Report on the thesis “Properties of Sobolev mappings”

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April 22, 2017

This thesis consist of three papers together with an introduction. The three articles cover two topics, the Lusin (N) condition for Sobolev mappings and the Sobolev embedding theorem.

With respect to the Lusin (N) condition, the case of first order Sobolev spaces is well understood. Hence, the author focuses on higher order Sobolev spaces. More precisely, he lies his focus on the negative results as the positive results follow from the Sobolev embedding theorem as he points out.

The Sobolev embedding theorem shows that if f is, say in the Sobolev space $W^{1,p}$ for some $1 \leq p < \infty$, then f is actually in L^q for some $q > p$. This result implies a Lusin (N) condition result for higher order Sobolev spaces. The stronger the Sobolev embedding theorem is the stronger is the Lusin (N) condition result. This can be seen as one of the motivations for the first paper. There, the author investigates the relation between the integrability of the first order weak derivative and the integrability of the function. For example in [EG92], the Gagliardo–Sobolev–Nirenberg inequality (the term Sobolev embedding theorem is a term that encompasses many different results) is stated for the Euclidean space. However, using an extension theorem, one can extend the result to Lipschitz domains. The author gives a domain where the Sobolev exponent exhibits a jump. It is now interesting to know how much worse the exponent can become. The main result of the first paper is that the gap can be arbitrarily large.

As we have seen, the Lusin (N) condition can be deduced from Sobolev embedding theorem statements. Moreover, the theorem is very important in the domain of partial differential equation, where it is used to show regularity properties of solutions.

The second and third papers study the Lusin (N) condition. It is known that functions (their continuous representatives) in $W^{1,p}(\Omega, \mathbb{R}^n)$ satisfy the Lusin (N) condition provided that $\Omega \subset \mathbb{R}^n$ is open and $p > n$. The statement is wrong for $p = n$. However, there are versions of the statement for $p = n$; one version requires the functions to be a homeomorphism. In the thesis this scenario is studied for higher order Sobolev spaces.

Moreover, the author investigates whether the Lusin (N) condition holds if additionally to being in $W^{1,n}$, the mapping is actually the derivative of a

C^1-function. Differently formulated, the mapping is assumed to be in $W^{2,n}$ but then the Lusin (N) condition is not studied for the mapping itself, but for its weak derivative. This question came up in the context of varifolds concerning the behaviour of the Gauss map.

The author briefly alludes how this result shows that certain assumptions in a result by Menne cannot be relaxed.

In my opinion, the thesis clearly shows the author's ability for creative scientific work. All three papers have the author as sole author and two of them have been published, both of them ranked B on the journal list by the Australian Mathematical Society. The papers contain many delicate estimates that the author carried out.

In this thesis, the author studies two important topics in functional analysis. The Lusin condition (N) in the form of generating mass out of nothing is important in physics, especially in nonlinear elasticity. It is of importance to know which deformations are admissible by not creating mass out of nothing. The author makes a contribution by showing in which ways the class of studied functions has to be restricted. Furthermore, the Lusin (N) condition is important for area formulas. Another application is the clarification of the assumptions in a result by Ulrich Menne.

I like the form of the thesis. The introduction gives a good overview of what to expect from the thesis. The papers are interesting contributions to the theory. All main results are new.

Here are some points the author might want to address:

Introduction

- Theorem 4: I guess $m \geq n$ is needed.

First paper:

- Remark 1.2 is confusing to me. What is the matter with max and sup?
- Proof of Lemma 3.1 (ii) and (iv): I wonder if the author could expand the proof a bit.
- (11): I could not quite see how it is possible to introduce $A_{i,j-1}$.

Second paper:

- In my opinion, Remark 1.3 is much too short to be able to convey the connection between the author's and Menne's result. For me, it was not possible to see the connection.
- In the computation of $f(b_i)$ on page 6, I get a slightly different result than the author.

Third paper:

- In equation (7), it seems to me that one should require A to be larger such that in (10), the definition of l_i makes sense.

- In equation (2), does one need $0 \leq \lambda(t) \leq 1$ as well to ensure that before (14), $\prod \lambda(|x_i|) \in (0, 1)$?

General comments

- When enumerating elements, the comma is often missing after $\ldots$
- Available is often spelled wrongly as aviable.

I would like to congratulate the author for his thesis.

References

[EG92] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.