R/A. In other words, every $r \in R$ satisfying $rI \subseteq A$ must belong to A.

To see this, use that I is finitely generated to find ideals $I_1, I_2, \ldots, I_l \in \mathcal{I}$ such that $rI \subseteq \sum_{j=1}^{l} \mathrm{Ann}(I_l)$. Then $r(II_1I_2 \cdots I_l) = 0$, and $r \in \mathrm{Ann}(II_1I_2 \cdots I_l) \subseteq A$.

Step II: An R/A-module M satisfies $\mathrm{Ext}^1_{R/A}(S_I, M) = 0$ if and only if $M = IM$.

Indeed, the map $\sigma_I : R \to R^{n_I}$, $r \mapsto (r\, x_0^I, r\, x_1^I, \ldots, r\, x_{n_I-1}^I)$ induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & R & \xrightarrow{\sigma_I} & R^{n_I} & \longrightarrow & \mathrm{Tr}(R/I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R/A & \xrightarrow{\overline{\sigma_I}} & (R/A)^{n_I} & \longrightarrow & S_I & \longrightarrow & 0
\end{array}$$

because the kernel of $\overline{\sigma_I}$, consisting of the elements $\overline{r} \in R/A$ with $rI \subset A$, is trivial by Step I. It is now an easy observation that $\mathrm{Ext}^1_{R/A}(S_I, M) = 0$ if and only if $M = (I + A)/A \cdot M = IM$.

Step III: Filtration of $C_\mathcal{G}$ and $C'_\mathcal{G}$.

For each $n < \omega$ denote by Λ_n the set of all sequences from Λ of length at most n. Let F_n be the span of Λ_n in F, and let G_n be the $\varphi_\mathcal{G}$-image of the span of $\Lambda_{n-1} \times \mathcal{I}$ in K. For $n = 0$ we have $F_0 = R\omega$, and we set $G_0 = \emptyset$. Let C_n be the span of the image of Λ_n in C, that is, $C_n = F_n/(F_n \cap G)$, where $G = \mathrm{Im}\,\varphi_\mathcal{G}$.

We claim that $F_n \cap G = AF_n + G_n$. For any $\lambda \in \Lambda_n$ and any $I \in \mathcal{I}$, the element $\varphi_\mathcal{G}((\lambda, I)) = \lambda - \sum_{k<n_I} x_k^I (\lambda \sqcup (I, k))$ lies in G. Therefore, by multiplying by any $r \in \mathrm{Ann}(I)$, we obtain $r\lambda \in G$. As clearly $G_n \subseteq F_n \cap G$, we have $AF_n + G_n \subseteq F_n \cap G$.

For the reverse inclusion, let $x \in F_n \cap G$. As $x \in G$, it is of the form

$$x = \sum_{j=1}^{m} r_j \varphi_\mathcal{G}((\lambda_j, I_j)) = \sum_{j=1}^{m} r_j(\lambda_j - \sum_{k<n_{I_j}} x_k^{I_j} (\lambda_j \sqcup (I_j, k)))$$

for some $r_j \in R$, and (λ_j, I_j) pairwise distinct elements of $\Lambda \times \mathcal{I}$. We claim that if the length of some λ_j exceeds $n-1$, then $r_j \in A$. We prove this claim by backward induction on the length of λ_j. If j is such that the length of λ_j is maximal and exceeding n, it is clear from $x \in F_n$ that $r_j \in \mathrm{Ann}(\{x_0^{I_j}, x_1^{I_j}, \ldots, x_{n_{I_j}-1}^{I_j}\}) = \mathrm{Ann}(I_j) \subseteq A$. Suppose now that λ_j is of length $k > n-1$, and that all coefficients r_i such that λ_i has length $> k$ are in A. Then, since $x \in F_n$, the induction premise yields $r_j x_k^{I_j} \subseteq A$ for each $k = 0, 1, \ldots, n_{I^j} - 1$. In other words, $r_j I \subseteq A$. By Step I, this implies that $r_j \in A$ as claimed. We proved that the coefficient r_j is in A for any j such that the length of λ_j exceeds $n - 1$, and thus $x \in AF + G_n$. But since $x \in F_n$, and $AF \cap F_n = AF_n$, we get $x \in AF_n + G_n$ as desired.

It follows that $C_n = F_n/(AF_n + G_n)$. Then $C_0 \simeq R/A$, and $C_{n+1}/C_n \simeq F_{n+1}/(F_n + AF_{n+1} + G_{n+1})$ for any $n \in \omega$. Therefore, $C_{n+1}/C_n \simeq F_{n+1}/(F_n + G_{n+1}) \otimes_R R/A$. The elements $\varphi_\mathcal{G}((\lambda, I))$, where λ is of length n, and $I \in \mathcal{I}$ generate G_{n+1} modulo $F_n \cap G_{n+1}$. We obtain that C_{n+1}/C_n is isomorphic to:

$$\bigoplus_{\lambda\in\Lambda_n\setminus\Lambda_{n-1}} \bigoplus_{I\in\mathcal{I}} (R^{(\lambda\sqcup(I,k)|k<n_I)}/(\sum_{k<n_I} x_k^I(\lambda \sqcup (I, k)))R) \otimes_R R/A \simeq \bigoplus_{\lambda\in\Lambda_n\setminus\Lambda_{n-1}} \bigoplus_{I\in\mathcal{I}} S_I.$$

In particular, $C_{\mathcal{G}}$ is $\{R/A\} \cup \mathcal{S}$-filtered, and the quotient $C_{\mathcal{G}}/C_0$, which is clearly isomorphic to $C'_{\mathcal{G}}$, is $\mathcal{S}$-filtered.

Step IV: We claim that $\operatorname{Gen}(T_{\mathcal{G}}) = \text{Div-}\mathcal{G}$.

Since $C'_{\mathcal{G}} = C_{\mathcal{G}}/C_0$, it is enough to show $\operatorname{Gen}(C_{\mathcal{G}}) = \text{Div-}\mathcal{G}$. In $C_{\mathcal{G}}$, the image of any basis element λ is identified with the linear combination $\sum_{k<n_I} x_k^I(\lambda \sqcup (I,k))$ with coefficients from I, so $C_{\mathcal{G}} \in \text{Div-}\mathcal{G}$. Note that this implies that $C_{\mathcal{G}}$ is an R/A-module. For the other inclusion, let $M \in \text{Div-}\mathcal{G}$. It is clear that $AM = 0$, and therefore there is a surjection $\pi : (C_0)^{(\varkappa)} \simeq (R/A)^{(\varkappa)} \to M$. Since $\operatorname{Ext}^1_{R/A}(S_I, M) = 0$ for every $I \in \mathcal{I}$ by Step II, we have by Step III and by the Eklof Lemma that $\operatorname{Ext}^1_{R/A}(C'_{\mathcal{G}}, M) = 0$, and thus the R/A-homomorphism π can be extended to a (surjective) map $C_{\mathcal{G}}^{(\varkappa)} \to M$, proving the claim.

Step V: The map $\varphi_{\mathcal{G}}$ induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & K & \xrightarrow{\varphi_{\mathcal{G}}} & F & \longrightarrow & C_{\mathcal{G}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K/AK & \xrightarrow{\overline{\varphi_{\mathcal{G}}}} & F/AF & \longrightarrow & C_{\mathcal{G}} & \longrightarrow & 0 \end{array}$$

and the analagous result holds for $\varphi'_{\mathcal{G}}$.

Indeed, the R/A-module $C_{\mathcal{G}}$ is the cokernel of $\overline{\varphi_{\mathcal{G}}}$. Further, since K/AK is a free R/A-module with basis $\Lambda \times \mathcal{I}$, injectivity of $\overline{\varphi_{\mathcal{G}}}$ amounts to showing that the elements $\overline{\varphi_{\mathcal{G}}((\lambda, I))}$ with $(\lambda, I) \in \Lambda \times \mathcal{I}$ form an R/A-linearly independent subset in F/AF. To this end, we prove in next paragraph that for each $n \in \omega$ the elements $\overline{\varphi_{\mathcal{G}}((\lambda, I))}$ where λ has length n form a linearly independent subset in the free R/A-module $F_{n+1}/F_n \otimes_R R/A$ with basis $\Lambda_{n+1} \setminus \Lambda_n$. Then indeed, as $\varphi_{\mathcal{G}}(\Lambda_{n-1} \times \mathcal{I}) \subseteq F_n$ for each $n > 0$, a simple induction argument shows the linear independence of the $\varphi_{\mathcal{G}}$-image of $\Lambda_n \times \mathcal{I}$ for each $n \in \omega$, and thus of the $\varphi_{\mathcal{G}}$-image of the whole basis $\Lambda \times \mathcal{I}$.

For any sequence $\lambda \in \Lambda$ of length n and any $I \in \mathcal{I}$, the image of (λ, I) in $F_{n+1}/F_n \otimes_R R/A$ is equal to $\sum_{k<n_I}(x_k^I + A)(\lambda \sqcup (I,k))$. As these elements are linear combinations of pairwise disjoint subsets of Λ, it is clear that their spans are independent in the free R/A-module $F \otimes_R R/A$ with basis Λ. To prove R/A-linear independency, it remains to show that these elements have zero annihilator over R/A. But that follows from Step I, as $\operatorname{Ann}_{R/A}(\sum_{k<n_I}(x_k^I + A)(\lambda \sqcup (I,k))) = \operatorname{Ann}_{R/A}((I+A)/A) = 0$.

So $\overline{\varphi_{\mathcal{G}}}$ is injective, and the proof of injectivity of $\overline{\varphi'_{\mathcal{G}}}$ is completely analogous.

Step VI: $\mathcal{D}_{\sigma_{\mathcal{G}}} = \text{Div-}\mathcal{G}$.

Let $M \in \mathcal{D}_{\sigma_{\mathcal{G}}}$. We first show that $AM = 0$. For any $m \in M$, define map $f : K \to M$ by setting $f((\lambda, I)) = m$ for each $(\lambda, I) \in \Lambda \times \mathcal{I}$. As $\mathcal{D}_{\sigma_{\mathcal{G}}} \subseteq \mathcal{D}_{\varphi'_{\mathcal{G}}}$, there is a map $g : F' \to M$ such that $f = g\varphi'_{\mathcal{G}}$. But $\varphi'_{\mathcal{G}}((w, I)) = \sum_{k<n_I} x_k^I(I,k)$ is annihilated by $\operatorname{Ann}(I)$. It follows that $\operatorname{Ann}(I)M = 0$ for all $I \in \mathcal{I}$, and thus $AM = 0$. Now, since $M \in \mathcal{D}_{\sigma_{\mathcal{G}}}$ also implies $\operatorname{Ext}^1_R(C'_{\mathcal{G}}, M) = 0$ by Lemma 3.2.1, we can conclude as in Step IV that $M \in \operatorname{Gen} C_{\mathcal{G}} = \text{Div-}\mathcal{G}$.

Conversely, let $M \in \text{Div-}\mathcal{G}$, and let $f : P_{-1} \to M$ be a map. By Step V we have a commutative diagram with exact rows

$$\begin{tikzcd}
 & P_{-1} \arrow[r, "\sigma_{\mathcal{G}}"] \arrow[d, "\pi"'] & P_0 \arrow[r] \arrow[d, "\psi"'] & T_{\mathcal{G}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_{-1}/AP_{-1} \arrow[r, "\overline{\sigma_{\mathcal{G}}}"] & P_0/AP_0 \arrow[r] & T_{\mathcal{G}} \arrow[r] & 0,
\end{tikzcd}$$

where the vertical maps π and ψ are the canonical projections. As M is an R/A-module, the map f can be factorized through π, say $f = f'\pi$. Now $T_{\mathcal{G}}$ is $\{R/A\} \cup \mathcal{S}$-filtered by Step III, so the Eklof Lemma and Step II imply $\operatorname{Ext}^1_{R/A}(T_{\mathcal{G}}, M) = 0$. Therefore, there is a map $h : P_0/AP_0 \to M$ such that $f' = h\,\overline{\sigma_{\mathcal{G}}}$. Then $f = h\psi\sigma_{\mathcal{G}}$, proving that $M \in \mathcal{D}_{\sigma_{\mathcal{G}}}$. □

Theorem 3.4.7. *Let R be a commutative ring. There is a 1-1 correspondence between*

(i) silting classes $\mathcal{D}$ in Mod-R,

(ii) Gabriel filters of finite type $\mathcal{G}$ over R.

The correspondence is given as follows:

$$\Theta : \mathcal{G} \mapsto \text{Div-}\mathcal{G},$$

$$\Xi : \mathcal{D} \mapsto \{I \subseteq R \mid M = IM \textit{ for all } M \in \mathcal{D}\}.$$

Proof. By Proposition 3.4.4 and Proposition 3.4.6, both maps of the correspondence are well defined. By Proposition 3.4.4, it is clear that $\Theta(\Xi(\mathcal{D})) = \mathcal{D}$ for any silting class $\mathcal{D}$. That $\Xi(\Theta(\mathcal{G})) = \mathcal{G}$ for any Gabriel topology of finite type follows from [18, Chapter VI, Theorem 5.1], and by character duality. □

In [17], it was asked whether any tilting torsion pair $(\mathcal{T}, \mathcal{F})$ of finite type is classical (that is, $\mathcal{T}$ is generated by a finitely presented tilting module). The answer turned out to be negative for general rings, but positive for commutative rings ([8]). We remind that for commutative rings, this means that $\mathcal{F}$ is closed under direct limits if and only if $\mathcal{F} = \{0\}$. We conclude this section with a generalization of this phenomenon for silting classes.

Proposition 3.4.8. *Let R be a commutative ring, T a silting R-module, and $(\mathcal{D}, \mathcal{F}) = (\operatorname{Gen}(T), \operatorname{Ker}\operatorname{Hom}_R(T, -))$ the associated torsion pair. The following are equivalent:*

(i) T is projective,

(ii) there is a finitely presented silting R-module generating $\mathcal{D}$,

(iii) $\mathcal{F}$ is closed under direct limits,

(iv) $\mathcal{D} = \operatorname{Gen}(Re)$ for a (central) idempotent $e \in R$.

Proof. Denote $A = \mathrm{Ann}(T) = \mathrm{Ann}(\mathcal{D})$. By [3, Proposition 3.13 and Lemma 3.4], T is a tilting R/A-module. Moreover, it is easy to check that $R/A \in \mathcal{D}$ if and only if $\mathcal{D} = \text{Mod–}R/A$, or equivalently, $\mathcal{D} = \operatorname{Gen} R/A$. In this case, A is an idempotent ideal with $\mathcal{D} = \operatorname{Ker}\operatorname{Hom}_R(A, -) \subset (R/A)^{\perp}$, cf. [4, Proposition 2.5].

(i) → (iv): As T is also projective as an R/A-module, $\mathcal{D} = \operatorname{Ker}\operatorname{Ext}^1_{R/A}(T, -) = \text{Mod–}R/A$. Then $R/A \in \operatorname{Add}(T)$ is a projective R-module. Hence, $R/A = Re$ for an idempotent $e \in R$, and (iv) follows.

(ii) → (iv): Let T' be a finitely presented silting module such that $\operatorname{Gen}(T') = \mathcal{D}$. Then T' is a finitely presented tilting R/A-module, which is projective by [11, Proposition 13.2]). Hence, $\mathcal{D} = \text{Mod–}R/A$, and A is an idempotent ideal. Also, $R/A \in \operatorname{Add}(T')$ is finitely presented, and thus A is finitely generated. It follows from [10, Proposition 1.10(i)] that $A = Rf$ for some idempotent $f \in R$, and thus $R/A = R(1 - f)$, proving (iv).

(iii) → (iv): Consider the (tilting) torsion pair $(\mathcal{D}, \mathcal{F}')$ in $\text{Mod–}R/A$, where $\mathcal{F}' = \operatorname{Ker}\operatorname{Hom}_{R/A}(T, -)$. Then $\mathcal{F}'$ is closed under direct limits, and thus $\mathcal{D} = \text{Mod–}R/A$ by [8] or [13, Theorem 4.6]. In particular, A is an idempotent ideal.

We claim that A is finitely generated. Let us write A as a direct union of its finitely generated subideals, $A = \varinjlim_{j \in J} I_j$. Denote by K_j the ideal such that R/K_j is the torsion-free quotient of R/I_j with respect to the torsion pair $(\mathcal{D}, \mathcal{F})$. Then $K_i \subseteq K_j$ whenever $i \le j \in J$. Since $\mathcal{F}$ is closed under direct limits, we have that $\varinjlim_{j \in J} R/K_j = R/\bigcup_{j \in J} K_j$ is in $\mathcal{F}$, and thus zero, because $R/A \in \mathcal{D}$, and $A \subseteq \bigcup_{j \in J} K_j$. It follows that there is $j \in J$ such that $R = K_j$, and therefore $R/I_j \in \mathcal{D}$. But $\mathcal{D} = \text{Mod–}R/A$, and $I_j \subseteq A$, which forces $I_j = A$.

We now conclude this implication as in (ii) → (iv).

(iv) → (i), (ii), (iii): As $\mathcal{F} = \operatorname{Gen}(R(1 - e))$, condition (iii) is clear. Consider the map $\sigma : R \to Re \oplus Re$ given by the canonical projection of R onto the first direct summand Re. Then $\operatorname{Ker}(\sigma) = R(1 - e)$, and clearly $\mathcal{D}_\sigma = \operatorname{Ker}\operatorname{Hom}_R(R(1 - e), -) = \operatorname{Gen}(Re)$. Hence, $\operatorname{Coker}(\sigma) = Re$ is a silting module generating $\operatorname{Gen}(Re)$. This proves (ii). Finally, $T \in \operatorname{Add} Re$, and thus T is projective. □

The following example shows that, in contrast with tilting modules over commutative rings, we cannot replace “finitely presented” by “finitely generated” in Proposition 3.4.8(ii).

Example 3.4.1. Let k be a field, $\varkappa$ an infinite cardinal, and $R = k^{\varkappa}$. Consider the Gabriel filter $\mathcal{G}$ over R with basis consisting of all principal ideals generated by elements of $k^{\varkappa}$, such that their support is cofinite in $\varkappa$. Let $\mathcal{D} = \text{Div-}\mathcal{G}$ be the associated silting class. Then $A = \mathrm{Ann}(\mathcal{D}) = \sum_{I \in \mathcal{G}} \mathrm{Ann}(I)$ is equal to $k^{(\varkappa)} \subseteq R$. Because $A + I = R$ for any $I \in \mathcal{G}$, we have that $R/A \in \mathcal{D}$, and therefore $\mathcal{D} = \operatorname{Gen} R/A = \operatorname{Ker}\operatorname{Hom}_R(A, -) \subset (R/A)^{\perp}$ (cf. the proof of Proposition 3.4.8). We claim that R/A is a silting module.

For each $a \in \varkappa$, consider the idempotent $e_a \in R$ with a-th component equal to 1, and all other components equal to zero. Taking the direct sum of the split exact sequences $0 \to e_a R \to R \to (1 - e_a)R \to 0$, we obtain a split exact sequence $0 \to A \xrightarrow{\iota} R \xrightarrow{\pi} \bigoplus_{a \in \varkappa}(1 - e_a)R \to 0$, where $\mathcal{D}_\pi = \operatorname{Ker}\operatorname{Hom}_R(A, -) = \mathcal{D} \subset (R/A)^{\perp}$. The map $\sigma = \iota \oplus \pi \in \text{Mor(Proj-R)}$ then satisfies $\operatorname{Coker}(\sigma) = \operatorname{Coker}(\iota) = R/A$, and as ι is monic, $\mathcal{D}_\sigma = (R/A)^{\perp} \cap \mathcal{D}_\pi = \mathcal{D}_\pi = \operatorname{Gen} R/A$. We proved that R/A is silting.

Finally, note that R/A is not finitely presented, and thus not projective.

3.5 Cosilting modules over commutative rings

If R is a commutative noetherian ring, then all Gabriel filters and all hereditary torsion pairs are of finite type, and they correspond bijectively to subsets of $\operatorname{Spec}(R)$ closed under specialization. Recall that a subset $P \subset \operatorname{Spec}(R)$ is *closed under specialization* if $\mathfrak{p} \in P$ implies that all prime ideals $\mathfrak{q} \supset \mathfrak{p}$ belong to P. Such P gives rise to a hereditary torsion pair $(\mathcal{T}(P), \mathcal{F}(P))$ where $\mathcal{F}(P) = \{M \in \text{Mod-}R \mid \operatorname{Hom}_R(R/p, M) = 0 \text{ for all } p \in P\}$, and the assignment $P \mapsto (\mathcal{T}(P), \mathcal{F}(P))$ defines the stated bijection. For details we refer to [18, Chapter VI, §6.6].

Theorem 3.5.1. *If R is a commutative noetherian ring, there are bijections between*

(i) silting classes $\mathcal{D}$ in Mod-R,

(ii) subsets $P \subseteq \operatorname{Spec}(R)$ closed under specialization,

(iii) Gabriel filters $\mathcal{G}$ over R,

(iv) cosilting classes $\mathcal{C}$ in Mod–R.

In particular, every cosilting class is of cofinite type.

Proof. Apply Corollary 3.3.7 and Theorem 3.4.7. □

Next, we provide a construction for a cosilting module cogenerating the $\mathcal{G}$-torsionfree modules for a given Gabriel filter $\mathcal{G}$. It is inspired by the construction of cotilting modules over commutative noetherian rings in [12].

Construction 3.5.2. Let R be commutative, and $\mathcal{G}$ be a Gabriel filter of finite type. Let $(\mathcal{T}_\mathcal{G}, \mathcal{F}_\mathcal{G})$ be the associated hereditary torsion pair from 3.4.3, that is, $\mathcal{F}_\mathcal{G} = \bigcap_{I \in \mathcal{G}} \operatorname{Ker} \operatorname{Hom}_R(R/I, -)$, and $\mathcal{T}_\mathcal{G}$ consists of the modules M for which every element $m \in M$ is annihilated by some $I \in \mathcal{G}$. Let us construct a cosilting module $C_\mathcal{G}$ such that $\operatorname{Cogen}(C_\mathcal{G}) = \mathcal{F}_\mathcal{G}$.

First, since $\mathcal{F}_\mathcal{G}$ is a hereditary torsion-free class, there is an injective module E with $\operatorname{Cogen}(E) = \mathcal{F}_\mathcal{G}$. Indeed, we can put $E = \prod\{E(R/J) \mid R/J \in \mathcal{F}_\mathcal{G}\}$. Then E is injective, $E \in \mathcal{F}_\mathcal{G}$, and any module from $\mathcal{F}_\mathcal{G}$ is easily seen to be cogenerated by E.

Next, we let $E_1 = \prod\{E(R/I) \mid I \in \mathcal{G}\}$. Since $\mathcal{G}$ is of finite type, $\mathcal{F}_\mathcal{G}$ is definable, and thus a precovering class. Let $f : F \to E_1$ be a $\mathcal{F}_\mathcal{G}$-precover of E_1. Since E_1 is injective, we can extend f to a map $\bar{f} : E(F) \to E_1$. As $E(F) \in \mathcal{F}_\mathcal{G}$, the map $\bar{f}$ is also an $\mathcal{F}_\mathcal{G}$-precover of E_1. Then $E_0 = E \oplus E(F)$ is an injective module in $\mathcal{F}_\mathcal{G}$. Denote by $k : K \to E(F)$ the kernel of $\bar{f}$, and consider the following exact sequence:

$$0 \to E \oplus K \xrightarrow{\begin{pmatrix} 1_E & 0 \\ 0 & k \end{pmatrix}} E_0 \xrightarrow{\begin{pmatrix} 0 & \bar{f} \end{pmatrix}} E_1.$$

We claim that $C_{\mathcal{G}} = E \oplus K$ is a cosilting module with respect to the map $\lambda = \begin{pmatrix} 0 & \bar{f} \end{pmatrix}$, and $\mathrm{Cogen}\,(C_{\mathcal{G}}) = \mathcal{F}_{\mathcal{G}}$.

Since $\mathrm{Cogen}\,(E) = \mathcal{F}_{\mathcal{G}}$, and K is isomorphic to a submodule of $E(F) \in \mathcal{F}_{\mathcal{G}}$, we have $\mathrm{Cogen}\,(C_{\mathcal{G}}) = \mathcal{F}_{\mathcal{G}}$. Further, if $M \in \mathcal{F}_{\mathcal{G}}$, then any map $g : M \to E_1$ factors through the $\mathcal{F}_{\mathcal{G}}$-precover $\bar{f}$ of E_1, so there is $h : M \to E(F)$ such that $g = \bar{f}h = \lambda \begin{pmatrix} 0 \\ h \end{pmatrix}$, and thus $M \in \mathcal{C}_{\lambda}$. This proves that $\mathcal{F}_{\mathcal{G}} \subseteq \mathcal{C}_{\lambda}$.

Let now M be an R-module such that the $\mathcal{T}_{\mathcal{G}}$-torsion part M' of M is non-zero. Choose any non-zero cyclic submodule R/I of M'. As necessarily $I \in \mathcal{G}$, there is a non-zero map $g : R/I \to E_1$, which extends to $\bar{g} : M \to E_1$. Suppose that there is $h : M \to E_0$ such that $\bar{g} = \lambda h$. Then $h_{\restriction M'}$ is a non-zero map $M' \to E_0$ with $E_0 \in \mathcal{F}_{\mathcal{G}}$, a contradiction. Therefore, $M \notin \mathcal{C}_{\lambda}$. We have $\mathcal{F}_{\mathcal{G}} = \mathcal{C}_{\lambda}$ as desired.

Corollary 3.5.3. *Let R be a commutative ring. With the notation of Constructions 3.4.5 and 3.5.2, $\{T_{\mathcal{G}} \mid \mathcal{G}$ a Gabriel filter of finite type $\}$ is a set of representatives, up to equivalence, of all silting R-modules, and*

$$\{C_{\mathcal{G}} \mid \mathcal{G} \text{ a Gabriel filter of finite type }\}$$

is a set of representatives, up to equivalence, of all cosilting R-modules of cofinite type.

We close this note with an example of a cosilting module which is not of cofinite type. The same module is also an example for a finendo quasitilting module which is not silting. Recall that all silting modules are finendo quasitilting [3, Proposition 3.10].

Example 3.5.1. Let R be a commutative local ring with a non-zero idempotent maximal ideal $\mathfrak{m}$ (e.g. any valuation domain with non-zero idempotent radical, such as the ring of Puiseux series over a field). We consider the module $R/\mathfrak{m}$.

Since $\mathfrak{m}$ is idempotent, the class $\mathcal{C} = \mathrm{Gen}\,(R/\mathfrak{m}) = \mathrm{Add}\,(R/\mathfrak{m})$ is a torsion class contained in $(R/\mathfrak{m})^{\perp}$. The natural projection $R \to R/\mathfrak{m}$ is easily seen to be a $\mathcal{C}$-preenvelope. The cokernel of this map is zero, and [3, Proposition 3.2] shows that $R/\mathfrak{m}$ is a finendo quasitilting module. On the other hand, $\mathcal{C}$ is not silting by Theorem 3.4.7. Indeed, the only ideal $R/\mathfrak{m}$ is divisible by is R. But $\mathcal{C} \neq \text{Mod-}R$, because $\mathfrak{m} \notin \mathcal{C}$, as $\mathfrak{m}^2 = \mathfrak{m} \neq 0$.

The same class $\mathcal{C}$ is a cosilting class not of cofinite type. Indeed, $\mathcal{C}$ is closed for direct products, and thus it coincides with $\mathrm{Cogen}\,(R/\mathfrak{m})$. We prove that $R/\mathfrak{m}$ is a cosilting module. Let $0 \to R/\mathfrak{m} \to E_0 \xrightarrow{\varphi} E_1$ be the begining of the minimal injective coresolution of $R/\mathfrak{m}$. Define an injective module $E = \prod\{E(R/J) \mid J \subseteq R$ such that $\mathrm{Soc}\, R/J = 0\}$. Let $\sigma : E_0 \to E_1 \oplus E$ be the direct sum of φ and the zero map $0 \to E$. We prove that $\mathcal{C}_{\sigma} = \mathcal{C}$.

Note that the image of any map $f : R/\mathfrak{m} \to E_1 \oplus E$ is contained in E_1 by the definition of E. By the essentiality of the image of φ in E_1, f is actually a map $R/\mathfrak{m} \to \mathrm{Im}\,\varphi$. Since $\mathrm{Ext}^1_R(R/\mathfrak{m}, R/\mathfrak{m}) = 0$ by the idempotency of $\mathfrak{m}$, we have that $R/\mathfrak{m} \in \mathcal{C}_{\sigma}$, and thus $\mathcal{C} \subseteq \mathcal{C}_{\sigma}$.

Let now $M \in \text{Mod-}R$ be such that $\mathfrak{m}M \neq 0$. Then M contains a cyclic submodule R/I with $\mathfrak{m} \not\subseteq I$. Using injectivity, it is enough to show that $R/I \notin \mathcal{C}_{\sigma}$. If $\mathrm{Soc}\, R/I = 0$, then R/I injects into E, and this injection clearly cannot be factorized through σ. If $\mathrm{Soc}\, R/I \neq 0$, let J be an ideal such that $(R/I)/\mathrm{Soc}\, R/I \simeq R/J$.

Then $J \neq R$, because in such case $\operatorname{Soc} R/I = R/I$, implying that $\operatorname{Ann}(R/I) = \mathfrak{m}$, and thus $R/I = R/\mathfrak{m}$, which is not the case. If $\operatorname{Soc} R/J \neq 0$, the full preimage of this socle in R/I would be a non-trivial extension of two semisimple modules, which does not exist by idempotency of $\mathfrak{m}$. Hence, $\operatorname{Soc} R/J = 0$, and the composition of the projection $R/I \to R/J$ with inclusion $R/J \to E$ is a non-zero map $R/I \to E$. Again, this map cannot be factorized through σ. Hence, $R/I \notin \mathcal{C}_\sigma$, and $\mathcal{C}_\sigma = \operatorname{Cogen}(R/\mathfrak{m})$.

Finally, the class $\mathcal{C}$ is not of cofinite type. Indeed, the only injective the class $\mathcal{C}$ contains is zero, and thus it is not of cofinite type by Lemma 3.4.2.

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Tilting classes over commutative rings

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Preprint: arXiv:1701.05534

Abstract. We classify all tilting classes over an arbitrary commutative ring via certain sequences of Thomason subsets of the spectrum, generalizing the classification for noetherian commutative rings by Angeleri-Pospíšil-Šťovíček-Trlifaj. We show that the n-tilting classes can be equivalently expressed as classes of all modules vanishing in the first n degrees of one of the following homology theories arising from a finitely generated ideal: $\mathrm{Tor}_*(R/I, -)$, Koszul homology, Čech homology, or local homology (even though in general none of those theories coincide). Cofinite-type n-cotilting classes are described by vanishing of the corresponding cohomology theories. For any cotilting class of cofinite type, we also construct a corresponding cotilting module, generalizing the construction of Šťovíček-Trlifaj-Herbera. Finally, we characterize cotilting classes of cofinite type amongst the general ones, and construct new examples of n-cotilting classes *not* of cofinite type, which are in a sense hard to tell apart from those of cofinite type.

Chapter 4

Tilting classes over commutative rings

4.1 Introduction

Infinitely generated tilting and cotilting modules were introduced in [15, 4] about two decades ago as a formal generalization of Miyashita tilting modules [33]. The main motivation for studying Miyashita tilting modules is that they represent equivalences of derived categories. In the last few years, it became clear that infinitely generated modules represent derived equivalences as well (see [10, 11, 44, 20, 38]), but also that there is more than that.

In the realm of commutative noetherian rings, it was shown [6] that tilting modules have a very close relation to the underlying geometry of the corresponding affine schemes. In fact, a full classification of tilting modules up to additive equivalence in terms of geometric data was obtained there. From a wider perspective, this was explained by Angeleri and Saorín [7] who exhibited a direct relation of the dual cotilting modules to t-structures of the derived category, and the resulting t-structures were known to have a similar classification from [2]. The outcome was that in the commutative noetherian setting, cotilting modules represent a class of t-structures (namely those, which are compactly generated and induce a derived equivalence to the heart).

The motivating question of the present work is, what remains true *for commutative rings which are not necessarily noetherian*. A posteriori, the general setting forced us to look for more conceptual proofs and gave more insight in the problem even for noetherian rings. Here is a short overview of highlights of the paper:

1. For a commutative ring R, we give a full classification of tilting R-modules up to additive equivalence in terms of filtrations of Thomason subsets of $\operatorname{Spec} R$ (Theorem 4.6.2).

2. We obtain more insight in R-modules which are perfect in the derived category as well as in resolving subcategories formed by such modules, in that every such resolving subcategory is generated by syzygies of Koszul complexes (this follows from Theorem 4.6.2(iv) combined with Proposition 4.5.12).

3. We observe that tilting modules up to additive equivalence are also classified by vanishing of the derived completion functors as well as by vanishing of the Čech homology (Theorem 4.7.7).

4. We give a construction of dual cotilting modules to tilting modules (Theorem 4.8.7) and present some intriguing examples in the last section.

A starting point for the classification in this paper (as well as for its predecessor [26] which focused on (co)tilting modules of homological dimensions at most one) is the 1997 paper of Thomason [43]. There he generalized Neeman-Hopkins's classification of thick subcategories of the derived category of perfect complexes over a commutative noetherian ring to a general commutative ring. The parametrizing family for this classification is the set of all sets of the spectrum of the ring, which are open with respect to the Hochster dual of the usual Zariski topology. If the ring is noetherian, these *Thomason* sets coincide precisely with the specialization closed subsets (that is, upper subsets of $\operatorname{Spec} R$ with respect to set-theoretic inclusion). It turns out that the main classification theorem for tilting modules and classes induced by them in [6] (but not its proof) remains valid in the non-noetherian setting provided that we simply change 'specialization closed set' to 'Thomason set' in the statement.

As was the case in the noetherian case, it is convenient to carry out the main steps of the proof in the dual setting of cotilting modules and cotilting classes induced by them first, and then transfer the results via elementary duality. The reason why the dual setting is more graspable seems to lie in the fact that, over a commutative ring, the cotilting classes dual to a tilting class are closed under injective envelopes (Proposition 4.5.5). In homological dimension one, this already leads to the well-understood theory of hereditary torsion pairs, as explained in [26]. We do not know of any analogous closure property for tilting classes.

In other respects, our approach differs considerably from [6]. For example, we cannot use the Matlis theory of injectives, the classical theory of associated primes or the Auslander-Bridger transpose anymore. Instead, we use the classification of hereditary torsion pairs of finite type from [26] and prove directly that a cotilting class is described cosyzygy-wise by a sequence of such torsion pairs (Lemma 4.5.8).

Another problem we have to tackle is that tilting classes bijectively correspond to resolving subcategories of modules having finite projective resolutions by finitely generated projectives (in the sense of Theorem 4.4.2). The description of such resolving subcategories in the noetherian case was obtained, independently of [6], by Dao and Takahashi [17]. It turns out that a sufficiently rich supply of R-modules with such a finite resolution, for any commutative ring R, comes from tails of classical Koszul complexes associated with finite sets of elements of R. The key point here is to understand when exactly such tails are exact, i.e. when high enough Koszul homologies of R vanish.

Apart from the homological description of tilting classes which was obtained for noetherian rings in [6], i.e. in terms of vanishing of certain degrees of homology $\mathrm{Tor}_*(R/I, -)$, we also obtain another one in terms of vanishing of the Koszul homology. We proceed further here to show that two other homology theories also fit in this place—the local and the Čech homology associated to a finitely generated ideal I. The interesting point is that, unless the ring is noetherian, these two homologies need not be isomorphic. However, the vanishing of the first n homologies of a module is always equivalent for all of the four homology theories in play. The analogous result holds for the dual setting too for more classical local and Čech cohomologies.

The paper is organized as follows. Section 4.2 gathers the required results about hereditary torsion pairs of commutative rings, focusing on those of finite type. In Section 4.3 we prove the key Proposition 4.3.9 which shows that vanishing of $\mathrm{Ext}^i_R(R/I, -)$ for $i = 0, 1, \dots, n-1$ is equivalent to vanishing of the corresponding Koszul cohomology for any commutative ring. We link this vanishing property with the notion of vaguely associated prime ideal, yielding results which are analogous to characterizations of the grade of a finitely generated module over a noetherian ring. The fourth section recalls the fundamentals of the theory of large tilting modules over an arbitrary ring. These three sections prepare the ground for the core Section 4.5, in which we classify the n-cotilting classes of cofinite type over a commutative ring. These results are then reformulated and translated for the tilting side of the story in Section 4.6. The aforementioned connection with the derived functors of torsion and completion, as well as the Čech (co)homology, is explained in Section 4.7. In the following Section 4.8, we show how a cotilting module associated to any cotilting class of cofinite type over a commutative ring can be constructed, building on the idea from [42]. In the final Section 4.9, we characterize the cofinite-type cotilting classes amongst the general ones, and we construct new examples of n-cotilting classes which are not of cofinite type, but which are in a sense difficult to tell apart from cofinite type cotilting classes.

4.2 Torsion pairs over commutative rings

In this section we give a classification of hereditary torsion pairs of finite type over commutative rings by Thomason subsets of the Zariski spectrum. This will be a key tool further in the paper. The material is probably known or not difficult to see for experts and almost all the fragments of the classification are present in the literature, but we have not been able to find a convenient reference.

Regarding our notation, R will always stand for an associative and unital ring and Mod-R for the category of right R-modules. We will always assume that R is also commutative unless stated otherwise. If $I \subseteq R$ is an ideal, we denote by

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subseteq \mathfrak{p}\}$$

the corresponding Zariski closed set. We begin our discussion with the so-called Hochster duality of spectral topological spaces and Thomason sets.

4.2.1 Spectral spaces, Hochster duality, and Thomason sets

The set of Thomason subsets of $\operatorname{Spec} R$ was used as an index set in Thomason's classification [43] of thick subcategories of the category of perfect complexes. Although Thomason sets already appear in the work of Hochster [25], their name is customary nowadays since they were used in connection with other classification problems (see e.g. [29] and the references therein) and, in particular, with tilting theory in [26]. Let us start the discussion with a definition, which is relatively simple:

Definition 4.2.1. Given a commutative ring R, a subset $X \subseteq \operatorname{Spec} R$ is a *Thomason set* if it can be expressed as a union of complements of quasi-compact Zariski open sets. That is, there exists a collection $\mathcal{G}$ of finitely generated ideals of R such that

$$X = \bigcup_{I \in \mathcal{G}} V(I),$$

where $U_I = \operatorname{Spec} R \setminus V(I)$ are the quasi-compact Zariski open sets.

To put this into the right context, one should note that Thomason sets define a topology on $\operatorname{Spec} R$ for which they are the open sets. This observation comes from Hochster's [25, Proposition 8] and, following [29, §1], it can be also explained as follows. First, $\operatorname{Spec} R$ with the Zariski topology is a so-called spectral space.

Definition 4.2.2. A quasi-compact topological space X is *spectral* (or *coherent* in the terminology of [27, §II.3.4, p. 65]) if

1. the class of quasi-compact open sets is closed under intersections and forms an open basis for the topology, and

2. every irreducible closed set is the closure of a unique point.

In fact, Hochster [25] proved that spectral spaces are up to homeomorphism precisely the ones of the form $\operatorname{Spec} R$ for a commutative ring R. If X is a spectral topological space, the collection of quasi-compact open sets with the operations of the set-theoretic union and intersection is a distributive lattice. In fact, X is fully determined by this distributive lattice and every distributive lattice arises like that. Recall that an ideal I in a lattice L is a prime ideal if $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

Proposition 4.2.3 ([27, Corollaries II.1.7(i) and II.3.4])**.** *There is a bijective correspondence between*

1. *homeomorphism classes of spectral spaces X, and*

2. *isomorphism classes of distributive lattices L,*

given as follows. To a space X we assign the lattice of quasi-compact open sets. On the other hand, given L, the underlying set of X is the set of all lattice prime ideals $\mathfrak{p} \subseteq L$, and the open basis of the topology is formed by the quasi-compact sets $U_x = \{\mathfrak{p} \in X \mid x \notin \mathfrak{p}\}$, $x \in L$ (this is a lattice theoretic version of the Zariski topology).

Remark 4.2.4. The lattice $\Omega(X)$ of all open sets in a topological space X is always a distributive lattice. If X is spectral and $L \subseteq \Omega(X)$ is the sublattice of all quasi-compact open sets, then $\Omega(X)$ is isomorphic to the lattice of all ideals L (see the proof [27, Proposition II.3.2]).

Now, the opposite lattice of L^{op} of a distributive lattice L is again distributive. Since the complement of a prime ideal in L is a prime filter in L and hence a prime ideal in L^{op} (see [27, Prop. I.2.2]), the space corresponding to L^{op} has (up to this canonical identification) the same underlying set as the one corresponding to L. The topology is different, however. Starting with a spectral space X, the dual topology will have as an open basis precisely the complements of the original quasi-compact open sets. The resulting space is called the *Hochster dual* of X.

If $X = \operatorname{Spec} R$ with the Zariski topology, the open sets in the dual topology are precisely the Thomason sets. The following immediate consequence of the present discussion will be useful to us.

Corollary 4.2.5. *Let R be a commutative ring and consider $X = \operatorname{Spec} R$ with the Zariski topology. Then there is a bijective correspondence between*

1. *Thomason subsets of* $\operatorname{Spec} R$*, and*
2. *filters in the lattice of quasi-compact open sets of X.*

Given a Thomason set $X \subseteq \operatorname{Spec} R$, we assign to it the filter of quasi-compact open sets $\{U \mid \operatorname{Spec} R \setminus U \subseteq X\}$. Conversely, given a filter $\mathcal{F}$, we assign to it $X = \bigcup_{U \in \mathcal{F}} (\operatorname{Spec} R \setminus U)$.

4.2.2 Hereditary torsion pairs of finite type

A torsion pair in Mod-R is a pair of classes $(\mathcal{T}, \mathcal{F})$ of R-modules such that $\operatorname{Hom}_R(T, F) = 0$ for each $T \in \mathcal{T}$ and $F \in \mathcal{F}$ and for each $X \in$ Mod-R there exists a short exact sequence $0 \to T \to X \to F \to 0$. This short exact sequence is unique up to a unique isomorphism. The class $\mathcal{T}$ is called a *torsion class* and torsion classes of a torsion pair $\mathcal{T}$ are characterized by the closure properties—they are closed under coproducts, extensions and factor modules. Dually, *torsion-free classes* $\mathcal{F}$ are characterized as those being closed under products, extensions and submodules.

A torsion pair is *hereditary* if $\mathcal{T}$ is closed under submodules. Equivalently, $\mathcal{F}$ is closed under injective envelopes, [41, §VI.3].

We will be mostly interested in the case when the torsion class is generated by a set of finitely presented modules. That is, $\mathcal{F} = \operatorname{Ker} \operatorname{Hom}_R(\mathcal{T}_0, -)$ for a set of finitely presented modules $\mathcal{T}_0$. Although all the concepts can be defined over any ring, commutative or not, the following is special in the commutative case.

Proposition 4.2.6. *Let R be a commutative ring and $(\mathcal{T}, \mathcal{F})$ be a torsion pair such that $\mathcal{F} = \operatorname{Ker} \operatorname{Hom}_R(\mathcal{T}_0, -)$ for a set of finitely presented modules $\mathcal{T}_0$. Then $(\mathcal{T}, \mathcal{F})$ is hereditary.*

Proof. We refer to (the proof of) [5, Lemma 4.2]. $\square$

A hereditary torsion pair is determined by the class of cyclic torsion modules, or equivalently by the set of ideals I for which R/I is torsion, [41, Proposition VI.3.6]. Such a set of ideals corresponding to a hereditary torsion pair is called a *Gabriel topology.* We refer to [41, §VI.5] for the general definition of a Gabriel topology, while a much simpler special case will be discussed below in Lemma 4.2.9.

Proposition 4.2.7. *Let R be a commutative ring. There is a bijective correspondence between hereditary torsion pairs $(\mathcal{T}, \mathcal{F})$ in* Mod-R *and Gabriel topologies $\mathcal{G}$ on R. Moreover, the following are equivalent for a hereditary torsion pair $(\mathcal{T}, \mathcal{F})$:*

1. *There is a set $\mathcal{T}_0$ of finitely presented modules with $\mathcal{F} = \operatorname{Ker} \operatorname{Hom}_R(\mathcal{T}_0, -)$.*
2. *$\mathcal{F}$ is closed under direct limits.*
3. *The corresponding Gabriel topology $\mathcal{G}$ has a basis of finitely generated ideals (i.e. each ideal in $\mathcal{G}$ contains a finitely generated ideal in $\mathcal{G}$).*

Proof. We refer to [41, §VI.6] for the bijective correspondence. The second part has been proved in [26, Lemma 2.4]. ☐

Definition 4.2.8. The torsion pairs satisfying the conditions (1)–(3) above will be called hereditary torsion pairs *of finite type.*

If $\mathcal{G}$ is a Gabriel topology on R, we denote by $\mathcal{G}^f$ the collection of all finitely generated ideals in $\mathcal{G}$. It is always closed under products of ideals (i.e. if $I_1, I_2 \in \mathcal{G}$, then $I_1 {\cdot} I_2 \in \mathcal{G}$) and finitely generated overideals. If Propositions 4.2.7(3) holds, $\mathcal{G}^f$ completely determines $\mathcal{G}$ and the two latter closure properties in fact completely characterize such Gabriel topologies.

Lemma 4.2.9 ([26, Lemma 2.3])**.** *Let R be a commutative ring. Then a filter $\mathcal{G}$ of ideals of R with a basis of finitely generated ideals is a Gabriel topology if and only if it is closed under products of ideals.*

The following correspondence is a consequence of standard commutative algebra.

Lemma 4.2.10. *For a commutative ring R, there is a bijective correspondence between*

1. *Gabriel topologies with a basis of finitely generated ideals, and*
2. *filters of the lattice of quasi-compact Zariski open subsets of* $\operatorname{Spec} R$*.*

Proof. Quasi-compact Zariski open sets are precisely those of the form $U_I = \operatorname{Spec} R \setminus V(I)$ for a finitely generated ideal $I \subseteq R$. Moreover, $U_I \subseteq U_{I'}$ if and only if $V(I) \supseteq V(I')$ if and only if

$$I \subseteq \bigcap_{\mathfrak{p} \supseteq I'} \mathfrak{p} = \sqrt{I'}$$

if and only if $I^n \subseteq I'$ for some $n \geq 1$. Since also $U_{I_1 \cdot I_2} = U_{I_1} \cap U_{I_2}$, the correspondence which assigns to a Gabriel topology $\mathcal{G}$ as in (1) the filter $\{U_I \mid I \in \mathcal{G}\}$ of the lattice of quasi-compact open sets is a bijective. ☐

If we combine the discussion above with Corollary 4.2.5, we obtain the following parametrization of hereditary torsion pairs of finite type (see [5, 26] for closely related results and compare also to [41, Proposition VI.6.15]). Here, if $M \in$ Mod-R, $\operatorname{Supp}(M) = \{\mathfrak{p} \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \neq 0\}$ denotes as usual the support of M.

Proposition 4.2.11. *Given a commutative ring R, there is a bijective correspondence between*

1. *hereditary torsion pairs $(\mathcal{T}, \mathcal{F})$ of finite type, and*
2. *Thomason subsets of* $\operatorname{Spec} R$.

Given a torsion pair $(\mathcal{T}, \mathcal{F})$, we assign to it the subset $X = \bigcup_{T \in \mathcal{T}} \operatorname{Supp}(T)$. Conversely, if X is a Thomason set, we put $\mathcal{T} = \{T \in$ Mod-R $\mid \operatorname{Supp}(T) \subseteq X\}$.

Proof. The fact that there is a bijective correspondence between (1) and (2) follows by combining Proposition 4.2.7, Lemma 4.2.10 and Corollary 4.2.5. The particular correspondence given by the three statements is rather explicit. To $(\mathcal{T}, \mathcal{F})$ we assign the Thomason set X of all prime ideals $\mathfrak{p} \in \operatorname{Spec} R$ for which there exists a finitely generated ideal $I \subseteq \mathfrak{p}$ with $R/I \in \mathcal{T}$. Conversely, to a Thomason set X we assign the unique hereditary torsion pair of finite type such that, given a finitely generated ideal I, R/I is torsion precisely when $V(I) \subseteq X$.

Suppose now that $(\mathcal{T}, \mathcal{F})$ is of finite type and X is the corresponding Thomason set. Since every object in $\mathcal{T}$ is an epimorphic image of a coproduct of cyclic modules contained in $\mathcal{T}$, we have $X = \bigcup_{R/I \in \mathcal{T}} \operatorname{Supp}(R/I) = \bigcup_{T \in \mathcal{T}} \operatorname{Supp}(T)$, as required. Next let $\mathcal{T}' = \{T \in$ Mod-R $\mid \operatorname{Supp}(T) \subseteq X\}$. This is clearly a hereditary torsion class and, given a finitely generated ideal $I \subseteq R$, we have $R/I \in \mathcal{T}'$ if and only if $\operatorname{Supp} R/I \subseteq X$. As $\operatorname{Supp} R/I = V(I)$ in this case, $\mathcal{T}'$ is also sent to X under the bijective correspondence, and hence $\mathcal{T} = \mathcal{T}'$. □

We conclude the discussion with a description of the torsion-free classes under the correspondence from Proposition 4.2.11. This was important in the classification of tilting classes in the noetherian case [6] and the current version comes from [26].

Definition 4.2.12 ([26, §3.2])**.** Let M be a module over a commutative ring. A prime ideal $\mathfrak{p}$ is *vaguely associated* to M if the smallest class of modules containing M and closed under submodules and direct limits contains $R/\mathfrak{p}$. The set of primes vaguely associated to M is denoted by $\operatorname{VAss}(M)$.

If R is in addition noetherian, $\operatorname{VAss}(M)$ coincides with the set of usual associated prime ideals by [26, Lemma 3.8]. The following proposition then generalizes [6, Proposition 2.3(iii)].

Proposition 4.2.13. *Let R be a commutative ring and $(\mathcal{T}, \mathcal{F})$ a hereditary torsion pair of finite type. If X is the Thomason set assigned to the torsion pair by Proposition 4.2.11, then*

$$\mathcal{F} = \{F \in \text{Mod-R} \mid \operatorname{VAss}(M) \cap X = \emptyset\}.$$

Proof. This has been proved in [26]. Namely, given $\mathfrak{p} \in \operatorname{Spec} R$ and a finitely generated R-module M, then $\operatorname{Hom}_R(M, R/\mathfrak{p}) \neq 0$ if and only if $\mathfrak{p} \in \operatorname{Supp}(M)$ by [26, Lemma 3.10]. Then necessarily $\operatorname{VAss}(M) \cap X = \emptyset$ for each $F \in \mathcal{F}$. Conversely, suppose that $F \notin \mathcal{F}$. Then there is an embedding $i\colon R/J \to F$ with $R/J \in \mathcal{T}$. By the proof of [26, Lemma 3.2] there exists $\mathfrak{p} \in \operatorname{VAss}(M)$ such that $\mathfrak{p} \supseteq I$. That is, $\mathfrak{p} \in \operatorname{VAss}(M) \cap X$. □

4.3 Generalized grade of a module

A very important concept in homological algebra for modules over commutative noetherian rings is the one of a regular sequence and of the grade of a module. The maximal length of a regular sequence in a given ideal has various homological characterizations. Appropriate forms of these characterizations still remain equivalent over non-noetherian commutative rings which will be useful for us. We shall give details in this section.

We shall use the following notation here. Given $M \in$ Mod-R and $i \geq 0$, we denote by $\Omega^i(M)$ the i-th syzygy of M (uniquely determined only up to projective equivalence). If $i < 0$, we let $\Omega^i M$ stand for the minimal $|i|$-th cosyzygy of M (determined uniquely as a module).

4.3.1 Derived categories and truncation of complexes

In this section it will also be useful to argue using the derived category $\mathbf{D}(\text{Mod-R})$ of Mod-R; see for instance [28, Chapter 13]. The suspension functor will be denoted by

$$\Sigma\colon \mathbf{D}(\text{Mod-R}) \to \mathbf{D}(\text{Mod-R})$$

and we will typically use the homological indexing of components of complexes:

$$X\colon \qquad \cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \to \cdots$$

In this context, we will also use the homological truncation functors with respect to the (suspensions of the) canonical t-structure on $\mathbf{D}(\text{Mod-R})$ (see [13, Examples 1.3.22] or [28, §12.3]). We shall slightly adapt our notation for it to be compatible with our homological indexing. Thus, given a complex $X \in \mathbf{D}(\text{Mod-R})$ and $n \in \mathbb{Z}$, we shall denote by $X_{\geq n}$ the complex

$$X_{\geq n}\colon \qquad \cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} \operatorname{Im} d_n \to 0 \to \cdots$$

The subcomplex inclusion $i\colon X_{\geq n}\colon \to X$ (when we consider the complexes in the usual category of complexes $\mathbf{C}(\text{Mod-R})$) induces an isomorphism on the k-th homology for each $k \geq n$ and $H_k(X_{\geq n}) = 0$ for all $k < n$. Dually, the projection morphism $p\colon X \to X/X_{\geq n}$ induces an isomorphism on the k-th homology for all $k < n$ and $H_k(X/X_{\geq n}) = 0$ whenever $k \geq n$.

In fact, $(X_{\geq n} \mid n \in \mathbb{Z})$ yields a filtration of X in $\mathbf{C}(\text{Mod-R})$. Since the homologies of $X_{\geq n}/X_{\geq n+1}$ in degrees different from n vanish, we have for each $n \in \mathbb{Z}$ a distinguished triangle

$$X_{\geq n+1} \to X_{\geq n} \to \Sigma^n H_n(X) \to \Sigma X_{\geq n+1}. \tag{4.1}$$

in $\mathbf{D}(\text{Mod-R})$. This observation, which formalizes how a complex can be built by extension from its homology modules will be especially useful in §4.5.

4.3.2 Koszul complexes and Koszul (co)homology

A particularly useful class of complexes here will be the so-called Koszul complexes. Here R will stand for a commutative ring.

Definition 4.3.1 ([14, §1.6], [35, §8.2])**.** Given $x \in R$, we define the *Koszul complex* with respect to x, denoted $K_\bullet(x)$, as follows

$$0 \to R \xrightarrow{-\cdot x} R \to 0,$$

by which we mean a complex concentrated in degree 1 and 0, with the only non-zero map $R \to R$ being the multiplication by x. Here we use the homological indexing (i.e. the differential has degree -1).

More generally, given a finite sequence $\mathbf{x} = (x_1, x_2, \dots, x_n)$ of elements of R, we define the complex $K_\bullet(\mathbf{x})$ as the tensor product $\bigotimes_{i=1}^n K_\bullet(x_i)$ of complexes of R-modules.

In particular, $K_\bullet(\mathbf{x})$ is a complex of finitely generated free R-modules concentrated in degrees 0 to n. Given an arbitrary module, we can define the corresponding Koszul homology and cohomology.

Definition 4.3.2. Given a finite sequence $\mathbf{x} = (x_1, x_2, \dots, x_n)$ of elements of R, a module $M \in$ Mod-R, and $i \in \mathbb{Z}$, the *i-th Koszul homology* of $\mathbf{x}$ with coefficients in M is defined as

$$H_i(\mathbf{x}; M) = H_i\Big(K_\bullet(\mathbf{x}) \otimes_R M\Big).$$

Similarly, the *i-th Koszul cohomology* of $\mathbf{x}$ with coefficients in M is defined as

$$H^i(\mathbf{x}; M) = H^i\Big(\operatorname{Hom}_R(K_\bullet(\mathbf{x}), M)\Big).$$

Remark 4.3.3. The Koszul cohomology has a particularly easy interpretation in the derived category $\mathbf{D}$(Mod-R). Namely, we have

$$H^i(\mathbf{x}; M) \cong \operatorname{Hom}_{\mathbf{D}(\text{Mod-R})}(K_\bullet(\mathbf{x}), \Sigma^i M).$$

Typically we will start with a finitely generated ideal $I = (x_1, \dots, x_n)$ of R and we will consider the Koszul complex or homology or cohomology of $\mathbf{x} = (x_1, x_2, \dots, x_n)$. These notions are *not* invariant under change of the generating set of I—see [14, Proposition 1.6.21] for a precise discussion on how the complex changes. However, the following consequence of this discussion will be crucial for us and will make the particular choice of a finite generating set of I unimportant.

Proposition 4.3.4. *Let R be a commutative ring, I be a finitely generated ideal and let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_m)$ be two systems of generators of I. Given any integer $\ell \in \mathbb{Z}$, one has $H^i(\mathbf{x}; M) = 0$ for all $i \le \ell$ if and only if $H^i(\mathbf{y}; M) = 0$ for all $i \le \ell$.*

Proof. This is an immediate consequence of [14, Corollary 1.6.22] and [14, Corollary 1.6.10(d)]. □

Hence, we introduce the following slightly abused notation.

Notation 4.3.5. Given a commutative ring R and a finitely generated ideal I, we always fix once and for ever a system of generators $\mathbf{x} = (x_1, x_2, \dots, x_n)$. We then say that $K_\bullet(\mathbf{x})$ is the *Koszul complex* of the ideal I and denote it by $K_\bullet(I)$.

Similarly for $M \in$ Mod-R and $i \in \mathbb{Z}$ we define the *i-th Koszul homology* of I with coefficients in M as $H_i(I; M) = H_i(\mathbf{x}; M)$, and similarly for the *i-the Koszul cohomology* $H^i(I; M) = H^i(\mathbf{x}; M)$.

The following two well-known properties of Koszul complexes which will be important in our application.

Lemma 4.3.6. *Let I be a finitely generated ideal. Then:*

(i) $H_0(K_\bullet(I)) = R/I$,

(ii) $I \subseteq \operatorname{Ann}\Big(H_j(K_\bullet(I)\Big)$ *for all* $k = 0, 1, \dots, n$.

Proof. See [35, p. 360, (8.2.7) and p. 364, Theorem 3] or [14, Prop. 1.6.5(b)]. □

In fact, the latter lemma allows us to relate the Koszul cohomology of I to ordinary Ext-groups. Namely, we have a map of complexes

$$q\colon K_\bullet(I) \to K_\bullet(I)/K_\bullet(I)_{\geq 1} \cong R/I, \tag{4.2}$$

and if M is an R-module, we can apply $\operatorname{Hom}_{\mathbf{D}(\text{Mod-R})}(-, \Sigma^i M)$ to obtain a map

$$q^i_M\colon \operatorname{Ext}^i_R(R/I, M) \to H^i(I; M). \tag{4.3}$$

This map is natural in M and compares the Ext-group to the Koszul cohomology with coefficients in M (see also [14, Proposition 1.6.9]). It is easy to see that q^0_M is always an isomorphism, but for $i > 0$ the relation is more complicated and will be studied in the next subsection. To get a quick impression of the potential difficulties, note that $H^i(I; M)$ always commutes with direct limits, while $\operatorname{Ext}^i_R(R/I, -)$ need not if R is not a coherent ring.

4.3.3 More on the relation between Koszul cohomology and Ext groups

Our strategy will be to try to quantify the difference between $K_\bullet(I)$ and R/I in the derived category. We start with an easy lemma.

Lemma 4.3.7. *Let I be a finitely generated ideal. Suppose that M is an R-module such that $M \in \bigcap_{i=0}^{n-1} \operatorname{Ker}\operatorname{Ext}^i_R(R/I, -)$. Then $\operatorname{Ext}^k_R(N, M) = 0$ for all $k = 0, 1, \dots, n-1$ and any R/I-module N.*

Proof. Since N is an R/I-module, there is an exact sequence

$$0 \to K \to R/I^{(\varkappa)} \to N \to 0.$$

The lemma is then proved by applying $\operatorname{Hom}_R(-, M)$ on this sequence and a straightforward induction on k, using the fact that K is also an R/I-module. □

The last result extends in a straightforward way to the derived category.

Lemma 4.3.8. *Let I be a finitely generated ideal and suppose that M is an R-module such that $M \in \bigcap_{i=0}^{n-1} \operatorname{Ker} \operatorname{Ext}_R^i(R/I, -)$. If $X \in \mathbf{D}(\text{Mod-R})$ is a complex such that $H_k(X) = 0$ for $k \le -n$ and $I \subseteq \operatorname{Ann}\Big(H_k(X)\Big)$ for $k = -n+1, \dots, -1, 0$. Then $\operatorname{Hom}_{\mathbf{D}(\text{Mod-R})}(X, M) = 0$.*

Proof. Using the notation from §4.3.1 for truncations of complexes, we first recall the well-known fact that $\operatorname{Hom}_{\mathbf{D}(\text{Mod-R})}(X_{\ge 1}, M) = 0$; see [28, Propositions 3.1.8 and 3.1.10]. Since by our assumption $\operatorname{Hom}_{\mathbf{D}(\text{Mod-R})}(\Sigma^k H_k(X), M) = 0$ for $k = -n+1, \dots, -1, 0$, a straightforward induction using the triangles from (4.1) in §4.3.1 shows that $\operatorname{Hom}_{\mathbf{D}(\text{Mod-R})}(X_{\ge -n+1}, M) = 0$. Finally, the inclusion $X_{\ge -n+1} \to X$ is an isomorphism in $\mathbf{D}(\text{Mod-R})$ since we assume that $H_k(X) = 0$ for $k \le -n$. □

The latter lemma implies that the comparison map in (4.3) in §4.3.2 between the i-th Koszul cohomology of I and $\operatorname{Ext}_R^i(R/I, M)$ is an isomorphism under certain assumptions.

Proposition 4.3.9. *Let R be a commutative ring and I be a finitely generated ideal of R. Suppose that M is a module such that $M \in \bigcap_{i=0}^{n-1} \operatorname{Ker} \operatorname{Ext}_R^i(R/I, -)$ for some $n \ge 0$. Then $q_M^n \colon \operatorname{Ext}_R^n(R/I, M) \to H^n(I; M)$ is an isomorphism.*

Proof. Since the two rightmost terms $F_1 \xrightarrow{d_1} F_0$ of $K_\bullet(I)$ constitute a projective presentation of R/I, it is clear that $q_M^0 \colon H^0(I; M) \to \operatorname{Hom}_R(R/I, M)$ is always an isomorphism. This proves the lemma for $n = 0$.

If $n > 0$, we consider the triangle

$$K_\bullet(I)_{\ge 1} \to K_\bullet(I) \xrightarrow{q} R/I \to \Sigma K_\bullet(I)_{\ge 1}$$

in $\mathbf{D}(\text{Mod-R})$ induced by the short exact sequence of complexes in the first three terms (see also (4.2) in §4.3.2). If we apply $\operatorname{Hom}_{\mathbf{D}(\text{Mod-R})}(-, \Sigma^n M)$, we obtain an exact sequence of abelian groups

$$\operatorname{Hom}(\Sigma K_\bullet(I)_{\ge 1}, \Sigma^n M) \to \operatorname{Ext}^n(R/I, M) \xrightarrow{q_M^n} H^n(I; M) \to \operatorname{Hom}(K_\bullet(I)_{\ge 1}, \Sigma^n M)$$

Since all the homologies of $K_\bullet(I)_{\ge 1}$ are R/I-modules, the leftmost and the rightmost term vanish by Lemma 4.3.8 provided that $M \in \bigcap_{i=0}^{n-1} \operatorname{Ker} \operatorname{Ext}_R^i(R/I, -)$. Hence q_M^n is an isomorphism in such a case. □

An immediate but particularly useful consequence is the following identification.

Corollary 4.3.10. *If R is a commutative ring, then*

$$\bigcap_{i=0}^{n-1} \operatorname{Ker} \operatorname{Ext}_R^i(R/I, -) = \bigcap_{i=0}^{n-1} \operatorname{Ker} H^i(I; -)$$

as subcategories of Mod-R *for each finitely generated ideal I and $n \ge 0$.*

Remark 4.3.11. The dual versions of results from §4.3.3 also hold, and will be used in §4.7. Explicitly, for each finitely generated ideal I and $n \ge 0$ we have

$$\bigcap_{i=0}^{n-1} \operatorname{Ker} \operatorname{Tor}_i^R(R/I, -) = \bigcap_{i=0}^{n-1} \operatorname{Ker} H_i(I; -),$$

and for any module M belonging to this class, there is a natural isomorphism

$$q_n^M : H_n(I; M) \to \mathrm{Tor}_n^R(R/I, M),$$

obtained by applying $H_n(- \otimes_R^{\mathbb{L}} M)$ onto (4.2). This can be proven either directly using similar arguments as in this section, or it follows by using the elementary duality $(-)^+$ (see §4.4 for the definition). Indeed, using the Hom-$\otimes$-adjunction and exactness of the elementary duality, we have for any $M \in \bigcap_{i=0}^{n-1} \operatorname{Ker} \mathrm{Tor}_i^R(R/I, -)$ that $M^+ \in \bigcap_{i=0}^{n-1} \operatorname{Ker} \mathrm{Ext}_R^i(R/I, -)$. It is straightforward to check that the same properties ensure that $(q_n^M)^+$ is naturally equivalent to $q_{M^+}^n$, which is an isomorphism by Proposition 4.3.9. Since $(-)^+$ is exact and faithful, we conclude that q_n^M is an isomorphism.

4.3.4 Vaguely associated primes revisited

Now we give another way to express the very same class as in Corollary 4.3.10 by giving a homological generalization of Proposition 4.2.13. We start with an easy observation

Lemma 4.3.12. *Let R be a commutative ring, I be a finitely generated ideal. Then $\mathcal{F} = \operatorname{Ker} \mathrm{Hom}_R(R/I, -)$ is a hereditary torsion-free class of finite type whose corresponding Thomason set (in the sense of Proposition 4.2.11) is $V(I)$.*

Proof. By Lemma 4.2.9, the smallest Gabriel topology $\mathcal{G}$ containing I consists of the ideals J such that $I^m \subseteq J$ for some $m \geq 1$. The corresponding cyclic module R/J then admits a filtration by R/I-modules of length m. Hence the smallest torsion class $\mathcal{T}$ containing R/I coincides with the smallest torsion class containing $\mathcal{G}$. In particular, $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion pair. The fact that it corresponds to the Thomason set $V(I)$ follows from the proof of Proposition 4.2.11. $\square$

Now can we state and prove the promised result.

Proposition 4.3.13. *Let R be a commutative ring, I be a finitely generated ideal, M be an R-module and $n \geq 1$. Then the following are equivalent:*

1. $\mathrm{Ext}_R^i(R/I, M) = 0$ *for each* $i = 0, 1, \ldots, n-1$.
2. $\mathrm{VAss}\left(\Omega^{-i}(M)\right) \cap V(I) = \emptyset$ *for each* $i = 0, 1, \ldots, n-1$.

Proof. We prove the proposition by induction on n. Let $\mathcal{F} = \operatorname{Ker} \mathrm{Hom}_R(R/I, -)$. The statement for $n = 1$ is precisely Proposition 4.2.13. Suppose now that $n > 1$, and consider the exact sequence

$$0 \to \Omega^{-(n-2)}(M) \to E \to \Omega^{-(n-1)}(M) \to 0, \tag{4.4}$$

where E is the injective envelope of $\Omega^{-(n-2)}(M)$. An application of $\mathrm{Hom}_R(R/I, -)$ yields an exact sequence

$$\mathrm{Hom}_R(R/I, E) \to \mathrm{Hom}_R(R/I, \Omega^{-(n-1)}(M)) \to \mathrm{Ext}_R^{n-1}(R/I, M) \to 0.$$

If (1) holds, we have $\mathrm{VAss}\left(\Omega^{-(n-2)}(M)\right) \cap V(I) = \emptyset$ by the inductive hypothesis, so $\Omega^{-(n-2)}(M) \in \mathcal{F}$ by Proposition 4.2.13. Since $\mathcal{F}$ is a hereditary torsion-free class, also $E \in \mathcal{F}$ and the leftmost term in (4.4) vanishes. As also $\mathrm{Ext}_R^{n-1}(R/I, M) = 0$ by the assumption, we have $\mathrm{Hom}_R(R/I, \Omega^{-(n-1)}(M)) = 0$. Therefore, we conclude that $\mathrm{VAss}\left(\Omega^{-(n-1)}(M)\right) \cap V(I) = \emptyset$ as required.

Suppose conversely that (2) holds. Then $\mathrm{Hom}_R(R/I, \Omega^{-(n-1)}(M)) = 0$ by Proposition 4.2.13 and hence $\mathrm{Ext}_R^{n-1}(R/I, M) = 0$. The other Ext-groups in (1) vanish by the inductive hypothesis. □

4.3.5 Characterizations of grade

Now we are in a position to state and prove the main result of the section. As the definition of the grade of a module is specific to the noetherian situation and will not be so important for the rest of the paper, we only refer to [14, §§I.1.1–I.1.2] for the corresponding standard definitions. The equivalences below are well-known under additional finiteness conditions (R noetherian, M finitely generated)—see for instance [14, Theorems I.2.5 and I.6.17].

Theorem 4.3.14. *Let R be a commutative ring, M be an R-module and $n \geq 1$. Then the following are equivalent:*

1. $H^i(I; M) = 0$ *for each* $i = 0, 1, \ldots, n-1$.
2. $\mathrm{Ext}_R^i(R/I, M) = 0$ *for each* $i = 0, 1, \ldots, n-1$.
3. $\mathrm{VAss}\left(\Omega^{-i}(M)\right) \cap V(I) = \emptyset$ *for each* $i = 0, 1, \ldots, n-1$.

If, moreover, R is noetherian and M finitely generated, the statements are further equivalent to:

(4) *The grade of I on M is at least n.*

Proof. The equivalence between (1) and (2) has been established in Corollary 4.3.10 and the equivalence between (2) and (3) in Proposition 4.3.13. For the equivalence between (2) and (4) see [14, Theorems I.2.5]. □

4.4 Infinitely generated tilting theory

At this point, we quickly recall basic terminology and facts about module approximations and cotorsion pairs, two essential tools of the forthcoming sections. We also remind the reader of the notion of (not necessarily finitely generated) n-tilting and n-cotilting module, as defined by [15] and [4], and the duality between those two.

4.4.1 Module approximations

We briefly recall the definitions of (pre)covers and (pre)envelopes of modules. Let $\mathcal{C}$ be a class of right R-modules, and $M \in$ Mod-R. We say that a map $f : C \to M$ is a *$\mathcal{C}$-precover* of M provided that $C \in \mathcal{C}$, and for any $C' \in \mathcal{C}$ the map $\mathrm{Hom}_R(C', f)$ is surjective. Furthermore, if any map $g \in \mathrm{End}_R(C)$ such that $f = fg$ is necessarily an automorphism, we say that f is a *$\mathcal{C}$-cover*. Finally, a surjective map $f : C \to M$ is called a *special $\mathcal{C}$-precover* if $C \in \mathcal{C}$ and $\mathrm{Ker}(f) \in$

${}^{\perp_1}\mathcal{C}$. It is easy to see that any special $\mathcal{C}$-precover is a $\mathcal{C}$-precover. Also, by the Wakamatsu Lemma ([21, Lemma 5.13]), any surjective $\mathcal{C}$-cover is a special $\mathcal{C}$-precover. Finally, we say that a class $\mathcal{C}$ is *(special) (pre)covering*, if any module $M \in$ Mod-R admits a (special) $\mathcal{C}$-(pre)cover.

The notions of $\mathcal{C}$-preenvelope, $\mathcal{C}$-envelope, and special $\mathcal{C}$-preenvelope are defined dually.

4.4.2 Cotorsion pairs

Given a class $\mathcal{C} \subseteq$ Mod-R, we fix the following notation:

$$\mathcal{C}^{\perp_1} = \{M \in \text{Mod-R} \mid \operatorname{Ext}^1_R(C, M) = 0 \text{ for all } C \in \mathcal{C}\},$$

$$\mathcal{C}^{\perp} = \{M \in \text{Mod-R} \mid \operatorname{Ext}^i_R(C, M) = 0 \text{ for all } C \in \mathcal{C} \text{ and } i > 0\},$$
$$\mathcal{C}^{\intercal} = \{M \in \text{R-Mod} \mid \operatorname{Tor}_i^R(C, M) = 0 \text{ for all } C \in \mathcal{C} \text{ and } i > 0\}.$$

We also define the "left-hand" version of those classes in an obviously analogous way. A pair of classes $(\mathcal{A}, \mathcal{B})$ is called a *cotorsion pair* if $\mathcal{B} = \mathcal{A}^{\perp_1}$, and $\mathcal{A} = {}^{\perp_1}\mathcal{B}$. Such a cotorsion pair is said to be *hereditary*, if furthermore $\mathcal{B} = \mathcal{A}^{\perp}$.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *complete* provided that $\mathcal{A}$ is a special precovering class (equivalently, $\mathcal{B}$ is a special preenveloping class, see [21, Lemma 5.20]). Complete cotorsion pairs are abundant - indeed, any cotorsion pair generated by a set is complete, and the left class of the pair can be described explicitly:

Theorem 4.4.1. ([21, Theorem 6.11], [21, Corollary 6.14]) *Let $\mathcal{S}$ be a set of modules. Then:*

(i) The cotorsion pair $({}^{\perp_1}(S^{\perp_1}), S^{\perp_1})$ is complete,

(ii) The class ${}^{\perp_1}(S^{\perp_1})$ consists precisely of all direct summands of all $\mathcal{S} \cup \{R\}$-filtered modules.

4.4.3 Tilting and cotilting modules and classes

Let T be a right R-module and $n \geq 0$. We say that T is *n-tilting* if the following three conditions hold:

(T1) $\operatorname{pd} T \leq n$,

(T2) $\operatorname{Ext}^i_R(T, T^{(X)}) = 0$ for all $i > 0$ and all sets X,

(T3) there is an exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$, where T_i is a direct summand of a direct sum of copies of T for all $i = 0, 1, \dots, n$.

A module T is *tilting* if it is n-tilting for some $n \geq 0$. An n-tilting module T induces a hereditary and complete cotorsion pair $(\mathcal{A}, \mathcal{T}) = ({}^{\perp_1}(T^{\perp}), T^{\perp})$. The class $\mathcal{T}$ is called an *(n-)tilting class*. Two tilting modules T, T' are *equivalent* if they induce the same tilting class. Even though the tilting modules in our setting are in general big (indeed, over a commutative ring, any finitely generated tilting module is projective), the tilting classes arise from small modules in the following sense. Let mod-R denote the full subcategory of Mod-R consisting of *strongly finitely presented modules*, that is, modules having finite projective resolution consisting of finitely generated projectives. A full subcategory $\mathcal{S}$ of mod-R is called *resolving* if it contains all finitely generated projectives, is closed under extensions and direct summands, and $A \in \mathcal{S}$ whenever there is an exact sequence

$$0 \to A \to B \to C \to 0,$$

with $B, C \in \mathcal{S}$.

Theorem 4.4.2. ([12], [21, Theorem 13.49]) *There is a bijective correspondence between n-tilting classes $\mathcal{T}$ and resolving subcategories $\mathcal{S}$ of* mod-R *consisting of R-modules of projective dimension $\leq n$. The correspondence is given by mutually inverse assignments $\mathcal{T} \mapsto ({}^{\perp}\mathcal{T}) \cap \text{mod-R}$, and $\mathcal{S} \mapsto \mathcal{S}^{\perp} = \mathcal{S}^{\perp_1}$.*

The cotilting modules have a formally dual definition - a left R-module C is *(n-)cotilting* if the following conditions hold:

(C1) $\operatorname{id} T \leq n$,

(C2) $\operatorname{Ext}^i_R(C^X, C) = 0$ for all $i > 0$ and all sets X,

(C3) there is an exact sequence $0 \to C_n \to \cdots \to C_0 \to W \to 0$, where C_i is a direct summand of a direct product of copies of T for all $i = 0, 1, \ldots, n$, and W is an injective cogenerator of R-Mod.

As for the tilting modules, a cotilting module C induces a *cotilting class* $\mathcal{C} = {}^{\perp}C$, and two cotilting modules C, C' are *equivalent* if they induce the same cotilting class. There is also an explicit duality between tilting and cotilting modules. If R is a k-algebra over a commutative ring k (e.g. $k = \mathbb{Z}$), we denote by $(-)^+ = \operatorname{Hom}_k(-, E)$ the duality with respect to an injective cogenerator E of k-Mod. Then for any right tilting R-module T, the dual T^+ is a left cotilting R-module. We say that a tilting class $\mathcal{T}$ in Mod-R and cotilting class $\mathcal{C}$ in R-Mod are *associated* if there is a tilting module T inducing $\mathcal{T}$ such that T^+ induces $\mathcal{C}$.

It is not true that every cotilting class is associated to a tilting class. We say that a class $\mathcal{C}$ is of *cofinite type* provided that there is a set of strongly finitely presented modules $\mathcal{S}$ of projective dimension bounded by n such that $\mathcal{C} = \mathcal{S}^{\intercal}$.

Theorem 4.4.3. ([21, Proposition 15.17], [21, Theorem 15.18]) *Any class of cofinite type is cotilting. A cotilting class is associated to some tilting class if and only it is of cofinite type. Furthermore, the assignment $T^{\perp} \mapsto {}^{\perp}(T^+)$ induces a bijection between tilting class in* Mod-R *and cotilting classes in* R-Mod *of cofinite type.*

An example of a cotilting class not of cofinite type was first exhibited in [9]. In §4.9, we show a more general construction of such classes.

All tilting and all cotilting classes are definable (i.e., closed under pure submodules, direct products, and direct limits). Furthermore, a pair of associated tilting class and cotilting class is dual definable in the following sense:

Lemma 4.4.4. *Let R be a ring, $\mathcal{T}$ a tilting class in* Mod-R, *and $\mathcal{C}$ the cotilting class of cofinite type in* R-Mod *associated to $\mathcal{T}$. Then for any $M \in$* Mod-R, *and $N \in$* R-Mod*:*

(i) $M \in \mathcal{T}$ if and only if $M^+ \in \mathcal{C}$,

(ii) $N \in \mathcal{C}$ *if and only if* $N^+ \in \mathcal{T}$.

Proof. The proof is the same as that of [5, Lemma 3.3]. As $\mathcal{T}$ is of finite type, there is a set $\mathcal{S} \subseteq$ mod-R such that $\mathcal{T} = \mathcal{S}^{\perp}$. By [21, Theorem 15.19, and the paragraph following it], also $\mathcal{C} = \mathcal{S}^{\intercal}$. Using the Hom-$\otimes$ adjunction, exactness of $(-)^+$, and [19, Theorem 3.2.10], we have $\mathrm{Tor}_i^R(S, N)^+ \simeq \mathrm{Ext}_R^i(S, N^+)$, which yields (ii). Since $\mathcal{T}$ is definable, $M \in \mathcal{T}$ if and only if $M^{++} \in \mathcal{T}$ by [39, 3.4.21]. Then we can apply (ii) tho obtain (i). □

Finally, as in [6], we fix the following notation for cotilting classes of lower dimensions induced by a cotilting class, which will be useful for arguing by induction on the dimension.

Notation 4.4.5. Given an n-cotilting class $\mathcal{C}$ induced by a cotilting module C, we let $\mathcal{C}_{(i)} = {}^{\perp}\Omega^{-i}C$ for all $i \geq 0$. In particular, $\mathcal{C}_{(0)} = \mathcal{C}$, and $\mathcal{C}_{(i)}$ is a $(n-i)$-cotilting class for all $i = 0, 1, \ldots, n$ (see [6, Lemma 3.5]).

4.5 Cotilting classes of cofinite type

In this section we classify cotilting classes of cofinite type over a commutative ring. The parametrizing set for this classification will consist of sequences of torsion-free classes of hereditary torsion-free pairs of finite type satisfying some extra conditions. Using results from Section 4.2, these classes are in bijective correspondence with certain Thomason sets, and thus generalize in a direct way the parametrizing sets used in the noetherian case in [6].

Definition 4.5.1. Let R be a commutative ring and $n \geq 0$. We say that

a sequence of torsion-free classes $\mathfrak{S} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{n-1})$ is a *characteristic sequence (of length n)* if

(i) $\mathcal{F}_i$ is hereditary and of finite type for each $i = 0, 1, \ldots, n-1$,

(ii) $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{n-1}$,

(iii) $\Omega^{-i}R \in \mathcal{F}_i$ for each $i = 0, 1, \ldots, n-1$.

Notation 4.5.2. Given a characteristic sequence $\mathfrak{S} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{n-1})$ and $i = 0, 1, \ldots, n-1$ we put

$$\mathcal{F}_i(\mathfrak{S}) = \mathcal{F}_i,$$

and we define a class

$$\mathcal{C}(\mathfrak{S}) = \{M \in \text{Mod-R} \mid \Omega^{-i}M \in \mathcal{F}_i(\mathfrak{S}) \text{ for each } i = 0, 1, \ldots, n-1\}.$$

Our goal in this section is to prove the following:

Theorem 4.5.3. *Let R be a commutative ring and $n \geq 0$. Then the assignment*

$$\Psi : \mathfrak{S} \mapsto \mathcal{C}(\mathfrak{S})$$

is a bijection between the set of all characteristic sequences of length n and all n-cotilting classes in Mod-R *of cofinite type.*

The proof of Theorem 4.5.3 will be done in several steps throughout this section, by proving subsequently that Ψ is surjective, well-defined, and injective.

4.5.1 Ψ is surjective

Lemma 4.5.4. *Let R be a commutative ring, $M \in$ Mod-R, and P a finitely generated projective R-module. If $\iota : M \to E(M)$ is the injective envelope of M, then $P \otimes_R \iota$ is the injective envelope of $P \otimes_R M$.*

Proof. Let $P' \in$ Mod-R be a finitely generated projective module such that $P \oplus P' \simeq R^n$ for some $n \in \omega$. As $E(M)^n$ is injective, the map $R^n \otimes_R \iota = \iota^n$ is the injective envelope of M^n by [3, Proposition 6.16(2)]. As $\iota^n = (P \otimes_R \iota) \oplus (P' \otimes_R \iota)$, we infer that $P \otimes_R \iota$ is the injective envelope of $P \otimes_R M$. $\square$

Proposition 4.5.5. *Let R be a commutative ring and $\mathcal{C}$ be a cotilting class in* Mod-R. *If $\mathcal{C}$ is of cofinite type, then it is closed under injective envelopes.*

Proof. Fix a module $M \in \mathcal{C}$, and let us show that $E(M) \in \mathcal{C}$, provided that $\mathcal{C}$ is of cofinite type. Under this assumption, there is a set $\mathcal{S}$ of strongly finitely presented modules of projective dimension bounded by n such that $\mathcal{C} = \mathcal{S}^\intercal$. We proceed by induction on n. If $n = 0$ the claim is clear as $\mathcal{C} =$ Mod-R; henceforth assume that we proved the claim for all $k < n$. Pick $S \in \mathcal{S}$ and fix its projective resolution

$$P_\bullet : \quad 0 \to P_n \xrightarrow{\sigma_n} P_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_1} P_0 \to 0,$$

consisting of finitely generated projectives. Tensoring the complex $P_\bullet$ with the injective envelope $\iota : M \to E(M)$ yields a commutative diagram in Mod-R (this is where we use the commutativity of R):

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
P_2 \otimes_R M \arrow[r, "\sigma_2 \otimes_R M"] \arrow[d] & P_1 \otimes_R M \arrow[r, "\sigma_1 \otimes_R M"] \arrow[d, "P_1 \otimes_R \iota"'] & P_0 \otimes_R M \arrow[d] \\
P_2 \otimes_R E(M) \arrow[r, "\sigma_2 \otimes_R E(M)"] & P_1 \otimes_R E(M) \arrow[r, "\sigma_1 \otimes_R E(M)"] & P_0 \otimes_R E(M)
\end{tikzcd}$$

(In the case of $n = 1$ we put $P_2 = 0$.)

Since $P_\bullet$ consists of projective modules, the columns are exact. We have

$$H_1(P_\bullet \otimes_R M) \simeq \mathrm{Tor}_1^R(S, M) = 0,$$

as $M \in \mathcal{C}$, and we want to show that $H_1(P_\bullet \otimes_R E(M)) \simeq \mathrm{Tor}_1^R(S, E(M) = 0$. If $\mathrm{Ker}(\sigma_1 \otimes_R E(M)) = 0$ there is nothing to prove. Otherwise, since $P_1 \otimes_R \iota$ is the injective envelope of $P_1 \otimes_R M$ by Lemma 4.5.4, $P_1 \otimes_R \iota$ is an essential monomorphism. Therefore, the module

$$\begin{aligned}\mathrm{Ker}(\sigma_1 \otimes_R E(M)) \cap \mathrm{Im}(P_1 \otimes_R \iota) = (P_1 \otimes_R \iota)(\mathrm{Ker}(\sigma_1 \otimes_R M)) = \\ = (P_1 \otimes_R \iota)(\mathrm{Im}(\sigma_2 \otimes_R M))\end{aligned}$$

is non-zero, and thus essential in $\mathrm{Ker}(\sigma_1 \otimes_R E(M))$. It follows that $\mathrm{Im}(\sigma_2 \otimes_R E(M))$ is essential in $\mathrm{Ker}(\sigma_1 \otimes_R E(M))$. Now we use the induction hypothesis, which implies that $\mathrm{Tor}_k^R(S, E(M)) \simeq \mathrm{Tor}_1^R(\Omega^{k-1} S, E(M)) = 0$ for all $k > 1$. This means that $H_k(P_\bullet \otimes_R E(M)) = 0$ for all $k > 1$, and as this complex consists of injectives and is left-bounded, the map $(\sigma_2 \otimes_R E(M))$ is a split monomorphism. The only case for which this is not a contradiction is when $\operatorname{Im}(\sigma_2 \otimes_R E(M)) = \operatorname{Ker}(\sigma_1 \otimes_R E(M))$, showing that $0 = H_1(P_\bullet \otimes_R E(M)) = \operatorname{Tor}_1^R(S, E(M))$, proving finally that $E(M) \in \mathcal{C}$. $\square$

Given a fixed module C, we can assign to any module X the canonical map $\eta_X : X \to C^{\operatorname{Hom}_R(X,C)} = C_X$. This map is in fact (covariantly) functorial, as we recall in the following Lemma.

Lemma 4.5.6. *The map η_X is functorial in X. That is, given any map $X \xrightarrow{f} Y$, there is a map $\eta_f : C_X \to C_Y$ such that $\eta_Y f = \eta_f \eta_X$, and such that $\eta_g \eta_f = \eta_{gf}$ for any map $g : Y \to Z$.*

Proof. For any $\beta \in \operatorname{Hom}_R(Y, C)$, let $\pi_\beta : C_Y \to C$ be the projection onto the β-th coordinate. We define η_f by the following rule: For any $\mathbf{c} = (c_\alpha)_{\alpha \in \operatorname{Hom}_R(X,C)} \in C_X$, and any $\beta \in \operatorname{Hom}_R(Y, C)$, we let

$$\pi_\beta \eta_f(\mathbf{c}) = \begin{cases} c_\alpha, & \alpha = \beta f \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

It is easy to see that η_f is an R-module homomorphism. Also, for any $\beta \in \operatorname{Hom}_R(Y, C)$ we have

$$\pi_\beta \eta_Y f = \beta f = \pi_\beta \eta_f \eta_X,$$

proving that indeed $\eta_Y f = \eta_f \eta_X$. Finally, the equality $\eta_g \eta_f = \eta_{gf}$ can be checked by direct computation from (4.5). $\square$

Lemma 4.5.7. *Let R be a ring and let $\mathcal{C}$ be an cotilting class in* R-Mod *closed under injective envelopes. Then there is a hereditary faithful torsion-free class of finite type $\mathcal{F}$ such that*

$$\mathcal{C} = \{M \in \text{Mod-R} \mid M \in \mathcal{F} \ \& \ \Omega^{-1}M \in \mathcal{C}_{(1)}\}.$$

Proof. Let C be the cotilting module associated to $\mathcal{C}$. Applying $\operatorname{Hom}_R(-, C)$ to the exact sequence $0 \to M \to E(M) \to \Omega^{-1}M \to 0$, and using that $\mathcal{C}$ is closed under injective envelopes by Proposition 4.5.5, we infer that $M \in \mathcal{C}$ if and only if $E(M) \in \mathcal{C}$ and $\Omega^{-1}M \in \mathcal{C}_{(1)}$.

We let $\mathcal{F}$ be the closure of $\mathcal{C}$ under submodules. As $\mathcal{C}$ is closed under injective envelopes, injectives of $\mathcal{C}$ and $\mathcal{F}$ coincide. From this it is easy to infer that $\mathcal{C} = \{M \in \text{Mod-R} \mid M \in \mathcal{F} \ \& \ \Omega^{-1}M \in \mathcal{C}_{(1)}\}$. We are left to show that $\mathcal{F}$ is a hereditary faithful torsion-free class of finite type. It is easy to check that $R \in \mathcal{F}$, and that $\mathcal{F}$ is closed under submodules, injective envelopes, extensions, and products.

It remains to show that $\mathcal{F}$ is closed under direct limits. Note that $\mathcal{F} = \operatorname{Cogen}(C)$. Let $(X_i)_{i \in I}$ be a directed system of modules from $\mathcal{F}$. As X_i is cogenerated by C, the canonical map $\eta_{X_i} : X_i \to C_{X_i}$ is monic. Using the functoriality proved in Lemma 4.5.6, we actually have a directed system $(X_i \to C_{X_i})_{i \in I}$ of monic maps. Taking the direct limit yields a monic map $\varinjlim_{i \in I} X_i \to \varinjlim_{i \in I} C_{X_i}$. As $\mathcal{C}$ is definable, the latter direct limit is in $\mathcal{C}$, proving that $\varinjlim_{i \in I} X_i$ is indeed in $\mathcal{F}$. $\square$

Lemma 4.5.8. *Let R be a commutative ring and $\mathcal{C}$ be an n-cotilting class in* Mod-R *of cofinite type. Then there is a characteristic sequence $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1})$ such that $\mathcal{C} = \{M \in \text{Mod-R} \mid \Omega^{-i}M \in \mathcal{F}_i \text{ for each } i = 0, 1, \dots, n-1\}$. In particular, map Ψ is surjective.*

Proof. First observe that $\mathcal{C}_{(1)}$ is of cofinite type provided that $\mathcal{C}$ is. Indeed, if $\mathcal{C} = \mathcal{S}^\intercal$ for some resolving subcategory of mod-R consisting of modules of bounded projective dimension, we have

$$\mathcal{C}_{(1)} = \{M \in \text{Mod-R} \mid \Omega M \in \mathcal{C}\} = \{\Omega S \mid S \in \mathcal{S}\}^\intercal,$$

demonstrating the cofinite type of $\mathcal{C}_{(1)}$. With Proposition 4.5.5 in mind, we can apply Lemma 4.5.7 inductively $(n-1)$-times in order to obtain the desired sequence $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1})$, where $\mathcal{F}_{n-1} = \mathcal{C}_{(n-1)}$. Using the same Lemma and Proposition, this sequence is indeed characteristic, and $\mathcal{C} = \mathcal{C}((\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1}))$ as desired. $\square$

4.5.2 Ψ is well-defined

Definition 4.5.9. Let $\mathfrak{S}$ be a characteristic sequence of length n. We let $\mathcal{G}_i(\mathfrak{S})$ denote the Gabriel topology associated to the torsion-free class $\mathcal{F}_i(\mathfrak{S})$ in the sense of Proposition 4.2.7 for each $i = 0, 1, \dots, n-1$.

In particular, $\mathcal{G}_i(\mathfrak{S})$ is a Gabriel topology of finite type, and

$$\mathcal{F}_i(\mathfrak{S}) = \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} \operatorname{Ker} \operatorname{Hom}_R(R/I, -).$$

Lemma 4.5.10. *Let $\mathfrak{S}$ be a characteristic sequence of length n. Then*

$$\mathcal{C}(\mathfrak{S}) = \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} \operatorname{Ker} \operatorname{Ext}_R^i(R/I, -)$$

.

Proof. Let $\mathfrak{S} = (\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1})$. We prove by induction on $0 < k \leq n$ that $\mathcal{C}((\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{k-1})) = \bigcap_{i=0}^{k-1} \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} \operatorname{Ker} \operatorname{Ext}_R^i(R/I, -)$. If $k = 1$, then indeed $C((\mathcal{F}_0)) = \bigcap_{I \in \mathcal{G}_0^f(\mathfrak{S})} \operatorname{Ker} \operatorname{Hom}_R(R/I, -)$. Suppose that the claim is valid up to $k-1$ for $0 < k-1 < n$, and let $I \in \mathcal{G}_k^f$. Let $M \in C((\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{k-1}))$. The long exact sequence obtained by applying $\operatorname{Hom}_R(R/I, -)$ on the exact sequence $0 \to M \to E(M) \to \Omega^{-1}M \to 0$ yields an isomorphism $\operatorname{Ext}_R^{k-1}(R/I, \Omega^{-1}M) \simeq \operatorname{Ext}_R^k(R/I, M)$ (for $k = 1$ we use the fact that $E(M) \in \mathcal{F}_0 \subseteq \mathcal{F}_k$).

By dimension shifting, we have $\operatorname{Ext}_R^{k-1}(R/I, \Omega^{-1}M) \simeq \operatorname{Ext}_R^1(R/I, \Omega^{-k+1}M)$. Finally, since $E(\Omega^{-k+1}M) \in \mathcal{F}_{k-1} \subseteq \mathcal{F}_k$, we have also the zero-dimension shift isomorphism $\operatorname{Ext}_R^1(R/I, \Omega^{-k+1}M) \simeq \operatorname{Hom}_R(R/I, \Omega^{-k}M)$. Putting the isomorphisms together, we have $\operatorname{Hom}_R(R/I, \Omega^{-k}M) \simeq \operatorname{Ext}_R^k(R/I, M)$, showing that $\Omega^{-k}M \in \mathcal{F}_k$ holds if and only if $M \in \bigcap_{I \in \mathcal{G}_k^f} \operatorname{Ker} \operatorname{Ext}_R^k(R/I, -)$ for any $M \in \mathcal{C}(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{k-1})$. Therefore, using the induction premise, an arbitrary module $M \in \text{Mod-R}$ satisfies $M \in \mathcal{C}(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_k)$ if and only if we have $M \in \bigcap_{i=0}^{k} \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} \operatorname{Ker} \operatorname{Ext}_R^i(R/I, -)$, establishing the induction step. $\square$

In what follows, we denote by $(-)^*$ the regular module duality $\operatorname{Hom}_R(-, R)$.

Definition 4.5.11. Let I be a finitely generated ideal, and let us denote the Koszul complex $K_\bullet(I)$ as follows

$$\cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0,$$

where F_k is in degree k for all $k = 0, 1, 2, \dots, n+1$. We denote by $S_{I,k}$ the cokernel of the map d_k^*. That is, $S_{I,k}$ is (up to stable equivalence) the Auslander-Bridger transpose of $\mathrm{Coker}(d_k)$.

Proposition 4.5.12. *Let I be a finitely generated ideal such that $\mathrm{Ext}_R^i(R/I, R) = 0$ for all $i = 0, 1, \dots, n-1$. Then:*

(i) $S_{I,n}$ is a strongly finitely presented module of projective dimension n,

(ii) $\bigcap_{i=0}^{n-1} \mathrm{Ker}\,\mathrm{Ext}_R^i(R/I, -) = (S_{I,n})^\intercal$,

(iii) $\bigcap_{i=0}^{n-1} \mathrm{Ker}\,\mathrm{Ext}_R^i(R/I, -)$ is an n-cotilting class of cofinite type.

Proof. Let us adopt the notation for $K_\bullet(I)$ as in Definition 4.5.11. Let $0 < k \leq n$. Applying $(-)^*$ to $K_\bullet(I)$ we obtain complex

$$0 \to F_0^* \to F_1^* \to \cdots \to F_{k-1}^* \xrightarrow{d_n^*} F_k^* \to S_{I,k} \to 0, \tag{4.6}$$

which is exact by our assumption and Proposition 4.3.9. This proves (i).

Denote by $C_\bullet$ the complex (4.6) with $S_{I,k}$ deleted, and F_k^* in the degree zero. Then $C_\bullet$ is a projective resolution of $S_{I,k}$, and thus $H_1(C_\bullet \otimes_R M) \simeq \mathrm{Tor}_1^R(S_{I,k}, M)$. But as F_j is finitely generated projective for all $j = 0, 1, \dots, n$, we have by [3, Proposition 20.10] a natural isomorphism $F_j^* \otimes_R M \simeq \mathrm{Hom}_R(F_j, M)$. Hence, using Proposition 4.3.9 we obtain

$$\mathrm{Tor}_1^R(S_{I,k}, M) \simeq H_1(C_\bullet \otimes_R M) \simeq \tag{4.7}$$
$$\simeq H^{k-1}(\mathrm{Hom}_R(K_\bullet(I), M) \simeq \mathrm{Ext}_R^{k-1}(R/I, M),$$

for any $M \in \bigcap_{i=0}^{k-2} \mathrm{Ker}\,\mathrm{Ext}_R^i(R/I, -)$.

Now we prove (ii). Note first that $S_{I,k}$ is an $(n-k)$-th syzygy of $S_{I,n}$. Hence, $\mathrm{Tor}_k^R(S_{I,n}, M) \simeq \mathrm{Tor}_1^R(S_{I,n-k+1}, M)$. Then $(S_{I,n})^\intercal = \bigcap_{k=1}^n \mathrm{Ker}\,\mathrm{Tor}_1^R(S_{I,k}, -)$. A straightforward induction on $k = 1, 2, \dots, n$ together with (4.7) proves that the latter class is equal to $\bigcap_{k=1}^n \mathrm{Ker}\,\mathrm{Ext}_R^{k-1}(R/I, -)$ as desired.

Finally, (iii) follows directly from (ii) by Theorem 4.4.3. □

Lemma 4.5.13. *Let $\mathfrak{S}$ be a characteristic sequence of length n, then $\mathcal{C}(\mathfrak{S})$ is a n-cotilting class of cofinite type. In particular, map Ψ is well-defined.*

Proof. We have the following chain of equalities:

$$\mathcal{C}(\mathfrak{S}) = \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} \mathrm{Ker}\,\mathrm{Ext}\,_R^i(R/I, -) = \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} \bigcap_{j=0}^{i} \mathrm{Ker}\,\mathrm{Ext}\,_R^j(R/I, -) =$$

$$= \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} (S_{I,i+1})^\intercal.$$

The first equality is Lemma 4.5.10, the second one follows easily from $\mathcal{G}_i^f \supseteq \mathcal{G}_{i+1}^f$ for each $i = 0, 1, \dots, n-2$, and the last one is an application of Proposition 4.5.12(ii). Then Theorem 4.4.3 yields the result. □

4.5.3 Ψ is injective

Lemma 4.5.14. *Let $\mathfrak{S} = (\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1})$ be a characteristic sequence, and $\mathcal{C} = \mathcal{C}(\mathfrak{S})$ the associated n-cotilting class. Then*

(i) $\Omega M \in \mathcal{C}((\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1}))$ if and only if $M \in \mathcal{C}((\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{n-1}))$,

(ii) $\mathcal{C}_{(i)} = \mathcal{C}((\mathcal{F}_i, \mathcal{F}_{i+1}, \dots, \mathcal{F}_{n-1}))$ for each $i = 0, 1, \dots, n$.

Proof. (i) Choose $I \in \mathcal{G}_i^f(\mathfrak{S})$ for some $i = 0, 1, \dots, n-1$. Consider the long exact sequence obtained by applying $\mathrm{Hom}_R(R/I, -)$ onto a projective presentation of M, say

$$0 \to \Omega M \to P \to M \to 0.$$

Since cotilting classes contain all projectives modules, this long exact sequence yields $\mathrm{Hom}_R(R/I, \Omega M) = 0$ and $\mathrm{Ext}_R^j(R/I, \Omega M) \simeq \mathrm{Ext}_R^{j-1}(R/I, M)$ for all $j = 1, 2, \dots, i$. Therefore,

$$\Omega M \in \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} \mathrm{Ker}\,\mathrm{Ext}_R^{\,i}(R/I, -)$$

if and only if

$$M \in \bigcap_{i=1}^{n-1} \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} \mathrm{Ker}\,\mathrm{Ext}_R^{\,i-1}(R/I, -).$$

This concludes (i) by Lemma 4.5.10.

(ii) It follows directly from the definition that $\mathcal{C}_{(i)} = \{M \in \text{Mod-R} \mid \Omega^i M \in \mathcal{C}\}$. Then (ii) is proved by (i) and a straightforward induction.

□

Lemma 4.5.15. *Let $\mathfrak{S}$ and $\mathfrak{S}'$ be two characteristic sequences. If $\mathfrak{S} \neq \mathfrak{S}'$, then $\mathcal{C}(\mathfrak{S}) \neq \mathcal{C}(\mathfrak{S}')$. In particular, map Ψ is injective.*

Proof. Let $i \geq 0$ be smallest such that $\mathcal{F}_i(\mathfrak{S}) \neq \mathcal{F}_i(\mathfrak{S}')$. Suppose without loss of generality that there is $M \in \mathcal{F}_i'(\mathfrak{S}) \setminus \mathcal{F}_i(\mathfrak{S})$. If $i = 0$, then clearly $E(M)$ is in $\mathcal{C}(\mathfrak{S}')$, but $E(M) \notin \mathcal{F}_0(\mathfrak{S})$ proving the statement for $i = 0$. Let now $i = 0, 1, \dots, n-1$ and suppose towards contradiction that $\mathcal{C}(\mathfrak{S}) = \mathcal{C}(\mathfrak{S}')$. Then also $\mathcal{C}(\mathfrak{S})_{(i)} = \mathcal{C}(\mathfrak{S}')_{(i)}$, but this is a contradiction using the case $i = 0$ and Lemma 4.5.14(ii). □

4.5.4 The result

Now we are ready to prove our classification theorem.

Proof of Theorem 4.5.3. The assignment $\Psi : \mathfrak{S} \mapsto \mathcal{S}(\mathfrak{S}))$ is a well-defined map from the set of all characteristic sequences of length n to n-cotilting classes of cofinite type by Lemma 4.5.13. This map is injective by Lemma 4.5.15 and surjective by Lemma 4.5.8. □

4.6 Main classification results

In this section we rephrase Theorem 4.5.3 in terms of Thomason sets, and state our characterization of tilting classes over commutative rings.

Theorem 4.6.1. *Let R be a commutative ring and $n \geq 0$. There is a 1-1 correspondence between n-cotilting classes $\mathcal{C}$ of cofinite type in* Mod-R *and finite sequences $(X_0, X_1, \ldots, X_{n-1})$ of Thomason subsets of* $\operatorname{Spec}(R)$ *satisfying:*

(i) $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_{n-1}$,

(ii) $X_i \cap \operatorname{VAss}(\Omega^{-j} R) = \emptyset$ *for all* $j = 0, 1, \ldots, i$ *and all* $i = 0, 1, \ldots, n-1$.

The correspondence is given by mutually inverse maps

$$\mathcal{C} \mapsto (\operatorname{Spec}(R) \setminus \operatorname{Ass}(\mathcal{C}_{(0)}), \operatorname{Spec}(R) \setminus \operatorname{Ass}(\mathcal{C}_{(1)}), \ldots, \operatorname{Spec}(R) \setminus \operatorname{Ass}(\mathcal{C}_{(n-1)})),$$

$$(X_0, X_1, \ldots, X_{n-1}) \mapsto \{M \in \text{Mod-R} \mid \operatorname{VAss}(\Omega^{-i} M) \cap X_i = \emptyset \textit{ for all } i < n\}.$$

Proof. Start with a cofinite-type cotilting class $\mathcal{C}$, and let $\mathfrak{S} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{n-1})$ be the characteristic sequence corresponding to $\mathcal{C}$ in the sense of Theorem 4.5.3. Note that $\operatorname{Ass}(\mathcal{C}_{(j)}) = \operatorname{Ass}(\mathcal{F}_j)$ for each $j = 0, 1, \ldots, n-1$. Indeed, one inclusion follows trivially, as $\mathcal{C}_{(j)} \subseteq \mathcal{F}_j$ by Lemma 4.5.14, while the second inclusion follows from $\mathcal{C}_{(j)}$ and $\mathcal{F}_j$ being both closed under injective envelopes and having the same injectives.

The rest of the proof is a combination of Theorem 4.5.3, Proposition 4.2.11, and Proposition 4.2.13. □

Theorem 4.6.2. *Let R be a commutative ring and $n \geq 0$. There are 1-1 correspondences between the following collections:*

(i) sequences $(\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{n-1})$ of Gabriel topologies of finite type satisfying:

  (a) $\mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \cdots \supseteq \mathcal{G}_{n-1}$,

  (b) $\operatorname{Ext}^j_R(R/I, R) = 0$ *for all* $I \in \mathcal{G}_i$*, all* $i = 0, 1, \ldots, n-1$*, and all* $j = 0, 1, \ldots, i$.

(ii) n-cotilting classes $\mathcal{C}$ in Mod-R *of cofinite type,*

(iii) n-tilting classes $\mathcal{T}$ in Mod-R,

(iv) resolving subcategories $\mathcal{S}$ of mod-R *consisting of modules of projective dimension $\leq n$.*

The correspondences are given as follows:

$$(i) \to (ii) : (\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{n-1}) \mapsto \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f} \operatorname{Ker} \operatorname{Ext}^i_R(R/I, -) = \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f} (S_{I,i+1})^{\intercal}$$

$$(i) \to (iii) : (\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{n-1}) \mapsto \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f} \operatorname{Ker} \operatorname{Tor}^R_i(R/I, -) = \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f} (S_{I,i+1})^{\perp}$$

$$(i) \to (iv) : (\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{n-1}) \mapsto \{M \in \text{mod-R} \mid M \textit{ is isomorphic}$$
$$\textit{to a direct summand of a finitely } \{R\} \cup \{S_{I,i+1} \mid I \in \mathcal{G}_i^f, i < n\}\textit{-filtered module}\}$$

Proof. Correspondence $(i) \to (ii)$ follows directly from Theorem 4.5.3, Gabriel correspondence between hereditary torsion pairs and Gabriel topologies (see [41, VI, Theorem 5.1 and XIII, Proposition 1.2]), and combination of Lemma 4.5.10 and Proposition 4.5.12. We show that $(i) \to (iii)$ is a composition of $(i) \to (ii)$ with the character duality correspondence between tilting and cofinite-type cotilting classes. Indeed, we have as in the proof of Lemma 4.4.4 that $\operatorname{Tor}_i^R(X, M) = 0$ if and only if $\operatorname{Ext}_R^i(X, M^+) = 0$ for any $i \geq 0$. Therefore, we have $M \in \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f} \operatorname{Ker} \operatorname{Tor}_i^R(R/I, -)$ if and only if $M^+ \in \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f} \operatorname{Ker} \operatorname{Ext}_R^i(R/I, -)$, and thus the former class is the tilting class associated to the latter cotilting class by Lemma 4.4.4(i). Similarly, $N \in \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f} (S_{I,i+1})^{\intercal}$ if and only if $N^+ \in \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f} (S_{I,i+1})^{\perp}$. As the latter class is of finite type, and thus definable, it is uniquely determined by its pure-injective objects ([21, Lemma 6.9]). Using Lemma 4.4.4(ii) and [21, Theorem 2.27(c)], we see that pure-injectives of this class coincide with pure-injectives of the tilting class associated to the cotilting class $\bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f} (S_{I,i+1})^{\intercal}$, and thus the two classes coincide.

Finally, by Theorem 4.4.2, resolving subcategories $\mathcal{S}$ as in (iv) correspond bijectively to tilting classes $\mathcal{T}$ via the assignment $\mathcal{T} \mapsto ({}^{\perp}\mathcal{T}) \cap \text{mod-R}$. Whenever $\mathcal{T} = \mathcal{S}^{\perp}$ for some set $\mathcal{S} \subseteq \text{mod-R}$, we have that ${}^{\perp}\mathcal{T}$ equals to all direct summands of all $\{R\} \cup \mathcal{S}$-filtered modules by Theorem 4.4.1, and thus ${}^{\perp}\mathcal{T} \cap \text{mod-R}$ equals to direct summands of all finitely $\{R\} \cup \mathcal{S}$-filtered modules. By $(i) \to (iii)$, we can chose $\mathcal{S} = \{S_{I,i+1} \mid I \in \mathcal{G}_i^f, i = 0, 1, \dots, n-1\}$, establishing $(i) \to (iv)$. $\square$

Remark 4.6.3. If the ring R is coherent, we can use a projective resolution of R/I in mod-R instead of the Koszul complex. Therefore, in this case $S_{I,i+1}$ can be replaced by module $\operatorname{Tr}(\Omega^i R/I)$ for each $i = 0, 1, \dots, n-1$, where Tr is the Auslander-Bridger transpose (cf. [6]).

Example 4.6.4. Let R be a commutative perfect ring. Then the only tilting class in Mod-R is the trivial one, that is the whole Mod-R. By Theorem 4.6.2, it is enough to show that if I is a finitely generated ideal such that $\operatorname{Hom}_R(R/I, R) = 0$, then $I = R$. Indeed, since I is finitely generated, the descending chain $(I^n \mid n \in \omega)$ stabilizes at some n (see [31, Theorem 23.20, p. 345]). Then either I is nilpotent, and thus $\operatorname{Hom}_R(R/I, R) \neq 0$ unless $I = R$, or I^n is idempotent. As I is finitely generated, the latter case implies that I^n is a direct summand of R, and thus again $\operatorname{Hom}_R(R/I, R) \neq 0$ unless $I = R$.

4.7 Derived functors of torsion and completion and Čech (co)homology

In Section 4.6, we characterized cofinite type n-cotilting classes over a commutative ring as classes of all modules which vanish in certain degrees of Koszul cohomologies, arising from a set of finitely generated ideals. In this section, we show that we can replace Koszul complexes by two kinds of more canonically defined cohomology theories associated to an ideal—Čech cohomology, and local cohomology. Our results are valid for a general commutative ring, even though the latter two cohomology theories do not in general coincide for non-noetherian rings. An analogous description of n-tilting classes via Čech homology and local homology will also be accomplished, after dealing with a few extra technical difficulties. The main sources we use in this section are [23, 37, 36, 40].

Throughout this section, let R be a commutative ring.

4.7.1 Local (co)homology

Given a finitely generated ideal I, there are two additive functors Mod-R $\to$ Mod-R arising from it—the I-torsion functor Γ_I and the I-adic completion functor Λ_I, defined for an R-module M as follows:

$$\Gamma_I(M) = \{m \in M \mid I^n m = 0 \text{ for some } n > 0\} = \varinjlim_{n\in\omega} \mathrm{Hom}_R(R/I^n, M),$$

$$\Lambda_I(M) = \varprojlim_{n\in\omega} M \otimes_R R/I^n.$$

A module M is said to be I-torsion if $\Gamma_I(M) = M$, and we denote the full subcategory of all I-torsion modules by $\mathcal{T}_I$. Then $\mathcal{T}_I$ is an abelian category with exact direct sums, and the embedding $\mathcal{T}_I \subseteq$ Mod-R is exact and clearly admits Γ_I as is its right adjoint.

$$\begin{tikzcd} \mathcal{T}_I \arrow[r, bend left, "\subseteq"] & \text{Mod-R} \arrow[l, bend left, "\Gamma_I"] \end{tikzcd} \tag{4.8}$$

In particular, Γ_I is left exact, and we can form the right derived functor $\mathbb{R}\Gamma_I$, called the *local cohomology* functor of I.

The situation is a bit more tricky in the case of completion functors. Following Positselski [37], we say that a module M is an I-contramodule provided that $\mathrm{Ext}^j_R(R[x_i^{-1}], M) = 0$ for $j = 0, 1$, and for $i = 1, 2, \ldots, n$, where $\{x_1, x_2, \ldots, x_n\}$ is a set of generators of I. By [37, p. 3880], the choice of generators does not matter, and this is a correct definition. Denote by $\mathcal{C}_I$ the full subcategory of all I-contramodules. Then $\mathcal{C}_I$ is an abelian category with exact products and the embedding $\mathcal{C}_I \subseteq$ Mod-R is exact and admits a left adjoint ([37, Proposition 2.1]). Following [37] we denote the left adjoint by Δ_I.

$$\begin{tikzcd} \mathcal{C}_I \arrow[r, bend right, "\subseteq"'] & \text{Mod-R} \arrow[l, bend right, "\Delta_I"'] \end{tikzcd} \tag{4.9}$$

However, usually Λ_I does not fit in this adjunction in place of Δ_I. Indeed, Λ_I can be neither left nor right exact, even over a noetherian ring. Nevertheless, we can compute the left derived functor $\mathbb{L}\Lambda_I$, and call it the *local homology*.

By [32, §I. Lemma 5.13], both adjunctions (4.8) and (4.9) survive passing to the (unbounded) derived category, and thus we have adjoint pairs:

$$\begin{tikzcd} \mathbf{D}(\mathcal{T}_I) \arrow[r, bend left] & \mathbf{D}(\text{Mod-R}) \arrow[l, bend left, "\mathbb{R}\Gamma_I"] \end{tikzcd} \tag{4.10}$$

and

$$
\begin{tikzcd}
\mathbf{D}(\mathcal{C}_I) \arrow[r, bend right] & \mathbf{D}(\text{Mod-R}). \arrow[l, bend right, "\mathbb{L}\Delta_I"']
\end{tikzcd}
\tag{4.11}
$$

However, note that in general not even in the derived picture we can swap $\mathbb{L}\Delta_I$ for $\mathbb{L}\Lambda_I$ (see [37, Example 2.6]). This will be further discussed in §4.7.4.

4.7.2 Čech (co)homology

As discussed in §4.3.2, the Koszul complex $K_\bullet(I)$ for a finitely generated ideal I is not a well-defined object even in the derived category, as the homology can change when passing from one generating system of I to another. This can be mended by stepping outside of the realm of perfect complexes and using generators of I to form a Čech cochain complex (also called a *stable Koszul complex*).

Definition 4.7.1. Let x be an element of R. The *Čech complex* with respect to x, denoted by $\check{C}^\bullet(x)$, which is defined as

$$0 \to R \xrightarrow{\iota} R_x \to 0,$$

where $R_x = R[x^{-1}]$, ι is the natural morphism, and the cochain complex is concentrated in (cohomological) degrees 0 and 1. Given a sequence $\mathbf{x} = (x_1, x_2, \dots, x_n)$ of elements of R, we define $\check{C}^\bullet(\mathbf{x})$ as the tensor product $\bigotimes_{i=1}^n \check{C}^\bullet(x_i)$.

Lemma 4.7.2. ([22, Corollary 3.12]) *Let I be a finitely generated ideal, and $\mathbf{x} = (x_1, x_2, \dots, x_n), \mathbf{y} = (y_1, y_2, \dots, y_m)$ two sequences of generators of I. Then the Čech complexes $\check{C}^\bullet(\mathbf{x})$ and $\check{C}^\bullet(\mathbf{y})$ are quasi-isomorphic.*

Lemma 4.7.2 legitimizes the following notation: Given a finitely generated ideal I with a finite sequence of generators $\mathbf{x}$, we denote $\check{C}^\bullet(I) = \check{C}^\bullet(\mathbf{x})$. The cochain complex $\check{C}^\bullet(I)$ is then well-defined as an object of the derived category.

Similar to Koszul complexes, we can compute Čech cohomology and homology. First we address the former, defined as follows:

$$\check{H}^i(I; -) = H^i(\check{C}^\bullet(I) \otimes_R^{\mathbb{L}} -).$$

As $\check{C}^\bullet(I)$ is a bounded complex of flat modules, we can drop the left derivation symbol $\mathbb{L}$ from the formula. This *Čech cohomology* can also be viewed as a limit version of Koszul cohomology in the following way. Given $x \in R$, the Koszul chain complexes $(K_\bullet(x^j) \mid j > 0)$ form an inverse system with the following connecting maps:

$$
\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\cdot x^j"] & R \arrow[r] & 0 \\
0 \arrow[r] & R \arrow[r, "\cdot x^{j+1}"] \arrow[u, "\cdot x"] & R \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}
\tag{4.12}
$$

Dualizing this with respect to R, we obtain a direct system $(K_\bullet(x^j)^* \mid j > 0)$ of cochain complexes, and it is easy to check that its limit is precisely $\check{C}^\bullet(x)$. As direct limit commutes with tensor product, we get $\check{C}^\bullet(I) = \varinjlim_{j>0} K_\bullet(\mathbf{x}^j)^*$, where we fix the notation

$$\mathbf{x}^j = (x_1^j, x_2^j, \dots, x_m^j)$$

for a set of generators $\mathbf{x} = \mathbf{x}^1$ of I. Given a module M, we infer from exactness of the direct limit functor the following isomorphism

$$\check{H}^i(I; M) \simeq \varinjlim_{j>0} H^i(\mathbf{x}^j; M). \tag{4.13}$$

Using this representation, we can already show that the Čech complexes classify the cofinite type cotilting classes (see also Theorem 4.7.7 below). In the following proofs, let always I_j denote the ideal generated by the sequence $\mathbf{x}^j$ (not to be confused with I^j).

Lemma 4.7.3. *Let R be a commutative ring, I a finitely generated ideal, and $n > 0$. Then*

$$\bigcap_{i=0}^{n-1} \operatorname{Ker} H^i(I; -) = \bigcap_{i=0}^{n-1} \operatorname{Ker} \check{H}^i(I; -).$$

Proof. We proceed by induction on $n \geq 0$. For the induction step, we fix throughout the proof a module

$$M \in \bigcap_{i=0}^{n-1} \operatorname{Ker} H^i(I; -) = \bigcap_{i=0}^{n-1} \operatorname{Ker} \check{H}^i(I; -)$$

(which is a vacuous assumption if $n = 0$), and prove that $H^n(I; M) = 0$ if and only if $\check{H}^n(I; M) = 0$. We recall the comparison map $q^n_M(j) : \operatorname{Ext}^n_R(R/I_j, M) \to H^n(\mathbf{x}^j; M)$ from §4.3.2. By Proposition 4.3.9, the map $q^n_M(j)$ is an isomorphism for any $j > 0$.

Assume first that $H^n(I; M) = 0$. By the pigeon hole principle, I_j/I_{j+1} is an $R/I^{m(j+1)}$-module for any $j > 0$, and thus R/I_j is finitely filtered by R/I-modules. Then $H^n(\mathbf{x}^j; M) \simeq \operatorname{Ext}^n_R(R/I_j, M) = 0$ by Lemma 4.3.7. Using (4.13), we infer $\check{H}^n(I; M) = 0$. This proves one implication.

To prove the other implication, assume that $\check{H}^n(I; M) = 0$. It is enough to show that the direct system $(H^n(\mathbf{x}^j; M) \mid j > 0)$ consists of monomorphisms, because then (4.13) immediately yields $H^n(I; M) = H^n(\mathbf{x}^1; M) = 0$. Observing that $H^0(\mathbf{x}^j; M) = \{m \in M \mid I_j m = 0\}$, where I_j is the ideal generated by $\mathbf{x}^j$, we infer that the directed system $(H^0(\mathbf{x}^j; M) \mid j > 0)$ consists of inclusions. For $n > 0$, we argue as follows. By Corollary 4.3.10, we have $M \in \bigcap_{i=0}^{n-1} \operatorname{Ker} \operatorname{Ext}^i_R(R/I, -)$. Consider the long exact sequence obtained by applying $\operatorname{Hom}_R(-, M)$ to the following exact sequence, where π is the natural projection:

$$0 \to I_j/I_{j+1} \to R/I_{j+1} \xrightarrow{\pi} R/I_j \to 0.$$

By the same argument using Lemma 4.3.7 as above, we have

$$\operatorname{Ext}^{n-1}_R(I_j/I_{j+1}, M) = 0.$$

It follows that the map $\operatorname{Ext}^n_R(\pi, M)$ is a monomorphism. From the construction and naturality of q^n_M, we infer that the there is a commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^n_R(R/I_j, R) \arrow[r, "q^n_M(j)"] \arrow[d, "\operatorname{Ext}^n_R(\pi, M)"] & H^n(\mathbf{x}^j; M) \arrow[d, "\phi_j"] \\
\operatorname{Ext}^n_R(R/I_{j+1}, R) \arrow[r, "q^n_M(j+1)"] & H^n(\mathbf{x}_{j+1}; M),
\end{tikzcd} \tag{4.14}$$

where ϕ_j is the j-th map from the direct system $(H^n(\mathbf{x}^j; M) \mid j > 0)$. Since the horizontal maps of (4.14) are isomorphisms, we finally infer that this direct system consists of monomorphisms, as desired. □

Now we treat the *Čech homology*, which we define, following [40], in this way:

$$\check{H}_i(I; -) = H_i(\mathbb{R}\operatorname{Hom}_R(\check{C}^\bullet(I), -)).$$

Because this functor *a priori* inhabitates strictly the derived category, we would like to replace $\check{C}^\bullet(I)$ by its projective resolution, in a way that respects the limit construction of $\check{C}^\bullet(I)$. To this end, we recall the *telescope cochain complex* (here we follow [36]). For any subset X of ω, let $F[X]$ be the free R-module with basis $\{\delta_j \mid j \in X\}$. Given an element $x \in R$ we let

$$\operatorname{Tel}(x) = (\cdots \to 0 \to F[\omega] \xrightarrow{d} F[\omega] \to 0 \to \cdots),$$

be the cochain complex concentrated in (cohomological) degrees 0 and 1, where the differential d is defined on the above basis as follows

$$d(\delta_j) = \begin{cases} \delta_0, & \text{if } j = 0, \\ \delta_{j-1} - x\delta_j, & \text{otherwise.} \end{cases}$$

For any $j > 0$, we let

$$\operatorname{Tel}_j(x) = (\cdots \to 0 \to F[j] \xrightarrow{d} F[j] \to 0 \to \cdots)$$

be the subcomplex of $\operatorname{Tel}(x)$, so that $\operatorname{Tel}(x) = \bigcup_{j>0} \operatorname{Tel}_j(x)$. More generally, given a sequence of elements $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of R, we let

$$\operatorname{Tel}_j(\mathbf{x}) = \bigotimes_{i=1}^{n} \operatorname{Tel}_j(x_i) \quad \text{and} \quad \operatorname{Tel}(\mathbf{x}) = \bigotimes_{i=1}^{n} \operatorname{Tel}(x_i).$$

Note that again $\operatorname{Tel}(\mathbf{x}) = \bigcup_{j>0} \operatorname{Tel}_j(\mathbf{x})$. It follows from ([36, Lemma 5.7]) that there are natural homotopy equivalences

$$w_{\mathbf{x},j} : \operatorname{Tel}_j(\mathbf{x}) \to K_\bullet(\mathbf{x}^j)^*,$$

such that their limit map

$$w_{\mathbf{x}} : \operatorname{Tel}(\mathbf{x}) \to \check{C}^\bullet(\mathbf{x})$$

is a quasi-isomorphism. If I is the ideal generated by $\mathbf{x}$, we can now represent the Čech homology as follows:

$$\begin{gathered} \check{H}_i(I; M) = H_i(\mathbb{R}\operatorname{Hom}_R(\check{C}^\bullet(I), M)) \simeq H_i(\operatorname{Hom}_R(\operatorname{Tel}(\mathbf{x}), M)) \simeq \\ \simeq H_i(\operatorname{Hom}_R(\varinjlim_{j>0} \operatorname{Tel}_j(\mathbf{x}), M)) \simeq H_i(\varprojlim_{j>0} \operatorname{Hom}_R(\operatorname{Tel}_j(\mathbf{x}), M)) \simeq \\ \simeq H_i(\varprojlim_{j>0} (\operatorname{Tel}_j(\mathbf{x})^* \otimes_R M)). \end{gathered}$$

Of course, in general, taking homology does not commute with inverse limits. On the other hand, the inverse system $(\operatorname{Tel}_j(\mathbf{x})^* \otimes_R M \mid j > 0)$ consists of epimorphisms, and thus satisfies the Mittag-Leffler condition (see [21, Definition

3.5 and Lemma 3.6]). Using [45, Theorem 3.5.8], we have for each $i \geq 0$ the following exact sequence:

$$0 \to \varprojlim{}^1_{j>0} H_{i+1}(\mathrm{Tel}_j(\mathbf{x})^* \otimes_R M) \to \check{H}_i(I; M) \to \varprojlim{}_{j>0} H_i(\mathrm{Tel}_j(\mathbf{x})^* \otimes_R M) \to 0.$$

Here, the symbol $\varprojlim^1_{j>0}$ stands for the first right derived functor of the inverse limit functor $\varprojlim_{j>0}$. Furthermore, because

$$w^*_{\mathbf{x},j} : K_\bullet(\mathbf{x}^j) \to \mathrm{Tel}_j(\mathbf{x})^*$$

is also a natural homotopy equivalence of complexes, we can rewrite this sequence as:

$$0 \to \varprojlim{}^1_{j>0} H_{i+1}(\mathbf{x}^j; M) \to \check{H}_i(I; M) \to \varprojlim{}_{j>0} H_i(\mathbf{x}^j; M) \to 0. \tag{4.15}$$

We are ready to prove that Čech complexes allow to classify tilting classes (see also Theorem 4.7.7 below).

Lemma 4.7.4. *Let R be a commutative ring, I a finitely generated ideal, and $n > 0$. Then*

$$\bigcap_{i=0}^{n-1} \mathrm{Ker}\, H_i(I; -) = \bigcap_{i=0}^{n-1} \mathrm{Ker}\, \check{H}_i(I; -).$$

Proof. We proceed again by induction on $n \geq 0$, and fix throughout the proof a module $M \in \bigcap_{i=0}^{n-1} \mathrm{Ker}\, H_i(I; -) = \bigcap_{i=0}^{n-1} \mathrm{Ker}\, \check{H}_i(I; -)$.

Suppose first that $H_n(I; M) = 0$. By Remark 4.3.11, $H_i(\mathbf{x}^j; M)$ is naturally isomorphic to $\mathrm{Tor}_i^R(R/I_j, M)$ for each $j > 0$ and $i = 0, 1, \ldots, n+1$. An argumentation analogous to the one in the proof of Lemma 4.7.3 then yields that $H_n(\mathbf{x}^j, M) = 0$ for each $j > 0$, and that the inverse system $(H_{n+1}(\mathbf{x}^j; M) \mid j > 0)$ consists of epimorphisms. In particular, this system is Mittag-Leffler, and thus [21, Lemma 3.6] implies that

$$\varprojlim{}^1_{j>0} H_{n+1}(\mathbf{x}^j; M) = 0.$$

Therefore, we can use (4.15) to infer that $\check{H}_n(I; M) \simeq \varprojlim_{j>0} H_n(\mathbf{x}^j; M) = 0$. This proves $\bigcap_{i=0}^{n-1} \mathrm{Ker}\, H_i(I; -) \subseteq \bigcap_{i=0}^{n-1} \mathrm{Ker}\, \check{H}_i(I; -)$.

To prove the other inclusion, suppose that $\check{H}_n(I; M) = 0$. By (4.15), this implies $\varprojlim_{j>0} H_n(\mathbf{x}^j; M) = 0$. Using again the same argument as above for homological degree shifted by -1, the inverse system $(H_n(\mathbf{x}^j; M) \mid j > 0)$ consists of epimorphisms (in the initial case of $n = 0$, it consists of projections $M/I_{j+1}M \to M/I_jM$). It follows that $H_n(\mathbf{x}; M) = 0$, and thus $H_n(I; M) = 0$. □

4.7.3 Main theorem revisited

In this section, we show that instead of Ext (Tor) or Koszul (co)homology, we can use either local (co)homology, or Čech (co)homology, in the formulation of Theorem 4.6.2. We prove the remaining parts in the following Lemmas, and then state the alternative classification Theorem.

Lemma 4.7.5. *Let R be a commutative ring and $\mathfrak{S} = (\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1})$ a characteristic sequence of length n. Then*

$$\mathcal{C}(\mathfrak{S}) = \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} \{M \in \text{Mod-R} \mid \mathbb{R}^i\Gamma_I(M) = 0\}.$$

Proof. This follows directly from the definition of $\mathcal{C}(\mathfrak{S})$ (see Notation 4.5.2). Indeed, if $I \in \mathcal{G}_m^f$ and $0 \leq m < n$, then a module $M \in \mathcal{C}(\mathfrak{S})$ must be in the class

$$\bigcap_{i=0}^{m} \operatorname{Ker} \operatorname{Hom}_R(R/I, \Omega^{-i}(-)) = \bigcap_{i=0}^{m} \operatorname{Ker} \Gamma_I(\Omega^{-i}(-)).$$

We prove by induction on $j \leq m$ that $\mathbb{R}^j\Gamma_I(M) \simeq \Gamma_I(\Omega^{-j}M)$ for each $M \in \operatorname{Ker} \operatorname{Hom}_R(R/I, \Omega^{-(j-1)}(-)) = \operatorname{Ker} \Gamma_I(\Omega^{-(j-1)}(-))$ (this condition is vacuous for $j = 0$). This follows from the definition of Γ_I for $j = 0$, and for $j > 0$ note that $\Gamma_I(\Omega^{-(j-1)}M) = 0$ implies $\Gamma_I(E(\Omega^{-(j-1)}M)) = 0$ and we have an exact sequence

$$0 = \Gamma_I(E(\Omega^{-(j-1)}M)) \to \Gamma_I(\Omega^j M) \to \mathbb{R}^j\Gamma_I(M) \to 0. \qquad \square$$

Lemma 4.7.6. *Let R be a commutative ring, I a finitely generated ideal, and $n > 0$. Then:*

$$\bigcap_{i=0}^{n-1} \operatorname{Ker} \operatorname{Tor}_i^R(R/I, -) = \bigcap_{i=0}^{n-1} \{M \in \text{Mod-R} \mid \mathbb{L}_i\Lambda_I(M) = 0\}.$$

Proof. The shape of the proof is the same as that of Lemma 4.7.4, using the exact sequence [23, Proposition 1.1] instead of (4.15). For the convenience of the reader, we provide details here. We proceed by induction on $n \geq 0$ (the case of $n = 0$ is a vacuous statement). For the induction step, we will assume that $M \in \bigcap_{i=0}^{n-1} \operatorname{Ker} \mathbb{L}_i\Lambda_I$. By [23, Proposition 1.1], there is an exact sequence:

$$0 \to \varprojlim{}^1_{j>0} \operatorname{Tor}{}^R_{n+1}(R/I^j, M) \to \mathbb{L}_n\Lambda_I(M) \to \varprojlim_{j>0} \operatorname{Tor}{}^R_n(R/I^j, M) \to 0. \quad (4.16)$$

If $\mathbb{L}_n\Lambda_I(M) = 0$, then rightmost term also vanishes. By the induction hypothesis we have $M \in \bigcap_{i=0}^{n-1} \operatorname{Ker} \operatorname{Tor}_i^R(R/I, -)$. Applying $- \otimes_R M$ to the exact sequence

$$0 \to I^j/I^{j+1} \to R/I^{j+1} \to R/I^j \to 0,$$

and noting that $\operatorname{Tor}_{n-1}^R(I^j/I^{j+1}, M) = 0$, we infer that the inverse system

$$(\operatorname{Tor}{}^R_n(R/I^j, M) \mid j > 0)$$

consists of epimorphisms. This shows that $\operatorname{Tor}_n^R(R/I, M) = 0$, proving one inclusion. To prove the other inclusion, suppose now that $\operatorname{Tor}_n^R(R/I, M) = 0$. It follows easily that $\operatorname{Tor}_n^R(R/I^j, M) = 0$ for all $j > 0$, and thus the rightmost term of (4.16) is zero. By repeating the same argument as above for a homological degree shifted by 1, we obtain that the inverse system $(\operatorname{Tor}{}^R_{n+1}(R/I^j, M) \mid j > 0)$ consists of epimorphisms, and thus is Mittag-Leffler. Hence, the leftmost term of (4.16) also vanishes by [21, Lemma 3.6], and thus $\mathbb{L}_n\Lambda_I(M) = 0$, finishing the proof. $\square$

Theorem 4.7.7. *Let R be a commutative ring. Consider the following collections:*

(i) characteristic sequences $\mathfrak{S} = (\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1})$ of length n,

(ii) n-tilting classes in Mod-R*,*

(iii) n-cotilting classes of cofinite type in Mod-R*,*

The following assignments are bijections $(i) \to (ii)$:

$$\mathfrak{S} \mapsto \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} \{M \in \text{Mod-R} \mid \mathbb{L}_i\Lambda_I(M) = 0\},$$

$$\mathfrak{S} \mapsto \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} \{M \in \text{Mod-R} \mid \check{H}_i(I; M) = 0\},$$

and the following assignments are bijections $(i) \to (iii)$:

$$\mathfrak{S} \mapsto \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} \{M \in \text{Mod-R} \mid \mathbb{R}_i\Gamma_I(M) = 0\},$$

$$\mathfrak{S} \mapsto \bigcap_{i=0}^{n-1} \bigcap_{I \in \mathcal{G}_i^f(\mathfrak{S})} \{M \in \text{Mod-R} \mid \check{H}^i(I; M) = 0\}.$$

Proof. Follows by putting together Theorem 4.6.2, Corollary 4.3.10, and Lemmas 4.7.5, 4.7.6, 4.7.3, and 4.7.4. □

4.7.4 The big picture

As the four homological and four cohomological theories used in the classification Theorems 4.6.2 and 4.7.7 may feel a little overwhelming, we devote this and the next subsection to a short explanation of the relations between these. In particular we show that the Čech (co)homology, analogously to the local (co)homology, also induces a pair of adjoint functors between derived categories, and that there are always comparison functors between the Čech and local (co)homologies which are equivalences under a technical condition.

Although in this material is not really new, it requires some effort to extract it from the existing literature [1, 23, 36, 40]. Here we especially rely on a recent and original presentation in [37, Theorem 3.4]. In fact, all we want to say is essentially a reformulation of [37, Theorem 3.4] and its proof, and we refer the reader to [37] for a more comprehensive treatment. We will freely use the theory of localization of the derived category, as well as recollements and their translation to TTF triples. For these we refer to [13], [30], or [34].

Given a full subcategory $\mathcal{A}$ of Mod-R, let us denote by $\mathbf{D}_{\mathcal{A}}$(Mod-R) the subcategory of $\mathbf{D}$(Mod-R) consisting of all complexes such that their homology modules live in $\mathcal{A}$. If $\mathcal{A}$ is an extension closed abelian subcategory of Mod-R, then $\mathbf{D}_{\mathcal{A}}$(Mod-R) is a triangulated subcategory of $\mathbf{D}$(Mod-R). Given a set of objects $\mathcal{S}$ in $\mathbf{D}$(Mod-R), let Loc($\mathcal{S}$) be the smallest localizing subcategory of $\mathbf{D}$(Mod-R) containing $\mathcal{S}$. Let I be an ideal generated by $\mathbf{x} = (x_1, x_2, \dots, x_n)$. By [37, Proposition 5.1], the category $\mathbf{D}_{\mathcal{T}_I}(\text{Mod-R})$ is generated (as a triangulated subcategory) by the compact object $\operatorname{Tel}_j(\mathbf{x})$ for any fixed $j > 0$. Using [29, Proposition 2.1.2] (see also [18, §6]), we have

$$\operatorname{Loc}(R/I) = \operatorname{Loc}(K_\bullet(I)) = \mathbf{D}_{\mathcal{T}_I}(\text{Mod-R}).$$

Using the machinery of localization theory of triangulated categories (see [30], namely [30, 4.9.1, 4.13.1, 5.3.1, 5.4.1, 5.5.1]) there is a recollement (we adopt the convention that going up amounts to taking *left adjoints*)

$$\begin{tikzcd}
\mathbf{D}_{\mathcal{T}_I}(\text{Mod-R})^{\perp} \arrow[r, "\subseteq"] & \mathbf{D}(\text{Mod-R}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "T"] & \mathbf{D}_{\mathcal{T}_I}(\text{Mod-R}), \arrow[l, bend right, "\iota"'] \arrow[l, bend left, "H"]
\end{tikzcd} \tag{4.17}$$

corresponding (as in [34, 2.1]) to the *TTF triple*

$$(\operatorname{Loc}(R/I), \mathcal{Y}, \mathcal{Z}),$$

where $\mathcal{Y} = \mathbf{D}_{\mathcal{T}_I}(\text{Mod-R})^{\perp}$. By [29, Theorem 2.2.4], we also have

$$\mathcal{Y} = \operatorname{Loc}(R_{x_1}, R_{x_2}, \dots, R_{x_n}). \tag{4.18}$$

In particular it is a localizing subcategory, and hence a *tensor ideal* by [29, Lemma 1.1.8]. Consider the triangle

$$\check{C}^\bullet(I) \xrightarrow{f} R \to \operatorname{Cone}(f) \to \Sigma\check{C}^\bullet(I), \tag{4.19}$$

induced by the identity map $R \to R$ in degree 0. Let M be a complex and apply $- \otimes^{\mathbb{L}}_R M$ to (4.19) in order to obtain a triangle

$$\check{C}^\bullet(I) \otimes_R M \xrightarrow{f \otimes_R M} M \longrightarrow \operatorname{Cone}(f) \otimes_R M \longrightarrow \Sigma\check{C}^\bullet(I) \otimes_R M. \tag{4.20}$$

Then $\check{C}^\bullet(I) \otimes_R M \in \mathbf{D}_{\mathcal{T}_I}(\text{Mod-R})$ by [37, Lemma 1.1]. Note that $\operatorname{Cone}(f)$ is quasi-isomorphic to the complex

$$\bigoplus_{i=1}^{n} R_{x_i} \to \bigoplus_{1 \le i < j \le n} R_{x_i, x_j} \to \cdots \to R_{x_1, x_2, \dots, x_n}.$$

Since $\mathcal{Y}$ is thick and a tensor ideal, it follows from 4.18 that $\operatorname{Cone}(f) \otimes_R M \in \mathcal{Y}$. Then (4.20) is the approximation triangle for M with respect to the torsion pair $(\operatorname{Loc}(R/I), \mathcal{Y})$, and thus by [13, Proposition 1.3.3] or [29, Theorem 1.1.9], the right adjoint T is equivalent to $\check{C}^\bullet(I) \otimes_R -$.

Since T composed with the inclusion ι is equivalent to $\check{C}^\bullet(I) \otimes^{\mathbb{L}}_R -$, passing to right adjoints we obtain $HT \simeq \mathbb{R}\operatorname{Hom}_R(\check{C}^\bullet(I), -)$. From the description of recollements arising from TTF triples (see [34, 2.1]), we get

$$\mathbb{R}\operatorname{Hom}_R(\check{C}^\bullet(I), -) \simeq HT \simeq \tau^{\mathcal{Z}}\iota T,$$

where $\tau^{\mathcal{Z}}$ is the left adjoint to the inclusion $\mathcal{Z} \subseteq \mathbf{D}(\text{Mod-R})$. Since ιT is a triangle equivalence $\operatorname{Loc}(R/I) \to \mathcal{Z}$, we finally infer that $\mathbb{R}\operatorname{Hom}_R(\check{C}^\bullet(I), -)$ is the left adjoint to the inclusion $\mathcal{Z} \subseteq \mathbf{D}(\text{Mod-R})$. Now it can be easily checked that the triangle obtained by applying $\mathbb{R}\operatorname{Hom}_R(\check{C}^\bullet(I), -)$ to (4.19) is the approximation triangle with respect to the torsion pair $(\mathcal{Y}, \mathcal{Z})$. This yields that a complex M is in $\mathcal{Z}$ if and only if the natural map $\mathbb{R}\operatorname{Hom}_R(\check{C}^\bullet(I), f) : M \mapsto \mathbb{R}\operatorname{Hom}_R(\check{C}^\bullet(I), M)$ is an isomorphism. Combining [37, Lemma 2.2 a),c)], and the proof of [37, Theorem 3.4], we conclude that $\mathcal{Z} = \mathbf{D}_{\mathcal{C}_I}(\text{Mod-R})$, where $\mathcal{C}_I$ is the subcategory of I-contramodules (see §4.7.1).

To summarize, since a composition of right (left) adjoints is a right (left) adjoint, respectively, we have two compositions of adjoint pairs depicted in (4.21) and (4.22). Here $F := \mathbb{R}\Gamma_I \restriction \mathbf{D}_{\mathcal{T}_I}(\text{Mod-R})$ is the right adjoint of the canonical functor $\mathbf{D}(\mathcal{T}_I) \to \mathbf{D}_{\mathcal{T}_I}(\text{Mod-R})$ and similarly G is the left adjoint of the canonical functor $\mathbf{D}(\mathcal{C}_I) \to \mathbf{D}_{\mathcal{C}_I}(\text{Mod-R})$. Both $\mathbb{R}\Gamma_I$ and $\mathbb{L}\Delta_I$ are then naturally equivalent to the compositions of the corresponding two "short" adjoints pointing to the left:

$$\begin{tikzcd}
\mathbf{D}(\mathcal{T}_I) \arrow[r, bend left] & \mathbf{D}_{\mathcal{T}_I}(\text{Mod-R}) \arrow[l, bend left, "F"] \arrow[r, bend left, "\subseteq"] & \mathbf{D}(\text{Mod-R}) \arrow[l, bend left, "\check{C}^\bullet(I)\otimes_R -"] \arrow[ll, bend left=60, "\mathbb{R}\Gamma_I"]
\end{tikzcd} \tag{4.21}$$

$$\begin{tikzcd}
\mathbf{D}(\mathcal{C}_I) \arrow[r, bend right] & \mathbf{D}_{\mathcal{C}_I}(\text{Mod-R}) \arrow[l, bend right, "G"'] \arrow[r, bend right, "\subseteq"'] & \mathbf{D}(\text{Mod-R}) \arrow[l, bend right, "\mathbb{R}\operatorname{Hom}_R(\check{C}^\bullet(I), -)"'] \arrow[ll, bend right=60, "\mathbb{L}\Delta_I"']
\end{tikzcd} \tag{4.22}$$

There is a technical condition on I, so-called weak proregularity of I, which ensures (and in fact is equivalent to) that both adjunctions on the left in (4.21) and (4.22) are in fact equivalences (see [37, Theorem 1.3, Corollary 2.10]). We will discuss this in §4.7.5

Here we conclude by noting that in such a case, the local (co)homology coincides with the Čech (co)homology. Furthermore, weak proregularity also implies that $\mathbb{L}\Delta_I$ is naturally equivalent to $\mathbb{L}\Lambda_I$ ([37, Lemma 2.5]) and, since $\mathbf{D}_{\mathcal{T}_I}(\text{Mod-R})$ and $\mathbf{D}_{\mathcal{C}_I}(\text{Mod-R})$ are always equivalent (cf. the recollement (4.17) or [37, Theorem 3.4]), weak proregularity also implies that:

$$\mathbf{D}(\mathcal{T}_I) \simeq \mathbf{D}(\mathcal{C}_I).$$

The latter statement is known as the *Matlis-Greenlees-May duality*, [18, 36, 37].

4.7.5 Weak proregularity

A classical result ([24]) says that, over a commutative noetherian ring, the local cohomology coincides with the Čech cohomology. The dual result for the left derived completion functor and Čech homology ([23, 36, 40, 1]) is a much more recent development. However, over a general commutative ring, the local and Čech (co)homologies need not be the same, despite the fact that we can use either of them in Theorem 4.7.7 to classify (co)tilting classes. Here we gather relevant results to understand the issue.

Definition 4.7.8 ([40, §2])**.**

1. An inverse system $(M_i, f_{ji} \mid j \geq i)$ of modules is *pro-zero* if for every i there is $j \geq i$ such that f_{ji} is zero.

2. Let R be $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a sequence of elements of a commutative ring R. We say that $\mathbf{x}$ is *weakly proregular* if the inverse system $(H_i(\mathbf{x}^j; R) \mid j > 0)$ (see §4.7.1) is pro-zero for each $i > 0$.

Fact 4.7.9. ([36, Corollary 6.2]) The weak proregularity of $\mathbf{x}$ depends only on the ideal I generated by $\mathbf{x}$ (in fact only on $\sqrt{I}$). This legitimizes us to define a finitely generated ideal I to be *weakly proregular*, if any of its finite generating sequences is weakly proregular.

If R is noetherian, then any ideal is weakly proregular ([36, Theorem 4.34]). On the other hand, over a general commutative rings, there can easily be non-weakly proregular finitely generated ideals ([23, Example 1.4]). It turns out that this property characterizes precisely when the local (co)homology of an ideal coincides with the Čech (co)homology. The result on the side of cohomology is [40, Theorem 3.2]. In the same paper, the homology analog is proved under the extra assumption that each element of the generator sequence forms itself a one-element weakly proregular sequence ([40, Theorem 4.5]). This extra assumption was removed in [36].

Theorem 4.7.10. *Let R be a commutative ring and I a finitely generated ideal. Then the following are equivalent:*

(a) I is weakly proregular,

(b) the functorial map

$$\mathbb{R}\Gamma_I(M) \to \check{C}^\bullet(I) \otimes_R M$$

is a quasi-isomorphism for each module M,

(c) the functorial map

$$\mathbb{R}\operatorname{Hom}_R(\check{C}^\bullet(I), -) \to \mathbb{L}_i\Lambda_I(M)$$

is a quasi-isomorphism for each module M.

Proof. Although all the difficult steps have already been carried out by the aforementioned authors, we need to make a few explanations to establish the full equivalence in this form.

The equivalence $(a) \leftrightarrow (b)$ is a slight reformulation of the one in [40]—our statement is a weakened version of statement (iii) of [40, Theorem 3.2], which easily implies (ii), and thus also (i) by the proof.

The equivalence $(a) \leftrightarrow (c)$ is explained as follows. The implication $(a) \rightarrow (c)$ is proved in [36, Corollary 5.25]. The converse implication follows from the proof of [40, Theorem 4.5]. Indeed, both the implications $(iii) \rightarrow (iv)$ and $(iv) \rightarrow (i)$ of [40, Theorem 4.5] do not use (or need) the assumption of "bounded torsion" imposed in the statement of [40, Theorem 4.5]. □

Therefore, any example of a non-weakly proregular ideal I (such as [23, Example 1.4]) gives a situation where both the local cohomology and homology are not computed via the Čech complex. Indeed, not only the functorial map from Theorem 4.7.10 fails to be a quasi-isomorphism, but inspecting the proofs in [40], some flat (injective) module has a non-zero higher Čech homology (cohomology), but the local homology (cohomology) will vanish, respectively.

4.8 Construction of the corresponding cotilting modules

In this section we construct to each cotilting class of cofinite type over a commutative ring a cotilting module inducing it. The construction generalizes ideas from [42].

Lemma 4.8.1. *Let R be a ring and $\mathcal{C}$ a n-cotilting class in* R-Mod. *Suppose that C is a left R-module satisfying ${}^{\perp}C = \mathcal{C}$, $C \in \mathcal{C}$, and $\mathcal{C} \subseteq \mathrm{Cogen}(C)$. Then C is a cotilting module.*

Proof. We prove that $\mathrm{Cogen}_n(C) = \mathcal{C}$, which implies that C is a n-cotilting module by [8, Theorem 3.11]. Since $\mathcal{C}$ is a n-cotilting class and $C \in \mathcal{C}$, we have inclusion $\mathrm{Cogen}_n(C) \subseteq \mathcal{C}$. To show the other inclusion, let $M \in \mathcal{C}$, put $I = \mathrm{Hom}_R(M, C)$, and let $\varphi : M \to C^I$ be the coevaluation map. Since $M \in \mathrm{Cogen}(C)$, we have that the map φ is injective. Applying $\mathrm{Hom}_R(-, C)$ onto the exact sequence $0 \to M \xrightarrow{\varphi} C^I \to X \to 0$ yields

$$\mathrm{Hom}_R(C^I, C) \xrightarrow{\mathrm{Hom}_R(\varphi, C)} \mathrm{Hom}_R(M, C) \to \mathrm{Ext}^1_R(X, C) \to \mathrm{Ext}^1_R(C^I, C) = 0.$$

As $\mathrm{Hom}_R(\varphi, C)$ is clearly surjective, we have that $\mathrm{Ext}^1_R(X, C) = 0$. Using the fact that $C^I, M \in \mathcal{C} = {}^{\perp}C$, we infer that $X \in \mathcal{C} = {}^{\perp}C$. Repeating this argument shows that indeed $M \in \mathrm{Cogen}_n(C)$. $\square$

Corollary 4.8.2. *Let R be a commutative ring and C a cotilting module such that the induced cotilting class $\mathcal{C} = {}^{\perp}C$ is of cofinite type. Let $\mathfrak{S}$ be the characteristic sequence such that $\mathcal{C} = \mathcal{C}(\mathfrak{S})$. Let W_j be an injective module such that $\mathrm{Cogen}(W_j) = \mathcal{F}_j(\mathfrak{S})$. Then $\Omega^{-j}(C) \oplus W_j$ is a cotilting module inducing the cotilting class $\mathcal{C}_{(j)}$.*

Proof. Put $C' = \Omega^{-j}C \oplus W_j$. We clearly have ${}^{\perp}C' = \mathcal{C}_{(j)}$, Lemma 4.5.14 gives $C' \in \mathcal{C}_{(j)}$, and $\mathcal{C}_{(j)} \subseteq \mathcal{F}_j(\mathfrak{S}) = \mathrm{Cogen}(C')$. Therefore, Lemma 4.8.1 implies that C' is a cotilting module. $\square$

In the rest of the section let R be a commutative ring and fix a characteristic sequence $\mathfrak{S} = (\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1})$. Our goal is to construct a cotilting module $C(\mathfrak{S})$ such that ${}^{\perp}C(\mathfrak{S}) = \mathcal{C}(\mathfrak{S})$.

Construction 4.8.1. We aim to construct a (co)complex of injective modules

$$0 \to E^0 \xrightarrow{\psi_0} E^1 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_{n-1}} E^n \xrightarrow{\psi_n} 0 (= E^{n+1}),$$

where E^i is in cohomological degree i, satisfying the following properties:

(i) the cohomology of the complex vanishes with the exception of degree 0,

(ii) for each $i = 0, 1, \dots, n$, the kernel C^i of ψ_i is a $(n-i)$-cotilting module such that ${}^\perp C^i = \mathcal{C}_{(i)}$.

We construct the complex by backwards induction on $i = n, n-1, \dots, 1, 0$. For the step $i = n$, we let E^n be an injective cogenerator of Mod-R, and ψ_n be the zero map.

Suppose that we have already constructed the complex down to degree $k+1$ for some $0 \leq k < n$ so that it is exact in degrees $> k+1$ and satisfies (ii). By [21, Theorem 15.9], there is a $\mathcal{C}_{(k)}$-cover $f \colon F^k \to C^{k+1}$ of C^{k+1}.

Lemma 4.8.3. *The module F^k is injective.*

Proof. Because $F^k \in \mathcal{C}_{(k)}$, we have by Lemma 4.5.14(ii) that $\Omega^{-1}F^k \in \mathcal{C}_{(k+1)}$. By the inductive premise, C^{k+1} is a cotilting module such that ${}^\perp C^{k+1} = \mathcal{C}_{(k+1)}$, and therefore $C^{k+1} \in \mathcal{C}_{(k+1)}^\perp$. It follows that the cover $f \colon F^k \to C^{k+1}$ can be extended to a map $f' \colon E(F^k) \to C^{k+1}$. As $E(F^k) \in \mathcal{C}_{(k)}$ by Proposition 4.5.5, it can be easily seen that f' is an $\mathcal{C}_{(k)}$-precover of C^{k+1}. Therefore, F^k is a direct summand of $E(F^k)$, proving that F^k is injective. □

Now we let $E^k = F^k \oplus W^k$, where W^k is any injective module such that $\operatorname{Cogen}(W^k) = \mathcal{F}_k$ (e.g. $W^k = \prod\{E(R/J) \mid R/J \in \mathcal{F}_k\}$). We define $\psi_k : E^k \to E^{k+1}$ by setting $\psi_k\restriction F^k = f$ and $\psi_k\restriction W^k = 0$. Since W^k is injective and belongs to $\mathcal{F}_k$, it is in $\mathcal{C}_{(k)}$, and then it easily follows that $\psi_k : E^k \to C^{k+1}$ is a $\mathcal{C}_{(k)}$-precover of $C^{k+1} = \operatorname{Ker}(\psi_{k+1}) = \operatorname{Im}(\psi_k)$.

Lemma 4.8.4. *We have ${}^\perp C^k = \mathcal{C}_{(k)}$ and $C^k \in \mathcal{C}_{(k)}$.*

Proof. By the inductive premise of the construction, we know ${}^\perp\Omega^{-1}C^k = \mathcal{C}_{(k+1)}$, and thus $\bigcap_{j>1} \operatorname{Ker}\operatorname{Ext}^j_R(-, C^k) = \mathcal{C}_{(k+1)}$. Let $M \in \mathcal{C}_{(k+1)}$. Then $\operatorname{Ext}^1_R(M, C^k) = 0$ if and only if any map $g \colon M \to C^{k+1}$ can be factorized through $\psi_k \colon E^k \to C^{k+1}$. If $M \in \mathcal{C}_{(k)}$, then this factorization is always possible, because ψ_k is an $\mathcal{C}_{(k)}$-precover. This proves that $\mathcal{C}_{(k)} \subseteq {}^\perp C^k$.

For the converse inclusion, suppose that $M \notin \mathcal{C}_{(k)}$. Consider first the case where even $M \notin \mathcal{F}_k$ and let T be the torsion part of M with respect to the torsion pair $(\mathcal{T}_k, \mathcal{F}_k)$. By construction, C^{k+1} has an injective direct summand W^{k+1} which cogenerates the torsion-free class $\mathcal{F}_{k+1}$. Since M and also T belong to $\mathcal{F}_{k+1}$, there is a non-zero map $T \to W^{k+1}$, which extends to a map $g \colon M \to W^{k+1} \subseteq C^{k+1}$. Such map does not factor through ψ_k since $\operatorname{Hom}_R(T, E^k) = 0$. The remaining case is where $M \in \mathcal{F}_k$, but $\Omega^{-1}M \notin \mathcal{C}_{(k+1)}$. Consider the long exact sequence obtained by applying $\operatorname{Hom}_R(-, C^k)$ to

$$0 \to M \to E(M) \to \Omega^{-1}M \to 0.$$

Since $M \in \mathcal{F}_k$, then necessarily $E(M) \in \mathcal{C}_{(k)}$, and thus we obtain isomorphisms $\operatorname{Ext}^i_R(M, C^k) \simeq \operatorname{Ext}^{i+1}_R(\Omega^{-1}M, C^k)$ for all $i > 0$. Hence, if $\Omega^{-1}M \notin \mathcal{C}_{(k+1)}$, then $M \notin {}^\perp C^k$. This finishes the proof of ${}^\perp C^k = \mathcal{C}_{(k)}$.

Finally, that $C^k \in \mathcal{C}_{(k)}$ follows from Lemma 4.5.14 and from $\Omega^{-j}C^k$ being a direct summand of C^{k+j} for each $j = 1, 2, \dots, n-k-1$. □

Lemma 4.8.5. *The module C^k is an $(n-k)$-cotilting module.*

Proof. Since $\mathrm{Cogen}(C^k) = \mathcal{F}_k \supseteq \mathcal{C}_{(k)}$, and by Lemma 4.8.4, the module C^k and the $(n-k)$-cotilting class $\mathcal{C}_{(k)}$ satisfy the hypothesis of Lemma 4.8.1. Therefore, C^k is $(n-k)$-cotilting by that lemma. □

This concludes the inductive step, and hence also the construction.

Notation 4.8.6. We put $C(\mathfrak{S}) = \mathrm{Ker}(\psi_0)$, and conclude:

Theorem 4.8.7. *Let R be a commutative ring. Then the set*

$$\{C(\mathfrak{S}) \mid \mathfrak{S} \text{ a characteristic sequence of length } n\}$$

parametrizes the equivalences classes of all n-cotilting modules of cofinite type in Mod-R*.*

4.9 Examples of cotilting classes not of cofinite type

We conclude the paper by providing intriguing examples of cotilting classes which are not of cofinite type, but which are in some sense difficult to tell apart from classes of cofinite type. To explain the issue, we first give a characterization of cotilting classes of cofinite type which follows from the proof of Theorem 4.5.3.

Theorem 4.9.1. *Let R be a commutative ring and $\mathcal{C}$ an n-cotilting class in* Mod-R*. Then $\mathcal{C}$ is of cofinite type if and only if $\mathcal{C}_{(i)}$ is closed under injective envelopes for all $i = 0, 1, \ldots, n-1$.*

Proof. If $\mathcal{C}$ is of cofinite type, then $\mathcal{C}_{(i)}$ is easily seen to be of cofinite type too for any $i = 0, 1, \ldots, n-1$. Then $\mathcal{C}_{(i)}$ is closed under injective envelopes by Proposition 4.5.5.

The other implication follows from Lemma 4.5.7, the proof of Lemma 4.5.8, and Lemma 4.5.13. □

In the rest of the section, we exhibit for each $n \geq 2$ a concrete example of an n-cotilting class which is *not* of cofinite type, but for which $\mathcal{C}_{(i)}$ is closed under injective envelopes for all $i = 0, 1, \ldots, n-2$. To this end, we first recall a characterization of cotilting classes which is valid for any (even non-commutative) ring:

Proposition 4.9.2. ([6, Proposition 3.14]) *Let R be a ring, $n \geq 0$, and $\mathcal{C}$ a class of left R-modules. Then $\mathcal{C}$ is n-cotilting if and only if the following conditions hold:*

(i) $\mathcal{C}$ is definable,

(ii) $R \in \mathcal{C}$ and $\mathcal{C}$ is closed under extensions and syzygies,

(iii) each n-th syzygy module is in $\mathcal{C}$,

In particular, a class of left R-modules is 1*-cotilting precisely when it is a definable torsion-free class containing R.*

In order to construct the aforementioned examples, we need a suitable family of examples of non-cofinite type 1-cotilting classes to start with. The following is a generalization of [9, Proposition 4.5]:

Example 4.9.3. Let R be a local commutative ring admitting a non-trivial idempotent ideal J. Let $\mathcal{G}$ be a Gabriel topology of finite type such that $J \in \mathcal{G}$, and such that $\mathcal{G}$ is *faithful*, i.e. $\operatorname{Hom}_R(R/I, R) = 0$ for all $I \in \mathcal{G}$. Let $\mathcal{F} = \bigcap_{I \in \mathcal{G}} \operatorname{Ker} \operatorname{Hom}_R(R/I, -)$ be the torsion-free class of the associated hereditary torsion pair of finite type. Given a module M, let $\operatorname{Soc}_J(M) = \{m \in M \mid Jm = 0\}$. We define a class $\mathcal{C}$ as follows:

$$\mathcal{C} = \{M \in \text{Mod-R} \mid M/\operatorname{Soc}_J(M) \in \mathcal{F}\}.$$

Alternatively,

$$\mathcal{C} = \{M \in \text{Mod-R} \mid Jt_{\mathcal{G}}(M) = 0\} = \{M \in \text{Mod-R} \mid t_{\mathcal{G}}(M) \in \text{Mod-}R/J\},$$

where $t_{\mathcal{G}}$ is the torsion radical associated to the torsion pair $(\mathcal{T}, \mathcal{F})$.

We will show that $\mathcal{C}$ is a 1-cotilting class, but not of cofinite type. First, we claim that $\mathcal{C} = \bigcap_{I \in \mathcal{G}} \operatorname{Ker} \operatorname{Hom}_R((J+I)/I, -)$. Let M be a module such that there is a non-zero map $f\colon (J+I)/I \to M$ for some $I \in \mathcal{G}$. Since $(J+I)/I$ is J-divisible (i.e. $((J+I)/I) \cdot J = (J+I)/I$), we have $\operatorname{Im} f \cap \operatorname{Soc}_J(M) = 0$. Thus, if we compose f with the surjection $M \to M/t_{\mathcal{G}}(M)$, we obtain a non-zero map $(J+I)/I \to M/t_{\mathcal{G}}(M)$. Since $(J+I)/I \in \mathcal{T}$, it follows that $M/t_{\mathcal{G}}(M) \notin \mathcal{F}$ and $M \notin \mathcal{C}$. This establishes one inclusion of the claim.

Let now M be such that $\operatorname{Hom}_R((J+I)/I, M) = 0$ for all $I \in \mathcal{G}$, and let us show that $M \in \mathcal{C}$. Towards contradiction, suppose that there is non-zero map $f\colon R/I \to M/\operatorname{Soc}_J(M)$ for some $I \in \mathcal{G}$. Because $\operatorname{Soc}_J(M/\operatorname{Soc}_J(M)) = 0$, applying $\operatorname{Hom}_R(-, M/\operatorname{Soc}_J(M))$ to $0 \to (J+I)/I \to R/I \to R/(J+I) \to 0$ yields an exact sequence

$$0 \to \operatorname{Hom}_R(R/I, M/\operatorname{Soc}_J(M)) \to \operatorname{Hom}_R((J+I)/I, M/\operatorname{Soc}_J(M)).$$

Hence f restricts to a non-zero map $\tilde{f}\colon (J+I)/I \to M/\operatorname{Soc}_J(M)$. Let us denote the image of $\tilde{f}$ by $\tilde{X}$ and by X the full preimage of $\tilde{X}$ in M with respect to the projection $M \to M/\operatorname{Soc}_J(M)$. We have $IX \subseteq \operatorname{Soc}_J(M)$, and thus $IJX = 0$. Observe that if $JX \neq 0$, we would have a non-zero morphism $R/I \to JX$ which would similarly as above restrict to a non-zero morphism $(J+I)/I \to JX$. This is, however, not possible since we assume that $\operatorname{Hom}_R((J+I)/I, M) = 0$. Thus $JX = 0$ and $X \subseteq \operatorname{Soc}_J(M)$. But then $\tilde{X} = 0$, contradicting that $\tilde{f}$ is a non-zero map. This establishes the other inclusion of the claim. In particular, we have proved that $\mathcal{C}$ is a torsion-free class.

As a torsion-free class, $\mathcal{C}$ is closed under products and (pure) submodules. To prove that $\mathcal{C}$ is definable, it remains to treat direct limits. Since $(\mathcal{T}, \mathcal{F})$ is of finite type, the torsion radical $t_{\mathcal{G}}$ commutes with direct limits. Given a directed system $M_i, i \in I$ with $M_i \in \mathcal{C}$, we compute:

$$t_{\mathcal{G}}(\varinjlim_i M_i) \simeq \varinjlim_i t_{\mathcal{G}} M_i.$$

The latter is a direct limit of R/J-modules, proving that $Jt_{\mathcal{G}}(\varinjlim_i M_i) = 0$ as desired. Finally, since $\mathcal{G}$ is faithful, we have that $R \in \mathcal{F} \subseteq \mathcal{C}$. Using Proposition 4.9.2, we infer that $\mathcal{C}$ is a 1-cotilting class.

Finally, we prove that $\mathcal{C}$ is not of cofinite type. Indeed, if it was of cofinite type, Theorem 4.6.2 would provide us with a Gabriel topology $\mathcal{H}$ of finite type such that $\mathcal{C} = \bigcap_{K \in \mathcal{H}} \operatorname{Ker} \operatorname{Hom}_R(R/K, -)$. Since Mod- $R/J \subseteq \mathcal{C}$, this implies that $J + K = R$ for all $K \in \mathcal{H}$. Since R is local, the only possibility is $\mathcal{H} = \{R\}$, which forces $\mathcal{C} =$ Mod-R. Recall that we assumed $J \in \mathcal{G}$, and thus there is a finitely generated ideal $I \in \mathcal{G}$ with $I \subseteq J$. If $R/I \in \mathcal{C}$, then necessarily $R/I = \operatorname{Soc}_J(R/I)$, which implies $I = J$. But J is a non-trivial idempotent ideal in a local commutative ring, so it cannot be finitely generated, a contradiction.

As a next step, we would like to extend the example to n-cotilting classes for $n \geq 1$. The strategy is to reverse (under suitable assumptions) the process of Lemma 4.5.7. That is, we would like to combine a hereditary faithful torsion-free class $\mathcal{F}$ and an n-cotilting class, which is *not* necessarily of cofinite type, to an $(n+1)$-cotilting class. Note that in Lemma 4.5.7 we have Inj- $R \cap \mathcal{F} \subseteq \mathcal{C}$, where Inj- $R \subseteq$ Mod-R is the class of injective modules. We will adopt this assumption for the next auxiliary result which can be viewed as an analog of [6, Lemma 3.10].

Lemma 4.9.4. *Let $\mathcal{F}$ be a hereditary torsion-free class and let $C \in$ Mod-R be a module such that the class $\mathcal{C} = {}^{\perp}C$ contains* Inj- $R \cap \mathcal{F}$. *Then the class*

$$\mathcal{D} = \{M \in \text{R-Mod} \mid M \in \mathcal{F} \text{ and } \Omega^{-1}M \in \mathcal{C}\}$$

satisfies the following property: If $0 \to L \to M \to N \to 0$ is a short exact sequence with $M \in \mathcal{D}$, then $L \in \mathcal{D}$ if and only if $N \in \mathcal{C}$.

Proof. First observe that $M \in \mathcal{D}$ if and only if $M \in \mathcal{F} \cap \mathcal{C}$ and each morphism $f\colon M \to C$ extends to $\tilde{f}\colon E(M) \to C$. Indeed, this follows at once from the long exact sequence obtained by applying $\operatorname{Hom}_R(-, C)$ to $0 \to M \to E(M) \to \Omega^{-1}(M) \to 0$.

Now let $\varepsilon\colon 0 \to L \xrightarrow{u} M \to N \to 0$ be exact with $M \in \mathcal{D}$. In particular $L \in \mathcal{F}$ and, if we apply $\operatorname{Hom}_R(-, C)$ to ε, we obtain isomorphisms $\operatorname{Ext}^i_R(L, C) \cong \operatorname{Ext}^{i+1}_R(N, C)$ for all $i \geq 1$.

Suppose next that $L \in \mathcal{D}$. In particular $L \in {}^{\perp}C$, so $\operatorname{Ext}^i_R(N, C) = 0$ for all $i \geq 2$. It remains to show that $\operatorname{Ext}^1_R(N, C) = 0$. To this end, let $v\colon L \to E(L)$ be an injective envelope and let $w\colon M \to E(L)$ be a morphism such that $wu = v$, obtained by the injectivity of $E(L)$. If $f\colon L \to C$ is any homomorphism, it extends to $\tilde{f}\colon E(L) \to C$ since $L \in \mathcal{D}$. In particular $f = \tilde{f}v = \tilde{f}wu$, showing that the leftmost morphism in the following exact sequence is surjective:

$$\operatorname{Hom}_R(M, C) \to \operatorname{Hom}_R(L, C) \to \operatorname{Ext}^1_R(N, C) \to \operatorname{Ext}^1_R(M, C) = 0$$

Thus $\operatorname{Ext}^1_R(N, C) = 0$ and $N \in \mathcal{C}$.

Conversely suppose that $N \in \mathcal{C}$. Then $L \in \mathcal{F} \cap \mathcal{C}$ by the above observations. To see that $L \in \mathcal{D}$, it remains to prove that each $f\colon L \to C$ extends to $\tilde{f}\colon E(L) \to C$. Consider the following commutative square with injective envelopes in rows, where the right vertical map is completed using the injectivity of E(M):

$$\begin{array}{ccc} L & \xrightarrow{v} & E(L) \\ {\scriptstyle u}\big\downarrow & & \big\downarrow \\ M & \xrightarrow{z} & E(M) \end{array}$$

Then $\mathrm{Hom}_R(u, C)$ is surjective since $\mathrm{Ext}^1_R(N, C) = 0$ and $\mathrm{Hom}_R(z, C)$ is surjective since $M \in \mathcal{D}$. It follows that $\mathrm{Hom}_R(v, C)$ is surjective, as required. □

Now we prove a result which, under more restrictive assumptions, allows us to combine a hereditary torsion-free class with an n-cotilting class.

Proposition 4.9.5. *Let R be a ring and $n \geq 1$. Suppose that all of the following conditions are satisfied:*

1. *$\mathcal{C}$ is an n-cotilting class in* Mod-R *and $\Omega^{-1}(R) \in \mathcal{C}$.*
2. *$(\mathcal{T}, \mathcal{F})$ is a faithful hereditary torsion pair of finite type in* Mod-R.
3. *$\bigcap_{i=0}^{n-1} \mathrm{Ker}\,\mathrm{Ext}^i_R(\mathcal{T}, -) \subseteq \mathcal{C}$.*

Then $\mathcal{D} = \{M \in \text{R-Mod} \mid M \in \mathcal{F}$ and $\Omega^{-1}M \in \mathcal{C}\}$ is an $(n+1)$-cotilting class which is closed under injective envelopes and $\mathcal{D}_{(1)} = \mathcal{C}$.

Proof. Let $C \in$ Mod-R be a cotilting module such that $\mathcal{C} = {}^{\perp}C$ and let us denote the class from condition (4) by $\mathcal{I}$. If $\mathcal{G}$ is the Gabriel topology corresponding to $(\mathcal{T}, \mathcal{F})$, we have

$$\bigcap_{i=0}^{n} \mathrm{Ker}\,\mathrm{Ext}^i_R(\mathcal{T}, -) = \bigcap_{i=0}^{n} \bigcap_{I \in \mathcal{G}^f} \mathrm{Ker}\,\mathrm{Ext}^i_R(R/I, -) = \bigcap_{i=0}^{n} \bigcap_{I \in \mathcal{G}^f} \mathrm{Ker}\, H^i(I; -).$$

Indeed, the first equality follows by an argument analogous to the one from Lemma 4.3.7 while the second equality follows from Corollary 4.3.10. Since all the $H^i(I; -)$ commute with direct products and direct limits, the class $\mathcal{I}$ is definable by [16, §§2.1–2.3].

Next we claim that $\text{Inj-}R \cap \mathcal{F} \subseteq \mathcal{I} \subseteq \mathcal{D}$. The first inclusion is trivial and to see the second one, let $M \in \mathcal{I}$ and consider the short exact sequence $0 \to M \to E(M) \to \Omega^{-1}(M) \to 0$. We must show that $M \in \mathcal{D}$. Clearly $M \in \mathcal{F}$ by assumption, and we also have $E(M) \in \mathcal{F}$. If $T \in \mathcal{T}$ is torsion, it follows that $\mathrm{Ext}^i_R(T, \Omega^{-1}(M)) \cong \mathrm{Ext}^{i+1}_R(T, M)$ for all $i \geq 0$. Thus,

$$\Omega^{-1}(M) \in \bigcap_{i=0}^{n-1} \mathrm{Ker}\,\mathrm{Ext}^i_R(\mathcal{T}, -) \subseteq \mathcal{C}$$

by condition (3), and the claim is proved.

Now we prove that $\mathcal{D}$ is $(n+1)$-cotilting by checking the assumptions of Proposition 4.9.2. If $M \in \mathcal{D}$ and $N \subseteq M$ is a pure submodule, then $M/N \in \mathcal{C}$ since $M \in \mathcal{C}$ and definable classes are closed under pure quotients; [39, Theorem 3.4.8]. Hence $N \in \mathcal{D}$ by Lemma 4.9.4. If $(M_i)_{i \in I}$ is a directed system in $\mathcal{D}$, we fix an injective module $F \in \mathcal{F}$ which cogenerates $\mathcal{F}$ (see [41, Proposition VI.3.7]) and consider the directed system of maps $(M_i \to F_{M_i} = F^{\mathrm{Hom}_R(M_i, F)})_{i \in I}$ given by Lemma 4.5.6. All the maps in the system are monomorphisms by the choice of F, and there is an exact sequence

$$0 \to \varinjlim_{i \in I} M_i \to \varinjlim_{i \in I} F_{M_i} \to \varinjlim_{i \in I} F_{M_i}/M_i \to 0.$$

As $F_{M_i} \in \mathcal{I}$ for each $i \in I$, we have $\varinjlim_{i \in I} F_{M_i} \in \mathcal{I} \subseteq \mathcal{D}$. Since $F_{M_i} \in \mathcal{D}$, we have $F_{M_i}/M_i \in \mathcal{C}$ for each $i \in I$, and so $\varinjlim_{i \in I} F_{M_i}/M_i \in \mathcal{C}$. It follows from

Lemma 4.9.4 that $\varinjlim_{i\in I} M_i \in \mathcal{D}$. Finally, since $\mathcal{D}$ is closed under products by its very definition, we have verified Proposition 4.9.2(i).

Conditions (1) and (2) imply that $R \in \mathcal{D}$. Since $\mathcal{D}$ is definable, it must contain all projective modules as well. Hence given any $M \in$ Mod-R, we have $\Omega(M) \in \mathcal{D}$ if and only if $M \in \mathcal{C}$ by Lemma 4.9.4. In particular $\mathcal{D}$ is closed under taking syzygies, and that $\mathcal{D}$ is closed under extensions follows from the Horseshoe Lemma. Thus Proposition 4.9.2(ii) holds for $\mathcal{D}$. Finally, if $M \in$ Mod-R, then $\Omega^n(M) \in \mathcal{C}$ since $\mathcal{C}$ is assumed to be n-cotilting, so $\Omega^{n+1}(M) \in \mathcal{D}$ by what we have just shown. Hence $\mathcal{D}$ is an $(n+1)$-cotilting class by Proposition 4.9.2 and we have also proved that

$$\mathcal{D}_{(1)} = \{M \in \text{Mod-R} \mid \Omega(M) \in \mathcal{D}\} = \mathcal{C}.$$

The closure of $\mathcal{D}$ under injective envelopes is obvious from the definition. $\square$

Now we formulate an easier way to apply this result for constructing n-cotilting classes not of cofinite type. The constructed classes are almost identical to what we obtained for classes of cofinite type in Lemma 4.5.8. The only difference is that the last torsion-free class in the sequence *need not* be of cofinite type (viewed as a 1-cotilting class).

Corollary 4.9.6. *Let R be a commutative ring, $n \geq 1$, and let $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1})$ be a sequence of definable torsion-free classes such that*

(i) $\mathcal{F}_i$ is hereditary for each $i = 0, 1, \dots, n-2$,

(ii) $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{n-1}$,

(iii) $\Omega^{-i}R \in \mathcal{F}_i$ for each $i = 0, 1, \dots, n-1$.

Then $\mathcal{D} = \{M \in \text{Mod-R} \mid \Omega^{-i}M \in \mathcal{F}_i \text{ for each } i = 0, 1, \dots, n-1\}$ is an n-cotilting class such that $\mathcal{D}_{(i)}$ is closed under injective envelopes for each $i = 0, 1, \dots, n-2$. In particular, $\mathcal{D}$ is of cofinite type if and only if the definable torsion-free class $\mathcal{F}_{n-1}$ is hereditary.

Proof. We will inductively apply Proposition 4.9.5 and prove the result along with the following equality:

$$\mathcal{D} = \bigcap_{i=0}^{n-1} \operatorname{Ker} \operatorname{Ext}^i_R(\mathcal{T}_i, -), \tag{4.23}$$

where for each i, $\mathcal{T}_i$ is the torsion class corresponding to $\mathcal{F}_i$.

If $n = 1$, there is nothing to prove, because a definable torsion-free class containing R is 1-cotilting by Proposition 4.9.2. If $n > 1$, we apply the inductive hypothesis to $(\mathcal{F}_1, \dots, \mathcal{F}_{n-1})$ and obtain an $(n-1)$-cotilting class

$$\mathcal{C} = \bigcap_{i=0}^{n-2} \operatorname{Ker} \operatorname{Ext}^i_R(\mathcal{T}_{i+1}, -). \tag{4.24}$$

We need to check conditions (1)–(3) of Proposition 4.9.5 for $\mathcal{F}_0$ and $\mathcal{C}$ and formula (4.23) for $\mathcal{D}$. However, (1) and (2) are straightforward and (3) follows immediately from (4.24) since $\mathcal{T}_0 \supseteq \mathcal{T}_i$ for each $i = 1, \dots, n-2$. Finally, to prove (4.23), consider $M \in \mathcal{F}_0$ and a short exact sequence $0 \to M \to E(M) \to \Omega^{-1}(M) \to 0$. Then $E(M) \in \mathcal{F}_0$ and $\operatorname{Ext}^i_R(T, \Omega^{-1}(M)) \cong \operatorname{Ext}^{i+1}_R(T, M)$ for each $T \in \mathcal{T}_0$ and $i \geq 0$. It follows that in that case $M \in \mathcal{D}$ if and only if $\Omega^{-1}(M) \in \mathcal{C}$ if and only if $\operatorname{Ext}^i_R(\mathcal{T}_i, M) = 0$ for each $i = 1, \ldots, n-1$. □

We conclude by an explicit construction of an n-cotilting class not of cofinite type, by combining Corollary 4.9.6 and Example 4.9.3.

Example 4.9.7. Let $n > 0$ and R be a local commutative ring satisfying:

(i) there is a non-trivial idempotent ideal J in R,

(ii) there is a finitely generated ideal $I \subseteq J$ satisfying $\operatorname{Ext}^i_R(R/I, R) = 0$ for all $i = 0, 1, \ldots, n-1$.

First we provide a concrete example of such a ring R. Let k be a field, and let R be the ring of Puiseux series in n variables. That is,

$$R = \bigcup_{m \in \mathbb{N}} k[[x_1^{\frac{1}{m}}, x_2^{\frac{1}{m}}, \ldots, x_n^{\frac{1}{m}}]]$$

is the ring of formal power series in n variables with exponents which are rational, but for each particular series the denominators are bounded. This ring has a unique maximal ideal J consisting of all series with zero coefficient in degree 0, and J is easily seen to be idempotent. Also, put $I = \operatorname{Span}(x_1, x_2, \ldots, x_n)$ and note that the elements $(x_1, x_2, \ldots, x_n)$ form a regular sequence. Then $K_\bullet(I)$ is a projective resolution of R/I, implying that $H^i(I; R) = 0$ for all $i = 0, 1, \ldots, n-1$. Therefore, we have $\operatorname{Ext}^i_R(R/I, R) = 0$ for all $i = 0, 1, \ldots, n-1$ by Corollary 4.3.10.

Put $\mathcal{F}_{n-1} = \{M \in \text{Mod-R} \mid M/\operatorname{Soc}_J(M) \in \operatorname{Ker}\operatorname{Hom}_R(R/I, -)\}$. Then $\mathcal{F}_{n-1}$ fits the construction of Example 4.9.3 (the Gabriel topology $\mathcal{G}$ is generated by the ideal I, and the choice of non-trivial idempotent ideal is J). This shows that $\mathcal{F}_{n-1}$ is a 1-cotilting class not of cofinite type, i.e. a definable torsion-free class containing R which is not hereditary. It also follows from the assumption on I that $\operatorname{Ker}\operatorname{Hom}_R(R/I, -) \subseteq \mathcal{F}_{n-1}$ contains $\Omega^{-i}R$ for all $i = 0, 1, \ldots, n-1$.

We put $\mathcal{F}_k = \operatorname{Ker}\operatorname{Hom}_R(R/I, -)$ for all $k = 0, 1, \ldots, n-2$, and note that those are hereditary torsion-free classes of finite type. Then it is straightforward to check that the sequence $(\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{n-1})$ satisfies conditions of Corollary 4.9.6. We conclude that

$$\mathcal{D} = \{M \in \text{Mod-R} \mid \Omega^{-i}M \in \mathcal{F}_i \text{ for all } i = 0, 1, \ldots, n-1\}$$

is an n-cotilting class such that $\mathcal{D}_{(i)}$ is closed under injective envelopes for all $i = 0, 1, \ldots, n-2$, but not for $i = n-1$. In particular, $\mathcal{D}$ is not of cofinite type.

Acknowledgement

Both authors were partially supported by the Grant Agency of the Czech Republic under the grant no. 14-15479S. The first named author was also partially supported by the project SVV-2016-260336 of the Charles University in Prague.

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