Report on the Doctoral Thesis

”Stochastic Evolution Equations”

by Petr Čoupek

This thesis mainly deals with the existence, uniqueness, smoothness and longtime properties of solutions of linear stochastic evolution equations driven by a Hilbert-valued Volterra-type noisy input

$$dX_t = AX_t dt + \Phi dB_t, \quad X_0 = x \tag{1}$$

on a separable Hilbert space V, where the operator A generates a strongly continuous semigroup S on V, $\Phi \in \mathcal{L}(U, V)$ (U is also a Hilbert space), the initial condition $x \in L^2(\Omega; V)$ and B is a U–cylindrical Volterra process. The solution X^x of (1) will be understood in the mild sense

$$X_t^x = S(t)x + \int_0^t S(t-r)\Phi dB_r, \quad t \geq 0. \tag{2}$$

The thesis is divided into an introduction, three chapters, and four short appendices on Sobolev spaces, operators (Hilbert-Schmidt and γ-radonifying operators), semigroups and Wiener Chaos.
The Introduction is very well-written. As a necessary first concept, the scalar Volterra process is introduced, which will be the basis to further construct Hilbert space-valued noise of Volterra type.
Furthermore, there is a concise description of the results that will be established in the memory: results concerning the existence and uniqueness of solutions to SEE's with Volterra type noise, regularity of the solution in terms of its Hölder regularity, and longtime behavior via the limit of the law of the solution.
The Introduction finishes with a detailed description of the contains of the following forthcoming chapters.
Chapter 1 begins with the fundamental notions of Volterra kernel and scalar Volterra process. For the latter there are 4 distinguished situations: one-sided or two-sided Volterra process, that could be *regular* or α*–regular*, depending on the set of assumptions satisfied by the corresponding Volterra kernel.
Let $\mathcal{H}$ be a real separable Hilbert space. Then the stochastic integral of $\mathcal{H}$-valued deterministic function against the Volterra process over the interval I is introduced. The first step consists of defining the integral for step functions, and via an Itô–like isometry, the definition is extended to the space of integrands denoted by $\mathcal{D}(I; \mathcal{H})$. This space is further characterized in the sense that function spaces that are subspaces of $\mathcal{D}(I; \mathcal{H})$ are studied, both for the cases of regular and α-regular Volterra processes. Moreover, basic properties of the integral are analyzed. This chapter ends with several examples which give a detailed picture of different Volterra processes, Gaussian and not Gaussian as well.
Chapter 2 treats existence, uniqueness, smoothness and longtime behavior of (1). To start with, the definition of a cylindrical Volterra process and its characterization in terms of scalar Volterra processes are given. Then the stochastic integral with a cylindrical Volterra process as integrator is constructed and, as it was done in the previous chapter, some subspaces of the space of admissible integrands are

analyzed. Further, there are a collection of results which ensures the existence of a mild solution X^x to (2), for regular and α–regular processes as well. The regularity of the solution is also investigated by using a Fubini-like theorem together with the celebrated factorization method. Under some conditions, the solution X^x to (2) is proved to belong to $C([0,T];V)$, and when S is analytic, to $C^\nu([0,T];V_\delta)$ for suitable values of the parameters ν, δ. When B is a two-sided α–regular cylindrical Volterra process with reflexible and stationary increments, the longtime behavior of X^x in terms of the existence of its limiting measure μ_∞ is addressed. This chapter ends presenting several examples, that in particular emphasize the compensation between the regularity of the kernel and that of the operator Φ.
SEE's in L^p spaces and driven by an one-side α–regular cylindrical Volterra process are investigated in the last chapter. As a first natural step, the stochastic integral is defined for elementary operators and further for γ-radonifying operators. Results regarding existence, uniqueness and regularity are considered in the case that A generates an analytic semigroup in L^p. This chapter also finishes with two illustrative examples.

In my opinion, this is an excellent thesis which covers a large amount of results for SEE's driven by Volterra processes, with clear proofs and a very interesting analysis of the regularity and longtime properties of the solutions of these equations. The exposition is self-contained and the results are applied to a wide range of examples of equations driven by Volterra processes (also the less known *non-Gaussian* Volterra processes). I have enjoyed reading this work, where clearly the author has learnt and applied very new techniques of Stochastic Analysis in order to deal with Volterra processes, starting with the understanding of the integration with respect to these processes. As far as I know, most of the described results are original, new and of a high mathematical quality.

Taking into account all my previous considerations, not only I recommend the thesis for Defense but also I consider that it deserves the best qualification in the Czech system.

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